Multiplicity for critical and overcritical equations

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Abstract

Let (M, g) be a compact Riemannian $n-$manifold, $n \geq 3$. We prove the existence of multiple solutions for equations like

$$\Delta u + \alpha u = f u^p, \qquad u > 0$$

where $\alpha \in \mathbb{R}^{+*}$, $f \in C^\infty(M)$ is positive, and the exponent p takes critical and overcritical values. General results are obtained and specific examples are discussed, like S^n, $S^1(t) \times S^{n-1}$, and $S^1(a) \times S^2(b) \times S^{n-3}$.

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1 Introduction

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Our paper is concerned with the question of the existence of multiple smooth solutions for the equation

$$\Delta u + \alpha u = f u^p, \qquad u > 0 \qquad (E_p)$$

where $\Delta = -\text{div}\,(\nabla\ )$ is the $g-$Laplacian, $\alpha \in \mathbb{R}^{+*}$, $f \in C^\infty(M)$ is positive and $p \geq \frac{n+2}{n-2}$. We say that the equation (E_p) is critical when $p = \frac{n+2}{n-2}$ and overcritical when $p > \frac{n+2}{n-2}$. Indeed, the exponent $\frac{n+2}{n-2}$ is the classical critical Sobolev growth exponent. It appears in particular in the equation one has to solve in the prescribed scalar curvature problem :

$$\Delta u + \frac{(n-2)\, S_g}{4(n-1)} u = f u^{\frac{n+2}{n-2}}, \qquad u > 0 \tag{1.1}$$

where S_g is the scalar curvature of g. More precisely, if for $f \in C^\infty(M)$ there exists $u \in C^\infty(M)$ a positive solution of (1.1), then f is the scalar curvature of the g-conformal metric $u^{\frac{4}{n-2}} g$. We are here interested in two particular cases of equation (1.1). On the standard sphere (S^n, h_n), this problem is referred to as the Nirenberg problem. Its resolution is equivalent to the resolution of (1.1) with $S_{h_n} = n(n-1)$. For references on the Nirenberg problem, see Hebey [11], Kazdan-Warner [15] and Li [16]. There is also the intensively studied Yamabe problem, which consists in the search for conformal metrics with constant scalar curvature. It corresponds to the resolution of (1.1) with $f = 1$. The Yamabe problem is completly solved.

Concerning multiplicity and uniqueness of positive solutions for such equations, we refer to Aubin [1, 2], Bidaut-Véron and Véron [3], Esposito [6], Hebey-Vaugon [13], Obata [17], Pollack [18], Schoen [19] and [20]. In particular, note that the Yamabe equation possesses a unique solution if there exists $\tilde{g} \in [g]$ such that $S_{\tilde{g}} \leq 0$ or if there exists an Einstein metric $\tilde{g} \in [g]$, where $[g]$ stands for the conformal class of g. We are here especially interested on results of Hebey-Vaugon [13] (see also Schoen [19]). In their work, the manifold is assumed to have big enough isometry groups and solutions are required to be invariant under the action of subgroups. Besides, all groups are finite which implies that the quotient space of all orbits can be equiped with a structure of manifold. In our results, this condition is not required. This is made possible thanks to the recent advances of Hebey-Vaugon [14] and Faget [7, 8] concerning the influence of isometry groups on Sobolev spaces and Sobolev inequalities.

Given G an isometry group, $\alpha \in \mathbb{R}^{+*}$, and $f \in C^\infty(M)$ positive and G-invariant, we consider G-invariant solutions of the equation

$$\Delta u + \alpha u = f u^{\frac{n+2-k}{n-2-k}}, \qquad u > 0, \tag{$E^k_{\alpha f}$}$$

where $k \geq 0$ is the minimum dimension of the G-orbits. The energy of a solution u of $(E^k_{\alpha f})$ is defined by

$$\mathcal{E}(u) = \int_M f u^{\frac{2(n-k)}{n-2-k}} \, dv_g. \tag{1.2}$$

We obtain multiplicity of energies for solutions of $(E^k_{\alpha f})$ where each solution is invariant by the action of an isometry group G_i such that all the G_i-orbits have the same minimal dimension k. When $k = 0$, the equation $(E^0_{\alpha f})$ is critical and when $k > 0$, one has $\frac{n+2-k}{n-2-k} > \frac{n+2}{n-2}$ and $(E^k_{\alpha f})$ turns out to be overcritical. The study of equation $(E^k_{\alpha f})$ is strongly related to the notion of first and second best constants in the Sobolev inequalities presented in section 2. The first best constant appears to be of importance in existence results and the second in multiplicity results.

2 Preliminaries

Let (M, g) be a compact Riemannian n-manifold, $Is(M, g)$ its isometry group ($Is(M, g)$ is a compact Lie group), and G a subgroup of $Is(M, g)$. By taking its closure $\bar{G}$ for the standard topology, we can assume that G is compact. We note for any $p \in [0, +\infty]$,

$$\begin{aligned} C^p_G(M) &= \{u \in C^p(M), \forall \sigma \in G, u \circ \sigma = u\} \\ H^2_{1,G}(M) &= \{u \in H^2_1(M), \forall \sigma \in G, u \circ \sigma = u\} \end{aligned}$$

where the Sobolev space $H^2_1(M)$ is the completion of $C^\infty(M)$ with respect to the norm $\|u\|^2_{H^2_1} = \|\nabla u\|^2_2 + \|u\|^2_2$. When no confusion is possible, we write $C^p_G, H^2_1, H^2_{1,G}$ instead of $C^p_G(M), H^2_1(M), H^2_{1,G}(M)$. If $n - k > 2$, we let $2^\sharp = \frac{2(n-k)}{n-2-k}$, and Hebey-Vaugon [14] proved that for any $1 \leq q \leq 2^\sharp$, the embedding $H^2_{1,G} \subset L^q$ is continuous, and compact if $q < 2^\sharp$. For $p < 2^\sharp - 1$, compactness of the embedding $H^2_{1,G} \subset L^{p+1}$ implies, thanks to the variational method, that there exists a C^∞_G solution for the equation

$$\Delta u + \alpha u = f u^p, \qquad u > 0 \qquad (E_p)$$

where $\Delta = -\operatorname{div}(\nabla\ )$ is the $g-$Laplacian, $\alpha \in \mathbb{R}^{+*}$, and $f \in C^\infty_G$ is positive. When $p = 2^\sharp - 1$, the existence of solutions is more difficult to obtain because of lack of compactness.

For convenience in what follows, we recall some results about the action of an isometry group G on a compact manifold. We refer to Bredon [4], Gallot-Hulin-Lafontaine [9] and Hebey-Vaugon [14] for more details. Since we can choose G compact, for any $x \in M$, $O^G_x = \{\sigma(x), \sigma \in G\}$ the G-orbit of x is a compact submanifold of M and $S^G_x = \{\sigma \in G, \sigma(x) = x\}$ the isotropy group of x is a Lie group of G. A G-orbit O^G_x is principal if for any $y \in M$, S^G_y possesses a subgroup which is conjugate to S^G_x. Principal orbits are of maximum dimension but the converse is false in general. Let Ω be the union of all principal orbits. Then Ω is a dense open subset of M, and Ω/G is a quotient manifold. More precisely, if π is the associated submersion, then $(\pi, \Omega, \Omega/G)$ is a fibration where each fiber is a G-orbit. Note that if all G-orbits are principal, there exists a unique manifold structure on the topological space M/G and the metric g induces a quotient metric $\tilde{g}$ on M/G such that $\pi_G : M \to M/G$ is a Riemannian submersion.

We consider here $\mathcal{C}^\infty_G$ solutions of (E_p) for $p = 2^\sharp - 1$. The equation is written as

$$\Delta u + \alpha u = f u^{\frac{n+2-k}{n-2-k}}, \qquad u > 0. \qquad (E^k_{\alpha f})$$

When $k > 0$, namely when there is no finite G-orbit, then $\frac{n+2-k}{n-2-k} > \frac{n+2}{n-2}$ and $(E^k_{\alpha f})$ is, in some sense, overcritical. The study of $(E^k_{\alpha f})$ is strongly related to the problem of the attainability of sharp constants in functional inequalities associated with the continuous embedding $H^2_{1,G} \subset L^{2^\sharp}$. Following Faget [8], we introduce two assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ given by :

$(\mathcal{H}_1)$: for any orbit $O^G_{x_0}$ of minimum dimension k and minimum volume A, there exists H a subgroup of $Is(M, g)$ and $\delta > 0$ such that
i) in $\mathcal{O}_{x_0,\delta} = \{x \in M / d_g(x, O^G_{x_0}) < \delta\}$, all H-orbits are principal,

ii) for any $x \in \mathcal{O}_{x_0,\delta},\ O_x^H \subset O_x^G$ and $O_{x_0}^H = O_{x_0}^G$,
iii) for any $x \in \mathcal{O}_{x_0,\delta},\ A = vol_g O_{x_0}^G \leq vol_g O_x^H$.

and

$(\mathcal{H}_2)$: for any orbit $O_{x_0}^G$ of minimum dimension k and minimum volume A, there exists H a normal subgroup of G and $\delta > 0$ such that
i) in $\mathcal{O}_{x_0,\delta} = \{x \in M / d_g(x, O_{x_0}) < \delta\}$, all H-orbits are principal,
ii) $O_{x_0}^H = O_{x_0}^G$.
iii) for any $x \in \mathcal{O}_{x_0,\delta},\ x \notin O_{x_0}^G,\ dim O_x^G > k = dim O_{x_0}^G$,
iv) for any $x \in \mathcal{O}_{x_0}$, x is a critical point of the function $v_H(y) = vol_g O_y^H$.

Faget [8] shows that :

Theorem F [Faget [8]] *Let (M, g) be a compact Riemannian n-manifold, G a compact subgroup of $Is(M, g)$, k the minimum G-orbit dimension, and A the minimum volume of G-orbits of dimension k. Assume that $n - k > 2$. If at least one of the assumptions $(\mathcal{H}_1)$ or $(\mathcal{H}_2)$ holds true, then there exists $B > 0$ such that for any $u \in H_{1,G}^2$,*

$$\|u\|_{2^\sharp}^2 \leq \frac{K_{n-k}}{A^{\frac{2}{n-k}}} \left[\|\nabla u\|_2^2 + B\|u\|_2^2 \right], \tag{2.1}$$

where $K_{n-k} = \frac{4}{(n-k)(n-2-k)\omega_{n-k}^{2/(n-k)}}$, and ω_{n-k} is the volume of the standard sphere (S^{n-k}, h_{n-k}). The value $K_{n-k}\ A^{-\frac{2}{n-k}}$ is the best possible in (2.1), i.e. the smallest constant such that (2.1) holds true for all $u \in H_{1,G}^2$.

When assumptions $(\mathcal{H}_1)$ or $(\mathcal{H}_2)$ hold true for a subgroup H we use in the sequel the following notations : π_H is the canonical submersion $\mathcal{O}_{x_0,\delta} \to \mathcal{O}_{x_0,\delta}/H$ and $\tilde{g}$ is the quotient metric induced by g on $\mathcal{O}_{x_0,\delta}/H$ such that π_H is a Riemannian submersion. For any $x \in \mathcal{O}_{x_0,\delta}$, we note $\tilde{x} = \pi_H(O_x^H)$ and $\tilde{v}_H$ the function defined for any $y \in \mathcal{O}_{x_0,\delta}/H$ by $\tilde{v}_H(y) = vol_g(\pi_H^{-1}(y))$.

When inequality (2.1) holds true, we define the second best constant by

$$B_{0,G}(M, g) := \inf\{B > 0, \forall u \in H_{1,G}^2,\ (2.1) \text{ is valid with } B\}.$$

If (2.1) holds true, we can take $B = B_{0,G}(M, g)$ in (2.1), so that for any $u \in H_{1,G}^2$,

$$\|u\|_{2^\sharp}^2 \leq \frac{K_{n-k}}{A^{\frac{2}{n-k}}} \left(\|\nabla u\|_2^2 + B_{0,G}(M, g)\ \|u\|_2^2 \right). \tag{$I_S^{G,opt}$}$$

This inequality is optimal with respect to the first and to the second constants, i.e. none of them can be improved. When no confusion is possible we write $B_{0,G}$ instead of $B_{0,G}(M, g)$. Note that Hebey-Vaugon [12] proved earlier that when $G = \{Id\}$, then $(I_S^{Id,opt})$ holds true on every compact Riemannian n-manifold, $n \geq 3$. As a remark, $(I_S^{G,opt})$ is true if all G-orbits are principal of constant volume, since we can take $H = G$ in $(\mathcal{H}_1)$. We then easily see that

$$B_{0,G}(M, g) = B_{0,Id}(M/G, \tilde{g}). \tag{2.2}$$

Now we discuss the role of the first best constant in (I_S^{opt}) with respect to the existence of solutions of $(E_{\alpha f}^k)$. G-invariant solutions of $(E_{\alpha f}^k)$ can be obtained by the variational method by minimizing I on $\mathcal{P}$ where :

$$I(u) = \frac{\|\nabla u\|_2^2 + \alpha \|u\|_2^2}{\left(\int_M f|u|^{2^\sharp}\, dv_g\right)^{2/2^\sharp}},$$

and

$$\mathcal{P} = \left\{ u \in H_{1,G}^2,\ \int_M f|u|^{2^\sharp}\, dv_g > 0 \right\}.$$

We note $\Upsilon_G := \inf_{u\in\mathcal{P}} I(u)$. The main difficulty is the lack of compactness coming from the critical exponent $2^\sharp$, but this is by now a classical problem. It was firstly solved for the Yamabe problem by working with subcritical exponent and then by passing to the limit exponent. Faget [7] proves that

$$\Upsilon_G \le \frac{A^{\frac{2}{n-k}}}{K_{n-k}\, (\max f)^{2/2^\sharp}}, \tag{2.3}$$

and that, if

$$\Upsilon_G < \frac{A^{\frac{2}{n-k}}}{K_{n-k}\, (\max f)^{2/2^\sharp}}, \tag{2.4}$$

then there exists a solution $u \in C_G^\infty$ for $(E_{\alpha f}^k)$ such that $\Upsilon_G = I(u)$. Such a solution is said to be G-minimizing. Let (E_α^k) be $(E_{\alpha f}^k)$ when $f = 1$. Propositions 1 and 2 below follow from the work of Faget [7].

Proposition 2.1 *Let (M,g) be a compact Riemannian n-manifold , $n \ge 3$, G an isometry group, k be the minimum G-orbit dimension. Assume that $n-k > 2$ and that $(I_S^{G,opt})$ holds true. If $\alpha \in]0, B_{0,G}[$, then there exists a C_G^∞ and G-minimizing solution for the equation (E_α^k).*

Proof. By the definition of $B_{0,G}$, the strict inequality (2.4) holds true, and we can apply the results in Faget [7]. ■

Proposition 2.2 *Let (M,g) be a compact Riemannian n-manifold , $n \ge 4$, G an isometry group, k be the minimum G-orbit dimension, and A be the minimum volume of G-orbits of dimension k. Assume that $n-k \ge 4$. Let $x_0 \in M$ such that $dim O_{x_0}^G = k$ and $vol_g O_{x_0}^G = A$ and let $f \in C_G^2$ maximal at x_0. Assume that one of the assumptions $(\mathcal{H}_1)$ or $(\mathcal{H}_2)$ holds true for a subgroup H. With the notations introduced above, if*

$$\begin{cases} (n-4-k)\ \Delta_g f(x_0) = 0 \\ \alpha < \frac{n-2-k}{4(n-1-k)} \left(\frac{3\Delta_{\tilde g} \tilde v_H(\tilde x_0)}{A} + S_{\tilde g}(\tilde x_0) \right), \end{cases} \tag{2.5}$$

then there exists a G-minimizing C_G^∞ solution for the equation $(E_{\alpha f}^k)$.

Proof. For any $\epsilon > 0$, let $\tilde{u}_\epsilon$ be defined on $\mathcal{O}_{x_0,\delta}$ by $\tilde{u}_\epsilon = (\epsilon + \tilde{r}^2)^{1-N/2} - (\epsilon + \delta^2)^{1-N/2}$ where $\tilde{r} = d_{\tilde{g}}(., \tilde{x}_0)$ and $N = n - k$. We set $u_\epsilon = \tilde{u}_\epsilon \circ \pi_H$, and after lengthy computations, we get that

$$I(u_\epsilon) \leq \frac{A^{2/N}}{K_N f(x_0)^{2/2^\sharp}}$$
$$\times \left[1 + \frac{\epsilon}{N(N-4)}\left(\frac{\alpha\, 4(N-1)}{N-2} + \frac{(N-4)\Delta_g f(x_0)}{2f(x_0)} - \frac{3\Delta_{\tilde{g}}\tilde{v}_H(\tilde{x}_0)}{A} - S_{\tilde{g}}(\tilde{x}_0)\right) + \circ(\epsilon)\right] \quad \text{if } N > 4$$
$$\times \left[1 + \frac{\epsilon \ln \epsilon}{8}\left(S_{\tilde{g}}(\tilde{x}_0) + \frac{3\Delta_{\tilde{g}}\tilde{v}_H(\tilde{x}_0)}{A} - 6\alpha\right) + \circ(\epsilon \ln \epsilon)\right] \quad \text{if } N = 4.$$

Thanks to (2.5), inequality (2.4) holds true and we can apply the results in Faget [7]. Proposition 2.2 is proved. ■

Now we briefly discuss estimates on $B_{0,G}(M, g)$. At the moment, the only compact Riemannian manifold where one knows its explicit value is the standard sphere (S^n, h_n) when no isometry invariance is requiered, i.e. when $G = \{Id\}$. Noting B_0 instead of $B_{0,Id}$, one has that

$$B_0(S^n, h_n) = \frac{n(n-2)}{4}. \tag{2.6}$$

Lower bounds for $B_{0,G}(M, g)$ have recently been obtained by Faget [8] : on a compact Riemannian n-manifold, $n \geq 4$, with the same G, k, A and notations as above, if $n-k > 4$ and if $(\mathcal{H}_1)$ or $(\mathcal{H}_2)$ holds true, then

$$B_{0,G}(M, g) \geq \max\left\{\frac{A^{\frac{2}{n-k}}}{V_g^{\frac{2}{n-k}} K_{n-k}}, \frac{n-2-k}{4(n-k-1)}\left(S_{\tilde{g}}(\tilde{x}_0) + \frac{3\Delta_{\tilde{g}}\tilde{v}_H(\tilde{x}_0)}{A}\right)\right\} \tag{2.7}$$

where V_g is the volume of (M, g). We do not know yet upper bounds for $B_{0,G}(M, g)$ in the general case. Hebey-Vaugon [13] computed upper bounds on specific conformally flat manifolds. On $(S^1(t) \times S^{n-1}, h_1 \times h_{n-1})$, with $t > 0, n \geq 3$ and when no isometry invariance is requiered, i.e. $G = \{Id\}$:

$$\frac{(n-2)^2}{4} \leq B_0(S^1(t) \times S^{n-1}, h_1 \times h_{n-1}) \leq \frac{1}{4t^2} + \frac{(n-2)^2}{4}. \tag{2.8}$$

Note that this approximation is optimal when $t \to \infty$. On the quotient manifold $(S^n/G, \tilde{g})$, $n \geq 3$, where $G \subset O(n+1)$ is a cyclic group of order A and acts freely on S^n and $\tilde{g}$ is the quotient metric induced by h_n,

$$\frac{A^{2/n} n(n-2)}{4} \leq B_0(S^n/G, \tilde{g}) \leq \left(1 + \frac{A^2}{4}\right)\left(\frac{n+1}{2}\right) - 1 + \frac{n(n-2)}{4}. \tag{2.9}$$

As we will see, these estimates on $B_{0,G}$, especially the upper bounds, are fundamental in the problem of multiplicity of solutions.

3 Multiplicity results 1

Assuming that there exists two invariant solutions for $(E^k_{\alpha f})$, we give general conditions to separat the energies in Theorems 4.1.a and 4.1.b. Then we illustrate these theorems on specific examples where existence and multiplicity are compatible. We postpone the proof of Theorems 4.1.a and 4.1.b to section 4.

Theorem 1a *Let (M,g) be a compact Riemannian n-manifold, $n>4$, G_1 and G_2 be two isometry groups such that the minimum dimensions of G_1- and G_2-orbits are the same. We denote by $k\geq 0$ this common minimum orbit dimension, and let $A_i>0$ be the minimum volume of G_i-orbits of dimension $k, i\in\{1,2\}$. We suppose that $n-k>2$, $A_1<A_2$ and that $(I_S^{G_2,opt})$ is valid. Assume that for $\alpha\in\mathbb{R}_+^*$ and $f\in C^\infty_{G_1\cup G_2}$ positive there exist two solutions of $(E^k_{\alpha f})$: $u_1\in C^\infty_{G_1}$ which is G_1-minimizing and $u_2\in C^\infty_{G_2}$ which is G_2-minimizing. If*

$$i) \qquad \alpha \leq B_{0,G_2}(M,g), \tag{3.1}$$

$$ii) \qquad \alpha \geq \frac{n(n-4)}{(n-2)^2}\, B_0(M,g), \ \ and \tag{3.2}$$

$$iii) \qquad \alpha > B_{0,G_2}(M,g) - \left[\left(\frac{A_2}{A_1}\right)^{\frac{2}{n-k}} - 1\right] \frac{A_2^{-\frac{4}{(n-k)(n-2)}}\, K_{n-k}^{\frac{2}{n-2}}}{V_g^{\frac{2(n-2-k)}{(n-k)(n-2)}}\, K_n^{\frac{n}{n-2}}} \left(\frac{n(n-4)}{(n-2)^2}\right)^{\frac{n}{n-2}} \left(\frac{\max f}{<f>}\right)^{\frac{2(n-2-k)}{(n-k)(n-2)}}, \tag{3.3}$$

where $<f>$ stands for the average value of f, then $\mathcal{E}(u_1)<\mathcal{E}(u_2)$. In particular, u_1 and u_2 are distinct.

With similar global arguments, and basically only one technical variation in the proof, we can prove a slightly different result :

Theorem 1b *Let (M,g) be a compact Riemannian n-manifold, $n>4$, G_1 and G_2 be two isometry groups such that the minimum dimensions of G_1- and G_2-orbits are the same. We denote by $k\geq 0$ this common minimum orbit dimension, and let $A_i>0$ be the minimum volume of G_i-orbits of dimension $k, i\in\{1,2\}$. We suppose that $n-k>4$, $A_1<A_2$ and that $(I_S^{G_2,opt})$ is valid. Assume that for $\alpha\in\mathbb{R}_+^*$ and $f\in C^\infty_{G_1\cup G_2}$ positive, there exist two solutions of $(E^k_{\alpha f})$: $u_1\in C^\infty_{G_1}$ which is*

G_1-minimizing and $u_2 \in C^\infty_{G_2}$ which is G_2-minimizing. If

$$i) \quad \alpha \leq B_{0,G_2}(M,g), \tag{3.4}$$

$$ii) \quad \alpha \geq \frac{(n-k)(n-4-k)}{(n-2-k)^2} B_{0,G_2}(M,g), \text{ and} \tag{3.5}$$

$$iii) \quad \alpha > B_{0,G_2}(M,g) - \left[\left(\frac{A_2}{A_1}\right)^{\frac{2}{n-k}} - 1\right] \frac{A_2^{\frac{2}{n-k}}}{V_g^{\frac{2}{n-k}} K_{n-k}} \left(\frac{(n-k)(n-4-k)}{(n-2-k)^2}\right)^{\frac{n-k}{n-2-k}} \left(\frac{\max f}{< f >}\right)^{\frac{2}{n-k}}, \tag{3.6}$$

where $< f >$ stands for the average value of f, then $\mathcal{E}(u_1) < \mathcal{E}(u_2)$. In particular, u_1 and u_2 are distinct.

As a remark, if in Theorems 1.a and 1.b, one of the solutions u_1 or u_2 satisfies (2.4), then inequality $iii)$ is not necessarily strict. We refer to the proof of Theorems 4.1.a and 4.1.b for more details on this claim. As a remark, the compatibility of conditions $i), ii)$ and $iii)$ is not automatic. In our examples, we choose f such that the right side in $iii)$ is nonpositive so that $iii)$ is valid. Then multiplicity holds true when α belongs to the interval defined by $i)$ and $ii)$. In the following Corollary of Theorem 4.1.a, we give general conditions in order to separate energies of an infinity of solutions.

Corollary 3.1 *Let (M,g) be a compact Riemannian n-manifold with $n \geq 3$, and $(G_i)_{i\in I}$ a family of isometry groups of $Is(M,g)$ such that for any $i \in I$, $(I_S^{G_i,opt})$ is valid. For any $i \in I$, let k_i be the minimum dimension of G_i-orbits, and A_i be the minimum volume of G_i-orbits of dimension k_i. We assume that $\forall i \in I, k_i = k$. Given $\alpha \in \mathbb{R}^{+*}$, and $f \in C^\infty_{\cup_{i\in I} G_i}$ positive, we suppose that for any $i \in I$, there exists a G_i-minimizing solution $u_i \in C^\infty_{G_i}$ for $(E^k_{\alpha f})$. If $\alpha \in \left[\frac{n(n-4)}{(n-2)^2} B_0(M,g);\ \min_{i\in I}(B_{0,G_i})\right]$ and if for any $i \in I$ and $j \in I$ such that $A_j < A_i$ we have that*

$$\left(\frac{A_i}{A_j}\right)^{\frac{2}{n-k}} > 1 + (B_{0,G_i} - \alpha) \frac{K_n^{\frac{n}{n-2}}}{K_{n-k}^{\frac{2}{n-2}}} A_i^{\frac{4}{(n-k)(n-2)}} \left(\frac{(n-2)^2}{n(n-4)}\right)^{\frac{n}{n-2}} \left(\frac{\int_M f \, dv_g}{\max f}\right)^{\frac{2(n-2-k)}{(n-k)(n-2)}},$$

then $\mathcal{E}(u_j) < \mathcal{E}(u_i)$.

Now we discuss specific examples. The two first examples concern critical equations and the third example concerns overcritical equations.

Example 3.1 Let (S^n, h_n) be the standard sphere of odd dimension $n \geq 5$ and G_1 and G_2 be two finite subgroups of $O(n+1)$ acting freely on S^n of respective cardinal $1 < A_1 < A_2$. Let $f \in C^\infty_{G_1 \cup G_2}$ positive and maximal at $x_0 \in S^n$ such that the derivatives at x_0 are zero up to the order $n-3$, and let $< f >$ be the average value of f. If

$$\left(\frac{\max f}{< f >}\right)^{2/n} \geq$$

$$\left(B_{0,G_2}(S^n,h_n) - \frac{n^2(n-4)}{4(n-2)}\right)\left(\frac{(n-2)^2}{n(n-4)}\right)^{\frac{n}{n-2}} \frac{4A_2^{\frac{4}{n(n-2)}}}{n(n-2)}\left[\left(\frac{A_2}{A_1}\right)^{\frac{2}{n}} - 1\right]^{-1}$$

then there exist at least two C^∞ solutions of different energies for the critical equation

$$\Delta u + \alpha u = f u^{\frac{n+2}{n-2}}, \qquad u > 0, \qquad (E^0_{\alpha f})$$

when α belongs to the interval

$$\alpha \in \left[\frac{n^2(n-4)}{4(n-2)}; \frac{n(n-2)}{4}\right].$$

One of these solutions is G_1-invariant and the other is G_2-invariant.

As a remark, when $\alpha = \frac{n(n-2)}{4}$, $(E^0_{\alpha f})$ is the Nirenberg equation and we recover a result of Hebey-Vaugon [13].

Proof of Example 3.1. Since G_i acts freely, S^n/G_i is a manifold. with a quotient metric induced by h_n noted $\tilde{g}_i$. As mentioned in section 2, since the G_i-orbits are principal of constant cardinal, $(I_S^{G_i,opt})$ holds true and with (2.2) and (2.9), we have that

$$B_{0,G_i}(S^n,h_n) = B_0(S^n/G_i,\tilde{g}_i) \geq \frac{n(n-2)}{4}. \qquad (3.7)$$

We claim that for $\alpha \leq \frac{n(n-2)}{4}$ there exist two solutions $u_i \in C^\infty_{G_i},\ i=1,2$, G_i-minimizing for $(E^0_{\alpha f})$. The existence for $\alpha = \frac{n(n-2)}{4}$ is given by Hebey-Vaugon [13] since the derivatives of f are zero up to the order $n-3$. Besides, thanks to Proposition 2.2, there exists $u_i \in C^\infty_{G_i}$ solution of $(E^0_{\alpha f})$ if

$$\alpha < \frac{n-2}{4(n-1)} S_{\tilde{g}_i}(S^n/G_i) = \frac{n(n-2)}{4}.$$

Our claim is proved. Now according to Theorem 4.1.a, u_1 and u_2 are distinct if the three assumptions (3.1), (3.2) and (3.3) hold true. The first condition (3.1) holds true if $\alpha \leq \frac{n(n-2)}{4}$, thanks to (3.7). Condition (3.2) is stated here, since $B_0(S^n,h_n) = \frac{n(n-2)}{4}$, as

$$\alpha \geq \frac{n^2(n-4)}{4(n-2)}.$$

With this lower bound on α, in order to get (3.3), it suffices that

$$\frac{n^2(n-4)}{4(n-2)} \geq B_{0,G_2}(S^n,h_n) - \left[\left(\frac{A_2}{A_1}\right)^{\frac{2}{n}} - 1\right] \frac{n(n-2)}{4A_2^{\frac{4}{n(n-2)}}} \left(\frac{n(n-4)}{(n-2)^2}\right)^{\frac{n}{n-2}} \left(\frac{\max f}{< f >}\right)^{\frac{2}{n}}$$

with a inequality which is not strict, thanks to the remark following Theorem 4.1.b. This is exactly the assumption made on f. Thus u_1 and u_2 exists and are distinct when

$$\alpha \in \left[\frac{n^2(n-4)}{4(n-2)} ; \frac{n(n-2)}{4} \right]$$

and Example 3.1 is proved. ■

Now we discuss the following example. Here, we apply Theorem 4.1.b and Theorem 4.1.a does not provide the result.

Example 3.2 On $(S^1(t) \times S^{n-1}, h_1 \times h_{n-1})$ with $n > 4$, and $t \geq \sqrt{\frac{n(n-4)}{4(n-2)^2}}$, let $G_1 = R_1 \times Id_{S^{n-1}}$ and $G_2 = R_2 \times Id_{S^{n-1}}$ be two isometry groups, where R_1 and R_2 are finite subgroups of $SO(2)$ with respective cardinal $A_1 < A_2$. Let $f \in C^\infty_{G_1 \cup G_2}$ positive and maximal at x_0 with derivatives at x_0 equal to 0 up to the order $n-2$ and such that

$$\left(\frac{\max f}{< f >} \right)^{2/n} \geq \left(\frac{(n-2)^2}{4} + \frac{1}{4t^2} \right) \frac{K_n A_2^{\frac{4}{n(n-2)}} \left(2\pi t \omega_{n-1}\right)^{2/n}}{\left(\frac{A_2}{A_1}\right)^{2/n} - 1} \left(\frac{(n-2)^2}{n(n-4)} \right)^{\frac{n}{n-2}} . \tag{3.8}$$

Then there exist at least two C^∞ solutions of different energies for the critical equation $(E^0_{\alpha f})$ when α belongs to the interval

$$\alpha \in \left[\frac{n(n-4)}{(n-2)^2} \left(\frac{(n-2)^2}{4} + \frac{1}{4t^2} \right) \; ; \; \frac{(n-2)^2}{4} \right].$$

One of these solutions is G_1-invariant and the other is G_2-invariant.

Proof of Example 3.2. The G_i-orbits are finite and principal and thus

$$\left(S^1(t) \times S^{n-1}\right) / \left(R_i \times Id_{S^{n-1}}\right) = S^1\left(\frac{t}{A_i}\right) \times S^{n-1}$$

with quotient metric $h_1 \times h_{n-1}$. As already mentioned in section 2, $(I_S^{G_i,opt})$ holds true and with (2.2) and (2.8)

$$B_{0,G_i}\left(S^1(t) \times S^{n-1}, h_1 \times h_{n-1}\right) \geq \frac{(n-2)^2}{4}. \tag{3.9}$$

We claim now that for

$$\alpha \leq \frac{(n-2)^2}{4} \tag{3.10}$$

there exist two $\mathcal{C}^\infty$ solutions for $(E^0_{\alpha f})$, minimizing for $G_i,\ i \in 1, 2$. Since the second derivatives of f at x_0 are zero and $S_{h_1 \times h_{n-1}}\left(S^1(t/A_i) \times S^{n-1}\right) = (n-1)(n-2)$, the existence condition (2.5) of Proposition 2.2 is written as $\alpha < \frac{(n-2)^2}{4}$. If $\alpha = \frac{(n-2)^2}{4}$, $(E^0_{\alpha f})$

is the equation of the prescribed scalar curvature problem and it is solved by Escobar-Schoen [5] on compact conformally flat manifolds if f has derivatives at a maximum point which turn out to be zero up to the order $n-2$. Thus on $(S^1(t/A_i)\times S^{n-1}, h_1\times h_{n-1})$ there exists $\tilde{u}_i$ a minimizing solution of the equation

$$\Delta\tilde{u}_i + \frac{(n-2)^2}{4}\tilde{u}_i = \tilde{f}\tilde{u}_i^{\frac{n+2}{n-2}}, \qquad \tilde{u}_i > 0.$$

If $\pi_i : S^1(t)\times S^{n-1} \to S^1(t/A_i \times S^{n-1})$ is the canonical submersion, then $u_i = \tilde{u}_i \circ \pi_i$ is a G_i-minimizing solution of $(E^0_{\alpha f})$ with $\alpha = \frac{(n-2)^2}{4}$. Our claim is proved. Then one has $\mathcal{E}(u_1) < \mathcal{E}(u_2)$ if the three assumptions (3.4), (3.5) and (3.6) of Theorem 4.1.b hold true. (3.4) is valid by (3.9) if $\alpha \le \frac{(n-2)^2}{4}$. (3.5) holds true thanks to the upper bound in (2.8) if

$$\alpha \ge \frac{n(n-4)}{(n-2)^2}\left(\frac{1}{4t^2} + \frac{(n-2)^2}{4}\right). \tag{3.11}$$

By (2.8) and (3.8), the right side of (3.6) is nonpositive. Then (3.6) is valid. Thus existence and multiplicity are compatible if α satisfies (3.10) and (3.11) which is possible if $t \ge \left(\frac{n(n-4)}{4(n-2)^2}\right)^{1/2}$. Example 3.2 is proved. ■

Now we discuss an example where there are non constant dimensions of orbits and the minimum dimension is 3.

Example 3.3 On $(S^1(a)\times S^2(b)\times S^{n-3}, h_1\times h_2\times h_{n-3})$ with $n\ge 10$ and

$$\frac{1}{4a} < b^2 < \frac{(n-5)^2}{(n-7)(3n^2-26n+57)}, \tag{3.12}$$

we consider the following isometry groups:

$$G_1 = Id_{S^1(a)\times S^2(b)}\times O(n-6)\times O(4) \qquad \text{and} \qquad G_2 = O(2)\times O(3)\times Id_{S^{n-3}}.$$

Let $x_0 = (\theta, 0_{\mathbb{R}^{n-6}}, z_0)$ where $\theta\in S^1(a)\times S^2(b)$ and $z_0\in S^3$ and let $f\in C^\infty_{G_1\cup G_2}$ be a positive function maximal at x_0 such that $\Delta f(x_0) = 0$ and

$$\frac{\max f}{< f >} \ge \left(\left(4ab^2\right)^{2/(n-3)} - 1\right)^{-\frac{n-3}{2}} \left(\frac{(n-5)^2}{(n-3)(n-7)}\right)^{\frac{(n-3)^2}{2(n-5)}}. \tag{3.13}$$

Then there exist at least two C^∞ solutions with different energies for the over critical equation $(E^3_{\alpha f})$ when α belongs to the interval

$$\left[\frac{(n-3)^2(n-7)}{4(n-5)}; \min\left\{\frac{(n-3)(n-5)}{4}, \frac{n-5}{4(n-4)}\left(\frac{2}{b^2} + (n-6)(n-7)\right)\right\}\right[.$$

One of these solutions is G_1-invariant and the other is G_2-invariant.

Proof of Example 3.3. The G_2-orbits are $S^1(a)\times S^2(b)\times\{z\}$, where $z\in S^{n-3}$, and thus they are principal of constant dimension 3 and constant volume $8\pi^2ab^2$. The quotient metric on $\left(S^1(a)\times S^2(b)\times S^{n-3}\right)/G_2 = S^{n-3}$ is h_{n-3}. According to section 2, $(I_S^{G_2,opt})$ holds true and with (2.2) and (2.6)

$$B_{0,G_2} = B_0(S^{n-3},h_{n-3}) = \frac{(n-3)(n-5)}{4}.$$

The G_1-orbit of $x=(\theta,y,z)\in\mathbb{R}^5\times\mathbb{R}^{n-6}\times\mathbb{R}^4$ where $\theta\in S^1(a)\times S^2(b)$, and $(y,z)\in S^{n-3}$, is

$$O_x^{G_1} = \{\theta\}\times S^{n-7}(\|y\|)\times S^3(\|z\|).$$

If $\|y\|\neq 0$ and $\|z\|\neq 0$, $dimO_x^{G_1}=n-4$ is maximum. For $x_0=(\theta,0_{\mathbb{R}^{n-6}},z_0)$, where $\theta\in S^1(a)\times S^2(b)$ and $z_0\in S^3$, we have

$$O_{x_0}^{G_1}=\{\theta\}\times\{0_{\mathbb{R}^{n-6}}\}\times S^3$$

and $dimO_{x_0}^{G_1}=3$ is minimum (thus $O_{x_0}^{G_1}$ is not a principal orbit) and $volO_{x_0}^{G_1}=2\pi^2$. We set $H = Id_{S^1(a)\times S^2(b)\times\mathbb{R}^{n-6}}\times O(4)$. H is a normal subgroup of G_1, and for any $x=(\theta,y,z)$ such that $z\neq 0$,

$$O_x^H=\{\theta\}\times\{y\}\times S^3(\|z\|),$$

where $\|z\|\in]0,1]$. The maximum volume for H-orbit is archieved at x_0. Moreover the H-orbits are principal and $O_{x_0}^H=O_{x_0}^{G_1}$. If $x\notin O_{x_0}^{G_1}$, then $O_x^{G_1}=\{\theta\}\times S^{n-7}(\|y\|)\times S^3(\|z\|)$ with $\|y\|\neq 0$ and $\|z\|\neq 0$ and $dimO_x^{G_1}=n-4>3$. Finally assumption $(\mathcal{H}_2)$ is true with H and $(I_S^{G_1,opt})$ is valid. Now in order to get G_i-invariant and -minimizing solutions of $(E^3_{\alpha f})$, we use Proposition 2.2. The condition (2.5) for G_2 is

$$\alpha<\frac{(n-3)(n-5)}{4}. \tag{3.14}$$

For G_1, we have $\Delta_{\tilde g}\tilde v_H(\tilde x_0)\geq 0$ and thus (2.5) holds true if

$$\alpha<\frac{n-5}{4(n-4)}S_{\tilde g}(\tilde x_0).$$

Thanks to Proposition 3.1 below, this inequality holds true if

$$\alpha<\frac{n-5}{4(n-4)}\left(\frac{2}{b^2}+(n-6)(n-7)\right). \tag{3.15}$$

Energies of both solutions obtained under conditions (3.14) and (3.15) are different if the three multiplicity conditions of Theorem 4.1.b hold true. The first condition (3.4) is $\alpha\leq\frac{(n-3)(n-5)}{4}$ and holds true if (3.14) does. The second one (3.5) is stated here as

$$\alpha\geq\frac{(n-3)^2(n-7)}{4(n-5)}. \tag{3.16}$$

The last condition (3.6) is stated here as

$\alpha >$

$$\frac{(n-3)(n-5)}{4}\left[1-\left((4ab^2)^{2/(n-3)}-1\right)\left(\frac{(n-3)(n-7)}{(n-5)^2}\right)^{\frac{n-3}{n-5}}\left(\frac{\max f}{< f >}\right)^{2/(n-3)}\right].$$

By (3.13), the right side of this inequality is nonpositive so that (3.6) holds true. Finally (3.14), (3.15) and (3.16) guarantee existence and multiplicity of two solutions for $(E^3_{\alpha f})$ when

$$\alpha \in \left[\frac{(n-3)^2(n-7)}{4(n-5)};\ \min\left\{\frac{(n-3)(n-5)}{4};\frac{n-5}{4(n-4)}\left(\frac{2}{b^2}+(n-6)(n-7)\right)\right\}\right[.$$

This interval is not empty thanks to (3.12). Example 3.3 is proved. ■

Proposition 3.1 below was used in the above proof.

Proposition 3.1 *On a product manifold $(V^m \times S^{n-m}, g \times h_{n-m})$ where (V^m, g) is a compact Riemannian m-manifold, we consider the isometry groups*

$$G = Id_V \times O(r_1) \times O(r_2), \qquad \text{and} \qquad H = Id_V \times Id_{\mathbb{R}^{r_1}} \times O(r_2)$$

where $r_1 \geq r_2$ et $r_1 + r_2 = n - m + 1$. Let $x_0 = (\theta_0, 0_{\mathbb{R}^{r_1}}, z_0)$ with $\theta_0 \in V$ and $z_0 \in S^{r_2-1}$. Then assumption $(\mathcal{H}_2)$ holds true and with the notations used above, we have that

$$S_{\tilde{g}}(\tilde{x}_0) \geq S_g(\theta_0) + r_1(r_1-1).$$

We postpone the proof of Proposition 3.1 to section 7.

4 Proofs of Theorems 1.a and 1.b

For convenience, we introduce a general inequality : for $crit > 2$ fixed, $\exists P > 0, \exists D > 0, \forall u \in H \subset H_1^2(M)$,

$$\|u\|^2_{crit} \leq P\left[\|\nabla u\|_2^2 + D\|u\|_2^2\right] \qquad (I_{PD})$$

where $H \subset H_1^2$ is a functional space such that the inclusion $H \subset L^{crit}$ is critical in sense of being continuous but not compact. Theorems 1.a and 1.b are direct corollaries of the following Theorem 4.1. In order to get Theorem 1.a from Theorem 4.1, it suffices to set $H = H_1^2$, $crit = \frac{2n}{n-2}$, $P = K_n$, and $D = B_0(M, g)$. In this case, (I_{PD}) is the optimal Sobolev inequality $(I_S^{Id,opt})$ which holds true according to Hebey-Vaugon [12] on every compact Riemanian $n-$manifold, $n \geq 3$. To get Theorem 1.b from Theorem 4.1, it suffices to set $H = H^2_{1,G_2}$, $crit = 2^\sharp$, $P = K_{n-k}\, A_2^{-\frac{2}{n-k}}$, and $D = B_{0,G_2}(M, g)$. In this case, (I_{PD}) is the optimal G_2-Sobolev inequality $(I_S^{G_2,opt})$ which holds true according to Theorem [F] when we assume $(\mathcal{H}_1)$ or $(\mathcal{H}_2)$.

Theorem 4.1 *Let (M,g) be a compact Riemannian $n-$manifold, $n\geq 3$, G_1 and G_2 be two isometry groups such that the minimum dimensions of G_1- and G_2-orbits are the same. We denote by $k\geq 0$ this common minimum orbit dimension, and let $A_i>0$ be the minimal volume of G_i-orbits of dimension k, $i=1,2$. We suppose that $n-k>2$, $A_1<A_2$ and that $(I_S^{G_2,opt})$ holds true. Assume that for $\alpha\in\mathbb{R}^{+*}$ and $f\in C^\infty_{G_1\cup G_2}$ positive, there exist two solutions of $(E^k_{\alpha f})$: $u_1\in C^\infty_{G_1}$ which is G_1-minimizing and $u_2\in C^\infty_{G_2}$ which is G_2-minimizing. If*

$$i)\qquad \alpha \leq B_{0,G_2}(M,g) \tag{4.1}$$

$$ii)\qquad \alpha \geq \frac{(4-crit)crit}{4}D \tag{4.2}$$

$$iii)\qquad \alpha > B_{0,G_2}(M,g)-\left[\left(\frac{A_2}{A_1}\right)^{\frac{2}{n-k}}-1\right]\frac{A_2^{\frac{2-crit}{n-k}}K_{n-k}^{\frac{crit-2}{2}}}{V_g^{\frac{(crit-2)(n-2-k)}{2(n-k)}}P^{\frac{crit}{2}}}\left(\frac{(4-crit)crit}{4}\right)^{\frac{crit}{2}}\left(\frac{\max f}{<f>}\right)^{\frac{(crit-2)(n-2-k)}{2(n-k)}} \tag{4.3}$$

then $\mathcal{E}(u_1)<\mathcal{E}(u_2)$. In particular u_1 and u_2 are distinct.

Proof of Theorem 4.1. Since u_i is G_i-minimizing, the strict inequality $\mathcal{E}(u_1)<\mathcal{E}(u_2)$ is equivalent to the strict inegality $\Upsilon_{G_1}<\Upsilon_{G_2}$. According to (2.3), it suffices then to prove that

$$\frac{A_1^{\frac{2}{n-k}}}{K_{n-k}(\max f)^{2/2^\sharp}}<\Upsilon_{G_2}. \tag{4.4}$$

Note that if u_1 satisfies (2.4), then the equality in (4.4) is sufficient to get $\mathcal{E}(u_1)<\mathcal{E}(u_2)$. Let us now search for a lower bound for Υ_{G_2}. Since u_2 is G_2-minimizing and with $(I_S^{G_2,opt})$, we get that

$$\frac{1}{\Upsilon_{G_2}} \leq \frac{(\max f)^{2/2^\sharp}}{\Upsilon_{G_2}^{\frac{n-k}{2}}}\frac{K_{n-k}}{A_2^{\frac{2}{n-k}}}\left[\|\nabla u_2\|_2^2+B_{0,G_2}\|u_2\|_2^2\right].$$

Thus

$$\frac{1}{\Upsilon_{G_2}}\leq(\max f)^{2/2^\sharp}\frac{K_{n-k}}{A_2^{\frac{2}{n-k}}}\left[1+\frac{B_{0,G_2}-\alpha}{\Upsilon_{G_2}^{\frac{n-k}{2}}}\|u_2\|_2^2\right]. \tag{4.5}$$

Since by (4.1), $B_{0,G_2}-\alpha\geq 0$, we search for an upper bound for $\|u_2\|_2^2$. Multiplying $(E^k_{\alpha f})$ by $u_2^{\frac{4}{crit}-1}$ and integrating over M gives :

$$\|\nabla u_2^{\frac{2}{crit}}\|_2^2=\frac{4}{crit(4-crit)}\left(\int_M fu_2^{2^\sharp-2+\frac{4}{crit}}\,dv_g-\alpha\int_M u_2^{\frac{4}{crit}}\,dv_g\right).$$

Then by Hölder's inequality

$$\int_M fu_2^{2^\sharp-2+\frac{4}{crit}}\,dv_g \leq \left(\int_M fu_2^{2^\sharp}\,dv_g\right)^{\frac{2^\sharp-2+\frac{4}{crit}}{2^\sharp}}\left(\int_M f\,dv_g\right)^{\frac{2-\frac{4}{crit}}{2^\sharp}}$$

and by (I_{PD})

$$\|\nabla u_2^{\frac{2}{crit}}\|_2^2 \geq \frac{1}{P}\|u_2^{\frac{2}{crit}}\|_{crit}^2 - D\,\|u_2^{\frac{2}{crit}}\|_2^2.$$

In particular, we have that

$$\frac{1}{P}\|u_2^{\frac{2}{crit}}\|_{crit}^2 \leq \frac{4}{crit(4-crit)}\left(\int_M f u_2^{2^\sharp}\, dv_g\right)^{\frac{2^\sharp-2+\frac{4}{crit}}{2^\sharp}}\left(\int_M f\, dv_g\right)^{\frac{2-\frac{4}{crit}}{2^\sharp}}$$

$$+\left(D-\frac{4\alpha}{crit(4-crit)}\right)\int_M u_2^{\frac{4}{crit}}\, dv_g.$$

Now by (4.2) and since u_2 is a G_2-minimizing solution we obtain that

$$\|u_2\|_2^2 \leq \left(\frac{4P}{(4-crit)crit}\right)^{\frac{crit}{2}} \Upsilon_{G_2}^{\frac{crit-2+n-k}{2}}\left(\int_M f\, dv_g\right)^{\frac{crit-2}{2^\sharp}}.$$

Reporting this inequality in (4.5)

$$\frac{1}{\Upsilon_{G_2}} \leq$$

$$(\max f)^{2/2^\sharp}\frac{K_{n-k}}{A_2^{\frac{2}{n-k}}}\left[1+(B_{0,G_2}-\alpha)\left(\frac{4P}{(4-crit)crit}\right)^{\frac{crit}{2}}\Upsilon_{G_2}^{\frac{crit-2}{2}}\left(\int_M f\, dv_g\right)^{\frac{crit-2}{2^\sharp}}\right]$$

and with the upper bound for Υ_{G_2} given by (2.3),

$$\frac{1}{\Upsilon_{G_2}} \leq (\max f)^{2/2^\sharp}\frac{K_{n-k}}{A_2^{\frac{2}{n-k}}}\times$$

$$\left[1+(B_{0,G_2}-\alpha)A_2^{\frac{crit-2}{n-k}}\frac{P^{\frac{crit}{2}}}{K_{n-k}^{\frac{crit-2}{2}}}\left(\frac{4}{(4-crit)crit}\right)^{\frac{crit}{2}}\left(\frac{\int_M f\, dv_g}{\max f}\right)^{\frac{crit-2}{2^\sharp}}\right].$$

Note that if u_2 satisfies (2.4), the above inequality is strict. Finally thanks to (4.4) we have $\mathcal{E}(u_1) < \mathcal{E}(u_2)$ if

$$(\max f)^{2/2^\sharp}\frac{K_{n-k}}{A_1^{\frac{2}{n-k}}} > (\max f)^{2/2^\sharp}\frac{K_{n-k}}{A_2^{\frac{2}{n-k}}}\times$$

$$\left[1+(B_{0,G_2}-\alpha)A_2^{\frac{crit-2}{n-k}}\frac{P^{\frac{crit}{2}}}{K_{n-k}^{\frac{crit-2}{2}}}\left(\frac{4}{(4-crit)crit}\right)^{\frac{crit}{2}}\left(\frac{\int_M f\, dv_g}{\max f}\right)^{\frac{crit-2}{2^\sharp}}\right]$$

or isolating α and introducing $< f >$ the average value of f :

$$\alpha > B_{0,G_2} - \left[\left(\frac{A_2}{A_1}\right)^{\frac{2}{n-k}} - 1\right] A_2^{\frac{2-crit}{n-k}}\frac{K_{n-k}^{\frac{crit-2}{2}}}{P^{\frac{crit}{2}}}\left(\frac{(4-crit)crit}{4}\right)^{\frac{crit}{2}}\left(\frac{\max f}{V_g < f >}\right)^{\frac{crit-2}{2^\sharp}}$$

which is exactly (4.3). Theorem 4.1 is proved. Note that the remark following Theorem 4.1.b is also proved since if u_1 or u_2 satisfies (2.4), then the previous inequality is not necessarily strict. ■

5 Multiplicity results 2

We provide another general result for multiplicity in Theorem 5.1 below. Then we illustrate the Theorem on specific examples. We postpone the proof of Theorem 5.1 to section 6.

Theorem 5.1 *Let (M,g) be a compact Riemannian n-manifold, $n \geq 3$, G_1 and G_2 be two isometry groups such that the minimum dimensions of G_1- and G_2-orbits are the same. We denote by $k \geq 0$ this common minimum orbit dimension, and let $A_i > 0$ be the minimum volume of G_i-orbits of dimension k, $i = 1, 2$. We suppose that $n-k > 2$ and $A_1 < A_2$, and that $(I_S^{G_2,opt})$ holds true. Assume that for $\alpha \in \mathbb{R}_+^*$ and $f \in C^\infty_{G_1 \cup G_2}$ positive, there exist two solutions of $(E^k_{\alpha f})$: $u_1 \in C^\infty_{G_1}$ which is G_1-minimizing, and $u_2 \in C^\infty_{G_2}$, which is G_2-minimizing. If*

$$i) \qquad \alpha \leq B_{0,G_2}(M,g), \text{ and} \tag{5.1}$$

$$ii) \qquad \alpha > B_{0,G_2}(M,g) - \frac{A_2^{\frac{2}{n-k}} - A_1^{\frac{2}{n-k}}}{K_{n-k}\ V_g^{\frac{2}{n-k}}} \ \frac{\inf f}{\max f^{\frac{2}{2^\sharp}} < f >^{\frac{2}{n-k}}}, \tag{5.2}$$

where $< f >$ stands for the average value of f, then $\mathcal{E}(u_1) < \mathcal{E}(u_2)$. In particular, u_1 and u_2 are distinct.

Here again, if u_1 satisfies (2.4), then inequality $ii)$ is not necessarily strict. In the following Corollary to Theorem 5.1, $f = 1$ and we obtain three different solutions for (E^k_α).

Corollary 5.1 *Let (M,g) be a compact Riemannian n-manifold, $n \geq 3$, G_1 and G_2 be two isometry groups such that the minimum dimensions of G_1- and G_2-orbits are the same. We denote by $k \geq 0$ this common minimum orbit dimension, and let $A_i > 0$ be the minimum volumes of G_i-orbits of dimension k, $i = 1, 2$. We suppose that $n-k > 2$ and $A_1 < A_2$, and that $(I_S^{G_1,opt})$ and $(I_S^{G_2,opt})$ hold true. Then :*
1) *If*

$$B_{0,G_2}(M,g) - \frac{A_2^{\frac{2}{n-k}}}{K_{n-k}\ V_g^{\frac{2}{n-k}}} < B_{0,G_1}(M,g) - \frac{A_1^{\frac{2}{n-k}}}{K_{n-k}\ V_g^{\frac{2}{n-k}}} \tag{5.3}$$

then there exist two solutions of different energies for the equation

$$\Delta u + \alpha u = u^{\frac{n+2-k}{n-2-k}} \tag{E^k_α}$$

when α belongs to the interval

$$\alpha \in \left] B_{0,G_2}(M,g) - \frac{A_2^{\frac{2}{n-k}} - A_1^{\frac{2}{n-k}}}{K_{n-k}\ V_g^{\frac{2}{n-k}}} \ ; \ \min_{i=1,2} B_{0,G_i}(M,g) \right[. \tag{5.4}$$

One of these solutions is non constant and G_1-invariant, the other is G_2-invariant.
2) *If moreover*

$$\frac{A_2^{\frac{2}{n-k}}}{K_{n-k} V_g^{\frac{2}{n-k}}} < \min_{i=1,2} B_{0,G_i}(M,g) \tag{5.5}$$

then the constant solution $\bar u_\alpha = \alpha^{\frac{n-2-k}{4}}$ *of* (E^k_α) *is different from the two previous solutions given in* ***1)*** *when* α *belongs to the interval*

$$\alpha \in \left[\max \left\{ B_{0,G_2}(M,g) - \frac{A_2^{\frac{2}{n-k}} - A_1^{\frac{2}{n-k}}}{K_{n-k}\ V_g^{\frac{2}{n-k}}}, \frac{A_2^{\frac{2}{n-k}}}{K_{n-k} V_g^{\frac{2}{n-k}}} \right\} \ ; \ \min_{i=1,2} B_{0,G_i}(M,g) \right[. \tag{5.6}$$

Proof of Corollary 5.1. Part **1)** is a corollary of Theorem 5.1 when $f = 1$ and where existence of solutions is given by Proposition 2.1. We have here $\alpha < B_{0,G_i}$. In particular (2.4) holds true and by the remark following Theorem 5.1, inequality $ii)$ in Theorem 5.1 is not necessarily strict. Theorem 5.1 claims that the two solutions have different energies when α belongs to the interval in (5.4). In particular, with (2.7), we have that

$$\alpha^{\frac{n-k}{2}} V_g \geq \frac{A_1}{K_{n-k}^{\frac{n-k}{2}}}.$$

But $\alpha^{\frac{n-k}{2}} V_g$ is the energy of constant solution $\alpha^{\frac{n-2-k}{4}}$. Since $\mathcal{E}(u_1) < A_1 K_{n-k}^{-\frac{n-k}{2}}$, we get that $\mathcal{E}(u_1) < \mathcal{E}(\alpha^{\frac{n-2-k}{4}})$ and u_1 is not constant. Part **1)** is proved and

$$\mathcal{E}(u_1) < \mathcal{E}(u_2) = \Upsilon_2^{\frac{n-k}{2}} < A_2\, K_{n-k}^{-\frac{n-k}{2}} .$$

Then $\mathcal{E}(u_2) < \mathcal{E}(\alpha^{\frac{n-2-k}{4}})$ if $\alpha \geq \frac{A_2^{\frac{2}{n-k}}}{K_{n-k}\ V_g^{\frac{2}{n-k}}}$. This is compatible with (5.4), thanks to (5.5), and part **2)** is proved. ■

Now we discuss specific examples. In the three following examples, the manifold is $S^1(t) \times S^{n-1}$ and we fix $f \equiv 1$. The first example concerns the critical equation (E^0_α) and the two other examples concern the overcritical equation (E^k_α) with $k = 1$. In the first example, we pass from the Yamabe multiplicity to an interval of multiplicity.

Example 5.1 On $(S^1(t) \times S^{n-1}, h_1 \times h_{n-1}), n \geq 3$, let $G_1 = R_1 \times Id_{S^{n-1}}$ and $G_2 = R_2 \times Id_{S^{n-1}}$ be two isometry groups, where R_1 and R_2 are finite subgroups of $SO(2)$ with respectif cardinals $A_1 < A_2$. If

$$t > \max \left\{ \frac{A_2 \omega_n}{2\pi\omega_{n-1}} \left(\frac{n}{n-2} \right)^{n/2} ; \left(\frac{A_2^2\ (2\pi\omega_{n-1})^{2/n}}{(A_2^{2/n} - A_1^{2/n})\ n(n-2)\omega_n^{2/n}} \right)^{\frac{n}{2(n-1)}} \right\}$$

then there exist at least three C^∞ solutions of different energies for the critical equation (E^0_α) when α belongs to the interval

$$\alpha \in \left[\max \left\{ \frac{(n-2)^2}{4} + \frac{A_2^2}{4t^2} - \frac{\left(A_2^{\frac{2}{n}} - A_1^{\frac{2}{n}}\right) n(n-2)\omega_n^{2/n}}{4(2\pi t\omega_{n-1})^{2/n}} \ , \ \frac{A_2^{2/n} n(n-2)\omega_n^{2/n}}{4(2\pi t \omega_{n-1})^{2/n}} \right\} \ ; \ \frac{(n-2)^2}{4} \right] . \tag{5.7}$$

One of these solutions is G_1-invariant, the other is G_2-invariant and the third one is the constant solution $\bar{u}_\alpha = \alpha^{\frac{n-2}{4}}$.

As a remark, when $\alpha = \frac{(n-2)^2}{4}$, (E^0_α) is the Yamabe equation on $S^1(t) \times S^{n-1}$ and we recover a multiplicity result of Hebey-Vaugon [13].

Proof of Example 5.1. The actions of the groups are already presented in Example 3.2. In particular, $(I_S^{G_i,opt})$ holds true and with (2.2) and (2.8) we have that

$$\frac{(n-2)^2}{4} \leq B_{0,G_i}\left(S^1(t) \times S^{n-1}, h_1 \times h_{n-1}\right) \leq \frac{A_i^2}{4t^2} + \frac{(n-2)^2}{4}. \tag{5.8}$$

We claim that there exist two solutions $u_i \in C^\infty_{G_i}$ for (E^0_α) if

$$\alpha \leq \frac{(n-2)^2}{4}.$$

The double existence for $\alpha < \frac{(n-2)^2}{4}$ is indeed given by (2.5). For $\alpha = \frac{(n-2)^2}{4}$ this is given by the the resolution of the Yamabe problem on $S^1(t/A_i) \times S^{n-1}$ and with similar arguments to the one used in Example 3.2. Now Corollary 5.1 guarantees that u_1, u_2 and the constant solution have different energies if (5.3) and (5.5) hold true. First by (5.8), (5.3) holds true if

$$\frac{A_2^2}{4t^2} + \frac{(n-2)^2}{4} - \frac{A_2^{2/n}}{K_n(2\pi t\omega_{n-1})^{2/n}} < \frac{(n-2)^2}{4} - \frac{A_1^{2/n}}{K_n(2\pi t\omega_{n-1})^{2/n}}$$

namely if

$$t > \left(\frac{A_2^2(2\pi\omega_{n-1})^{2/n}}{n(n-2)(A_2^{2/n} - A_1^{2/n})\omega_n^{2/n}}\right)^{\frac{n}{2(n-1)}}.$$

Since $B_{0,G_i} \geq \frac{(n-2)^2}{4}$, (5.5) holds true if

$$\frac{A_2^{2/n} n(n-2)\omega_n^{2/n}}{4(2\pi t\omega_{n-1})^{2/n}} < \frac{(n-2)^2}{4}$$

namely if

$$t > \frac{A_2\omega_n}{2\pi\omega_{n-1}}\left(\frac{n}{n-2}\right)^{n/2}.$$

Under these two conditions on t, Corollary 5.1 gives the triple multiplicity when α belongs to the interval in (5.6) which contains the interval in (5.7, thanks to (5.8). Example 5.1 is proved. ∎

The next example involves the Hopf fibration and concerns overcritical equations on $S^1(t) \times S^3$.

Example 5.2 On $(S^1(t) \times S^3, h_1 \times h_3)$, where $t > 1$, let

$$G_1 = Id_{S^1(t)} \times \{(\sigma, \sigma)/\sigma \in SO(2)\} \qquad \text{and} \qquad G_2 = O(2) \times Id_{S^3}$$

be two isometry groups. There exist at least two C^∞ solutions of different energies for the overcritical equation

$$\Delta u + \alpha u = u^5, \qquad u > 0 \tag{E_α^1}$$

when α belongs to the interval

$$\alpha \in \left[\frac{3}{4\, t^{2/3}}; \frac{3}{4}\right[. \tag{5.9}$$

One of these solutions is G_1-invariant and nonconstant, the other is G_2-invariant. Besides if u_2 is not the constant solution, then there exist at least three different solutions when α belongs to the interval in (5.9). On the other hand, if u_2 is the constant solution, the interval of multiplicity for α extends to $[\frac{3}{4\ t^{2/3}}; 1[$.

Proof of Example 5.2. The G_2-orbits are $S^1(t) \times \{\theta\}$, where $\theta \in S^3$. Thus they are principal of dimension 1 and constant volume $2\pi t$ and we have that $\left(S^1(t) \times S^3\right)/G_2 = S^3$ with quotient metric h_3. As already mentioned, $(I_S^{G_2,opt})$ holds true and with (2.2) and (2.6)

$$B_{0,G_2}(S^1(t) \times S^3, h_1 \times h_3) = \frac{3}{4}.$$

The group $\{(\sigma,\sigma), \sigma \in SO(2)\}$ gives the Hopf fibration $S^3 \to S^2(1/2)$ with fiber S^1 and h_2 as quotient metric on $S^2(1/2)$. The G_1-orbits are $\{\rho\} \times S^1$ where $\rho \in S^1(t)$. Thus they are principal of dimension 1 and constant volume 2π and we have $(S^1(t) \times S^3)/G_1 = S^1(t) \times S^2(1/2)$ with quotient metric $h_1 \times h_2$. Here again $(I_S^{G_1,opt})$ holds true and

$$B_{0,G_1}(S^1(t) \times S^3, h_1 \times h_3) = B_0(S^1(t) \times S^2(1/2), h_1 \times h_2).$$

Part **1)** of Corollary 5.1 gives a multiplicity interval for α if (5.3) holds true. We easily check that

$$B_{0,G_2} - \frac{A_2^{2/3}}{K_3 V_{h_1 \times h_3}^{2/3}} = 0$$

and that

$$B_{0,G_1} - \frac{A_1^{2/3}}{K_3 V h_1 \times h_3{}^{2/3}} = B_0(S^1(t) \times S^2(1/2), h_1 \times h_2) - \frac{3}{4t^{2/3}}.$$

Thus (5.3) becomes here

$$B_{0,G_1} > \frac{3}{4t^{2/3}}.$$

By (2.7) we know that $B_{0,G_1} \geq \max\left\{\frac{3}{4t^{2/3}}, 1\right\} = 1$ since $t > 1$, and thus (5.3) holds true. Part **1)** of Corollary 5.1 guarantees then a double multiplicity when α belongs to the interval in (5.4). We easily see that this interval is here

$$\alpha \in \left[\frac{3}{4\, t^{2/3}}; \frac{3}{4}\right[.$$

In this example, (5.5) does not hold true, so part **2)** of Corollary 5.1 does not apply. The constant solution $\bar{u}_\alpha = \alpha^{\frac{1}{4}}$ exists for any $\alpha > 0$. If $u_2 \neq \bar{u}_\alpha$ then there exist at least three solutions of different energies when α belongs to the interval in (5.9). Now if $u_2 = \bar{u}_\alpha$ then u_2 exists for any $\alpha > 0$. The solution u_1 exists when $\alpha < 1 \leq B_{0,G_1}$ and its energy verifies $\mathcal{E}(u_1) < A_1 K_3^{-\frac{3}{2}}$. Thus u_1 is not constant if

$$\frac{A_1}{K_3^{\frac{2}{3}}} \leq \mathcal{E}(\bar{u}_\alpha) = \alpha^{\frac{3}{2}} V_{h_1 \times h_3}$$

namely if $\alpha \geq \frac{3}{4t^{2/3}}$. The interval of double multiplicity is here $[\frac{3}{4t^{\frac{2}{3}}}, 1[$. Example 5.2 is proved. ∎

The last example involves infinite non principal orbits.

Example 5.3 On $(S^1(t) \times S^{n-1}, h_1 \times h_{n-1})$ with $n \geq 4$ and $t > \left(\frac{n-1}{n-3}\right)^{\frac{n-1}{2}}$, let

$$G_1 = Id_{S^1(t)} \times O(n-2) \times O(2) \qquad \text{and} \qquad G_2 = O(2) \times Id_{S^{n-1}}$$

be two isometry groups. There exist at least two C^∞ solutions of different energies for the overcritical equation

$$\Delta u + \alpha u = u^{\frac{n+1}{n-3}} \tag{E_α^1}$$

when α belongs to the interval

$$\alpha \in \left[\frac{(n-1)(n-3)}{4t^{\frac{2}{n-1}}} ; \frac{(n-3)^2}{4} \right[.$$

One of these solutions is G_1-invariant and nonconstant, the other one is G_2-invariant.

Proof of Example 5.3. The group G_2 is the same as in Example 5.2. The G_2-orbits are $S^1(t) \times \{\theta\}$ where $\theta \in S^{n-1}$, of dimension 1 and constant volume $2\pi t$. The quotient manifold is (S^{n-1}, h_{n-1}) and $(I_S^{G_2,opt})$ holds true with

$$B_{0,G_2}\left(S^1(t) \times S^{n-1}, h_1 \times h_{n-1}\right) = \frac{(n-1)(n-3)}{4}.$$

We easily check that

$$B_{0,G_2} - \frac{A_2^{\frac{2}{n-k}}}{K_{n-k} V_{h_1 \times h_{n-1}}^{\frac{2}{n-k}}} = 0.$$

The G_1-orbits are sphere products possibly reduced to a point :
$\forall x = (\theta, y, z) \in S^1(t) \times \mathbb{R}^{n-2} \times \mathbb{R}^2 \subset S^1(t) \times S^{n-1}$,

$$O_x^{G_1} = \{\theta\} \times S^{n-3}(\|y\|) \times S^1(\|z\|).$$

For $x_0 = (\theta, 0_{\mathbb{R}^{n-2}}, z_0)$, where $\theta \in S^1(t)$, and $z_0 \in S^1$, we have that $O^{G_1}_{x_0} = \{\theta\} \times \{0_{\mathbb{R}^{n-2}}\} \times S^1$. Thus $dim O^{G_1}_{x_0} = 1$ is minimum and $vol O^{G_1}_{x_0} = 2\pi$. Similar arguments as in the proof of Example 3.3 show that $(\mathcal{H}_2)$ holds true if we choose the normal subgroup H of G_1 as $H = Id_{S^1(t)\times\mathbb{R}^{n-2}} \times O(2)$. Thus $(I^{G_1,opt}_S)$ holds true. Now assumption (5.3) of Corollary 5.1 becomes

$$B_{0,G_1} > \frac{(n-1)(n-3)}{4t^{\frac{2}{n-1}}}.$$

By (2.7) we know that

$$B_{0,G_1} \geq \max\left\{\frac{(n-1)(n-3)}{4t^{2/(n-1)}};\ \frac{(n-3)}{4(n-2)}\left(S_{\tilde{g}}(\tilde{x}_0) + \frac{3\Delta_{\tilde{g}}\tilde{v}_H(\tilde{x}_0)}{A_1}\right)\right\}.$$

Since $vol O^H_{x_0} = vol O^{G_1}_{x_0}$ is maximal on H-orbits we have $\Delta_{\tilde{g}}\tilde{v}_H(\tilde{x}_0) \geq 0$ and according to Proposition 3.1, $S_{\tilde{g}}(\tilde{x}_0) \geq (n-2)(n-3)$. In particular

$$B_{0,G_1} \geq \max\left\{\frac{(n-1)(n-3)}{4t^{2/(n-1)}};\ \frac{(n-3)^2}{4}\right\} = \frac{(n-3)^2}{4}$$

since $t > \left(\frac{n-1}{n-3}\right)^{\frac{n-1}{2}}$. Thus (5.3) holds true. Finally part **1)** of Corollary 5.1 guarantees a double multiplicity when α belongs to the interval in (5.4) whose endpoints are

$$B_{0,G_2} - \frac{\left(A_2^{\frac{2}{n-1}} - A_1^{\frac{2}{n-1}}\right)}{K_{n-1}V^{\frac{2}{n-1}}_{h_1\times h_{n-1}}} = \frac{(n-1)(n-3)}{4t^{\frac{2}{n-1}}}$$

and

$$\min\{B_{0,G_1}, B_{0,G_2}\} \geq \min\left\{\frac{(n-3)^2}{4}, \frac{(n-1)(n-3)}{4}\right\} = \frac{(n-3)^2}{4}.$$

Example 5.3 is proved. ■

6 Proof of Theorem 5.1

The proofs of Theorems 4.1 and 5.1 are similar but with an important difference in the way we find an upper bound for $\|u_2\|_2$. In order to prove Theorem 5.1 it suffices, as in the proof of Theorem 4.1, to prove that

$$\frac{A_1^{\frac{2}{n-k}}}{K_{n-k}(\max f)^{2/2^\sharp}} < \Upsilon_{G_2}. \tag{6.1}$$

We search for a lower bound for Υ_{G_2} and similar arguments as in proof of Theorem 4.1 lead us to inequality (4.5)

$$\frac{1}{\Upsilon_{G_2}} \leq (\max f)^{2/2^\sharp}\ \frac{K_{n-k}}{A_2^{\frac{2}{n-k}}}\left[1 + \frac{B_{0,G_2} - \alpha}{\Upsilon_{G_2}^{\frac{n-k}{2}}}\|u_2\|_2^2\right]. \tag{6.2}$$

Thanks to (5.1), $B_{0,G_2} - \alpha \geq 0$, and we search now for an upper bound for $\|u_2\|_2$. Here is where the proof diverges from the proof of Theorem 4.1. We obtain with Hölder's inequality and since u_2 is G_2-minimizing that

$$\int_M u_2^2 \, dv_g \leq \frac{\Upsilon_{G_2}^{\frac{n-2-k}{2}}}{\min f} \left(\int_M f \, dv_g \right)^{\frac{2}{n-k}} .$$

Reporting this inequality in (6.2) and isolating Υ_{G_2} gives :

$$\Upsilon_{G_2} \geq \frac{A_2^{\frac{2}{n-k}}}{(\max f)^{2/2^\sharp} K_{n-k}} - (B_{0,G_2} - \alpha) \frac{\left(\int_M f \, dv_g\right)^{\frac{2}{n-k}}}{\min_M f}.$$

Finally (6.1), and thus also the strict inequality $\mathcal{E}(u_1) < \mathcal{E}(u_2)$, hold true if

$$\frac{A_1^{\frac{2}{n-k}}}{(\max f)^{2/2^\sharp} K_{n-k}} < \frac{A_2^{\frac{2}{n-k}}}{(\max f)^{2/2^\sharp} K_{n-k}} - (B_{0,G_2} - \alpha) \frac{\left(\int_M f \, dv_g\right)^{\frac{2}{n-k}}}{\min f},$$

or else

$$\alpha > B_{0,G_2} - \frac{A_2^{\frac{2}{n-k}} - A_1^{\frac{2}{n-k}}}{K_{n-k}} \, \frac{\min f}{(\max f)^{2/2^\sharp} \left(V_g < f >\right)^{\frac{2}{n-k}}}.$$

The last inequality is not necessarily strict when u_1 satisfies (2.4). Theorem 5.1 is proved.■

7 Proof of Proposition 3.1

We start with the following Lemma.

Lemma 7.1 *Let (M, g) be a compact Riemannian n-manifold, $n \geq 3$, of constant sectional curvature $K_g(M)$, and G be an isometry group such that all G-orbits are principal, and thus of constant dimension k. Assume that $k < n$. Then*

$$S_{\tilde{g}}(y) \geq K_g(M) \; (n-k)(n-k-1), \tag{7.1}$$

for all $y \in M/G$, where $\tilde{g}$ is the quotient metric induced by g on M/G.

As a remark, if the G-orbits are finite, the canonical submersion $\pi : M \to M/G$ is a local isometry and inequality (7.1) is an equality.

Proof of Lemma 7.1. On $(M/G, \tilde{g})$, which has dimension $n - k$, we have the following relation between the sectional $K_{\tilde{g}}$ and the scalar curvature: $S_{\tilde{g}}$

$$S_{\tilde{g}}(y) = \sum_{(i,j) \in [1, n-k]^2, i \neq j} K_{\tilde{g}}(\tilde{e}_i, \tilde{e}_j) \tag{7.2}$$

for all $y \in M/G$, where $(\tilde{e}_1, ..., \tilde{e}_{n-k})$ is an orthonormal basis of $T_y(M/G)$. O'Neil's formula links the sectional curvatures K_g of M and $K_{\tilde{g}}$ of (M/G) by

$$K_{\tilde{g}}(\tilde{e}_i, \tilde{e}_j) = K_g(e_i, e_j) + \frac{3}{4} \left| [e_i \; e_j]^v \right|^2 \geq K_g(e_i, e_j)$$

where $e_i = \left(d\pi_x\backslash_{(Kerd\pi_x)^\perp}\right)^{-1}(\tilde{e}_i) \in (Ker\ d\pi_x)^\perp$, and where $[e_i\ e_j]^v \in Ker\ d\pi_x$ is the vertical composant of $[e_i\ e_j] \in T_x(M)$. Since K_g is constant and with (7.2), we finally obtain

$$S_{\tilde{g}}(y) \geq K_g(M)\ (n-k)(n-k-1)$$

and Lemma 7.1 is proved. ■

Now we prove Proposition 3.1.

Proof of Proposition 3.1. On the open set

$$\Omega = \{x = (\theta, y, z) \in V^m \times S^{n-m}, \|z\| \neq 0\},$$

all H-orbits are principal and $(\mathcal{H}_2)$ holds true. We have that Ω contains $O^H_{x_0} = \{\theta_0\} \times \{0_{\mathbb{R}^{r_1}}\} \times S^{r_2-1}$; thus there exist an open set $\Omega_1 \ni \theta_0$ of V^m and an open set $\Omega_2 \ni \{0_{\mathbb{R}^{r_1}}\} \times S^{r_2-1}$ of S^{n-m} such that

$$O^H_{x_0} \in \Omega_1 \times \Omega_2 \subset \Omega$$

and we have

$$(\Omega_1 \times \Omega_2)/H = \Omega_1\ \times\ (\Omega_2/\ H')$$

where $H' = Id_{\mathbb{R}^{r_1}} \times O(r_2)$. The metric on $(\Omega_1 \times \Omega_2)/H$ is the quotient metric $\tilde{g} = g \times \tilde{h}_{n-m}$ where $\tilde{h}_{n-m}$ is the quotient metric induced by h_{n-m} on S^{n-m}/H'. Now

$$\tilde{x}_0 = \pi_H\left(\{\theta_0\} \times \{0_{\mathbb{R}^{r_1}}\} \times S^{r_2-1}\right) = \{\theta_0\} \times \{t_0\}$$

with $t_0 = \pi_{H'}\left(\{0_{r_1}\} \times S^{r_2-1}\right) \in \Omega_2/H'$ and where $\pi_{H'} : \Omega_2 \to \Omega_2/H'$ is the canonical submersion. Thus

$$S_{\tilde{g}}(\tilde{x}_0) = S_g(\theta_0) + S_{\tilde{h}_{n-m}}(t_0)\,.$$

Since the H'-orbits are principal on $\Omega_2 \subset S^{n-m}$, thanks to lemma 7.1, and since $dim\,\Omega_2/H' = n - m - r_2 + 1 = r_1$ and $K_{h^{n-m}}(S^{n-m}) = 1$, we have $S_{\tilde{h}_{n-m}}(t_0) \geq r_1(r_1-1)$. Finally

$$S_{\tilde{g}}(\tilde{x}_0) \geq S_g(\theta_0) + r_1(r_1-1),$$

Proposition 3.1 is proved. ■

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