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# Dense granular flows: interpolating between grain inertia and fluid viscosity based constitutive laws

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A scalar constitutive law was recently obtained for dense granular flows from a two-grain argument, both in the inertial regime (grain inertia) and in the viscous regime. As the resulting law is not exactly the same in both regimes, we here provide an expression for the crossover between both regimes.

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## I. INTRODUCTION

The deformations of granular materials are usually categorized into quasistatic deformations, dense flows and collisional flows [1, 2].

In the quasistatic and dense regimes, the shear-rate dependence can be expressed [2] in terms of a parameter  $I = T\dot{\gamma}$ , where  $T$  is the typical time for a single grain, accelerated by the pressure  $P$  (or force  $Pd^2$ ), to move over a distance comparable to its own size  $d$ . For dry grains of mass  $m$ , the surrounding fluid can often be neglected and the pressure is resisted only by grain inertia (“gi”) [3]:

$$m \frac{d}{T_{\text{gi}}^2} \simeq P d^2 \quad \text{i.e.,} \quad I_{\text{gi}} = \dot{\gamma} T_{\text{gi}} = \dot{\gamma} \sqrt{\frac{m}{Pd}} \quad (1)$$

For granular pastes, if the grain inertia is negligible as compared to the viscosity of the surrounding fluid (“fv”), the force  $Pd^2$  exerted by the pressure is resisted by a typical Stokes viscous drag force felt by a single grain moving at the velocity  $d/T_{\text{fv}}$  [4]:

$$\eta d \frac{d}{T_{\text{fv}}} \simeq P d^2 \quad \text{i.e.,} \quad I_{\text{fv}} = \dot{\gamma} T_{\text{fv}} = \dot{\gamma} \frac{\eta}{P} \quad (2)$$

Empirically, the dimensionless parameter  $I$  has been used in the form of a frictional constitutive law, in terms of the ratio between the shear stress  $\tau$  and the pressure  $P$ :

$$\frac{\tau}{P} = \mu(I) \quad (3)$$

Empirically again, a universal function  $\mu(I)$  seems to account for existing data in both regimes [4]:

$$\frac{\mu(I)}{\mu_c} - 1 = \frac{\mu_2/\mu_c - 1}{I_0/I + 1} \quad (4)$$

Recently, a two-grain argument (see Fig. 1), based on the same physics, led to a clear distinction between inter-grain approach and separation times [5]:

$$\frac{1}{\dot{\gamma}} \simeq T^{\text{app}} + T^{\text{sep}} \quad (5)$$

As a result, two different functions arose for the regime where the grain inertia dominates and for the regime governed by the fluid viscosity:

$$\frac{1}{1 + \sqrt{\frac{\tau - P}{\tau + P}}} \sqrt{\frac{\tau}{P} - 1} = I_{\text{gi}} \quad (6)$$

$$\frac{\tau}{P} - \frac{P}{\tau} = I_{\text{fv}} \quad (7)$$

They do not include the saturation ( $\mu(I) \rightarrow \mu_2$  at large  $I$ ) that is present in Eq. (4) and which reflects the onset of the collisional regime.

In the present note, we derive an interpolation between both regimes represented by Eqs. (6) and (7).

## II. CROSSOVER FROM GRAIN INERTIA TO FLUID VISCOSITY

The regimes discussed here are dominated either by the grain inertia or by the fluid viscosity. In other or in future experiments, it may happen that the system lie in the crossover between regime “gi” and regime “fv”. In the present section, we discuss how it is possible to interpolate between both behaviours.

Cassar *et al.* [4] proposed a universal behaviour, given by Eq. (4), which they showed to be compatible both with the data in the “gi” regime and with that in the “fv” regime. Nevertheless, the dimensionless parameter  $I$  in Eq. (4) does not have the same meaning, as it is given either by Eq. (1) or by Eq. (2). Hence, in the crossover region, there is no obvious interpolation between both definitions of  $I$ .

In the present approach, we not only have this difficulty with  $I$ , but we additionally have two different constitutive relations, namely Eqs. (6) and (7). We therefore need to go back to the equation introduced by Courrech du Pont [6] for freely falling grains in avalanches, and used by Cassar *et al.* [4] with a pressure  $P$ . Omitting all numerical coefficients:

$$m \dot{v} \simeq P d^2 - d \eta v \quad (8)$$

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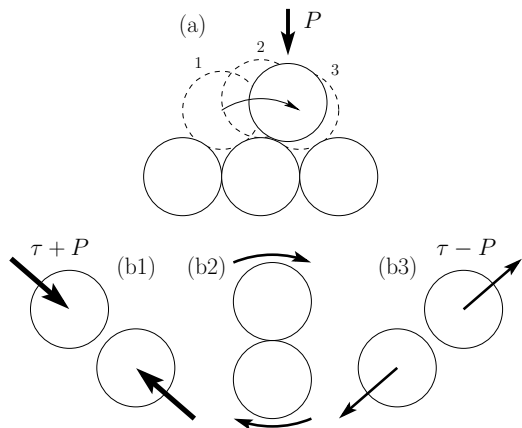


FIG. 1: Schematic evolution within the granular material during shear. (a) One grain is transported quasistatically from position 1 to position 2, then falls into position 3 due to the applied pressure  $P$ . In a recent work [5], we rather consider a pair of grains during the period of time when they are close neighbours (b1-3). First, the deviatoric (typically, shear) component of the stress,  $\tau$ , helps the pressure  $P$  establish the contact (b1). Then, the contact rotates due to the overall material deformation (b2). Finally, the deviatoric stress overcomes the pressure to break the contact (b3). Because the pressure is compressive in a non-cohesive granular material, the typical magnitude of the force transmitted between both grains is stronger when the contact forms than when it breaks.

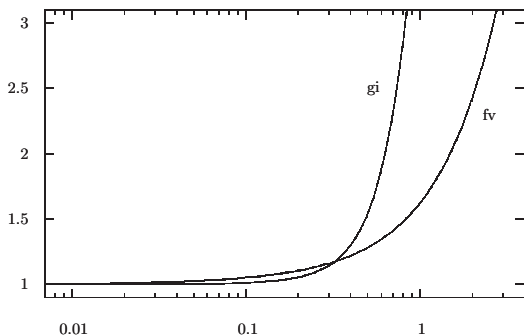


FIG. 2: Raw predictions of the two-particle argument [5] in the regimes dominated by grain inertia and by fluid viscosity, see Eqs. (6-7).

where  $v$  is the grain velocity,  $Pd^2$  is the typical force resulting from the pressure, and  $d\eta v$  is the Stokes drag force of the grain in the fluid. With vanishing initial velocity  $v(0)$  and position  $x(0)$ , we can derive the grain position:

$$x(t) = \frac{dP}{\eta} t - \frac{mP}{\eta^2} \left( 1 - \exp \left\{ -\frac{\eta d}{m} t \right\} \right) \quad (9)$$

In the present two-grain approach, we will use this re-

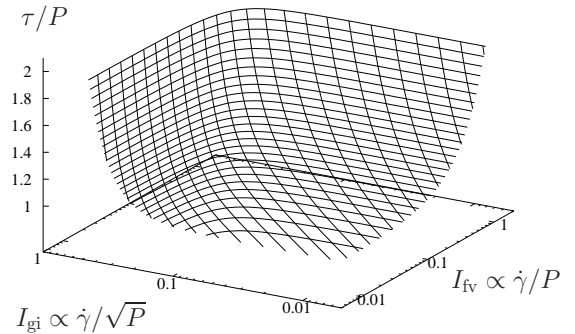


FIG. 3: Predicted response for a granular material in the crossover between the grain inertia regime and the fluid viscosity regime.  $\tau/P$  is the effective friction.  $I_{\text{gi}}$  and  $I_{\text{fv}}$  are given by Eq. (16). The “gi” and “fv” limits ( $\tau/P$  as a function of  $I_{\text{gi}}$  or as a function of  $I_{\text{fv}}$ ), represented on Fig. 2, are (approximately) given by the edges of the meshed surface.

sult with a stress  $\sigma = \tau \pm P$  instead of  $P$ . Moreover, let us define:

$$T^* = \frac{m}{\eta d} \quad (10)$$

$$P^* = \frac{\eta^2 d}{m} \quad (11)$$

which are the values taken respectively by  $T$  and  $P$  at the crossover between both regimes, see Eqs. (1) and (2).

With these notations, Eq. (9) provides the time at which a grain submitted to a stress  $\sigma$  has traveled a distance  $d$  to meet another grain:

$$\frac{P^*}{\sigma} = f(T/T^*) \quad (12)$$

$$f(x) = x - 1 + e^{-x} \quad (13)$$

The lifetime of a contact, given by Eq. (5), can be rewritten using Eq. (12):

$$\begin{aligned} \frac{1}{\dot{\gamma} T^*} &= \frac{T^{\text{app}}}{T^*} + \frac{T^{\text{sep}}}{T^*} \\ &\simeq g \left( \frac{P^*/P}{\mu + 1} \right) + g \left( \frac{P^*/P}{\mu - 1} \right) \end{aligned} \quad (14)$$

where  $\mu = \tau/P$  and where  $g$  is an approximation (precise up to within two percent) for the inverse of function  $f$  defined by Eq. (13):

$$g(x) = x + 1 - e^{-(\sqrt{2x+x/3})} \simeq f^{-1}(x) \quad (15)$$

With the notations of Eqs. (10-11), both limits of the dimensionless parameter  $I$  can be expressed as:

$$I_{\text{gi}} = \frac{\dot{\gamma} T^*}{\sqrt{P/P^*}} \quad I_{\text{fv}} = \frac{\dot{\gamma} T^*}{P/P^*} \quad (16)$$

Hence, in order to show how it is possible to interpolate between Eqs. (6) and (7), let us use Eq. (14) to plot  $\mu = \tau/P$  as a function of both  $\dot{\gamma}T^*/\sqrt{P/P^*}$  and  $\dot{\gamma}T^*/(P/P^*)$ . This is shown on Fig. 3.

### III. CONCLUSION

A recent two-grain argument provided two distinct constitutive laws for the rheology of granular media in dense regimes: one for the regime (gi) where the grain inertia is dominant, and one for the regime (fv) governed by the fluid viscosity. In the present note, we followed

the same arguments to derive an interpolation between both laws.

Further studies are now needed in order to (i) test whether the distinct predictions are compatible with the experiments, and (ii) develop theoretical arguments that could rationalize the saturation of the apparent frictional response as the collisional regime is approached.

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