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The Schramm-Loewner evolution (SLE) is a powerful tool to describe fractal interfaces in 2D critical statistical systems, yet the application of SLE is well established for statistical systems described by quantum field theories satisfying only conformal invariance, the so-called minimal conformal field theories (CFTs). We consider interfaces in $Z(N)$ spin models at their self-dual critical point for $N = 4$ and $N = 5$. These lattice models are described in the continuum limit by nonminimal CFTs where the role of a $Z_N$ symmetry, in addition to the conformal one, should be taken into account. We provide numerical results on the fractal dimension of the interfaces which are SLE candidates for nonminimal CFTs. Our results are in excellent agreement with some recent theoretical predictions.

*Introduction*— The description of phase transitions in terms of geometrical objects is a long-standing problem [1] which has provided a different conceptual framework to study critical phenomena. In this respect, the two dimensional (2D) systems are particularly interesting as an extensive variety of theoretical tools is available. In particular, the approach based on the so called Schramm-Loewner evolutions (SLEs), which are growth processes defined via stochastic evolution of conformal maps, has been proven an efficient tool to study fractal shapes in 2D critical statistical systems and unveiled geometrical properties of critical systems that were missing before [2, 3, 4].

The SLE approach has been applied to different problems as the critical percolation [5], the domain boundaries in magnetic systems at the phase transition [4] or the 2D turbulence [6]. The theoretical ideas behind this approach often combines the probability theory, the complex analysis and the quantum field theory. The conformal field theories (CFTs) play a key role for understanding the universal properties of 2D systems [7]. If SLEs consider directly the geometrical characterization of non-local objects, the CFTs focus on the computation of the correlation function of local variables by fully exploiting the symmetries of the system under consideration. The first solutions of CFTs, the so called minimal CFTs, were constructed by demanding the correlation functions to satisfy the conformal symmetry alone [8]. So far the SLE interfaces have been identified and studied in statistical models (critical percolation, self-avoiding walks, loop erased random walks, etc.), which are described in the continuum limit by minimal CFTs. One of the most important results is the relation between the SLEs and the minimal CFTs which has been worked out in [9, 10, 11]. Yet, there are other solutions of quantum fields theories which satisfy, in addition to the conformal symmetry, additional symmetries. These theories, called non-minimal CFTs, describe many condensed matter and statistical problems characterized in general by some internal sym-

metry such as, e.g., the $SU(2)$ spin-rotational symmetry in spin chains [12] or replica permutational symmetry in disordered systems [13, 14]. The connection between SLEs and non-minimal CFTs has been first addressed in [15, 16], where the relation between stochastic evolutions and superconformal field theory was investigated. More recently, the connection between SLE and Wess-Zumino-Witten models, i.e. CFTs with additional Lie-group symmetries, has been studied by very different approaches [17, 18]. These results concern mainly some particular properties of the CFTs under consideration which generalize the ones on which the link between SLE and minimal CFT is based. However, an interpretation in terms of the continuum limit of lattice interfaces, necessary to give the SLE a physical meaning, was missing. In this respect, an interesting model is the $Z(N)$ spin model (defined below) [19], i.e. a lattice of spins which can take $N$-values. The nearest-neighbor interaction defining the model is invariant under a cyclic permutation of the states. For $N = 2$ and $N = 3$ one finds respectively the Ising and the three-state Potts model. The phase diagrams of these $Z(N)$ spin models present self-dual critical points [20, 21, 22] described in the continuum limit by CFTs with $Z_N$ additional symmetries, the so called $Z(N)$ parafermionic theories [23]. For $N \geq 4$ the parafermionic theories are non-minimal CFTs where the role of the $Z_N$ symmetry beside the one of conformal symmetry must be taken into account (for $N = 2, 3$ these theories coincide with minimal models). In [19] the interfaces expected to be described in the continuum limit by SLE have been identified on the lattice. Further, combining CFT results with the idea, suggested in [18], of an additional stochastic motion in the internal symmetry group space, the geometric properties of these interfaces was predicted to be described by some specific SLE process. In this letter, we will investigate this model further and we will check the prediction against numerical simulations for the self dual critical $Z(4)$ and $Z(5)$ spin models. We present the first numerical results on critical interfaces on the lat-

tice which are SLE candidates for non minimal conformal field theories. Before presenting the model that we simulate and the results we give some more definitions on SLEs.

*Schramm-Loewner evolution.* Here we consider chordal SLE which describes random curves joining two boundary points of a connected planar domain. For a detailed introduction to SLE, see e.g. [2, 3, 4]. The definition of SLE is most conveniently given in the upper half complex plane $\mathbb{H}$: it describes a fluctuating self-avoiding curve $\gamma_t$ which emanates from the origin ($z = 0$) and progresses in a properly chosen time t. If $\gamma_t$ is a simple curve, this evolution is defined via the conformal map $g_t(z)$ from the domain $\mathbb{H}_t = \mathbb{H}/\gamma_{]0,t]}$, i.e. the upper half plane from which the curve is removed, to $\mathbb{H}$. In the more general case of non-simple curves, the function $g_t(z)$ produce conformal maps from $\mathbb{H}_t = \mathbb{H}/K_t$ to $\mathbb{H}$ where $K_t$ is the SLE hull at time $t$. The SLE map $g_t(z)$, where the curve parametrization $t$ is chosen so that $g_t(z) = z + 2t/z + \cdots$ near $z = \infty$, is a solution of the Loewner equation:

$$\frac{d}{dt}g_t(z) = \frac{2}{g_t(z) - \xi_t} \quad g_{t=0}(z) = z, \qquad (1)$$

where $\xi_t$ is a real valued process, $\xi_t \in \mathbb{R}$, which drives the evolution of the curve. For a system which satisfies the Markovian and conformal invariance properties, together with the left-right symmetry, the process $\xi_t$ is shown [24] to be proportional to a Brownian motion: $\mathbf{E}[\xi_t] = 0$ and $\mathbf{E}[\xi_t \xi_s] = \kappa \min(s, t)$. The symbol $\mathbf{E}[...]$ indicates the stochastic average over the Brownian motion. The SLE curves are fractal objects and their length, $S$, measured in units of lattice spacing $a$, scales as a function of the system size $L$ as $S \sim a(L/a)^{d_f}$ where $d_f$ is the fractal dimension given by:

$$d_f = 1 + \frac{\kappa}{8} . \qquad (2)$$

*The lattice model and the interface*— In this letter we consider the model defined on a square lattice with spin variable $\sigma_j = \exp i2\pi/Nn(j)$ at each site $j$ taking $N$ possible values, $n(j) = 0, 1, \cdots, N-1$. The most general $Z_N$ invariant spin model with nearest-neighbor interactions is defined by the reduced Hamiltonian [25, 26]:

$$H[n] = -\sum_{m=1}^{\lfloor N/2 \rfloor} J_m \left[ \cos\left(\frac{2\pi mn}{N}\right) - 1 \right], \qquad (3)$$

where $\lfloor N/2 \rfloor$ denotes the integer part of $N/2$. The associated partition function reads:

$$Z = \sum_{\{\sigma\}} \exp\left[ -\beta \sum_{<ij>} H[n(i) - n(j)] \right] . \qquad (4)$$

For $J_m = J$, for all $m$, one recovers the $N-$state Potts model, invariant under a permutational $S_N$ symmetry

while the case $J_m = J\delta_{m,1}$ defines the clock model [27]. For $N = 2$ and $N = 3$ these models coincide with the Ising and the three-state Potts model respectively, while the case $N = 4$ is isomorphic to the Ashkin-Teller model [28, 29]. Defining the Boltzmann weights:

$$x_n = \exp\left[-\beta H(n)\right], \quad n = 0, 1, \cdots, N-1 , \qquad (5)$$

the most general $Z_N$ spin model is then described by $\lfloor N/2 \rfloor$ independent parameters $x_n$ as $x_0 = 1$ and $x_n = x_{N-n}$. The general properties of these models for $N = 5, 6, 7$ have been studied long time ago (see e.g. [30] and references therein). The associated phase diagrams turn out to be particularly rich as they contain in general first-order, second-order and infinite-order phase transitions. For all the $Z_N$ spin models it is possible to construct a duality transformation (Kramers-Wannier duality). In the self-dual subspace of (3)-(4), which also contains the Potts and the clock model, the $Z_N$ spin model are critical and completely integrable at the points [20, 21] :

$$x_0^* = 1 \; ; \; x_n^* = \prod_{k=0}^{n-1} \frac{\sin\left(\frac{\pi k}{N} + \frac{\pi}{4N}\right)}{\sin\left(\frac{\pi(k+1)}{N} - \frac{\pi}{4N}\right)} . \qquad (6)$$

There is strong evidence that the self-dual critical points (6), referred usually as Fateev-Zamolodchikov (FZ) points, are described in the continuum limit by $Z(N)$ parafermionic theories [31]. Very recently, a further strong support for this picture has been given in [32] where the lattice candidates for the chiral currents generating the $Z_N$ symmetry of the continuum model has been constructed.

Consider now the model, at the self-dual critical point, defined on a simple connected domain. By choosing some specific boundary conditions, for each spin configuration there is a domain wall connecting two fixed points on the boundaries (see below for some specific example). In general one is interested in the conformally invariant boundary conditions which, for a given bulk CFT, represent a finite set into which, under renormalization group, any uniform boundary condition will flow [33]. The change of conformally boundary conditions at some point of the boundary is implemented in CFT by the insertion at that point of a given boundary conditions changing (b.c.c.) operator [33].

By carefully choosing the boundary conditions, the associated domain wall connecting the two points at the boundaries is then expected to be described by measures which are invariant under conformal transformation. This can be understood from the fact that the expectation values describing the curve correspond in the continuum limit to the correlation functions of the CFT with the insertion of the two b.c.c operators.

In order to establish the SLE/CFT connection, the b.c.c. operator associated to the interface have to satisfy particular relations under the action of the symmetry

generators, the so-called null state condition. In [19] the existence of such operators in the $Z(N)$ parafermionic was pointed out. One of these b.c.c. operator, inserted at a point $x_0$, generate the condition where the spins are fixed to (say) the value $A$ on the left side of $x_0$ while they can take the other $N-1$ values $B, C, ..$ with equal probability on the right side (in the following we indicate the possible values of the spins with the letters $A, B \cdots$). Interpreting the b.c.c. null state condition via the introduction of an additional stochastic motion in the $Z_N$ internal space independent from (1), the geometric property of the interface generated by such boundary conditions was predicted to be described for $N \geq 4$ by an SLE with $\kappa = 4(N+1)/(N+2)$ [19], thus the prediction

$$d_f = 1 + \frac{1}{2} \frac{(N+1)}{(N+2)} \; . \tag{7}$$

We will test this relation in the following.

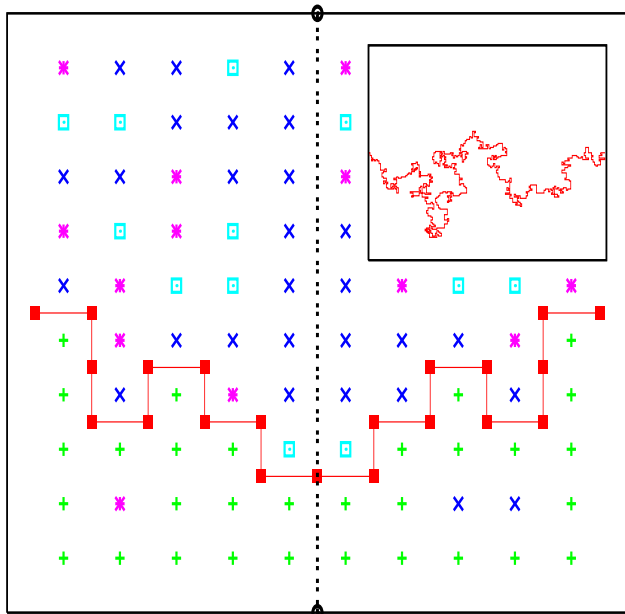*Numerical simulation* Our goal is to compute the in-



FIG. 1: Definition of the interface. The interface separetes the spins with a fixed value connected to the bottom boundary from spins with other colors. The vertical dashed line corresponds to the test of the crossing probabilities against Schramm's formula. The inset contains a typical $320 \times 320$ configuration for the $Z(5)$ parafermionic theory.

terface and check the validity of eq.(7) for the two cases $N = 4$ and $N = 5$ which are the simplest lattice models described by non-minimal conformal field theories.

We are going to compute the fractal dimension associated to the interface which crosses the lattice. To create

this interface, we impose that half of the spins on the boundary take a fixed value $A$, these spins being connected two by two, while the remaining boundary spins are forced to take values different from $A$. Then the interface will be the border of the geometric cluster of spins taking a value $A$ and connected to the spins on the boundary with fixed spins. We show an example of such a configuration in Fig. 1. In this figure, the spins with a fixed value are the ones which touch the bottom boundary. The interface is shown as the line which connects the left boundary to the right boundary. Similar conditions were considered in a recent work by Gamsa and Cardy for the $Q = 2$ and $Q = 3$ Potts model case[34] who obtained a good agreement with the prediction of the corresponding formula (2) for the Potts models. This type of boundary condition, which was called fluctuating in [34], ensures that there is a unique interface which crosses the lattice. We should also mention that on the square lattice, the definition of the interface can contain some ambiguities. There are different ways of dealing with these ambiguities but the large size results will not be affected by them[35]. For the simulation of the $Z(4)$ and $Z(5)$ model at the FZ point, we employed a standard Monte Carlo algorithm. One can also use a cluster algorithm but with the boundary conditions that we consider, it turns out to be less efficient than Monte Carlo. We performed simulations on square lattices of rectangle geometry $L_x \times L_y$, the interface being created along the $y$ direction. We simulate the size $L_x = 10, 20, 30, 40, 60, 80$ and 160 with for each size $L_y = L_x$ and $L_y = 3L_x$. For the larger linear size $L_x$ that we consider, we see very little difference between these two cases.

For $L_x = L_y$, we simulated 1 million independent configurations up to $L_x = 80$ and 200000 configurations for $L_x = 160$. For $L_y = 3L_x$, we simulated 1 million independant configurations up to $L_x = 60$, 300000 configurations for $L_x = 80$ and 50000 configurations for $L_x = 160$. The autocorrelation time grows as $\tau \simeq L^z$ with $z \simeq 2.2(1)$ for both $N = 4$ and $N = 5$ and for the two ratios $L_y/L_x$ that we considered. For the largest sizes, we obtain $\tau \simeq 8000$ for $L_y = L_x$ while $\tau \simeq 15000$ for $L_y = 3L_y$ for both $N = 4$ and $N = 5$.

Fig. 2 contains our main results. In this figure, one shows the exponent $d_f$ obtained by doing a fit of $S \simeq L_x^{d_f}$ with data in the range $L_x = [L_{min}, \cdots, 160]$. For $N = 4$, the measured value moves close the predicted value $d_f = 1 + 5/12$. The deviation for the larger size that we can measure is of order $1/4$ % and from the figure, we expect that this deviation will decrease for larger size. For $N = 5$, the agreement is already perfect for the larger sizes and for $L_y = 3L_x$. For $L_y = L_x$, there is still a small deviation (of order $1/10$ %) but again this deviation decrease while increasing the size. The fact that the agreement is better for $N = 5$ than for $N = 4$ is not surprising since the $Z(4)$ parafermionic field theory has a $c = 1$ central charge. CFTs with such a central
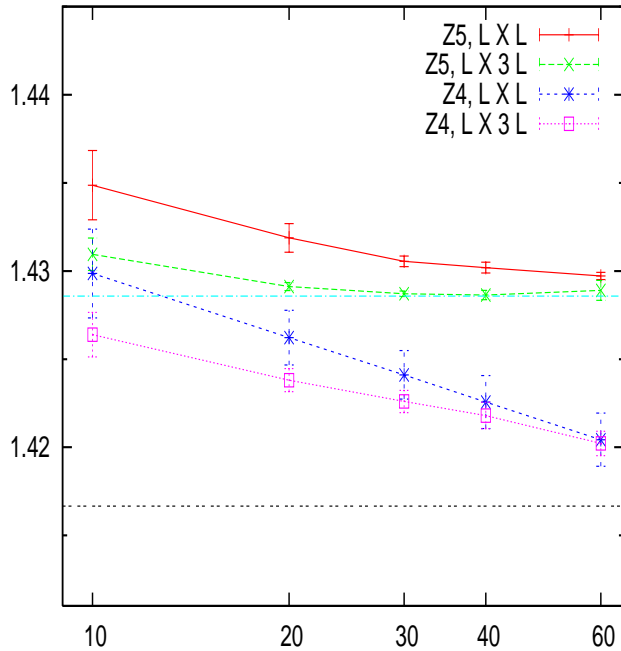
geometry on the triangular lattice was suitable and provided good results [34]. This configuration is not possible in our case since the location of the critical point is not known for the triangular lattice. In this respect we mention that a method to find these critical points for $Z(N)$ spin models on different lattices has been proposed in [32].

In this letter we obtained the first results on the geometry on the interfaces which are expected to be described by SLE in non minimal CFTs. We provide strong numerical support to the validity of the exponent eq.(7) first obtained in [19]. The agreement is excellent for both cases that we considered with $N = 4$ and $N = 5$. We believe that these results give support to the theoretical approach proposed in [18, 19] to describe non minimal CFTs by SLE with additional stochastic motion in the internal degrees of freedom.



FIG. 2: $d_f$ vs. $L_{min}$ for $N = 4$ and $N = 5$. The straight lines correspond to the predictions of eq.(7).

charge are known to contain marginal operators which may produce strong finite size effects.

Further tests can also be done like in [34, 36]. These authors made additional checks like the test against Schramm's formula or the computations of $\kappa$ from the statistics of the Loewner driving function obtained by "unfolding" the interfaces. Actually, for our purposes, these measurements turn out to be not very practical and precise due to the fluctuating boundary conditions and the geometry that we employed. Indeed, concerning Schramm's formula, these boundary conditions explicitly breaks the $Z_N$ symmetry and the left-right symmetry is expected to be recovered only in the very large scale limit. One observes then strong finite size corrections as already observed by Gamsa and Cardy for the $Q = 3$ Potts model with the same type of boundary conditions. Note that in our case we have more states (4 or 5) and thus the boundary conditions are even more asymmetrical. We tested crossing probabilities against the Schramm's formula along the line indicated in Fig. 1. The best fit gives a value of $\kappa = 3.41(2)$ for $Z(4)$ and $\kappa = 3.42(2)$ for $Z(5)$ which is close to the expected results. The agreement in both cases of the numerical data compared to Schramm's formula is of order 1% which is comparable to the result in [34]. Concerning the direct extraction of $\kappa$ the situation is even worse since the unfolding transformation is singular. To bypass the problem, one should use a different geometry. For the $Q = 3$ Potts model, the disk

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[1] see, for instance, B. Duplantier, *Conformal Random Geometry*, Proceedings of the Les Houches Summer School, Session LXXXIII (Elsevier, New York, 2006), p.101, math-ph/0608053.
[2] W. Kager and B. Nienhuis, J. Stat. Phys. **115**, 1149 (2004).
[3] J. Cardy, Annals Phys. **318**, 81 (2005).
[4] M. Bauer and D. Bernard, Phys. Rep. **432**, 115 (2006).
[5] S. Smirnov, C.R. Acad. Sci. Ser. I, Math. **333**, 239 (2001).
[6] D. Bernard, G. Boffetta, A. Celani and G. Falkovich, Nature Physics **2**, 124 (2006).
[7] P. Di Francesco, P. Mathieu, D. Sénéchal, *Conformal Field Theory*, Springer Verlag, (1997).
[8] A. Belavin, A. Polyakov and A. Zamolodchikov, Nucl. Phys. B **241**, 333 (1984).
[9] M. Bauer and D. Bernard, Comm. Math. Phys. **239**, 493 (2003).
[10] M. Bauer and D. Bernard, Phys. Lett. B**543**, 135 (2002).
[11] M. Bauer and D. Bernard, Phys. Lett. B**557**, 309 (2003).
[12] I. Affleck and F. D. M. Haldane, Phys. Rev. B. **36**, 5291 (1987).
[13] A. W. W. Ludwig, *Nucl. Phys.* B **285** 97 (1987).
[14] V. Dotsenko, M. Picco and P. Pujol, *Nucl. Phys.* B **455**, 701 (1995).
[15] J. Rasmussen, Lett. Math. Phys. **68**, 41 (2004).
[16] J. Nagi and J. Rasmussen, Nucl. Phys. B **704**, 475 (2005).
[17] J. Rasmussen, arXiv:hep-th/0409026.
[18] E. Bettelheim, I. A. Gruzberg, A. W. W. Ludwig and P. Wiegmann, Phys. Rev. Lett. **95**, 251601 (2005).
[19] R. Santachiara, Nucl. Phys. B, in press, arXiv:0705.2749
[20] V. A. Fateev and A. B. Zamolodchikov, Phys. Lett. A **92**, 37 (1982).
[21] F. C. Alcaraz and R. Köberle, J. Phys. A: Math. Gen. **13**, L153 (1980).
[22] J. Cardy, J. Phys. A: Math. Gen. **13**, 1507 (1980).
[23] V. A. Fateev and A. B. Zamolodchikov,

Zh. Eksp. Teor. Fiz. **89**, 380 (1986) [Sov. Phys. JETP **62**, 215 (1986)].

[24] O. Schramm, Israel. J.Math. **118**, 221 (2000).

[25] A. B. Zamolodchikov, Zh. Eksp. Teor. Fiz., **75** (1978), 341 [Sov. Phys. JETP, **48**, 168 (1978)]

[26] V. S. Dotsenko, Zh. Eksp. Teor. Fiz., **75** (1978), 1083 [Sov. Phys. JETP **48**, 546 (1978)]

[27] F. Y. Wu, Reviews of Modern Physics, **54**, 235 (1982).

[28] J. Ashkin and E. Teller, Phys. Rev.**64**, 178 (1943).

[29] K. .Y. Lin and F. Y. Wu, J. Phys. C: Solid St. Phys. **7**, L181 (1974).

[30] F. C. Alcaraz and R. Köberle, J. Phys. A: Math. Gen.

[31] F. C. Alcaraz, J. Phys. A: Math. Gen. **20**, 2511 (1987); J. Phys. A: Math. Gen. **20**, 623 (1987).

[32] M. A. Rajabpour and J. Cardy, arXiv:0708.3772

[33] J. Cardy, Encyclopedia of Mathematical Physics (Elsevier 2005), hep-th/0411189.

[34] A. Gamsa and J. Cardy, J. Stat. Mech. P08020 (2007), arXiv:0705.1510.

[35] M. Picco and R. Santachiara, in preparation.

[36] D. Bernard, P. Le Doussal and A. A. Middleton, Phys. Rev. **B** 76, 020403(R) (2007)

**14**, 1169 (1981).