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Thème NUM

Rapport de recherche

Derivation of a non-hydrostatic shallow water model;
Comparison with Saint-Venant and Boussinesq systems

Jacques Sainte-Marie[*][†], Marie-Odile Bristeau[†]

Abstract: From the free surface Navier-Stokes system, we derive the non-hydrostatic Saint-Venant system for the shallow waters including friction and viscosity. The derivation leads to two formulations of growing complexity depending on the level of approximation chosen for the fluid pressure. The obtained models are compared with the Boussinesq models.

Key-words: Navier-Stokes equations, Saint-Venant equations, Boussinesq equations, Free surface, Dispersive terms

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Dérivation d'un modèle de type Saint-Venant non hydrostatique; Comparaison avec les modèles de type Boussinesq

Résumé : A partir des équations de Navier-Stokes à surface libre, on obtient deux modèles moyennés sur la verticale, non hydrostatiques qui étendent le système de Saint-Venant et incluent le frottement et la viscosité. La complexité des formulations obtenues dépend du niveau d'approximation retenu pour la pression du fluide. Les modèles obtenus sont comparés aux formulations de type Boussinesq.

Mots-clés : Equations de Navier-Stokes, équations de Saint-Venant, équations de Boussinesq, surface libre, termes dispersifs

Contents

1 Introduction

Despite the available numerical results obtained by the simulation of the Navier-Stokes equations, there exists a demand for models of reduced complexity such as shallow waters type models.

Non-linear shallow water equations model the dynamics of a shallow, rotating layer of homogeneous incompressible fluid and are typically used to describe vertically averaged flows in two or three dimensional domains, in terms of horizontal velocity and depth variation, see Fig. 1. This set of equations is particularly well-suited for the study and numerical simulations of a large class of geophysical phenomena, such as rivers, coastal domains, oceans, or even run-off or avalanches when modified with adapted source terms [7].

The classical Saint-Venant system [3] with viscosity and friction [14, 17, 13] is well suited for modeling of dam breaks or hydraulic jump but due to the hydrostatic assumption it is not well adapted for the modeling of gravity waves propagation.

For the modeling of long wavelength, small amplitude, gravity waves, the Boussinesq system [8, 9, 10] is used. The Boussinesq equations are obtained from the Euler equations i.e. ignoring rotational and dissipative effects [4, 11, 12, 19, 20, 24]. In practice, the use of such models ignoring rotational and friction effects at the bottom may be very restrictive. Furthermore, even when well posed, the Boussinesq models often exhibit a lack of conservation energy that is odd since they are derived from Euler equations [5, 6].

The objective of this paper is twofold. First, we want to extend the Saint-Venant system so that the long waves propagation can be modeled and second we aim at comparing/unifying the obtained formulation with the Boussinesq system, see Fig. 1. The paper is organized as follows. In section 2, we recall the Navier-Stokes system with a free moving boundary and its closure. We also
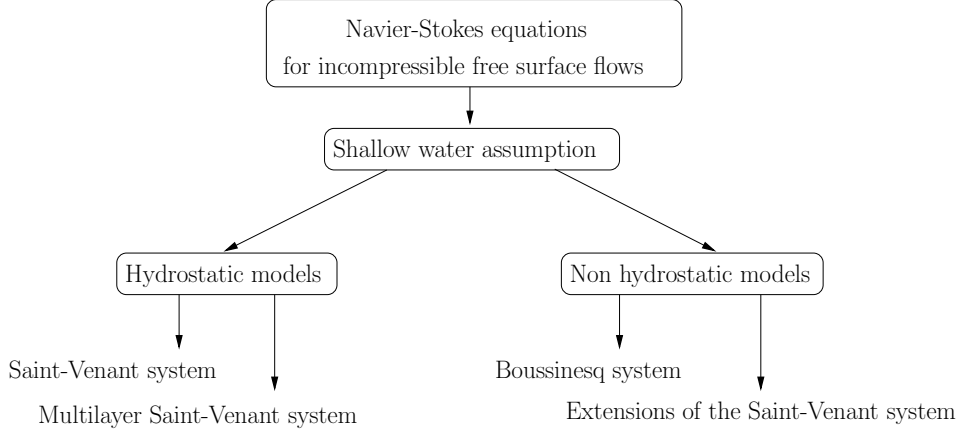
Figure 1: Averaged models derived from Navier-Stokes equations.

present the Saint-Venant and Boussinesq assumptions and the associated rescaling. In section 3 we recall the Shallow Water system and show the hydrostatic Boussinesq system assumption corresponds to the classical Saint-Venant system. In section 4, the hydrostatic assumption is relaxed and we obtain two formulations of growing complexity extending the Saint-Venant system and depending on the level of approximation chosen for the fluid pressure.

2   The Navier-Stokes system

Let start with the Navier-Stokes system [16] restricted to two dimensions with gravity in which the z axis represents the vertical direction. For simplicity, the viscosity will be kept constant throughout the paper. Therefore we have the following general formulation expression:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial \Sigma_{xx}}{\partial x} + \frac{\partial \Sigma_{xz}}{\partial z}, \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = -g + \frac{\partial \Sigma_{zx}}{\partial x} + \frac{\partial \Sigma_{zz}}{\partial z}, \tag{3}$$

and we consider this system for

$$t > t_0, \quad x \in \mathbb{R}, \quad z_b(x,t) \leq z \leq \eta(x,t),$$

where $\eta(x,t)$ represents the free surface elevation, $\mathbf{u} = (u,w)^T$ the horizontal and vertical velocities. The water height is $H = \eta - z_b$, see Fig. 2. We consider the bathymetry z_b can vary with respect to abscissa x and also with respect to time t. The chosen form of the viscosity tensor is

$$\Sigma_{xx} = 2\nu\frac{\partial u}{\partial x}, \qquad \Sigma_{xz} = \nu\big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\big),$$

$$\Sigma_{zz} = 2\nu\frac{\partial w}{\partial z}, \qquad \Sigma_{zx} = \nu\big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\big),$$

with ν the viscosity coefficient. For a more complex form of the viscosity tensor using eddy and bulk viscosities, the reader can refer to [15].
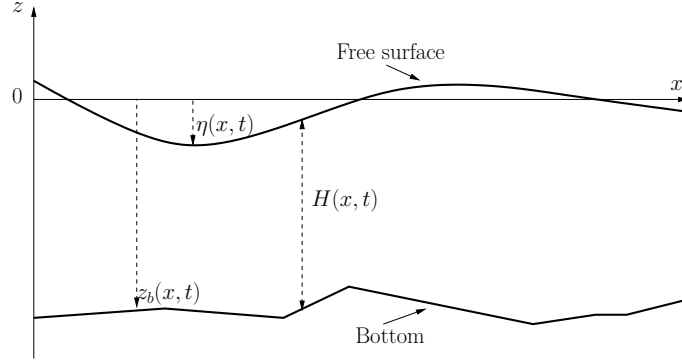

Figure 2: Notations: water height $H(x,t)$, free surface $\eta(x,t)$ and bottom $z_b(x,t)$.

2.1 Boundary conditions

The system (1)-(3) is complete with boundary conditions. The outward and upward unit normals to the free surface $\mathbf{n}_s$ and to the bottom $\mathbf{n}_b$ are given by

$$\mathbf{n}_s = \frac{1}{\sqrt{1 + \left(\frac{\partial \eta}{\partial x}\right)^2}} \begin{pmatrix} -\frac{\partial \eta}{\partial x} \\ 1 \end{pmatrix}, \quad \mathbf{n}_b = \frac{1}{\sqrt{1 + \left(\frac{\partial z_b}{\partial x}\right)^2}} \begin{pmatrix} -\frac{\partial z_b}{\partial x} \\ 1 \end{pmatrix}.$$

Let Σ_T be the total stress tensor with

$$\Sigma_T = -\frac{1}{\rho} p I_d + \begin{pmatrix} \Sigma_{xx} & \Sigma_{xz} \\ \Sigma_{zx} & \Sigma_{zz} \end{pmatrix}.$$

2.1.1 At the free surface

Classically at the free surface we have the kinematic boundary condition

$$\frac{\partial \eta}{\partial t} + u_s \frac{\partial \eta}{\partial x} - w_s = 0, \tag{4}$$

where the subscript s denotes the value of the considered quantity at the free surface. Considering the air viscosity is negligible, the continuity of stresses at the free boundary imposes

$$\Sigma_T \mathbf{n}_s = -\frac{p^a}{\rho} \mathbf{n}_s, \tag{5}$$

where $p^a = p^a(x,t)$ is a given function corresponding to the atmospheric pressure. Relation (5) is equivalent to

$$\mathbf{n}_s . \Sigma_T \mathbf{n}_s = -\frac{p^a}{\rho}, \quad \text{and} \quad \mathbf{t}_s . \Sigma_T \mathbf{n}_s = 0,$$

$\mathbf{t}_s$ being orthogonal to $\mathbf{n}_s$.

2.1.2   At the bottom

Since we consider the bottom can vary with respect to time t, the kinematic boundary condition is

$$\frac{\partial z_b}{\partial t} + u_b \frac{\partial z_b}{\partial x} - w_b = 0, \tag{6}$$

where the subscript b denotes the value of the considered quantity at the bottom and $(x,t) \mapsto z_b(x,t)$ is a given function. Note that Eq. (6) reduces to a classical no-penetration condition when z_b does not depend on time t.

For the stresses at the bottom we consider a wall law under the form

$$\Sigma_T \mathbf{n_b} - (\mathbf{n}_b . \Sigma_T \mathbf{n}_b)\mathbf{n}_b = \kappa(\mathbf{v_b}, H)\mathbf{v}_b, \tag{7}$$

with $\mathbf{v}_b = \mathbf{u}_b - (0, \frac{\partial z_b}{\partial t})^T$ the relative velocity between the water and the bottom. If $\kappa(\mathbf{v_b}, H)$ is constant then we recover a Navier friction condition as in [14]. Introducing laminar k_l and turbulent k_t friction, we use the expression

$$\kappa(\mathbf{v_b}, H, \nu) = k_l + k_t H |\mathbf{v_b}|,$$

corresponding to the boundary condition used in [17]. Another form of $\kappa(\mathbf{v_b}, H)$ is used in [7] and for other wall laws, the reader can also refer to [18]. Due to thermomechanical considerations, in the sequel we suppose $\kappa(\mathbf{v_b}, H) \geq 0$ and $\kappa(\mathbf{v_b}, H)$ is often simply denoted κ.

Let $\mathbf{t}_b$ satisfying $\mathbf{t}_b . \mathbf{n}_b = 0$ then when multiplied by $\mathbf{t}_b$ and $\mathbf{n}_b$, Eq. (7) leads to

$$\mathbf{t}_b . \Sigma_T \mathbf{n}_b = \kappa \mathbf{v}_b . \mathbf{t}_b, \quad \text{and} \quad \mathbf{v}_b . \mathbf{n}_b = 0.$$

Remark 1 *If the boundary condition (7) was written under the form $\Sigma_T . n_b = \kappa(\mathbf{v_b}, H)\mathbf{v}_b$ as in Ferrari et al. [13, Eq. (2.25), p. 217], then in absence of friction and viscosity, this would give*

$$p_b = 0,$$

that is not correct.

2.2   The rescaled system

The physical system is rescaled using the quantities

- h and λ, two characteristic dimensions along the z and x axis respectively,

- a_s the typical wave amplitude, a_b the typical bathymetry variation,

- $C = \sqrt{gh}$ the typical horizontal wave speed.

Classically for the derivation of the Saint-Venant system, we introduce the small parameter

$$\varepsilon = \frac{h}{\lambda}.$$

When considering long waves propagation, another important parameter needs be considered, namely

$$\delta = \frac{a_s}{h},$$

and we consider for the bathymetry $\frac{a_b}{h} = \mathcal{O}(\delta)$. Depending on the application, δ can be considered or not as a small parameter. For finite amplitude wave theory and assuming $z_b(x,t) = z_b^0$, one considers $\varepsilon \ll 1$, $\delta = \mathcal{O}(1)$ whereas the Boussinesq waves theory requires

$$\delta \ll 1, \quad \varepsilon \ll 1 \quad \text{and} \quad U_r = \mathcal{O}(1).$$

where U_r is the Ursell number defined by $U_r = \frac{\delta}{\varepsilon^2}$, see [23]. All along this work, we consider $\varepsilon \ll 1$ whereas, even if the parameter δ is introduced in the rescaling, the assumption $\delta \ll 1$ is not considered (paragraphs 4.1, 4.2 and 4.3) except when explictly mentioned.

As for the Saint-Venant system [14, 17], we introduce some characteristic quantities : $T = \lambda/C$ for the time, $W = a_s/T = a_b/T = \varepsilon\delta C$ for the vertical velocity, $U = W/\varepsilon = \delta C$, for the horizontal velocity, $P = \rho C^2$ for the pressure. This leads to the following dimensionless quantities

$$\tilde{x} = \frac{x}{\lambda}, \quad \tilde{z} = \frac{z}{h}, \quad \tilde{\eta} = \frac{\eta}{a_s}, \quad \tilde{t} = \frac{t}{T},$$

$$\tilde{p} = \frac{p}{P}, \quad \tilde{u} = \frac{u}{U}, \quad \text{and} \quad \tilde{w} = \frac{w}{W}.$$

Note that the definition of the charateristic velocites implies $\delta = \frac{U}{C}$ so δ also corresponds to the Froude number. When $\delta = \mathcal{O}(1)$ we have $U \approx C$ and we recover the classical rescaling used for the Saint-Venant system. For the bathymetry z_b we write $z_b(x,t) = Z_b(x) + b(t)$ and we introduce $\tilde{z}_b = Z_b/h$ and $\tilde{b} = b/a_b$. This leads to

$$\frac{\partial z_b}{\partial t} = \varepsilon\delta C \frac{\partial \tilde{b}}{\partial \tilde{t}} = W \frac{\partial \tilde{b}}{\partial \tilde{t}}, \quad \text{and} \quad \frac{\partial z_b}{\partial x} = \varepsilon \frac{\partial \tilde{z}_b}{\partial \tilde{x}}.$$

The different rescaling applied to the time and space derivatives of z_b means that a classical shallow water assumption is made concerning the space variations of the bottom profile whereas we assume the time variations of z_b lie in the framework of the Boussinesq assumption and are consistent with the rescaling applied to the velocity w.

We also introduce $\tilde{\nu} = \frac{\nu}{\lambda C}$ and we set $\tilde{\kappa} = \frac{\kappa}{C}$. Note that the definitions for the dimensionless quantities are consistent with the one used for the Boussinesq system [20, 24]. Note also that the rescaling used by Nwogu [19] differs from the preceding one since Nwogu uses $\tilde{w} = \frac{\varepsilon^2}{W} w$.

As in [14, 17], we suppose we are in the following asymptotic regime

$$\tilde{\nu} = \varepsilon\nu_0, \qquad \text{and} \qquad \tilde{\kappa} = \varepsilon\kappa_0,$$

with $\kappa_0 = \kappa_{l,0} + \varepsilon\kappa_{t,0}(\tilde{\mathbf{v}}_b, \tilde{H})$, $\kappa_{l,0}$ being constant.

This non-dimensionalization of the system (1)-(3) leads to

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0, \tag{8}$$

$$\varepsilon\delta\frac{\partial \tilde{u}}{\partial \tilde{t}} + \varepsilon\delta^2\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \varepsilon\delta^2\tilde{w}\frac{\partial \tilde{u}}{\partial \tilde{z}} + \varepsilon\frac{\partial \tilde{p}}{\partial \tilde{x}} = \varepsilon^2\delta\frac{\partial}{\partial \tilde{x}}\left(2\nu_0\frac{\partial \tilde{u}}{\partial \tilde{x}}\right)$$

$$+ \frac{\partial}{\partial \tilde{z}}\left(\delta\nu_0\frac{\partial \tilde{u}}{\partial \tilde{z}} + \varepsilon^2\delta\nu_0\frac{\partial \tilde{w}}{\partial \tilde{x}}\right), \tag{9}$$

$$\varepsilon^2\delta\left(\frac{\partial \tilde{w}}{\partial \tilde{t}} + \delta\tilde{u}\frac{\partial \tilde{w}}{\partial \tilde{x}} + \delta\tilde{w}\frac{\partial \tilde{w}}{\partial \tilde{z}}\right) + \frac{\partial \tilde{p}}{\partial \tilde{z}} = -1$$

$$+ \frac{\partial}{\partial \tilde{x}}\left(\varepsilon\delta\nu_0\frac{\partial \tilde{u}}{\partial \tilde{z}} + \nu_0\varepsilon^3\delta\frac{\partial \tilde{w}}{\partial \tilde{x}}\right) + \varepsilon\delta\frac{\partial}{\partial \tilde{z}}\left(2\nu_0\frac{\partial \tilde{w}}{\partial \tilde{z}}\right), \tag{10}$$

with the boundary conditions (4), (5), (6) and (7) becoming

$$\frac{\partial \tilde{\eta}}{\partial \tilde{t}} + \delta\tilde{u}_s\frac{\partial \tilde{\eta}}{\partial \tilde{x}} - \tilde{w}_s = 0, \tag{11}$$

$$2\varepsilon\delta\nu_0\left.\frac{\partial \tilde{w}}{\partial \tilde{z}}\right|_s - \tilde{p}_s - \varepsilon^2\delta^2\nu_0\frac{\partial \tilde{\eta}}{\partial \tilde{x}}\left(\left.\frac{\partial \tilde{u}}{\partial \tilde{z}}\right|_s + \varepsilon^2\left.\frac{\partial \tilde{w}}{\partial \tilde{x}}\right|_s\right) = -\delta\tilde{p}^a, \tag{12}$$

$$\delta\nu_0\left(\left.\frac{\partial \tilde{u}}{\partial \tilde{z}}\right|_s + \varepsilon^2\left.\frac{\partial \tilde{w}}{\partial \tilde{x}}\right|_s\right) - \varepsilon\delta\frac{\partial \tilde{\eta}}{\partial \tilde{x}}\left(2\varepsilon\delta\nu_0\left.\frac{\partial \tilde{w}}{\partial \tilde{z}}\right|_s - \tilde{p}_s\right) = \varepsilon\delta^2\frac{\partial \tilde{\eta}}{\partial \tilde{x}}\tilde{p}^a, \tag{13}$$

$$\frac{\partial \tilde{b}}{\partial \tilde{t}} + \tilde{u}_b\frac{\partial \tilde{z}_b}{\partial \tilde{x}} - \tilde{w}_b = 0, \tag{14}$$

$$\delta\nu_0\left(\varepsilon^2\left.\frac{\partial \tilde{w}}{\partial \tilde{x}}\right|_b + \left.\frac{\partial \tilde{u}}{\partial \tilde{z}}\right|_b\right) - \varepsilon\frac{\partial \tilde{z}_b}{\partial \tilde{x}}\left(2\varepsilon\delta\nu_0\left.\frac{\partial \tilde{u}}{\partial \tilde{x}}\right|_b - p_b\right)$$

$$+ \varepsilon\frac{\partial \tilde{z}_b}{\partial \tilde{x}}\left(2\varepsilon\delta\nu_0\left.\frac{\partial \tilde{w}}{\partial \tilde{z}}\right|_b - p_b - \varepsilon\nu_0\frac{\partial \tilde{z}_b}{\partial \tilde{x}}\left(\delta\left.\frac{\partial \tilde{u}}{\partial \tilde{z}}\right|_b + \varepsilon^2\delta\left.\frac{\partial \tilde{w}}{\partial \tilde{x}}\right|_b\right)\right)$$

$$= \varepsilon\delta\kappa_0\sqrt{1 + \varepsilon^2\left(\frac{\partial \tilde{z}_b}{\partial \tilde{x}}\right)^2}\left(\tilde{u}_b + \varepsilon^2\frac{\partial \tilde{z}_b}{\partial \tilde{x}}\left(\tilde{w}_b - \frac{\partial \tilde{f}}{\partial \tilde{t}}\right)\right). \tag{15}$$

For the sake of clarity, in the sequel we drop the symbol ˜ and we denote $\frac{\partial b}{\partial t} = \frac{\partial z_b}{\partial t}$.

3   The Shallow Water system

In this section we first derive the expression of the fluid pressure p in the context of the Shallow Water assumption and then show the combination of the Boussinesq and hydrostatic assumption leads to the classical Saint-Venant system.

     The process used hereafter is similar to the technique employed by Gerbeau and Perthame [14] to derive a formulation for the viscous Saint-Venant system.

3.1 The vertically averaged system

Using the divergence free condition, the system (8)-(10) is rewritten under the form

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{16}$$

$$\varepsilon\delta\frac{\partial u}{\partial t} + \varepsilon\delta^2\frac{\partial u^2}{\partial x} + \varepsilon\delta^2\frac{\partial uw}{\partial z} + \varepsilon\frac{\partial p}{\partial x} = \varepsilon^2\delta\frac{\partial}{\partial x}\left(2\nu_0\frac{\partial u}{\partial x}\right)$$
$$+ \frac{\partial}{\partial z}\left(\delta\nu_0\frac{\partial u}{\partial z} + \varepsilon^2\delta\nu_0\frac{\partial w}{\partial x}\right), \tag{17}$$

$$\varepsilon^2\delta\left(\frac{\partial w}{\partial t} + \delta\frac{\partial uw}{\partial x} + \delta\frac{\partial w^2}{\partial z}\right) + \frac{\partial p}{\partial z} = -1$$
$$+ \frac{\partial}{\partial x}\left(\varepsilon\delta\nu_0\frac{\partial u}{\partial z} + \varepsilon^3\delta\nu_0\frac{\partial w}{\partial x}\right) + \varepsilon\delta\frac{\partial}{\partial z}\left(2\nu_0\frac{\partial w}{\partial z}\right). \tag{18}$$

Due to the applied rescaling some terms of the viscosity tensor e.g.

$$\varepsilon^3\delta\frac{\partial}{\partial x}\left(\nu_0\frac{\partial w}{\partial x}\right)$$

are very small and could be neglected. But, as mentioned in [2, Remarks 1 and 2], the approximation of the viscous terms have to preserve the dissipation energy that is an essential property of the Navier-Stokes and averaged Navier-Stokes equations. Since we privilege this stability requirement and in order to keep a symmetric form of the viscosity tensor, we consider in the sequel a modified version of (16)-(18) under the form

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{19}$$

$$\varepsilon\delta\frac{\partial u}{\partial t} + \varepsilon\delta^2\frac{\partial u^2}{\partial x} + \varepsilon\delta^2\frac{\partial uw}{\partial z} + \varepsilon\frac{\partial p}{\partial x} = \varepsilon^2\delta\frac{\partial}{\partial x}\left(2\nu_0\frac{\partial u}{\partial x}\right)$$
$$+ \frac{\partial}{\partial z}\left(\delta\nu_0\frac{\partial u}{\partial z}\right), \tag{20}$$

$$\varepsilon^2\delta\left(\frac{\partial w}{\partial t} + \delta\frac{\partial uw}{\partial x} + \delta\frac{\partial w^2}{\partial z}\right) + \frac{\partial p}{\partial z} = -1 + \frac{\partial}{\partial x}\left(\varepsilon\delta\nu_0\frac{\partial u}{\partial z}\right)$$
$$+ \frac{\partial}{\partial z}\left(2\varepsilon\delta\nu_0\frac{\partial w}{\partial z}\right), \tag{21}$$

corresponding to a viscosity tensor of the form

$$\Sigma_{xx} = 2\nu\frac{\partial u}{\partial x}, \quad \Sigma_{xz} = \Sigma_{zx} = \nu\frac{\partial u}{\partial z}, \quad \Sigma_{zz} = 2\nu\frac{\partial w}{\partial z}.$$

Remark 2 *If we strictly follow Audusse [2, Lemma 2.1], the chosen form of the viscosity tensor will not allow us to include under the form of a square term in the energy equality the quantity*

$$\nu_0\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}.$$

But we will see in paragraph 4.3 that due to the shallow water assumption, this quantity appear as a friction term.

From Eqs. (12), it comes

$$p_s = \delta p^a + 2\varepsilon\delta \left.\frac{\partial w}{\partial z}\right|_s + \mathcal{O}(\varepsilon^2\delta^2),$$

so using Eqs. (13) and (15) one obtains

$$\left.\frac{\partial u}{\partial z}\right|_s = \mathcal{O}(\varepsilon^2), \quad \left.\frac{\partial u}{\partial z}\right|_b = \mathcal{O}(\varepsilon), \tag{22}$$

and an integration of Eq. (21) from $\delta\eta$ to z gives

$$p - \delta p^a = \delta\eta - z + \mathcal{O}(\varepsilon\delta), \tag{23}$$

leading to

$$\frac{\partial p}{\partial x} = \mathcal{O}(\delta).$$

The preceding relation inserted in (20) leads to

$$\nu_0\frac{\partial^2 u}{\partial z^2} = \mathcal{O}(\varepsilon), \tag{24}$$

and Eqs. (22) and (24) mean that

$$u(x, z, t) = u_0(x, t) + \mathcal{O}(\varepsilon),$$

i.e. we recognize the so-called "motion by slices" of the usual Saint-Venant system. Then we introduce the averaged quantities

$$\bar{u} = \frac{1}{\delta\eta - z_b}\int_{z_b}^{\delta\eta} u\,dz, \quad \overline{u^2} = \frac{1}{\delta\eta - z_b}\int_{z_b}^{\delta\eta} u^2\,dz,$$

and the previous definitions involve

$$u(x, z, t) = \bar{u} + \mathcal{O}(\varepsilon), \quad \text{and} \quad \overline{u^2} = \bar{u}^2(x, z, t) + \mathcal{O}(\varepsilon). \tag{25}$$

Note that the velocity $\bar{u}$ is exactly the one arising in the conservation law for the water height since an integration of Eq. (19) from z_b to $\delta\eta$ with boundary conditions (11) and (14) leads to

$$\frac{\partial\eta}{\partial t} - \frac{\partial z_b}{\partial t} + \frac{\partial}{\partial x}\big(H_\delta\bar{u}\big) = 0, \tag{26}$$

with $H_\delta = \delta\eta - z_b$. Conversely an integration of Eq. (19) from z_b to z with boundary conditions (11) and (14) leads to

$$w = \frac{\partial z_b}{\partial t} - \frac{\partial}{\partial x}\int_{z_b}^z u\,dz = \frac{\partial z_b}{\partial t} - z\frac{\partial\bar{u}}{\partial x} + \frac{\partial(z_b\bar{u})}{\partial x} + \mathcal{O}(\varepsilon). \tag{27}$$

We use the approximations obtained in this paragraph to simplify the boundary conditions (11)-(15) and retaining only the high order terms we obtain

$$\frac{\partial \eta}{\partial t} + \delta u_s \frac{\partial \eta}{\partial x} - w_s = 0, \tag{28}$$

$$p_s = \delta p^a + 2\varepsilon \delta \nu_0 \left. \frac{\partial w}{\partial z} \right|_s + \mathcal{O}(\varepsilon^3 \delta), \tag{29}$$

$$\delta \nu_0 \left. \frac{\partial u}{\partial z} \right|_s - \varepsilon \delta \frac{\partial \eta}{\partial x} \left(2\varepsilon \delta \nu_0 \left. \frac{\partial u}{\partial x} \right|_s - p_s \right) = \varepsilon \delta^2 \frac{\partial \eta}{\partial x} p^a, \tag{30}$$

$$\frac{\partial z_b}{\partial t} + u_b \frac{\partial z_b}{\partial x} - w_b = 0, \tag{31}$$

$$\delta \nu_0 \left. \frac{\partial u}{\partial z} \right|_b - \varepsilon \frac{\partial z_b}{\partial x} \left(2\varepsilon \delta \nu_0 \left. \frac{\partial u}{\partial x} \right|_b - p_b \right) = -\varepsilon \frac{\partial z_b}{\partial x} \left(2\varepsilon \delta \nu_0 \left. \frac{\partial w}{\partial z} \right|_b - p_b \right.$$
$$\left. -\varepsilon \delta \nu_0 \frac{\partial z_b}{\partial x} \left. \frac{\partial u}{\partial z} \right|_b \right) + \varepsilon \delta \kappa_0 \left(1 + \frac{3\varepsilon^2}{2} \left(\frac{\partial z_b}{\partial x} \right)^2 \right) u_b + \mathcal{O}(\varepsilon^4 \delta). \tag{32}$$

Using the Leibniz rule i.e.

$$\frac{\partial}{\partial x} \int_{a(x)}^{b(x)} g \; dx_1 = \int_{a(x)}^{b(x)} \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial b}{\partial x} g(a(x)) - \frac{\partial a}{\partial x} g(b(x)),$$

and the kinematic boundary conditions (28) and (31), an integration of Eq. (20) from z_b to $\delta \eta$ shows that a solution to (19)-(21) satisfies

$$\varepsilon \delta \frac{\partial}{\partial t} \int_{z_b}^{\delta \eta} u \; dz + \varepsilon \frac{\partial}{\partial x} \int_{z_b}^{\delta \eta} \left(\delta^2 u^2 + p \right) dz = \varepsilon^2 \delta \frac{\partial}{\partial x} \int_{z_b}^{\delta \eta} 2\nu_0 \frac{\partial u}{\partial x} dz$$
$$+\delta \nu_0 \left. \frac{\partial u}{\partial z} \right|_s - \varepsilon \delta \frac{\partial \eta}{\partial x} \left(2\varepsilon \delta \nu_0 \left. \frac{\partial u}{\partial x} \right|_s - p_s \right)$$
$$-\delta \nu_0 \left. \frac{\partial u}{\partial z} \right|_b + \varepsilon \frac{\partial z_b}{\partial x} \left(2\varepsilon \delta \nu_0 \left. \frac{\partial u}{\partial x} \right|_b - p_b \right),$$

and using Eqs. (30) and (32), we obtain

$$\delta \frac{\partial}{\partial t} \int_{z_b}^{\delta \eta} u \; dz + \frac{\partial}{\partial x} \int_{z_b}^{\delta \eta} \left(\delta^2 u^2 + p \right) dz = \varepsilon \delta \frac{\partial}{\partial x} \int_{z_b}^{\delta \eta} 2\nu_0 \frac{\partial u}{\partial x} dz$$
$$+\delta^2 \frac{\partial \eta}{\partial x} p^a + \frac{\partial z_b}{\partial x} \left(2\varepsilon \delta \nu_0 \left. \frac{\partial w}{\partial z} \right|_b - p_b - \varepsilon \delta \nu_0 \frac{\partial z_b}{\partial x} \left. \frac{\partial u}{\partial z} \right|_b \right)$$
$$-\delta \kappa_0 \left(1 + \frac{3\varepsilon^2}{2} \left(\frac{\partial z_b}{\partial x} \right)^2 \right) u_b + \mathcal{O}(\varepsilon^3 \delta), \tag{33}$$

An expression for the pressure p can be obtained as follows. An integration of Eq. (21) from z to $\delta \eta$ gives

$$\varepsilon^2 \delta \int_z^{\delta \eta} \left(\frac{\partial w}{\partial t} + \delta \frac{\partial (uw)}{\partial x} \right) dz + \varepsilon^2 \delta^2 (w_s^2 - w^2) + p_s - p = -(\delta \eta - z)$$
$$+\varepsilon \delta \int_z^{\delta \eta} \frac{\partial}{\partial x} \left(\nu_0 \frac{\partial u}{\partial z} \right) dz - 2\varepsilon \delta \nu_0 \frac{\partial w}{\partial z} + 2\varepsilon \delta \nu_0 \left. \frac{\partial w}{\partial z} \right|_s$$

and using the boundary conditions (28) and (29), it comes

$$\varepsilon^2\delta\left(\frac{\partial}{\partial t}\int_z^{\delta\eta} w\ dz + \delta\frac{\partial}{\partial x}\int_z^{\delta\eta}(uw)\ dz\right) - \varepsilon^2\delta^2 w^2 + \delta p^a - p = -(\delta\eta - z)$$
$$+\varepsilon\delta\int_z^{\delta\eta}\frac{\partial}{\partial x}\left(\nu_0\frac{\partial u}{\partial z}\right)dz - 2\varepsilon\delta\nu_0\frac{\partial w}{\partial z}.$$

Classically we have

$$\frac{\partial u_s}{\partial x} = \left.\frac{\partial u}{\partial x}\right|_s + \delta\frac{\partial \eta}{\partial x}\left.\frac{\partial u}{\partial z}\right|_s = \left.\frac{\partial u}{\partial x}\right|_s + \mathcal{O}(\varepsilon^2\delta), \tag{34}$$

and using relations (30), (34) and the Liebniz rule we have

$$\varepsilon\delta\int_z^{\delta\eta}\frac{\partial}{\partial x}\left(\nu_0\frac{\partial u}{\partial z}\right)dz - 2\varepsilon\delta\nu_0\frac{\partial w}{\partial z} = \varepsilon\delta\nu_0\frac{\partial u}{\partial x} + \varepsilon\delta\nu_0\left.\frac{\partial u}{\partial x}\right|_s + \mathcal{O}(\varepsilon^3\delta).$$

This leads to the expression for the pressure p

$$p = \delta p^a + (\delta\eta - z) + \varepsilon^2\delta\left(\frac{\partial}{\partial t}\int_z^{\delta\eta} w\ dz + \delta\frac{\partial}{\partial x}\int_z^{\delta\eta}(uw)\ dz\right)$$
$$-\varepsilon^2\delta^2 w^2 - \varepsilon\delta\nu_0\frac{\partial u}{\partial x} - \varepsilon\delta\nu_0\left.\frac{\partial u}{\partial x}\right|_s + \mathcal{O}(\varepsilon^3\delta). \tag{35}$$

Hereafter several models of growing accuracy and complexity will be derived, depending on the level of approximation chosen for Eq. (35). In the hydrostatic case, we will consider an approximation of p in $\mathcal{O}(\varepsilon^2\delta)$, then in section 4 we will use two expressions of p respectively in $\mathcal{O}(\varepsilon^2\delta^2, \varepsilon^3\delta)$ and in $\mathcal{O}(\varepsilon^3\delta)$.

Remark 3 *For the derivation of Eq. (33) note that due to the rescaling applied to the time derivative of z_b, we have*

$$\int_{z_b}^{\delta\eta}\frac{\partial u}{\partial t}dz = \frac{\partial}{\partial t}\int_{z_b}^{\delta\eta} u\ dz - \delta\frac{\partial\eta}{\partial t}u_s + \varepsilon\delta\frac{\partial z_b}{\partial t}u_b.$$

Remark 4 *The second relation in (22) is crucial for the derivation of shallow water models. When considering large friction coefficients then the assumption of asymptotic regime $\hat\kappa = \varepsilon\kappa_0$ no more holds and relation (32) leads to*

$$\frac{\partial u}{\partial z} = \mathcal{O}(1),$$

meaning the assumption of motion by slices has to be justified by other arguments.

3.2   Hydrostatic approximation

We begin with the classical hydrostatic approximation. The objectives of this paragraph are twofold. First we want to obtain the expression of $\bar{u}$ as a function of δ, ε, ν_0, κ_0 and H_δ. And second, we aim at verifying that despite the parameter δ, we recover the well-known formulation of the viscous Saint-Venant system with friction as expressed in the following proposition

Proposition 1 *The viscous Saint-Venant system defined by*

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\big(H\bar{u}\big) = 0, \tag{36}$$

$$\frac{\partial(H\bar{u})}{\partial t} + \frac{\partial(H\bar{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = -H\frac{\partial p^a}{\partial x} - gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\Big(4\nu H\frac{\partial \bar{u}}{\partial x}\Big)$$
$$-\frac{\kappa(\bar{\mathbf{v}}, H)}{1 + \frac{\kappa(\bar{\mathbf{v}},H)}{3\nu}H}\bar{u}, \tag{37}$$

where $H = \eta - z_b$ and $\bar{\mathbf{v}} = (1, \frac{\partial z_b}{\partial x})^T \bar{u}$, results from an hydrostatic approximation in $\mathcal{O}(\varepsilon^2\delta)$ of the Navier-Stokes equations.

Proof of prop. 1: we retain only the terms up to $\varepsilon\delta$ in the expression (35) for the pressure p i.e. we have

$$p = \delta p^a + (\delta\eta - z) - \varepsilon\delta\nu_0\frac{\partial u}{\partial x} - \varepsilon\delta\nu_0 \left.\frac{\partial u}{\partial x}\right|_s + \mathcal{O}(\varepsilon^2\delta). \tag{38}$$

And Eq. (33) with Eqs. (25) and (38) gives

$$\varepsilon\delta\frac{\partial\big(H_\delta\bar{u}\big)}{\partial t} + \varepsilon\delta^2\frac{\partial\big(H_\delta\overline{u^2}\big)}{\partial x} + \frac{\varepsilon}{2}\frac{\partial H_\delta^2}{\partial x} =$$
$$-\varepsilon\delta\kappa_0 u_b - \varepsilon\delta\frac{\partial}{\partial x}(H_\delta p^a) + \varepsilon\delta^2\frac{\partial\eta}{\partial x}p^a - \varepsilon\frac{\partial z_b}{\partial x}p_b + \mathcal{O}(\varepsilon^2\delta),$$

that is also using the expression of p obtained in Eq. (38)

$$\delta\frac{\partial\big(H_\delta\bar{u}\big)}{\partial t} + \delta^2\frac{\partial\big(H_\delta\overline{u^2}\big)}{\partial x} + \frac{1}{2}\frac{\partial H_\delta^2}{\partial x} =$$
$$-\delta\kappa_0 u_b - \delta H_\delta\frac{\partial p^a}{\partial x} - \frac{\partial z_b}{\partial x}p_b + \mathcal{O}(\varepsilon\delta). \tag{39}$$

Note that due to the assumption concerning the time derivative of z_b and the associated rescaling, the first term in the left hand side of (39) reads

$$\frac{\partial\big(H_\delta\bar{u}\big)}{\partial t} = H_\delta\frac{\partial\bar{u}}{\partial t} + \delta\frac{\partial(\eta - z_b)}{\partial t}\bar{u},$$

and (39) coupled with (26) gives

$$\delta\frac{\partial\bar{u}}{\partial t} + \delta^2\bar{u}\frac{\partial\bar{u}}{\partial x} + \delta\frac{\partial\eta}{\partial x} = -\frac{\delta\kappa_0}{H_\delta}u_b - \delta\frac{\partial p^a}{\partial x} + \mathcal{O}(\varepsilon\delta).$$

Now we come back to Eq. (20), using (25), (38) and (39) we get

$$\begin{aligned}
\delta\frac{\partial}{\partial z}\Big(\nu_0\frac{\partial u}{\partial z}\Big) &= \varepsilon\delta\frac{\partial u}{\partial t} + \varepsilon\delta^2 u\frac{\partial u}{\partial x} + \varepsilon\delta^2 w\frac{\partial u}{\partial z} + \varepsilon\frac{\partial p}{\partial x} - \varepsilon^2\delta\frac{\partial}{\partial x}\Big(\nu_0\frac{\partial u}{\partial x}\Big) \\
&= \varepsilon\delta\frac{\partial\bar{u}}{\partial t} + \varepsilon\delta^2\bar{u}\frac{\partial\bar{u}}{\partial x} + \varepsilon\delta\frac{\partial}{\partial x}(\eta + p^a) + \mathcal{O}(\varepsilon^2\delta) \\
&= -\frac{\varepsilon\delta\kappa_0}{H_\delta}u_b + \mathcal{O}(\varepsilon^2\delta). \tag{40}
\end{aligned}$$

Integrating from z_b to z and taking into account the boundary condition (32), we deduce

$$\frac{\partial u}{\partial z} = \frac{\varepsilon \kappa_0}{\nu_0}\left(1 - \frac{z - z_b}{H_\delta}\right)u_b + \mathcal{O}(\varepsilon^2), \tag{41}$$

and we obtain the following formula which gives an expression of the vertical velocity though a parabolic correction

$$u = \left(1 + \frac{\varepsilon \kappa_0}{\nu_0}\left(z - z_b - \frac{(z - z_b)^2}{2H_\delta}\right)\right)u_b + \mathcal{O}(\varepsilon^2). \tag{42}$$

Then integrating from z_b to $\delta\eta$, we obtain

$$\bar{u} = \left(1 + \frac{\varepsilon \kappa_0}{3\nu_0}H_\delta\right)u_b + \mathcal{O}(\varepsilon^2). \tag{43}$$

Moreover

$$u^2 = \left(1 + \frac{2\varepsilon \kappa_0}{\nu_0}\left(z - z_b - \frac{(z - z_b)^2}{2H_\delta}\right)\right)u_b^2 + \mathcal{O}(\varepsilon^2),$$

which yields

$$\overline{u^2} = \left(1 + \frac{2\varepsilon \kappa_0}{3\nu_0}H_\delta\right)u_b^2 + \mathcal{O}(\varepsilon^2),$$

meaning

$$\overline{u^2} = \bar{u}^2 + \mathcal{O}(\varepsilon^2). \tag{44}$$

Using (38), (42) and (43), the right hand side of Eq. (33) can be written

$$\varepsilon\delta\frac{\partial}{\partial x}\int_{z_b}^{\delta\eta} 2\nu_0\frac{\partial u}{\partial x}dz + \delta^2\frac{\partial \eta}{\partial x}p^a - \frac{\partial z_b}{\partial x}\left(p_b + 2\varepsilon\delta\nu_0\left.\frac{\partial u}{\partial x}\right|_b\right)$$

$$-\delta\kappa_0\left(1 + \frac{5\varepsilon^2}{2}\left(\frac{\partial z_b}{\partial x}\right)^2\right)u_b = -\delta\kappa_0 u_b - H_\delta\frac{\partial z_b}{\partial x}$$

$$+\delta\frac{\partial H_\delta}{\partial x}p^a + \varepsilon\delta\frac{\partial}{\partial x}\left(2\nu_0 H_\delta\frac{\partial \bar{u}}{\partial x}\right) + \mathcal{O}(\varepsilon^2\delta). \tag{45}$$

Finally from Eqs. (26), (33), (43), (44) and (45), we obtain the model

$$\frac{\partial \eta}{\partial t} - \frac{\partial z_b}{\partial t} + \frac{\partial}{\partial x}\left(H_\delta\bar{u}\right) = 0,$$

$$\delta\frac{\partial(H_\delta\bar{u})}{\partial t} + \delta^2\frac{\partial(H_\delta\bar{u}^2)}{\partial x} + \frac{1}{2}\frac{\partial H_\delta^2}{\partial x} = -H_\delta\frac{\partial}{\partial x}(z_b + \delta p^a) - \frac{\delta\kappa_0}{1 + \frac{\varepsilon\kappa_0}{3\nu_0}H_\delta}\bar{u}$$

$$+\varepsilon\delta\frac{\partial}{\partial x}\left(4\nu_0 H_\delta\frac{\partial \bar{u}}{\partial x}\right) + \mathcal{O}(\varepsilon^2\delta).$$

In terms of the initial variables, the preceding model becomes (36)-(37) that complete the proof of prop. 1. Note that when the bathymetry is constant $z_b(x,t) = z_b^0$, this formulation is equivalent to the viscous Saint-Venant system obtained by Gerbeau *et al.* [14] and Ferrari *et al.* [13].

4 Two non-hydrostatic shallow water models

In the previous paragraph we have obtained an approximation of the Navier-Stokes equations up to $\varepsilon\delta$ terms using an hydrostatic approximation of the pressure p. In this section we consider two more acurate approximations of the pressure p respectively in $\mathcal{O}(\varepsilon^2\delta^2)$ and $\mathcal{O}(\varepsilon^3\delta)$ leading to two non-hydrostatic extensions of the Saint-Venant system.

4.1 First extension, $\delta \ll 1$

The first refinement of the classical Saint-Venant model (36)-(37) is achieved by considering the pressure p given by Eq. (35) with the terms up to $\mathcal{O}(\varepsilon^2\delta^2)$. This means we consider the momentum equation along z is no more reduced to

$$\frac{\partial p}{\partial z} = -1 + \frac{\partial}{\partial x}\left(\varepsilon\delta\nu_0\frac{\partial u}{\partial z}\right) + \varepsilon\delta\frac{\partial}{\partial z}\left(2\nu_0\frac{\partial w}{\partial z}\right) + \mathcal{O}(\varepsilon^2\delta),$$

but given by

$$\varepsilon^2\delta\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = -1 + \frac{\partial}{\partial x}\left(\varepsilon\delta\nu_0\frac{\partial u}{\partial z}\right) + \varepsilon\delta\frac{\partial}{\partial z}\left(2\nu_0\frac{\partial w}{\partial z}\right) + \mathcal{O}(\varepsilon^2\delta^2),$$

and the convective terms are still neglected. Since we keep the terms in $\varepsilon^2\delta$ and drop those in $\varepsilon^2\delta^2$, this means we assume $\delta \ll 1$ and due to the applied rescaling this implies $U \ll C$ so we are in a fluvial regime. The following result holds.

Proposition 2 *The system defined by*

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\big(H\bar{u}\big) = 0, \tag{46}$$

$$\frac{\partial}{\partial t}(H\bar{u}) + \frac{\partial}{\partial x}(H\bar{u}^2) + \frac{\partial}{\partial x}\left(\frac{g}{2}H^2 - \frac{z_b^3}{6}\frac{\partial^2\bar{u}}{\partial x\partial t} + \frac{z_b^2}{2}\frac{\partial^2(z_b\bar{u})}{\partial x\partial t}\right) =$$
$$-H\frac{\partial p^a}{\partial x} + \frac{\partial}{\partial x}\left(4\nu H\frac{\partial\bar{u}}{\partial x} + \frac{\kappa(\bar{\mathbf{v}}, H)}{6}z_b\left(z_b\frac{\partial\bar{u}}{\partial x} + 7\frac{\partial z_b}{\partial x}\bar{u}\right)\right)$$
$$-\frac{\kappa(\bar{\mathbf{v}}, H)}{2}\frac{\partial z_b}{\partial x}\left(z_b\frac{\partial\bar{u}}{\partial x} - \frac{\partial z_b}{\partial x}\bar{u}\right) - \frac{\partial z_b}{\partial x}\left(gH + \frac{z_b^2}{2}\frac{\partial^2\bar{u}}{\partial x\partial t} - z_b\frac{\partial^2(z_b\bar{u})}{\partial x\partial t}\right)$$
$$-\frac{\kappa(\bar{\mathbf{v}}, H)}{1 + \frac{\kappa(\bar{\mathbf{v}}, H)H}{3\nu}}\left(1 + \frac{5}{2}\left(\frac{\partial z_b}{\partial x}\right)^2\right)\bar{u} - \frac{z_b^2}{2}\frac{\partial^3 z_b}{\partial x\partial t^2}, \tag{47}$$

where $\bar{\mathbf{v}} = (1, \frac{\partial z_b}{\partial x})^T\frac{\bar{u}}{1 + \frac{\kappa_l H}{3\nu}}$ results from an approximation in $\mathcal{O}(\varepsilon^2\delta^2, \varepsilon^3\delta)$ of the Navier-Stokes equations.

The proof of proposition 2 is given in the next paragraph, we examine here some properties of the model (46)-(47).

Note that except for the dissipative terms corresponding to viscosity or friction, all the terms added in the non-hydrostatic model (46)-(47) compared to the original Saint-Venant model (36)-(37) appear as time derivative of the variables z_b, η or $\bar{u}$. This means in a stationary regime, the solutions of (36)-(37) and (46)-(47) are identical.

We first examine the system (46)-(47) without friction and viscosity. Starting from the Euler equations instead of the Navier-Stokes equations does not allow to account for the motion by slices as obtained in relations (24) and (41). So if one wants to neglect the viscosity and friction effects in the model (46)-(47), it is necessary to consider an asymptotic regime for example under the form $\nu = \beta\nu_{nv}$, $\kappa = \beta^2\kappa_{nf}$ – and conversely $\nu_0 = \beta\nu_{0,nv}$, $\kappa_0 = \beta^2\kappa_{0,nf}$ – with $\beta \ll 1$. Introducing the preceding asymptotic regime and considering $\beta \to 0$, the formulation of (46)-(47) reads

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\big(H\bar{u}\big) = 0,$$
$$\frac{\partial}{\partial t}(H\bar{u}) + \frac{\partial}{\partial x}(H\bar{u}^2) + \frac{\partial}{\partial x}\left(\frac{g}{2}H^2 - \frac{z_b^3}{6}\frac{\partial^2\bar{u}}{\partial x\partial t} + \frac{z_b^2}{2}\frac{\partial^2(z_b\bar{u})}{\partial x\partial t}\right) =$$
$$\frac{\partial z_b}{\partial x}\left(-gH - \frac{z_b^2}{2}\frac{\partial^2\bar{u}}{\partial x\partial t} + z_b\frac{\partial^2(z_b\bar{u})}{\partial x\partial t}\right) - H\frac{\partial p^a}{\partial x} - \frac{z_b^2}{2}\frac{\partial^3 z_b}{\partial x\partial t^2}.$$

or equivalently in a non-conservative form

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\big(H\bar{u}\big) = 0,$$
$$\frac{\partial\bar{u}}{\partial t} + \bar{u}\frac{\partial\bar{u}}{\partial x} + g\frac{\partial\eta}{\partial x} + \frac{z_b^2}{6}\frac{\partial^3\bar{u}}{\partial x^2\partial t} - \frac{z_b}{2}\frac{\partial^3(z_b\bar{u})}{\partial x^2\partial t} = -\frac{\partial p^a}{\partial x} + \frac{z_b}{2}\frac{\partial^3 z_b}{\partial x\partial t^2},$$

that is analogous to the expression obtained by Peregrine [20]. It is worth being noticed that, in any case, the formulations obtained by Nwogu [19], Walkley [24], Saut *et al.* [5] and Soares Frazao *et al.* [22] are different from the preceding ones. The differences lie either in the continuity equation or in the momentum equation.

The mathematical and numerical analysis of the obtained model is not in the scope of this paper but let us mention some interesting works in the literature. The Sobolev equation

$$-\frac{\partial}{\partial x}(a(x)\frac{\partial^2 u}{\partial x\partial t}) + c(x)\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(\alpha(x)\frac{\partial u}{\partial x}) + \beta(x)\frac{\partial u}{\partial x},$$

has been studied by several authors [1, 4] as an alternative to the Korteweg-de Vries equations. Perotto and Saleri [21] proposed an *a posteriori* error analysis for the Peregrine formulation of the Boussinesq system with constant bathymetry. Bona *et al.* [5, 6] have studied the well-posedness of several high-order generalizations of the Boussinesq equations.

4.2   Derivation

Proof of prop. 2: the refinement of the classical Saint-Venant model (36)-(37) is achieved by improving the approximation for the pressure p. Actually, if we only drop the terms in $\mathcal{O}(\varepsilon^2\delta^2)$ in the momentum equation along z so the system

(8)-(10) becomes

$$w = \frac{\partial z_b}{\partial t} - \frac{\partial}{\partial x} \int_{z_b}^z u \ dz,$$

$$\varepsilon\delta\frac{\partial u}{\partial t} + \varepsilon\delta^2\frac{\partial u^2}{\partial x} + \varepsilon\delta^2\frac{\partial uw}{\partial z} + \varepsilon\frac{\partial p}{\partial x} = \varepsilon^2\delta\frac{\partial}{\partial x}\left(2\nu_0\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z}\left(\delta\nu_0\frac{\partial u}{\partial z}\right),$$

$$\varepsilon^2\delta\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = -1 + \frac{\partial}{\partial x}\left(\varepsilon\delta\nu_0\frac{\partial u}{\partial z}\right) + \varepsilon\delta\frac{\partial}{\partial z}\left(2\nu_0\frac{\partial w}{\partial z}\right) + \mathcal{O}(\varepsilon^2\delta^2),$$

with the boundary conditions (28)-(32). This means we consider the pressure p is given by (35) where we retain only the terms up to $\varepsilon^2\delta^2$ and $\varepsilon^3\delta$ i.e.

$$
\begin{aligned}
p_{nh} = & \ \delta p^a + (\delta\eta - z) - \varepsilon\delta\nu_0\left.\frac{\partial u}{\partial x}\right|_s - \varepsilon\delta\nu_0\frac{\partial u}{\partial x} + \varepsilon^2\delta\frac{\partial}{\partial t}\int_z^{\delta\eta} w \ dz \\
& + \mathcal{O}(\varepsilon^2\delta^2, \varepsilon^3\delta),
\end{aligned}
$$

leading to

$$
\begin{aligned}
p_{nh} = & \ \delta p^a + (\delta\eta - z) - \varepsilon\delta\nu_0\left.\frac{\partial u}{\partial x}\right|_s - \varepsilon\delta\nu_0\frac{\partial u}{\partial x} + \varepsilon^2\delta(\delta\eta - z)\frac{\partial^2 z_b}{\partial t^2} \\
& -\varepsilon^2\delta\frac{\partial}{\partial t}\int_z^{\delta\eta}\frac{\partial}{\partial x}\int_{z_b}^z u dz_1 dz + \mathcal{O}(\varepsilon^2\delta^2, \varepsilon^3\delta).
\end{aligned}
\tag{48}
$$

Retaining only the terms up to $\mathcal{O}(\varepsilon^2\delta^2, \varepsilon^3\delta)$, relation (33) gives

$$
\begin{aligned}
& \delta\frac{\partial}{\partial t}\int_{z_b}^{\delta\eta} u \ dz + \delta^2\frac{\partial}{\partial x}\int_{z_b}^{\delta\eta} u^2 \ dz + \frac{\partial}{\partial x}\int_{z_b}^{\delta\eta} p_{nh} \ dz \\
& = \varepsilon\delta\frac{\partial}{\partial x}\int_{z_b}^{\delta\eta} 2\nu_0\frac{\partial u}{\partial x}dz + \delta^2\frac{\partial\eta}{\partial x}p^a + \frac{\partial z_b}{\partial x}\left(2\varepsilon\delta\nu_0\left.\frac{\partial w}{\partial z}\right|_b - p_{nh}|_b\right) \\
& -\delta\kappa_0\left(1 + \frac{5\varepsilon^2}{2}(\frac{\partial z_b}{\partial x})^2\right)u_b + \mathcal{O}(\varepsilon^2\delta^2, \varepsilon^3\delta).
\end{aligned}
\tag{49}
$$

Now we derive the expressions for the quantities appearing in (48) and (49) and depending on u, w and p. Since $\kappa_0 = \kappa_{0,l} + \mathcal{O}(\varepsilon)$, from Eqs. (42) and (43) we have

$$
\begin{aligned}
\frac{\partial u}{\partial x} = & \ \left(1 + \frac{\varepsilon\kappa_0}{\nu_0}\left(z - z_b - \frac{(z - z_b)^2}{2H_\delta}\right)\right)\frac{\partial u_b}{\partial x} \\
& + \frac{\varepsilon\kappa_0}{\nu_0}\left(\frac{\partial z_b}{\partial x}(-1 + \frac{z - z_b}{H_\delta}) + \frac{\partial H_\delta}{\partial x}\frac{(z - z_b)^2}{2H_\delta^2}\right)u_b + \mathcal{O}(\varepsilon^2) \\
= & \ \left(1 + \frac{\varepsilon\kappa_0}{\nu_0}\left(z - z_b - \frac{(z - z_b)^2}{2H_\delta}\right)\right)\frac{\partial u_b}{\partial x} \\
& - \frac{\varepsilon\kappa_0}{\nu_0}\frac{\partial z_b}{\partial x}\left(1 - \frac{z - z_b}{H_\delta} + \frac{(z - z_b)^2}{2H_\delta^2}\right)u_b + \mathcal{O}(\varepsilon\delta),
\end{aligned}
$$

so

$$\left.\frac{\partial u}{\partial x}\right|_s \quad = \quad \left(1 + \frac{\varepsilon\kappa_0}{2\nu_0}H_\delta\right)\frac{\partial u_b}{\partial x} + \frac{\varepsilon\kappa_0}{2\nu_0}\frac{\partial H_\delta}{\partial x}u_b + \mathcal{O}(\varepsilon^2) \tag{50}$$

$$= \quad \left(1 + \frac{\varepsilon\kappa_0}{6\nu_0}H_\delta\right)\frac{\partial \bar{u}}{\partial x} + \frac{\varepsilon\kappa_0}{6\nu_0}\frac{\partial H_\delta}{\partial x}\bar{u} + \mathcal{O}(\varepsilon^2), \tag{51}$$

$$\left.\frac{\partial u}{\partial x}\right|_b \quad = \quad \frac{\partial u_b}{\partial x} - \frac{\varepsilon\kappa_0}{\nu_0}\frac{\partial z_b}{\partial x}u_b + \mathcal{O}(\varepsilon^2) \tag{52}$$

$$= \quad \left(1 - \frac{\varepsilon\kappa_0}{3\nu_0}H_\delta\right)\frac{\partial \bar{u}}{\partial x} - \frac{\varepsilon\kappa_0}{\nu_0}\left(\frac{\partial z_b}{\partial x} + \frac{1}{3}\frac{\partial H_\delta}{\partial x}\right)\bar{u} + \mathcal{O}(\varepsilon^2), \tag{53}$$

and

$$\int_{z_b}^{\delta\eta} \nu_0\frac{\partial u}{\partial x} \quad = \quad \nu_0 H_\delta\left(1 + \frac{\varepsilon\kappa_0}{3\nu_0}H_\delta\right)\frac{\partial u_b}{\partial x} - \frac{\varepsilon\kappa_0 H_\delta}{2}\left(\frac{\partial z_b}{\partial x} - \frac{1}{3}\frac{\partial H_\delta}{\partial x}\right)u_b + \mathcal{O}(\varepsilon^2)$$

$$= \quad \nu_0 H_\delta\frac{\partial \bar{u}}{\partial x} - \frac{\varepsilon\kappa_0}{3}H_\delta\frac{\partial H_\delta}{\partial x}\bar{u} - \frac{\varepsilon\kappa_0 H_\delta}{2}\left(\frac{\partial z_b}{\partial x} - \frac{1}{6}\frac{\partial H_\delta}{\partial x}\right)\bar{u} + \mathcal{O}(\varepsilon^2)$$

$$= \quad \nu_0 H_\delta\frac{\partial \bar{u}}{\partial x} - \frac{\varepsilon\kappa_0}{2}H_\delta\left(\frac{\partial z_b}{\partial x} + \frac{1}{3}\frac{\partial H_\delta}{\partial x}\right)\bar{u} + \mathcal{O}(\varepsilon^2), \tag{54}$$

and finally from (34) we get

$$\int_{z_b}^{\delta\eta} \nu_0\left.\frac{\partial u}{\partial x}\right|_s \quad = \quad \int_{z_b}^{\delta\eta}\nu_0\frac{\partial u_s}{\partial x} + \mathcal{O}(\varepsilon^2\delta)$$

$$= \quad \nu_0 H_\delta\left(1 + \frac{\varepsilon\kappa_0}{6\nu_0}H_\delta\right)\frac{\partial \bar{u}}{\partial x} + \frac{\varepsilon\kappa_0}{6}\frac{\partial H_\delta}{\partial x}H_\delta\bar{u} + \mathcal{O}(\varepsilon^2).$$

From (51) and (53) we have

$$p_h|_b - 2\varepsilon\delta\nu_0\left.\frac{\partial w}{\partial z}\right|_b \quad = \quad \delta p^a + H_\delta + \varepsilon\delta\nu_0\left.\frac{\partial u}{\partial x}\right|_b - \varepsilon\delta\nu_0\left.\frac{\partial u}{\partial x}\right|_s$$

$$= \quad \delta p^a + H_\delta - \varepsilon^2\delta\frac{\kappa_0}{2}H_\delta\frac{\partial \bar{u}}{\partial x}$$

$$-\varepsilon^2\delta\kappa_0\left(\frac{\partial z_b}{\partial x} + \frac{1}{2}\frac{\partial H_\delta}{\partial x}\right)\bar{u} + \mathcal{O}(\varepsilon^3\delta), \tag{55}$$

and

$$\int_{z_b}^{\delta\eta}\left(2\varepsilon\delta\nu_0\frac{\partial u}{\partial x} - p_h\right)dz = -H_\delta\delta p^a - \frac{H_\delta^2}{2} + 4\varepsilon\delta\nu_0 H_\delta\frac{\partial \bar{u}}{\partial x}$$

$$+\varepsilon^2\delta\kappa_0 H_\delta\left(\frac{H_\delta}{6}\frac{\partial \bar{u}}{\partial x} - \frac{7}{6}\frac{\partial z_b}{\partial x}\bar{u} - \frac{\delta}{3}\frac{\partial \eta}{\partial x}\bar{u}\right) + \mathcal{O}(\varepsilon^3\delta), \tag{56}$$

where p_h corresponds to the gravitational, viscous and friction part of the pressure p given by Eq. (48) i.e.

$$p_h = \delta p^a + (\delta\eta - z) - \varepsilon\delta\nu_0\left.\frac{\partial u}{\partial x}\right|_s - \varepsilon\delta\nu_0\frac{\partial u}{\partial x}. \tag{57}$$

Inserting (54), (55) and (56) in equilibrium (49) leads to

$$
\delta\frac{\partial}{\partial t}(H_\delta \bar{u}) + \delta^2\frac{\partial}{\partial x}(H_\delta \bar{u}^2) + \frac{1}{2}\frac{\partial H_\delta^2}{\partial x} + \frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\Delta p_{nh}\ dz =
$$
$$
-H_\delta\frac{\partial}{\partial x}(\delta p^a + z_b) + \varepsilon\delta\frac{\partial}{\partial x}\big(4\nu_0 H_\delta\frac{\partial\bar{u}}{\partial x}\big)
$$
$$
+\frac{\varepsilon^2\delta}{6}\frac{\partial}{\partial x}\left(\kappa_0 z_b\left(z_b\frac{\partial\bar{u}}{\partial x} + 7\frac{\partial z_b}{\partial x}\bar{u}\right)\right) - \frac{\varepsilon^2\delta\kappa_0}{2}\frac{\partial z_b}{\partial x}\left(-\frac{\partial z_b}{\partial x}\bar{u} + z_b\frac{\partial\bar{u}}{\partial x}\right)
$$
$$
-\frac{\partial z_b}{\partial x}\Delta p_{nh}|_b - \delta\kappa_0\big(1 + \frac{5\varepsilon^2}{2}\big(\frac{\partial z_b}{\partial x}\big)^2\big)u_b + \mathcal{O}(\varepsilon^2\delta^2, \varepsilon^3\delta),
$$

where $\Delta p_{nh} = p_{nh} - p_h$. And using the expression for the pressure p_{nh} given in Eq. (48) it comes

$$
\begin{aligned}
\int_{z_b}^{\delta\eta}\Delta p_{nh}\ dz &= -\varepsilon^2\delta\frac{H_\delta^2}{6}(2\delta\eta + z_b)\frac{\partial^2\bar{u}}{\partial x\partial t} + \varepsilon^2\delta\frac{H_\delta^2}{2}\frac{\partial^2(z_b\bar{u})}{\partial x\partial t} \\
&\quad -\varepsilon^2\delta^2 H_\delta\frac{\partial\eta}{\partial t}\left(\delta\eta\frac{\partial\bar{u}}{\partial x} - \frac{\partial(z_b\bar{u})}{\partial x}\right) + \varepsilon^2\delta\frac{H_\delta^2}{2}\frac{\partial^2 z_b}{\partial t^2} \qquad (58) \\
&= \varepsilon^2\delta\frac{z_b^2}{2}\left(-\frac{z_b}{3}\frac{\partial^2\bar{u}}{\partial x\partial t} + \frac{\partial^2(z_b\bar{u})}{\partial x\partial t} + \frac{\partial^2 z_b}{\partial t^2}\right) + \mathcal{O}(\varepsilon^2\delta^2, \varepsilon^3\delta),
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta p_{nh}|_b &= -\varepsilon^2\delta\frac{\delta^2\eta^2 - z_b^2}{2}\frac{\partial^2\bar{u}}{\partial x\partial t} + \varepsilon^2\delta H_\delta\frac{\partial^2(z_b\bar{u})}{\partial x\partial t} \\
&\quad -\varepsilon^2\delta^2\frac{\partial\eta}{\partial t}\left(\delta\eta\frac{\partial\bar{u}}{\partial x} - \frac{\partial(z_b\bar{u})}{\partial x}\right) + \varepsilon^2\delta H_\delta\frac{\partial^2 z_b}{\partial t^2} \qquad (59) \\
&= \frac{\varepsilon^2\delta}{2}\left(z_b^2\frac{\partial^2\bar{u}}{\partial x\partial t} - 2z_b\frac{\partial^2(z_b\bar{u})}{\partial x\partial t}\right) - \varepsilon^2\delta z_b\frac{\partial^2 z_b}{\partial t^2} + \mathcal{O}(\varepsilon^2\delta^2, \varepsilon^3\delta).
\end{aligned}
$$

We finally obtain the model

$$
\frac{\partial H_\delta}{\partial t} + \frac{\partial}{\partial x}\big(H_\delta\bar{u}\big) = 0, \qquad\qquad (60)
$$
$$
\delta\frac{\partial}{\partial t}(H_\delta\bar{u}) + \delta^2\frac{\partial}{\partial x}(H_\delta\bar{u}^2) + \frac{1}{2}\frac{\partial H_\delta^2}{\partial x} - \varepsilon^2\delta\frac{\partial}{\partial x}\left(\frac{z_b^3}{6}\frac{\partial^2\bar{u}}{\partial x\partial t} - \frac{z_b^2}{2}\frac{\partial^2(z_b\bar{u})}{\partial x\partial t}\right) =
$$
$$
-H_\delta\frac{\partial}{\partial x}(\delta p^a + z_b) + \frac{\partial}{\partial x}\left(4\varepsilon\delta\nu_0 H_\delta\frac{\partial\bar{u}}{\partial x} + \frac{\varepsilon\kappa_0}{6}z_b\left(z_b\frac{\partial\bar{u}}{\partial x} + 7\frac{\partial z_b}{\partial x}\bar{u}\right)\right)
$$
$$
-\frac{\varepsilon^2\delta\kappa_0}{2}\frac{\partial z_b}{\partial x}\left(-\frac{\partial z_b}{\partial x}\bar{u} + z_b\frac{\partial\bar{u}}{\partial x}\right) + \varepsilon^2\delta\frac{\partial z_b}{\partial x}\left(-\frac{z_b^2}{2}\frac{\partial^2\bar{u}}{\partial x\partial t} + z_b\frac{\partial^2(z_b\bar{u})}{\partial x\partial t}\right)
$$
$$
-\delta\kappa_0\left(1 + \frac{5\varepsilon^2}{2}\big(\frac{\partial z_b}{\partial x}\big)^2\right)u_b - \varepsilon^2\delta\frac{z_b^2}{2}\frac{\partial^3 z_b}{\partial x\partial t^2} + \mathcal{O}(\varepsilon^2\delta^2, \varepsilon^3\delta), \qquad (61)
$$

that complete the proof of proposition 2. When the terms in $\mathcal{O}(\varepsilon^2\delta)$ are dropped in (61), we verify that we recover the classical viscous hydrostatic Saint-Venant model with friction (36)-(37).

4.3 Energy equality

Until now, we have not verified the derived models satisfy an energy equality. The system (36)-(37) that is equivalent to the Saint-Venant system, admits a

dissipation energy [2, 7]. Indeed we have

$$\frac{\partial E_h}{\partial t} + \frac{\partial}{\partial x}\left(\bar{u}\big(E_h + g\frac{H^2}{2}\big) - 4\nu H\bar{u}\frac{\partial \bar{u}}{\partial x}\right) = -H\frac{\partial p^a}{\partial t} - 4\nu H\big(\frac{\partial \bar{u}}{\partial x}\big)^2$$
$$-\frac{\kappa(\bar{\mathbf{v}}, H)}{1 + \frac{\kappa(\bar{\mathbf{v}},H)H}{3\nu}}\bar{u}^2 + gH\frac{\partial z_b}{\partial t}, \qquad (62)$$

with $E_h = \frac{H\bar{u}^2}{2} + \frac{gH(\eta+z_b)}{2} + Hp^a$. The energy equality (62) associated with the hydrostatic Saint-Venant model can be obtained using classical computations by multiplying Eq. (33) when $p = p_h$ by the velocity $\bar{u}$.

The only differences between the hydrostatic Saint-Venant model (36)-(37) and its extended version (46)-(47) comes from

- the non hydrostatic terms of the pressure p_{nh},

- the terms involving the viscosity and the friction at the bottom,

so the energy equality for (60)-(61) will differ from Eq. (62) only by the terms

$$\mathcal{C}_1 = \bar{u}\frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\Delta p_{nh} + \bar{u}\frac{\partial z_b}{\partial x}\,\Delta p_{nh}|_b,$$

$$\mathcal{C}_2 = \bar{u}\frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\left(2\varepsilon\delta\nu_0\frac{\partial u}{\partial x}\right),$$

$$\mathcal{C}_3 = \bar{u}\frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}p_{v,f},$$

$$\mathcal{C}_4 = \bar{u}\frac{\partial z_b}{\partial x}\left(2\varepsilon\delta\nu_0\left.\frac{\partial w}{\partial z}\right|_b - p_{v,f}|_b - \varepsilon\delta\nu_0\frac{\partial z_b}{\partial x}\left.\frac{\partial u}{\partial z}\right|_b\right),$$

where $\Delta p_{nh} = p_{nh} - p_h$ and $p_{v,f} = p_h - \delta p^a$ denotes the terms in the pressure p containing the viscosity and friction. The quantities $\mathcal{C}_1$-$\mathcal{C}_4$ corresponding to the non-hydrostatic terms, come from the multiplication of Eq. (33) by $\bar{u}$ and have to be added to (62). Since $\bar{u} = u + \mathcal{O}(\varepsilon) = u_b + \mathcal{O}(\varepsilon)$ and $\Delta p_{nh} = \mathcal{O}(\varepsilon^2\delta^2)$, we rewrite $\mathcal{C}_1$ under the form

$$\mathcal{C}_1 = \bar{u}\frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\Delta p_{nh} + u_b\frac{\partial z_b}{\partial x}\,\Delta p_{nh}|_b + \mathcal{O}(\varepsilon^2\delta^2)$$

$$= \frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}up_{nh} - \int_{z_b}^{\delta\eta}\frac{\partial u}{\partial x}\Delta p_{nh} + u_b\frac{\partial z_b}{\partial x}\,\Delta p_{nh}|_b + \mathcal{O}(\varepsilon^2\delta^2)$$

$$= \frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}u\Delta p_{nh} + [w\Delta p_{nh}]_{z_b}^{\delta\eta} - \int_{z_b}^{\delta\eta}w\frac{\partial\Delta p_{nh}}{\partial z} + u_b\frac{\partial z_b}{\partial x}\,\Delta p_{nh}|_b + \mathcal{O}(\varepsilon^2\delta^2)$$

$$= \frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}u\Delta p_{nh} + w_s\Delta p_{nh}|_s - \int_{z_b}^{\delta\eta}w\frac{\partial\Delta p_{nh}}{\partial z} - \frac{\partial z_b}{\partial t}\,\Delta p_{nh}|_b + \mathcal{O}(\varepsilon^2\delta^2),$$

where relation (31) has been used. From Eqs. (48) and (57), we have

$$\Delta p_{nh}|_s = \mathcal{O}(\varepsilon^2\delta^2),$$

$$\Delta p_{nh}|_b = \varepsilon^2\delta\int_{z_b}^{\delta\eta}\frac{\partial w}{\partial t} + \mathcal{O}(\varepsilon^2\delta^2),$$

$$\frac{\partial\Delta p_{nh}}{\partial z} = -\varepsilon^2\delta\frac{\partial w}{\partial t} + \mathcal{O}(\varepsilon^2\delta^2),$$

leading to

$$
\begin{aligned}
\mathcal{C}_1 &= \frac{\partial}{\partial x} \int_{z_b}^{\delta\eta} u \Delta p_{nh} + \varepsilon^2 \delta \int_{z_b}^{\delta\eta} w \frac{\partial w}{\partial t} - \frac{\partial z_b}{\partial t} \left. \Delta p_{nh} \right|_b + \mathcal{O}(\varepsilon^2 \delta^2) \\
&= \frac{\partial}{\partial x} \int_{z_b}^{\delta\eta} u \Delta p_{nh} + \varepsilon^2 \delta \frac{\partial}{\partial t} \int_{z_b}^{\delta\eta} \frac{w^2}{2} - \frac{\partial z_b}{\partial t} \left. \Delta p_{nh} \right|_b + \mathcal{O}(\varepsilon^2 \delta^2).
\end{aligned}
$$

Due to the rescaling applied to the time derivative of z_b (see paragraph 2.2), the Leibniz rule applied to obtain the preceding relation reads

$$
\int_{z_b}^{\delta\eta} w \frac{\partial w}{\partial t} = \delta \frac{\partial z_b}{\partial t} \frac{w_b^2}{2} - \delta \frac{\partial \eta}{\partial t} \frac{w_s^2}{2} + \frac{\partial}{\partial t} \int_{z_b}^{\delta\eta} \frac{w^2}{2} = \frac{\partial}{\partial t} \int_{z_b}^{\delta\eta} \frac{w^2}{2} + \mathcal{O}(\delta).
$$

And finally we have for $\mathcal{C}_1$

$$
\mathcal{C}_1 = \frac{\partial}{\partial x} \int_{z_b}^{\delta\eta} u \Delta p_{nh} + \varepsilon^2 \delta \frac{\partial}{\partial t} \int_{z_b}^{\delta\eta} \frac{w^2}{2} - \frac{\partial z_b}{\partial t} \left. \Delta p_{nh} \right|_b + \mathcal{O}(\varepsilon^2 \delta^2).
$$

From relations (42) and (43) we obtain

$$
u = \left(1 + \frac{\varepsilon \kappa_0}{\nu_0} \left(z - z_b - \frac{(z - z_b)^2}{2 H_\delta} - \frac{H_\delta}{3} \right) \right) \bar{u} = \left(1 + \varepsilon f(z - z_b, H_\delta) \right) \bar{u} + \mathcal{O}(\varepsilon^2),
$$

so we have for $\mathcal{C}_2$ and $\mathcal{C}_3$

$$
\begin{aligned}
\mathcal{C}_2 &= \frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\bar{u}\left(2\varepsilon\delta\nu_0\frac{\partial u}{\partial x}\right) - 2\varepsilon\delta\nu_0\int_{z_b}^{\delta\eta}\frac{\partial\bar{u}}{\partial x}\left(\frac{\partial u}{\partial x}\right), \\
&= \frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\bar{u}\left(2\varepsilon\delta\nu_0\frac{\partial u}{\partial x}\right) - 2\varepsilon\delta\nu_0\left(\int_{z_b}^{\delta\eta}\left(\frac{\partial\bar{u}}{\partial x}\right)^2 + \varepsilon\frac{\partial\bar{u}}{\partial x}\int_{z_b}^{\delta\eta}\frac{\partial(f\bar{u})}{\partial x}\right) + \mathcal{O}(\varepsilon^3\delta), \\
\mathcal{C}_3 &= \frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\bar{u}p_{v,f} - \int_{z_b}^{\delta\eta}\frac{\partial\bar{u}}{\partial x}p_{v,f}, \\
&= \frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\bar{u}p_{v,f} + \int_{z_b}^{\delta\eta}\frac{\partial w}{\partial z}p_{v,f} + \varepsilon\int_{z_b}^{\delta\eta}\frac{\partial(f\bar{u})}{\partial x}p_{v,f} + \mathcal{O}(\varepsilon^3\delta), \\
&= \frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\bar{u}p_v + [wp_{v,f}]_{z_b}^{\delta\eta} - \varepsilon\delta\nu_0\int_{z_b}^{\delta\eta}w\frac{\partial^2 u}{\partial x\partial z} - 2\varepsilon\delta\nu_0\int_{z_b}^{\delta\eta}w\frac{\partial^2 w}{\partial z^2} \\
&\quad + \varepsilon\int_{z_b}^{\delta\eta}\frac{\partial(f\bar{u})}{\partial x}p_{v,f} + \mathcal{O}(\varepsilon^3\delta), \\
&= \frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\bar{u}p_{v,f} + [wp_{v,f}]_{z_b}^{\delta\eta} - 2\varepsilon\delta\nu_0\int_{z_b}^{\delta\eta}\frac{\partial}{\partial z}\left(w\frac{\partial w}{\partial z}\right) + 2\varepsilon\delta\nu_0\int_{z_b}^{\delta\eta}\left(\frac{\partial w}{\partial z}\right)^2 \\
&\quad -\varepsilon\delta\nu_0\frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\left(w\frac{\partial u}{\partial z}\right) + \varepsilon\delta^2\nu_0\frac{\partial\eta}{\partial x}w_s\left.\frac{\partial u}{\partial z}\right|_s - \varepsilon\delta\nu_0\frac{\partial z_b}{\partial x}w_b\left.\frac{\partial u}{\partial z}\right|_b \\
&\quad +\varepsilon\delta\nu_0\int_{z_b}^{\delta\eta}\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + \varepsilon\int_{z_b}^{\delta\eta}\frac{\partial(f\bar{u})}{\partial x}p_{v,f} + \mathcal{O}(\varepsilon^3\delta), \\
&= \frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\bar{u}p_{v,f} + [wp_{v,f}]_{z_b}^{\delta\eta} - 2\varepsilon\delta\nu_0\left(w_s\left.\frac{\partial w}{\partial z}\right|_s - w_b\left.\frac{\partial w}{\partial z}\right|_b\right) \\
&\quad +2\varepsilon\delta\nu_0\int_{z_b}^{\delta\eta}\left(\frac{\partial w}{\partial z}\right)^2 - \varepsilon\delta\nu_0\frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\left(w\frac{\partial u}{\partial z}\right) + \varepsilon\delta^2\nu_0\frac{\partial\eta}{\partial x}w_s\left.\frac{\partial u}{\partial z}\right|_s \\
&\quad -\varepsilon\delta\nu_0\frac{\partial z_b}{\partial x}w_b\left.\frac{\partial u}{\partial z}\right|_b + \varepsilon\delta\nu_0\int_{z_b}^{\delta\eta}\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + \varepsilon\int_{z_b}^{\delta\eta}\frac{\partial(f\bar{u})}{\partial x}p_{v,f} + \mathcal{O}(\varepsilon^3\delta),
\end{aligned}
$$

and from relation (41) we also have

$$
\begin{aligned}
\nu_0\int_{z_b}^{\delta\eta}\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} &= \varepsilon\kappa_0\int_{z_b}^{\delta\eta}\frac{\partial w}{\partial x}\left(1 - \frac{z-z_b}{H_\delta}\right)u_b + \mathcal{O}(\varepsilon^2), \\
&= \varepsilon\kappa_0\frac{H_\delta}{2}\frac{\partial^2 z_b}{\partial x\partial t}u_b + \varepsilon\kappa_0\int_{z_b}^{\delta\eta}\left(-z\frac{\partial^2\bar{u}}{\partial x^2} + \frac{\partial^2(z_b\bar{u})}{\partial x^2}\right)\left(1 - \frac{z-z_b}{H_\delta}\right)\bar{u} + \mathcal{O}(\varepsilon^2), \\
&= \varepsilon\kappa_0\frac{H_\delta}{2}\frac{\partial^2 z_b}{\partial x\partial t}u_b - \varepsilon\kappa_0\frac{H_\delta^2}{6}\frac{\partial^2\bar{u}}{\partial x^2}\bar{u} + \varepsilon\kappa_0\frac{H_\delta}{2}\frac{\partial}{\partial x}\left(\frac{\partial z_b}{\partial x}\bar{u}^2\right) + \mathcal{O}(\varepsilon^2), \\
&= \varepsilon\kappa_0\frac{H_\delta}{2}\frac{\partial^2 z_b}{\partial x\partial t}u_b - \varepsilon\kappa_0\frac{\partial}{\partial x}\left(\frac{H_\delta^2}{6}\frac{\partial\bar{u}}{\partial x}\bar{u}\right) + \varepsilon\kappa_0\frac{H_\delta^2}{6}\left(\frac{\partial\bar{u}}{\partial x}\right)^2 \\
&\quad +\varepsilon\kappa_0\frac{H_\delta}{3}\frac{\partial H_\delta}{\partial x}\frac{\partial\bar{u}}{\partial x}\bar{u} + \varepsilon\kappa_0\frac{\partial}{\partial x}\left(\frac{H_\delta}{2}\frac{\partial z_b}{\partial x}\bar{u}^2\right) - \frac{\varepsilon\kappa_0}{2}\frac{\partial H_\delta}{\partial x}\frac{\partial z_b}{\partial x}\bar{u}^2 + \mathcal{O}(\varepsilon^2).
\end{aligned}
$$

The preceding expression shows that due to relation (41), the term

$$
\nu_0\int_{z_b}^{\delta\eta}\frac{\partial w}{\partial x}\frac{\partial u}{\partial z},
$$

has to be treated as a friction term in the energy equality. We finally have for $\mathcal{R} = \mathcal{C}_2 - \mathcal{C}_3 + \mathcal{C}_4$

$$
\begin{aligned}
\mathcal{R} &= \frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\left(2\varepsilon\delta\nu_0\bar{u}\frac{\partial u}{\partial x} - p_{v,f}\right) - 2\varepsilon\delta\nu_0\int_{z_b}^{\delta\eta}\left(\left(\frac{\partial\bar{u}}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right) \\
&\quad + \varepsilon\delta\nu_0\frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\left(w\frac{\partial u}{\partial z}\right) + \varepsilon^2\delta\frac{\partial}{\partial x}\left(\kappa_0\frac{H_\delta}{2}\left(\frac{H_\delta}{3}\frac{\partial\bar{u}}{\partial x}\bar{u} - \frac{\partial z_b}{\partial x}\bar{u}^2\right)\right) \\
&\quad - \frac{\varepsilon^2\delta\kappa_0}{6}\left(\left(H_\delta\frac{\partial\bar{u}}{\partial x} + \frac{\partial H_\delta}{\partial x}\bar{u}\right)^2 - \left(-2\left(\frac{\partial z_b}{\partial x}\right)^2 + \delta\frac{\partial\eta}{\partial x}\frac{\partial z_b}{\partial x}\right.\right. \\
&\quad \left.\left. + \delta^2\left(\frac{\partial\eta}{\partial x}\right)^2\right)\bar{u}^2\right) + \frac{\partial z_b}{\partial t}\left(p_{v,f}|_b + 2\varepsilon\delta\nu_0\left.\frac{\partial u}{\partial x}\right|_b\right) \\
&\quad - \varepsilon^2\delta\kappa_0\frac{H_\delta}{2}\frac{\partial^2 z_b}{\partial x\partial t}u_b + \mathcal{O}(\varepsilon^3\delta).
\end{aligned}
$$

Returning to the initial variables and integrating $\mathcal{C}_1$ and $\mathcal{R}$ into relation (62) gives an energy equality for the model (46)-(47) under the form

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(E_h + \frac{H\overline{w^2}}{2}\right) + \frac{\partial}{\partial x}\left(\bar{u}\left(E_h + H\bar{p}_{nh}\right) - \nu\int_{z_b}^{\eta}\left(2H\bar{u}\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right)\right) \\
&\quad + \frac{\partial}{\partial x}\left(\kappa\left(\frac{z_b^2}{6}\frac{\partial\bar{u}}{\partial x}\bar{u} + \frac{z_b}{2}\frac{\partial z_b}{\partial x}\bar{u}^2\right)\right) \\
&= -2\nu\int_{z_b}^{\eta}\left(\left(\frac{\partial\bar{u}}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right) - \frac{\kappa}{6}\left(z_b\frac{\partial\bar{u}}{\partial x} + \frac{\partial z_b}{\partial x}\bar{u}\right)^2 \\
&\quad - \frac{\kappa}{1 + \frac{\kappa H}{3\nu}}\left(1 + \frac{11}{6}\left(\frac{\partial z_b}{\partial x}\right)^2\right)\bar{u}^2 \\
&\quad - H\frac{\partial p^a}{\partial t} + \left(p_{nh}|_b + 2\nu\frac{\partial u_b}{\partial x}\right)\frac{\partial z_b}{\partial t} + \kappa\frac{z_b}{2}\frac{\partial^2 z_b}{\partial x\partial t}\bar{u},
\end{aligned}
$$

where

$$
\begin{aligned}
H\overline{w^2} &= \int_{z_b}^{\eta}w^2 = \int_{z_b}^{\eta}\left(\frac{\partial z_b}{\partial t} - z\frac{\partial\bar{u}}{\partial x} + \frac{\partial(z_b\bar{u})}{\partial x}\right)^2 = -\frac{z_b^2}{3}\left(\frac{\partial\bar{u}}{\partial x}\right)^2 \\
&\quad - z_b\left(\frac{\partial(z_b\bar{u})}{\partial x}\right)^2 + z_b^2\frac{\partial\bar{u}}{\partial x}\frac{\partial(z_b\bar{u})}{\partial x} - z_b\left(\frac{\partial z_b}{\partial t}\right)^2 \\
&\quad + 2z_b\frac{\partial z_b}{\partial t}\left(\frac{z_b}{2}\frac{\partial\bar{u}}{\partial x} - \frac{\partial(z_b\bar{u})}{\partial x}\right), \\
H\bar{p}_{nh} &= \int_{z_b}^{\eta}p_{nh}.
\end{aligned}
$$

When the time derivatives of p^a and z_b are dropped, the right hand side of the preceding energy equality is always negative.

4.4 A more complex approximation, $\delta = \mathcal{O}(1)$

Now we return to the dimensionless and rescaled variables. The assumption that the elevation of the free surface is small done in paragraph 4.2 is now relaxed

i.e. $\delta = \mathcal{O}(1)$. This means that no assumption is made concerning the hydraulic regime. We consider for the pressure p the complete expression obtained in (35) and the following proposition is a refinement of the Proposition 2.

Proposition 3 *The system defined by*

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0, \tag{63}$$

$$\frac{\partial}{\partial t}(H\bar{u}) + \frac{\partial}{\partial x}(H_m\bar{u}^2) + \frac{1}{2}\frac{\partial H^2}{\partial x} + \frac{\partial(H\bar{p}_{ng,nv})}{\partial x} = -H\frac{\partial p^a}{\partial x} - gH\frac{\partial z_b}{\partial x}$$

$$+ \frac{\partial}{\partial x}\left(4\nu H\frac{\partial \bar{u}}{\partial x}\right) + \frac{\partial}{\partial x}\left(\kappa(\mathbf{v}_b,H)H\left(\frac{H}{6}\frac{\partial \bar{u}}{\partial x} - \left(\frac{7}{6}\frac{\partial z_b}{\partial x} + \frac{1}{3}\frac{\partial \eta}{\partial x}\right)\bar{u}\right)\right)$$

$$- \frac{\kappa(\mathbf{v}_b,H)}{1 + \frac{\kappa(\mathbf{v}_b,H)H}{3\nu}}\left(1 + \frac{5}{2}\left(\frac{\partial z_b}{\partial x}\right)^2\right)\bar{u} + \kappa(\mathbf{v}_b,H)\frac{\partial z_b}{\partial x}\left((\frac{1}{2}\frac{\partial H}{\partial x} + \frac{\partial z_b}{\partial x})\bar{u}\right.$$

$$\left. + \frac{H}{2}\frac{\partial \bar{u}}{\partial x}\right) - \frac{\partial z_b}{\partial x}\,p_{ng,nv}|_b + z_b\frac{\partial z_b}{\partial x}\frac{\partial^2 z_b}{\partial t^2} - \frac{1}{2}\frac{\partial}{\partial x}\left(H^2\frac{\partial^2 z_b}{\partial t^2}\right), \tag{64}$$

where $\bar{\mathbf{v}} = (1, \frac{\partial z_b}{\partial x})^T\frac{\bar{u}}{1 + \frac{\kappa_l H}{3\nu}}$ *results from an approximation in* $\mathcal{O}(\varepsilon^3\delta)$ *of the Navier-Stokes equations. In the previous expressions,* H_m *is a modified water height taking into account the Coriolis-Boussinesq coefficient and* $\bar{p}_{ng,nv}$*,* $p_{ng,nv}|_b$ *corresponds to the vertically averaged and bottom value of the non gravitational and non viscous part of the pressure* p *given by (35).*

Proof of prop. 3: we still start from the averaged momentum equation (33) where, compared to the first extension of the Saint-Venant model detailed in paragraphs 4.1, 4.2 and 4.3, the expressions of

$$\int_{z_b}^{\delta\eta} p\,dz \quad \text{and} \quad \int_{z_b}^{\delta\eta} u^2 dz,$$

have to be refined. The approximation $\overline{u^2} = \bar{u}^2 + \mathcal{O}(\varepsilon^2)$ obtained in paragraph 3.2 is no more sufficient. From (27), (38), (42) and (43) we get

$$u = \left(1 + \frac{\varepsilon\kappa_0}{\nu_0}\left(z - z_b - \frac{(z - z_b)^2}{2H_\delta} - \frac{H_\delta}{3}\right)\right)\bar{u} + \mathcal{O}(\varepsilon^2)$$

$$= \left(1 + \varepsilon f(z - z_b, H_\delta)\right)\bar{u} + \mathcal{O}(\varepsilon^2),$$

$$w = \frac{\partial z_b}{\partial t} - \frac{\partial}{\partial x}\left(\left(z - z_b + \varepsilon\int_{z_b}^z f(z, z_b, H_\delta)dz\right)\bar{u}\right) + \mathcal{O}(\varepsilon^2)$$

$$= \frac{\partial z_b}{\partial t} - \frac{\partial}{\partial x}\left(g(z, z_b, H_\delta)\bar{u}\right) + \mathcal{O}(\varepsilon^2),$$

$$\frac{\partial p}{\partial x} = \delta\frac{\partial}{\partial x}(p^a + \eta) - 2\varepsilon\delta\frac{\partial}{\partial x}\left(\nu_0\frac{\partial \bar{u}}{\partial x}\right) + \mathcal{O}(\varepsilon^2\delta),$$

and Eq. (46) is equivalent to

$$\delta\frac{\partial \bar{u}}{\partial t} + \delta^2\bar{u}\frac{\partial \bar{u}}{\partial x} + \delta\frac{\partial \eta}{\partial x} = -\delta\frac{\partial}{\partial x}p^a - \frac{\delta\kappa_0}{H_\delta\left(1 + \frac{\varepsilon\kappa_0}{3\nu_0}H_\delta\right)}\bar{u}$$

$$+ \frac{\varepsilon\delta}{H_\delta}\frac{\partial}{\partial x}\left(4\nu_0 H_\delta\frac{\partial \bar{u}}{\partial x}\right) + \mathcal{O}(\varepsilon^2\delta).$$

Now we can improve the approximation (40) in the following way

$$
\begin{aligned}
\delta\frac{\partial}{\partial z}\left(\nu_0\frac{\partial u}{\partial z}\right) &= \varepsilon\delta\frac{\partial u}{\partial t} + \varepsilon\delta^2 u\frac{\partial u}{\partial x} + \varepsilon\delta^2 w\frac{\partial u}{\partial z} + \varepsilon\frac{\partial p}{\partial x} - \varepsilon^2\delta\frac{\partial}{\partial x}\left(\nu_0\frac{\partial u}{\partial x}\right) \\
&= \varepsilon\delta\frac{\partial}{\partial t}\big((1+\varepsilon f)\bar{u}\big) + \varepsilon\delta^2(1+\varepsilon f)\bar{u}\frac{\partial}{\partial x}\big((1+\varepsilon f)\bar{u}\big) \\
&\quad + \varepsilon\delta\frac{\partial}{\partial x}(\eta + p^a) + \varepsilon\delta^2 w\frac{\partial u}{\partial z} - 3\varepsilon^2\delta\frac{\partial}{\partial x}\left(\nu_0\frac{\partial \bar{u}}{\partial x}\right) + \mathcal{O}(\varepsilon^3\delta) \\
&= -\frac{\varepsilon\delta\kappa_0}{H_\delta\left(1+\frac{\varepsilon\kappa_0}{3\nu_0}H_\delta\right)}\bar{u} + \varepsilon\delta^2 w\frac{\partial u}{\partial z} + \varepsilon^2\delta\frac{\partial}{\partial t}\big(f\bar{u}\big) + \varepsilon^2\delta^2\frac{\partial}{\partial x}\big(f\bar{u}^2\big) \\
&\quad + \varepsilon^2\delta\frac{\partial}{\partial x}\left(\nu_0\frac{\partial \bar{u}}{\partial x}\right) + \varepsilon^2\delta\frac{4\nu_0}{H_\delta}\frac{\partial H_\delta}{\partial x}\frac{\partial \bar{u}}{\partial x} + \mathcal{O}(\varepsilon^3\delta).
\end{aligned}
$$

Taking into acount the boundary condition (32), an integration of the preceding relation from z_b to z gives

$$
\begin{aligned}
\nu_0\frac{\partial u}{\partial z} &= \frac{\varepsilon\kappa_0}{\left(1+\frac{\varepsilon\kappa_0}{3\nu_0}H_\delta\right)}\left(1-\frac{z-z_b}{H_\delta}\right)\bar{u} + \varepsilon^2\delta\left(\delta\bar{u}fw + \delta\bar{u}\frac{\partial \bar{u}}{\partial x}\int_{z_b}^z f\right. \\
&\quad \left.+ \frac{\partial}{\partial t}\left(\bar{u}\int_{z_b}^z f\right) + \delta\frac{\partial}{\partial x}\left(\bar{u}^2\int_{z_b}^z f\right) + (z-z_b)\frac{\partial}{\partial x}\left(\nu_0\frac{\partial \bar{u}}{\partial x}\right)\right. \\
&\quad \left.+ \frac{4\nu_0(z-z_b)}{H_\delta}\frac{\partial H_\delta}{\partial x}\frac{\partial \bar{u}}{\partial x}\right) + \mathcal{O}(\varepsilon^3\delta), \qquad (65)
\end{aligned}
$$

where the relation

$$
\int_{z_b}^z w\frac{\partial u}{\partial z} = \varepsilon\bar{u}\int_{z_b}^z w\frac{\partial f}{\partial z} = \varepsilon\bar{u}(fw - f|_b w|_b) + \varepsilon\bar{u}\frac{\partial \bar{u}}{\partial x}\int_{z_b}^z f,
$$

has been used. Another integration of relation (65) between z_b and z gives

$$
\begin{aligned}
u &= \left(1+\frac{\varepsilon\kappa_0}{\nu_0}\left(z-z_b-\frac{(z-z_b)^2}{2H_\delta}\right)\right)u_b + \frac{\varepsilon^2\delta^2}{\nu_0}\bar{u}\int_{z_b}^z fw \\
&\quad + \frac{\varepsilon^2\delta^2}{\nu_0}\bar{u}\frac{\partial \bar{u}}{\partial x}\int_{z_b}^z\int_{z_b}^{z_1} f + \frac{\varepsilon^2\delta}{\nu_0}\frac{\partial}{\partial t}\left(\bar{u}\int_{z_b}^z\int_{z_b}^{z_1} f\right) + \frac{\varepsilon^2\delta^2}{\nu_0}\frac{\partial}{\partial x}\left(\bar{u}^2\int_{z_b}^z\int_{z_b}^{z_1} f\right) \\
&\quad + \frac{\varepsilon^2\delta}{2}(z-z_b)^2\frac{\partial^2 \bar{u}}{\partial x^2} + \varepsilon^2\delta\frac{2(z-z_b)^2}{H_\delta}\frac{\partial H_\delta}{\partial x}\frac{\partial \bar{u}}{\partial x} + \mathcal{O}(\varepsilon^3\delta), \\
&= \left(1+\frac{\varepsilon\kappa_0}{\nu_0}\left(z-z_b-\frac{(z-z_b)^2}{2H_\delta}\right)\right)u_b + \varepsilon^2\delta\Delta u + \mathcal{O}(\varepsilon^3\delta),
\end{aligned}
$$

so we obtain the new expressions for $\bar{u}$, $\bar{u}^2$ and u^2

$$
\begin{aligned}
\bar{u} &= \left(1+\frac{\varepsilon\kappa_0}{3\nu_0}H_\delta\right)u_b + \varepsilon^2\delta\overline{\Delta u} + \mathcal{O}(\varepsilon^3\delta), \\
\bar{u}^2 &= \left(1+\frac{2\varepsilon\kappa_0}{3\nu_0}H_\delta\right)u_b^2 + 2\varepsilon^2\delta\overline{\Delta u} + \mathcal{O}(\varepsilon^3\delta), \\
u^2 &= \left(1+\frac{2\varepsilon\kappa_0}{\nu_0}\left(z-z_b-\frac{(z-z_b)^2}{2H_\delta}\right) + \frac{\varepsilon^2\kappa_0^2}{\nu_0}\left(z-z_b-\frac{(z-z_b)^2}{2H_\delta}\right)^2\right)u_b \\
&\quad + 2\varepsilon^2\delta\Delta u + \mathcal{O}(\varepsilon^3\delta),
\end{aligned}
$$

so finally

$$\overline{u^2} = \left(1 + \frac{2\varepsilon\kappa_0}{3\nu_0}H_\delta + \frac{2\varepsilon^2\kappa_0^2}{15\nu_0^2}H_\delta^2\right)u_b^2 + 2\varepsilon^2\delta\overline{\Delta u}$$

$$= \left(1 + \frac{2\varepsilon^2\kappa_0^2}{15\nu_0^2}H_\delta^2\right)\bar{u}^2 + \mathcal{O}(\varepsilon^3\delta).$$

Now concerning the expression of the pressure terms, it has to be noticed that Eqs. (35) and (48) only differ by the terms

$$\mathcal{A} = \varepsilon^2\delta^2\frac{\partial}{\partial x}\int_z^{\delta\eta} uw\ dz - \varepsilon^2\delta^2 w^2.$$

Using

$$u = \bar{u} + \mathcal{O}(\varepsilon), \quad w = \frac{\partial z_b}{\partial t} - \frac{\partial}{\partial x}\int_{z_b}^z u\ dz,$$

it comes

$$\mathcal{A} = \varepsilon^2\delta^2\left(-\frac{\delta^2\eta^2-z^2}{2}\frac{\partial}{\partial x}\big(\bar{u}\frac{\partial\bar{u}}{\partial x}\big) - \delta^2\eta\frac{\partial\eta}{\partial x}\frac{\partial\bar{u}}{\partial x}\bar{u} + (\delta\eta-z)\frac{\partial}{\partial x}\big(\bar{u}\frac{\partial(z_b\bar{u})}{\partial x}\big)\right.$$
$$\left. + \delta\frac{\partial\eta}{\partial x}\frac{\partial(z_b\bar{u})}{\partial x}\bar{u} - \left(-z\frac{\partial\bar{u}}{\partial x} + \frac{\partial(z_b\bar{u})}{\partial x}\right)^2\right).$$

This leads to the new expression for the fluid pressure p appearing in (33)

$$\int_{z_b}^{\delta\eta} p\ dz = \int_{z_b}^{\delta\eta}(p_{nh}+\mathcal{A})dz$$
$$= \int_{z_b}^{\delta\eta} p_{nh}dz + \frac{\varepsilon^2\delta^2 H_\delta}{6}\left(-4H_\delta^2\big(\frac{\partial\bar{u}}{\partial x}\big)^2 - 2H_\delta^2\bar{u}\frac{\partial^2\bar{u}}{\partial x^2} - 6H_\delta\frac{\partial H_\delta}{\partial x}\frac{\partial\bar{u}}{\partial x}\bar{u}\right.$$
$$\left. + 9H_\delta\frac{\partial z_b}{\partial x}\frac{\partial\bar{u}}{\partial x}\bar{u} + 3H_\delta\frac{\partial^2 z_b}{\partial x^2}\bar{u}^2 + 6\frac{\partial z_b}{\partial x}\frac{\partial H_\delta}{\partial x}\bar{u}^2\right) + \mathcal{O}(\varepsilon^3\delta),$$

where $\int_{z_b}^{\delta\eta} p_{nh} = \int_{z_b}^{\delta\eta}(p_h + \Delta p_{nh})$ is given by (58). Conversely using (35) we obtain

$$p_b = p_{nh}|_b + \frac{\varepsilon^2\delta^2}{2}\left(-\frac{\partial}{\partial x}\big(H_\delta^2\frac{\partial\bar{u}}{\partial x}\bar{u}\big) + 4H_\delta\frac{\partial z_b}{\partial x}\frac{\partial\bar{u}}{\partial x}\bar{u} + 2\frac{\partial}{\partial x}\big(H_\delta\frac{\partial z_b}{\partial x}\big)\bar{u}^2\right) + \mathcal{O}(\varepsilon^3\delta),$$

where $p_{nh}|_b$ is given by (59). Inserting (54), (55) and (56) in equilibrium (33) leads to the system

$$\frac{\partial H_\delta}{\partial t} + \frac{\partial}{\partial x}\big(H_\delta\bar{u}\big) = 0, \tag{66}$$

$$\delta\frac{\partial}{\partial t}(H_\delta\bar{u}) + \delta^2\frac{\partial}{\partial x}(H_{\delta,m}\bar{u}^2) + \frac{1}{2}\frac{\partial H_\delta^2}{\partial x} + \frac{\partial(H_\delta\bar{p}_{ng,nv})}{\partial x} = -H_\delta\frac{\partial}{\partial x}(\delta p^a + z_b)$$

$$+ \varepsilon\delta\frac{\partial}{\partial x}\big(4\nu_0 H_\delta\frac{\partial\bar{u}}{\partial x}\big) + \varepsilon^2\delta\frac{\partial}{\partial x}\left(\kappa_0 H_\delta\big(\frac{H_\delta}{6}\frac{\partial\bar{u}}{\partial x} - \big(\frac{7}{6}\frac{\partial z_b}{\partial x} + \frac{\delta}{3}\frac{\partial\eta}{\partial x}\big)\bar{u}\big)\right)$$

$$+ \varepsilon^2\delta\kappa_0\frac{\partial z_b}{\partial x}\left(\big(\frac{1}{2}\frac{\partial H_\delta}{\partial x} + \frac{\partial z_b}{\partial x}\big)\bar{u} + \frac{H_\delta}{2}\frac{\partial\bar{u}}{\partial x}\right) - \frac{\partial z_b}{\partial x}p_{ng,nv}|_b$$

$$- \delta\kappa_0\left(1 + \frac{5\varepsilon^2}{2}\big(\frac{\partial z_b}{\partial x}\big)^2\right)u_b + \varepsilon^2\delta z_b\frac{\partial z_b}{\partial x}\frac{\partial^2 z_b}{\partial t^2} - \frac{1}{2}\frac{\partial}{\partial x}\left(H_\delta^2\frac{\partial^2 z_b}{\partial t^2}\right) + \mathcal{O}(\varepsilon^3\delta) \tag{67}$$

where

$$H_\delta^m = H_\delta\left(1 + \frac{2\varepsilon^2\kappa_0^2}{15\nu_0^2}H_\delta^2\right),$$

$$
\begin{aligned}
H_\delta \bar{p}_{ng,nv} &= \int_{z_b}^{\delta\eta} (p - p_h)\,dz \\
&= \varepsilon^2\delta\frac{\partial}{\partial x}\left(\frac{H_\delta^3}{6}\frac{\partial^2\bar{u}}{\partial x\partial t} + \frac{H_\delta^2}{2}\frac{\partial^2(z_b\bar{u})}{\partial x\partial t} - \delta\eta\frac{H_\delta^2}{2}\frac{\partial^2\bar{u}}{\partial x\partial t}\right. \\
&\quad \left. -\delta H_\delta\frac{\partial\eta}{\partial t}\left(\delta\eta\frac{\partial\bar{u}}{\partial x} - \frac{\partial(z_b\bar{u})}{\partial x}\right)\right) + \frac{\varepsilon^2\delta^2 H_\delta}{6}\left(-4H_\delta^2\left(\frac{\partial\bar{u}}{\partial x}\right)^2\right. \\
&\quad -2H_\delta^2\bar{u}\frac{\partial^2\bar{u}}{\partial x^2} - 6H_\delta\frac{\partial H_\delta}{\partial x}\frac{\partial\bar{u}}{\partial x}\bar{u} + 9H_\delta\frac{\partial z_b}{\partial x}\frac{\partial\bar{u}}{\partial x}\bar{u} \\
&\quad \left. +3H_\delta\frac{\partial^2 z_b}{\partial x^2}\bar{u}^2 + 6\frac{\partial z_b}{\partial x}\frac{\partial H_\delta}{\partial x}\bar{u}^2\right) + \varepsilon^2\delta\frac{H_\delta^2}{2}\frac{\partial^2 z_b}{\partial t^2} + \mathcal{O}(\varepsilon^3\delta),
\end{aligned}
$$

and

$$
\begin{aligned}
p_{ng,nv}|_b &= (p - p_h)|_b \\
&= \frac{\varepsilon^2\delta}{2}\left(-\frac{\partial}{\partial t}\left(H_\delta^2\frac{\partial\bar{u}}{\partial x}\right) + 2H_\delta\frac{\partial}{\partial t}\left(\frac{\partial z_b}{\partial x}\bar{u}\right) + 2\delta\frac{\partial\eta}{\partial t}\frac{\partial z_b}{\partial x}\bar{u}\right) \\
&\quad +\varepsilon^2\delta\left(H_\delta\frac{\partial^2 z_b}{\partial t^2} + \delta\frac{\partial\eta}{\partial t}\frac{\partial z_b}{\partial t}\right) + 4H_\delta\frac{\partial z_b}{\partial x}\frac{\partial\bar{u}}{\partial x}\bar{u} + 2\frac{\partial}{\partial x}\left(H_\delta\frac{\partial z_b}{\partial x}\right)\bar{u}^2 \\
&\quad +\frac{\varepsilon^2\delta^2}{2}\left(-\frac{\partial}{\partial x}\left(H_\delta^2\frac{\partial\bar{u}}{\partial x}\bar{u}\right)\right) + \varepsilon^2\delta H_\delta\frac{\partial^2 z_b}{\partial t^2} + \mathcal{O}(\varepsilon^3\delta), \\
&= \frac{\varepsilon^2\delta}{2}\left(H_\delta^2\frac{\partial^2\bar{u}}{\partial x\partial t} + 2H_\delta\frac{\partial^2(z_b\bar{u})}{\partial x\partial t} + 2\delta\frac{\partial\eta}{\partial t}\frac{\partial(z_b\bar{u})}{\partial x} - 2\delta\eta\left(\delta\frac{\partial\eta}{\partial t}\frac{\partial\bar{u}}{\partial x}\right.\right. \\
&\quad \left.\left. +H_\delta\frac{\partial^2\bar{u}}{\partial x\partial t}\right)\right) + \varepsilon^2\delta\left(H_\delta\frac{\partial^2 z_b}{\partial t^2} + \delta\frac{\partial\eta}{\partial t}\frac{\partial z_b}{\partial t}\right) + \varepsilon^2\delta H_\delta\frac{\partial^2 z_b}{\partial t^2} \\
&\quad +\frac{\varepsilon^2\delta^2}{2}\left(-\frac{\partial}{\partial x}\left(H_\delta^2\frac{\partial\bar{u}}{\partial x}\bar{u}\right)\right) + 2\frac{\partial}{\partial x}\left(H_\delta\frac{\partial z_b}{\partial x}\bar{u}^2\right) + \mathcal{O}(\varepsilon^3\delta).
\end{aligned}
$$

In terms of the initial variables, the model (66)-(67) corresponds to the one depicted in proposition 3 with obvious expressions for H_m, $H\bar{p}_{ng,nv}$ and $p_{ng,nv}|_p$.

In order to obtain the energy equality for the model (63)-(64), we use the same process and the same notations as in paragraph 4.3 but the approximation order is now $\mathcal{O}(\varepsilon^3\delta)$ instead of $\mathcal{O}(\varepsilon^2\delta^2)$. Still using $\bar{u} = u + \mathcal{O}(\varepsilon) = u_b + \mathcal{O}(\varepsilon)$, we have

$$
\begin{aligned}
\tilde{\mathcal{C}}_1 &= \bar{u}\frac{\partial}{\partial x}\int_{z_b}^{\delta\eta}\Delta p + \bar{u}\frac{\partial z_b}{\partial x}\Delta p|_b \\
&= \frac{\partial}{\partial x}\left(\int_{z_b}^{\delta\eta}u\Delta p\right) + [w\Delta p]_{z_b}^{\delta\eta} - \int_{z_b}^{\delta\eta}w\frac{\partial\Delta p}{\partial z} + u_b\frac{\partial z_b}{\partial x}\Delta p|_b + \mathcal{O}(\varepsilon^3\delta),
\end{aligned}
$$

with $\Delta p = p - p_{nh}$, p being given by (35). From Eqs. (35), (57) and the boundary condition (28), we get

$$\Delta p|_s = \mathcal{O}(\varepsilon^3\delta), \quad \Delta p|_b = \varepsilon^2\delta^2\int_{z_b}^{\delta\eta}\frac{\partial(uw)}{\partial x} + \varepsilon^2\delta^2(w_s^2 - w_b^2) + \mathcal{O}(\varepsilon^3\delta),$$

$$\frac{\partial \Delta p}{\partial z} = -\varepsilon^2 \delta^2 \frac{\partial(uw)}{\partial x} - 2\varepsilon^2 \delta^2 w \frac{\partial w}{\partial z} + \mathcal{O}(\varepsilon^3 \delta),$$

leading to

$$
\begin{aligned}
\tilde{\mathcal{C}}_1 &= \frac{\partial}{\partial x} \int_{z_b}^{\delta\eta} u \Delta p + \varepsilon^2 \delta^2 \int_{z_b}^{\delta\eta} w \frac{\partial uw}{\partial x} + \frac{2}{3}\varepsilon^2 \delta^2 (w_s^3 - w_b^2) \\
&\quad - \frac{\partial z_b}{\partial t} \Delta p|_b + \mathcal{O}(\varepsilon^3 \delta), \\
&= \frac{\partial}{\partial x} \left(\int_{z_b}^{\delta\eta} u \big(\Delta p + \frac{w^2}{2}\big) \right) - \frac{\partial z_b}{\partial t} \Delta p|_b + \mathcal{O}(\varepsilon^3 \delta).
\end{aligned}
$$

Returning to the initial variables, the preceding relation and the expression of $\mathcal{R}$ obtained in paragraph 4.3 allows us to write an energy equality for the model (63)-(64) under the form

$$
\begin{aligned}
&\frac{\partial \bar{E}}{\partial t} + \frac{\partial}{\partial x} \left(\bar{u} \left(\bar{E} + H\bar{p} \right) - \nu \int_{z_b}^{\eta} \left(2H\bar{u}\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \right) + \kappa \left(\frac{H^2}{6}\frac{\partial \bar{u}}{\partial x}\bar{u} - \frac{H}{2}\frac{\partial z_b}{\partial x}\bar{u}^2 \right) \right) \\
&= -2\nu \int_{z_b}^{\eta} \left(\left(\frac{\partial \bar{u}}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 \right) - \frac{\kappa}{6}\left(H\frac{\partial \bar{u}}{\partial x} + \frac{\partial H}{\partial x}\bar{u} \right)^2 \\
&\quad - \frac{\kappa}{3}\left(\left(\frac{\partial z_b}{\partial x} - \frac{1}{4}\frac{\partial \eta}{\partial x}\right)^2 - \frac{1}{8}\left(\frac{\partial \eta}{\partial x}\right)^2 \right)\bar{u}^2 - \frac{\kappa}{1 + \frac{\kappa H}{3\nu}}\left(1 + \frac{3}{2}\left(\frac{\partial z_b}{\partial x}\right)^2 \right)\bar{u}^2 \\
&\quad - H\frac{\partial p^a}{\partial t} + \left(p_{nh}|_b + 2\nu\frac{\partial u_b}{\partial x} \right)\frac{\partial z_b}{\partial t} - \kappa\frac{H}{2}\frac{\partial^2 z_b}{\partial x \partial t}\bar{u},
\end{aligned}
$$

with

$$
\begin{aligned}
\bar{E} &= \frac{H\overline{u^2}}{2} + \frac{H\overline{w^2}}{2} + \frac{gH(\eta + z_b)}{2}, \\
H\bar{p} &= \int_{z_b}^{\eta} p \, dz, \quad H\overline{u^2} = H\left(1 + \frac{2\kappa^2 H^2}{15\nu^2} \right)\bar{u}^2, \\
H\overline{w^2} &= \int_{z_b}^{\eta} w^2 = H\left(\frac{\eta^2 + \eta z_b + z_b^2}{3}\left(\frac{\partial \bar{u}}{\partial x}\right)^2 - (\eta + z_b)\frac{\partial \bar{u}}{\partial x}\frac{\partial(z_b\bar{u})}{\partial x} \right. \\
&\quad \left. + \left(\frac{\partial(z_b\bar{u})}{\partial x}\right)^2 \right) + H\left(\frac{\partial z_b}{\partial t}\right)^2 + 2\frac{\partial z_b}{\partial t}\left(-\frac{\eta^2 - z_b^2}{2}\frac{\partial \bar{u}}{\partial x} + H\frac{\partial(z_b\bar{u})}{\partial x} \right)
\end{aligned}
$$

Note that except for the friction terms, the previous expression is analogous to the energy equality for the Navier-Stokes system [16] but expressed with the vertically averaged variables. When the time derivatives of p^a and z_b are dropped, the right hand side of the preceding energy equality is negative when $\frac{\partial \eta}{\partial x}$ is enough small.

5   Conclusion

In this paper we have derived two extensions of the Saint-Venant system when the hydrostatic assumption is relaxed. The obtained models, especially in section 4, are similar to Boussinesq type models but derived in a more rigourous context and satisfying an energy equality.

On one hand the averaged models of shallow water type presented in this paper reduce the complexity of the discretization of the Navier-Stokes equations since they are written over a fixed domain. But on the other hand their mathematical formulation is more complex since high order derivatives – especially in space – appear.

The preliminary numerical simulations and comparison with experimental measurements performed with the proposed models are promising. They are not presented in this paper and will be described in a forthcoming publication.

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