On a model for the storage of files on a hardware I : Statistics at a fixed time and asymptotics.

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January 24, 2008

Abstract

We consider a generalized version in continuous time of the parking problem of Knuth. Files arrive following a Poisson point process and are stored on a hardware identified with the real line. We specify the distribution of the space of unoccupied locations at a fixed time and give its asymptotics when the hardware is becoming full.

Key words. Parking problem. Data storage. Random covering. Lévy processes. Regenerative sets. Queuing theory.

A.M.S. Classification. 60D05, 60G51, 68B15.

1 Introduction

We consider a generalized version in continuous time of the original parking problem of Knuth. Knuth was interested by the storage of data on a hardware represented by a circle with n spots. Files arrive successively at locations chosen uniformly among these n spots. They are stored in the first free spot at the right of their arrival point (at their arrival point if it is free). Initially Knuth worked on the hashing of data (see e.g. [9, 11, 12]) : he studied the distance between the spots where the files arrive and the spots where they are stored. Later Chassaing and Louchard [8] have described the evolution of the largest block of data in such coverings when n tends to infinity. They observed a phase transition at the stage where the hardware is almost full, which is related to the additive coalescent. Bertoin and Miermont [5] have extended these results to files of random sizes which arrive uniformly on the circle.

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We consider here a continuous time version of this model where the hardware is large and now identified with the real line. A file labelled i of length (or size) l_i arrives at time $t_i \geq 0$ at location $x_i \in \mathbb{R}$. The storage of this file uses the free portion of size l_i of the real line at the right of x_i as close to x_i as possible (see Figure 1). That is, it covers $[x_i, x_i + l_i[$ if this interval is free at time t_i. Otherwise it is shifted to the right until a free space is found and it may be split into several parts which are stored in the closest free portions. We require absence of memory for the storage of files, uniformity of the location where they arrive and identical distribution of the sizes. Thus, we model the arrival of files by a Poisson point process (PPP) : $\{(t_i, x_i, l_i) \ : \ i \in \mathbb{N}\}$ is a PPP with intensity $\mathrm{d}t \otimes \mathrm{d}x \otimes \nu(\mathrm{d}l)$ on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$. We denote $\mathrm{m} := \int_0^\infty l\nu(\mathrm{d}l)$ and assume $\mathrm{m} < \infty$. So m is the mean of the total sizes of files which arrive during a unit interval time on some interval with unit length.

We begin by constructing this random covering (Section 2.1). The first questions which arise and are treated here concern statistics at a fixed time. What is the distribution of the covering at a fixed time ? At what time the hardware becomes full ? What are the asymptotics of the covering at this saturation time ? What is the length of the largest block on a part of the hardware ?

It is quite easy to see that the hardware becomes full at a deterministic time equal to $1/\mathrm{m}$. In Section 3, we give some geometric properties of the covering and characterize the distribution of the covering at a fixed time by giving the joint distribution of the block of data straddling 0 and the free spaces on both sides of this block. The results given in this section will be useful for the problem of the dynamic of the covering considered in [1], where we characterize the evolution in time of a typical data block. Moreover, using this characterization, we determine the asymptotics of the covering $\mathcal{C}(t)$ at the saturation time $1/\mathrm{m}$ (Theorem 3).
By the same method, we determine the asymptotic regime of the hardware restricted to $[0, x]$ rescaled to $[0, 1]$ at saturation time (Theorem 4). We derive then the asymptotic of the largest block of the hardware restricted to $[0, x]$ when x tends to infinity. As expected, we recover the phase transition observed by Chassaing and Louchard in [8].

As we look at $\mathcal{C}(t)$ at a fixed t, it does not depend on the order of of files before time t. Thus if ν is finite, we can view the files which arrive before time t as customers : the size of the file l is the service time and the location x where the file arrives is the arrival time of the customer. We are then in the framework of $M/G/1$ queuing model in the stationary regime and the covering $\mathcal{C}(t)$ is the union of busy periods (see e.g. Chap 3 in [21]). Thus, results of Section 3 for finite ν follow easily from known results on $M/G/1$. For infinite ν, results are the same but busy cycle is not defined and proofs are different and proving asymptotics on random sets need results about Lévy processes. Moreover, as far as we know, the longest busy period and more generally asymptotic regimes on $[0, x]$ when x tends to infinity and t tends to the saturation time has not been considered in queuing model.

2 Preliminaries

Throughout this paper, we use the classical notation δ_x for the Dirac mass at x and $\mathbb{N} = \{1, 2, ..\}$. If $\mathcal{R}$ is a measurable subset of $\mathbb{R}$, we denote by $| \mathcal{R} |$ its Lebesgue measure and by $\mathcal{R}^{cl}$ its closure. For every $x \in \mathbb{R}$, we denote by $\mathcal{R} - x$ the set $\{y - x : x \in \mathcal{R}\}$ and

$$g_x(\mathcal{R}) = \sup\{y \leq x : y \in \mathcal{R}\}, \qquad d_x(\mathcal{R}) = \inf\{y > x : y \in \mathcal{R}\}. \tag{1}$$

If I is a closed interval of $\mathbb{R}$, we denote by $\mathcal{H}(I)$ the space of closed subset of I. We endow $\mathcal{H}(I)$ with the Hausdorff distance d_H defined for all $A, B \subset \mathbb{R}$ by :

$$d_H(A, B) = \max\big(\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\big), \quad \text{where } d(x, A) = \inf\{1 - e^{-|x-y|} : y \in A\}.$$

The topology induced by this distance is the topology of Matheron [18] : a sequence $\mathcal{R}_n$ in $\mathcal{H}(I)$ converges to $\mathcal{R}$ iff for each open set G and each compact K,

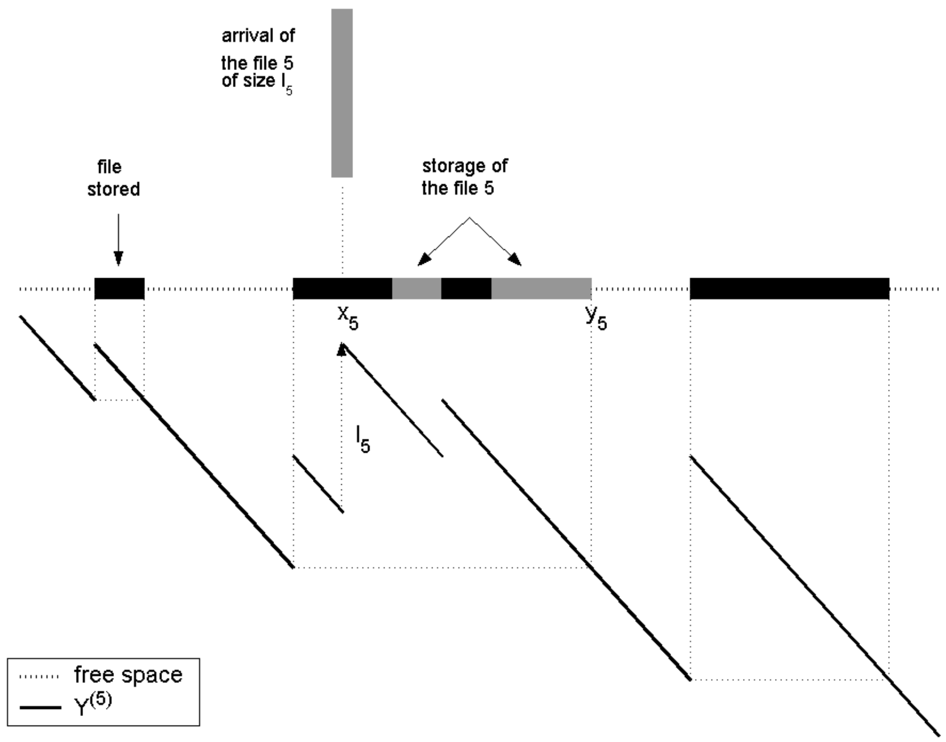
$$\begin{array}{lll} \mathcal{R} \cap G \neq \varnothing & \text{implies} & \mathcal{R}_n \cap G \neq \varnothing \quad \text{for } n \text{ large enough} \\ \mathcal{R} \cap K = \varnothing & \text{implies} & \mathcal{R}_n \cap K = \varnothing \quad \text{for } n \text{ large enough} \end{array}$$

It is also the topology induced by the Hausdorff metric on a compact using $\arctan(\mathcal{R} \cup \{-\infty, \infty\})$ or the Skorokhod metric using the class of 'descending saw-tooth functions' (see [18] and [13] for details).

2.1 Construction of the covering $\mathcal{C}(t)$

First, we present a deterministic construction of the covering $\mathcal{C}$ associated with a given sequence of files labeled by $i \in \mathbb{N}$. The file labeled by $i \in \mathbb{N}$ has size l_i and arrives after the files labeled by $j \leq i - 1$, at location x_i on the real line. Files are stored following the process described in the Introduction and $\mathcal{C}$ is the portion of line which is used for the storage. We begin by constructing the covering $\mathcal{C}^{(n)}$ obtained by considering only the first n files, so that $\mathcal{C}$ is obtained as the increasing union of these coverings. A short though (see Remark 1) enables us to see that the covering $\mathcal{C}$ does not depend on the order of arrival of the files. This construction of $\mathcal{C}$ will then be applied to the construction of our random covering at a fixed time $\mathcal{C}(t)$ by considering files arrived before time t.

Figure 1. Arrival and storage of the 5-file and representation of $Y^{(5)}$. The first four files have been stored without splitting and are represented by the black rectangles.



We define $\mathcal{C}^{(n)}$ by induction :

$$\mathcal{C}^{(0)} := \varnothing \qquad \mathcal{C}^{(n+1)} := \mathcal{C}^{(n)} \cup [x_{n+1}, y_{n+1}[$$

where $y_{n+1} = \inf\{y \geq 0, \mid \mathcal{R}^{(n)} \cap [x_{n+1}, y[| = l_{n+1}\}$ and $\mathcal{R}^{(n)}$ is the complementary set of $\mathcal{C}^{(n)}$ (i.e. the free space of the real line). So y_{n+1} is the right-most point which is used for storing the $(n+1)$-th file.

Now we consider the quantity of data over x, $R_x^{(n)}$, as the quantity of data which we have tried to store at the location x (successfully or not) when n files are stored. These data are the data fallen in $[g_x(\mathcal{R}^{(n)}), x]$ which could not be stored in $[g_x(\mathcal{R}^{(n)}), x]$, so $R_x^{(n)}$ is defined by

$$R_x^{(n)} := \sum_{\substack{i \leq n \\ x_i \in [g_x(\mathcal{R}^{(n)}), x]}} l_i \quad - \quad (x - g_x(\mathcal{R}^{(n)})).$$

This quantity can be expressed using the function $Y^{(n)}$, which sums the sizes of the files arrived at the left of a point x minus the drift term x. It is thus defined by $Y_0^{(n)} = 0$ and

$$Y_b^{(n)} - Y_a^{(n)} = \sum_{\substack{i \leq n \\ x_i \in]a, b]}} l_i \quad - \quad (b - a) \qquad \text{for } a < b.$$

Introducing also its infimum function defined for $x \in \mathbb{R}$ by $I_x^{(n)} := \inf\{Y_y^{(n)} : y \leq x\}$, we get the following expression.

Lemma 1. *For every $n \geq 1$, we have $R^{(n)} = Y^{(n)} - I^{(n)}$.*

Proof. Let $x \in \mathbb{R}$. For every $y \leq x$, the quantity of data over x is at least the quantity of data fallen in $[y, x]$ minus $y - x$, i.e.

$$R_x^{(n)} \geq \sum_{\substack{i \leq n \\ x_i \in [y,x]}} l_i \ - (x - y)$$

and by definition of $R_x^{(n)}$, we get :

$$R_x^{(n)} = \sup\{ \sum_{\substack{i \leq n \\ x_i \in [y,x]}} l_i \ - \ (x - y) : \ y \leq x\} = \sup\{Y_x^{(n)} - Y_y^{(n)} : \ y \leq x\}.$$

Then $R_x^{(n)} = Y_x^{(n)} - I_x^{(n)}$. □

As a consequence, the covered set when the first n files are stored is given by

$$\mathcal{C}^{(n)} = \{x \in \mathbb{R} : \ Y^{(n)} - I^{(n)} > 0\}. \tag{2}$$

We are now able to investigate the situation when n tends to infinity under the following mild condition

$$\forall \ L \geq 0, \quad \sum_{x_i \in [-L,L]} l_i < \infty, \tag{3}$$

which means that the quantity of data arriving on a compact set is finite. We introduce the function Y defined on $\mathbb{R}$ by $Y_0 = 0$ and

$$Y_b - Y_a = \sum_{x_i \in]a,b]} l_i \ - \ (b - a) \qquad \text{for } a < b$$

and its infimum I defined for $x \in \mathbb{R}$ by $\quad I_x := \inf\{Y_y : y \leq x\}$.

As expected, the covering $\mathcal{C} := \cup_{n \in \mathbb{N}} \mathcal{C}^{(n)} = \cup_{n \in \mathbb{N}} \{x \in \mathbb{R} : \ Y^{(n)} - I^{(n)} > 0\}$ is given by

Proposition 1. *- If $\lim_{x \to -\infty} Y_x = +\infty$, then $\mathcal{C} = \{x \in \mathbb{R} : \ Y_x - I_x > 0\} \neq \mathbb{R}$.*

- If $\liminf_{x \to -\infty} Y_x = -\infty$, then $\mathcal{C} = \{x \in \mathbb{R} : \ Y_x - I_x > 0\} = \mathbb{R}$.

Remark 1. This result ensures that the covering does not depend on the order of arrival of files.

Proof. Condition (3) ensures that $Y^{(n)}$ converges to Y uniformly on every compact set of $\mathbb{R}$.

- If $\lim_{x \to -\infty} Y_x = +\infty$, then for every $L \geq 0$, there exists $L' \geq L$ such that $I_{-L'} = Y_{-L'}$. Moreover $Y_x \leq Y_x^{(n)}$ if $x \leq 0$. So :

$$Y_{-L'}^{(n)} \overset{n \to \infty}{\longrightarrow} Y_{-L'} = I_{-L'} \quad \text{and} \quad I_{-L'} \leq I_{-L'}^{(n)} \leq Y_{-L'}^{(n)}$$

Then $I_{-L'}^{(n)} \stackrel{n\to\infty}{\longrightarrow} I_{-L'}$. As $Y^{(n)}$ converges to Y uniformly on $[-L', L']$, this entails that for every x in $[-L, L]$, $\inf\{Y_y^{(n)}, -L' \le y \le x\} \stackrel{n\to\infty}{\longrightarrow} \inf\{Y_y, -L' \le y \le x\}$. Then,

$$I_x^{(n)} = I_{-L'}^{(n)} \wedge \inf\{Y_y^{(n)}, -L' \le y \le x\} \stackrel{n\to\infty}{\longrightarrow} I_{-L'} \wedge \inf\{Y_y, -L' \le y \le x\} = I_x.$$

So $Y_x^{(n)} - I_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} Y_x - I_x$ and $Y_x^{(n)} - I_x^{(n)}$ increases when n increases since it is equal to $R_x^{(n)}$, the quantity of data over x (see Lemma 1). We conclude that there is the identity

$$\{x \in \mathbb{R},\ Y_x - I_x > 0\} = \cup_{n\in\mathbb{N}}\{x \in \mathbb{R},\ Y_x^{(n)} - I_x^{(n)} > 0\} = \mathcal{C}.$$

Moreover $-L' \notin \{x \in \mathbb{R} :\ Y_x - I_x > 0\}$, so $\mathcal{C} = \{x \in \mathbb{R},\ Y_x - I_x > 0\} \neq \mathbb{R}$.

- If $\liminf_{x\to-\infty} Y_x = -\infty$, then for every $x \in \mathbb{R}$,

$$I_x = -\infty \qquad \text{and} \qquad I_x^{(n)} \stackrel{n\to\infty}{\longrightarrow} -\infty$$

The first identity entails that $\{x \in \mathbb{R},\ Y_x - I_x > 0\} = \mathbb{R}$. As $(Y_x^{(n)})_{n\in\mathbb{N}}$ is bounded, the second one implies that there exists n in $\mathbb{N}$ such that $Y_x^{(n)} - I_x^{(n)} > 0$. Then we have also $\cup_{n\in\mathbb{N}}\{x \in \mathbb{R} :\ Y^{(n)} - I^{(n)} > 0\} = \mathbb{R}$, which gives the result. □

Finally, we can construct the random covering associated with a PPP. As the order of arrival of files has no importance, the random covering $\mathcal{C}(t)$ at time t described in Introduction is obtained by the deterministic construction above by taking the subfamily of files i which verifies $t_i \le t$.

When files arrive according to a PPP, $(Y_x)_{x\ge0}$ is a Lévy process, and we recall now some results about Lévy processes and their fluctuations which will be useful in the rest of this work.

2.2 Background on Lévy processes

The results given in this section can be found in the Chapters VI and VII in [4] (there, statements are made in terms of the dual process $-Y$). We recall that a Lévy process is càdlàg process starting from 0 which has iid increments. A subordinator is an increasing Lévy process.

We consider in this section a Lévy process $(X_x)_{x\ge0}$ which has no negative jumps (spectrally positive Lévy process). We denote by Ψ its Laplace exponent which verifies for every $\rho \ge 0$:

$$\mathbb{E}(\exp(-\rho X_x)) = \exp(-x\Psi(\rho)). \tag{4}$$

We stress that this is not the classical choice for the sign of the Laplace exponent of Lévy processes with no negative jumps and a negative drift such as the process $(Y_x)_{x\ge0}$ introduced in the previous section. However it is the classical choice for subordinators, which we will need. It is then convenient to use this same definition for all Lévy processes which appear in this text.

First, we consider the case when $(X_x)_{x\geq 0}$ has bounded variations. That is,

$$X_x := dx + \sum_{x_i \leq x} l_i,$$

where $\{(x_i, l_i) : i \in \mathbb{N}\}$ is a PPP on $[0,\infty[\times[0,\infty]$ with intensity measure $\mathrm{d}x \otimes \nu$ such that $\int_0^\infty x\nu(\mathrm{d}x) < \infty$. We call ν the Lévy measure and $d \in \mathbb{R}$ the drift. Note that $(Y_x)_{x\geq 0}$ is a subordinator iff $d \geq 0$.

Writing $\bar{\nu}$ for the tail of the measure ν, the Lévy-Khintchine formula gives

$$\Psi(\rho) = d\rho + \int_0^\infty (1 - e^{-\rho x})\nu(\mathrm{d}x), \tag{5}$$

$$\frac{\Psi(\rho)}{\rho} = d + \int_0^\infty e^{-\rho x}\bar{\nu}(x)\mathrm{d}x, \tag{6}$$

$$\Psi'(0) = d + \int_0^\infty x\nu(\mathrm{d}x), \tag{7}$$

$$\lim_{\rho\to\infty} \frac{\Psi(\rho)}{\rho} = d \quad \text{and} \quad \lim_{\rho\to\infty} (\Psi(\rho) - d\rho) = \bar{\nu}(0). \tag{8}$$

Second, we consider the case when Ψ has a right derivative at 0 with

$$\Psi'(0) < 0 \tag{9}$$

meaning that $\mathbb{E}(X_1) < 0$. And we consider the infimum process which has continuous path and the first passage time defined for $x \geq 0$ by

$$I_x = \inf\{X_y : 0 \leq y \leq x\} \quad ; \quad \tau_x = \inf\{z \geq 0 : X_z < -x\}.$$

As $-\Psi$ is strictly convex and $-\Psi'(0) > 0$, $-\Psi$ is strictly increasing from $[0,\infty[$ to $[0,\infty[$ and so is strictly positive on $]0,\infty]$. We write $\kappa : [0,\infty[\to \mathbb{R}$ for the inverse function of $-\Psi$ and we have (see [4] Theorem 1 on page 189 and Corollary 3 on page 190) :

Theorem 1. *$(\tau_x)_{x\geq 0}$ is a subordinator with Laplace exponent κ.*

Moreover the following identity holds between measures on $[0,\infty[\times[0,\infty[$:

$$x\mathbb{P}(\tau_l \in dx)dl = l\mathbb{P}(-X_x \in dl)dx. \tag{10}$$

Note that if $(X_x)_{x\geq 0}$ has bounded variations, using (8), we can write

$$\forall\ \rho \geq 0, \quad \kappa(\rho) = -\frac{\rho}{d} + \int_0^\infty (1 - e^{-\rho z})\Pi(\mathrm{d}z), \tag{11}$$

where Π is a measure on $\mathbb{R}^+$ verifying (use (8) and Wald's identity or (7)) :

$$\bar{\Pi}(0) = -\frac{\bar{\nu}(0)}{d}, \qquad \int_0^\infty x\Pi(\mathrm{d}x) = \frac{1}{d} - \frac{1}{d + \int_0^\infty x\nu(\mathrm{d}x)}. \tag{12}$$

Now we introduce the supremum process defined for $x \geq 0$ by

$$S_x := \sup\{X_y : 0 \leq y \leq x\},$$

and the a.s unique instant at which X reaches this supremum on $[0, x]$:

$$\gamma_x := \inf\{y \in [0, x] : X_y = S_x\}.$$

By duality, we have $(S_x, \gamma_x) \stackrel{d}{=} (X_x - I_x, x - g_x)$ where g_x denotes the a.s unique instant at which $(X_{x^-})_{x \geq 0}$ reaches its overall infimum on $[0, x]$ (see Proposition 3 in [4] or [3] on page 25). If T is an exponentially distributed random time with parameter $q > 0$ which is independent of X and $\lambda, \mu > 0$, then we have (use [4] Theorem 5 on page 160 and Theorem 4 on page 191) :

$$\begin{aligned}
\mathbb{E}\big(\exp(-\mu S_T - \lambda\gamma_T)\big) &= \frac{q(\kappa(\lambda + q) - \mu)}{\kappa(q)(q + \lambda + \Psi(\mu))} \\
&= \exp\Big(\int_0^\infty \mathrm{d}x \int_0^\infty \mathbb{P}(Y_x \in \mathrm{d}y)(e^{-\lambda x - \mu y} - 1)x^{-1}e^{-qx}\Big)
\end{aligned}$$

which gives

$$\mathbb{E}\big(\exp(-\mu S_\infty - \lambda\gamma_\infty)\big) = \frac{1}{\kappa'(0)}\frac{\kappa(\lambda) - \mu}{\lambda + \Psi(\mu)} = -\Psi'(0)\frac{\kappa(\lambda) - \mu}{\lambda + \Psi(\mu)} \tag{13}$$

$$\mathbb{E}\big(\exp(-\mu S_\infty)\big) = \mu\frac{\Psi'(0)}{\Psi(\mu)} \tag{14}$$

$$\mathbb{E}\big(\exp(-\lambda\gamma_\infty)\big) = \exp\Big(\int_0^\infty (e^{-\lambda x} - 1)x^{-1}\mathbb{P}(X_x > 0)\mathrm{d}x\Big) \tag{15}$$

3 Properties of the covering at a fixed time

3.1 Statement of the results

Our purpose in this section is to specify the distribution of the covering $\mathcal{C}(t)$ and we will use the characterization of Section 2.1 and results of Section 2.2. In that view, following the previous section, we consider the process $(Y_x^{(t)})_{x \in \mathbb{R}}$ associated to the PPP $\{(t_i, l_i, x_i), i \in \mathbb{N}\}$ and defined by

$$Y_0^{(t)} := 0 \quad ; \quad Y_b^{(t)} - Y_a^{(t)} = \sum_{\substack{t_i \leq t \\ x_i \in]a, b]}} l_i \quad - \quad (b - a) \quad \text{for } a < b,$$

which has independent and stationary increments, no negative jumps and bounded variation. Introducing also its infimum process defined for $x \in \mathbb{R}$ by

$$I_x^{(t)} := \inf\{Y_y^{(t)} : y \leq x\},$$

we can give now a handy expression for the covering at a fixed time and obtain that the hardware becomes full at a deterministic time equal to 1/m (see below for the proofs).

Proposition 2. *For every $t < 1/\mathrm{m}$, we have $\mathcal{C}(t) = \{x \in \mathbb{R} : \ Y_x^{(t)} > I_x^{(t)}\} \neq \mathbb{R}$ a.s. For every $t \geq 1/\mathrm{m}$, we have $\mathcal{C}(t) = \mathbb{R}$ a.s.*

To specify the distribution of $C(t)$, it is equivalent and more convenient to describe its complementary set, denoted by $\mathcal{R}(t)$, which corresponds to the free space of the hardware. By the previous proposition, there is the identity :

$$\mathcal{R}(t) = \{x \in \mathbb{R} : Y_x^{(t)} = I_x^{(t)}\}. \tag{16}$$

We begin by giving some classical geometric properties which will be useful.

Proposition 3. *For every $t \geq 0$, $\mathcal{R}(t)$ is stationary, its closure is symmetric in distribution and it enjoys the regeneration property : For every $x \in \mathbb{R}$, $(\mathcal{R}(t) - d_x(\mathcal{R}(t))) \cap [0, \infty[$ is independent of $\mathcal{R}(t) \cap] - \infty, x]$ and is distributed as $(\mathcal{R} - d_0(\mathcal{R}(t))) \cap [0, \infty[$. Moreover for every $x \in \mathbb{R}$, $\mathbb{P}(x \in \mathcal{C}(t)) = \min(1, \mathrm{m}t)$.*

Remark 2. Even though the distribution of $\mathcal{R}(t)^{cl}$ is symmetric, the processes $(\mathcal{R}(t)^{cl} : t \in [0, 1/\mathrm{m}])$ and $(-\mathcal{R}(t)^{cl} : t \in [0, 1/\mathrm{m}])$ are quite different. For example, we shall observe in [1] that the left extremity of the data block straddling 0 is a Markov process but the right extremity is not.

We want now to characterize the distribution of the free space $\mathcal{R}(t)$. For this purpose, we need some notation. The drift of the Lévy process $(Y_x^{(t)})_{x \geq 0}$ is equal to -1, its Lévy measure is equal to $t\nu$ and its Laplace exponent $\Psi^{(t)}$ is then given by (see (5))

$$\Psi^{(t)}(\rho) := -\rho + \int_0^\infty \left(1 - e^{-\rho x}\right) t\nu(\mathrm{d}x). \tag{17}$$

For sake of simplicity, we write, recalling (1),

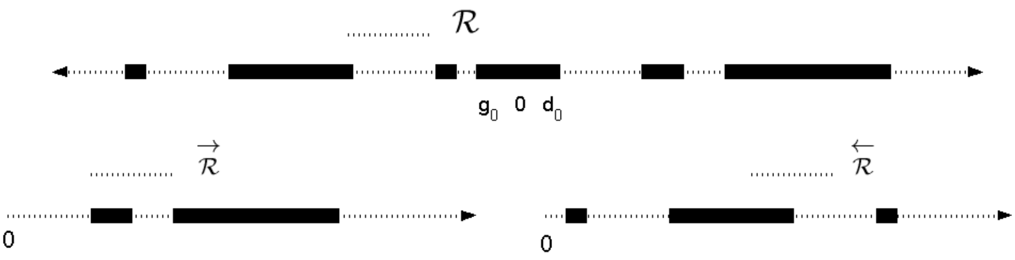
$$g(t) := g_0(\mathcal{R}(t)), \qquad d(t) = d_0(\mathcal{R}(t)), \qquad l(t) = d(t) - g(t),$$

which are respectively the left extremity, the right extremity and the length of the data block straddling 0, $\mathbf{B}_0(t)$. Note that $g(t) = d(t) = 0$ if $\mathbf{B}_0(t) = \varnothing$.
We work with $\mathcal{R}$ subsets of $\mathbb{R}$ of the form $\sqcup_{n \in \mathbb{N}} [a_n, b_n[$ and we denote by $\widetilde{\mathcal{R}} := \sqcup_{n \in \mathbb{N}} [-b_n, -a_n[$ the symmetrical of $\mathcal{R}$ with respect to 0 closed at the left, open at the right. We consider the positive part (resp. negative part) of $\mathcal{R}$ defined by

$$\overrightarrow{\mathcal{R}} := (\mathcal{R} - d_0(\mathcal{R})) \cap [0, \infty] = \bigsqcup_{n \in \mathbb{N}:\ a_n \geq d_0(\mathcal{R})} [a_n - d_0(\mathcal{R}), b_n - d_0(\mathcal{R})[,$$

$$\overleftarrow{\mathcal{R}} := \overrightarrow{\widetilde{\mathcal{R}}} = \bigsqcup_{n \in \mathbb{N}:\ b_n \leq g_0(\mathcal{R})} [g_0(\mathcal{R}) - b_n, g_0(\mathcal{R}) - a_n[.$$

Example 1. For a given $\mathcal{R}$ represented by the dotted lines, we give below $\overrightarrow{\mathcal{R}}$ and $\overleftarrow{\mathcal{R}}$, which are also represented by dotted lines. Moreover the endpoints of the data blocks containing 0 are denoted by g_0 and d_0.



Thus $\overrightarrow{\mathcal{R}(t)}$ (resp. $\overleftarrow{\mathcal{R}(t)}$) is the free space at the right of $\mathbf{B}_0(t)$ (resp. at the left of $\mathbf{B}_0(t)$, turned over, closed at the left and open at the right). We have then the identity

$$\mathcal{R}(t) = (d(t) + \overrightarrow{\mathcal{R}(t)}) \sqcup \widetilde{(-g(t) + \overleftarrow{\mathcal{R}(t)})}. \tag{18}$$

Introducing also the processes $(\overrightarrow{\tau}_x^{(t)})_{x\geq 0}$ and $(\overleftarrow{\tau}_x^{(t)})_{x\geq 0}$ defined by

$$\overrightarrow{\tau}_x^{(t)} := \inf\{y \geq 0 : |\overrightarrow{\mathcal{R}(t)} \cap [0,y]| > x\}, \qquad \overleftarrow{\tau}_x^{(t)} := \inf\{y \geq 0 : |\overleftarrow{\mathcal{R}(t)} \cap [0,y]| > x\},$$

enables us to describe $\mathcal{R}(t)$ in the following way :

Theorem 2. *(i) The random sets $\overrightarrow{\mathcal{R}(t)}$ and $\overleftarrow{\mathcal{R}(t)}$ are independent, identically distributed and independent of $(g(t), d(t))$.*
(ii) $\overrightarrow{\mathcal{R}(t)}$ and $\overleftarrow{\mathcal{R}(t)}$ are the range of the subordinators $\overrightarrow{\tau}^{(t)}$ and $\overleftarrow{\tau}^{(t)}$ respectively whose Laplace exponent $\kappa^{(t)}$ is the inverse function of $-\Psi^{(t)}$.
(iii) The distribution of $(g(t), d(t))$ is specified by :

$$(g(t), d(t)) = (-Ul(t), (1-U)l(t)),$$

$$\mathbb{P}(l(t) \in \mathrm{d}x) = (1 - \mathrm{m}t)\big(\delta_0(\mathrm{d}x) + \mathbb{1}_{\{x>0\}} x \Pi^{(t)}(\mathrm{d}x)\big)$$

where U uniform random variable on $[0,1]$ independent of $l(t)$ and $\Pi^{(t)}$ is the Lévy measure of $\kappa^{(t)}$.

We can then estimate the number of data blocks on the hardware. If ν has a finite mass, we write $N_x^{(t)}$ the number of data blocks of the hardware restricted to $[-x,x]$ at time t. This quantity has a deterministic asymptotic as x tends to infinity which is maximum at time $1/(2\mathrm{m})$. In this sense, the number of blocks of the hardware reaches a.s. its maximal at time $1/(2\mathrm{m})$. More precisely,

Corollary 1. *If $\bar{\nu}(0) < \infty$, then for every $t \in [0, 1/\mathrm{m}[$,*

$$\lim_{x\to\infty} \frac{N_x^{(t)}}{2x} = \bar{\nu}(0)t(1-\mathrm{m}t) \quad a.s.$$

Moreover, we can describe here the hashing of data. We recall that a file labelled by i is stored at location x_i. In the hashing problem, one is interested by the time needed to recover the file i knowing x_i. By stationarity, we can take $x_i = 0$. Thus we consider a file of size l which we store at time t at location 0 on the hardware whose free space space is equal to $\mathcal{R}(t)$. The first point (resp. the last point) of the hardware occupied for the storage of this file is equal to $d(t)$ (resp. to $d(t) + \vec{\tau}_l^{(t)}$). This gives the distribution of the extremities of the portion of the hardware used for the storage of a file.

Before the proofs, we make some useful observations and give examples. First, we have for every $\rho \geq 0$ (use (11)),

$$\kappa^{(t)}(\rho) = \rho + \int_0^\infty (1 - e^{-\rho x})\Pi^{(t)}(\mathrm{d}x) \tag{19}$$

and using (12)

$$\bar{\Pi}^{(t)}(0) = t\bar{\nu}(0), \qquad \int_0^\infty x\Pi^{(t)}(\mathrm{d}x) = \frac{\mathrm{m}t}{1 - \mathrm{m}t}. \tag{20}$$

Using (10), we have also the following identity of measures on $[0, \infty[\times[0, \infty[$

$$x\mathbb{P}(\vec{\tau}_l^{(t)} \in \mathrm{d}x)\mathrm{d}l = l\mathbb{P}(-Y_x^{(t)} \in \mathrm{d}l)\mathrm{d}x. \tag{21}$$

Finally, we give the distribution of the extremities of $\mathbf{B}_0$:

$$\mathbb{P}(-g(t) \in \mathrm{d}x) = \mathbb{P}(d(t) \in \mathrm{d}x) = (1 - \mathrm{m}t)\big(\delta_0(\mathrm{d}x) + \mathbb{1}_{\{x>0\}}\bar{\Pi}^{(t)}(x)\mathrm{d}x\big). \tag{22}$$

Let us consider three explicit examples

Example 2. (1) The basic example is when $\nu = \delta_1$ (all files have the same unit size as in the original parking problem in [8]). Then for all $x \in \mathbb{R}_+$ and $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}(Y_x^{(t)} + x = n) &= e^{-tx}\frac{(tx)^n}{n!}, \\ \mathbb{P}(\vec{\tau}_x^{(t)} = x + n) &= \frac{x}{x+n}e^{-t(x+n)}\frac{(t(n+x))^n}{n!}, \end{aligned} \tag{23}$$

where the second identity follows from integrating (21) on $\{(x, l) : l \in [z, z+h], x-z = n\}$ and letting h tend to 0. Then,

$$\Pi^{(t)}(n) = \frac{(tn)^n}{n.n!}e^{-tn}$$

and $l(t)$ follows a size biased Borel law :

$$\mathbb{P}(l(t) = n) = (1 - t)\frac{(tn)^n}{n!}e^{-tn}.$$

(2) An other example where calculus can be made explicitly is the gamma case when $\nu(\mathrm{d}l) = \mathbb{1}_{\{l \geq 0\}}l^{-1}e^{-l}\mathrm{d}l$. Note that $\bar{\nu}(0) = \infty$ and $\mathrm{m} = 1$. Then, for every $x \in \mathbb{R}_+$,

$$\mathbb{P}(Y_x^{(t)} \in \mathrm{d}z) = \mathbb{1}_{[-x,\infty[}(z)\Gamma(tx)^{-1}e^{-(z+x)}(z+x)^{tx-1}\mathrm{d}z,$$

$$\mathbb{P}(\overrightarrow{\tau}_x^{(t)} \in \mathrm{d}z) = \mathbb{1}_{[x,\infty[}(z)x(z\Gamma(tz))^{-1}e^{-(z-x)}(z-x)^{tz-1}\mathrm{d}z. \tag{24}$$

Further

$$\Pi^{(t)}(\mathrm{d}z) = (z\Gamma(tz))^{-1}e^{-z}z^{tz-1}\mathrm{d}z$$

and

$$\mathbb{P}(l(t) \in \mathrm{d}x) = (1-t)\big(\delta_0(\mathrm{d}x) + \Gamma(tz)^{-1}e^{-x}x^{tx-1}\mathrm{d}x\big).$$

(3) For the exponential distribution $\nu(\mathrm{d}l) = \mathbb{1}_{\{l\geq 0\}}e^{-l}\mathrm{d}l$, we can get :

$$\Psi^{(t)}(\lambda) = \lambda(-1+\frac{t}{\lambda+1}), \quad \kappa^{(t)}(\lambda) = (\lambda+t-1+\sqrt{(\lambda+t-1)^2+4\lambda})/2.$$

Finally, we specify two distributions involved in the storage of the data.

Writing $-g(t) = \gamma(t)$ (see (27) and (28)) and using the identity of fluctuation (15) gives an other expression for the Laplace transform of $g(t)$: For all $t \in [0, 1/\mathrm{m}[$ and $\lambda \geq 0$, we have

$$\mathbb{E}\big(\exp\big(\lambda g(t)\big)\big) = \exp\bigg(\int_0^\infty (e^{-\lambda x}-1)x^{-1}\mathbb{P}(Y_x^{(t)} > 0)\mathrm{d}x\bigg). \tag{25}$$

As a consequence, we see that the law of $g(t)$ is infinitively divisible. Moreover this expression will be useful to study the process $(g(t))_{t\in[0,1/\mathrm{m}[}$ in [1].

The quantity of data over 0, $R_0^{(t)}$ (see Section 2.1), is an increasing process equal to $(-I_0^{(t)})_{t\geq 0}$. Its law is given by $S(t)$ (see (27)) and, by (14), its Laplace transform is then equal to

$$\lambda \longrightarrow \frac{(1-\mathrm{m}t)\lambda}{\Psi^{(t)}(\lambda)}.$$

3.2 Proofs

Proof of Proposition 2. First $\mathrm{m} < \infty$ entails that $\forall L \geq 0$, $\sum_{t_i\leq t, x_i\in[-L,L]} l_i < \infty$ and condition (9) is satisfied a.s.

- If $t < 1/\mathrm{m}$, then $\mathbb{E}(Y_{-1}^{(t)}) = 1-\mathrm{m}t > 0$ and the càdlàg version of $(Y_{(-x)^-}^{(t)})_{x\geq 0}$ is a Lévy process. So we have (see [4] Corollary 2 on page 190) :

$$Y_x^{(t)} \overset{x\to-\infty}{\longrightarrow} \infty \quad \text{a.s.}$$

Then Proposition 1 ensures that for every $t < 1/\mathrm{m}$, $\mathcal{C}(t) = \{x \in \mathbb{R} : Y_x^{(t)} > I_x^{(t)}\} \neq \mathbb{R}$ a.s.

- If $t \geq 1/\mathrm{m}$, then $\mathbb{E}(Y_{-1}^{(t)}) \leq 0$ ensures (see [4] Corollary 2 on page 190) :

$$Y_x^{(t)} \overset{x \to -\infty}{\longrightarrow} -\infty \text{ a.s} \quad \text{or} \quad (Y_x^{(t)})_{x \leq 0} \text{ oscillates a.s in } -\infty$$

Similarly, we get that for every $t \geq 1/\mathrm{m}$, $\mathcal{C}(t) = \mathbb{R}$ a.s. □

For the other proofs, we fix $t \in [0, 1/\mathrm{m}[$, which is omitted from the notation of processes for simplicity.

To prove the next proposition and the theorem, we need to establish first a regeneration property at the right extremities of the data blocks. In that view, we consider for every $x \geq 0$, the files arrived at the left/at the right of x before time t :

$$\mathcal{P}_x := \{(t_i, x_i, l_i) : t_i \leq t,\ x_i \leq x\}, \quad \mathcal{P}^x := \{(t_i, x_i - x, l_i) : t_i \leq t,\ x_i > x\}.$$

Lemma 2. *For every $x \geq 0$, $\mathcal{P}^{d_x(\mathcal{R}(t))}$ is independent of $\mathcal{P}_{d_x(\mathcal{R}(t))}$ and distributed as $\mathcal{P}^0$.*

Proof. The simple Markov property for PPP states that for every $x \in \mathbb{R}$, $\mathcal{P}^x$ is independent of $\mathcal{P}_x$ and distributed as $\mathcal{P}^0$. Clearly this extends to simple stopping times in the filtration $\sigma(\mathcal{P}_x)_{x \in \mathbb{R}}$ and further to any stopping time in this filtration using the classical argument of approximation of stopping times by a decreasing sequence of simple stopping times (see also [19]). As $d_x(\mathcal{R}(t))$ is a stopping time in this filtration, $\mathcal{P}^{d_x(\mathcal{R}(t))}$ is independent of $\mathcal{P}_{d_x(\mathcal{R}(t))}$ and distributed as $\mathcal{P}^0$. □

Proof of Proposition 3. • The free space at the right of $d_x(\mathcal{R}(t))$ at time t is given by the point process of files arrived at the right of $d_x(\mathcal{R}(t))$ before time t. That is, there exists a measurable functional F such that for all $x \in \mathbb{R}$,

$$(\mathcal{R}(t) - d_x(\mathcal{R}(t))) \cap [0, \infty[= F\big(\mathcal{P}^{d_x(\mathcal{R}(t))}\big).$$

Similarly $\mathcal{R}(t) \cap] - \infty, x]$ is $\mathcal{P}_{d_x(\mathcal{R}(t))}$ measurable. The previous lemma ensures then that $(\mathcal{R}(t) - d_x(\mathcal{R}(t))) \cap [0, \infty[$ is independent of $\mathcal{R}(t) \cap] - \infty, x]$ and is distributed as $(\mathcal{R} - d_0(\mathcal{R}(t))) \cap [0, \infty[$.

- The stationarity of $\mathcal{C}(t)$ should be plain from the construction of the covering and the fact that the law of a PPP with intensity $\mathrm{d}x \otimes \nu$ is invariant by translation of the first coordinate. Stationarity can also be viewed as a consequence of regeneration and $\inf \mathcal{R}(t) = -\infty$ (see Remark (4.11) in [17]).

- The symmetry of $\mathcal{R}(t)^{cl}$ is a consequence of the regeneration property and stationarity (see Lemma 6.5 in [22] or Corollary (7.19) in [23]).

- As a consequence of stationarity, $\mathbb{P}(x \in \mathcal{C}(t))$ does not depend on x and is equal to $\mathbb{P}(0 \in \mathcal{C}(t))$. Following Section 2.1, we write $R_x := Y_x - I_x$ the quantity of data over x so that the quantity of data stored in $[-L, L]$ is given for every $L > 0$ by

$$| \mathcal{C}(t) \cap [-L, L] | = R_{-L} + \Big(\sum_{t_i \le t,\ x_i \in]-L,L]} l_i \Big) - R_L.$$

By invariance of the PPP $\{(t_i, x_i, l_i) : i \in \mathbb{N}\}$ by translation of the second coordinate,

$$\mathbb{P}\big((2L)^{-1} R_L \ge \epsilon\big) = \mathbb{P}\big((2L)^{-1} R_{-L} \ge \epsilon\big) = \mathbb{P}\big((2L)^{-1} R_0 \ge \epsilon\big) \stackrel{L\to\infty}{\longrightarrow} 0.$$

Moreover using (5), $(2L)^{-1} \sum_{t_i \le t,\ x_i \in]-L,L[} l_i \stackrel{L\to\infty}{\longrightarrow} \mathrm{m}t$ in probability. So

$$\mathbb{E}\big((2L)^{-1} \mid \mathcal{C}(t) \cap [-L, L] \mid \big) \stackrel{L\to\infty}{\longrightarrow} \mathrm{m}t$$

and we conclude with

$$\mathbb{E}\big(\mid \mathcal{C}(t) \cap [-L, L] \mid \big) = \mathbb{E}\Big(\int_{-L}^{L} \mathbb{1}_{\{x \in \mathcal{C}(t)\}} \mathrm{d}x \Big) = \int_{-L}^{L} \mathbb{P}(x \in \mathcal{C}(t)) \mathrm{d}x = 2L\mathbb{P}(0 \in \mathcal{C}(t)).$$

One can also give a formal argument using Theorem 1 in [22] or $\mathbb{P}(0 \in \mathcal{C}(t)) = \mathbb{P}(l(t) > 0)$ and Theorem 2. □

Proof of Theorem 2. (i) By symmetry of $\mathcal{R}(t)^{cl}$, $\overrightarrow{\mathcal{R}(t)}$ and $\overleftarrow{\mathcal{R}(t)}$ are identically distributed. The regeneration property ensures that $\overrightarrow{\mathcal{R}(t)}$ is independent of $(\overleftarrow{\mathcal{R}(t)}, g(t), d(t))$. By symmetry, $\overleftarrow{\mathcal{R}(t)}$ is independent of $(g(t), d(t), \overrightarrow{\mathcal{R}(t)})$. So $\overrightarrow{\mathcal{R}(t)}$, $\overleftarrow{\mathcal{R}(t)}$ and $(g(t), d(t))$ are independent.

(ii) As $\overrightarrow{\mathcal{R}}(t)$ is a.s. the union of intervals of the form $[a, b[$, then for every $x \ge 0$,

$$\overrightarrow{\tau}_{|\mathcal{R}(t) \cap [0,x]|} = d_x(\mathcal{R}(t)), \qquad \overrightarrow{\tau}_x = d_{\overrightarrow{\tau}_x}(\mathcal{R}(t)) \qquad a.s.$$

So the range of $\overrightarrow{\tau}$ is equal to $\overrightarrow{\mathcal{R}(t)}$. The fact $\overrightarrow{\tau}$ is a subordinator will be proved below but could be derived now from the regeneration property of $\overrightarrow{\mathcal{R}}(t)$. Similarly the range of $\overleftarrow{\tau}$ is equal to $\overleftarrow{\mathcal{R}(t)}$.
Moreover, $\mathrm{d}Y = -1$ on $\mathcal{R}(t)$ and $Y_{a-} = Y_b$ if $[a, b[$ is an interval component of $\mathcal{C}(t)$. By integrating on $[d(t), d(t) + y]$, we have a.s for every $y \ge 0$ such that $d(t) + y \in \mathcal{R}(t)$,

$$Y_{y+d(t)} - Y_y = -|\mathcal{R}(t) \cap [d(t), d(t) + y]|.$$

Then using again the definition of $\overrightarrow{\tau}$ given in Section 3.1 and that $\overrightarrow{\mathcal{R}(t)}$ is the range of $\overrightarrow{\tau}$,

$$\begin{aligned} \overrightarrow{\tau}_x &= \inf\{y \ge 0 : y \in \overrightarrow{\mathcal{R}(t)},\ |\overrightarrow{\mathcal{R}(t)} \cap [0, y]| > x\} \\ &= \inf\{y \ge 0 : d(t) + y \in \mathcal{R}(t),\ |\mathcal{R}(t) \cap [d(t), d(t) + y]| > x\} \\ &= \inf\{y \ge 0 : Y_{y+d(t)} - Y_{d(t)} < -x\} \qquad (26) \end{aligned}$$

Moreover,

$$Y_{y+d(t)} - Y_{d(t)} = -y + \sum_{\substack{(t_i,x_i,l_i)\in\mathcal{P}^{d(t)} \\ 0\le x_i\le y}} l_i$$

and Lemma 2 entails that $\mathcal{P}^{d(t)}$ is distributed as a PPP on $[0,t]\times\mathbb{R}_+\times\mathbb{R}_+$ with intensity $\mathrm{d}s\otimes\mathrm{d}x\otimes\nu(\mathrm{d}l)$. So $(Y_{y+d(t)} - Y_{d(t)})_{y\ge 0}$ is a Lévy process with bounded variation and drift -1 which verifies condition (9) (use (7) and $-1+\mathrm{m}t<0$). Then Theorem 1 entails that $\overrightarrow{\tau}$ is a subordinator whose Laplace exponent is the inverse function of $-\Psi^{(t)}$.

As $\overleftarrow{\mathcal{R}(t)}$ is distributed as $\overrightarrow{\mathcal{R}(t)}$, $\overleftarrow{\tau}$ is distributed as $\overrightarrow{\tau}$ by definition.

(iii) We determine now the distribution of $(g(t),d(t))$ using fluctuation theory, which enables us to get identities useful for the rest of the work. We write $(\widetilde{Y}_x)_{x\ge 0}$ for the càdlàg version of $(-Y_{-x})_{x\ge 0}$ and

$$S(t) := \sup\{\widetilde{Y}_x, x\ge 0\} = -I_0, \qquad \gamma(t) := \arg(S(t)) = \inf\{x\ge 0 : \widetilde{Y}_x = S(t)\}. \quad (27)$$

Using (16) and the fact that Y has no negative jumps, we have

$$\begin{aligned} g(t) &= g_0(\mathcal{R}(t)) = \sup\{x\le 0 :\ Y_x = I_x\} \\ &= \sup\{x\le 0 :\ Y_{x-} = I_0\} = -\inf\{x\ge 0 :\ \widetilde{Y}_x = -I_0\} \\ &= -\gamma(t) \end{aligned} \quad (28)$$

Using again (16) and the fact that $(Y_x)_{x\ge 0}$ is regular for $]-\infty,0[$ (see [4] Proposition 8 on page 84), we have also a.s.

$$\begin{aligned} d(t) &= \inf\{x>0 : Y_x = I_x\} = \inf\{x>0 : Y_x = I_0\} \\ &= \inf\{x>0 : Y_x < I_0\} = \inf\{x>0 : Y_x < -S(t)\} = T_{S(t)} \end{aligned}$$

where $(T_x)_{x\ge 0}$ is distributed as $(\overrightarrow{\tau}_x)_{x\ge 0}$ by (26) and $(T_x)_{x\ge 0}$ is independent of $(S(t),\gamma(t))$ since $(Y_x)_{x\ge 0}$ is independent of $(Y_x)_{x\le 0}$. Then for all $\lambda,\mu\ge 0$ with $\lambda\neq\mu$:

$$\begin{aligned} \mathbb{E}\big(\exp(\lambda g(t) - \mu d(t))\big) &= \mathbb{E}\big(\exp(-\lambda\gamma(t))\mathbb{E}(\exp(-\mu T_{S(t)}))\big) \\ &= \mathbb{E}\big(\exp(-\lambda\gamma(t) - \kappa^{(t)}(\mu)S(t))\big) \\ &= -[\Psi^{(t)}]'(0)\frac{\kappa^{(t)}(\lambda)-\kappa^{(t)}(\mu)}{\lambda-\mu} \quad \text{using (13)} \quad (29) \\ &= (1-\mathrm{m}t)\frac{\kappa^{(t)}(\lambda)-\kappa^{(t)}(\mu)}{\lambda-\mu} \quad \text{using (7)} \quad (30) \end{aligned}$$

which gives the distributions of $d(t)$, $g(t)$ and $l(t)$ letting respectively $\lambda=0$, $\mu=0$ and $\lambda\to\mu$. Computing then the Laplace transform of $(-Ul(t),(1-U)l(t))$ where U is a uniform random variable on $[0,1]$ independent of $l(t)$ gives the right hand side of (30). So $(g(t),d(t)) = (-U'l(t),(1-U')l(t))$, where U' is a uniform random variable on $[0,1]$ independent of $l(t)$. □

Remark 3. We have proved above that $\overleftarrow{\mathcal{R}(t)}$ is distributed as $\overrightarrow{\mathcal{R}(t)}$, which entails that the last passage-time-process of the post-infimum process of $(-Y_x)_{x\geq 0}$ is distributed as the first-passage-time process of $(-Y_x)_{x\geq 0}$.
This result is also a consequence of the fact that the post-infimum process of $(-Y_x)_{x\geq 0}$ is distributed as the Lévy process $(-Y_x)_{x\geq 0}$ conditioned to stay positive [20], whose last-passage-time process is a subordinator with Laplace exponent κ (see Exercise 3 on page 213 in [4]).

Proof of Corollary 1. As $\bar{\nu}(0) < \infty$, then $\bar{\Pi}(0) = t\bar{\nu}(0) < \infty$ (see (20)). So $\vec{\tau}$ is the sum of a drift and a compound Poisson process. That is, there exists a Poisson process $(N_x)_{x\geq 0}$ of intensity $t\bar{\nu}(0)$ and a sequence $(X_i)_{i\in\mathbb{N}}$ of iid variables of law $\nu/\bar{\nu}(0)$ independent of $(N_x)_{x\geq 0}$ such that

$$\vec{\tau}_x = x + \sum_{i=1}^{N_x} X_i, \qquad x \geq 0.$$

As $\overrightarrow{\mathcal{R}(t)}$ is the range of $\vec{\tau}$, the number of data blocks of $\mathcal{C}(t)$ between $d(t)$ and $d(t) + \vec{\tau}_x$ is equal to the number of jumps of $\vec{\tau}$ before x, that is N_x. Thus,

$$\frac{\text{number of data blocks in } [d(t), d(t) + \vec{\tau}_x]}{\vec{\tau}_x} = \frac{N_x}{\vec{\tau}_x} \overset{x\to\infty}{\longrightarrow} \frac{\mathbb{E}(N_1)}{\mathbb{E}(\vec{\tau}_1)} = t\bar{\nu}(0)(1 - \mathrm{m}t) \quad \text{a.s.}$$

by the law of large numbers (see [4] on page 92). This completes the proof. □

4 Asymptotics at saturation of the hardware

We focus now on the asymptotic behavior of $\mathcal{R}(t)$ when t tends to $1/\mathrm{m}$, that is when the hardware is becoming full. First, note that if ν has a finite second moment, then

$$\mathbb{E}\big(l(t)\big) = \frac{\int_0^\infty l^2 \nu(\mathrm{d}l)}{(1 - \mathrm{m}t)^2}.$$

Thus we may expect that if ν has a finite second moment, then $(1 - \mathrm{m}t)^2 l(t)$ should converge in distribution as t tends to $1/\mathrm{m}$. Indeed, in the particular case $\nu = \delta_1$ or in the conditions of Corollary 2.4 in [3], we have an expression of $\Pi^{(t)}(\mathrm{d}x)$ and we can prove that $(1 - \mathrm{m}t)^2 l(t)$ does converge in distribution to a gamma variable. More generally, we shall prove that the rescaled free space $(1 - \mathrm{m}t)^2 \mathcal{R}(t)$ converges in distribution as t tends to $1/\mathrm{m}$. In that view, we need to prove that the process $(Y^{(t)}_{(1-\mathrm{m}t)^{-2}x})_{x\in\mathbb{R}}$ converges after suitable rescaling to a random process. Thanks to (16), $(1 - \mathrm{m}t)^2\mathcal{R}(t)$ should then converge to the set of points where this limiting process coincides with its infimum process. We shall also handle the case where ν has an infinite second moment and find the correct normalization.

In queuing systems, asymptotics at saturation are known as heavy traffic approximation ($\rho = t\mathrm{m} \to 1$), which depend similarly on the tail of ν. And for ν finite, results given here could be directly derived from results in queuing theory (See III.7.2 in [10] or [16] if ν has a second moment order and [7] for heavy tail of ν). The main difference here is that ν can be infinite and we consider the whole random set of occupied locations. Moreover, as explained below, asymptotics of $\mathcal{R}(t)$ can not be directly derived from asymptotics of Y or the workload R.

Following the notation in [5], we say that $\nu \in \mathcal{D}_{2+}$ if ν has a finite second moment $\mathrm{m}_2 := \int_0^\infty l^2 \nu(\mathrm{d}l)$. For $\alpha \in]1,2]$, we say that $\nu \in \mathcal{D}_\alpha$ whenever

$$\exists C > 0 \ \text{ such that } \ \bar{\nu}(x) \overset{x\to\infty}{\sim} Cx^{-\alpha}$$

and we put for $\alpha \in]1,2[$:

$$C_\alpha := \left(\frac{C\Gamma(2-\alpha)}{\mathrm{m}_2(\alpha-1)}\right)^{1/\alpha}.$$

We denote by $(B_z)_{z\in\mathbb{R}}$ a two-sided Brownian motion, i.e. $(B_x)_{x\geq 0}$ and $(B_{-x})_{x\geq 0}$ are independent standard Brownian motions. For $\alpha \in]1,2[$, we denote by $(\sigma_z^{(\alpha)})_{z\in\mathbb{R}}$ a càdlàg process with independent and stationary increments such that $(\sigma_x^{(\alpha)})_{x\geq 0}$ is a standard spectrally positive stable Lévy process with index α :

$$\forall\ x \geq 0, \lambda \geq 0, \quad \mathbb{E}(\exp(-\lambda\sigma_x^{(\alpha)}) = \exp(x\lambda^\alpha).$$

We introduce now the following functions and processes defined for all $(t,x,z) \in [0, 1/\mathrm{m}[\times \mathbb{R}_+^* \times \mathbb{R}$ and $\alpha \in]1,2[$ by

$$\begin{aligned}
&\epsilon_{2+}(t) = (1-\mathrm{m}t)^2 && f_{2+}(x) = 1/\sqrt{x} && Y_z^{2+,\lambda} = -\lambda z + \sqrt{\mathrm{m}_2/\mathrm{m}}B_z \\
&\epsilon_2(t) = 2\frac{(1-\mathrm{m}t)^2}{-\log((1-\mathrm{m}t))} && f_2(x) = \sqrt{\log(x)/x} && Y_z^{2,\lambda} = -\lambda z + \sqrt{C/\mathrm{m}}B_z \\
&\epsilon_\alpha(t) = (1-\mathrm{m}t)^{\frac{\alpha}{\alpha-1}} && f_\alpha(x) = x^{1/\alpha-1} && Y_z^{\alpha,\lambda} = -\lambda z + C_\alpha\sigma_z^{(\alpha)}
\end{aligned}$$

and the infimum process defined for $x \in \mathbb{R}$ by $I_x^{\alpha,\lambda} := \inf\{Y_y^{\alpha,\lambda} : y \leq x\}$.

We have the following weak convergence result for the Hausdorff metric defined in Section 2.

Theorem 3. *If $\nu \in \mathcal{D}_\alpha$ ($\alpha \in [1,2] \cup \{2+\}$), then $\epsilon_\alpha(t).\mathcal{R}(t)^{cl}$ converges weakly in $\mathcal{H}(\mathbb{R})$ as t tends to $1/\mathrm{m}$ to $\{x \in \mathbb{R} : Y_x^{\alpha,1} = I_x^{\alpha,1}\}^{cl}$.*

First we prove the convergence of the Laplace exponent $\Psi^{(t)}$ after suitable rescaling as t tends to $1/\mathrm{m}$, which ensures the convergence of the Lévy process $Y^{(t)}$ after suitable rescaling (see Lemma 3). These convergences will not a priori entail the convergence of the random set $\epsilon_\alpha(t).\mathcal{R}^{cl}(t)$ since they do not entail the convergence of excursions. Nevertheless, they will entail the convergence of $\kappa^{(t)}$ since $\kappa^{(t)} \circ (-\Psi^{(t)}) = \mathrm{Id}$ (Lemma 4). Then we get the convergence of $\tau^{(t)}$ as t tends to infinity and thus of its range $\epsilon_\alpha(t).\mathcal{R}^{cl}(t)$.

Remark 4. More generally, if $\bar{\nu}$ is regularly varying at infinity with index $-\alpha \in]-1,-2[$, then we have the following weak convergence in $\mathcal{H}(\mathbb{R})$

$$x^{-1}\mathcal{R}((1-x\bar{\nu}(x))/\mathrm{m})^{cl} \overset{x\to\infty}{\Longrightarrow} \{x\in\mathbb{R} : Y_x^{\alpha,1} = I_x^{\alpha,1}\}^{cl} \quad \text{with } C=1.$$

For instance, the case $\bar{\nu}(x) \overset{x\to\infty}{\sim} cx^{-\alpha}\log(x)^{\beta}$ with $(\alpha,\beta,c) \in]1,2[\times\mathbb{R}\times\mathbb{R}_+^*$ leads to

$$\big((1-\mathrm{m}t)\log(1/(1-\mathrm{m}t))^{-\beta}\big)^{\frac{1}{\alpha-1}}\mathcal{R}(t)^{cl} \overset{t\to 1/\mathrm{m}}{\Longrightarrow} \{x\in\mathbb{R} : Y_x^{\alpha,1} = I_x^{\alpha,1}\}^{cl} \quad \text{with } C = c/(\alpha-1)^{\beta}.$$

If $\bar{\nu}$ is regularly varying at infinity with index -2, there are many cases to consider.

We get then the asymptotic of $(g(t),d(t))$:

Corollary 2. *If $\nu \in \mathcal{D}_\alpha$ ($\alpha \in [1,2]\cup\{2+\}$), then $\epsilon_\alpha(t).(g(t),d(t))$ converges weakly as t tends to $1/\mathrm{m}$ to $(\sup\{x\le 0 : Y_x^{\alpha,1} = I_0^{\alpha,1}\}, \inf\{x\ge 0 : Y_x^{\alpha,1} = I_0^{\alpha,1}\})$.*
If $\nu\in\mathcal{D}_{2+}$ (resp. $\mathcal{D}_2$), $\epsilon_\alpha(t).l(t)$ converges weakly to a gamma variable with parameter $(1/2, \mathrm{m}/(4\mathrm{m}_2))$ (resp. $(1/2,\mathrm{m}/4)$).

Remark 5. The density of data blocks of size $\mathrm{d}x$ in $\epsilon_\alpha(t).\mathcal{R}(t)^{cl}$ is equal to $\frac{\mathrm{m}t}{1-\mathrm{m}t}\Pi^{(t)}(\mathrm{d}x)$. By the previous theorem or corollary, this density converges weakly as t tends to $1/\mathrm{m}$ to the density of data block of size $\mathrm{d}x$ of the limit covering $\{x\in\mathbb{R} : Y_x^{\alpha,1} = I_x^{\alpha,1}\}^{cl}$. This limit density, denoted by $\Pi^{\alpha,1}(\mathrm{d}x)$, can be computed explicitely in the cases $\nu \in \mathcal{D}_\alpha$ ($\alpha\in\{2,2+\}$), thanks to the last corollary :

$$\Pi^{2+,1}(\mathrm{d}x) = \sqrt{\frac{\mathrm{m}}{4\pi\mathrm{m}_2 x^3}}\exp\big(-\frac{\mathrm{m}}{4\mathrm{m}_2}x\big), \qquad \Pi^{2,1}(\mathrm{d}x) = \sqrt{\frac{\mathrm{m}}{4\pi x^3}}\exp\big(-\frac{\mathrm{m}}{4}x\big).$$

Note that is also the Lévy measure of the limit covering $\{x\in\mathbb{R} : Y_x^{\alpha,1} = I_x^{\alpha,1}\}^{cl}$.

If we look at $\mathcal{C}(t)$ in a window of size x and let x tend to infinity, we observe :

Theorem 4. *If $\nu\in\mathcal{D}_\alpha$ ($\alpha\in[1,2]\cup\{2+\}$), x tends to infinity and t to $1/\mathrm{m}$ such that $1-\mathrm{m}t \sim \lambda f_\alpha(x)$ with $\lambda>0$, then $x^{-1}(\mathcal{R}(t)^{cl}\cap[0,x])$ converges weakly in $\mathcal{H}([0,1])$ to $\{x\in[0,1] : Y_x^{\alpha,\lambda} = I_x^{\alpha,\lambda}\}^{cl}$.*

Thus as in [8], we observe a phase transition of the size of largest block of data in $[0,x]$ as $x\to\infty$ according to the rate of filling of the hardware. More precisely, denoting $B_1(x,t) = \mid I_1(x,t)\mid$ where $(I_j(x,t))_{j\ge 1}$ is the sequence of component intervals of $\mathcal{C}(t)\cap[0,x]$ ranked by decreasing order of size, we have :

Corollary 3. *Let $\nu\in\mathcal{D}_\alpha$ ($\alpha\in[1,2]\cup\{2+\}$), x tend to infinity and t to $1/\mathrm{m}$:*
- If $1-\mathrm{m}t\sim\lambda f_\alpha(x)$ with $\lambda>0$, then $B_1(x,t)/x$ converges in distribution to the largest length of excursion of $(Y_x^{\alpha,\lambda} - I_x^{\alpha,\lambda})_{x\in[0,1]}$.
- If $f_\alpha(x) = o(1-\mathrm{m}t)$, then $B_1(x,t)/x \overset{\mathbb{P}}{\longrightarrow} 0$.
- If $1-\mathrm{m}t = o(f_\alpha(x))$, then $B_1(x,t)/x \overset{\mathbb{P}}{\longrightarrow} 1$.

The phase transition occurs at time t such that $1-\mathrm{m}t \sim \lambda f_\alpha(x)$ with $\lambda > 0$. The more data arrive in small files (i.e. the faster $\bar{\nu}(x)$ tends to zero as x tends to infinity), the later the phase transition occurs. The phase transition in [8] or [5] uses the bridges of the processes involved here. A consequence is that in our model, $B_1(t,x)/x$ tends to zero or one with a positive probability at phase transition, which is not the case for the parking problem in [8] or [5]. More precisely, denoting by $B_{\alpha,\lambda}$ the law of the largest length of excursion of $(Y_x^{\alpha,\lambda} - I_x^{\alpha,\lambda})_{x\in[0,1]}$, we have :

$$\forall\ (\lambda,\alpha) \in \mathbb{R}_+^* \times]1,2[\cup \{2+\}, \qquad \mathbb{P}(B_{\alpha,\lambda}=0) > 0, \qquad \mathbb{P}(B_{\alpha,\lambda}=1) > 0.$$

For the proofs of the theorems, we introduce $\Psi^{\alpha,\lambda}$ the Laplace exponent (see (4)) of $Y^{\alpha,\lambda}$ given for $y \geq 0$, $\lambda \geq 0$ and $\alpha \in]1,2[$ by

$$\Psi^{2+,\lambda}(y) = -\lambda y - \frac{\mathrm{m}_2}{\mathrm{m}}\frac{y^2}{2}, \qquad \Psi^{2,\lambda}(y) = -\lambda y - \frac{C}{\mathrm{m}}\frac{y^2}{2}, \qquad \Psi^{\alpha,\lambda}(y) = -\lambda y - (C_\alpha y)^\alpha.$$

We denote by $\mathbb{D}$ the space of càdlàg function from $\mathbb{R}_+$ to $\mathbb{R}$ which we endow with the Skorokhod topology (see [14] on page 292). First, we prove the weak convergence of $Y^{(t)}$ after suitable rescaling.

Lemma 3. *If $\nu \in \mathcal{D}_\alpha$ ($\alpha \in [1,2] \cup \{2+\}$), then for all $y \geq 0$ and $\lambda > 0$:*

$$\epsilon_\alpha(t)^{-1}\Psi^{(t)}(\epsilon_\alpha(t)(1-\mathrm{m}t)^{-1}y) \overset{t\to 1/\mathrm{m}}{\longrightarrow} \Psi^{\alpha,1}(y),$$
$$x\Psi^{((1-\lambda f_\alpha(x))/\mathrm{m})}((xf_\alpha(x))^{-1}y) \overset{x\to\infty}{\longrightarrow} \Psi^{\alpha,\lambda}(y),$$

which entail the following weak convergences of processes in $\mathbb{D}$:

$$\left(\epsilon_\alpha(t)(1-\mathrm{m}t)^{-1}Y^{(t)}_{\epsilon_\alpha(t)^{-1}y}\right)_{y\geq 0} \overset{t\to 1/\mathrm{m}}{\Longrightarrow} (Y_y^{\alpha,1})_{y\geq 0},$$

$$\left((xf_\alpha(x))^{-1}Y^{((1-\lambda f_\alpha(x))/\mathrm{m})}_{xy}\right)_{y\geq 0} \overset{x\to\infty}{\Longrightarrow} (Y_y^{\alpha,\lambda})_{y\geq 0}.$$

Remark 6. If $\bar{\nu}$ is regularly varying at infinity with index $-\alpha \in]-1,-2[$, then $\bar{\nu}(x)^{-1}\Psi^{((1-\lambda x\bar{\nu}(x))/\mathrm{m})}(x^{-1}y)$ converges to $\Psi^{\alpha,\lambda}(y)$ as x tends to infinity.

Proof of Lemma 3. Using (6), we have

$$x\Psi^{(t)}(y) = xy(\mathrm{m}t - 1 - t\int_0^\infty (1-e^{-yu})\bar{\nu}(u)\mathrm{d}u). \tag{31}$$

We handle now the different cases :

• Case $\nu \in \mathcal{D}_{2+}$. Using $|1-e^{-yu}|/y \leq u$ $(u \geq 0)$ and dominated convergence theorem gives :

$$\int_0^\infty (1-e^{-yu})\bar{\nu}(u)\mathrm{d}u \overset{y\to 0}{\sim} y\int_0^\infty u\bar{\nu}(u)\mathrm{d}u = \frac{y\mathrm{m}_2}{2}$$

which proves the first part of the lemma using (31).

• Case $\nu \in \mathcal{D}_\alpha$ with $\alpha \in]1,2[$. Using that $(u/y)^\alpha \bar{\nu}(u/y)$ is bounded, we apply dominated convergence theorem and get

$$\begin{aligned} \int_0^\infty (1-e^{-yu})\bar{\nu}(u)\mathrm{d}u &= y^{-1}\int_0^\infty (1-e^{-u})\bar{\nu}(u/y)\mathrm{d}u \\ &\overset{y\to 0}{\sim} y^{-1}C(y^{-1})^{-\alpha}\int_0^\infty (1-e^{-u})u^{-\alpha}\mathrm{d}u \\ &\overset{y\to 0}{\sim} C\frac{\Gamma(2-\alpha)}{\alpha-1}y^{\alpha-1} \end{aligned} \tag{32}$$

which proves the first part of the lemma using (31).

• Case ν is regularly varying at infinity with index $-\alpha \in]-1,-2[$. First,

$$\int_0^{1/\sqrt{y}} (1-e^{-yu})\bar{\nu}(u)\mathrm{d}u \le y\int_0^{1/\sqrt{y}} u\bar{\nu}(u)\mathrm{d}u \overset{y\to 0}{\sim} y(1/\sqrt{y})^{2-\alpha} = y^{\alpha/2}$$

Moreover for every $u > 0$, $\bar{\nu}(u/y) \overset{y\to 0}{\sim} \bar{\nu}(y)u^{-\alpha}$. Let $\delta > 0$ such that $-2 < -\alpha-\delta < -\alpha+\delta < -1$. By Potter's theorem (page 25 in [6]) ensures that for all y small enough and u large enough,

$$\frac{\bar{\nu}(u/y)}{\bar{\nu}(1/y)} \le 2\max(u^{-\alpha+\delta}, u^{-\alpha-\delta}).$$

So we can apply the dominated convergence theorem to get

$$\int_0^{1/\sqrt{y}} (1-e^{-yu})\bar{\nu}(u)\mathrm{d}u = y^{-1}\int_{\sqrt{y}}^\infty (1-e^{-u})\bar{\nu}(u/y)\mathrm{d}u \overset{y\to 0}{\sim} \frac{\Gamma(2-\alpha)}{\alpha-1}y^{-1}\bar{\nu}(1/y).$$

As $y^{\alpha/2} = o(y^{-1}\bar{\nu}(1/y))$ $(y\to 0)$, we can complete the proof with

$$\int_0^\infty (1-e^{-yu})\bar{\nu}(u)\mathrm{d}u \overset{y\to 0}{\sim} \frac{\Gamma(2-\alpha)}{\alpha-1}y^{-1}\bar{\nu}(1/y).$$

• Case $\nu \in \mathcal{D}_2$. We split the integral. First, we have

$$\int_0^{1/\sqrt{y}} (1-e^{-yu})\bar{\nu}(u)\mathrm{d}u \overset{y\to 0}{\sim} y\int_0^{1/\sqrt{y}} u\bar{\nu}(u)\mathrm{d}u \overset{y\to 0}{\sim} Cy\log(1/y)/2.$$

since $\int_0^{1/\sqrt{y}}(1-e^{-yu}+yu)\bar{\nu}(u)\mathrm{d}u = o(y\int_0^{1/\sqrt{y}} u\bar{\nu}(u)\mathrm{d}u)$. Moreover,

$$\begin{aligned} \int_{1/\sqrt{y}}^\infty (1-e^{-yu})\bar{\nu}(u)\mathrm{d}u &= y^{-1}\int_{\sqrt{y}}^\infty (1-e^{-u})\bar{\nu}(u/y)\mathrm{d}u \\ &\overset{y\to 0}{\sim} Cy\int_{\sqrt{y}}^\infty (1-e^{-u})u^{-2}\mathrm{d}u \quad \text{using } \nu \in \mathcal{D}_2 \\ &\overset{y\to 0}{\sim} Cy\int_{\sqrt{y}}^1 u^{-1}\mathrm{d}u = Cy\log(1/y)/2 \end{aligned}$$

Then

$$\int_0^\infty (1-e^{-yu})\bar\nu(u)\mathrm{d}u \overset{y\to\infty}{\sim} Cy\log(1/y)$$

which proves the first part of the lemma using (31).

These convergences ensure the convergence of the finite-dimensional distributions of the processes. The weak convergence in $\mathbb{D}$, which is the second part of the lemma, follows from Theorem 13.17 in [15]. □

In the spirit of Section 3, we introduce the expected limit set, that is the free space of the covering associated with $Y^{\alpha,\lambda}$, and the extremities of the block containing 0.

$$\mathcal{R}(\alpha,\lambda) := \{x\in\mathbb{R} : Y_x^{\alpha,\lambda} = I_x^{\alpha,\lambda}\}, \qquad g(\alpha,\lambda) := g_0(\mathcal{R}(\alpha,\lambda)), \qquad d(\alpha,\lambda) := d_0(\mathcal{R}(\alpha,\lambda)).$$

We have the following analog of Theorem 2. $\overrightarrow{\mathcal{R}(\alpha,\lambda)}$ and $\overleftarrow{\mathcal{R}(\alpha,\lambda)}$ are independent, identically distributed and independent of $(g(\alpha,\lambda), d(\alpha,\lambda))$. Moreover $\overrightarrow{\mathcal{R}(\alpha,\lambda)}$ and $\overleftarrow{\mathcal{R}(\alpha,\lambda)}$ are respectively the range of the subordinators $\overrightarrow{\tau}^{\alpha,\lambda}$ and $\overleftarrow{\tau}^{\alpha,\lambda}$, whose Laplace exponent $\kappa^{\alpha,\lambda}$ is the inverse function of $-\Psi^{\alpha,\lambda}$. Finally, using $\big[\Psi^{\alpha,\lambda}\big]'(0) = -\lambda$, the counterpart of (29) gives for $\rho,\mu\geq 0$ and $\rho\neq\mu$:

$$\mathbb{E}\big(\exp(\rho g(\alpha,\lambda) - \mu d(\alpha,\lambda))\big) = \lambda\frac{\kappa^{\alpha,\lambda}(\rho)-\kappa^{\alpha,\lambda}(\mu)}{\rho-\mu}. \tag{33}$$

The proof of these results follow the proof of Section 3.2, except for two points :

1) We cannot use the point process of files to prove the stationarity and regeneration property of $\mathcal{R}(\alpha,\lambda)$ and we must use the process $Y^{\alpha,\lambda}$ instead. The stationarity is a direct consequence of the stationarity of $\big(Y_x^{\alpha,\lambda} - I_x^{\alpha,\lambda}\big)_{x\in\mathbb{R}}$. The regeneration property is a consequence of the counterpart of Lemma 2 which can be stated as follows. For all $x\in\mathbb{R}$,

$$\big(Y^{\alpha,\lambda}_{d_x(\mathcal{R}(\alpha,\lambda))+y} - Y^{\alpha,\lambda}_{d_x(\mathcal{R}(\alpha,\lambda))}\big)_{y\geq 0} \text{ is independent of } \big(Y^{\alpha,\lambda}_{d_x(\mathcal{R}(\alpha,\lambda))-y} - Y^{\alpha,\lambda}_{d_x(\mathcal{R}(\alpha,\lambda))}\big)_{y\geq 0}$$

and distributed as $\big(Y_y^{\alpha,\lambda}\big)_{y\geq 0}$. As Lemma 2, this property is an extension to the stopping time $d_x(\mathcal{R}(\alpha,\lambda))$ of the following obvious result : $\big(Y_{x+y}^{\alpha,\lambda} - Y_x^{\alpha,\lambda}\big)_{y\geq 0}$ is independent of $\big(Y_{x-y}^{\alpha,\lambda} - Y_x^{\alpha,\lambda}\big)_{y\geq 0}$ and distributed as $\big(Y_y^{\alpha,\lambda}\big)_{y\geq 0}$.

2) It is convenient to define directly $(\overrightarrow{\tau}_x^{\alpha,\lambda})_{x\geq 0}$ by

$$\overrightarrow{\tau}_x^{\alpha,\lambda} := \inf\{y\geq 0 : Y^{\alpha,\lambda}_{d(\alpha,\lambda)+y} - Y^{\alpha,\lambda}_{d(\alpha,\lambda)} < -x\}.$$

For $\lambda>0$, $[\Psi^{\alpha,\lambda}]'(0) = -\lambda < 0$ so we can apply Theorem 1 and $\overrightarrow{\tau}^{\alpha,\lambda}$ is a subordinator whose Laplace $\kappa^{\alpha,\lambda}$ is the inverse function of $-\Psi^{\alpha,\lambda}$. Moreover its range is a.s. equal to $\overrightarrow{\mathcal{R}(\alpha,\lambda)}$, since the Lévy process $(Y^{\alpha,\lambda}_{d(\alpha,\lambda)+y} - Y^{\alpha,\lambda}_{d(\alpha,\lambda)})_{y\geq 0}$ is regular for $]-\infty,0[$ (Proposition

8 on page 84 in [4]).

To prove the theorems, we need a final lemma, which states the convergence of the Laplace exponent of $\overrightarrow{\mathcal{R}(t)}$.

Lemma 4. *If $\nu \in \mathcal{D}_\alpha$ $(\alpha \in [1,2] \cup \{2+\})$, then for all $z \geq 0$ and $\lambda > 0$,*

$$(1-\mathrm{m}t)\epsilon_\alpha(t)^{-1}\kappa^{(t)}(\epsilon_\alpha(t)z) \overset{t\to 1/\mathrm{m}}{\longrightarrow} \kappa^{\alpha,1}(z),$$
$$x f_\alpha(x)\kappa^{((1-\lambda f_\alpha(x))/\mathrm{m})}(x^{-1}z) \overset{x\to\infty}{\longrightarrow} \kappa^{\alpha,\lambda}(z).$$

Remark 7. If $\bar{\nu}$ is regularly varying at infinity of index $-\alpha \in]-1,-2[$, we have similarly

$$\bar{\nu}(x)^{-1}\kappa^{((1-\lambda x\bar{\nu}(x))/\mathrm{m})}(x^{-1}z) \overset{x\to\infty}{\longrightarrow} \kappa^{\alpha,\lambda}(z).$$

Proof. First we prove that

$$\alpha(t) \overset{t\to 1/\mathrm{m}}{\sim} \beta(t) \quad \Rightarrow \quad \kappa^{(t)}(\alpha(t)) \overset{t\to 1/\mathrm{m}}{\sim} \kappa^{(t)}(\beta(t)). \tag{34}$$

Indeed the function $u \in \mathbb{R}_+^* \mapsto \frac{1-e^{-u}}{u}$ decreases so for all $x \geq 0$ and $u, v > 0$, we have :

$$\min(\frac{u}{v}, 1) \leq \frac{1-e^{-ux}}{1-e^{-vx}} \leq \max(\frac{u}{v}, 1),$$

which gives

$$\min(\frac{\alpha(t)}{\beta(t)}, 1) \leq \frac{\int_0^\infty (1-e^{-\alpha(t)x})\Pi^{(t)}(\mathrm{d}x)}{\int_0^\infty (1-e^{-\beta(t)x})\Pi^{(t)}(\mathrm{d}x)} \leq \max(\frac{\alpha(t)}{\beta(t)}, 1)$$

and proves (34) recalling (19).

Then the first part of Lemma 3 and the identity $\kappa^{(t)} \circ (-\Psi^{(t)}) = \mathrm{Id}$ give the first part of Lemma 4. Indeed for every $y \geq 0$, $\Psi^{(t)}(\epsilon_\alpha(t)(1-\mathrm{m}t)^{-1}y) \overset{t\to 1/\mathrm{m}}{\sim} \epsilon_\alpha(t)\Psi^{\alpha,1}(y)$. So (34) entails

$$\epsilon_\alpha(t)(1-\mathrm{m}t)^{-1}y \overset{t\to 1/\mathrm{m}}{\sim} \kappa^{(t)}(-\epsilon_\alpha(t)\Psi^{\alpha,1}(y)).$$

Put $y = \kappa^{\alpha,1}(z)$ to get the first limit of the lemma and follow the same way to get the second one. □

Proof of Theorem 3. First, we prove that $\epsilon_\alpha(t).(g(t), d(t))$ converges weakly as t tends to $1/\mathrm{m}$ to $(g(\alpha,1), d(\alpha,1))$. Indeed by (30), we have

$$\mathbb{E}\big(\exp(\rho\epsilon_\alpha(t)g(t) - \mu\epsilon_\alpha(t)d(t))\big) = (1-\mathrm{m}t)\frac{\kappa^{(t)}(\epsilon_\alpha(t)\rho) - \kappa^{(t)}(\epsilon_\alpha(t)\mu)}{\epsilon_\alpha(t)(\rho-\mu)}.$$

Let $t \to 1/\mathrm{m}$ using Lemma 4 and find the right hand side of (33) to conclude.

Moreover $\epsilon_\alpha(t)\overrightarrow{\mathcal{R}(t)}^{cl}$ (resp. $\epsilon_\alpha(t)\overleftarrow{\mathcal{R}(t)}^{cl}$) converges weakly in $\mathcal{H}(\mathbb{R}_+)$ as t tends to $1/\mathrm{m}$ to $\overrightarrow{\mathcal{R}(\alpha,1)}^{cl}$ (resp. $\overleftarrow{\mathcal{R}(\alpha,1)}^{cl}$). Indeed, by Proposition (3.9) in [13], this is a consequence of the convergence of the Laplace exponent of $\epsilon_\alpha(t)\overrightarrow{\mathcal{R}(t)}$ given by Lemma 4. Informally, $\epsilon_\alpha(t)\overrightarrow{\mathcal{R}(t)}^{cl}$ is the range of $\big(\epsilon_\alpha(t)\overrightarrow{\tau}^{(t)}_{(1-\mathrm{m}t)\epsilon_\alpha(t)^{-1}z}\big)_{z\geq 0}$ whose convergence in $\mathbb{D}$ follows from Lemma 4.

We can now prove the theorem. We know from (18) that

$$\epsilon_\alpha(t)\mathcal{R}(t) = \epsilon_\alpha(t).(d(t) + \overrightarrow{\mathcal{R}(t)}) \ \sqcup \ (\epsilon_\alpha(t).\widetilde{(-g(t) + \overleftarrow{\mathcal{R}(t)})})$$

where $\epsilon_\alpha(t)\overleftarrow{\mathcal{R}(t)}$, $\epsilon_\alpha(t)(-g(t), d(t))$ and $\epsilon_\alpha(t)\overrightarrow{\mathcal{R}(t)}$ are independent by Theorem 2. Similarly

$$\mathcal{R}(\alpha,1) = (d(\alpha,1) + \overrightarrow{\mathcal{R}(\alpha,1)}) \ \sqcup \ \widetilde{(-g(\alpha,1) + \overleftarrow{\mathcal{R}(\alpha,1)})}$$

where $\overleftarrow{\mathcal{R}(\alpha,1)}$, $(-g(\alpha,1), d(\alpha,1))$ and $\overrightarrow{\mathcal{R}(\alpha,1)}$ are independent. As remarked above, we have also the following weak convergences as t tends to $1/\mathrm{m}$:

$$\epsilon_\alpha(t)\overleftarrow{\mathcal{R}(t)}^{cl} \Rightarrow \overleftarrow{\mathcal{R}(\alpha,1)}^{cl}, \quad \epsilon_\alpha(t)(-g(t), d(t)) \Rightarrow (-g(\alpha,1), d(\alpha,1)), \quad \epsilon_\alpha(t)\overrightarrow{\mathcal{R}(t)}^{cl} \Rightarrow \overrightarrow{\mathcal{R}(\alpha,1)}^{cl}.$$

So $\epsilon_\alpha(t)\mathcal{R}(t)^{cl}$ converges weakly to $\mathcal{R}(\alpha,1)^{cl}$ in $\mathcal{H}(\mathbb{R})$ as t tends to $1/\mathrm{m}$. □

Proof of Corollary 2. The first result is a direct consequence of Theorem 3. We have then

$$\epsilon_\alpha(t)l(t) \overset{t\to 1/\mathrm{m}}{\Longrightarrow} d(\alpha,1) - g(\alpha,1).$$

Moreover, as $\kappa^{2+,1} \circ (-\Psi^{2+,1}) = \mathrm{Id}$, we can compute $\kappa^{2+,1}$ and (33) gives

$$\mathbb{E}\big(\exp(-\mu(d(2+,1)) - g(2+,1))\big) = (\kappa^{2+,1})'(\mu) = \Big(\frac{1+\sqrt{1+2\frac{\mathrm{m}_2}{\mathrm{m}}\mu}}{\frac{\mathrm{m}_2}{\mathrm{m}}}\Big)'(\mu) = \frac{1}{\sqrt{-1+2\frac{\mathrm{m}_2}{\mathrm{m}}\mu}}.$$

So, by identification of Laplace transform, $d(\alpha,1) - g(\alpha,1)$ is a gamma variable of parameter $(1/2, \mathrm{m}/(4\mathrm{m}_2))$ and we get the result. The argument is similar in the case $\alpha = 2$. □

Proof of Theorem 4. The argument is similar to that of the proof the previous theorem and use the others limits of Lemma 4 to get that if $x \to \infty$ and $1 - \mathrm{m}t \sim \lambda f_\alpha(x)$ with $\lambda > 0$, then $x^{-1}\mathcal{R}(t)$ converges weakly in $\mathcal{H}(\mathbb{R})$ to $\{x \in \mathbb{R} : Y^{\alpha,\lambda}_x = I^{\alpha,\lambda}_x\}^{cl}$. The theorem follows by restriction to $[0,1]$. □

To prove the corollary of Theorem 4, we need the following result.

Lemma 5. *The largest length of excursion of* $(Y^{\alpha,\lambda}_x - I^{\alpha,\lambda}_x)_{x\in[0,1]}$*, denoted by* $B_{\alpha,\lambda}$*, converges in probability to* 0 *as* λ *tends to infinity and to* 1 *as* λ *tends to* 0*.*

Proof. • Let $0 \le a < b \le 1$. Note that for all $\lambda' \ge 1$ and $x \ge 0$, $Y^{\alpha,\lambda'}_x - Y^{\alpha,1}_x = (1-\lambda')x$ ensures that $I^{\alpha,\lambda'}_x - I^{\alpha,1}_x \ge (1-\lambda')x$. Then,

$$Y^{\alpha,\lambda'}_{a+2\frac{b-a}{3}} - I^{\alpha,\lambda'}_{a+\frac{b-a}{3}} \le Y^{\alpha,1}_{a+2\frac{b-a}{3}} - I^{\alpha,1}_{a+\frac{b-a}{3}} + (1-\lambda')\frac{b-a}{3}.$$

So a.s there exists λ' such that

$$Y^{\alpha,\lambda'}_{a+2\frac{b-a}{3}} < I^{\alpha,\lambda'}_{a+\frac{b-a}{3}}.$$

As $Y^{\alpha,\lambda'}$ has no negative jumps, it reaches its infimum on $]-\infty, 2(b-a)/3]$ in a point $c \in [a+(b-a)/3, a+2(b-a)/3]$. Then a.s there exists $c \in [a+(b-a)/3, a+2(b-a)/3]$ and $\lambda' > 0$ such that $c \in \mathcal{R}(\alpha,\lambda')$, which entails that c does not belong to the interior of $B_{\alpha,\lambda'}$. Adding that $B_{\alpha,\lambda}$ decreases as λ increases, this property ensures that $B_{\alpha,\lambda}$ converges in probability to 0 as λ tends to infinity.

• As $(Y^{\alpha,0}_x)_{x\in\mathbb{R}}$ oscillates when x tends to $-\infty$ (see [4] Corollary 2 on page 190), then

$$I^{\alpha,\lambda}_0 \overset{\lambda\to 0}{\longrightarrow} -\infty,$$

which ensures that $B_{\alpha,\lambda}$ converges in probability to 1 as λ tends to 0. □

Proof of Corollary 3. The first result is a direct consequence of Theorem 4.

If $o(1-\mathrm{m}t) = f_\alpha(x) \quad (x\to\infty)$, then for every $\lambda > 0$ and x large enough, $t \le (1-\lambda f_\alpha(x))/\mathrm{m}$ and

$$B_1(x,t)/x \le B_1(x, \frac{1-\lambda f_\alpha(x)}{\mathrm{m}})/x.$$

The right hand side converges weakly to $B_{\alpha,\lambda}$ as x tends to infinity. Letting λ tend to infinity, the lemma above entails that $B_1(x,t)/x \overset{x\to\infty}{\longrightarrow} 0$ in $\mathbb{P}$.

Similarly if $1-\mathrm{m}t = o(f_\alpha(x)) \quad (x\to\infty)$, then for every $\lambda > 0$ and x large enough,

$$B_1(x,t)/x \ge B_1(x, \frac{1-\lambda f_\alpha(x)}{\mathrm{m}})/x.$$

Letting λ tend to 0, Lemma 5 entails that $B_1(x,t)/x \overset{x\to\infty}{\longrightarrow} 1$ in $\mathbb{P}$. □

Acknowledgments : I wish to thank Jean Bertoin for introducing me in this topic and guiding me along the different steps of this work.
I am very grateful to three anonymous referees for their insightful comments and for pointing at some important connections which were missed in the first place.

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