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Summary. In this paper we consider a d-dimensional continuous Itô process, which is observed at n regularly spaced times on a given time interval $[0, T]$. This process is driven by a multidimensional Wiener process, and our aim is to provide asymptotic statistical procedures which give the minimal dimension of the driving Wiener process, which is between 0 (a pure drift) and d. We exhibit several different procedures, which are all similar to asymptotic testing hypotheses.

Keywords. Asymptotic testing, Brownian dimension, Discrete observations, Itô processes.

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1 Introduction

In numerous applications, one chooses to model a complex dynamical phenomenon by stochastic differential equations or, more generally, by semimartingales, either because random forces excite a mechanical system, or because time–dependent uncertainties disturb a deterministic trend, or because one aims to reduce the dimension of a large scale system by considering that some components contribute stochastically to the evolution of the system. Examples of applications respectively concern mechanical oscillators submitted to random loading, prices of financial assets, molecular dynamics.

Of course the calibration of the model is a crucial issue. A huge literature deals with the statistics of stochastic processes, particularly of diffusion processes: Parametric and non-parametric estimators of the coefficients of stochastic differential equations have been intensively studied; see for example the books [6] and [7] of Prakasa Rao, in which a large number of papers are quoted and analyzed. However, somewhat astonishingly, it seems to us that most of the papers consider that the dimension of the noise is known by the observer. This hypothesis is often questionable: there is no reason to *a priori* fix this dimension when one observes a basket of assets, or a complex mechanical structure in a random environment. Actually the last two authors of this paper were motivated to study this question by modelling and simulation issues related to the pricing of contracts based on baskets of energy prices (see O. Bardou's thesis [3]). There was no determining financial reason to fix the Brownian motion dimension to a particular value. In addition, the interest to find an as small dimension as possible was two-fold: first, one then avoids the calibration of useless diffusion matrix components; second, practitioners need that the simulation of the model, and thus the computation of contract prices and of corresponding risk measures by means of Monte Carlo simulations, are as quick as possible.

We thus try, in this paper, to tackle the question of estimating the Brownian dimension of an Itô process from the observation of one trajectory during a finite time interval. More precisely, we aim to build estimators which provide an "explicative Brownian dimension" r_B: a model driven by a r_B Brownian motion satisfyingly fits the information conveyed by the observed path, whereas increasing the Brownian dimension does not allow to fit the data any better. Stated this way, the problem is obviously ill posed, hence our first step consists in defining a reasonable framework to develop our study.

Suppose that we observe a continuous d–dimensional Brownian semimartingale $X = (X^i)_{1\leq i\leq d}$ on some space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$. The observation time interval is $[0, T]$ with T finite. The process X is a continuous Itô process, meaning that it satisfies the following assumption:

Hypothesis (H): We have

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s \, dW_s, \tag{1}$$

where W is a standard q–dimensional BM, a is a predictable $\mathbf{R}^d$–valued locally bounded process, and σ is a $d \times q$ matrix–valued adapted and càdlàg processes (in (1) one can replace σ_s by the left limit σ_{s-}, so as to have a predictable integrand, if one wishes).

We say that we are in the "pure diffusion case" when $\sigma_s = \sigma(X_s)$. We set $c_s = \sigma_s \sigma_s^\star$ (so $c_s = c(X_s)$ where $c = \sigma\sigma^\star$ in the pure diffusion case). The process c takes its values in the set $\mathcal{M}_d$ of all $d \times d$ symmetric nonnegative matrices. We denote by $\mathrm{rank}(\Sigma)$ the rank of any $\Sigma \in \mathcal{M}_d$.

As it is well known, the same process X can be written as (1) with many different Wiener processes; namely if (Π_s) is a progressively measurable process taking its values in the set of $q \times q$ orthogonal matrices, then $W'_t = \int_0^t \Pi_s dW_s$ is another q-dimensional Wiener process and X is of the form (1) with W' and $\sigma'_s = \sigma_s \Pi_s^{-1}$. Then the *"Brownian dimension"* r_B of our model is defined as being the smallest integer r such that, after such a transformation, the last $q - r$ columns of $\sigma'_s(\omega)$ vanish (outside a $\mathbf{P}(d\omega)ds$-null set, and for $s \leq T$ of course). In this case, we can forget about the last $q - r$ components of W' and in fact write (1) with an r-dimensional Wiener process. Obviously $r_B \leq d$ always, so one could start with a model (1) with $q \leq d$ always, but it is convenient for the discussion in this paper to take q arbitrary.

Our aim is to make some kind of inference on this Brownian dimension r_B, which is also the maximal rank of c_s (up to a $\mathbf{P}(d\omega)ds$-null set), on the basis of the observation of the variables $X_{iT/n}$ for $i = 0, 1, \ldots, n$, where $[0, T]$ is the time interval on which the process is available. Let us make some preliminary comments, in which we refer to the "ideal" and "actual" observation schemes when one observes X completely over $[0, T]$, or at times iT/n only, respectively.

1) Suppose we are in the pure diffusion case, that is $c_s = c(X_s)$ with c a continuous function, and that the range of the process is the whole of $\mathbf{R}^d$ (that is, every open subset of $\mathbf{R}^d$ is visited by X on the time interval $[0, T]$ with a positive probability). Set $r(x) := \mathrm{rank}(c(x))$, and let $A(\omega)$ be the subset of $\mathbf{R}^d$ which is visited by the path $(X(\omega)_t : t \in [0, T])$. The Brownian dimension is $r_B = \sup_{x \in \mathbf{R}^d} r(x)$, but in the ideal scheme we observe $R(\omega) = \sup_{x \in A(\omega)} r(x)$ and so we can only assert that $r \geq R(\omega)$. The situation is similar to what happens in the non-parametric estimation of the function c: in the ideal scheme this function c is known on $A(\omega)$ and hopelessly unknown on $\mathbf{R}^d \backslash A(\omega)$.

2) More generally, the only relevant quantity we might hope to "estimate" is the (random) maximal rank

$$R(\omega) = \sup_{s \in [0,T)} \mathrm{rank}(c_s(\omega)) \tag{2}$$

(we should take the essential supremum rather than the supremum, but the two agree since c_s is right-continuous in s). The variable R is integer–valued, so its "estimation" is more akin to testing that $R = r$ for any particular $r \in \{0, \ldots, d\}$, although it will not be a test in the ordinary sense because R is random. Note that in many models we will have that $\mathrm{rank}(c_s(\omega))$ is independent of s and ω; then R is non–random, but this property does not really makes the analysis any easier.

3) In the actual scheme we will construct an integer–valued statistics $\widehat{R}_n$ which serves as an "estimator" for R. We have to somehow maximize (and evaluate) the probability that $\widehat{R}_n = R$, or perhaps this probability conditional on the value taken by R,

or conditional on the whole path of X over $[0,T]$. That is, we perform a kind of "conditional test".

4) We might also take a different look at the problem. Considering the model (1), we can introduce a kind of "distance" Δ_r between the true process X and the class of all processes X' of the same form, but with a diffusion coefficient c'_s satisfying identically $\text{rank}(c'_s) \leq r$. Then we construct estimators Δ_r^n for Δ_r, for all values of r, and decide on the basis of these Δ_r^n which Brownian dimension r_B is reasonable to consider for the model. The mathematical problem is then similar to the semi–parametric estimation of a parameter in the diffusion coefficient for a discretely observed diffusion with unknown drift: here the "parameter" is the collection of all Δ_r, and the unknown (nuisance) parameters are the processes a_s and c_s (or σ_s).

The paper is organized as follows: In Section 2 we explain in a more precise way the "distances" mentioned above. Section 3 is a collection of simple linear algebra results, and Section 4 contains the basic limiting results needed. Then in Sections 5 and 6 we put the previous results in use to develop some statistical applications, and finally we provide some numerical experiments in Section 7.

2 An instructive but non effective approach

In this section we measure the discrepancy between the model (1), and models of the same type but with a different Brownian dimension. We denote by $\mathcal{S}_r$ the set of all càdlàg adapted $d \times q$ matrix–valued processes σ' such that $c'_s = \sigma'_s \sigma'^{\star}_s$ satisfies $\text{rank}(c'_s) \leq r$ a.s. for all s. In particular $\mathcal{S}_0$ contains only $\sigma' \equiv 0$. With any $\sigma' \in \mathcal{S}_r$ we associate the process

$$X'_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma'_s \, dW_s, \tag{3}$$

with the same a and the same W than in (1).

A measure of the "distance" between the two processes X and X' of (1) and (3), measured on the time interval $[0,t]$, is the random variable $\Delta(X,X')_t$ defined below: in the next formula H ranges through all predictable d-dimensional process with $\|H_t(\omega)\| \leq 1$ for all (ω,t) and $H^\star$ is the transpose; $\langle M \rangle$ is the quadratic variation process of the semimartigale M and $\bullet$ denotes stochastic integration:

$$\Delta(X,X')_t = \sup_{H \,:\|H\| \leq 1} \langle H^\star \bullet (X - X') \rangle_t. \tag{4}$$

This measurement of the discrepancy between X and X' is particularly well suited to finance, where $\mathbf{E}(\Delta(X,X')_t)$ is a measure of the difference in the $\mathcal{L}^2$ sense between the portfolio evaluations when one takes the model with X or the model with X'. Then set

$$\Delta(r;X)_t = \inf(\Delta(X,X')_t : \ X' \text{ is given by (3), with } \ \sigma' \in \mathcal{S}_r) \tag{5}$$

for the "distance" from X to the set of semimartingales with Brownian dimension not more than r, again on the time interval $[0,t]$.

Remark 1 Of course $\Delta(X,X')_t$ is not a genuine distance, for two reasons: it does not satisfy the triangle inequality (it is rather the square of a distance), and more important it is random. The genuine distance (which is one of Emery's distances, see [5]) is $\sqrt{\mathbf{E}(\Delta(X,X')_t)}$, provided we identify two processes which are a.s. equal, as usual.

Also, note that the two approaches, here where W is kept fixed, and in the previous section where W may be changed into another Wiener process W', look different but are actually the same. □

The next proposition shows how to compute "explicitly" $\Delta(r;X)_t$. We denote by $\lambda(1)_s \geq \lambda(2)_s \geq \ldots \geq \lambda(d)_s \geq 0$ the eigenvalues of the matrix c_s, and we set

$$L(r)_t = \int_0^t \lambda(r)_s \, ds. \tag{6}$$

Proposition 2 *For any $r = 0, \ldots, d-1$ we have $\Delta(r;X)_t = L(r+1)_t$, and the infimum in (5) is attained.*

Proof. It is no restriction to suppose that $q \geq d$ (if not, we can always add independent components to W, and accordingly components to σ which are 0). Let J be the $d \times q$ matrix with (i,i) entry equal to 1 when $1 \leq i \leq d$, and all other entries equal to 0. Then we can find two càdlàg adapted processes Π_s and Q_s, with values in the sets of $d \times d$ and $q \times q$ orthogonal matrices respectively, and such that $\sigma_s = \Pi_s \Lambda_s^{1/2} J Q_s$, where Λ_s is the diagonal matrix with entries $\lambda(i)_s$. Note that $c_s = \Pi_s \Lambda_s \Pi_s^\star$.

Let I_r (resp. I'_r) be the $d \times d$ matrix with (i,i) entry equal to 1 when $1 \leq i \leq r$ (resp. $r+1 \leq i \leq d$), and all other entries equal to 0. Then we set $\sigma'_s = \Pi_s \Lambda_s^{1/2} I_r J Q_s$ and we associate X' by (3). Then $\sigma_s - \sigma'_s = \Pi_s \Lambda_s^{1/2} I'_r J Q_s$, and thus

$$\langle H^\star \bullet (X - X') \rangle_t = \int_0^t (H_s^\star \Pi_s \Lambda_s^{1/2} I'_r J Q_s Q_s^\star J^\star I'_r \Lambda_s^{1/2} \Pi_s^\star H_s) ds.$$

The integrand above is simply $H_s^\star \Pi_s \Lambda_s^{1/2} I'_r \Lambda_s^{1/2} \Pi_s^\star H_s$ and, if $\|H_s\| = 1$, we also have $\|\Pi_s^\star H_s\| = 1$ and thus this integrand is not bigger than $\lambda(r+1)_s$: therefore $\Delta(X,X')_t \leq L(r+1)_t$. Furthermore $\sigma'_s \sigma'^\star_s = \Pi_s \Lambda_s I_r \Pi_s^\star$ is of rank $\leq r$, so $\sigma' \in \mathcal{S}_r$.

Now, let σ' be any process in $\mathcal{S}_r$ and put $c'_s = \sigma'_s \sigma'^\star_s$. The kernel K'_s of the linear map on $\mathbf{R}^d$ associated with the matrix c'_s is of dimension at least $d - r$. The subspace K_s of $\mathbf{R}^d$ generated by all eigenvectors of this linear map, which are associated with the eigenvalues $\lambda(1)_s, \ldots, \lambda(r+1)_s$, is of dimension at least $r+1$ (it is strictly bigger than $r+1$ if $\lambda(r+1)_s = \lambda(r+2)_s$). Then $K_s \cap K'_s$ is not reduced to $\{0\}$, and we thus can find a process $H = (H_s)_{s \geq 0}$ with $\|H_s\| = 1$ and $H_s \in K_s \cap K'_s$ identically, and obviously this process can be chosen to be progressively measurable. Then $c'_s H_s = 0$ (because $H_s \in K'_s$) and $H_s^\star c_s H_s \geq \lambda(r+1)_s \|H_s\| = \lambda(r+1)_s$ (because $H_s \in K_s$). The first property above yields

$$H^\star \bullet X'_t = \int_0^t H_s^\star a_s ds,$$

hence

$$\langle H^\star \bullet (X - X') \rangle_t = \langle H^\star \bullet X \rangle_t = \int_0^t (H_s^\star c_s^\star H_s) ds,$$

which by the second property above is not less than $L(r+1)_t$: hence we are done. $\square$

In particular, $L(1)_t = \Delta(0;X)_t$ measures the "distance" between the process X and the pure drift process $X_0 + \int_0^t a_s ds$. The following is obvious, with the convention $L(d+1)_t = 0$:

$$R_t \leq r \quad \Longleftrightarrow \quad L(r+1)_t = \Delta(r;X)_t = 0, \tag{7}$$

where, similar to (2), we have set for all $t > 0$:

$$R(\omega)_t = \sup_{s\in[0,t]} \operatorname{rank}(c_s(\omega)). \tag{8}$$

Hence the random value $L(r+1)_t$ measures the distance between X and the set of all processes with Brownian dimension r, over the time interval $[0,t]$, and *for our particular path* of X. Note also that it is an "absolute" measure of the distance, which is multiplied by u^2 if we multiply the process X by u.

Unfortunately, the variables $L(r)_t$ do not seem to be easy to "estimate" from discrete observations since they involve eigenvalues. Hence we will construct below estimators which are easier to handle.

3 Linear algebra preliminaries

Consider for a moment the toy model $X = \sigma W$, where σ is a non-random $d \times q$ matrix. That is, we have (1) with $X_0 = 0$ and $a_s = 0$ and $\sigma_s(\omega) = \sigma$, or equivalently X is a Wiener process with covariance Σt at time t, where $\Sigma = \sigma\sigma^\star$. The observation scheme amounts to observing n i.i.d. random vectors G_i, all of them $\mathcal{N}(0,\Sigma)$–distributed (namely $G_i = \sqrt{n/T}\ \Delta_i^n X$, with the notation $\Delta_i^n X = X_{iT/n} - X_{(i-1)T/n}$). To infer $\operatorname{rank}(\Sigma)$ from the observations of the n first variables G_k, we can use the empirical covariance

$$\widehat{\Sigma}_n = \frac{1}{n}\sum_{k=1}^n G_k G_k^\star. \tag{9}$$

Indeed, the variables G_k have a density over their support, which is a linear subspace with dimension $\operatorname{rank}(\Sigma)$: hence $\operatorname{rank}(\widehat{\Sigma}_n)$ is almost surely equal to n when $n < \operatorname{rank}(\Sigma)$, and to $\operatorname{rank}(\Sigma)$ otherwise, and the problem is solved in a trivial way.

If we are in the same setting, except that c_s depends on s (and is still deterministic) and has a constant rank r, then typically the eigenspaces "rotate" when s varies, and the rank of $\widehat{\Sigma}_n$ above is a.s. equal to d as soon as $n \geq d$. Therefore $\widehat{\Sigma}_n$ gives no insight on the rank. So the problem for nonhomogeneous Wiener process, and a fortiori for general diffusions like (1), is actually more complex.

Despite the uselessness of the toy model consisting of an homogeneous Wiener process, let us give a couple of formulas about it, for further reference. We denote by $\mathcal{A}_r$ the family of all subsets of $\{1,\ldots,d\}$ with r elements ($r = 1,\ldots,d$). If $K \in \mathcal{A}_r$ and $\Sigma = (\Sigma^{ij}) \in \mathcal{M}_d$ we denote by $\det_K(\Sigma)$ the determinant of the $r \times r$ sub–matrix $(\Sigma^{kl} : k,l \in K)$, and we set

$$\det(r;\Sigma) = \sum_{K\in\mathcal{A}_r} \det_K(\Sigma). \tag{10}$$

Observe that $\det(d;\Sigma) = \det(\Sigma)$, while $\det(1;\Sigma)$ is the trace of Σ.

Lemma 3 *If $\Sigma \in \mathcal{M}_d$ has eigenvalues $\lambda(1) \geq \ldots \lambda(d) \geq 0$, we have for $r = 1, \ldots, d$:*

$$\frac{1}{d(d-1)\ldots(d-r+1)}\ \det(r;\Sigma)\ \leq\ \lambda(1)\lambda(2)\ldots\lambda(r)\ \leq\ \det(r;\Sigma). \tag{11}$$

Notice that both inequalities in (11) may be equalities. It follows from (11) that, with the convention $0/0 = 0$, we have

$$1 \leq r \leq d \quad \Longrightarrow \quad \begin{cases} r \leq \mathrm{rank}(\Sigma) & \Longrightarrow \quad \det(r;\Sigma) > 0 \\ r > \mathrm{rank}(\Sigma) & \Longrightarrow \quad \det(r;\Sigma) = 0, \end{cases} \tag{12}$$

$$2 \leq r \leq d \quad \Longrightarrow \quad \frac{r!}{d!}\ \frac{\det(r;\Sigma)}{\det(r-1;\Sigma)} \leq \lambda(r) \leq \frac{d!}{(r-1)!}\ \frac{\det(r;\Sigma)}{\det(r-1;\Sigma)}. \tag{13}$$

Proof. We expand the characteristic polynomial of Σ as

$$\det(\Sigma - \lambda I) = (-\lambda)^d + \sum_{r=1}^{d} (-\lambda)^{d-r} \det(r;\Sigma).$$

In view of the well known expressions for the "symmetrical functions" of the roots of a polynomial, we get

$$\sum_{1 \leq i_1 < \ldots < i_r \leq d} \lambda(i_1)\lambda(i_2)\ldots\lambda(i_r) = \det(r;\Sigma),$$

and thus both sides of (11) are obvious. □

Next, we consider a sequence $(G_i)_{i\geq 1}$ of i.i.d. $\mathcal{N}(0,\Sigma)$-distributed random vectors. For all $j = 1, \ldots, d$ we define two random elements of $\mathcal{M}_d$ by

$$\zeta_j = \sum_{i=1}^{j} G_i G_i^\star, \qquad \zeta'_j = \sum_{i=d+1}^{d+j} G_i G_i^\star, \tag{14}$$

and we consider the mean and covariance of the random vector $(\det(r;\zeta_r)/r! : 1 \leq r \leq d)$:

$$\left.\begin{array}{l} \gamma(r;\Sigma) = \frac{1}{r!}\ \mathbf{E}(\det(r;\zeta_r)), \\ \Gamma(r,r';\Sigma) = \frac{1}{r!r'!}\ \mathbf{E}(\det(r;\zeta_r)\ \det(r';\zeta_{r'})) - \gamma(r;\Sigma)\gamma(r';\Sigma). \end{array}\right\} \tag{15}$$

Since $(\zeta_j, \zeta'_j : 1 \leq j \leq d)$ are i.i.d. we also have

$$\Gamma(r,r';\Sigma) = \frac{1}{r!r'!}\ \mathbf{E}\Big(\det(r;\zeta_r)\ \det(r';\zeta_{r'}) - \det(r;\zeta_r)\ \det(r';\zeta'_{r'})\Big). \tag{16}$$

Lemma 4 *If $r \in \{1, \dots, d\}$, we have*

$$\gamma(r;\Sigma) = \det(r;\Sigma). \tag{17}$$

Moreover

$$\left.\begin{array}{lll} r \le \operatorname{rank}(\Sigma) & \Rightarrow & \gamma(r;\Sigma) > 0, \quad \Gamma(r,r;\Sigma) > 0 \\ r > \operatorname{rank}(\Sigma) & \Rightarrow & \gamma(r;\Sigma) = 0, \quad \Gamma(r,r;\Sigma) = 0. \end{array}\right\} \tag{18}$$

Proof. For proving (17) it is enough to show that for any $K \in \mathcal{A}_r$, we have $\mathbf{E}(\det_K(\zeta_r)) = r!\, \det_K(\Sigma)$, and for this it is no restriction to assume that $K = \{1, \dots, r\}$. We denote by $\mathcal{P}_r$ the set of all permutations of the set $\{1, \dots, r\}$, and $\varepsilon(\tau)$ is the signature of the permutation τ. Then

$$\det_K(\zeta_r) = \sum_{\tau \in \mathcal{P}_r} (-1)^{\varepsilon(\tau)} \prod_{l=1}^{r} \zeta_r^{l\tau(l)} = \sum_{1 \le k_1, \dots, k_r \le r} \; \sum_{\tau \in \mathcal{P}_r} (-1)^{\varepsilon(\tau)} \;\; \prod_{l=1}^{r} G_{k_l}^{l}\, G_{k_l}^{\tau(l)},$$

and each summand of the first sum of the extreme right side is the determinant of a matrix with rank less than r, unless all k_l are distinct. So it is enough to sum over all r–uples $(k_1, \dots, k_r)$ with distinct entries between 1 and r, that is for r–uples with $k_i = \tau'(i)$ for some $\tau' \in \mathcal{P}_r$. In other words, we have

$$\det_K(\zeta_r) = \sum_{\tau' \in \mathcal{P}_r} \; \sum_{\tau \in \mathcal{P}_r} (-1)^{\varepsilon(\tau)} \;\; \prod_{l=1}^{r} G_{\tau'(l)}^{l}\, G_{\tau'(l)}^{\tau(l)}. \tag{19}$$

Since the variables G_n are independent and $\mathbf{E}(G_n^k G_n^l) = \Sigma^{kl}$, we deduce that

$$\mathbf{E}(\det_K(\zeta_r)) = r! \;\; \sum_{\tau \in \mathcal{P}_r} (-1)^{\varepsilon(\tau)} \;\; \prod_{l=1}^{r} \Sigma^{l\tau(l)} = r!\, \det_K(\Sigma),$$

and we have (17).

If $r > \operatorname{rank}(\Sigma)$ we have $\det(r;\Sigma) = 0$ (see (12)): the nonnegative variable $\det(r;\zeta_r)$ has zero expectation, so it is a.s. null, and we have the second part of (18). Finally let $r \le \operatorname{rank}(\Sigma)$. By (12) again we have $\mathbf{E}(\det(r;\zeta_r)) > 0$. Also, observe that $\det(r;\zeta_r)$ is a continuous function of the random vectors G_n for $n = 1 \dots, r$, which vanishes if all these G_n are 0. Thus $\det(r;\zeta_r)$ can take arbitrarily small values with a positive probability, and it has a positive expectation, so it is not degenerate and we get the first part of (18). □

4 Limit theorems for estimators of the Brownian dimension

It turns out that determinants or "integrated determinants", are much easier to estimate than eigenvalues or integrated eigenvalues. So in view of (11) and (12) one might replace the variable $L(r)_t$ of (6) by

$$L'(r)_t = \begin{cases} \int_0^t \det(1; c_s)\, ds & \text{if } \; r = 1 \\ \int_0^t \frac{\det(r;c_s)}{\det(r-1;c_s)}\, ds & \text{if } \; r \ge 2. \end{cases}$$

However, $L'(r)_t$ for $r \geq 2$ is still not so easy to estimate: for example for the toy model of Section 3 the variable $\det(r;\zeta_r)/r!$ is an unbiased estimator of $\det(r;\Sigma)$ (see (17)), but we have no explicit unbiased estimator for a quotient like $\det(r;\Sigma)/\det(r-1;\Sigma)$.

So we propose to measure *the distance between X and the set of models with multiplicity r*, over the time interval $[0,t]$, by the following random variable:

$$\overline{L}(r)_t = \int_0^t \det(r;c_s)\ ds. \tag{20}$$

Up to multiplicative constants, this more or less amounts to replace the "natural" distance $L(r)_t$ by $\int_0^t \lambda(1)_s \dots \lambda(r)_s\ ds$. The variables $L(r)_t$, $L'(r)_t$ and $\overline{L}(r)_t$ convey essentially the same information as far as rank is concerned, and in particular they vanish simultaneously, which is the most important property for our purposes. In other words, exactly as in (7) we have

$$R_t \leq r \quad \Longleftrightarrow \quad \overline{L}(r+1)_t = 0, \tag{21}$$

By virtue of (17) we can rewrite $\overline{L}(r)$ as follows (we also introduce additional variables $Z(r,r')$, using the notation (15)):

$$\overline{L}(r)_t = \int_0^t \gamma(r;c_s)\ ds, \qquad Z(r,r')_t = \int_0^t \Gamma(r,r';c_s)\ ds. \tag{22}$$

Now, we need to approximate the variables in (22) by variables which depend on our discrete observations only. To this end we introduce the random matrices

$$\zeta(r)_i^n = \sum_{j=1}^{r} (\Delta_{i+j-1}^n X)\ (\Delta_{i+j-1}^n X)^\star, \quad \text{where } \ \Delta_i^n X = X_{iT/n} - X_{(i-1)T/n}. \tag{23}$$

We have $\zeta(r)_i^n \in \mathcal{M}_d$ and $\mathrm{rank}(\zeta(r)_i^n) \leq r$. Then we set (with $[x]$ being the integer part of x):

$$\overline{L}(r)_t^n = \frac{n^{r-1}}{T^{r-1}\ r!} \sum_{i=1}^{[nt/T]-r+1} \det(r;\zeta(r)_i^n), \tag{24}$$

and

$$\begin{aligned} Z(r,r')_t^n \ = \ & \frac{n^{r+r'-1}}{T^{r+r'-1}\ r!\ r'!} \sum_{i=1}^{[nt/T]-d-r'+1} \Big(\det(r;\zeta(r)_i^n)\ \det(r';\zeta(r')_i^n) \\ & \qquad -\det(r;\zeta(r)_i^n)\ \det(r';\zeta(r')_{d+i}^n)\Big). \end{aligned} \tag{25}$$

The first key theorem is the "consistency" of these variables:

Theorem 5 *Under (H) the variables $\overline{L}(r)_t^n$ and $Z(r,r')_t^n$ converge in probability to $\overline{L}(r)_t$ and $Z(r,r')_t$ respectively, uniformly in $t \in [0,T]$.*

This is not enough for our purposes, and we need rates of convergence. For this (H) is not sufficient, and some additional regularity on the coefficients a and σ is necessary. A first set of sufficient conditions is simple enough:

Hypothesis (H1): We have (H) with a càdlàg process a, and a process σ which is Hölder continuous (in time) with index $\rho > 1/2$, in the sense that

$$\sup_{0\leq s<t\leq T} \frac{\|\sigma_t - \sigma_s\|}{(t-s)^\rho} < \infty \quad \text{a.s.} \tag{26}$$

The assumption on a above is quite mild, and the assumption on σ is reasonable when σ is deterministic. However, in the pure diffusion case we have $\sigma_s = \sigma(X_s)$ for, say, a Lipschitz or locally Lipschitz function σ, and of course (26) fails for any $\rho \geq 1/2$. This assumption also fails when σ_s is a "stochastic volatility" driven by an Itô equation, and even more if this equation has jumps !

Therefore, for practical purposes which are especially relevant in finance, we need to replace (H1) by a different assumption. This assumption looks (is ?) complicated to state, but it essentially says that a is as in (H1), and that the process σ follows a jump-diffusion Itô equation, or in other words that it is driven by a Wiener process and a Poisson random measure); in particular it is satisfied in the pure diffusion case when $\sigma_s = \sigma(X_s)$ with a C^2 function σ.

Hypothesis (H2): We have (H), the process a is càdlàg, and the process σ is a (possibly discontinuous) Itô semimartingale on $[0,T]$, that is for $t \leq T$ we have

$$\begin{aligned}\sigma_t &= \sigma_0 + \int_0^t a'_s ds + \int_0^t \sigma'_{s-} dW_s \\ &\quad + \int_0^t\int_E \varphi\circ w(s-,x)(\mu-\nu)(ds,dx) + \int_0^t\int_E (w-\varphi\circ w)(s-,x)\mu(ds,dx). \end{aligned}\tag{27}$$

Here σ' is $\mathbf{R}^d \otimes \mathbf{R}^q \otimes \mathbf{R}^q$-valued adapted càdlàg, and a' is $\mathbf{R}^d \otimes \mathbf{R}^q$–valued predictable and locally bounded; μ is a Poisson random measure on $(0,\infty)\times E$ independent of W and V, with intensity measure $\nu(dt,dx) = dtF(dx)$ with F a σ–finite measure on some Polish space $(E,\mathcal{E})$; φ is a continuous function on $\mathbf{R}^{dq}$ with compact support, which coincides with the identity on a neighborhood of 0); finally $w(\omega,s,x)$ is a map $\Omega\times[0,\infty)\times E \to \mathbf{R}^d\otimes\mathbf{R}^q$ which is $\mathcal{F}_s\otimes\mathcal{E}$–measurable in (ω,x) for all s and càdlàg in s, and such that $\int_E (1 \bigwedge \sup_{\omega\in\Omega, s\leq T} \|w(\omega,s,x)\|^2)\ F(dx) < \infty$.

These conditions are indeed quite easy to check in practice. They accommodate the case of a stochastic volatility driven by a Wiener process having some (or all) components independent of X: since W has an "arbitrary" dimension q in this paper, possibly $q > d$, there might be components used for X in (1) and other components used in (27).

Theorem 6 *Assume either (H1) or (H2). The d–dimensional processes $(V(r)^n_t)_{1\leq r\leq d}$ with components*

$$V(r)^n_t = \sqrt{n}\ (\overline{L}(r)^n_t - \overline{L}(r)_t) \tag{28}$$

converge stably in law to a limiting process $(V(r)_t)_{1\leq r\leq d}$, which is defined on an extension of the original space and which, conditionally on $\mathcal{F}$, is a non-homogeneous Wiener process with quadratic variation process $t \mapsto (TZ(r,r')_t)_{1\leq r,r'\leq d}$.

Proof of Theorems 5 and 6. The proof goes through several steps:

1) It is based on the following two results of [4]. Take N functions g_j on $\mathbf{R}^d$, which are C^2 and with polynomial growth and even. Set

$$Y(g_1,\dots,g_N)^n_t = \frac{T}{n}\sum_{i=1}^{[nt/T]-N+1}\prod_{k=1}^{N} g_k(\sqrt{n/T}\ \Delta^n_{i+k-1}X).$$

Then under (H) we have

$$Y(g_1,\dots,g_N)^n_t \ \stackrel{\mathbf{P}}{\longrightarrow}\ Y(g_1,\dots,g_N)_t := \int_0^t y(g_1,\dots,g_N;c_s)\ ds,$$

where the convergence in uniform in $t\in[0,T]$ and where $y(g_1,\dots,g_N;\Sigma)$ is, for any $d\times d$ covariance matrix Σ, the expectation of the variable

$$\gamma(g_1,\dots,g_N) = \prod_{k=1}^{N} g_k(G_k)$$

and the G_n's are i.i.d. random vectors with law $\mathcal{N}(0,\Sigma)$, as in Section 3.

If further (H2) holds, then for any array $\Big((g^j_1,\dots,g^j_{N_j}) : 1\le j\le J\Big)$ with g^j_i as above, the J–dimensional processes $\Big(\sqrt{n/T}\ (Y(g^j_1,\dots,g^j_{N_j})^n_t - Y(g^j_1,\dots,g^j_{N_j})_t\Big)_{1\le j\le J}$ converge stably in law to a limiting process which, conditionally on $\mathcal{F}$, is a non-homogeneous Wiener process with quadratic variation process $\int_0^t\Gamma(c_s)ds$, and where $\Gamma(\Sigma)$ is the covariance matrix of the random vector $\Big(\gamma(g^j_1,\dots,g^j_{N_j})\Big)_{1\le j\le J}$ as defined above. Under (H1) instead of (H2) the same result holds: it is not explicitely stated in [4], but the proof is similar and technically much simpler.

2) These results extend by "linearity" in an obvious way. More precisely, for $1\le j\le J$ set

$$Y(j)^n_t = \frac{T}{n}\sum_{i=1}^{[nt/T]-N_j+1} h_j\left(\sqrt{n/T}\ \Delta^n_i X, \sqrt{n/T}\ \Delta^n_{i+1}X,\dots,\sqrt{n/T}\ \Delta^n_{i+N_j-1}X\right), \quad (29)$$

where each h_j is a linear combinations of tensor products $g_1\otimes\ldots\otimes g_{N_j}$, where the g_i's are C^2 functions on $\mathbf{R}^d$, even and with polynomial growth. Let also denote by $M(\Sigma)$ and $C(\Sigma)$ the mean vector and the covariance matrix of the J–dimensional random vector $(h_j(G_1,\dots,G_{N_j})_{1\le j\le J}$, with G_i as above. Then:

1. under (H) we have

$$Y(j)^n_t \ \stackrel{\mathbf{P}}{\longrightarrow}\ Y(j)_t := \int_0^t M^j(c_s)ds, \quad \text{uniformly in } \ t\in[0,T]; \quad (30)$$

2. under (H1) or (H2) the J-dimensional processes with components $\sqrt{n/T}\ (Y(j)^n_t - Y(j)_t)$ converge stably in law to a limiting process which, conditionally on $\mathcal{F}$, is a non-homogeneous Wiener process with quadratic variation process $\int_0^t C(c_s)\ ds$.

3) The theorem is now almost trivial. The determinants entering (24) and (25) are sums of even monomials of the components of $\Delta^n_{i+j-1}X$ for $1 \le j \le 2d$, each one with degree $2r$, resp. $2(r+r')$. More specifically, $\overline{L}(r)^n$ is of type (29), with $N_r = r$ and the function

$$h_r(x_1, \cdots, x_r) = \frac{1}{r!}\,\det\Big(r, \sum_{j=1}^{r} x_j x_j^\star\Big),$$

whereas $Z(r,r')^n$ is of type (29), with $N_{r,r'} = d + r'$ and the function

$$\begin{aligned} h_{r,r'}(x_1, \cdots, x_{d+r'}) \;=\; & \frac{1}{r!\;r'!}\;\Bigg(\det\Big(r, \sum_{j=1}^{r} x_j x_j^\star\Big)\;\det\Big(r', \sum_{j=1}^{r'} x_j x_j^\star\Big) \\ & \qquad - \det\Big(r, \sum_{j=1}^{r} x_j x_j^\star\Big)\;\det\Big(r', \sum_{j=d+1}^{d+r'} x_j x_j^\star\Big)\Bigg). \end{aligned}$$

Then Theorem 5 readily follows from Step 2 and from the relations (15). □

In the sequel we will need also some estimates on the moments of $V(r)^n_t$, uniform in n. It follows from the proofs in [4] that, under (H2) and for each $t \in (0,T]$, there is a sequence $A_{p,t}$ of $\mathcal{F}_t$–measurable sets such that

$$\left.\begin{array}{l} A_{p,t} \uparrow \Omega \;\text{ as }\; p \to \infty \\ p \ge 1, \;\; n \ge 1, \;\; t \in (0,T] \quad \Longrightarrow \quad \mathbf{E}(|V(r)^n_t|^2 1_{A_{p,t}}) \le C_p, \end{array}\right\} \tag{31}$$

for a suitable sequence of constants C_p (depending on T). This very same result also holds under (H1).

Now, if the coefficient a is bounded, and if we have (H1) with (26) holding uniformly in ω, or (H2) with $a'_s(\omega)$ and $\sigma'_s - \omega$) bounded and $\|w(\omega,s,x)\| \le h(x)$ for some function having $\int_E (1 \bigwedge h(x)^2) F(dx) < \infty$, one can (easily) prove that (31) holds for $A_{1,t} = \Omega$, and so there is a constant C, depending on T, such that

$$n \ge 1, \;\; t \in (0,T] \quad \Longrightarrow \quad \mathbf{E}(|V(r)^n_t|^2) \le C. \tag{32}$$

5 Tests based on thresholds

5.1 A test based on an absolute threshold

We come back to the initial problem, in the light of the second comment of Section 1: namely, we want to decide which integer value (between 0 and d) the variable R of (2) takes for the particular path ω which is known only through the observations $X_{iT/n}$. In principle we have our observations $X_{iT/n}$ for $i = 0, \ldots, n$, but it may be interesting to determine how our estimators behave as time changes; this is why we also give estimators for the variable R_t of (8), based on the observation of $X_{iT/n}$ for $i = 0, \ldots, [nt/T]$.

Let us say once more that in the ideal scheme (the whole path of X is known over $[0,T]$) we also know $R = R(\omega)$, whereas we have the equivalence (21). In view of this, and

taking into account the convergence result in Theorem 5, it seems natural to operate as follows: we choose a sequence of positive numbers ρ_n such that

$$\rho_n \to 0, \qquad \rho_n \sqrt{n} \to \infty. \tag{33}$$

Then we take the following "estimator" for R_t:

$$\widehat{R}_{n,t} = \inf \left(r \in \{0, \ldots, d-1\} : \ \overline{L}(r+1)_t^n < \rho_n t \right), \tag{34}$$

with $\inf(\emptyset) = d$. That this estimator is *a priori* reasonable comes from the fact that if we set $R_{n,t} = \inf \left(r \in \{0, \ldots, d-1\} : \ \overline{L}(r+1)_t < \rho_n t \right)$, then by (21) and the property $\rho_n \to 0$, we have $\mathbf{P}(R_{n,t} = R_t) \to 1$ as $n \to \infty$. We take a threshold of the form $\rho_n t$ because $\overline{L}(r+1)_t^n$ is roughly proportional to t.

Remark 7 Another equally reasonable estimator, which is a kind of "dual" of $\widehat{R}_{n,t}$, is the following one:

$$\widehat{R}'_{n,t} = \sup \left(r \in \{1, \ldots, d\} : \ \overline{L}(r)_t^n \geq \rho_n t \right), \tag{35}$$

with $\sup(\emptyset) = 0$. The analysis of $\widehat{R}_{n,t}$ made below carries over for $\widehat{R}'_{n,t}$ in pretty much the same way. □

Remark 8 The choice of the threshold ρ_n is arbitrary, upon the fact that (33) holds: asymptotically all choices are equivalent. In practice, though, it is of primary importance because n, albeit large, is given and is of course not infinite ! Even worse, an absolute threshold like in (34) is sensitive to the unit in which the values of the X_t^i are expressed: for example if we multiply all components by the same (known) constant the estimator of the Brownian dimension provides a different value. So using an absolute threshold is probably *not* advisable in general. Nevertheless we pursue here the analysis of tests based on an absolute threshold, since they may serve as a case study and are somewhat simpler to study than the tests based on relative thresholds which are introduced later. □

The integer–valued estimator $\widehat{R}_{n,t}$ should be analyzed using the testing methodology rather than as a usual estimator: we test the hypothesis $R_t = r$ with the critical region $\{\widehat{R}_{n,t} \neq r\}$. The "power function" is in principle the probability of rejection, a function of the underlying probability measure. Here we have a single $\mathbf{P}$, and R_t is (possibly) random. We thus develop two different substitutes to the power function.

5.2 A first substitute to the power function

A seemingly acceptable version of the power function is

$$\widehat{\beta}_{n,t}^r(r') = \mathbf{P}(\widehat{R}_{n,t} \neq r \mid R_t = r'), \qquad r' = 0, 1, \ldots, d, \tag{36}$$

provided $\mathbf{P}(R_t = r') > 0$. We explicitly mention the number n of observations and the number r, but it also depends on the sequence ρ_n. The index r indicates the "test" with null hypothesis $R_t = r$ which we are performing, while the index r' indicates the "true" value or R_t: so $\widehat{\beta}_{n,t}^r(r)$ should be small, and $\widehat{\beta}_{n,t}^r(r')$ should be close to 1 when $r' \neq r$.

We have a first – quite simple – result:

Theorem 9 *Under (33) and either (H1) or (H2) we have for all r, r' in $\{1, \dots, d\}$, and provided $\mathbf{P}(R_t = r') > 0$:*

$$\widehat{\beta}^r_{n,t}(r') \;\longrightarrow\; \begin{cases} 1 & \text{if } \; r \neq r' \\ 0 & \text{if } \; r = r'. \end{cases} \tag{37}$$

Another equivalent (simpler) way of stating this result consists in writing

$$\mathbf{P}(\widehat{R}_{n,t} \neq R_t) \;\to\; 0. \tag{38}$$

This is more intuitive, but somehow farther away from the way results on tests are usually stated.

Proof. For each $s = 1, \dots, (r'+1) \wedge d$ we set $\delta^s_{n,t}(r') = \mathbf{P}(\overline{L}(s)^n_t < \rho_n, R_t = r')$. Observe that, if $\rho'_n = \rho_n \sqrt{n}$, and with the notation (28),

$$\left.\begin{aligned} &\delta^s_{n,t}(r') \leq \mathbf{P}(\overline{L}(s)_t < 2\rho_n t, R_t = r') + \mathbf{P}(|V(s)^n_t| > \rho'_n t), \\ &\mathbf{P}(R_t = r') - \delta^s_{n,t}(r') \leq \mathbf{P}(\overline{L}(s)_t > \rho_n t/2, R_t = r') + \mathbf{P}(|V(s)^n_t| \geq \rho'_n t/2). \end{aligned}\right\} \tag{39}$$

Theorem 6 yields that the sequence $V(s)^n_t$ converges in law when n goes to infinity, whereas on the set $\{R_t = r'\}$ we have $\overline{L}(s)_t > 0$ if $s \leq r'$, and $\overline{L}(s)_t = 0$ if $s > r'$. Therefore it follows from (33) that

$$\left.\begin{aligned} s \leq r' \quad &\Rightarrow \quad \delta^s_{n,t}(r') \to 0 \\ s = r'+1 \leq d \quad &\Rightarrow \quad \delta^s_{n,t}(r') \to \mathbf{P}(R_t = r'). \end{aligned}\right\} \tag{40}$$

Now $\{\widehat{R}_{n,t} \neq r'\} = (\cup_{1 \leq s \leq r'} \{\overline{L}(s)^n_t < \rho_n t\}) \cup \{\overline{L}(r'+1)^n_t \geq \rho_n t\}$, with the convention $\{\overline{L}(d+1)^n_t \geq \rho_n t\} = \emptyset$. Then

$$\mathbf{P}(\widehat{R}_{n,t} \neq r' = R_t) \leq \begin{cases} \sum_{s=1}^{r'} \delta^s_{n,t}(r') + \mathbf{P}(R_t = r') - \delta^{r'+1}_{n,t}(r') & \text{if } \; r' \leq d-1 \\ \sum_{s=1}^{r'} \delta^s_{n,t}(r') & \text{if } \; r' = d. \end{cases}$$

Then it readily follows from (40) that $\widehat{\beta}^{r'}_{n,t}(r') \to 0$ as soon as $\mathbf{P}(R_t = r') > 0$. Under this assumption and if $r \neq r'$ we clearly have $\widehat{\beta}^r_{n,t}(r') = 1 - \mathbf{P}(\widehat{R}_{n,t} = r \mid R_t = r') \geq 1 - \widehat{\beta}^{r'}_{n,t}(r)$, hence $\widehat{\beta}^r_{n,t}(r') \to 1$. □

The previous result seems to settle the matter. However, it is not as nice as it may look, because it gives no "rate" for the convergence in (37) or (38) and is thus impossible to put in use in practice. The impossibility of getting a rate is apparent in (39): the second terms on the right may be more or less controlled through estimates like (31), but the first terms on the right cannot be controlled at all; indeed if $R_t = r$ the variable $\overline{L}(r)_t$ is positive, but may be arbitrarily close to 0.

5.3 A second substitute to the power function

As emphasized in Comment 4 of Section 1, one may reasonably decide that $R_t = r$ if r is the "true" Brownian dimension in a "significant" way, which means in particular that the "distance" between the model X and the set of models with Brownian dimension $r' < r$ is not "infinitesimal". This may be interpreted as the property that $\overline{L}(r)_t$ exceeds some positive level for all $r \leq R_t$.

In other words, we set $B_{r',\varepsilon,t} = \{R_t = r', \overline{L}(r)_t \geq \varepsilon t \text{ for } r = 1, \ldots, r'\}$, and we define the "power function" as being

$$\widehat{\beta}^r_{n,t}(r', \varepsilon) = \mathbf{P}(\widehat{R}_{n,t} \neq r \mid B_{r',\varepsilon,t}), \tag{41}$$

provided $\mathbf{P}(B_{r',\varepsilon,t}) > 0$.

Evaluating $\beta^r_{n,t}(r', \varepsilon)$ is still difficult, because it involves the unknown quantity $\mathbf{P}(B_{r',\varepsilon,t})$. So we provide a result which does not directly give the power function itself, but which is probably more relevant for applications.

Theorem 10 *Under (H1) or (H2) there are $\mathcal{F}_t$-measurable sets $(A_{p,t})$ increasing to Ω as $p \to \infty$, and constants C_p such that, for all r in $\{1, \ldots, d\}$, and provided $\rho_n < \varepsilon/2$:*

$$\mathbf{P}(\{\widehat{R}_{n,t} \neq r\} \cap B_{r,\varepsilon,t} \cap A_{p,t}) \leq \frac{C_p}{n\rho_n^2} \tag{42}$$

for all $t \in (0, T]$. If further (32) holds, we can find a constant such that

$$\mathbf{P}(\{\widehat{R}_{n,t} \neq r\} \cap B_{r,\varepsilon,t}) \leq \frac{C}{n\rho_n^2}, \tag{43}$$

or in other words, the "level" satisfies $\widehat{\beta}^r_{n,t}(r, \varepsilon) \leq C/\Big(n\rho_n^2 \mathbf{P}(B_{r,\varepsilon,t})\Big)$.

Note that we can choose ρ_n above at will, provided it satisfies (33), and neither A_p nor $B_{r',\varepsilon,t}$ nor C_p depend on this choice (the estimator $\widehat{R}_{n,t}$ does, though): so we can obtain a rate $1/n^\theta$ for any $\theta \in (0, 1)$, as close to 1 as one wishes.

Proof. We consider the sets $A_{p,t}$ for which (31) holds, and we denote by C'_p the constants occurring in that formula. For $s = 1, \ldots, (r+1) \wedge d$ we set ($\varepsilon > 0$ being fixed) $\delta^s_{n,t}(r, p) = \mathbf{P}(\{\overline{L}(s)^n_t < \rho_n t\} \cap B_{r,\varepsilon,t} \cap A_{p,t})$. Exactly as for (39), we have

$$\delta^s_{n,t}(r, p) \leq \mathbf{P}(\{\overline{L}(s)_t < 2\rho_n t\} \cap B_{r,\varepsilon,t})) + \mathbf{P}(\{|V(s)^n_t| > \rho'_n t\} \cap A_{p,t}),$$

$$\mathbf{P}(B_{r,\varepsilon,t} \cap A_{p,t}) - \delta'^s_{n,t}(r, p) \leq \mathbf{P}(\{\overline{L}(s)_t > \rho_n t/2\} \cap B_{r,\varepsilon,t}) + \mathbf{P}(\{|V(s)^n_t| \geq \rho'_n t/2\} \cap A_{p,t}).$$

Taking into account (31) and $\rho_n < \varepsilon/2$ and the facts that $\overline{L}(s)_t = 0$ if $R_t = r < s$ and that $\overline{L}(r)_t \geq \varepsilon t$ on $B(r, \varepsilon, t)$, we deduce from Tchebycheff inequality that

$$s \leq r \quad \Rightarrow \quad \delta^s_{n,t}(r, p) \leq \frac{C'_p}{\rho'^2_n}, \qquad \mathbf{P}(B_{r,\varepsilon,t} \cap A_{p,t}) - \delta^{r+1}_{n,t}(r, p) \leq \frac{4C'_p}{\rho'^2_n}, \tag{44}$$

where the second equality makes sense when $r < d$ only. Applying once more the identity $\{\widehat{R}_{n,t} \neq r\} = \{\overline{L}(r+1)^n_t \geq \rho_n t\} \cup (\cup_{1\leq s\leq r}\{\overline{L}(s)^n_t < \rho_n t\}$, we get

$$\mathbf{P}(\{\widehat{R}_{n,t} \neq r\} \cap B_{r,\varepsilon,t} \cap A_{p,t}) \leq \begin{cases} \sum_{s=1}^r \delta^s_{n,t}(r,p) + \mathbf{P}(B_{r,\varepsilon,t} \cap A_{p,t}) - \delta^{r+1}_{n,t}(r,p) & \text{if } \ r < d \\ \sum_{s=1}^r \delta^s_{n,t}(r,p) & \text{if } \ r = d. \end{cases}$$

Then we deduce (42) form (44) if we put $C_p = (4+d)C'_p$. Finally under (32) we may choose $A_{1,t} = \Omega$ above, and thus (43) with $C = (4+d)C'_1$. □

Of course (42) is not useful in general, although it gives us a rate, because we do not know the sets $A_{p,t}$. In case (32) holds the result appears much more satisfactory; however, we still do not know the constant C in (43), and have no mean to guess what it is from the observations.

5.4 Tests based on a relative threshold

In practice the previous tests are not recommended, see Remark 8. Now we exhibit other tests which are scale-invariant.

If we multiply X by a constant $\delta > 0$, then c_s is multiplied by δ^2 and both $\overline{L}(r)^n_t$ and $\overline{L}(r)_t$ are multiplied by δ^{2r}. Then, for any given sequence $\rho_n \in (0,1]$ satisfying (33) the following two "estimators" of R, which are candidates to be *explicative Brownian dimensions*, are scale-invariant:

$$\left.\begin{aligned} \widetilde{R}_{n,t} &= \inf\left(r \in \{0\ldots,d-1\} :\ \overline{L}(r+1)^n_t < \rho_n t^{-1/r}(\overline{L}(r)^n_t)^{(r+1)/r}\right), \\ \widetilde{R}'_{n,t} &= \inf\left(r \in \{0,\ldots,d-1\} :\ \overline{L}(r+1)^n_t < \rho_n t^{-r}(\overline{L}(1)^n_t)^{r+1}\right), \end{aligned}\right\} \tag{45}$$

with the convention that $\overline{L}(0)^n_t = 1$, and again $\inf(\emptyset) = d$. The presence of $t^{1/r}$ or t^r above accounts for the fact that $\overline{L}(r)^n_t$ is roughly proportional to t, as in (34).

Note that $\widetilde{R}'_{n,t} \geq 1$, even when $R_t = 0$: so if $R_t = 0$ this estimator is bad, but in this case our problem is essentially meaningless anyway ! When $R_t \geq 1$, the significance of these two estimators is essentially as follows: $\widetilde{R}_{n,t}$ is the smallest integer r for which there is a "large" drop between the explicative powers of the models with Brownian dimensions r and $r+1$, whereas $\widetilde{R}'_{n,t}$ is the smallest integer r at which the ratio between the contributions of the $(r+1)$th and the first Brownian dimension is smaller than ρ_n. Clearly there exist other estimators of the same kind, with slightly different meanings, the above two being the extremes. All such estimators are amenable to essentially the same mathematical analysis.

In practice the choice of ρ_n is relative to the physical phenomenon under consideration and to the use which is made of the model (prediction, simulation, computation of extreme values, etc.). Roughly speaking the choice should reflect the physical effects which are modelled as the driving noise, and the intensity of components of the noise which are considered as important to capture essential properties of the model.

Here again the substitutes to the power functions are

$$\widetilde{\beta}^r_{n,t}(r') = \mathbf{P}(\widetilde{R}_{n,t} \neq r \mid R_t = r'), \qquad \widetilde{\beta}'^r_{n,t}(r') = \mathbf{P}(\widetilde{R}'_{n,t} \neq r \mid R_t = r'). \tag{46}$$

We now aim to a result similar to Theorem 9 :

Theorem 11 *Under (33) and either (H1) or (H2) we have for all r, r' in $\{1, \ldots, d\}$, and provided $\mathbf{P}(R_t = r') > 0$:*

$$\widetilde{\beta}^r_{n,t}(r') \;\longrightarrow\; \begin{cases} 1 & \text{if } \; r \neq r' \\ 0 & \text{if } \; r = r', \end{cases} \tag{47}$$

and the same for $\widetilde{\beta}'^r_{n,t}(r')$.

Proof. We prove the result for $\widetilde{\beta}^r_{n,t}(r')$ only, the other case being similar. For each $s = 1, \ldots, r' \wedge d$ we set

$$\delta^s_{n,t}(r') = \mathbf{P}(\overline{L}(s)^n_t < \rho_n t^{-1/s} (\overline{L}(s-1)^n_t)^{s/(s-1)},\; R_t = r').$$

As in Theorem 9 we write $\rho'_n = \rho_n \sqrt{n}$, and with the convention $0/0 = 1$ and with $t \in (0, T]$ fixed we put

$$\widetilde{V}(s)_n = t^{1/s} \sqrt{n} \left(\frac{\overline{L}(s)^n_t}{(\overline{L}(s-1)^n_t)^{s/(s-1)}} - \frac{\overline{L}(s)_t}{(\overline{L}(s-1)_t)^{s/(s-1)}} \right). \tag{48}$$

Observe that, similar to (39), we have

$$\delta^s_{n,t}(r') \leq \mathbf{P}(\overline{L}(s)_t < 2\rho_n t^{-1/s} (\overline{L}(s-1)_t)^{(s-1)/s}, R_t = r') + \mathbf{P}(|\widetilde{V}(s)_n| > \rho'_n, R_t = r'),$$

$$\begin{aligned} \mathbf{P}(R_t = r') - \delta^s_{n,t}(r') \;\; &\leq \;\; \mathbf{P}(\overline{L}(s)_t > \rho_n t^{-1/s} (\overline{L}(s-1)_t)^{(s-1)/s}/2, R_t = r') \\ &\quad + \mathbf{P}(|\widetilde{V}(s)_n| \geq \rho'_n/2, R_t = r'). \end{aligned}$$

Moreover Theorem 6 yields that, on the set $\{\overline{L}(s-1)_t > 0\} = \{R_t \geq s-1\}$, the variables $\widetilde{V}(s)_n$ converge stably in law to the variable

$$\widetilde{V}(s) = \frac{t^{1/s} \overline{L}(s)_t}{(\overline{L}(s-1)_t)^{s/(s-1)}} \left(\frac{V(s)_t}{\overline{L}(s)_t} - \frac{s}{s-1} \, \frac{V(s-1)_t}{\overline{L}(s-1)_t} \right).$$

Therefore, since (33) holds, we get (40). At this stage, we can reproduce the end of the proof of Theorem 9 to obtain (47). □

Obviously the comments made after Theorem 9 for the estimators $\widehat{R}_{n,t}$ apply to $\widetilde{R}_{n,t}$ or $\widetilde{R}'_{n,t}$, and in particular the fact that (47) gives no rates. Moreover there is nothing like Theorem 10 here, because we have no moment estimates like (31) or (32) for the variables $\widetilde{V}(s)_n$ of (48) (such estimates seem to be out of reach, because of the denominators).

Remark 12 On could also think of estimators similar to (35), for example

$$\widetilde{R}''_{n,t} = \sup \left(r \in \{1 \ldots, d\} : \; \overline{L}(r)^n_t \geq \rho_n t^{-1/(r-1)} (\overline{L}(r-1)^n_t)^{r/(r-1)} \right). \tag{49}$$

However, the previous analysis does *not* carry over to $\widetilde{R}''_{n,t}$, again because we do not know whether the variables $\widetilde{V}(s)^n_t$ converge stably in law on the set $\{R_t < s-1\}$ (due once more to the presence of the denominators). □

Remark 13 Here again the problem of choosing the threshold ρ_n is crucial in practice, despite the fact that a relative threshold is - at least - insensitive to the scale. This is illustrated somehow by the numerical experiments conducted in Section 7. For real data, only the experience of the statistician, at this point, can help for the choice of ρ_n. We hope in a future work to be able to derive some tests based on the consideration of the variables $\overline{L}(r)_t^n$ again, but which do not necessitate an arbitrary threshold. The next section is a kind of first and somewhat incomplete attempt in this direction. □

6 A test based on confidence intervals

Finally, we can take full advantage of the fourth comment in the Introduction. Namely, instead of trying to directly evaluate R_t, we can try to evaluate the variables $\overline{L}(s)_t$ for all $s = 1, \ldots, d$.

In view of Theorem 6, this is quite simple: in restriction to the set $\{R_t \geq r\}$, the variables $\sqrt{n}\ (\overline{L}(r)_t^n - \overline{L}(r)_t)$ are asymptotically mixed normal, with a (conditional) variance $TZ(r,r)_t$, which in turn can be estimated by $TZ(r,r)_t^n$ because of Theorem 5. And on the set $\{R_t < r\}$, the variables $\sqrt{n}\ (\overline{L}(r)_t^n - \overline{L}(r)_t) = \sqrt{n}\ \overline{L}(r)_t^n$ go to 0 in law.

This allows one to derive (asymptotic) confidence intervals for $\overline{L}(s)_t$. More precisely, we get the following:

$$\lim_n \mathbf{P}\left(\left|\sqrt{nTZ(r,r)_t^n}\ (\overline{L}(r)_t^n - \overline{L}(r)_t)\right| \geq \gamma \mid R_t = r'\right) = \mathbf{P}(|G| \geq \gamma) \tag{50}$$

for any $\gamma > 0$, where G is an $\mathcal{N}(0,1)$ random variable, and provided $\mathbf{P}(R_t = r') > 0$ and $r' \geq r$. This is quite satisfactory because $Z(r,r)_t^n$ is observable. On the other hand, as soon as $\mathbf{P}(R_t = r') > 0$ and $r' < r$ and $\gamma > 0$, we get

$$\lim_n \mathbf{P}\left(\left|\sqrt{n}\ \overline{L}(r)_t^n\right| \geq \gamma \mid R_t = r'\right) = 0. \tag{51}$$

This is less satisfactory, because the confidence intervals based on this are not sharp. And it is difficult to obtain non trivial limit theorems for the sequence $\overline{L}(r)_t^n$ suitably normalized, when $R_t < r$: it seems linked to the speed with which the eigenspaces associated with the positive eigenvalues of c_s rotates in $\mathbf{R}^d$.

An example. Let us consider for instance the problem of "testing" whether the scale-invariant variable $S_t := t^{r-1}\overline{L}(r)_t/(\overline{L}(1)_t)^r$ exceeds some prescribed level $\varepsilon > 0$ (with, say, $0/0 = 0$). This variable is naturally estimated by $S_{n,t} = t^{r-1}\overline{L}(r)_t^n/(\overline{L}(1)_t^n)^r$.

Although once again this is not a testing problem in the usual sense, one can do as if S_t were a (deterministic) parameter and the null hypothesis is $S_t \geq \varepsilon$. Critical regions on which we reject this hypothesis are naturally of the form

$$C_{n,t}(\eta) = \{S_{n,t} < \eta\}. \tag{52}$$

The "level" of this test is

$$\alpha_{n,t}^{\eta} = \sup_{x \geq \varepsilon} \mathbf{P}(C_{n,t}(\eta) \mid S_t \geq x), \tag{53}$$

and its "power function" for $x \in (0, \varepsilon)$ is

$$\beta^{\eta}_{n,t}(x) = \mathbf{P}(C_{n,t}(\eta) \mid S_t \leq x). \tag{54}$$

(it would perhaps be more suitable to use $\mathbf{P}(C_n(\eta) \mid S = x)$ as the power function, but the later cannot be evaluated properly below). We also need the following variable

$$Z_{n,t} = Tt^{2(r-1)} \frac{(\overline{L}(r)^n_t)^2}{\overline{L}(1)^{2r}_n} \left(\frac{Z(r,r)^n_t}{(\overline{L}(r)^n_t)^2} - \frac{2rZ(1,r)^n_t}{\overline{L}(r)^n_t \overline{L}(1)^n_t} + \frac{r^2 Z(1,1)^n_t}{(\overline{L}(1)^n_t)^2} \right), \tag{55}$$

which looks complicated but is actually computable at stage n from our observations. Then we get the following result:

Theorem 14 *Assume (H1) or (H2). Let $\alpha \in (0,1)$ and take $\gamma \in \mathbf{R}$ to be such that $\mathbf{P}(G > \gamma) = \alpha$, where G is an $\mathcal{N}(0,1)$ variable.*

(i) If $\mathbf{P}(S_t \geq \varepsilon) > 0$, the "tests" with critical regions $C_{n,t}(\eta_{n,t})$ have an asymptotical level less than or equal to α (that is, $\limsup_n \alpha^{\eta_{n,t}}_{n,t} \leq \alpha$), if we take

$$\eta_{n,t} = \varepsilon - \gamma \frac{\sqrt{|Z_{n,t}|}}{\sqrt{n}}. \tag{56}$$

(ii) If $\mathbf{P}(\overline{L}(1)_t > 0) = 1$, the power function $\beta^{\eta_{n,t}}_{n,t}$ of the above test, with $\eta_{n,t}$ given by (56), satisfies $\beta^{\eta_{n,t}}_{n,t}(x) \to 1$ for any $x \in (0, \varepsilon)$.

The assumption $\mathbf{P}(\overline{L}(1)_t > 0) = 1$ in (ii) is very mild: it rules out the case where the function $s \mapsto c_s(\omega)$ vanishes on $[0,t]$ on a subset of Ω with positive probability. If it fails, then the variable S_t is not well defined on this set anyway and the problem is essentially meaningless.

Proof. The result is based on the following consequences of Theorems 5 and 6. We fix $t \in (0,T]$ and we introduce the variables

$$\widehat{V}_n := \sqrt{n}\ (S_{n,t} - S_t) = t^{r-1}\sqrt{n} \left(\frac{\overline{L}(r)^n_t}{(\overline{L}(1)^n_t)^r} - \frac{\overline{L}(r)_t}{(\overline{L}(1)_t)^r} \right). \tag{57}$$

Theorem 6 yields that, in restriction to the set $A = \{\overline{L}(1)_t > 0\}$, the variables $\widehat{V}_n$ converge stably in law to the variable

$$\widehat{V} = t^{r-1} \frac{\overline{L}(r)_t}{(\overline{L}(1)_t)^r} \left(\frac{V(r)_t}{\overline{L}(r)_t} - r \frac{V(1)_t}{\overline{L}(1)_t} \right).$$

Conditionally on the σ-field $\mathcal{F}$, and again in restriction to A, the variable $\widehat{V}$ is centered normal with variance

$$Z = Tt^{2(r-1)} \frac{\overline{L}(r)^2}{\overline{L}(1)^{2r}} \left(\frac{Z(r,r)_t}{(\overline{L}(r)_t)^2} - \frac{2rZ(1,r)_t}{\overline{L}(r)_t \overline{L}(1)_t} + \frac{r^2 Z(1,1)_t}{(\overline{L}(1)_t)^2} \right).$$

Finally, the above variable Z is the limit in probability of the sequence Z_n defined in (55), by virtue of Theorem 5. To summarize, we deduce that the variables $T_n = \widehat{V}_n/\sqrt{|Z_{n,t}|}$ converge stably in law, in restriction to A again, to an $\mathcal{N}(0,1)$ variable, say G, which is independent of $\mathcal{F}$. In particular, for all $y \in \mathbf{R}$ we have:

$$B \in \mathcal{F},\ B \subset A \quad \Rightarrow \quad \mathbf{P}(\{T_n < y\} \cap B) \to \mathbf{P}(B)\ \mathbf{P}(G < y). \tag{58}$$

It remains to observe that $S_{n,t} = S_t + T_n\sqrt{Z_{n,t}/n}$. Then with the choice (56) for $\eta_{n,t}$ we have $T_n < -\gamma$ on $C_{n,t}(\eta_{n,t})$ as soon as $S_t \geq \varepsilon$, and (i) follows from (58) applied to $B = \{S_t \geq \varepsilon\}$, which is included into A. Finally the assumption in (ii) is $\mathbf{P}(A) = 1$. Let $y, z > 0$ and $x < \varepsilon$, and observe that if $T_n < y$ and $S \leq x < \varepsilon$ and $\sqrt{|Z_{n,t}|} \leq z\sqrt{n}$, we have $S_n < x + yz$ and so we are in $C_{n,t}(\eta_{n,t})$ as soon as $yz < \varepsilon - x$. Then (58) with $B = \{S \leq x\}$ and the fact that $Z_{n,t}$ converges in probability to Z yield for any $y, z > 0$ with $yz < \varepsilon - x$:

$$\mathbf{P}(C_{n,t}(\eta_{n,t}) \cap B) \geq \mathbf{P}(\{T_n < y\} \cap B) - \mathbf{P}(Z_{n,t} > nz^2)\ \to\ \mathbf{P}(B)\ \mathbf{P}(G < y).$$

Since y is arbitrarily large, we conclude that $\mathbf{P}(C_{n,t}(\eta_{n,t}) \cap B) \to \mathbf{P}(B)$, and (ii) follows. □

7 Numerical experiments

In this section we present numerical results for three different families of models. The two first ones concern financial applications, namely, the calibration of baskets of standard stock prices, or energy indices, with stochastic volatilities. Our last example is not motivated by Finance; however it presents a kind of degeneracy which illustrates an (unsurprising) limitation of our estimation procedure.

All the numerical results below concern the test based on a relative threshold described in section 5.4.

7.1 Models with Stochastic Volatilities

In Finance the calibration of models is a difficult issue. One has to handle missing data in statistical analysis; the frequency of prices observations is often too weak to allow one to estimate quadratic variations and thus volatilities with good accuracies, and if it is high then microstructure noises tend to blur the picture; moreover because of market instabilities, any particular model with fixed parameters or coefficients can pretend to describe market prices over only short periods of time. Consequently the practitioners are used to calibrate *implicit* parameters of their stock price models by solving PDE inverse problems (see, e.g., Achdou and Pironneau [1] and the references therein), minimizing entropies (see, e.g., Avellaneda et al. [2]), etc. Such procedures use instantaneous market information on the stocks under consideration, particularly derivative prices, rather than historical data. In all these approaches, as soon as one deals with a portfolio with several assets, the Brownian dimension is a parameter of prime importance. We thus have studied the performances of our estimation procedure within the commonly used Black, Scholes and Samuelson framework with stochastic volatilities.

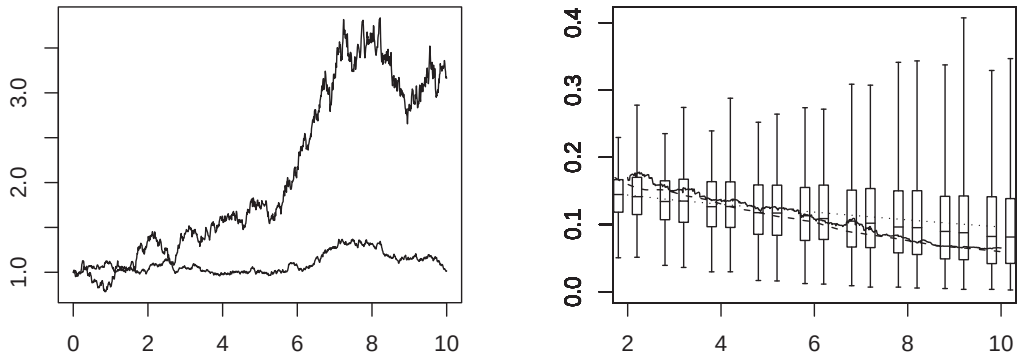


Figure 1: Test case 7-1: $\rho = 0$

Consider

$$\left.\begin{array}{l} X_t^1 = 1 + r_1 \int_0^t X_s^1 ds + \sigma_1 \int_0^t X_s^1 dB_s^1 \\ X_t^2 = 1 + r_2 \int_0^t X_s^2 ds + \sigma_2 \int_0^t X_s^2 (\rho dB_s^1 + \sqrt{1-\rho^2} dB_s^2) \end{array}\right\}$$

with $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $r_1 = 0.05$ and $r_2 = 0.15$.

To simulate paths of (X_t^1, X_t^2) we have used the Euler scheme with stepsize 10^{-4}. The final time is $T = 10$. The observations are at times $k \cdot 10^{-2}, 1 \leq k \leq 1000$. In view of (45) we consider the estimators

$$\overline{\xi}_t^n(1) = t\frac{\overline{L}_t^n(2)}{\overline{L}_t^n(1)^2}, \qquad \overline{\xi}_t(1) = t\frac{\overline{L}_t(2)}{\overline{L}_t(1)^2}.$$

Figs. 1-4 are organized as follows: the left picture displays a particular sample path of the pair (X_t^1, X_t^2), and in the right picture we have plotted the paths of $\overline{\xi}_t^n(1)$ (solid line) and $\overline{\xi}_t(1))$ (dashed line) corresponding to the path of the left display. Moreover at each integer time $t = 2, 3, \cdots, 10$ the right picture also displays two boxes and whiskers: the box and whiskers on the right plots the empirical quartiles and extends upward and downward to the extremal values of 500 independent samples of the random variables $\overline{\xi}_t^n(1)$; the box and whiskers on the left provides a similar information on $\overline{\xi}_t(1)$. Moreover the left-hand paths in all Figs 1-4 correspond to the same simulated path of the Brownian motion (B^1, B^2).

On this example we see that the paths of (X^1, X^2) for different values of ρ (and corresponding to the same Brownian motion (B^1, B^2)) are difficult to distinguish, whereas the values taken by $\overline{\xi}_t^n(1)$ clearly allows one to distinguish the strongly correlated and weakly correlated cases.

Figs. 5 and 6 display the same box and whiskers pictures than previously, but with $\rho = 0.00$ and $\rho = 0.99$ and for two sampling frequencies T/n: for each integer time t the left box and whiskers are the same on the left and right displays (they both are for $\overline{\xi}_t(1)$, beware the change of scale between the two displays), but unsurprisingly the spread of $\overline{\xi}_t^n(1)$ is bigger at low frequency (right display). However even at the lowest frequency (with only 100 observations) it allow to correctly estimate the real Brownian dimension.

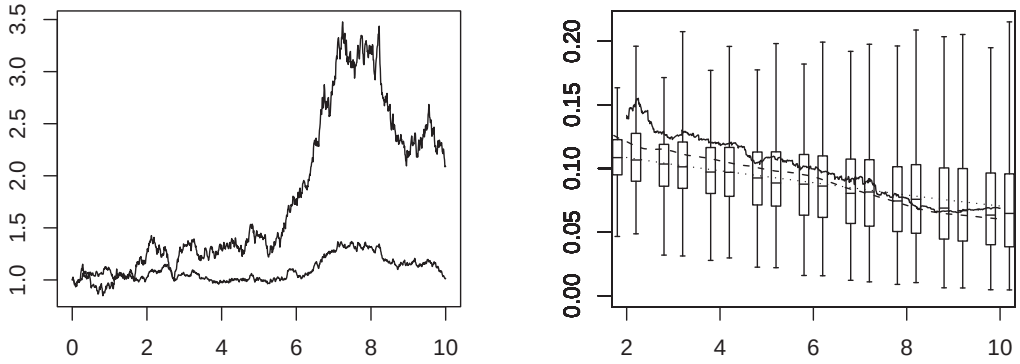


Figure 2: Test case 7.1: $\rho = 0.5$

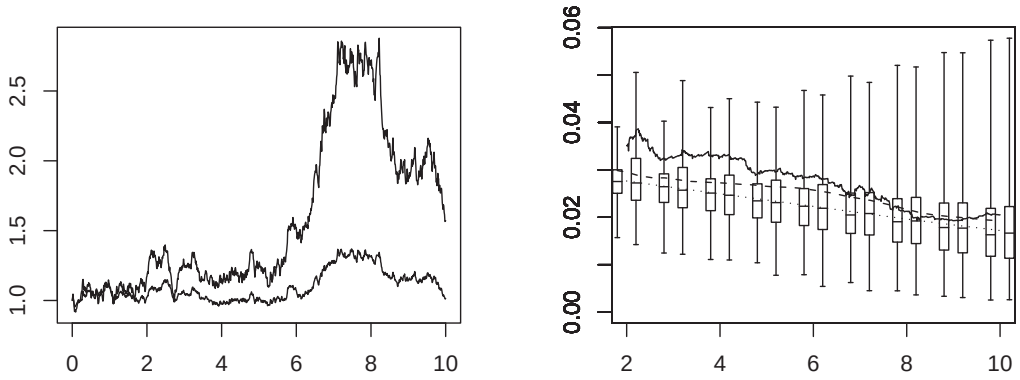


Figure 3: Test case 7.1: $\rho = 0.9$

7.2 A simplified model for energy indices

We now present a toy model for oil prices. In his Ph.D. thesis within a collaboration between INRIA and Gaz de France, O. Bardou [3] has studied modelling and simulation questions related to energy contract pricing problems. One question was to identify the coefficients of a stochastic differential system which could satisfyingly describe the dynamics of about ten energy indices.

Here, for the sake of simplicity, we consider a three-dimensional system whose coefficients resemble those identified by O. Bardou: for $1 \leq i \leq 3$ we set

$$dX_t^i = [\alpha_i (X_t^i - K_i)^+ + \beta_i] dB_t^i + \nu_i (\mu_i - X_t^i) dt,$$

with $X_0^1 = 0.29$, $X_0^2 = 0.89$, $X_0^3 = 0.62$.

Fixed this way, the diffusion term does not satisfy Hypothesis (H2). We thus slightly modify the equation and consider

$$dX_t^i = [\alpha_i \phi(X_t^i - K_i) + \beta_i] dB_t^i + \nu_i (\mu_i - X_t^i) dt,$$

where $\phi(x) = 0$ if $x \leq 0$, $\phi(x) = 2.5\ x^2$ if $0 < x < 0.2$, and $\phi(x) = x - 0.1$ if $x \geq 0.2$.

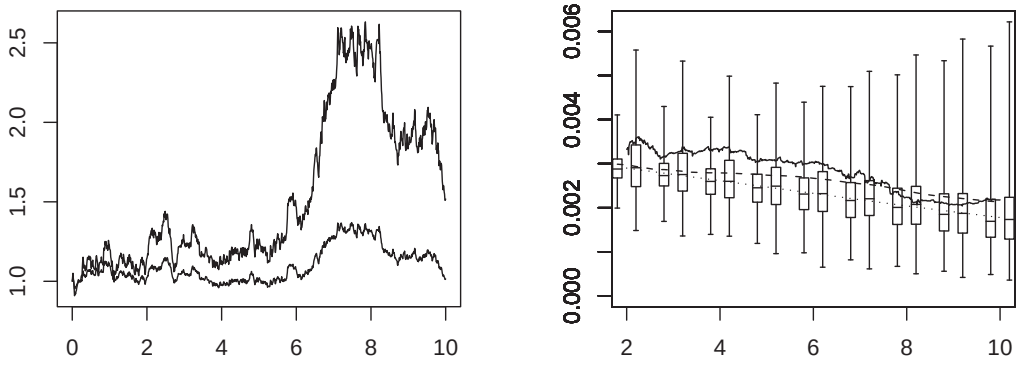
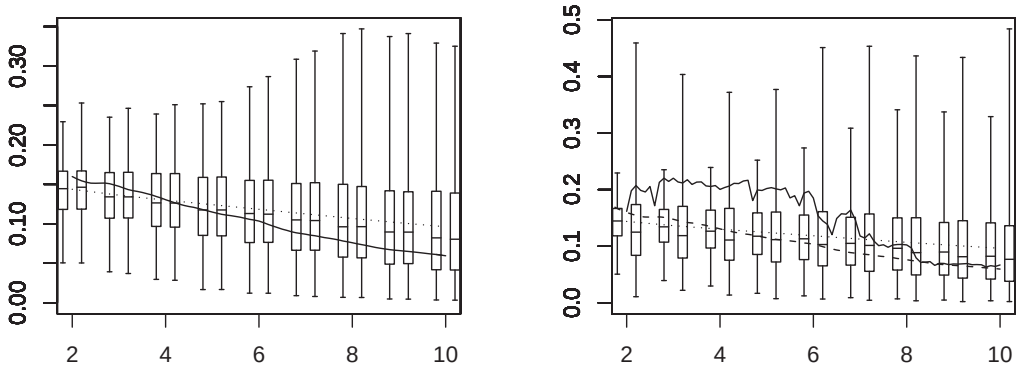


Figure 4: Test case 7.1: $\rho = 0.99$



(a) $T/n = 10^{-3}$ (b) $T/n = 10^{-1}$

Figure 5: Test case 7.1: $\overline{\xi}^n_t(1)$ in terms of T/n ($\rho = 0.00$)

In this very simplified model the components of X are independent; in the real situation where one observes energy indices, one should take correlated Brownian motions B^i, as in Subsection 7.1.

We set $\nu_i = \mu_i = \alpha_i = 1$. The drift term then stabilizes the process around the value 1. If $\beta_i = 0$, the process (X^i_t) diffuses only when X^i_t is above the threshold K_i.

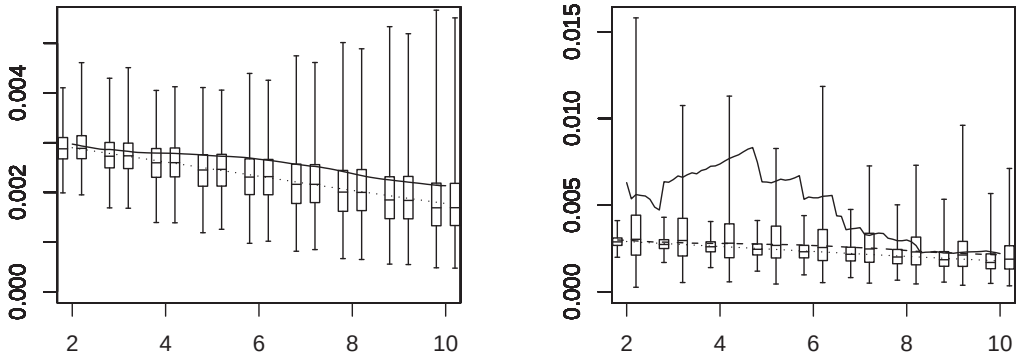
As above, we have approximated a path of the system by simulating the Euler scheme with stepsize 10^{-4} between times 0 and 10. The observations are at times $k \cdot 10^{-2}, 1 \leq k \leq 1000$. In view of (45) we consider the estimators

$$\overline{\xi}^n_t(1) = t\frac{\overline{L}^n_t(2)}{\overline{L}^n_t(1)^2}, \quad \overline{\xi}^n_t(2) = \sqrt{t}\frac{\overline{L}^n_t(3)}{\overline{L}^n_t(2)^{3/2}},$$

and

$$\overline{\xi}_t(1) = t\frac{\overline{L}_t(2)}{\overline{L}_t(1)^2}, \quad \overline{\xi}_t(2) = \sqrt{t}\frac{\overline{L}_t(3)}{\overline{L}_t(2)^{3/2}}.$$

In Figs.7-10 the left boxes show a particular path of (X^1, X^2, X^3); the right boxes show



(a) $T/n = 10^{-3}$ (b) $T/n = 10^{-1}$

Figure 6: Test case 7.1: $\overline{\xi}_t^n(1)$ in terms of T/n ($\rho = 0.99$)

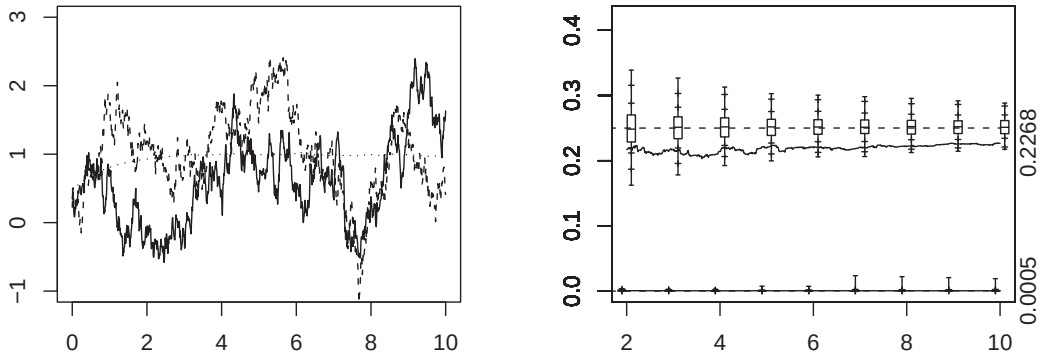


Figure 7: Test case 7.2: $\beta_1 = \beta_2 = 1$, $\beta_3 = 0$, $K_1 = K_2 = 3$, $K_3 = 0.9$

the corresponding paths of $\overline{\xi}_t^n(j)$ (solid line) and $\overline{\xi}_t(j)$ (dashed line), for $j = 1$ and $j = 2$: respectively the top and the bottom curves; values on the right hand-side vertical axes denote $\overline{\xi}_T^n(1)$ and $\overline{\xi}_T^n(2)$. Moreover on the right we have box and whiskers for the empirical quartiles of $\overline{\xi}_t^n(1)$ (top) and $\overline{\xi}_t^n(2)$ (bottom), computed from 500 independent paths and for all integer times $t = 2, 3, \cdots, 10$. The whiskers extend to the extremal values of the samples, the other ticks denoting the 1 %, 10 %, 90 % and 99 % quantiles.

In Fig.7, the two first components diffuse from time 0 to time 10 since K_1 and K_2 are large. The third component diffuses a little only since it is attracted to 1 and $\phi(1 - K_3) = \phi(0.1)$ is small. Given the threshold $\rho_n = 0.01$, the explicative Brownian dimension $\widetilde{R}_{n,t}$ is 2 since $\overline{\xi}_t^n(1)$ takes values around 0.2 whereas $\overline{\xi}_t^n(2)$ takes values around 2 10^{-3}.

In Fig. 8, the two last components have a small diffusion term. As both $\overline{\xi}_t^n(1)$ and $\overline{\xi}_t^n(2)$ take values less than 0.02, according to the same threshold $\rho_n = 0.01$ as above, the explicative Brownian dimension $\widetilde{R}_{n,t}$ is 1.

In Fig. 9, we have K_1 large, and $\phi(1 - K_2) = \phi(1 - K_3) = \phi(0.4) = 0.3$. Therefore none of the diffusion terms can be neglected, but the first component 'oscillates' more

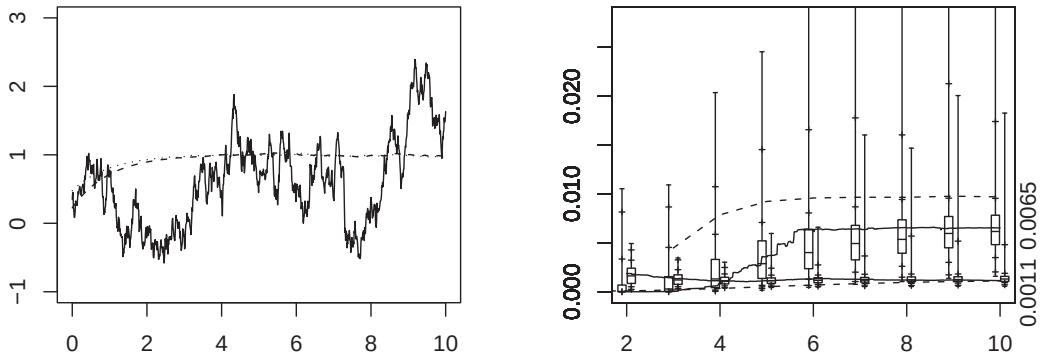


Figure 8: Test case 7.2: $\beta_1 = 1$, $\beta_2 = \beta_3 = 0$, $K_1 = 3$, $K_2 = K_3 = 0.9$

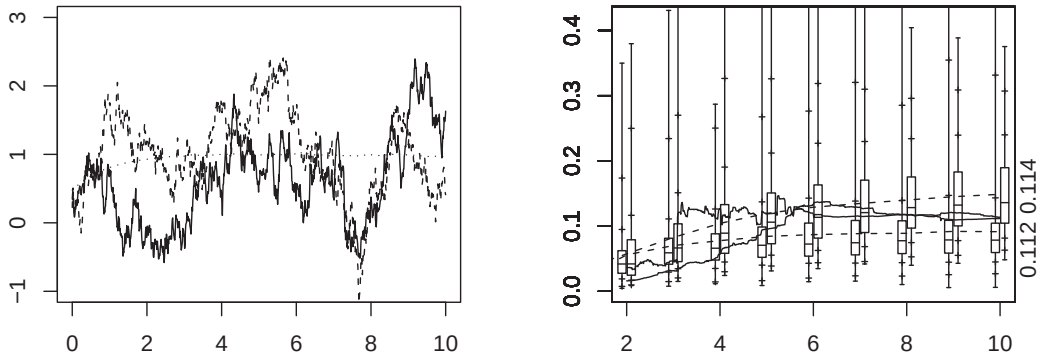


Figure 9: Test case 7.2: $\beta_1 = 1$, $\beta_2 = \beta_3 = 0$, $K_1 = 3$, $K_2 = K_3 = 0.6$

than the two other ones. It is a case where observed paths, for which both $\overline{\xi}^n_t(1)$ and $\overline{\xi}^n_t(2)$ take values around 0.1, may make difficult to decide whether the explicative Brownian dimension should be chosen as 1 or 2.

Finally, in Fig. 10, we keep the two first components as in Fig. 9, but change the third one into an almost constant process. Of course, as above, we have difficulties to decide whether the Brownian dimension is 1 or 2. However it is clear that it cannot be 3 since $\overline{\xi}^n_t(2)$ fluctuates around 5 10^{-3}.

7.3 Sensitivity to a drift term close to be a martingale

We now consider a model with a strongly oscillating drift term. These oscillations significantly decrease the efficiency of our estimator. In certain circumstances the explicative Brownian dimension over-estimates the real dimension.

The system under consideration is

$$\left.\begin{array}{l} X^1_t = \int_0^t \eta \cos(\theta X^2_s) ds, \\ X^2_t = B_t, \end{array}\right\},$$

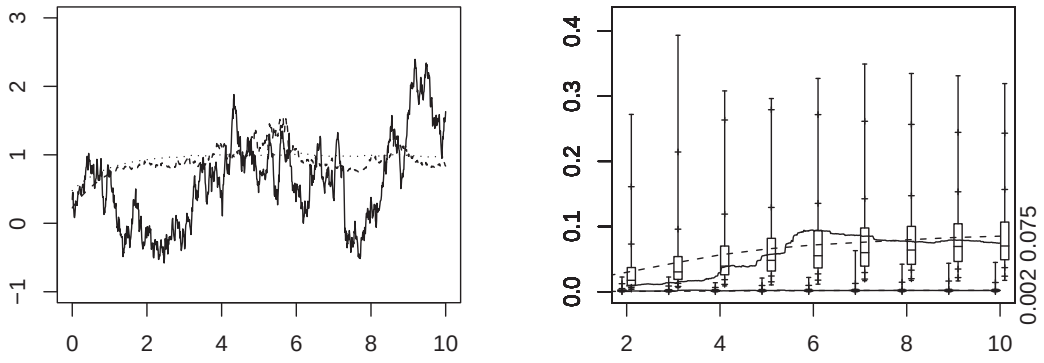


Figure 10: Test case 7.2: $\beta_1 = 1$, $\beta_2 = \beta_3 = 0$, $K_1 = 3$, $K_2 = 0.6$, $K_3 = 0.9$

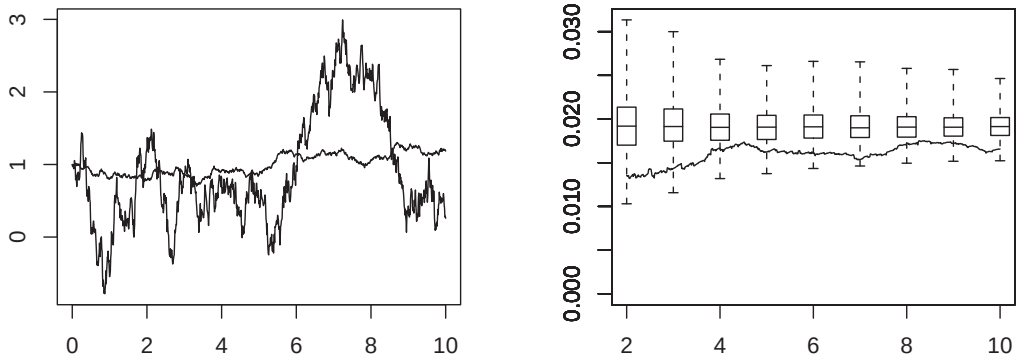


Figure 11: Test case 7.3: $\eta = 10$, $\theta = 100$

where B is a one-dimensional standard Brownian motion, and η, θ are positive real numbers.

As above, we have approximated a path of X by simulating the Euler scheme with stepsize 10^{-4} between times 0 and 10. The observations are at times $k \cdot 10^{-2}, 1 \leq k \leq 1000$. In view of (45) we consider the estimator $\overline{\xi}^n_t(1) = t\frac{\overline{L}^n_t(2)}{\overline{L}^n_t(1)^2}$. Observe that $\overline{L}_t(2) = 0$, so that $\overline{\xi}^n(1)$ should be close to 0.

The boxes and whiskers denote the empirical quartiles of $\overline{\xi}^n_t(1)$ and extends to the extremal values, at times $t = 2, 3, \ldots, 10$ and for 500 estimations of $\overline{\xi}^n_t(1)$.

Fig. 11 shows a case with so a highly oscillating coefficient that the first component is close to 0. The reason is clear since

$$X^1_t = \frac{2\eta}{\theta^2} - \frac{2\eta}{\theta^2}\cos(\theta B_t) - \frac{2\eta}{\theta}\int_0^t \sin(\theta B_s)dB_s, \tag{59}$$

and, here, $\eta = 10$ and $\theta = 100$. As our estimator takes values around 0.02, choosing $\rho_n = 0.02$ leads one to choose $\widetilde{R}_{n,t} = 1$ as the explicative Brownian dimension.

In Fig. 12 we fix $\eta = 10$ and $\theta = 10$. In view of (59), (X^1_t) is close to be a stochastic

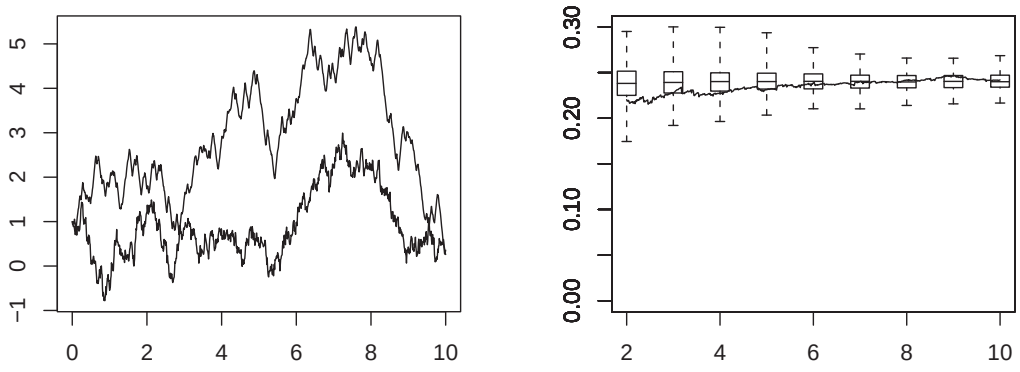


Figure 12: Test case 7.3: $\eta = 10$, $\theta = 10$

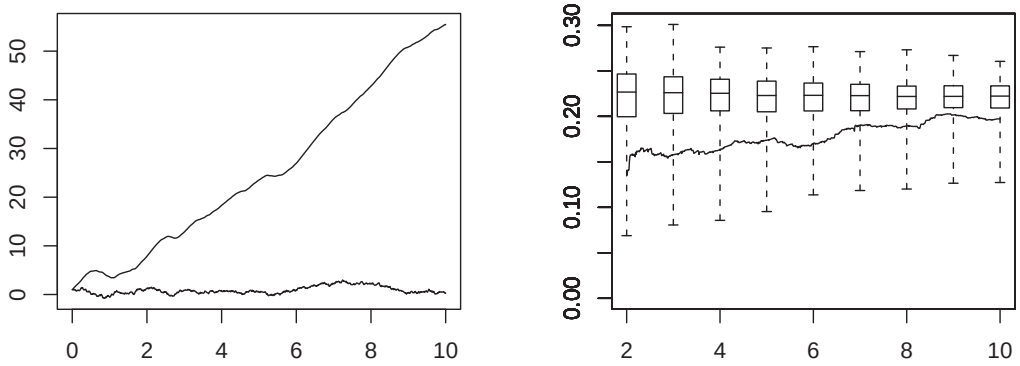


Figure 13: Test case 7.3: $\eta = 10$, $\theta = 1$

integral whose co-variation with B on the time interval $[0, 10]$ is small. As our estimator now takes values around 0.2, we are led to choose 2 as the explicative Brownian dimension and thus over-estimate the real Brownian dimension.

In Fig. 13 we fix $\eta = 10$ and $\theta = 1$. As our estimator takes values around 0.2, again we over-estimate the real Brownian dimension.

In Fig. 14, as $\eta = 1$ and $\theta = 1$, the first component oscillates "reasonably". The estimator takes values less than 0.01, and we are led to correctly choose 1 as the explicative Brownian dimension.

Fig. 15 shows histograms of $\overline{\xi}_t^n(1)$ in the various preceding situations. Finally, Fig. 16 shows the influence of the sampling frequency in the case exhibited in Fig. 14: we see that in this case the Brownian dimension remains correctly estimated to 1 when the step size remains smaller than 10^{-1}.

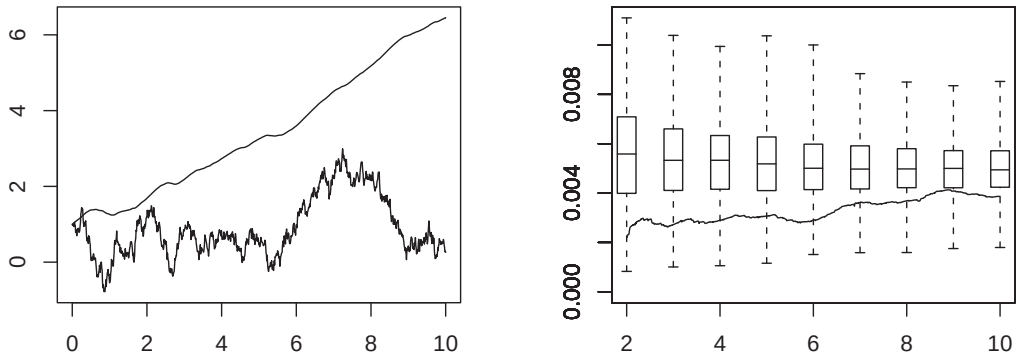


Figure 14: Test case 7.3: $\eta = 1$, $\theta = 1$

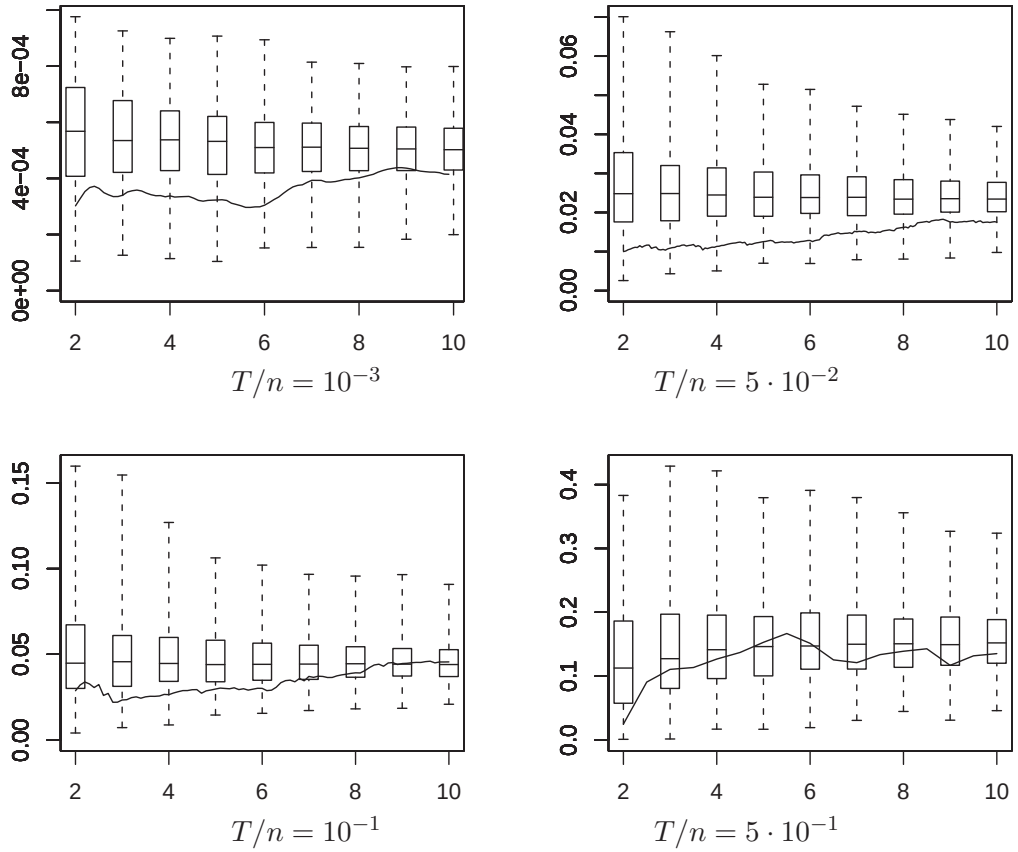
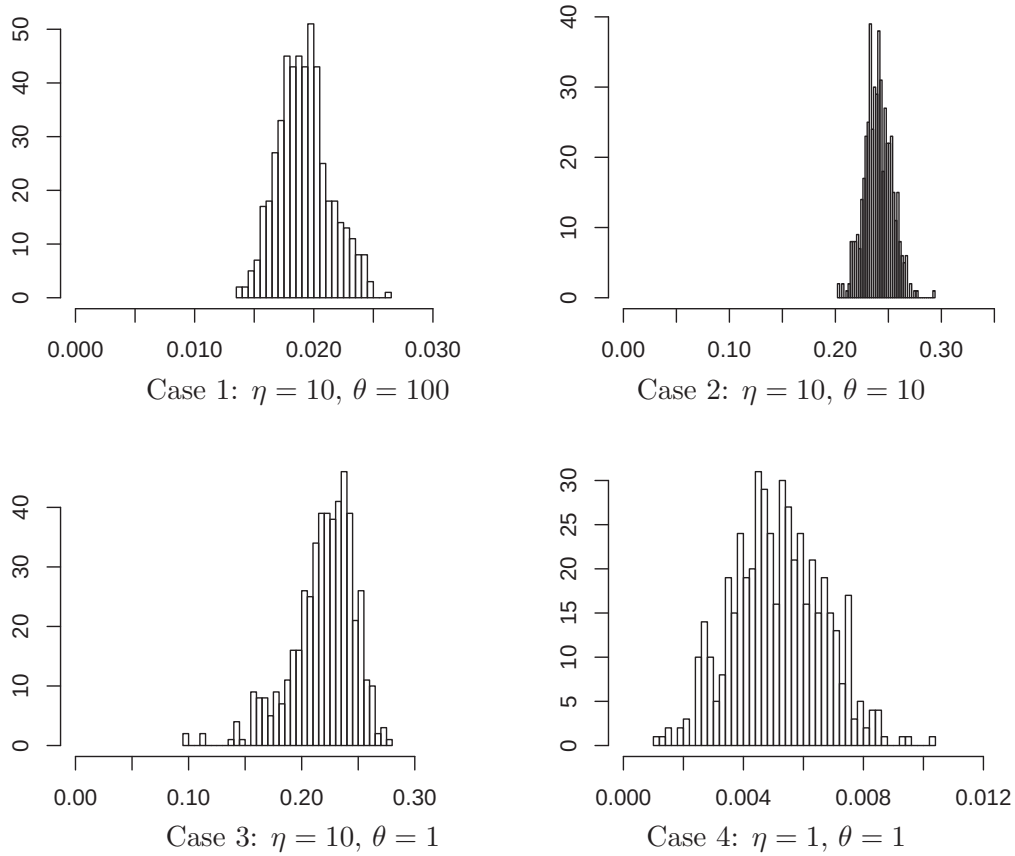


Figure 16: Test case 7.3: $\overline{\xi}_t^n(1)$ in terms of T/n ($\eta = 1$, $\theta = 1$)



Case 1: $\eta = 10$, $\theta = 100$ Case 2: $\eta = 10$, $\theta = 10$

Case 3: $\eta = 10$, $\theta = 1$ Case 4: $\eta = 1$, $\theta = 1$

Figure 15: Test case 7.3: Histograms of $\overline{\xi}_t^n(1)$ at time $t = 5$.

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