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Presentation of a Game Semantics for First-Order Propositional Logic

Samuel Mimram[†]

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Game semantics aim at describing the interactive behaviour of proofs by interpreting formulas as games on which proofs induce strategies. In this article, we introduce a game semantics for a fragment of first order propositional logic. One of the main difficulties that has to be faced when constructing such semantics is to make them precise by characterizing definable strategies – that is strategies which actually behave like a proof. This characterization is usually done by restricting to the model to strategies satisfying subtle combinatory conditions such as innocence, whose preservation under composition is often difficult to show. Here, we present an original methodology to achieve this task which requires to combine tools from game semantics, rewriting theory and categorical algebra. We introduce a diagrammatic presentation of definable strategies by the means of generators and relations: those strategies can be generated from a finite set of "atomic" strategies and that the equality between strategies generated in such a way admits a finite axiomatization. These generators satisfy laws which are a variation of bialgebras laws, thus bridging algebra and denotational semantics in a clean and unexpected way.

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Contents

Denotational semantics were introduced to provide useful abstract invariants of proofs and programs modulo cut-elimination or reduction. In particular, game semantics, introduced in the nineties, have been very successful in capturing precisely the interactive behaviour of programs. In these semantics, every type is interpreted as a *game*, that is as a set of *moves* that can be played during the game, together with the rules of the game, formalized by a partial order on the moves of the game indicating the dependencies between the moves. Every move in these games is to be played by one of the two players, called *Proponent* and *Opponent*, who should be thought respectively as the program and its environment. Interactions between these two players are sequences of moves respecting the partial order of the game, called *plays*. Every program is characterized by the set of such interactions that it can have with its environment during an execution and thus defines a *strategy* reflecting the interactive behaviour of the program inside the game specified by the type of the program.

In particular, the notion of *pointer game*, introduced by Hyland and Ong (Hyland and Ong(2000)) and independently by Nickau (Nickau(1994)), gave a fully abstract model of PCF – a simply-typed λ-calculus extended with recursion, conditional branching and arithmetical constants. It has revealed that PCF programs generate strategies with partial memory, called *innocent* because they react to Opponent moves according to their own *view* of the play. Thus innocence – together with another condition called *well-bracketing* – is in their setting a characterization of *definable* strategies, that is strategies which are the interpretation of a PCF term. This seminal work has lead to an extremely successful series of semantics: by relaxing in various ways the innocence constraint on strategies, it became suddenly possible to characterize the behaviour of PCF programs extended with imperative features like states, references, etc.

Unfortunately, these constraints are very specific to game semantics and remain difficult to link with other areas of computer science or algebra. Moreover, the conditions used to characterize definable strategies are very subtle and combinatorial and are thus sometimes difficult to work with. In particular, showing that these conditions are preserved under composition of strategies usually requires a fairly large amount of work.

Generating instead of restricting. In this paper, we introduce a game semantics for a fragment of first-order propositional logic and describe a monoidal category **Games** of games and strategies in which the proofs can be interpreted. Instead of characterizing definable strategies of the model by restricting the strategies we consider to strategies satisfying particular conditions, we show that we can equivalently use here a kind of converse approach: we explain how to *generate* definable strategies by giving a *presentation* of those strategies, i.e. we show that a finite set of definable strategies can be used to generate all definable strategies by composition and tensoring and finitely axiomatize the equality between strategies obtained this way.

We we mean precisely by a presentation is a generalization of the usual notion of presentation of a monoid (or a group, ...) presentation to monoidal categories. For example, consider the bicyclic monoid B whose set of elements is $\mathbb{N} \times \mathbb{N}$ and whose multiplication $*$ is defined by

$$(m_1, n_1) * (m_2, n_2) = (m_1 - n_1 + \max(n_1, m_2), n_2 - m_2 + \max(n_1, m_2))$$

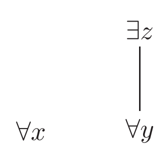
This monoid admits the presentation $\langle\, p, q \mid pq = 1 \,\rangle$, where p and q are two *generators* and $pq = 1$ is an equation between two elements of the free monoid M on $\{p, q\}$. This means that B is isomorphic to the free monoid M on two generators p and q quotiented by the smallest congruence $\equiv$ (with respect to multiplication) such that $pq \equiv 1$, where 1 is the unit of the free monoid. More explicitly, the morphism of monoids $\varphi : M \to B$ defined by $\varphi(p) = (1, 0)$ and $\varphi(q) = (0, 1)$ is surjective and induces an injective functor from $M/\equiv$ to B: two words w and w' have the same image under φ if and only if $w \equiv w'$.

Similarly, we give in this paper a finite set of typed generators from which we can generate a free monoidal category $\mathcal{G}$ by composing and tensoring generators. We moreover give a finite set of typed equations between morphisms of $\mathcal{G}$ and write $\equiv$ for the smallest congruence (with respect to composition and tensoring) on morphisms. Then we show that the category $\mathcal{G}/\equiv$, which is the category $\mathcal{G}$ whose morphisms are quotiented by the congruence $\equiv$, is equivalent to the category **Games** of definable strategies. As a by-product we obtain the fact that the composite of two definable strategies is well-defined which was not obvious from the definition we gave.

Strategies as refinements of the game. Game semantics has revealed that proofs in logics describe particular strategies to explore formulas. A formula A is a syntactic tree expressing in which order its connectives must be introduced in cut-free proofs of A: from the root to leaves. In this sense, it can be seen as the rules of a game whose moves correspond to introduction rules of connectives in logics. For instance, consider a formula A of the form

$$\forall x.P \quad \Rightarrow \quad \forall y.\exists z.Q \tag{1}$$

where P and Q are propositional formulas which may contain free variables. When searching for a proof of A, the $\forall y$ connective must be introduced before the $\exists z$ connective and the $\forall x$ connective can be introduced independently. The game – whose moves are first-order connectives – associated to this formula is therefore a partial order on the first-order connectives of the formula which can be depicted as the following diagram (to be read from the bottom to the top)



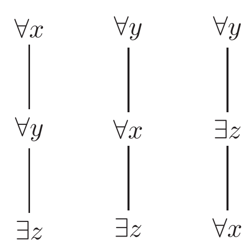
(2)

Existential connectives should be thought as Proponent moves (the strategy gives a witness for which the formula holds) and the universal connectives as Opponent moves (the strategy receives a term from its environment, for which it has to show that the formula holds).

Informally, in a first-order propositional logic, the formula (1) can have proofs of the three following shapes

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{P[t/x] \vdash Q[t'/z]}}{P[t/x] \vdash \exists z.Q}}{P[t/x] \vdash \forall y.\exists z.Q}}{\forall x.P \vdash \forall y.\exists z.Q} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{P[t/x] \vdash Q[t'/z]}}{P[t/x] \vdash \exists z.Q}}{\forall x.P \vdash \exists z.Q}}{\forall x.P \vdash \forall y.\exists z.Q} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{P[t/x] \vdash Q[t'/z]}}{\forall x.P \vdash Q[t'/z]}}{\forall x.P \vdash \exists z.Q}}{\forall x.P \vdash \forall y.\exists z.Q}$$

Here $P[t/x]$ denotes the formula P where every occurrence of the free variable x has been replaced by the term t. These proofs introduce the connectives in the orders depicted respectively below



It should be noted that they are all refinements of the partial order (2) corresponding to the formula, in the sense that they have more dependencies between moves: *proofs add causal dependencies between connectives.*

To understand exactly what dependencies which are added by proofs we are interested in, we shall examine precisely proofs of the formula

$$\exists x.P \quad \Rightarrow \quad \exists y.Q \tag{3}$$

which induces the following game

$$\exists x \qquad \exists y$$

By permuting the use of introduction rules, a proof of the formula (3)

$$\dfrac{\dfrac{\dfrac{\vdots}{P \vdash Q[t/y]}}{P \vdash \exists y.Q}}{\exists x.P \vdash \exists y.Q}$$

might be reorganized as the proof

$$\dfrac{\dfrac{\dfrac{\vdots}{P \vdash Q[t/y]}}{\exists x.P \vdash Q[t/y]}}{\exists x.P \vdash \exists y.Q}$$

if and only if the term t used in the introduction rule of the $\exists y$ connective does not have

x as free variable. If the variable x is free in t then the rule introducing $\exists y$ can only be done after the rule introducing the $\exists x$ connective. This will be reflected by a causal dependency in the strategy corresponding to the proof, depicted by an oriented wire:

$$\exists x \longrightarrow \exists y$$

We thus build a monoidal category **Games** of games and strategies. Its objects are *games*, that is total orders on a set whose elements (the *moves*) are polarized (they are either Proponent or Opponent moves). Its morphisms $\sigma : A \to B$ between two objects A and B are the partial orders $\leq_\sigma$ on the moves of A (with polarities inverted) and B which are compatible with both the partial orders of A and of B, i.e. does not create cycle with those partial orders.

The logic we have chosen to model here (the fragment of first-order propositional logic without connectives) is deliberately very simple in order to simplify our presentation of the category **Games**. We believe however that the techniques used here are very general and could extend to more expressive logics.

1. Presentations of monoidal categories

1.1. *Monoidal categories*

A *monoidal category* $(\mathcal{C}, \otimes, I)$ is a category $\mathcal{C}$ together with a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

and natural isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C), \quad \lambda_A : I \otimes A \to A \quad \text{and} \quad \rho_A : A \otimes I \to A$$

satisfying coherence axioms (MacLane(1971)). A symmetric monoidal category $\mathcal{C}$ is a monoidal category $\mathcal{C}$ together with a natural isomorphism

$$\gamma_{A,B} : A \otimes B \to B \otimes A$$

satisfying coherence axioms and such that $\gamma_{B,A} \circ \gamma_{A,B} = \mathrm{id}_{A \otimes B}$. A monoidal category $\mathcal{C}$ is strictly monoidal when the natural isomorphisms α, λ and ρ are identities. To simplify our presentation, in the rest of this paper we only consider strict monoidal categories. Formally, it can be shown that it is not restrictive, using MacLane's coherence theorem (MacLane(1971)): every monoidal category is monoidally equivalent to a strict one.

A (strict) *monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ between two strict monoidal categories $\mathcal{C}$ and $\mathcal{D}$ if a functor F between the underlying categories $\mathcal{C}$ and $\mathcal{D}$ such that $F(A \otimes B) = F(A) \otimes F(B)$ for every objects A and B of $\mathcal{C}$, and $F(I) = I$. A monoidal functor F between two strict symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is *symmetric* when it transports the symmetry of $\mathcal{C}$ to the symmetry of $\mathcal{D}$, that is when $F(\gamma_A) = \gamma_{F(A)}$.

A *monoidal natural transformation* $\theta : F \to G$ between two monoidal functors $F, G : \mathcal{C} \to \mathcal{D}$ is a natural transformation between the underlying functors F and G such that $\theta_{A \otimes B} = \theta_A \otimes \theta_B$ for every objects A and B of $\mathcal{C}$, and $\theta_I = \mathrm{id}_I$. A monoidal natural transformation $\theta : F \to G$ between two strict symmetric monoidal functors is said to be *symmetric*.

Two monoidal categories $\mathcal{C}$ and $\mathcal{D}$ are *monoidally equivalent* when there exists a pair of monoidal functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and two invertible monoidal natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to \mathrm{Id}_{\mathcal{D}}$.

1.2. *Monoidal theories*

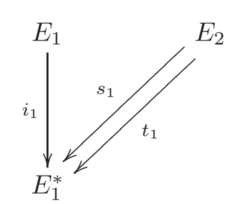
A *monoidal theory* $\mathcal{T}$ is a strict monoidal category whose objects are the natural integers such that the tensor product on objects is given by addition of integers. By an integer n, we mean here the finite ordinal $\underline{n} = \{0, 1, \ldots, n-1\}$ and the addition is given by $\underline{m} + \underline{n} = \underline{m+n}$. A *symmetric monoidal theory* is a monoidal theory where the category is moreover required to be symmetric. An algebra F of a monoidal theory $\mathcal{T}$ in a strict monoidal category $\mathcal{C}$ is a strict monoidal functor from $\mathcal{T}$ to $\mathcal{C}$. Every monoidal theory $\mathcal{T}$ and strict monoidal category $\mathcal{C}$ give rise to a category $\mathbf{Alg}^{\mathcal{C}}_{\mathcal{T}}$ of algebras of $\mathcal{T}$ in $\mathcal{C}$ and monoidal natural transformations between them. Examples of such categories are given in Section 3. Monoidal theories and symmetric monoidal theories are sometimes called respectively PRO and PROP, these terms were introduced by Mac Lane in (MacLane(1965)) as abbreviations for respectively "category with products" and "category with products and permutations".

Monoidal theories generalize equational theories: in this setting, operations are typed, and can moreover have multiple outputs as well as multiple inputs.

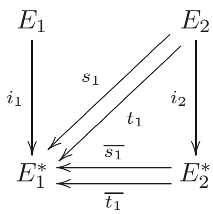
1.3. *Presentations of monoidal categories*

In this section, we recall the notion of *presentation* of a monoidal category by the means of 2-dimensional generators and relations.

Suppose that we are given a set E_1 whose elements are called *atomic types*. We write E_1^* for the free monoid on the set E_1 and $i_1 : E_1 \to E_1^*$ for the corresponding injection; the product of this monoid is written $\otimes$ and its unit is written I. The elements of E_1^* are called *types*. Suppose moreover that we are given a set E_2, whose elements are called *generators*, together with two functions $s_1, t_1 : E_2 \to E_1^*$ which to every generator associate a type called respectively its *source* and *target*. We call a *signature* such a 4-uple (E_1, s_1, t_1, E_2):



In particular, every strict monoidal category $\mathcal{C}$ generates a signature by taking E_1 to be the objects of the category $\mathcal{C}$, E_2 its morphisms, such that for every morphism $f : A \to B$, we have $s_1(f) = i_1(A)$ and $t_1(f) = i_1(B)$. Conversely, every signature (E_1, s_1, t_1, E_2) generates a free strict monoidal category $\mathcal{E}$ described as follows. If we write E_2^* for the morphisms of this category and $i_2 : E_2 \to E_2^*$ for the injection of the generators into this category, we get a diagram



in **Set** together with a structure of monoidal category on the graph

$$E_1^* \underset{\overline{t_1}}{\overset{\overline{s_1}}{\leftleftarrows}} E_2^*$$

where the morphisms $\overline{s_1}, \overline{t_1} : E_2^* \to E_1^*$ are the morphisms (unique by universality of E_2^*) such that $s_1 = \overline{s_1} \circ i_2$ and $t_1 = \overline{t_1} \circ i_2$. More explicitly, the category $\mathcal{E}$ has E_1^* as objects and its set E_2^* of morphisms is the smallest set such that

1 there is a morphism $f : A \to B$ in E_2^* for every element f of E_2 such that $s_1(f) = A$ and $t_1(f) = B$ (this is the image by i_2 of f),
2 there is a morphism $\mathrm{id}_A : A \to A$ in E_2^* for every element A of E_1^*,
3 for every morphisms $f : A \to B$ and $g : B \to C$ in E_2^* there is a morphism $g \circ f : A \to C$ in E_2^*,
4 for every morphisms $f : A \to B$ and $g : C \to D$ in E_2^* there is a morphism $f \otimes g : A \otimes C \to B \otimes D$ in E_2^*,

quotiented by equalities imposing that

1 composition is associative and admits identities as neutral element,
2 the tensor product is associative and admits id_I as neutral element,
3 identities form a monoidal natural transformation $\mathrm{id} : \mathrm{Id} \to \mathrm{Id}$: for every objects A and B,

$$\mathrm{id}_A \otimes \mathrm{id}_B \quad = \quad \mathrm{id}_{A \otimes B}$$

4 tensor product and composition are compatible in the sense that for every morphisms $f : A \to B$, $g : B \to C$, $f' : A' \to B'$ and $g' : B' \to C'$,

$$(g \circ f) \otimes (g' \circ f') \quad = \quad (g \otimes g') \circ (f \otimes f')$$

The *size* $|f|$ of a morphism $f : A \to B$ in $\mathcal{E}$ is defined inductively by

$$|\mathrm{id}| = 0 \qquad |f| = 1 \text{ if } f \text{ is a generator}$$
$$|f_1 \otimes f_2| = |f_1| + |f_2| \qquad |f_2 \circ f_1| = |f_1| + |f_2|$$

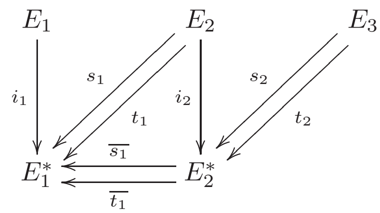
In particular, a morphism is of size 0 if and only if it is an identity.

This construction is a particular case of Street's 2-computads (Street(1976)) and Burroni's polygraphs (Burroni(1993)) who made precise the sense in which the generated monoidal category is free on the signature. In particular, the following notion of equational theory is a specialization of the definition of a 3-polygraph to the case where there is only one 0-cell.

Definition 1. A **monoidal equational theory** is a 7-uple

$$\mathfrak{E} = (E_1, s_1, t_1, E_2, s_2, t_2, E_3)$$

where (E_1, s_1, t_1, E_2) is a signature together with a set E_3 of *equations* and two morphisms $s_2, t_2 : E_3 \longrightarrow E_2^*$, as pictured in the diagram



such that

$$\overline{s_1} \circ s_2 = \overline{s_1} \circ t_2 \quad \text{and} \quad \overline{t_1} \circ s_2 = \overline{t_1} \circ t_2.$$

Every equational theory defines a monoidal category $\mathcal{E}/\equiv$ obtained from the monoidal category $\mathcal{E}$ generated by the signature (E_1, s_1, t_1, E_2) by quotienting the morphisms by the congruence $\equiv$ generated by the equations of the equational theory $\mathfrak{E}$: it is the smallest congruence (with respect to both composition and tensoring) such that $s_2(e) \equiv t_2(e)$ for every element $e \in E_3$. We say that a monoidal equational theory $\mathfrak{E}$ is a *presentation* of a strict monoidal category $\mathcal{M}$ when $\mathcal{M}$ is monoidally equivalent to the category $\mathcal{E}$ generated by $\mathfrak{E}$.

We sometimes informally say that an equational theory

$$\mathfrak{E} = (E_1, s_1, t_1, E_2, s_2, t_2, E_3)$$

has a *generator*

$$f \quad : \quad A \to B$$

to mean that f is an element of E_2 such that $s_1(f) = A$ and $t_1(f) = B$. We also say that the equational theory has an *equation*

$$f \quad = \quad g$$

to mean that there exists an element e of E_2 such that $s_2(e) = f$ and $t_2(e) = g$.

We say that two equational theories are *equivalent* when they generate monoidally equivalent categories. A generator f in an equational theory $\mathfrak{E}$ is *superfluous* when the equational theory $\mathfrak{E}'$ obtained from $\mathfrak{E}$ by removing the generator f and all equations involving f, is equivalent to $\mathfrak{E}$. Similarly, an equation e is *superfluous* when the equational theory $\mathfrak{E}'$ obtained from $\mathfrak{E}$ by removing the equation e is equivalent to $\mathfrak{E}$. An equational theory is *minimal* when it does not contain any superfluous generator or equation.

Remark 2. An equational presentation $(E_1, s_1, t_1, E_2, s_2, t_2, E_3)$ where E_1 is reduced to a set with only one object $\{1\}$ generates a monoidal category which is a monoidal theory.

1.4. *Presented categories as models*

Suppose that a strict monoidal category $\mathcal{M}$ is presented by an equational theory $\mathfrak{E}$. We write $\mathcal{E}/\equiv$ for the category generated by $\mathfrak{E}$. The proof that $\mathfrak{E}$ presents $\mathcal{M}$ can generally be decomposed in three parts:

1 $\mathcal{M}$ *is a model of the equational theory* $\mathfrak{E}$*:* there exists a functor $\widetilde{-}$ from the category $\mathcal{E}/\equiv$ to $\mathcal{M}$. This amounts to check that there exists a functor $F : \mathcal{E} \to \mathcal{M}$ such that for every morphisms $f, g : A \to B$ in $\mathcal{E}$, $f \equiv g$ implies $Ff = Fg$.
2 $\mathcal{M}$ *is a fully-complete model of the equational theory* $\mathfrak{E}$*:* the functor $\widetilde{-}$ is full.
3 $\mathcal{M}$ *is the initial model of the equational theory* $\mathfrak{E}$*:* the functor $\widetilde{-}$ is faithful.

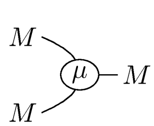
We say that a morphism $f : A \to B$ of $\mathcal{E}/\equiv$ *represents* the morphism $\widetilde{f} : \widetilde{A} \to \widetilde{B}$ of $\mathcal{M}$.

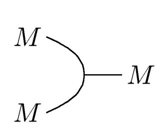
Usually, the first point is a straightforward verification and the second point is easy to show. Proving that the functor $\widetilde{-}$ is faithful often requires more work. In this paper, we use the methodology introduced by Lafont in (Lafont(2003)). We first define *canonical forms* which are (not necessarily unique) canonical representatives of the equivalence classes of morphisms of $\mathcal{E}$ under the congruence $\equiv$ generated by the equations of $\mathfrak{E}$ – proving that every morphism is equal to a canonical form can be done by induction on the size of the morphisms. Then we show that the functor $\widetilde{-}$ is faithful by showing that all the canonical forms which have the same image under $\widetilde{-}$ are equal.

It should be noted that this is not the only technique to prove that an equational theory presents a monoidal category. In particular, Joyal and Street have used topological methods (Joyal and Street(1991)) by giving a geometrical construction of the category generated by a signature, in which morphisms are equivalence classes under continuous deformation of progressive plane diagrams (their construction is detailed a bit more in Section 1.5). Their work is for example extended by Baez and Langford in (Baez and Langford(2003)) to give a presentation of the 2-category of 2-tangles in 4 dimensions. The other general methodology the author is aware of, is given by Lack in (Lack(2004)), by constructing elaborate monoidal theories from simpler monoidal theories. Namely, a monoidal theory can be seen as a monad in a particular span category and monoidal theories can therefore be composed, given a distributive law between their corresponding monads. We chose not to use those methods because, even though they can be very helpful to build intuitions, they are difficult to formalize and even more to mechanize – we believe indeed that some of the tedious proofs given in this paper could be somewhat automated.

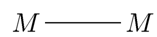
1.5. *String diagrams*

String diagrams provide a convenient way to represent the morphisms in the category generated by a presentation. Given an object M in a category $\mathcal{C}$, a morphism $\mu : M \otimes M \to M$ can be drawn graphically as a device with two inputs and one output of type M as follows:

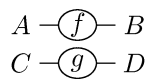


or simply as


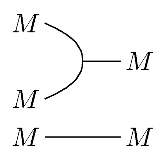
when it is clear from the context which morphism of type $M \otimes M \to M$ we are picturing (we sometimes even omit the source and target of the morphisms). Similarly, the identity $\mathrm{id}_M : M \to M$ can be pictured as



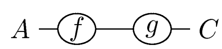
The tensor $f \otimes g$ of two morphisms $f : A \to B$ and $g : C \to D$ is obtained by putting the diagram corresponding to f above the diagram corresponding to g:



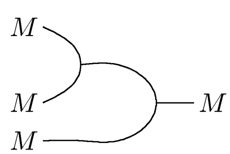
So, for instance, the morphism $\mu \otimes M : M \otimes M \otimes M \to M \otimes M$ can be drawn diagrammatically as



Finally, the composite $g \circ f$ of two morphisms $f : A \to B$ and $g : B \to C$ can be drawn diagrammatically by putting the diagram corresponding to g at the right of the diagram corresponding to f and by "linking the wires".



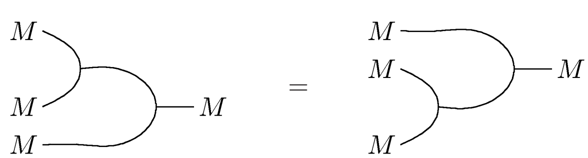
Thus, the diagram corresponding to the morphism $\mu \circ (\mu \otimes M) : M \otimes M \to M$ is



The associativity law for monoids (see Section 2.2)

$$\mu \circ (\mu \otimes M) \quad = \quad \mu \circ (M \otimes \mu)$$

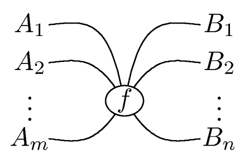
can therefore be represented graphically as



Suppose that (E_1, s_1, t_1, E_2) is a signature. Every element f of E_2 such that

$$s_1(f) = A_1 \otimes \cdots \otimes A_m \quad \text{and} \quad t_1(f) = B_1 \otimes \cdots \otimes B_n$$

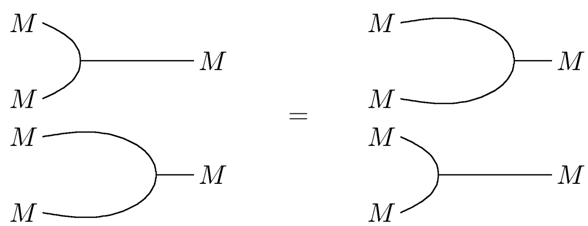
where the A_i and B_i are elements of E_1, can be represented by a diagram



Bigger diagrams can be constructed from these diagrams by composing and tensoring them, as explained above. Joyal and Street have shown in details in (Joyal and Street(1991)) that the category of those diagrams, modulo continuous deformations, is precisely the free category generated by a signature (which they call a tensor scheme). For example, the equality

$$(M \otimes \mu) \circ (\mu \otimes M \otimes M) \quad = \quad (\mu \otimes M \otimes M) \circ (M \otimes \mu)$$

in the category $\mathcal{C}$ given in the example above; this can be shown by continuously deforming the diagram on the left-hand side below into the diagram on the right-hand side:



All the equalities, given in Section 1.3, satisfied by the monoidal category generated by a signature have a similar geometrical interpretation.

2. Some algebraic structures

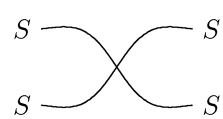
In this section, we recall the categorical formulation of some well-known algebraic structures (monoids, bialgebras, ...). It should be noted that we give those definitions in the setting of a monoidal category which is *not* required to be symmetric. We suppose that $(\mathcal{C}, \otimes, I)$ is a strict monoidal category, fixed throughout the section.

2.1. *Symmetric objects*

A *symmetric object* of $\mathcal{C}$ is an object S together with a morphism

$$\gamma : S \otimes S \to S \otimes S$$

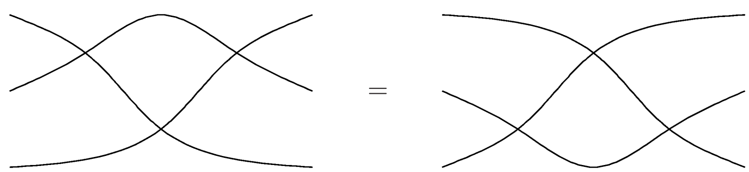
called *symmetry* and pictured as



such that the diagrams

$$\begin{array}{ccccc} S\otimes S\otimes S & \xrightarrow{\gamma\otimes S} & S\otimes S\otimes S & \xrightarrow{S\otimes\gamma} & S\otimes S\otimes S \\ {\scriptstyle S\otimes\gamma}\downarrow & & & & \downarrow{\scriptstyle \gamma\otimes S} \\ S\otimes S\otimes S & \xrightarrow[\gamma\otimes S]{} & S\otimes S\otimes S & \xrightarrow[S\otimes\gamma]{} & S\otimes S\otimes S \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} & & S\otimes S & & \\ & {\scriptstyle\gamma}\nearrow & & \searrow{\scriptstyle\gamma} & \\ S\otimes S & & \xrightarrow[S\otimes S]{\hspace{4em}} & & S\otimes S \end{array}$$

commute. Graphically,



and

These equations are called the Yang-Baxter equations.

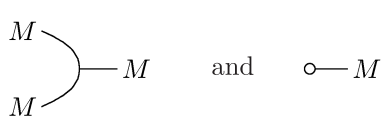
Remark 3. When the monoidal category $\mathcal{C}$ is symmetric, every object S has a symmetry $\gamma = \gamma_{S,S}$ induced by the symmetry of the category.

2.2. *Monoids*

A *monoid* (M, μ, η) in $\mathcal{C}$ is an object M together with two morphisms

$$\mu : M \otimes M \to M \quad \text{and} \quad \eta : I \to M$$

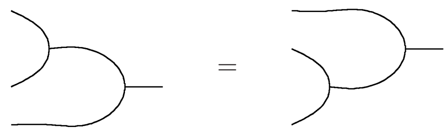
called respectively *multiplication* and *unit* and pictured respectively as



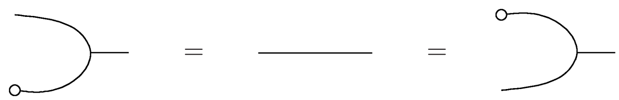
such that the diagrams

$$\begin{array}{ccc} M\otimes M\otimes M & \xrightarrow{\mu\otimes M} & M\otimes M \\ {\scriptstyle M\otimes\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ M\otimes M & \xrightarrow[\mu]{} & M \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} I\otimes M & \xrightarrow{\eta\otimes M} & M\otimes M & \xleftarrow{M\otimes\eta} & M\otimes I \\ & {\scriptstyle M}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle M} & \\ & & M & & \end{array}$$

commute. Graphically,



and



(4)

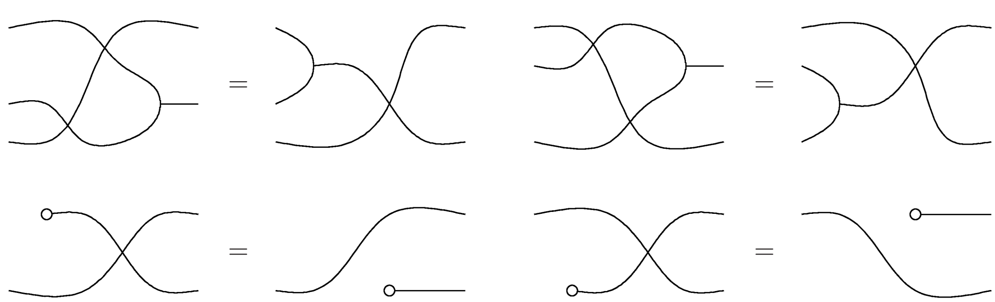
A *symmetric monoid* is a monoid which admits a symmetry $\gamma : M \otimes M \to M \otimes M$ which is compatible with the operations of the monoid in the sense that it makes the diagrams

$$\begin{array}{ccccc}
M \otimes M \otimes M & \xrightarrow{M\otimes\gamma} & M \otimes M \otimes M & \xrightarrow{\gamma\otimes M} & M \otimes M \otimes M \\
\downarrow{\scriptstyle \mu\otimes M} & & & & \downarrow{\scriptstyle M\otimes\mu} \\
M \otimes M & & \xrightarrow[\gamma]{} & & M \otimes M
\end{array}$$

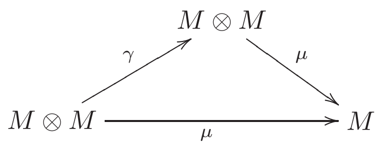
$$\begin{array}{ccccc}
M \otimes M \otimes M & \xrightarrow{\gamma\otimes M} & M \otimes M \otimes M & \xrightarrow{M\otimes\gamma} & M \otimes M \otimes M \\
\downarrow{\scriptstyle M\otimes\mu} & & & & \downarrow{\scriptstyle \mu\otimes M} \\
M \otimes M & & \xrightarrow[\gamma]{} & & M \otimes M
\end{array} \tag{5}$$

$$\begin{array}{ccc}
& M \otimes M & \\
{\scriptstyle \eta\otimes M}\nearrow & & \searrow{\scriptstyle \gamma} \\
M & \xrightarrow[\eta\otimes M]{} & M \otimes M
\end{array}
\qquad
\begin{array}{ccc}
& M \otimes M & \\
{\scriptstyle M\otimes\eta}\nearrow & & \searrow{\scriptstyle \gamma} \\
M & \xrightarrow[M\otimes\eta]{} & M \otimes M
\end{array}$$

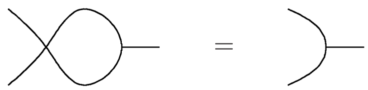
commute. Graphically,



A *commutative monoid* is a symmetric monoid such that the diagram



commutes. Graphically,

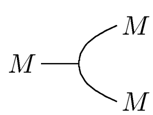

(6)

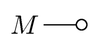
A commutative monoid in a symmetric monoidal category is a commutative monoid whose symmetry corresponds to the symmetry of the category: $\gamma = \gamma_{M,M}$. In this case, the equations (5) can always be deduced from the naturality of the symmetry of the monoidal category.

A *comonoid* (M, δ, ε) in $\mathcal{C}$ is an object M together with two morphisms

$$\delta : M \to M \otimes M \quad \text{and} \quad \varepsilon : M \to I$$

respectively drawn as



and


satisfying dual coherence diagrams. An similarly, the notions symmetric comonoid, co-commutative comonoid and cocommutative comonoid can be defined by duality.

2.3. *An equational theory of monoids*

The definition of a monoid can be reformulated internally using the notion of equational theory.

Definition 4. The *equational theory of monoids* $\mathfrak{M}$ has only one object 1 and two generators $\mu : 2 \to 1$ and $\eta : 0 \to 1$ subject to the equations

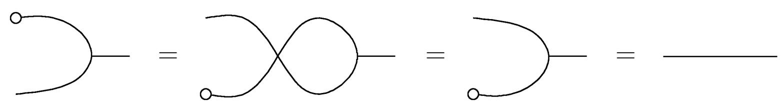
$$\mu \circ (\mu \otimes \mathrm{id}_1) = \mu \circ (\mathrm{id}_1 \otimes \mu) \quad \text{and} \quad \mu \circ (\eta \otimes \mathrm{id}_1) = \mathrm{id}_1 = \mu \circ (\mathrm{id}_1 \otimes \eta)$$

We write $\mathcal{M}$ for the monoidal category generated by the equational theory $\mathfrak{M}$. It can easily be seen that a monoid M in a strict monoidal category $\mathcal{C}$ is essentially the same as a functor from $\mathcal{M}$ to $\mathcal{C}$. More precisely,

Property 5. The category $\mathbf{Alg}^{\mathcal{C}}_{\mathcal{M}}$ of algebras of the monoidal theory $\mathcal{M}$ in $\mathcal{C}$ is equivalent to the category of monoids in $\mathcal{C}$.

Similarly, all the algebraic structures introduced in this section can be defined using algebraic theories.

Remark 6. The presentations given here are not necessarily minimal. For example, in the theory of commutative monoids the equation on the right-hand side of (4) is derivable from the equation (6), the equation on the left-hand side of (4) and one of the equations (5):



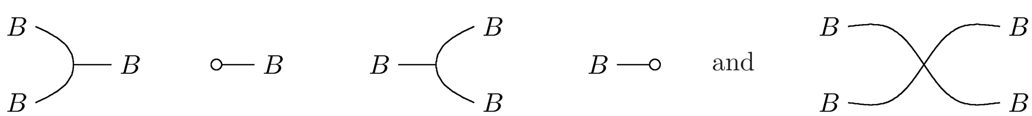
A minimal presentation of this equational theory with three generators and seven equations is given in (Massol(1997)). However, not all the equational theories introduced in this paper have a known presentation which is proved to be minimal.

2.4. *Bialgebras*

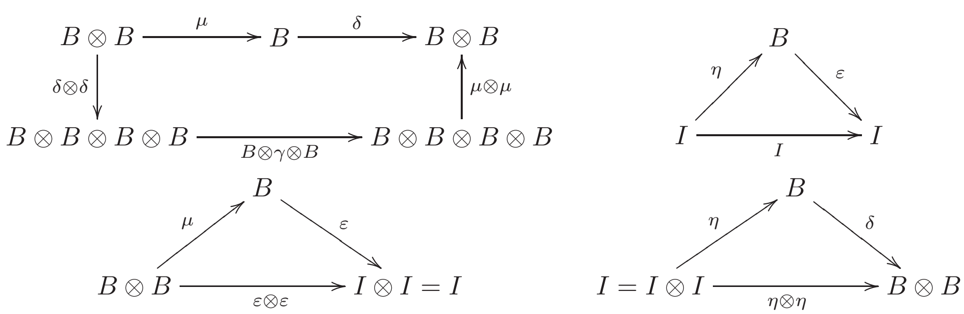
A *bialgebra* $(B, \mu, \eta, \delta, \varepsilon, \gamma)$ in $\mathcal{C}$ is an object B together with four morphisms

$$\begin{array}{rcl} \mu & : & B \otimes B \to B \\ \eta & : & I \to B \\ \delta & : & B \to B \otimes B \\ \varepsilon & : & B \to I \\ \gamma & : & B \otimes B \to B \otimes B \end{array}$$

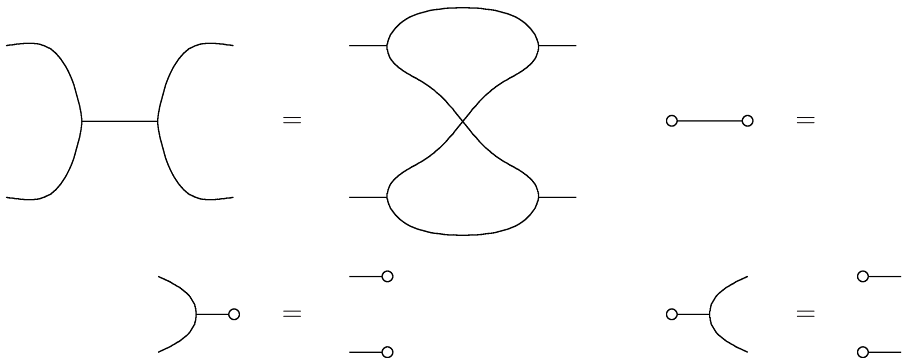
respectively drawn as



such that $\gamma : B \otimes B \to B \otimes B$ is a symmetry for B, (B, μ, η, γ) is a symmetric monoid and $(B, \delta, \varepsilon, \gamma)$ is a symmetric comonoid. Those two structures should be coherent, in the sense that the diagrams



should commute. Graphically,



A morphism of bialgebras of $\mathcal{C}$

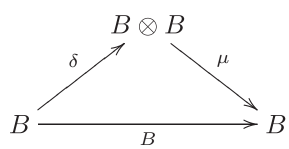
$$f : (A, \mu_A, \eta_A, \delta_A, \varepsilon_A, \gamma_1) \to (B, \mu_B, \eta_B, \delta_B, \varepsilon_B, \gamma_B)$$

is a morphism $f : A \to B$ of $\mathcal{C}$ preserving the structure of bialgebra, that is

$$(f \otimes f) \circ \mu_A = \mu_B \circ f, \qquad f \circ \eta_A = \eta_B, \qquad \text{etc.}$$

The symmetric bialgebra is *commutative* (resp. *cocommutative*) when the induced symmetric monoid (B, μ, η, γ) (resp. symmetric comonoid $(B, \delta, \varepsilon, \gamma)$) is commutative (resp. cocommutative), and *bicommutative* when it is both commutative and cocommutative.

A *qualitative bialgebra* is a bialgebra $(B, \mu, \eta, \delta, \varepsilon, \gamma)$ such that the diagram



commutes. Graphically,

A bialgebra B in a symmetric monoidal category is a bialgebra whose symmetry morphism γ corresponds with the symmetry of the category $\gamma_{B,B}$.

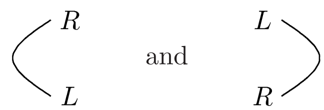
Similarly to what has been explained for monoids in Section 2.3, an equational theory of bialgebras, etc. can be defined. We write $\mathfrak{B}$ for the equational theory of bicommutative bialgebras and $\mathfrak{R}$ for the equational theory of bicommutative qualitative bialgebras.

2.5. *Dual objects*

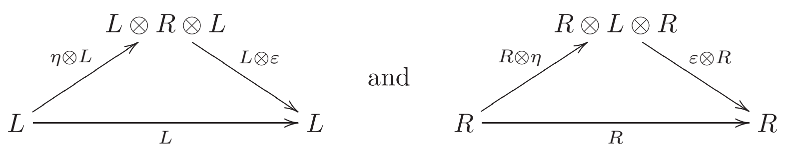
An object L of $\mathcal{C}$ is said to be *left dual* to an object R when there exists two morphism

$$\eta : I \to R \otimes L \quad \text{and} \quad \varepsilon : L \otimes R \to I$$

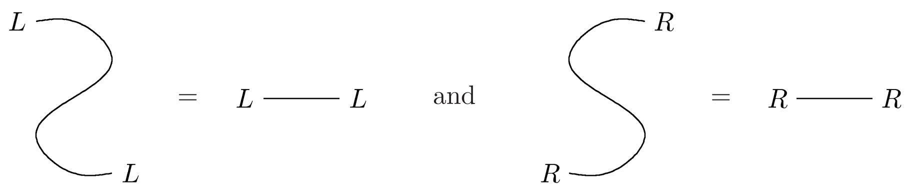
called respectively the *unit* and the *counit* of the duality and respectively pictured as



making the diagrams



commute. Graphically,



We write $\mathfrak{D}$ for the equational theory associated to dual objects and $\mathcal{D}$ for the generated monoidal category.

If $\mathcal{C}$ is category, two dual objects in the monoidal category $\mathrm{End}(\mathcal{C})$ of endofunctors of $\mathcal{C}$, with tensor product given on objects by composition of functors, are adjoint endofunctors of $\mathcal{C}$. More generally, the theory of adjoint functors in a 2-category is given in (Schanuel and Street(1986)), the definition of $\mathfrak{D}$ is a specialization of this construction to the case where there is only one 0-cell.

3. A presentation of relations

We now introduce a presentation for the category **Rel** of finite ordinals and relations. This result is mentioned in Examples 6 and 7 of (Hyland and Power(2000)) and is proved in three different ways in (Lafont(1995)), (Pirashvili(2002)) and (Lack(2004)). The proof we give here has the advantage of being simple to check and can be extended to give a presentation of the category of games and strategies, see Section 4.4.

3.1. *The simplicial category*

The simplicial category Δ is the strict monoidal category whose objects are the finite ordinals and whose morphisms $f : \underline{m} \to \underline{n}$ are the monotone functions from $\underline{m}$ to $\underline{n}$.

It has been known for a long time that this category is closely related to the notion of monoid, see (MacLane(1971)) or (Lafont(2003)) for example. This result can be formulated as follows:

Property 7. The monoidal category Δ is presented by the equational theory of monoids $\mathfrak{M}$.

In this sense, the simplicial category Δ impersonates the notion of monoid.

Dually, the monoidal category Δ^{op}, which is isomorphic to the category of finite ordinals and (weakly) monotonic functions $f : \underline{m} \to \underline{n}$ such that $f(0) = 0$, impersonates the notion of comonoid:

Property 8. The monoidal category Δ^{op} is presented by the equational theory of comonoids.

In the next Section, we show how to extend these results to the monoidal category of multirelations.

3.2. *Multirelations*

A *multirelation* R between two sets A and B is a function from $A \times B \to \mathbb{N}$. It can be equivalently be seen as a multiset whose elements are in $A \times B$, or as a matrix over $\mathbb{N}$, or as a span

$$\begin{array}{ccc} & R & \\ {}^{s}\swarrow & & \searrow^{t} \\ A & & B \end{array}$$

in the category **Set** – for the latest case, the multiset representation can be recovered from the span by

$$R(a,b) \quad = \quad |\{\ e \in R \quad | \quad s(e) = a \text{ and } t(e) = b\ \}|$$

for every element $(a,b) \in A \times B$. If $R_1 : A \to B$ and $R_2 : B \to C$ are two multirelations, their composition is defined by

$$R_2 \circ R_1 \quad = \quad (a,c) \mapsto \sum_{b \in B} R_1(a,b) \times R_2(b,c).$$

Again, this corresponds to the usual composition of matrices if we see R_1 and R_2 as matrices over $\mathbb{N}$, and as the span obtained by computing the pullback

$$\begin{array}{ccccccc} & & & R_2 \circ R_1 & & & \\ & & \swarrow & & \searrow & & \\ & R_1 & & & & R_2 & \\ {}^{s_1}\swarrow & & \searrow^{t_1} & & {}^{s_2}\swarrow & & \searrow^{t_2} \\ A & & & B & & & C \end{array}$$

if we see R_1 and R_2 as spans in **Set**.

The cardinal $|R|$ of a multirelation $R : A \to B$ is defined by

$$|R| \quad = \quad \sum_{(a,b) \in A \times B} R(a,b).$$

We write **MRel** for the monoidal theory of multirelations: its objects are finite ordinals and morphisms are multirelations between them. It is a strict symmetric monoidal category with the tensor product $\otimes$ defined on two morphisms $R_1 : \underline{m_1} \to \underline{n_1}$ and $R_2 : \underline{m_2} \to \underline{n_2}$ by

$$R_1 \otimes R_2 = R_1 \cup R_2 : \underline{m_1} + \underline{m_2} \to \underline{n_1} + \underline{n_2}$$

and the morphisms

$$R^{\gamma}_{\underline{m},\underline{n}} = (\underline{m} \times \underline{n}) \cup (\underline{n} \times \underline{m}) : \underline{m} + \underline{n} \to \underline{n} + \underline{m}$$

as symmetry. In particular, the following multirelations are morphisms in **MRel**:

$$\begin{array}{rcl} R^{\mu} = (i,j) \mapsto 1 & : & \underline{2} \to \underline{1} \\ R^{\eta} = (i,j) \mapsto 1 & : & \underline{0} \to \underline{1} \\ R^{\delta} = (i,j) \mapsto 1 & : & \underline{1} \to \underline{2} \\ R^{\varepsilon} = (i,j) \mapsto 1 & : & \underline{1} \to \underline{0} \\ R^{\gamma} = (i,j) \mapsto \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{otherwise.} \end{cases} & : & \underline{2} \to \underline{2} \end{array}$$

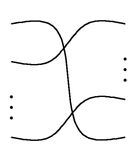
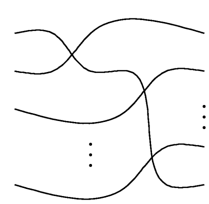
We now show that multirelations are presented by the equational theory $\mathfrak{B}$ bicommutative bialgebras. We write $\mathcal{B}$ for the monoidal category generated by $\mathfrak{B}$.

Lemma 9. In **MRel**, $(1, R^{\mu}, R^{\eta}, R^{\delta}, R^{\varepsilon})$ is a bicommutative bialgebra.

For every morphism $\phi : m \to n$ in $\mathcal{B}$, where $m > 0$, we define a morphism $S\phi : m+1 \to n$ by

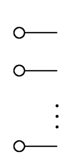
$$S\phi \quad = \quad \phi \circ (\gamma \otimes \mathrm{id}_{m-1})$$

We introduce the following notation which is defined inductively by

 is either or 

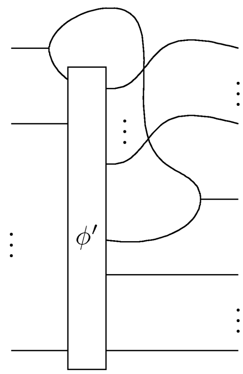
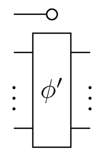
These morphisms are called *stairs*: a stair is therefore either id_1 or $S\phi'$ where ϕ' is a stairs. The *length* of a stairs is defined as 0 if its of the first form and the length of the stairs plus one if it is of the second form.

We define the following notion of *canonical form* inductively: ϕ is either



(7)

or there exists a canonical form ϕ' such that ϕ is either

$D_i\phi' \quad =$  or $\quad E\phi' \quad =$ 

In the latter case we write respectively ϕ as $D_i\phi'$ (where the index i is the length of the stairs) or as $E\phi'$.

Showing that identities are equal to canonical forms require the slightly more general following lemma.

Lemma 10. Any morphism $f = \eta \otimes \cdots \otimes \eta \otimes \mathrm{id}_n : m \to m$ is equal to a canonical form.

Proof. By induction on n. The result is immediate when $n = 0$. Otherwise, we have the equalities of Figure 2 in Appendix which show that f is equal to a morphism of the form $D_i(E\phi)$, where ϕ is equal to a canonical form by induction hypothesis. □

Lemma 11. For every morphism $\phi : m \to n$, where $m > 0$, for all indices i and j such that $0 \le i \le n$ and $0 \le j \le n$, we have

$$D_j(D_i\phi) \quad = \quad D_i(D_j\phi)$$

Proof. The proof is done by examining separately the cases $j < i$, $i \ne j$ and $j > i$ and showing the result for each case using in particular the derivable equalities shown in Figure 3 in Appendix. □

From this we deduce that

Lemma 12. Every multirelation $R : m \to n$ is represented by a canonical form and two canonical forms representing R are equal.

Proof. This is proved by induction on m and on the cardinal $|R|$ of R.

1 If $m = 0$ then R is represented by a unique normal form which is of the form (7).

2 If $m > 0$ and for every $j < n$, $R(0, j) = 0$ then R is of the form $R = R^\varepsilon \otimes R'$ and R is necessarily represented by a canonical form $E\phi'$ where ϕ' is a canonical form representing $R' : (m-1) \to n$, which exists by induction hypothesis.

3 Otherwise, R is necessarily represented by a canonical form of the form $D_k\phi'$, where k is such that $R(0, k) > 0$ and ϕ' is a canonical form represented by the relation

$R' : m \to n$ defined by

$$R'(i,j) = \begin{cases} R(i,j) - 1 & \text{if } i = 0 \text{ and } j = k, \\ R(i,j) & \text{else.} \end{cases}$$

and such a canonical form exists by induction hypothesis.

By Lemma 11, two canonical forms ϕ_1 and ϕ_2 representing R, obtained by choosing different values for k in case 3 during the construction of the canonical form are equal. □

Lemma 13. Every morphism $f : m \to n$ in $\mathcal{B}$ is equal to a canonical form.

Proof. The proof is done by induction on the size $|f|$ of f.

— If $|f| = 0$ then $m = n$ and $f = \mathrm{id}_m$ which is equal to a canonical form by Lemma 10.
— If $|f| > 0$ then f is of the form $f = h \circ g$ where $|h| = 1$ and $|g| = |f| - 1$. By induction hypothesis, g is equal to a canonical form ϕ. Since h is of size 1, it is of the form $h = \mathrm{id}_{m_2} \circ h' \circ \mathrm{id}_{m_1}$ where h is either μ, η, δ, ε or γ. We show the result by case analysis. For the lack of space, we only detail the case where $h' = \mu$. There are four cases to handle which are shown in Figures 4, 5, 6 and 7.

□

Theorem 14. The category **MRel** of multirelations is presented by the equational theory $\mathfrak{B}$ of bicommutative bialgebras.

3.3. *Relations*

The monoidal category **Rel** has finite ordinals as objects and relations as morphisms. This category can be obtained from **MRel** by quotienting the morphisms by the equivalence relation $\sim$ on multirelations defined as follows. Two multirelations $R_1, R_2 : m \to n$ are such that $R_1 \sim R_2$ whenever

$$\forall i < m, \forall j < n, \quad R_1(i,j) \neq 0 \text{ iff } R_2(i,j) \neq 0 \tag{8}$$

This induces a full monoidal functor F from **MRel** to **Rel**. We still write R^μ, R^η, R^δ, R^ε and R^γ for the images by this functor of the corresponding multirelations. We denote $\mathcal{R}$ for the monoidal category generated by the equational theory $\mathfrak{R}$ of qualitative bicommutative bialgebras.

Lemma 15. For every morphism $\phi : m \to n$ in $\mathcal{R}$, where $m > 0$, for every index i such that $0 \leq i \leq n$, we have

$$D_i(D_i\phi) \quad = \quad D_i\phi$$

Proof. See Figure 8 in Appendix. □

From this Lemma, we deduce that:

Theorem 16. The category **Rel** of relations is presented by the equational theory $\mathfrak{R}$ of qualitative bicommutative bialgebras.

Proof. Since **Rel** can be obtained from **MRel** by quotienting morphisms, by Lemma 9, $(1, R^\mu, R^\eta, R^\delta, R^\varepsilon, R^\gamma)$ is still a bialgebra in **Rel** and moreover it satisfies the additional equation making it a qualitative bialgebra. Therefore **Rel** is a model of the equational theory $\mathfrak{R}$. Moreover, $\mathfrak{R}$ is a complete axiomatization of **Rel**. In order to show this, we use the same notion of canonical form as in the previous Section: we have to show that two canonical forms representing the same relation are equal. This amounts to check that two canonical forms representing two multirelations R_1 and R_2, which are equivalent by the relation (8), are equal. This can easily be done using Lemmas 11 and 15. □

4. A game semantics for first-order propositional logic

4.1. *First-order propositional logic*

Suppose that we are given a fixed first-order language $\mathcal{L}$, that is

1 a set of proposition symbols $P, Q, \ldots$ with given arities,
2 a set of function symbols $f, g, \ldots$ with given arities,
3 a set of first-order variables $x, y, \ldots$.

Terms t and *formulas* A are respectively generated by the following grammars:

$$t \quad ::= \quad x \quad | \quad f(t, \ldots, t) \qquad\qquad A \quad ::= \quad P(t, \ldots, t) \quad | \quad \forall x.A \quad | \quad \exists x.A$$

We suppose that application of propositions and functions always respect arities. Moreover, we suppose here that there are proposition and function symbols of any arity (this is needed for the definability result of Proposition 19). Formulas are considered modulo renaming of variables. Substitution $A[t/x]$ of a free variable x by a term t in a formula A is defined as usual, avoiding capture of variables. We consider the logic associated to these formulas, where proofs are generated by the following inference rules:

$$\frac{A[t/x] \vdash B}{\forall x.A \vdash B}(\forall\text{-L}) \qquad\qquad \frac{A \vdash B}{A \vdash \forall x.B}(\forall\text{-R})$$
(with x not free in A)

$$\frac{A \vdash B}{\exists x.A \vdash B}(\exists\text{-L}) \qquad\qquad \frac{A \vdash B[t/x]}{A \vdash \exists x.B}(\exists\text{-R})$$
(with x not free in B)

$$\frac{}{P(t_1, \ldots, t_n) \vdash P(t_1, \ldots, t_n)}(\text{Ax}) \qquad \frac{A \vdash B \qquad B \vdash C}{A \vdash C}(\text{Cut})$$

4.2. *Games and strategies*

Definition 17. A *game* $A = (M_A, \lambda_A, \leq_A)$ consists of a set of moves M_A, a polarization function $\lambda_A : M_A \to \{-1, +1\}$ which to every move m associates its polarity, and a partial order $\leq_A$ on moves such that every move $m \in M_A$ defines a finite downward closed set

$$m{\downarrow} \quad = \quad \{\, n \in M_A \quad | \quad n \leq_A m \,\}.$$

A move m is said to be a Proponent move when $\lambda_A(m) = +1$ and an Opponent move else.

Suppose that A and B are two games. Their tensor product $A \otimes B$ is defined by

$$M_{A\otimes B} = M_A \uplus M_B, \quad \lambda_{A\otimes B} = \lambda_A + \lambda_B \qquad \text{and} \qquad \leq_{A\otimes B} = \leq_A \cup \leq_B \ .$$

The opposite game A^* of the game A is defined by

$$A^* = (M_A, -\lambda_A, \leq_A).$$

Finally, the arrow game $A \multimap B$ is defined by

$$A \multimap B \quad = \quad A^* \otimes B.$$

A game A is *filiform* when the associated partial order is total.

Two partial orders $\leq$ and $\leq'$ on a set M are *compatible* when their relational union $\leq \cup \leq'$ is still an order (i.e. is acyclic).

Definition 18. A *strategy* σ on a game A is a partial order $\leq_\sigma$ on the moves of A which is compatible with the order of the game and is moreover such that for every moves $m, n \in M_A$,

$$m <_\sigma n \quad \text{implies} \quad \lambda_A(m) = -1 \text{ and } \lambda_A(n) = +1. \tag{9}$$

The *size* $|A|$ of a game A is the cardinal of M_A and the *size* $|\sigma|$ of a strategy $\sigma : A$ is the cardinal of the set

$$\{ \ (m, n) \in M_A \times M_A \quad | \quad m <_\sigma n \ \}.$$

If $\sigma : A \multimap B$ and $\tau : B \multimap C$ are two strategies, their composite $\tau \circ \sigma : A \multimap C$ is the partial order $\leq_{\tau\circ\sigma}$ on the moves of $A \multimap C$, defined as the restriction to the set of moves of $A \multimap C$ of the transitive closure of the union $\leq_\sigma \cup \leq_\tau$ of the partial orders $\leq_\sigma$ and $\leq_\tau$ considered as relations. The identity strategy $\mathrm{id}_A : A \multimap A$ on a game A is the strategy such that for every move m of A we have $m_L \leq_{\mathrm{id}_A} m_R$ if $\lambda(m) = -1$ and $m_R \leq_{\mathrm{id}_A} m_L$ if $\lambda(m) = +1$ when m_L (resp. m_R) is the instance of the move m in the left-hand side (resp. right-hand side) copy of A. It can easily be checked that for every strategy $\sigma : A \to B$ we have $\mathrm{id}_B \circ \sigma = \sigma = \sigma \circ \mathrm{id}_A$.

Since the composition of strategies is defined in the category of relations, we still have to check that the composite of two strategies σ and τ is actually a strategy. Preservation of the polarization condition (9) by composition is easily checked. However, proving that the relation $\leq_{\tau\circ\sigma}$ corresponding to the composite strategy is acyclic is more difficult: a direct proof of this property is combinatorial and a bit lengthy. For now, we define the category **Games** as the smallest category whose objects are filiform games, whose morphisms between two games A and B contain the strategies on the game $A \multimap B$ and is moreover closed under composition. We will deduce in Corollary 28 that strategies are in fact the only morphisms of this category from our presentation of the category.

If A and B are two games, the game $A \oslash B$ (to be read A *before* B) is the game defined by

$$M_{A\oslash B} = M_A \uplus M_B, \quad \lambda_{A\oslash B} = \lambda_A + \lambda_B$$

and $\leq_{A \oslash B}$ is the transitive closure of the relation

$$\leq_A \cup \leq_B \cup \{\ (a,b) \quad | \quad a \in M_A \text{ and } b \in M_B\ \}$$

This operation is extended as a bifunctor on strategies as follows. If $\sigma : A \to B$ and $\tau : C \to D$ are two strategies, the strategy $\sigma \oslash \tau : A \oslash C \to B \oslash D$ is defined as the transitive closure of the relation

$$\leq_{\sigma \oslash \tau} \quad = \quad \leq_\sigma \cup \leq_\tau$$

This bifunctor induces a monoidal structure $(\mathbf{Games}, \oslash, I)$ on the category **Games**, where I denotes the empty game.

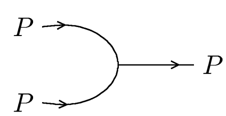
We write O for a game with only one Opponent move and P for a game with only one Proponent move. It can be easily remarked that filiform games A are generated by the following grammar

$$A \qquad ::= \qquad I \quad | \quad O \oslash A \quad | \quad P \oslash A$$

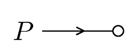
A game $X_1 \oslash \cdots \oslash X_n \oslash I$ where the X_i are either O or P is represented graphically as

$$\begin{array}{c} X_1 \\ \vdots \\ X_n \end{array}$$

A strategy $\sigma : A \to B$ is represented graphically by drawing a line from a move m to a move n whenever $m \leq_\sigma n$. For example, the strategy $\mu^P : P \oslash P \to P$



is the strategy on $(O \oslash O) \otimes P$ in which both Opponent move of the left-hand game justify the Proponent move of the right-hand game. When a move does not justify (or is not justified by) any other move, we draw a line ended by a small circle. For example, the strategy $\varepsilon^P : P \to I$ drawn as



is the unique strategy from P to I.

With these conventions, we introduce notations for some morphisms which are depicted in Figure 1 (perhaps a bit confusingly, the tensor product $\otimes$ on this figure is the $\oslash$ tensor).

4.3. *A game semantics for proofs*

A formula A is interpreted as a game $[\![A]\!]$ by

$$[\![P]\!] = I \qquad [\![\forall x.A]\!] = O \oslash [\![A]\!] \qquad [\![\exists x.A]\!] = P \oslash [\![A]\!]$$

$$
\begin{array}{lclclclcl}
\mu^O & : & O \otimes O \to O & \mu^P & : & P \otimes P \to P & \eta^{OP} & : & I \to O \otimes P \\
\eta^O & : & I \to O & \eta^P & : & I \to P & & & \\
\delta^O & : & O \to O \otimes O & \delta^P & : & P \to P \otimes P & \varepsilon^{OP} & : & P \otimes O \to I \\
\varepsilon^O & : & O \to I & \varepsilon^P & : & P \to I & & & \\
\gamma^O & : & O \otimes O \to O \otimes O & \gamma^P & : & P \otimes P \to P \otimes P & \gamma^{OP} & : & P \otimes O \to O \otimes P
\end{array}
$$

respectively drawn as

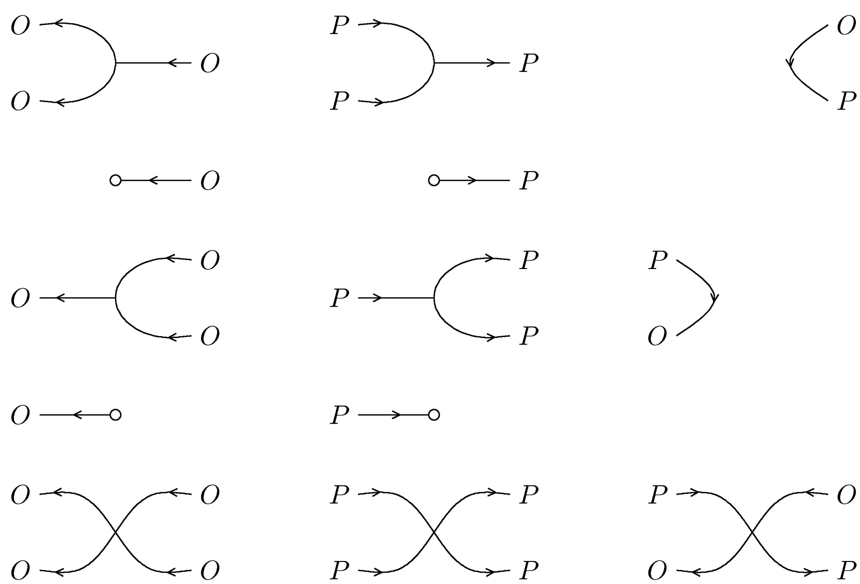


Figure 1. Generators of the strategies.

A proof $\pi : A \vdash B$ is interpreted as the strategy $\sigma : A \multimap B$. The corresponding partial order $\leq_\sigma$ is defined as follows. For every Proponent move P interpreting a quantifier which is introduced by a rule

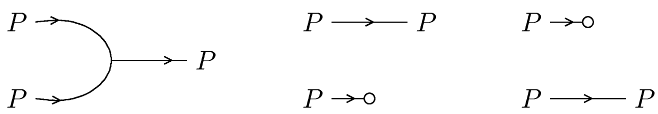
$$
\frac{A[t/x] \vdash B}{\forall x.A \vdash B}(\forall\text{-L}) \qquad \text{or} \qquad \frac{A \vdash B[t/x]}{A \vdash \exists x.B}(\exists\text{-R})
$$

every Opponent move O interpreting an universal quantification $\forall x$ on the right-hand side of a sequent, or an existential quantification $\exists x$ on the left-hand side of a sequent, is such that $O \leq_\sigma P$ whenever the variable x is free in the term t. The partial order interpreting a proof π can easily be shown to be a strategy.

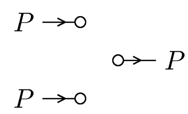
For example, a proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash Q[t/z]}(\text{Ax})}{P \vdash \exists z.Q}(\exists\text{-R})}{\exists y.P \vdash \exists z.Q}(\exists\text{-L})}{\exists x.\exists y.P \vdash \exists z.Q}(\exists\text{-L})
$$

is interpreted respectively by the strategies



and



when the free variables of t are $\{x, y\}$, $\{x\}$, $\{y\}$ and $\emptyset$.

The following Proposition shows that our game semantics contains only definable strategies.

Proposition 19. For every strategy $\sigma : A \to B$ in **Games**, there exists two propositions P and Q such that $A = [\![\sqcap_1 \dots \sqcap_k P]\!]$, $B = [\![\sqcap_1 \dots \sqcap_l Q]\!]$ and there exists a proof $\pi : \sqcap_1 \dots \sqcap_k P \vdash \sqcap'_1 \dots \sqcap'_l Q$ such that $[\![\pi]\!] = \sigma$, where $\sqcap_i$ and $\sqcap'_i$ is either $\forall$ or $\exists$.

4.4. *An equational theory of strategies*

Definition 20. The *equational theory of strategies* is the equational theory $\mathfrak{G}$ with two types O and P and 13 generators depicted in Figure 1 such that

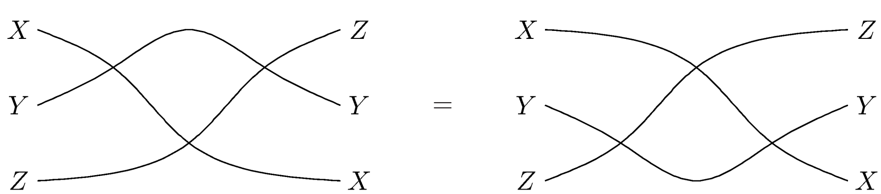
— $(O, \mu^O, \eta^O, \delta^O, \varepsilon^O, \gamma^O)$ is a bicommutative qualitative bialgebra,
— the Proponent structure is adjoint to the Opponent structure in the sense that the equations of Figure 9 hold.

We write $\mathcal{G}$ for the monoidal category generated by $\mathfrak{G}$.

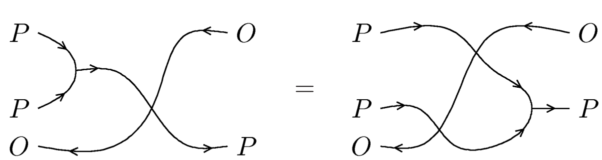
Remark 21. The generators μ^P, η^P, δ^P, ε^P, γ^P and γ^{OP} are superfluous in this presentation. However, removing them would seriously complicate the proofs.

Lemma 22. With the notations of 20, we have:

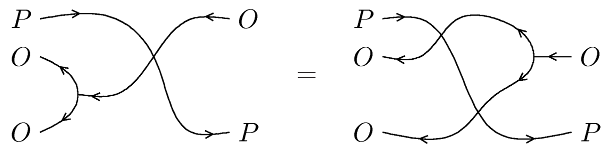
— $(P, \mu^P, \eta^P, \delta^P, \varepsilon^P, \gamma^P)$ is a qualitative bicommutative bialgebra,
— the Yang-Baxter equalities



hold whenever (X, Y, Z) is either (O, O, O), (P, O, O), (P, P, O) or (P, P, P),
— the equalities

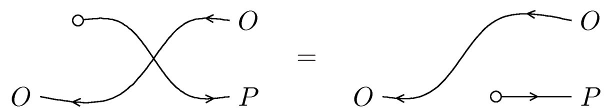


and



hold (and dually for comultiplications),

— the equalities



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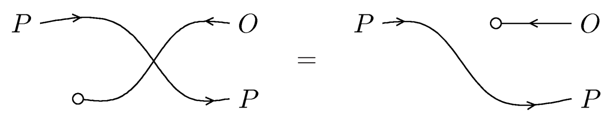
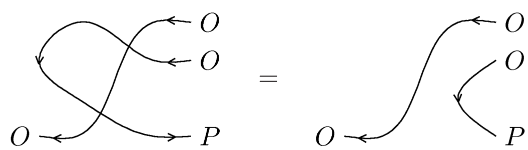
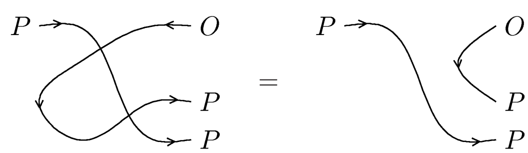


hold (and dually for counits),

— the equalities



and

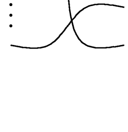


hold (and dually for the counit of duality).

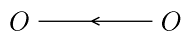
Property 23. In the category **Games** with the monoidal structure induced by $\olessthan$, the games O and P together with the morphisms introduced at the end of Section 4.2 induce a strategy structure in the sense of Definition 20.

We extend the proofs of Section 3 to show that $\mathfrak{G}$ is a presentation of the category **Games**.

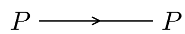
Stairs are defined inductively by



is either



or



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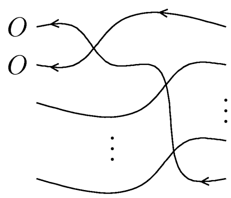
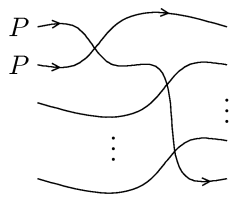


or



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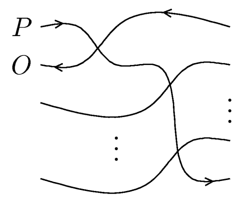
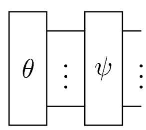
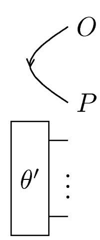


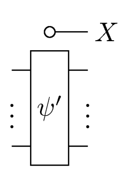
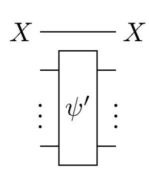
A *canonical form* is either of the form $\phi = \psi \circ \theta$

ϕ is  (10)

where a morphism of the form θ is defined inductively by

θ is either void or  (11)

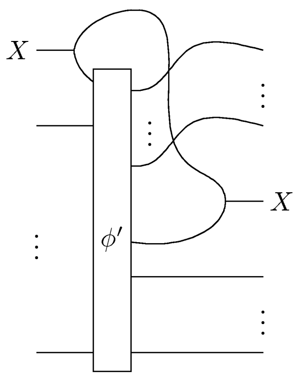
where θ' is of the form (11), and ψ is defined inductively by

ψ is either void or  or  (12)

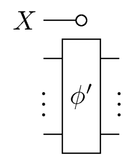
where X is either P or O and ψ' is of the form (12), or there exists a canonical form ϕ'

such that ϕ is either

$D_i^X\phi' \quad =$

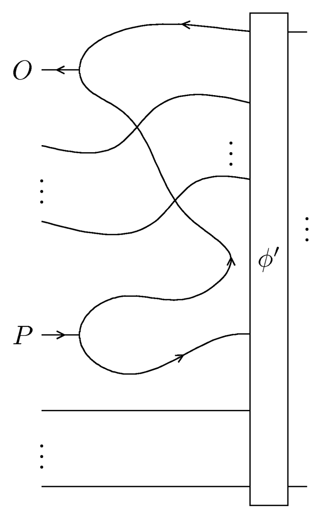


or $\quad E^X\phi' \quad =$



or

$A_i\phi' \quad =$



where X is either P or O. In the latter case, we write respectively ϕ as $D_i^X\phi'$ (where i is the length of the stairs), or as $E^X\phi'$ or as $A_i\phi'$ (where i is the length of the stairs).

Lemma 24. For any morphism ϕ, we have

$$\begin{array}{rclcrcl} D_j^X(D_i^X\phi) & = & D_i^X(D_j^X\phi) & \qquad & A_i(A_j\phi) & = & A_j(A_i\phi) \\ D_i^X(D_i^X\phi) & = & D_i^X\phi & & D_i^O(A_j\phi) & = & A_j(D_i^O\phi) \end{array}$$

whenever both members of the equalities are defined, where X is either P or O.

Lemma 25. Every strategy $\sigma : A \to B$ is represented by a canonical form and two canonical forms representing the same strategy are equal.

Proof. This is proved by induction on the respective sizes $|A|$ and $|\sigma|$ of A and σ.

1 If $|A| = 0$ then σ is necessarily represented by canonical form of the form (10), which is unique.

2 If $A = X \oslash A'$, where X is either P or O and $M_X = \{m\}$, and for every move $n \in M_{A \multimap B}$ we have $m \not<_\sigma n$ then σ is necessarily represented by a canonical form $E^X \phi'$ where ϕ' is a canonical form representing the restriction of σ to $A' \multimap B$.

3 Otherwise, A is of the form $A = X \oslash A'$, where X is either P or O and $M_X = \{m\}$. A canonical form ϕ of σ is necessarily of one of the two following forms.

— $\phi = D_i^X \phi'$ where n is the i-th move of B and is such that $m <_\sigma n$, and ϕ' is a canonical form representing either the strategy σ or the strategy σ' which is the same strategy as σ excepting that $m \not<_{\sigma'} n$ – for the construction part of the lemma we obviously chose the second possibility in order for the induction to work.

— $\phi = A_i \phi'$ where n is the i-th move of A and is such that $m <_\sigma n$, and ϕ' is a canonical form representing the strategy σ' which is the same strategy as σ excepting that $m \not<_{\sigma'} n$.

By Lemma 24, two canonical forms ϕ_1 and ϕ_2 representing σ, obtained by choosing different values for n in case 3 are equal.

□

Lemma 26. Every morphism $f : A \to B$ of $\mathcal{G}$ is equal to a canonical form.

Proof. The proof is similar to the proof of Lemma 13. □

Theorem 27. The category **Games** is presented by the equational theory $\mathfrak{G}$.

As a direct consequence of this Theorem, we deduce that

Corollary 28. The composite of two strategies is a strategy.

In particular, acyclicity is preserved by composition.

5. Conclusion

We have constructed a game semantics for the fragment of first-order propositional logic without connectives and given a presentation of the category **Games** of games and definable strategies. Our methodology has proved very useful to ensure that the composition of strategies was well-defined.

We consider this work much more as a starting point to bridge semantics and algebra than as a final result. The methodology presented here seem to be very general and many tracks remain to be explored.

First, we would like to extend the presentation to a game semantics for richer logic systems like first-order propositional logic with conjunction and disjunction. Whilst we do not expect many technical complications, this case is much more difficult to grasp and manipulate since a presentation of such a semantics would be a 4-polygraph (one dimension is added since games would be trees instead of lines) and corresponding diagrams now live in a 3-dimensional space.

It would be interesting to know whether it is possible to orient the equalities in the presentations in order to obtain strongly normalizing rewriting systems for the algebraic structures described in the paper. Such rewriting systems are given in (Lafont(2003)) – for monoids and commutative monoids for example – but finding a strongly normalizing rewriting system presenting the theory of bialgebras is still an open problem.

Finally, many of the proofs given here are repetitive and we believe that many of them could be (at least partly) automated or mechanically checked. However, finding a good representation of diagrams, in order for a program to be able to manipulate them, is a difficult task that we should address in subsequent works.

Acknowledgements

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Appendix A. Figures

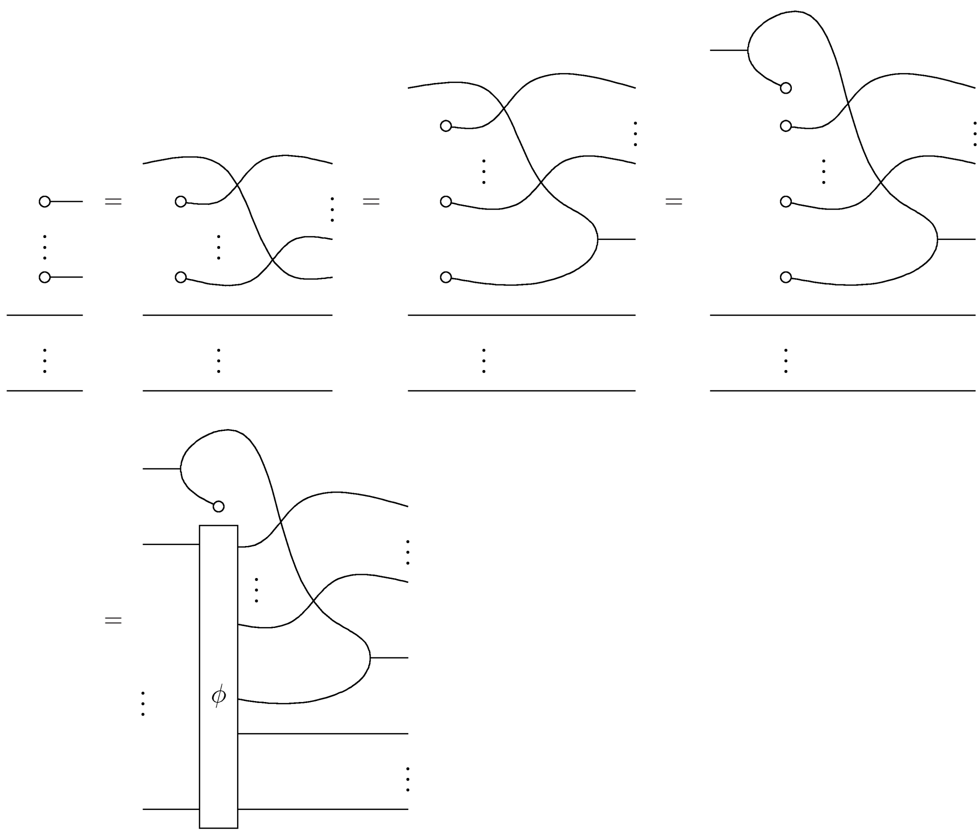


Figure 2. Induction step in proof of Lemma 10.

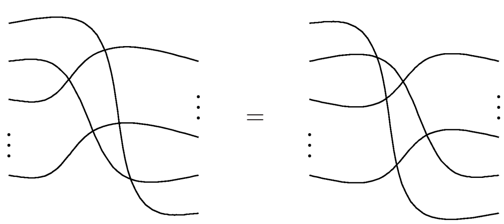
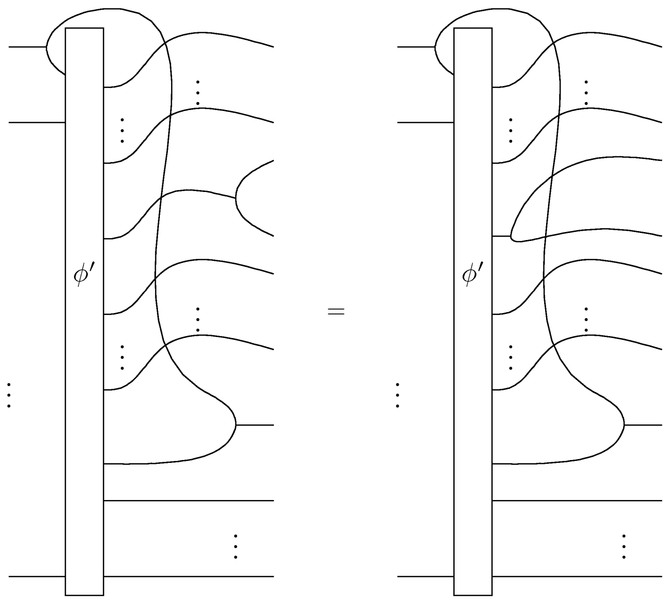


Figure 3. A generalization of the Yang-Baxter equality to stairs.

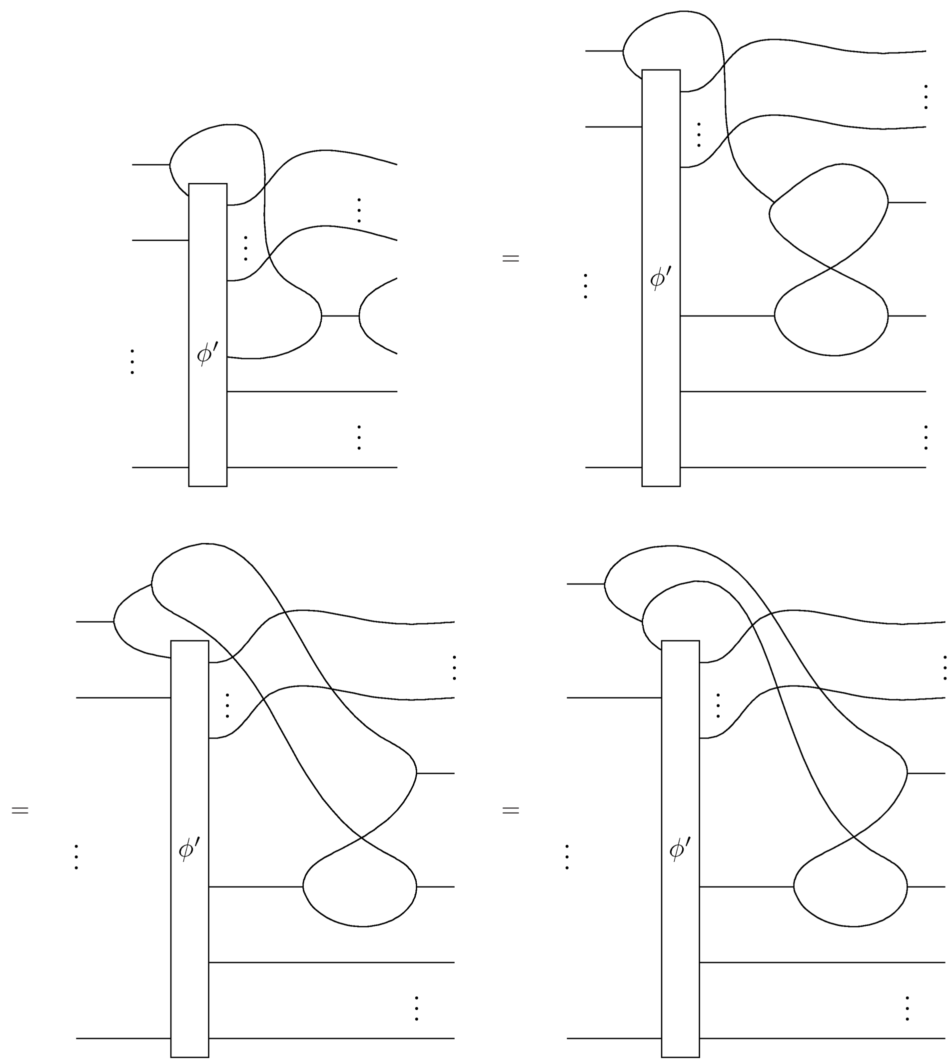
g is of the form $D_i\phi'$ and f is equal to



which is of the form $D_j\phi$ where ϕ is equal to a canonical form by induction hypothesis.

Figure 4. First case in proof of Lemma 13.

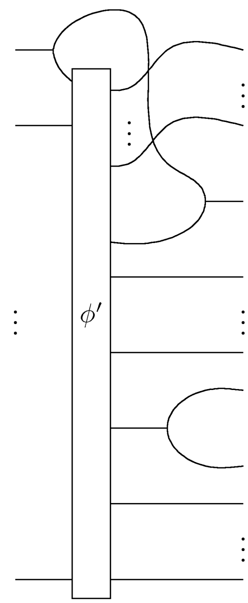
g is of the form $D_i\phi'$ and and f is equal to



which is of the form $D_j(D_k\phi)$ where ϕ is equal to a canonical form by induction hypothesis.

Figure 5. Second case in proof of Lemma 13.

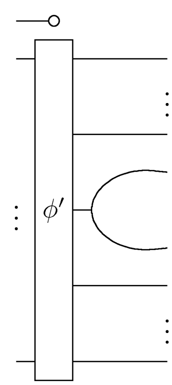
g is of the form $D_i\phi'$ and and f is equal to



which is of the form $D_j\phi$ where ϕ is equal to a canonical form by induction hypothesis.

Figure 6. Third case in proof of Lemma 13.

g is of the form $E\phi'$ and and f is equal to



which is of the form $E\phi$ where ϕ is equal to a canonical form by induction hypothesis.

Figure 7. Fourth case in proof of Lemma 13.

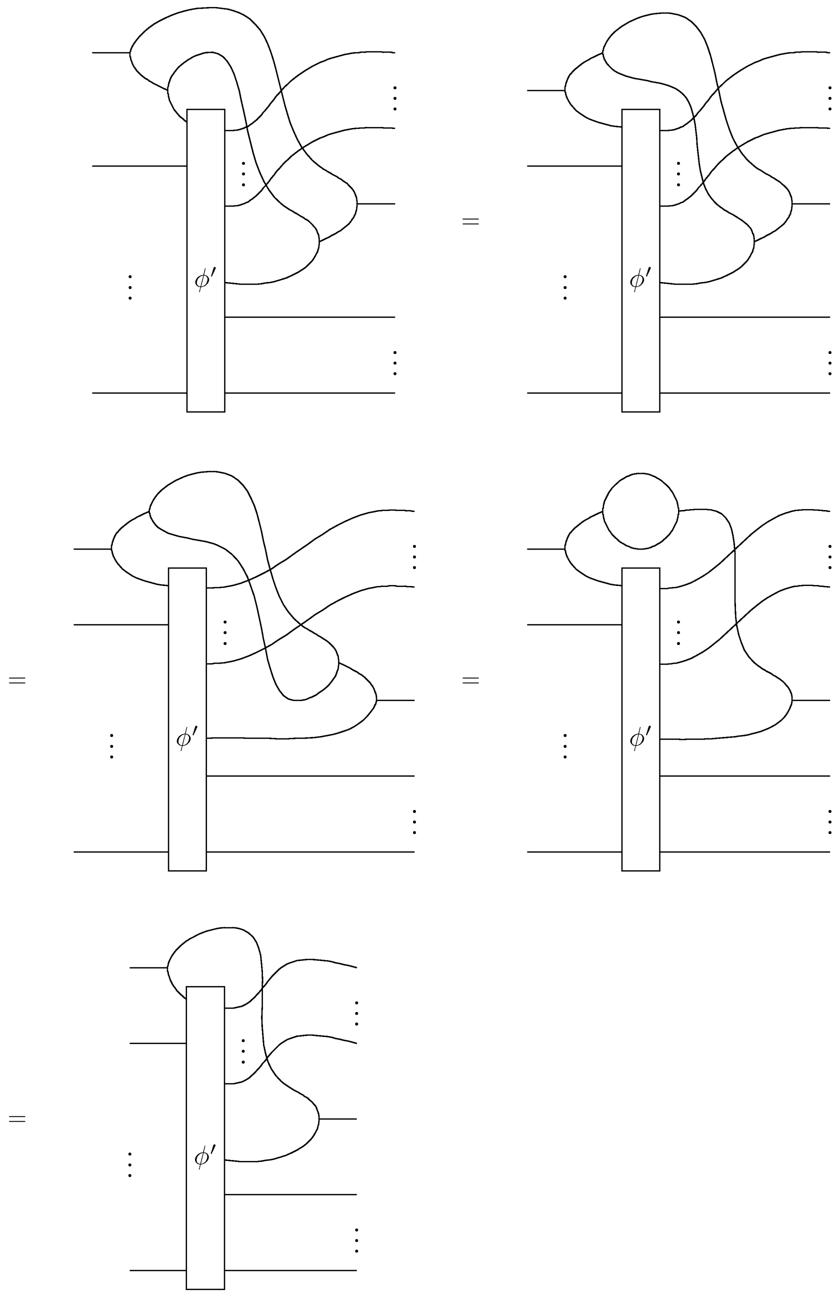


Figure 8. Proof of Lemma 15.

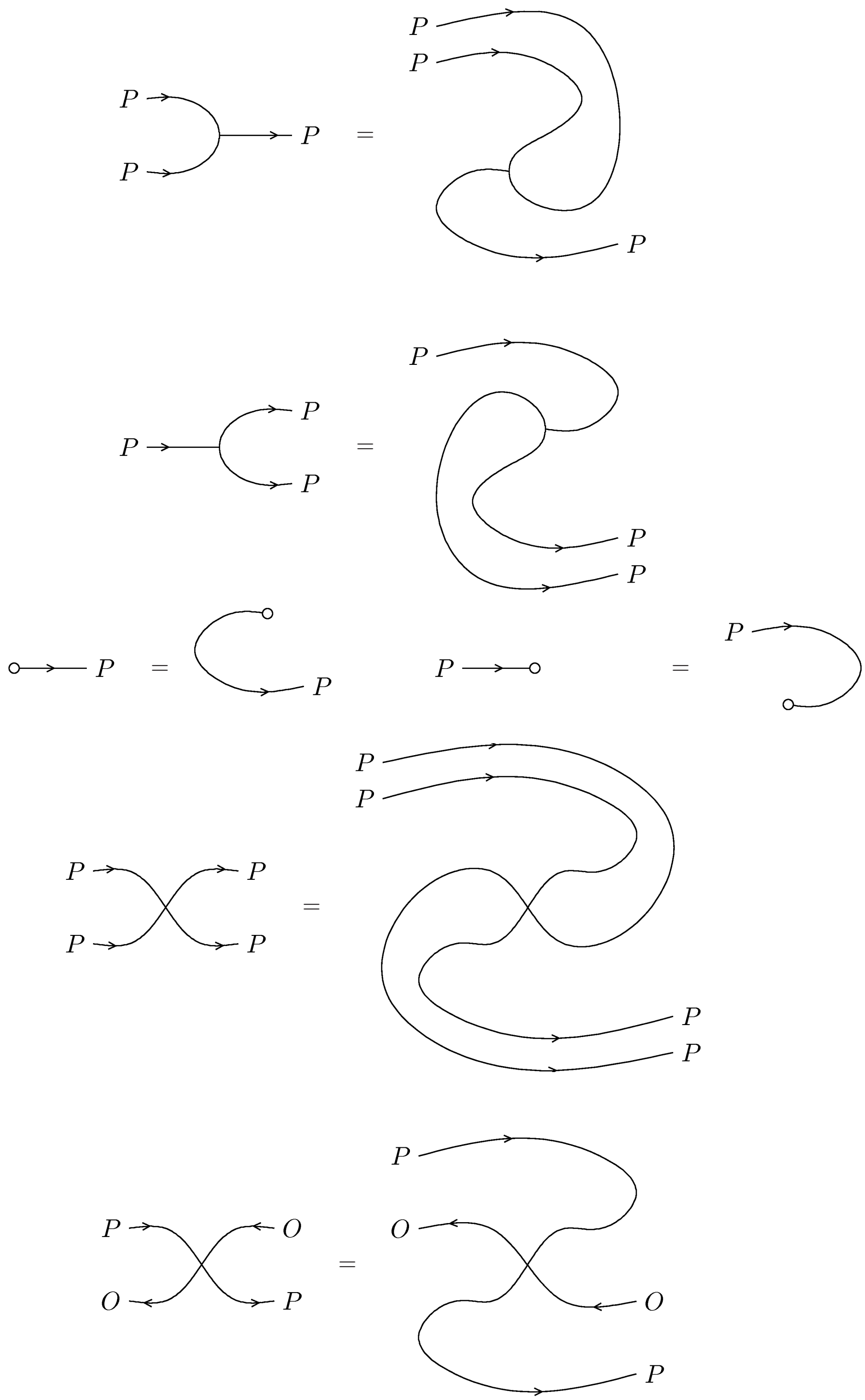


Figure 9. Proponent is left dual to Opponent.