Quillen's relative Chern character is multiplicative

Paul-Emile Paradan, Michèle Vergne

QUILLEN'S RELATIVE CHERN CHARACTER IS MULTIPLICATIVE

PAUL-EMILE PARADAN AND MICHÈLE VERGNE

Abstract. In the first part of this paper we prove the multiplicative property of the relative Quillen Chern character. Then we obtain a Riemann-Roch formula between the relative Chern character of the Bott morphism and the relative Thom form.

Contents

1. Introduction

The relative Chern character was defined by Atiyah and al. in [2, 5] as a map

$$\mathbf{Ch}_{X\setminus Y} : \mathbf{K}^0(X,Y) \longrightarrow \mathrm{H}^*(X,Y). \tag{1}$$

Here $Y \subset X$ are *finite* CW-complexes, $\mathbf{K}^0(X,Y)$ is the relative K-group and $\mathrm{H}^*(X,Y)$ is the singular relative cohomology group.

Date: September 2008.

The relative Chern character enjoys various functorial properties. In particular, **Ch** is multiplicative: the following diagram

$$
\begin{tikzcd}
\mathbf{K}^0(X,Y) \arrow[d, "\mathbf{Ch}_{X\setminus Y}"] & \times & \mathbf{K}^0(X,Y') \arrow[d, "\mathbf{Ch}_{X\setminus Y'}"] \arrow[r, "\odot"] & \mathbf{K}^0(X,Y\cup Y') \arrow[d, "\mathbf{Ch}_{X\setminus Y\cup Y'}"'] \\
\mathrm{H}^*(X,Y) & \times & \mathrm{H}^*(X,Y') \arrow[r, "\diamond"] & \mathrm{H}^*(X,Y\cup Y')
\end{tikzcd} \tag{2}
$$

is commutative. Here $Y, Y' \subset X$ are *finite* CW-complexes and $\odot$ and $\diamond$ denote the products. This property was extended to the case where X is a paracompact topological space and Y any *open* subset of X by Iversen in [14] (see also [12, 13]). Iversen deduces the existence of the local Chern character from functorial properties, but his construction is not explicit.

In this article, we work in the context of manifolds and differential forms. Indeed, in this framework, Quillen [19] constructed a very natural de Rham relative cohomology class associated to a smooth morphism between vector bundles, that we call the relative Quillen Chern character. Let N be a manifold, and let $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ be a morphism of complex vector bundles over N. Let $\mathrm{Supp}(\sigma)$ be the *support* of σ : it is the set of points $n \in N$ where $\sigma(n)$ is not invertible. We do not suppose that $\mathrm{Supp}(\sigma)$ is compact. Quillen [19] associates to σ a couple (α, β) of differential forms, where α is given by the usual Chern-Weil construction, and β is also constructed à la Chern-Weil, via super-connections. The form α is a closed differential form on N representing the difference of Chern characters $\mathrm{Ch}(\mathcal{E}^+) - \mathrm{Ch}(\mathcal{E}^-) \in \mathcal{H}^*(N)$, and β is a differential form on $N \setminus \mathrm{Supp}(\sigma)$ such that

$$\alpha|_{N\setminus \mathrm{Supp}(\sigma)} = d\beta.$$

The couple (α, β) defines then an explicit relative de Rham cohomology class

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma) \in \mathcal{H}^*(N, N \setminus \mathrm{Supp}(\sigma)).$$

The main purpose of this note is to show that Quillen's relative Chern character $\mathrm{Ch}_{\mathrm{rel}}$ is multiplicative. If σ_1, σ_2 are two morphisms on N, then the product $\sigma_1 \odot \sigma_2$ is a morphism on N with support equal to $\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)$. We prove in Section 4 that the following equality

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2) = \mathrm{Ch}_{\mathrm{rel}}(\sigma_1) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma_2) \tag{3}$$

holds in $\mathcal{H}^*(N, N \setminus \mathrm{Supp}(\sigma_1 \odot \sigma_2))$.

Intuitively (and true in many analytic cases), the relative Chern class could also be represented as a current supported on $\mathrm{Supp}(\sigma)$, but currents do not usually multiply. Thus another procedure, involving a choice of partition of unity, is needed to define the product $\diamond$ of relative classes in de Rham relative cohomology. The multiplicativity property (3) can also be deduced from the fact that Quillen's Chern character gives an explicit representative of Iversen's local Chern character, due to Schneiders functorial characterization of Iversen's class (see [20]). Our proof does not use Iversen's construction, and our explicit argument can be extended to the case of equivariant Chern characters with generalized coefficients (see [18]).

When $\mathrm{Supp}(\sigma)$ is *compact*, there is a natural homomorphism from $\mathcal{H}^*(N, N \setminus \mathrm{Supp}(\sigma))$ into the compactly supported cohomology algebra $\mathcal{H}^*_c(N)$ and the image of the Quillen relative Chern character $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ is the Chern character $\mathrm{Ch}_c(\sigma)$ with compact support. The equality (3) implies the relation

$$\mathrm{Ch}_c(\sigma_1 \odot \sigma_2) = \mathrm{Ch}_c(\sigma_1) \wedge \mathrm{Ch}_c(\sigma_2) \quad \text{in} \quad \mathcal{H}_c^*(N). \tag{4}$$

This last relation is well known and follows also from the fact that $\mathrm{Ch}_c(\sigma)$ is the Chern character of a difference bundle on a compactification of N.

As an important example, we consider σ_b the Bott morphism on a complex vector bundle $\sigma_b : \Lambda^+\mathcal{V} \to \Lambda^-\mathcal{V}$ over $\mathcal{V}$, given by the exterior product by $v \in \mathcal{V}$. This morphism has support the zero section M of $\mathcal{V}$. It leads to a relative class $\mathrm{Ch}_{\mathrm{rel}}(\sigma_b)$ in $\mathcal{H}^*(\mathcal{V}, \mathcal{V} \setminus M)$. One can give a similar construction of the relative Thom form $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V}) \in \mathcal{H}^*(\mathcal{V}, \mathcal{V} \setminus M)$ of the vector bundle $\mathcal{V} \to M$, using the Berezin integral instead of a super-trace. The explicit formulae for $\mathrm{Ch}_{\mathrm{rel}}(\sigma_b)$ and $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V})$ allows us to derive the "Riemann-Roch" relation between these two relative classes at the level of differential forms. Our proof follows the same scheme than the proof of the relation between the Chern character and the Thom class with Gaussian looks constructed by Mathai-Quillen [17].

Acknowledgements: We are grateful to M. Karoubi, J. Lannes, P. Schapira and J.P. Schneiders for enlightening discussions on these topics.

We wish to thank the referee for his careful reading, and suggestions for improvements.

2. Cohomological structures

Let N be a manifold. We denote by $\mathcal{A}^*(N)$ the algebra of differential forms on N and by $\mathcal{H}^*(N)$ the de Rham cohomology algebra of N. We denote by $\mathcal{H}_c^*(N)$ its compactly supported cohomology algebra.

In this paper, we work with differential forms with *complex* or *real* coefficients, depending on the context. In order to simplify the notation, we use the same notation for $\mathcal{A}^*$, $\mathcal{H}^*$ and $\mathcal{H}_c^*$ viewed as *complex* or *real* vector space : we speak of $\mathbb{K}$-differential forms, $\mathbb{K}$-cohomology classes or $\mathbb{K}$-algebras with

$$\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}.$$

2.1. Relative cohomology.

Let F be a closed subset of N. To a cohomology class on N vanishing on $N \setminus F$, we associate a relative cohomology class. Let us explain the construction (see [9]). Consider the $\mathbb{K}$-complex $\mathcal{A}^*(N, N \setminus F)$ with

$$\mathcal{A}^k(N, N \setminus F) := \mathcal{A}^k(N) \oplus \mathcal{A}^{k-1}(N \setminus F)$$

and differential $d_{\mathrm{rel}}\,(\alpha, \beta) = \big(d\alpha, \alpha|_{N\setminus F} - d\beta\big)$.

Definition 2.1. *The cohomology of the complex $(\mathcal{A}^*(N, N \setminus F), d_{\mathrm{rel}})$ is the relative $\mathbb{K}$-cohomology space $\mathcal{H}^*(N, N \setminus F)$.*

The class defined by a d_{rel}-closed element $(\alpha, \beta) \in \mathcal{A}^*(N, N \setminus F)$ will be denoted by $[\alpha, \beta]$. There is a natural $\mathbb{K}$-linear map $\mathcal{H}^*(N, N \setminus F) \to \mathcal{H}^*(N)$.

If F_1 and F_2 are closed subsets of N, there is a natural product

$$\begin{array}{ccc} \mathcal{H}^*(N, N \setminus F_1) \times \mathcal{H}^*(N, N \setminus F_2) & \longrightarrow & \mathcal{H}^*(N, N \setminus (F_1 \cap F_2)) \\ (\ a\ ,\ b\ ) & \longmapsto & a \diamond b\ , \end{array} \tag{5}$$

which is $\mathbb{K}$-bilinear.

We will use an explicit formula for $\diamond$ that we recall. Let $U_1 := N\setminus F_1$, $U_2 := N\setminus F_2$ so that $U := N \setminus (F_1 \cap F_2) = U_1 \cup U_2$. Let $\Phi := (\Phi_1, \Phi_2)$ be a partition of unity subordinate to the covering $U_1 \cup U_2$ of U. With the help of Φ, we define a bilinear map $\diamond_\Phi : \mathcal{A}^*(N, N \setminus F_1) \times \mathcal{A}^*(N, N \setminus F_2) \to \mathcal{A}^*(N, N \setminus (F_1 \cap F_2))$ as follows. For $a_i := (\alpha_i, \beta_i) \in \mathcal{A}^{k_i}(N, N \setminus F_i)$, $i = 1, 2$, we define

$$a_1 \diamond_\Phi a_2 := \Big(\alpha_1 \wedge \alpha_2, \Phi_1\beta_1 \wedge \alpha_2 + (-1)^{k_1}\alpha_1 \wedge \Phi_2\beta_2 - (-1)^{k_1} d\Phi_1 \wedge \beta_1 \wedge \beta_2\Big).$$

Remark that all forms $\Phi_1\beta_1 \wedge \alpha_2$, $\alpha_1 \wedge \Phi_2\beta_2$ and $d\Phi_1 \wedge \beta_1 \wedge \beta_2$ are well defined on $U_1 \cup U_2$. Indeed the support of the form $d\Phi_1$ is contained in $U_1 \cap U_2$, as $d\Phi_1 = -d\Phi_2$. So $a_1 \diamond_\Phi a_2 \in \mathcal{A}^{k_1+k_2}(N, N \setminus (F_1 \cap F_2))$. It is immediate to verify that $d_{\rm rel}(a_1 \diamond_\Phi a_2)$ is equal to $(d_{\rm rel}a_1) \diamond_\Phi a_2 + (-1)^{k_1} a_1 \diamond_\Phi (d_{\rm rel}a_2)$. Thus $\diamond_\Phi$ defines a bilinear map $\mathcal{H}^*(N, N \setminus F_1) \times \mathcal{H}^*(N, N \setminus F_2) \to \mathcal{H}^*(N, N \setminus (F_1 \cap F_2))$.

Let us see that this product do not depend on the choice of the partition of unity. If we have another partition $\Phi' = (\Phi'_1, \Phi'_2)$, then $\Phi_1 - \Phi'_1 = -(\Phi_2 - \Phi'_2)$. It is immediate to verify that, if $d_{\rm rel}(a_1) = 0$ and $d_{\rm rel}(a_2) = 0$, one has

$$a_1 \diamond_\Phi a_2 - a_1 \diamond_{\Phi'} a_2 = d_{\rm rel}\Big(0, (-1)^{k_1}(\Phi_1 - \Phi'_1)\beta_1 \wedge \beta_2\Big).$$

So the product on the relative cohomology spaces will be denoted by $\diamond$.

2.2. **Inverse limit of cohomology with support.** Let F be a closed subset of N. We consider the set $\mathcal{F}_F$ of all open neighborhoods U of F which is ordered by the relation $U \leq V$ if and only if $V \subset U$. For any $U \in \mathcal{F}_F$, we consider the $\mathbb{K}$-algebra $\mathcal{A}^*_U(N)$ of differential forms on N with support contained in U (that is vanishing on a neighborhood of $N \setminus U$): this algebra is stable under the de Rham differential d, and we denote by $\mathcal{H}^*_U(N)$ the corresponding cohomology $\mathbb{K}$-algebra. If $U \leq V$, we have then an inclusion map $\mathcal{A}^*_V(N) \hookrightarrow \mathcal{A}^*_U(N)$ which gives rise to a $\mathbb{K}$-linear map $f_{U,V} : \mathcal{H}^*_V(N) \to \mathcal{H}^*_U(N)$.

Definition 2.2. *We denote by $\mathcal{H}^*_F(N)$ the inverse limit of the inverse system $(\mathcal{H}^*_U(N), f_{U,V}; U, V \in \mathcal{F}_F)$. It is a $\mathbb{K}$-vector space.*

If F_1, F_2 are two closed subsets of N, there is a $\mathbb{K}$-bilinear map

$$\mathcal{H}^*_{F_1}(N) \times \mathcal{H}^*_{F_2}(N) \xrightarrow{\wedge} \mathcal{H}^*_{F_1 \cap F_2}(N) \tag{6}$$

which is defined via the wedge product on forms.

Now we define a $\mathbb{K}$-linear map from $\mathcal{H}^*(N, N \setminus F)$ into $\mathcal{H}^*_F(N)$.

Let $\beta \in \mathcal{A}^*(N \setminus F)$. If χ is a function on N which is identically 1 on a neighborhood of F, note that $d\chi\beta$ defines a differential form on N, since $d\chi$ is equal to 0 in a neighborhood of F.

Proposition 1. *For any open neighborhood U of F, we choose $\chi \in \mathcal{C}^\infty(N)$ with support in U and equal to 1 in a neighborhood of F.*

- *The map*

$$\mathrm{p}^\chi_U(\alpha, \beta) = \chi\alpha + d\chi\beta \tag{7}$$

defines a homomorphism of complexes $\mathrm{p}^\chi_U : \mathcal{A}^*(N, N \setminus F) \to \mathcal{A}^*_U(N)$.

Let $\alpha \in \mathcal{A}^*(N)$ be a closed *form and $\beta \in \mathcal{A}^*(N \setminus F)$ such that $\alpha|_{N \setminus F} = d\beta$. Then $\mathrm{p}^\chi_U(\alpha, \beta)$ is a* closed *differential form supported in U.*

- *The cohomology class of $\mathrm{p}^\chi_U(\alpha, \beta)$ in $\mathcal{H}^*_U(N)$ does not depend on χ. We denote this class by $\mathrm{p}_U(\alpha, \beta) \in \mathcal{H}^*_U(N)$.*
- *For any neighborhoods $V \subset U$ of F, we have $f_{U,V} \circ \mathrm{p}_V = \mathrm{p}_U$.*

Proof. The equation $\mathrm{p}_U^\chi \circ d_{\mathrm{rel}} = d \circ \mathrm{p}_U^\chi$ is immediate to check. In particular $\mathrm{p}_U^\chi(\alpha, \beta)$ is closed, if $d_{\mathrm{rel}}(\alpha, \beta) = 0$. This proves the first point. For two different choices χ and χ', we have $\mathrm{p}_U^\chi(\alpha, \beta) - \mathrm{p}_U^{\chi'}(\alpha, \beta) = d((\chi - \chi')\beta)$. Since $\chi - \chi' = 0$ in a neighborhood of F, the term $(\chi - \chi')\beta$ is a well defined element of $\mathcal{A}_U^*(N)$. This proves the second point. Finally, the last point is immediate, since $\mathrm{p}_U^\chi(\alpha, \beta) = \mathrm{p}_V^\chi(\alpha, \beta)$ for $\chi \in \mathcal{C}^\infty(N)$ with support in $V \subset U$. □

Definition 2.3. *Let $\alpha \in \mathcal{A}^*(N)$ be a* closed *form and $\beta \in \mathcal{A}^*(N \setminus F)$ be such that $\alpha|_{N \setminus F} = d\beta$. We denote by $\mathrm{p}_F(\alpha, \beta) \in \mathcal{H}_F^*(N)$ the element defined by the sequence $\mathrm{p}_U(\alpha, \beta) \in \mathcal{H}_U^*(N)$, $U \in \mathcal{F}_F$. We have then a morphism of $\mathbb{K}$-vector spaces*

$$\mathrm{p}_F : \mathcal{H}^*(N, N \setminus F) \to \mathcal{H}_F^*(N). \tag{8}$$

Proposition 2. *If F_1, F_2 are closed subsets of N, then we have*

$$\mathrm{p}_{F_1 \cap F_2}(a_1 \diamond a_2) = \mathrm{p}_{F_1}(a_1) \wedge \mathrm{p}_{F_2}(a_2) \tag{9}$$

for any $a_k \in \mathcal{H}^*(N, N \setminus F_k)$.

Proof. Let W be a neighborhood of $F_1 \cap F_2$. Let V_1, V_2 be respectively neighborhoods of F_1 and F_2 such that $V_1 \cap V_2 \subset W$. Let $\chi_i \in \mathcal{C}^\infty(N)$ be supported in V_i and equal to 1 in a neighborhood of F_i. Then $\chi_1\chi_2$ is supported in W and equal to 1 in a neighborhood of $F_1 \cap F_2$. Let $\Phi_1 + \Phi_2 = 1_{N \setminus (F_1 \cap F_2)}$ be a partition of unity relative to the decomposition $N \setminus (F_1 \cap F_2) = N \setminus F_1 \cup N \setminus F_2$.

Then one checks easily that

$$\mathrm{p}_{V_1}^{\chi_1}(a_1) \wedge \ \mathrm{p}_{V_2}^{\chi_2}(a_2) - \mathrm{p}_W^{\chi_1\chi_2}(a_1 \diamond_\Phi a_2)$$

is equal to

$$d\Big((-1)^{k_1+1}\chi_1 d\chi_2(\Phi_1\beta_1\beta_2) + (-1)^{k_1}\chi_2 d\chi_1(\beta_1\Phi_2\beta_2)\Big)$$

for d_{rel}-closed forms $a_i = (\alpha_i, \beta_i) \in \mathcal{A}^{k_i}(N, N \setminus F_i)$. Remark that $\Phi_1\beta_1\beta_2$ is defined on $N \setminus F_2$, so that $d\chi_2(\Phi_1\beta_1\beta_2)$ is well defined on N and supported in V_2. Thus the form $(-1)^{k_1+1}\chi_1 d\chi_2(\Phi_1\beta_1\beta_2) + (-1)^{k_1}\chi_2 d\chi_1(\beta_1\Phi_2\beta_2)$ is well defined on N and supported in $V_1 \cap V_2 \subset W$. This proves that $\mathrm{p}_{F_1}(a_1) \wedge \mathrm{p}_{F_2}(a_2) = \mathrm{p}_{F_1 \cap F_2}(a_1 \diamond a_2)$. □

The map $\mathrm{p}_F : \mathcal{H}^*(N, N \setminus F) \longrightarrow \mathcal{H}_F^*(N)$ factors the natural map $\mathcal{H}^*(N, N \setminus F) \to \mathcal{H}^*(N)$.

2.3. **Integration.** We consider first the case where F is a *compact* subset of an oriented manifold N. Let $\mathcal{H}^*(N, N \setminus F)$ be the relative $\mathbb{K}$-cohomology group. Let $\pi : N \to \{\bullet\}$ be the projection to the point. We will describe an integration morphism $\pi_* : \mathcal{H}^*(N, N \setminus F) \to \mathbb{K}$.

We have a $\mathbb{K}$-linear map

$$\mathrm{p}_c : \mathcal{H}^*(N, N \setminus F) \longrightarrow \mathcal{H}_c^*(N) \tag{10}$$

which is equal to the composition of p_F with the natural map $\mathcal{H}_F^*(N) \to \mathcal{H}_c^*(N)$. If $a \in \mathcal{H}^*(N, N \setminus F)$ is represented by the d_{rel}-closed differential form $(\alpha, \beta) \in \mathcal{A}^*(N, N \setminus F)$, the class $\mathrm{p}_c(a) \in \mathcal{H}_c^*(N)$ is represented by the differential form $\mathrm{p}_U^\chi(\alpha, \beta) = \chi\alpha + d\chi\beta$ where χ is a function with *compact* support.

Definition 2.4. *If $a \in \mathcal{H}^*(N, N \setminus F)$, then $\pi_*(a) \in \mathbb{K}$ is defined by*

$$\pi_*(a) := \int_N \mathrm{p}_c(a).$$

If N is *compact*, the elements α and $\mathrm{p}_c(a)$ coincide in $\mathcal{H}^*(N)$, hence $\pi_*(a) = \int_N \alpha$. When N is *non-compact*, an interesting situation is the case of a relative class $a = [\alpha, \beta]$ where the closed form α is *integrable*. The two terms $\pi_*(a)$ and $\int_N \alpha$ are defined. However, it is usually not true that they coincide. An interesting case is the relative Thom form $\mathrm{Th}_{\mathrm{rel}}(V)$ of a real oriented vector space V (see Section 6.2). Here $N = V$, $F = \{0\}$, and the relative class $\mathrm{Th}_{\mathrm{rel}}(V)$ is represented by $[0, \beta]$ with β a particular closed real form on $V \setminus \{0\}$. Here the integral of $\alpha = 0$ is equal to 0, while $\pi_*(\mathrm{Th}_{\mathrm{rel}}(V)) = 1$. See Example 5.3.2.

In some important cases studied in Subsection 5.2.2, we will however prove that the integral of $\mathrm{p}_c(a)$ is indeed the same as the integral of α. As we have

$$\begin{aligned} \mathrm{p}_U^\chi(\alpha, \beta) - \alpha &= (\chi - 1)\alpha + d\chi\beta \\ &= d\Big((\chi - 1)\beta\Big), \end{aligned} \tag{11}$$

the comparison between the integral of $\mathrm{p}_c(a)$ and the one of α will follow from the careful study of the behavior on N of the form $(\chi - 1)\beta$.

We consider now the case of an oriented real vector bundle $\pi : \mathcal{V} \to M$. We will describe a push-forward $\mathbb{K}$-linear map $\pi_* : \mathcal{H}^*(\mathcal{V}, \mathcal{V} \setminus M) \to \mathcal{H}^*(M)$.

Let $\mathcal{A}^*_{\text{fiber cpt}}(\mathcal{V})$ be the $\mathbb{K}$-subalgebra of $\mathcal{A}^*(\mathcal{V})$ formed by the differential forms which have a compact support in the fibers of π. Let $\mathcal{H}^*_{\text{fiber cpt}}(\mathcal{V})$ be the corresponding $\mathbb{K}$-cohomology space. We have a morphism $\int_{\text{fiber}} : \mathcal{H}^*_{\text{fiber cpt}}(\mathcal{V}) \to \mathcal{H}^*(M)$ of integration along the fibers.

We define a $\mathbb{K}$-linear map

$$\mathrm{p}_{\text{fiber cpt}} : \mathcal{H}^*(\mathcal{V}, \mathcal{V} \setminus M) \longrightarrow \mathcal{H}^*_{\text{fiber cpt}}(\mathcal{V}) \tag{12}$$

by setting that $\mathrm{p}_{\text{fiber cpt}}([\alpha, \beta])$ is the class represented by $\chi\alpha + d\chi\beta$, where χ is a function on $\mathcal{V}$ with *compact* support in the fibers, and equal to 1 in a neighborhood of the zero section.

Definition 2.5. *If $a \in \mathcal{H}^*(\mathcal{V}, \mathcal{V} \setminus M)$, the class $\pi_*(a) \in \mathcal{H}^*(M)$ is defined by*

$$\pi_*(a) := \int_{\text{fiber}} \mathrm{p}_{\text{fiber cpt}}(a).$$

In Section 6.2, we will describe a relative Thom class $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V})$ which is characterized by the fact that $\pi_*(\mathrm{Th}_{\mathrm{rel}}(\mathcal{V})) = 1$ in $\mathcal{H}^*(M)$.

3. Quillen's relative Chern Character

In this section, we work with differential forms with *complex* coefficients.

3.1. **Chern form of a super-connection.**

For an introduction to the Quillen's notion of super-connection, see [7].

If $\mathcal{E}$ is a complex vector bundle on a manifold N, we denote by $\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))$ the complex algebra of $\mathrm{End}(\mathcal{E})$-valued differential forms on N.

Let ∇ be a connection on $\mathcal{E}$. The curvature ∇^2 of ∇ is a $\mathrm{End}(\mathcal{E})$-valued two-form on N. Recall that the Chern character of $\mathcal{E}$ is the de Rham cohomology class of the closed differential form $\mathrm{Chern}(\mathcal{E}) := \mathrm{Tr}(\exp(\frac{-\nabla^2}{2i\pi}))$. Here we simply denote by

$\mathrm{Ch}(\mathcal{E}) \in \mathcal{H}^*(N)$ the de Rham cohomology class of $\mathrm{Tr}(\exp(\nabla^2))$. We will call it the (non normalized) Chern character of $\mathcal{E}$.

More generally, let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a $\mathbb{Z}_2$-graded complex vector bundle on a manifold N. Taking in account the $\mathbb{Z}_2$-grading of $\mathrm{End}(\mathcal{E})$, the algebra $\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))$ is a $\mathbb{Z}_2$-graded algebra: for example $[\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))]^+$ is equal to $\mathcal{A}^+(N, \mathrm{End}(\mathcal{E})^+) \oplus \mathcal{A}^-(N, \mathrm{End}(\mathcal{E})^-)$. The super-trace on $\mathrm{End}(\mathcal{E})$ extends to a $\mathbb{C}$-linear map $\mathrm{Str} : \mathcal{A}^*(N, \mathrm{End}(\mathcal{E})) \to \mathcal{A}^*(N)$.

Let $\mathbb{A}$ be a super-connection on $\mathcal{E}$ and $\mathbf{F} = \mathbb{A}^2$ be its curvature, an element of $[\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))]^+$. *The Chern form* of $(\mathcal{E}, \mathbb{A})$ is the closed differential form

$$\mathrm{Ch}(\mathbb{A}) := \mathrm{Str}(\mathrm{e}^{\mathbf{F}}).$$

We will use the following transgression formulaes.

Proposition 3. • *Let $\mathbb{A}_t$, for $t \in \mathbb{R}$, be a one parameter family of super-connections on E, and let $\frac{d}{dt}\mathbb{A}_t \in [\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))]^-$. Let $\mathbf{F}_t$ be the curvature of $\mathbb{A}_t$. Then one has*

$$\frac{d}{dt}\,\mathrm{Ch}(\mathbb{A}_t) = d\left(\mathrm{Str}\left((\frac{d}{dt}\mathbb{A}_t)\,\mathrm{e}^{\mathbf{F}_t}\right)\right). \tag{13}$$

• *Let $\mathbb{A}(s,t)$ be a two-parameter family of super-connections. Here $s, t \in \mathbb{R}$. We denote by $\mathbf{F}(s,t)$ the curvature of $\mathbb{A}(s,t)$. Then:*

$$\begin{aligned} &\frac{d}{ds}\,\mathrm{Str}\left((\frac{d}{dt}\mathbb{A}(s,t))\ \mathrm{e}^{\mathbf{F}(s,t)}\right) - \frac{d}{dt}\,\mathrm{Str}\left((\frac{d}{ds}\mathbb{A}(s,t))\ \mathrm{e}^{\mathbf{F}(s,t)}\right) \\ &\quad = d\left(\int_0^1 \mathrm{Str}\left((\frac{d}{ds}\mathbb{A}(s,t))\,\mathrm{e}^{u\mathbf{F}(s,t)}(\frac{d}{dt}\mathbb{A}(s,t))\ \mathrm{e}^{(1-u)\mathbf{F}(s,t)}\right)du\right). \end{aligned}$$

Proof. These formulae are well known, and are derived easily from the two identities: $\mathbf{F} = \mathbb{A}^2$, and $d\,\mathrm{Str}(\alpha) = \mathrm{Str}[\mathbb{A}, \alpha]$ for any $\alpha \in \mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))$ (see [7]). ☐

In particular, the cohomology class defined by $\mathrm{Ch}(\mathbb{A})$ in $\mathcal{H}^*(N)$ is independent on the choice of the super-connection $\mathbb{A}$ on $\mathcal{E}$. By definition, this is the Chern character $\mathrm{Ch}(\mathcal{E})$ of $\mathcal{E}$. By choosing $\mathbb{A} = \begin{pmatrix} \nabla^+ & 0 \\ 0 & \nabla^- \end{pmatrix}$ where $\nabla^\pm$ are connections on $\mathcal{E}^\pm$, this class is just $\mathrm{Ch}(\mathcal{E}^+) - \mathrm{Ch}(\mathcal{E}^-)$. However, different choices of $\mathbb{A}$ define very different looking representatives of $\mathrm{Ch}(\mathcal{E})$.

3.2. **Quillen's relative Chern character of a morphism.**

Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a $\mathbb{Z}_2$-graded complex vector bundle on a manifold N and $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ be a smooth morphism. At each point $n \in N$, $\sigma(n) : \mathcal{E}^+_n \to \mathcal{E}^-_n$ is a linear map. The *support* of σ is the closed subset of N

$$\mathrm{Supp}(\sigma) = \{n \in N \mid \sigma(n) \text{ is not invertible}\}.$$

Recall that the morphism σ is *elliptic* when $\mathrm{Supp}(\sigma)$ is compact : in this situation the data $(\mathcal{E}^+, \mathcal{E}^-, \sigma)$ defines an element of the $\mathbf{K}^0$-theory of N.

In the following, *we do not* assume σ elliptic. We recall Quillen's construction [19] of a $\mathbb{C}$-cohomology class $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ in $\mathcal{H}^*(N, N \setminus \mathrm{Supp}(\sigma))$. The definition will involve several choices. We choose Hermitian structures on $\mathcal{E}^\pm$ and a super-connection $\mathbb{A}$ on $\mathcal{E}$ *without* 0 *exterior degree term.*

We associate to the morphism σ the odd Hermitian endomorphism of $\mathcal{E}$ defined by

$$v_\sigma = \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix}. \tag{14}$$

Then $v_\sigma^2 = \begin{pmatrix} \sigma^*\sigma & 0 \\ 0 & \sigma\sigma^* \end{pmatrix}$ is a non negative even Hermitian endomorphism of $\mathcal{E}$. The support of σ coincides with the set of elements $n \in N$ where the spectrum of $v_\sigma^2(n)$ contains 0.

Definition 3.1. *Let E be a finite dimensional Hermitian vector space. If H is an Hermitian endomorphism of E and $h \in \mathbb{R}$, we write $H \geq h$ when $(Hw, w) \geq h\|w\|^2$ for any $w \in E$. Then $H \geq h$ if and only if the smallest eigenvalue of H is larger than h.*

Consider the family of super-connections

$$\mathbb{A}^\sigma(t) = \mathbb{A} + it\, v_\sigma, \quad t \in \mathbb{R}. \tag{15}$$

The curvature of $\mathbb{A}^\sigma(t)$ is the even element $\mathbf{F}(\sigma, \mathbb{A}, t) \in [\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))]^+$ defined by :

$$\mathbf{F}(\sigma, \mathbb{A}, t) = (itv_\sigma + \mathbb{A})^2 = -t^2 v_\sigma^2 + it[\mathbb{A}, v_\sigma] + \mathbb{A}^2. \tag{16}$$

Here $-t^2 v_\sigma^2$ is the term of exterior degree 0. As the super-connection $\mathbb{A}$ do not have 0 exterior degree term, both elements $it[\mathbb{A}, v_\sigma]$ and $\mathbb{A}^2$ are sums of terms with strictly positive exterior degrees. For example, if $\mathbb{A} = \nabla^+ \oplus \nabla^-$ is a direct sum of connections, then $it[\mathbb{A}, v_\sigma] \in \mathcal{A}^1(N, \mathrm{End}(\mathcal{E})^-)$ and $\mathbb{A}^2 \in \mathcal{A}^2(N, \mathrm{End}(\mathcal{E})^+)$.

Definition 3.2. *We denote by* $\mathrm{Ch}(\sigma, \mathbb{A}, t)$ *the Chern form of* $(\mathcal{E}, \mathbb{A}^\sigma(t))$*, that is*

$$\mathrm{Ch}(\sigma, \mathbb{A}, t) := \mathrm{Str}\left(\mathrm{e}^{\mathbf{F}(\sigma, \mathbb{A}, t)}\right).$$

As $iv_\sigma = \frac{d}{dt}\mathbb{A}^\sigma(t)$, we have the transgression formula $\frac{d}{dt}\mathrm{Ch}(\sigma, \mathbb{A}, t) = -d(\eta(\sigma, \mathbb{A}, t))$ with

$$\eta(\sigma, \mathbb{A}, t) := -\,\mathrm{Str}\left(iv_\sigma\, \mathrm{e}^{\mathbf{F}(\sigma, \mathbb{A}, t)}\right). \tag{17}$$

After integration, the transgression formula gives the following equality of differential forms on N

$$\mathrm{Ch}(\mathbb{A}) - \mathrm{Ch}(\sigma, \mathbb{A}, t) = d\left(\int_0^t \eta(\sigma, \mathbb{A}, s) ds\right), \tag{18}$$

since $\mathrm{Ch}(\mathbb{A}) = \mathrm{Ch}(\sigma, \mathbb{A}, 0)$.

Proposition 4. *Let $\mathcal{K}$ be a compact subset of N and let $h \geq 0$ be such that $v_\sigma^2(n) \geq h$ when $n \in \mathcal{K}$. There exists a polynomial $\mathcal{P}_\mathcal{K}$ of degree* $\dim N$ *such that, on $\mathcal{K}$,*

$$\left\| \mathrm{e}^{\mathbf{F}(\sigma, \mathbb{A}, t)} \right\| \leq \mathcal{P}_\mathcal{K}(t)\, \mathrm{e}^{-ht^2} \quad \text{for all } t \geq 0. \tag{19}$$

In particular, when $\mathcal{K}$ is contained in $N \setminus \mathrm{Supp}(\sigma)$, then $\mathrm{Ch}(\sigma, \mathbb{A}, t)$ *and* $\eta(\sigma, \mathbb{A}, t)$ *tends to* 0 *exponentially fast when t tends to infinity.*

Proof. We work on $\mathrm{End}(E)\otimes\mathcal{A}$, where $\mathcal{A}=\oplus_{k=0}^{\dim N}\mathcal{A}^k(N)$. To estimate $\|\,\mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)}\,\|$, we employ Lemma 9 of the Appendix, with $H=t^2v_\sigma^2$, and $R=-it[\mathbb{A},v_\sigma]-\mathbb{A}^2$. Here R is a sum of $\mathrm{End}(E)$-valued differential forms on N with strictly positive exterior degrees. Remark that R is a polynomial in t of degree 1. Lemma 9 gives us the estimate $\|\,\mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)}\,\|\leq\mathcal{P}(\|R\|)\,\mathrm{e}^{-ht^2}$ with $\mathcal{P}$ an explicit polynomial with positive coefficients of degree $\dim N$. Using the fact that $\|R\|\leq at+b$ on $\mathcal{K}$, we obtain the estimate (19) on N.

If $\mathcal{K}$ is contained in $N\setminus\mathrm{Supp}(\sigma)$, we can find $h>0$ such that $v_\sigma^2(n)\geq h$ when $n\in\mathcal{K}$. Thus we see that $\|\,\mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)}\,\|$ decreases exponentially fast, when t tends to infinity. □

The former estimates allows us to take the limit $t\to\infty$ in (18) on the open subset $N\setminus\mathrm{Supp}(\sigma)$. There, the differential form $\mathrm{Ch}(\sigma,\mathbb{A},t)=\mathrm{Str}\left(\mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)}\right)$ tends to 0 as t goes to ∞, and we get the following important lemma due to Quillen.

Lemma 1. [19] *We can define on $N\setminus\mathrm{Supp}(\sigma)$ the differential form*

$$\beta(\sigma,\mathbb{A})=\int_0^\infty\eta(\sigma,\mathbb{A},t)dt \tag{20}$$

and we have $\mathrm{Ch}(\mathbb{A})|_{N\setminus\mathrm{Supp}(\sigma)}=d\left(\beta(\sigma,\mathbb{A})\right)$.

We are in the situation of Definition 2.1. The closed form $\mathrm{Ch}(\mathbb{A})$ on N and the form $\beta(\sigma,\mathbb{A})$ on $N\setminus\mathrm{Supp}(\sigma)$ define an even relative cohomology class $[\mathrm{Ch}(\mathbb{A}),\beta(\sigma,\mathbb{A})]$ in $\mathcal{H}^*(N,N\setminus\mathrm{Supp}(\sigma))$.

Proposition 5. • *The class $[\mathrm{Ch}(\mathbb{A}),\beta(\sigma,\mathbb{A})]\in\mathcal{H}^*(N,N\setminus\mathrm{Supp}(\sigma))$ does not depend on the choice of $\mathbb{A}$, nor on the Hermitian structure on $\mathcal{E}$. We denote it by $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ and we call it the* Quillen Chern character.

• *Let F be a closed subset of N. For $s\in[0,1]$, let $\sigma_s:\mathcal{E}^+\to\mathcal{E}^-$ be a family of smooth morphisms such that $\mathrm{Supp}(\sigma_s)\subset F$. Then all classes $\mathrm{Ch}_{\mathrm{rel}}(\sigma_s)$ coincide in $\mathcal{H}^*(N,N\setminus F)$.*

Proof. Let us prove the first point. Let $\mathbb{A}_s, s\in[0,1]$, be a smooth one parameter family of super-connections on $\mathcal{E}$ without 0 exterior degree terms. Let $\mathbb{A}(s,t)=\mathbb{A}_s+itv_\sigma$. Thus $\frac{d}{ds}\mathbb{A}(s,t)=\frac{d}{ds}\mathbb{A}_s$ and $\frac{d}{dt}\mathbb{A}(s,t)=iv_\sigma$. Let $\mathbf{F}(s,t)$ be the curvature of $\mathbb{A}(s,t)$. We have $\frac{d}{ds}\mathrm{Ch}(\mathbb{A}_s)=d\gamma_s$ with $\gamma_s=\mathrm{Str}\left((\frac{d}{ds}\mathbb{A}_s)\,\mathrm{e}^{\mathbf{F}(s,0)}\right)$ and $\eta(\sigma,\mathbb{A}_s,t)=-\mathrm{Str}\left((\frac{d}{dt}\mathbb{A}(s,t))\,\mathrm{e}^{\mathbf{F}(s,t)}\right)$. We apply the double transgression formula of Proposition 3, and we obtain

$$\frac{d}{ds}\eta(\sigma,\mathbb{A}_s,t)=-\frac{d}{dt}\mathrm{Str}\left((\frac{d}{ds}\mathbb{A}(s,t))\,\mathrm{e}^{\mathbf{F}(s,t)}\right)-d\nu(s,t) \tag{21}$$

with

$$\begin{aligned}\nu(s,t)&=\int_0^1\mathrm{Str}\left((\frac{d}{ds}\mathbb{A}(s,t))\,\mathrm{e}^{u\mathbf{F}(s,t)}(\frac{d}{dt}\mathbb{A}(s,t))\,\mathrm{e}^{(1-u)\mathbf{F}(s,t)}\right)du\\&=\int_0^1\mathrm{Str}\left((\frac{d}{ds}\mathbb{A}_s)\,\mathrm{e}^{u\mathbf{F}(s,t)}(iv_\sigma)\,\mathrm{e}^{(1-u)\mathbf{F}(s,t)}\right)du.\end{aligned}$$

For $u,s\in[0,1]$ and $t\geq0$, we consider the element of $\mathcal{A}^*(N,\mathrm{End}(\mathcal{E}))$ defined by $I(u,s,t)=(\frac{d}{ds}\mathbb{A}_s)\,\mathrm{e}^{u\mathbf{F}(s,t)}(iv_\sigma)\,\mathrm{e}^{(1-u)\mathbf{F}(s,t)}$.

On a compact subset $\mathcal{K}$ of $N\setminus F$, σ is invertible and we can find $h>0$ such that $v_\sigma^2(n)\geq h$ when $n\in\mathcal{K}$. We have $u\mathbf{F}(s,t)=-t^2uv_\sigma^2-uR_{t,s}$, with $R_{t,s}=$

$-it[\mathbb{A}_s, v_\sigma] - \mathbb{A}_s^2$ which is a sum of terms with strictly positive exterior degrees. Remark that $R_{t,s}$ is a polynomial of degree 1 in t. By the estimate of Lemma 9 of the Appendix, we obtain

$$\begin{aligned}\left\|I(u,s,t)\right\| &\leq \left\|\frac{d}{ds}\mathbb{A}_s\right\| \left\|v_\sigma\right\| \left\| \mathrm{e}^{u\mathbf{F}(s,t)}\right\| \left\| \mathrm{e}^{(1-u)\mathbf{F}(s,t)}\right\| \\ &\leq \left\|\frac{d}{ds}\mathbb{A}_s\right\| \left\|v_\sigma\right\| \mathrm{e}^{-ht^2}\, \mathcal{P}(u\|R_{t,s}\|)\mathcal{P}((1-u)\|R_{t,s}\|)\end{aligned}$$

where $\mathcal{P}$ is a polynomial of degree less or equal to $\dim N$. So, we can find a constant C such that $\|I(u,s,t)\| \leq C(1+t^2)^{\dim N}\,\mathrm{e}^{-ht^2}$ for all $u, s \in [0,1]$ and $t \geq 0$. Thus we can integrate Equation (21) in t, from 0 to ∞. Since $-\int_0^\infty \frac{d}{dt}[\mathrm{Str}((\frac{d}{ds}\mathbb{A}(s,t))\,\mathrm{e}^{\mathbf{F}(s,t)})]dt = \gamma_s$, it follows that

$$\frac{d}{ds}\,\mathrm{Ch}(\mathbb{A}_s) = d\gamma_s, \qquad \frac{d}{ds}\beta(\sigma,\mathbb{A}_s) = \gamma_s - d\epsilon_s \tag{22}$$

where $\epsilon_s = \int_0^\infty \nu(s,t)dt$ and $\nu(s,t) = \int_0^1 I(u,s,t)du$. The first equality in Equations (22) holds on N, and the second on $N \setminus \mathrm{Supp}(\sigma)$. These equations (22) exactly mean that

$$\frac{d}{ds}\left(\mathrm{Ch}(\mathbb{A}_s), \beta(\sigma,\mathbb{A}_s)\right) = d_{\mathrm{rel}}\left(\gamma_s, \epsilon_s\right).$$

So the cohomology class $[\mathrm{Ch}(\mathbb{A}), \beta(\sigma,\mathbb{A})]$ in $\mathcal{H}^*(N, N\setminus \mathrm{Supp}(\sigma))$ does not depend on the choice of $\mathbb{A}$. With a similar proof, we see that this cohomology class does not depend on the choice of Hermitian structure on $\mathcal{E}$.

The proof of the second point is similar. □

We have defined a representative of the relative Chern class $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ using the one-parameter family $\mathbb{A}^\sigma(t)$ of super-connections, for t varying between 0 and ∞. We can define another representative as follows. We have $\mathrm{Ch}(\sigma,\mathbb{A},t) = d(\beta(\sigma,\mathbb{A},t))$ with

$$\beta(\sigma,\mathbb{A},t) = \int_t^\infty \eta(\sigma,\mathbb{A},s)ds. \tag{23}$$

Lemma 2. *For any $t \in \mathbb{R}$, the relative Chern character $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ satisfies $\mathrm{Ch}_{\mathrm{rel}}(\sigma) = [\mathrm{Ch}(\sigma,\mathbb{A},t), \beta(\sigma,\mathbb{A},t)]$ in $\mathcal{H}^*(N, N\setminus\mathrm{Supp}(\sigma))$.*

Proof. It is easy to check that

$$\Big(\mathrm{Ch}(\mathbb{A}), \beta(\sigma,\mathbb{A})\Big) - \Big(\mathrm{Ch}(\sigma,\mathbb{A},t), \beta(\sigma,\mathbb{A},t)\Big) = d_{\mathrm{rel}}(\delta(t), 0), \tag{24}$$

with $\delta(t) = \int_0^t \eta(\sigma,\mathbb{A},s)ds$. □

Remark 1. *Quillen relative Chern character seems to be very related to the "multiplicative K-theory" defined by Connes-Karoubi (see* [15], [16]*). For example, even if $\mathcal{E}^+, \mathcal{E}^-$ are flat bundles, the Quillen Chern character is usually non zero, as it also encodes the odd closed differential form $\omega = \beta(\sigma,\mathbb{A})$.*

4. Multiplicative property of $\mathrm{Ch}_{\mathrm{rel}}$

Let $\mathcal{E}_1, \mathcal{E}_2$ be two $\mathbb{Z}_2$-graded complex vector bundles on a manifold N. The space $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a $\mathbb{Z}_2$-graded complex vector bundle with even part $\mathcal{E}_1^+ \otimes \mathcal{E}_2^+ \oplus \mathcal{E}_1^- \otimes \mathcal{E}_2^-$ and odd part $\mathcal{E}_1^- \otimes \mathcal{E}_2^+ \oplus \mathcal{E}_1^+ \otimes \mathcal{E}_2^-$.

The complex super-algebra $\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}_1 \otimes \mathcal{E}_2))$ can be identified with $\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}_1)) \otimes \mathcal{A}^*(N, \mathrm{End}(\mathcal{E}_2))$ where the tensor is taken in the sense of super-algebras. Then, if $v_k \in \mathcal{A}^0(N, \mathrm{End}(\mathcal{E}_k)^-)$ are odd endomorphisms, we have $(v_1 \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes v_2)^2 = (v_1)^2 \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes (v_2)^2$.

Let $\sigma_1 : \mathcal{E}_1^+ \to \mathcal{E}_1^-$ and $\sigma_2 : \mathcal{E}_2^+ \to \mathcal{E}_2^-$ be two smooth morphisms. With the help of Hermitian structures, we define the morphism

$$\sigma_1 \odot \sigma_2 : (\mathcal{E}_1 \otimes \mathcal{E}_2)^+ \longrightarrow (\mathcal{E}_1 \otimes \mathcal{E}_2)^-$$

by $\sigma_1 \odot \sigma_2 := \sigma_1 \otimes \mathrm{Id}_{\mathcal{E}_2^+} + \mathrm{Id}_{\mathcal{E}_1^+} \otimes \sigma_2 + \mathrm{Id}_{\mathcal{E}_1^-} \otimes \sigma_2^* + \sigma_1^* \otimes \mathrm{Id}_{\mathcal{E}_2^-}$.

Let v_{σ_1} and v_{σ_2} be the odd Hermitian endomorphisms of $\mathcal{E}_1, \mathcal{E}_2$ associated to σ_1 and σ_2 (see (14)). Then $v_{\sigma_1 \odot \sigma_2} = v_{\sigma_1} \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes v_{\sigma_2}$ and $v^2_{\sigma_1 \odot \sigma_2} = v^2_{\sigma_1} \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes v^2_{\sigma_2}$. It follows that

$$\mathrm{Supp}(\sigma_1 \odot \sigma_2) = \mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2).$$

We can now state one of the main result of this paper.

Theorem 4.1. (Quillen's Chern character is multiplicative) *Let σ_1, σ_2 be two morphisms over N. The relative cohomology classes*

- $\mathrm{Ch}_{\mathrm{rel}}(\sigma_k) \in \mathcal{H}^*(N, N \setminus \mathrm{Supp}(\sigma_k))$*,* $k = 1, 2$*,*
- $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2) \in \mathcal{H}^*(N, N \setminus (\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)))$*,*

satisfy the following equality

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2) = \mathrm{Ch}_{\mathrm{rel}}(\sigma_1) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma_2)$$

in $\mathcal{H}^*(N, N \setminus (\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)))$. Here $\diamond$ is the product of relative classes (see (5)).

Proof. For $k = 1, 2$, we choose super-connections $\mathbb{A}_k$, without 0 exterior degree terms. We consider the closed forms $c_k(t) := \mathrm{Ch}(\sigma_k, \mathbb{A}_k, t)$ and the transgression forms $\eta_k(t) := \eta(\sigma_k, \mathbb{A}_k, t)$ so that $\frac{d}{dt}(c_k(t)) = -d(\eta_k(t))$. Let $\beta_k = \int_0^\infty \eta_k(t)dt$. A representative of $\mathrm{Ch}_{\mathrm{rel}}(\sigma_k)$ is $(c_k(0), \beta_k)$.

For the symbol $\sigma_1 \odot \sigma_2$, we consider $\mathbb{A}(t) = \mathbb{A} + itv_{\sigma_1 \odot \sigma_2}$ where $\mathbb{A} = \mathbb{A}_1 \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes \mathbb{A}_2$. Then $\mathrm{Ch}(\mathbb{A}) = c_1(0)c_2(0)$. Furthermore, it is easy to see that the transgression form for the family $\mathbb{A}(t)$ is $\eta(t) = \eta_1(t)c_2(t) + c_1(t)\eta_2(t)$. Let $\beta_{12} = \int_0^\infty \eta(t)dt$. A representative of $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2)$ is $(c_1(0)c_2(0), \beta_{12})$.

We consider the open subsets $U = N \setminus (\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2))$, and $U_k = N \setminus \mathrm{Supp}(\sigma_k)$. Let $\Phi_1 + \Phi_2 = 1_U$ be a partition of unity subordinate to the decomposition $U = U_1 \cup U_2$. The proof will be completed if one shows that

$$(c_1(0), \beta_1) \diamond_\Phi (c_2(0), \beta_2) - (c_1(0)c_2(0), \beta_{12}) \tag{25}$$

is d_{rel}-exact. We need the following lemma.

Lemma 3. *The integrals*

$$\begin{aligned} B_1 &= \iint_{0 \le t \le s} \Phi_1 \eta_1(s) \wedge \eta_2(t) ds\, dt, \\ B_2 &= \iint_{0 \le s \le t} \Phi_2 \eta_1(s) \wedge \eta_2(t) ds\, dt \end{aligned}$$

are well defined differential forms on U.

Proof. The function $(s,t) \mapsto \Phi_1\eta_1(s) \wedge \eta_2(t)$ is a function on $\mathbb{R}^2$ with values in $\mathcal{A}^*(U)$. We have to see that the integral B_1 is convergent on the domain $0 \le t \le s$. This fact follows directly from the estimates of Proposition 4. Indeed, let $\mathcal{K}$ be a compact subset of U. Since Φ_1 is supported on $U_1 = N \setminus \mathrm{Supp}(\sigma_1)$, there exists $h > 0$, and a polynomial $\mathcal{P}_1$ in s such that, on $\mathcal{K}$,

$$\|\Phi_1\eta_1(s)\| \le \mathcal{P}_1(s)\,\mathrm{e}^{-hs^2} \quad \text{for} \quad s \ge 0.$$

On the other hand, there exists a polynomial $\mathcal{P}_2$ in t such that, on $\mathcal{K}$,

$$\|\eta_2(t)\| \le \mathcal{P}_2(t) \quad \text{for} \quad t \ge 0.$$

Then, when $0 \le t \le s$, we have, on $\mathcal{K}$: $\|\Phi_1\eta_1(s) \wedge \eta_2(t)\| \le \mathcal{P}_1(s)\mathcal{P}_2(t)\ \mathrm{e}^{-hs^2}$ and the integral B_1 is absolutely convergent on $0 \le t < s$. Reversing the role $1 \leftrightarrow 2$, we prove in the same way that B_2 is well defined. □

We now prove that (25) is equal to $d_{\mathrm{rel}}\Big(0, B_1 - B_2\Big)$. Indeed

$$(c_1(0),\beta_1) \diamond_\Phi (c_2(0),\beta_2) = \Big(c_1(0)c_2(0), \Phi_1\beta_1 c_2(0) + c_1(0)\Phi_2\beta_2 - d\Phi_1\beta_1\beta_2\Big),$$

so that (25) is equal to $\Big(0, \Phi_1\beta_1 c_2(0) + c_1(0)\Phi_2\beta_2 - d\Phi_1\beta_1\beta_2 - \beta_{12}\Big)$. Thus we need to check that $dB_2 - dB_1 = \Phi_1\beta_1 c_2(0) + c_1(0)\Phi_2\beta_2 - d\Phi_1\beta_1\beta_2 - \beta_{12}$. We have $dB_1 = R_1 + S_1$ with

$$\begin{aligned} R_1 &= d\Phi_1 \wedge \iint_{0\le t\le s} \eta_1(s)\eta_2(t)\,ds\,dt, \\ S_1 &= \Phi_1 \iint_{0\le t\le s} \Big(d\eta_1(s)\eta_2(t) - \eta_1(s)d\eta_2(t)\Big)ds\,dt. \end{aligned}$$

Now we use $d\eta_j(s) = -\frac{d}{ds}c_j(s)$, so that we obtain

$$\begin{aligned} S_1 &= \Phi_1 \iint_{0\le t\le s} \Big((-\frac{d}{ds}c_1(s))\eta_2(t) + \eta_1(s)(\frac{d}{dt}c_2(t))\Big)\,ds\,dt \\ &= \Phi_1\Big(\int_0^\infty c_1(t)\eta_2(t)dt + \int_0^\infty \eta_1(s)c_2(s)ds\Big) - c_2(0)\Phi_1\beta_1, \end{aligned}$$

that is $S_1 = \Phi_1\beta_{12} - c_2(0)\Phi_1\beta_1$. Similarly, we compute that dB_2 is equal to $d\Phi_2 \wedge \iint_{0\le s\le t} \eta_1(s)\eta_2(t)\,ds\,dt - \Phi_2\beta_{12} + c_1(0)\Phi_2\beta_2$. So finally, as $d\Phi_1 = -d\Phi_2$, we get

$$dB_2 - dB_1 = -d\Phi_1 \int_0^\infty\int_0^\infty \eta_1(s)\eta_2(t)dsdt - \beta_{12} + c_2(0)\Phi_1\beta_1 + c_1(0)\Phi_2\beta_2$$

which was the equation to prove. □

5. Chern character of a morphism

We employ notations of Section 3.2. We work here with differential forms with *complex* coefficients.

5.1. **The Chern Character.** Let $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ be a morphism on N. Following Subsection 2.2, we consider the image of $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ through the map $\mathcal{H}^*(N, N \setminus \mathrm{Supp}(\sigma)) \to \mathcal{H}^*_{\mathrm{Supp}(\sigma)}(N)$. Applying Propositions 1 and 5, we obtain the following theorem.

Theorem 5.1. • *For any neighborhood U of* $\mathrm{Supp}(\sigma)$*, take $\chi \in \mathcal{C}^\infty(N)$ which is equal to 1 in a neighborhood of* $\mathrm{Supp}(\sigma)$ *and with support contained in U. The differential form*

$$c(\sigma, \mathbb{A}, \chi) = \chi\ \mathrm{Ch}(\mathbb{A}) + d\chi\, \beta(\sigma, \mathbb{A}) \tag{26}$$

is closed *and supported in U. Its cohomology class $c_U(\sigma) \in \mathcal{H}^*_U(N)$ does not depend on the choice of $\mathbb{A}, \chi$ and the Hermitian structures on $\mathcal{E}^\pm$. Furthermore, the inverse family $c_U(\sigma)$ when U runs over the neighborhoods of* $\mathrm{Supp}(\sigma)$ *defines a class*

$$\mathrm{Ch}_{\mathrm{sup}}(\sigma) \in \mathcal{H}^*_{\mathrm{Supp}(\sigma)}(N).$$

The image of this class in $\mathcal{H}^*(N)$ is the Chern character $\mathrm{Ch}(\mathcal{E})$ *of $\mathcal{E}$.*

• *Let F be a closed subset of N. For $s \in [0,1]$, let $\sigma_s : \mathcal{E}^+ \to \mathcal{E}^-$ be a family of smooth morphisms such that* $\mathrm{Supp}(\sigma_s) \subset F$*. Then all classes* $\mathrm{Ch}_{\mathrm{sup}}(\sigma_s)$ *coincide in $\mathcal{H}^*_F(N)$.*

Using Lemma 2 we get

Lemma 4. *For any $t \geq 0$, the class* $\mathrm{Ch}_{\mathrm{sup}}(\sigma)$ *can be defined with the forms $c(\sigma, \mathbb{A}, \chi, t) = \chi\ \mathrm{Ch}(\sigma, \mathbb{A}, t) + d\chi\, \beta(\sigma, \mathbb{A}, t)$.*

Proof. It is due to the following transgression

$$c(\sigma, \mathbb{A}, \chi) - c(\sigma, \mathbb{A}, \chi, t) = d(\chi \delta(t)), \tag{27}$$

which follows from (24). □

In some situations, *Quillen's Chern character* $\mathrm{Ch}_{\mathrm{Q}}(\sigma) = \mathrm{Ch}(\sigma, \mathbb{A}, 1)$ enjoys good properties relative to the integration. So it is natural to compare the differential forms $c(\sigma, \mathbb{A}, \chi)$ and $\mathrm{Ch}_{\mathrm{Q}}(\sigma)$.

Lemma 5. *We have*

$$c(\sigma, \mathbb{A}, \chi) - \mathrm{Ch}_{\mathrm{Q}}(\sigma) = d\left(\chi \int_0^1 \eta(\sigma, \mathbb{A}, s) ds\right) + d\Big((\chi - 1)\beta(\sigma, \mathbb{A}, 1)\Big).$$

Proof. This follows immediately from the transgressions (11) and (27). □

Definition 5.1. *When σ is elliptic, we denote by*

$$\mathrm{Ch}_c(\sigma) \in \mathcal{H}^*_c(N) \tag{28}$$

the cohomology class with compact support which is the image of $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ *through the map* $\mathrm{p}_c : \mathcal{H}^*(N, N \setminus \mathrm{Supp}(\sigma)) \to \mathcal{H}^*_c(N)$ *(see (10)).*

A representative of $\mathrm{Ch}_c(\sigma)$ is given by $c(\sigma, \mathbb{A}, \chi)$, where $\chi \in \mathcal{C}^\infty(N)$ is chosen with a compact support, and equal to 1 in a neighborhood of $\mathrm{Supp}(\sigma)$ and $c(\sigma, \mathbb{A}, \chi)$ is given by Formula (26).

We will now rewrite Theorem 4.1 for the Chern classes $\mathrm{Ch}_{\mathrm{sup}}$ and Ch_c. Let $\sigma_1 : \mathcal{E}_1^+ \to \mathcal{E}_1^-$ and $\sigma_2 : \mathcal{E}_2^+ \to \mathcal{E}_2^-$ be two smooth morphisms. Let $\sigma_1 \odot \sigma_2 : (\mathcal{E}_1 \otimes \mathcal{E}_2)^+ \to (\mathcal{E}_1 \otimes \mathcal{E}_2)^-$ be their product.

Following (6), the product of the elements $\mathrm{Ch}_{\mathrm{sup}}(\sigma_k) \in \mathcal{H}^*_{\mathrm{Supp}(\sigma_k)}(N)$ for $k = 1, 2$ belongs to $\mathcal{H}^*_{\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)}(N) = \mathcal{H}^*_{\mathrm{Supp}(\sigma_1 \odot \sigma_2)}(N)$.

Theorem 5.2. • *We have the equality*

$$\mathrm{Ch}_{\sup}(\sigma_1) \wedge \mathrm{Ch}_{\sup}(\sigma_2) = \mathrm{Ch}_{\sup}(\sigma_1 \odot \sigma_2) \quad \text{in} \quad \mathcal{H}^*_{\mathrm{Supp}(\sigma_1 \odot \sigma_2)}(N).$$

• *If the morphisms σ_1, σ_2 are elliptic, we have*

$$\mathrm{Ch}_c(\sigma_1) \wedge \mathrm{Ch}_c(\sigma_2) = \mathrm{Ch}_c(\sigma_1 \odot \sigma_2) \quad \text{in} \quad \mathcal{H}^*_c(N).$$

Proof. The second point is a consequence of the first point. Theorem 4.1 tells us that $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2) = \mathrm{Ch}_{\mathrm{rel}}(\sigma_1) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma_2)$ holds in $\mathcal{H}^*(N, N \setminus (\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)))$. We apply now the morphism "p" and use the relation (9) to get the first point. □

The second point of Theorem 5.2 has the following interesting refinement. Let σ_1, σ_2 be two morphisms on N which are **not elliptic**, and assume that the product $\sigma_1 \odot \sigma_2$ is **elliptic**. Since $\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)$ is compact, we consider neighborhoods U_k of $\mathrm{Supp}(\sigma_k)$ such that $\overline{U_1} \cap \overline{U_2}$ is compact. Let $\chi_k \in \mathcal{C}^\infty(N)$ be supported on U_k and equal to 1 in a neighborhood of $\mathrm{Supp}(\sigma_k)$. Then, the differential form $c(\sigma_1, \mathbb{A}_1, \chi_1) \wedge c(\sigma_2, \mathbb{A}_2, \chi_2)$ is compactly supported on N, and we have

$$\mathrm{Ch}_c(\sigma_1 \odot \sigma_2) = \Big[c(\sigma_1, \mathbb{A}_1, \chi_1) \wedge c(\sigma_2, \mathbb{A}_2, \chi_2) \Big] \quad \text{in} \quad \mathcal{H}^*_c(N).$$

Note that the differential forms $c(\sigma_k, \mathbb{A}_k, \chi_k)$ are not compactly supported.

5.2. **Comparison with other constructions.**

5.2.1. *Trivialization outside* $\mathrm{Supp}(\sigma)$. Outside the support of σ, the complex vector bundles $\mathcal{E}^+$ and $\mathcal{E}^-$ are "the same", so that it is natural to construct representatives of $\mathrm{Ch}(\mathcal{E}) = \mathrm{Ch}(\mathcal{E}^+) - \mathrm{Ch}(\mathcal{E}^-)$ which are zero "outside" the support of σ by the following identifications of bundles with connections. For simplicity, we assume in this section that σ is elliptic.

A connection $\nabla = \nabla^+ \oplus \nabla^-$ is said "adapted" to the morphism σ when the following holds

$$\begin{aligned} \nabla^- \circ \sigma + \sigma \circ \nabla^+ &= 0, \\ \nabla^+ \circ \sigma^* + \sigma^* \circ \nabla^- &= 0, \end{aligned} \tag{29}$$

outside a compact neighborhood of $\mathrm{Supp}(\sigma)$. An adapted connection is denoted by ∇^{adap}. It is easy to construct an adapted connection.

Proposition 6. *Let ∇^{adap} be a connection* adapted *to $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$. Then the differential form $\mathrm{Ch}(\nabla^{adap})$ is compactly supported and its cohomology class coincides with $\mathrm{Ch}_c(\sigma)$ in $\mathcal{H}^*_c(N)$.*

Proof. Suppose that ∇^{adap} satisfies (29) outside a compact neighborhood C of $\mathrm{Supp}(\sigma)$. We verify that the forms $\mathrm{Ch}(\nabla^{\mathrm{adap}})$ as well as the form $\beta(\sigma, \nabla^{\mathrm{adap}})$ are supported on C. Thus if $\chi \in \mathcal{C}^\infty(N)$ is equal to 1 on C, we see that the differential forms $c(\sigma, \nabla^{\mathrm{adap}}, \chi)$ and $\mathrm{Ch}(\nabla^{\mathrm{adap}})$ coincide. □

Remark 2. *If F is* compact*, the closed differential form $\mathrm{Ch}(\nabla^{adap})$ represents the Chern character of a difference bundle $[\mathcal{D}^+] - [\mathcal{D}^-]$, where $[\mathcal{D}^+]$ and $[\mathcal{D}^-]$ are complex vector bundles (isomorphic outside F) on a compactification N_F of N (see for example* [8]*). Thus $\mathrm{Ch}(\nabla^{adap})$ is a representative of the Chern character as defined by Atiyah and al. in* [2, 5]*. In this case, Theorem 5.2 is just the multiplicativity property of the Chern character in absolute theories.*

5.2.2. *Gaussian look.* In [17], Mathai-Quillen gives an explicit representative with "Gaussian look" of the Bott class of a complex vector bundle $N \to B$. The purpose of this paragraph is to compare the Mathai-Quillen construction of Chern characters with "Gaussian look" and the Quillen relative construction.

Let N be a real vector bundle over a manifold B. We denote by $\pi : N \to B$ the projection. We denote by (x, ξ) a point of N with $x \in B$ and $\xi \in N_x$. Let $\mathcal{E}^\pm \to B$ be two Hermitian vector bundles. We consider a morphism $\sigma : \pi^*\mathcal{E}^+ \to \pi^*\mathcal{E}^-$.

We choose a metric on the fibers of the fibration $N \to B$. We work under the following assumption on σ.

Assumption 1. *The morphism $\sigma : \pi^*\mathcal{E}^+ \to \pi^*\mathcal{E}^-$ and all its partial derivatives have at most a polynomial growth along the fibers of $N \to B$. Moreover we assume that, for any compact subset $\mathcal{K}_B$ of B, there exist $R \geq 0$ and $c > 0$ such that*[1] *$v_\sigma^2(x, \xi) \geq c\|\xi\|^2$ when $\|\xi\| \geq R$ and $x \in \mathcal{K}_B$.*

We may define the sub-algebra $\mathcal{A}^*_{\text{dec-rap}}(N)$ of forms on N such that all partial derivatives are rapidly decreasing along the fibers. Let $\mathcal{H}^*_{\text{dec-rap}}(N)$ be the corresponding cohomology algebra. Under Assumption 1, the support of σ intersects the fibers of π in compact sets. We have then a canonical map from $\mathcal{H}^*_{\text{Supp}(\sigma)}(N)$ into $\mathcal{H}^*_{\text{dec-rap}}(N)$. We will now compute the image of $\text{Ch}_{\text{sup}}(\sigma)$ under this map.

Let $\nabla = \nabla^+ \oplus \nabla^-$ be a connection on $\mathcal{E} \to B$, and consider the super-connection $\mathbb{A} = \pi^*\nabla$ so that $\mathbb{A}^\sigma(t) = \pi^*\nabla + itv_\sigma$. Then, the *Quillen Chern character* form $\text{Ch}_{\text{Q}}(\sigma) := \text{Ch}(\sigma, \mathbb{A}, 1)$ has a "Gaussian" look.

Lemma 6. *The differential forms $\text{Ch}(\sigma, \mathbb{A}, 1)$ and $\beta(\sigma, \mathbb{A}, 1)$ are rapidly decreasing along the fibers.*

Proof. The curvature of $\mathbb{A}^\sigma(t)$ is

$$\mathbf{F}(t) = \pi^*\mathbf{F} - t^2 v_\sigma^2 + it[\pi^*\nabla, v_\sigma].$$

Here $\mathbf{F} \in \mathcal{A}^2(B, \text{End}(\mathcal{E}))$ is the curvature of ∇. Assumption 1 implies that $[\pi^*\nabla, v_\sigma] \in \mathcal{A}^1(N, \text{End}(\pi^*\mathcal{E}))$ has at most a polynomial growth along the fibers. Furthermore, for any compact subset $\mathcal{K}_B$ of the basis, there exists $R \geq 0$ and $c > 0$ such that $v_\sigma^2(x, \xi) \geq c\|\xi\|^2$ when $\|\xi\| \geq R$ and $x \in \mathcal{K}_B$.

To estimate $\mathrm{e}^{\mathbf{F}(t)}$, we apply Lemma 9 of the Appendix, with $H = t^2 v_\sigma^2$, and $R = -\pi^*\mathbf{F} - it[\pi^*\nabla, v_\sigma]$. The smallest eigenvalue of H is greater or equal to $t^2 c\|\xi\|^2$, when $\|\xi\| \geq R$, and R is a sum of terms with strictly positive exterior degrees. Remark that R is a polynomial in t of degree 1 and is bounded in norm by a polynomial in $\|\xi\|$ along the fibers. It follows that from Lemma 9 that, for $t \geq 0$, we have

$$\left\| \mathrm{e}^{\mathbf{F}(t)} \right\|(x, \xi) \leq \mathcal{P}(\|R\|)\, \mathrm{e}^{-t^2 c\|\xi\|^2}.$$

Our estimates on the polynomial growth of R in t and $\|\xi\|$ implies that there exists a polynomial $\mathcal{Q}$ such that, for $t \geq 0$,

$$\left\| \mathrm{e}^{\mathbf{F}(t)} \right\|(x, \xi) \leq \mathcal{Q}(t\|\xi\|)\, \mathrm{e}^{-t^2 c\|\xi\|^2}, \tag{30}$$

for $(x, \xi) \in N$, $x \in \mathcal{K}_B$, $\|\xi\| \geq R$.

This implies that $\text{Ch}(\sigma, \mathbb{A}, 1) = \text{Str}(\mathrm{e}^{\mathbf{F}(1)})$ is rapidly decreasing along the fibers. Consider now $\beta(\sigma, \mathbb{A}, 1) = -i \int_1^\infty \text{Str}(v_\sigma\, \mathrm{e}^{\mathbf{F}(t)}) dt$ which is defined (at least) for $\|\xi\| \geq$

[1] This inequality means that $\|\sigma(x, \xi)w\|^2 \geq c\|\xi\|^2\|w\|^2$ for any $w \in \mathcal{E}_x$.

R. The estimate (30) shows also that $\beta(\sigma, \mathbb{A}, 1)$ is rapidly decreasing along the fibers. We can prove in the same way that all partial derivatives of $\mathrm{Ch}(\sigma, \mathbb{A}, 1)$ and $\beta(\sigma, \mathbb{A}, 1)$ are rapidly decreasing along the fibers: hence $\mathrm{Ch}(\sigma, \mathbb{A}, 1) \in \mathcal{A}^*_{\text{dec-rap}}(N)$ and $\beta(\sigma, \mathbb{A}, 1) \in \mathcal{A}^*_{\text{dec-rap}}(N \setminus \mathrm{Supp}(\sigma))$. □

Proposition 7. *Quillen's Chern character form* $\mathrm{Ch}_{\mathrm{Q}}(\sigma) \in \mathcal{A}^*_{\text{dec-rap}}(N)$ *represents the image of the class* $\mathrm{Ch}_{\mathrm{sup}}(\sigma) \in \mathcal{H}^*_{\mathrm{Supp}(\sigma)}(N)$ *in* $\mathcal{H}^*_{\text{dec-rap}}(N)$.

Proof. Choosing χ supported on $\|\xi\| \leq R + 1$ and equal to 1 in a neighborhood of $\|\xi\| \leq R$, the transgression formula of Lemma 5: $c(\sigma, \mathbb{A}, \chi) - \mathrm{Ch}(\sigma, \mathbb{A}, 1) = d(\chi \int_0^1 \eta(\sigma, \mathbb{A}, s) ds) + d\,((\chi - 1)\beta(\sigma, \mathbb{A}, 1))$ implies our proposition, since the form $c(\sigma, \mathbb{A}, \chi)$ represents $\mathrm{Ch}_{\mathrm{sup}}(\sigma)$ in $\mathcal{H}^*_{\text{dec-rap}}(N)$. □

When the fibers of $\pi : N \to B$ are oriented, we have an integration morphism $\int_{\text{fiber}} : \mathcal{H}^*_{\text{dec-rap}}(N) \to \mathcal{H}^*(B)$.

Corollary 1. *We have* $\int_{\text{fiber}} \mathrm{Ch}_{\mathrm{Q}}(\sigma) = \int_{\text{fiber}} \mathrm{Ch}_{\mathrm{sup}}(\sigma)$ *in* $\mathcal{H}^*(B)$.

5.3. **Examples.** If $\mathcal{E}$ is a trivial bundle $N \times E$ on a manifold N, an endomorphism of $\mathrm{End}(\mathcal{E})$ is determined by a map from N to $\mathrm{End}(E)$. We employ the same notation for both objects, so that if σ is a map from N to $\mathrm{End}(E)$, we also denote by σ the bundle map $\sigma[n, v] = [n, \sigma(n)v]$, for $n \in N$ and $v \in E$.

We will use the following convention. Let $E = E^+ \oplus E^-$ be a $\mathbb{Z}_2$-graded finite dimensional complex vector space. Let $\mathcal{A}$ be a super-commutative algebra (the ring of differential forms on a manifold for example). The elements of the super-algebra $\mathcal{A} \otimes \mathrm{End}(E)$ will be represented by matrices with coefficients in $\mathcal{A}$. This algebra operates on the space $\mathcal{A} \otimes E$. We take the following convention: the forms are always considered as operating first: for example, if $E^+ = \mathbb{C}$ and $E^- = \mathbb{C}$, the matrix $\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$ represents the operator

$$\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \alpha + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \beta \tag{31}$$

on $\mathcal{A} \otimes E$.

5.3.1. *The cotangent bundle* $\mathbf{T}^*S^1$. We consider $\mathbf{T}^*S^1 := S^1 \times \mathbb{R}$ the cotangent bundle to the circle S^1. The group $\mathbf{K}^0(\mathbf{T}^*S^1)$ of K-theory is generated, as a $\mathbb{Z}$-module, by the class $[\sigma]$ of the following elliptic symbol.

Take $\mathcal{E}^+ = \mathcal{E}^-$ the trivial bundles $\mathbf{T}^*S^1 \times \mathbb{C}$ over $\mathbf{T}^*S^1$. Let $u \in \mathcal{C}^\infty(\mathbb{R})$ be a function satisfying $u(\xi) = 1$ if $|\xi| > 1$ and $u(\xi) = 0$ if $|\xi| < 1/2$. The symbol $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ is defined by the map $\sigma : \mathbf{T}^*S^1 \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}$:

$$\sigma(\mathrm{e}^{i\theta}, \xi) = \begin{cases} u(\xi)\,\mathrm{e}^{i\theta}, & \text{if} \quad \xi \geq 0; \\ u(\xi), & \text{if} \quad \xi \leq 0. \end{cases}$$

Here $\mathrm{Supp}(\sigma) = \{(\mathrm{e}^{i\theta}, \xi);\ u(\xi) = 0\}$ is compact. Note that the class $[\sigma] \in \mathbf{K}^0(\mathbf{T}^*S^1)$ does not depend on the choice of the function u.

We choose on $\mathcal{E}^\pm$ the trivial connections $\nabla^+ = \nabla^- = d$ and we let $\mathbb{A} = \nabla^+ \oplus \nabla^-$ be the trivial connection on $\mathcal{E}^+ \oplus \mathcal{E}^-$. Then $\mathrm{Ch}(\mathbb{A}) = 0$. The curvature $\mathbf{F}(\sigma, \mathbb{A}, t)$ of the super-connection

$$\mathbb{A}^\sigma(t) = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} + \begin{pmatrix} 0 & itu(\xi)\,\mathrm{e}^{-i\theta} \\ itu(\xi)\,\mathrm{e}^{i\theta} & 0 \end{pmatrix}$$

is represented by the matrix $\begin{pmatrix} a & -\overline{b} \\ b & a \end{pmatrix}$ where $a(t,(\mathrm{e}^{i\theta},\xi)) = -t^2u(\xi)^2$ and

$$b(t,(\mathrm{e}^{i\theta},\xi)) = \begin{cases} -it\,\mathrm{e}^{-i\theta}(u'(\xi)d\xi - iu(\xi)d\theta), & \text{if} \quad \xi \geq 0; \\ -itu'(\xi)d\xi, & \text{if} \quad \xi \leq 0. \end{cases}$$

Then $\mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)}$ is represented by the matrix $\mathrm{e}^{-t^2u(\xi)^2}\begin{pmatrix} A & -\overline{b} \\ b & \overline{A} \end{pmatrix}$, where

$$A(t,(\mathrm{e}^{i\theta},\xi)) = \begin{cases} 1 + it^2u(\xi)u'(\xi)d\xi\, d\theta, & \text{if} \quad \xi \geq 0; \\ 1, & \text{if} \quad \xi \leq 0. \end{cases}$$

Thus $\eta(\sigma,\mathbb{A},t) = -\operatorname{Str}\left(iv_\sigma\ \mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)}\right)$ is given by

$$\eta(\sigma,\mathbb{A},t)(\mathrm{e}^{i\theta},\xi) = \begin{cases} -2it\,\mathrm{e}^{-t^2u(\xi)^2}\ u(\xi)^2d\theta, & \text{if} \quad \xi \leq 0; \\ 0, & \text{if} \quad \xi \geq 0. \end{cases}$$

Finally, integrating $\eta(\sigma,\mathbb{A},t)$ in t from 0 to ∞, we find that $\beta(\sigma,\mathbb{A})$ (which is defined on $\{(\mathrm{e}^{i\theta},\xi);\ u(\xi)\neq 0\} = \mathbf{T}^*S^1 \setminus \operatorname{Supp}(\sigma))$ is equal to

$$\beta(\sigma,\mathbb{A})(\mathrm{e}^{i\theta},\xi) = \begin{cases} -id\theta, & \text{if} \quad \xi \geq 0, u(\xi)\neq 0 \\ 0, & \text{if} \quad \xi \leq 0, u(\xi) \neq 0. \end{cases} \tag{32}$$

We have then proved the following

Proposition 8. • *The relative Chern class* $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ *is represented* $(0,\beta(\sigma,\mathbb{A}))$.

• *The Chern class with compact support* $\mathrm{Ch}_{\mathrm{c}}(\sigma)$ *is represented by the differential form* $-i\,\mathbf{1}_{\geq 0}\,d\chi\wedge d\theta$ *where* $\chi \in \mathcal{C}^\infty(\mathbb{R})$ *is compactly supported and equal to* 1 *on* $[-1,1]$, *and* $\mathbf{1}_{\geq 0}$ *is the characteristic function of the interval* $[0,\infty[$

Note that the differential form $-i\,\mathbf{1}_{\geq 0}\,d\chi\wedge d\theta$ is of integral $-2i\pi$ on $\mathbf{T}^*S^1$ (which is oriented by $d\theta\wedge d\xi$).

5.3.2. *The space* $\mathbb{R}^2$. Now we consider the case where $N = \mathbb{R}^2 \simeq \mathbb{C}$. Take $\mathcal{E}^+ = \mathcal{E}^-$ the trivial bundles $N\times\mathbb{C}$ over N. We consider Bott's symbol $\sigma_b : \mathcal{E}^+ \to \mathcal{E}^-$ which is given by the map $\sigma_b(z) = z$ for $z \in N \simeq \mathbb{C}$. The support of σ_b is reduced to the origin $\{0\}$, thus σ_b defines an element of $\mathbf{K}^0(\mathbb{R}^2)$. Recall that the Bott isomorphism tells us that $\mathbf{K}^0(\mathbb{R}^2)$ is a free $\mathbb{Z}$-module with base σ_b.

We choose on $\mathcal{E}^\pm$ the trivial connections $\nabla^+ = \nabla^- = d$. Let $\mathbb{A} = \nabla^+\oplus\nabla^-$ be the trivial connection on $\mathcal{E}^+\oplus\mathcal{E}^-$. The curvature $\mathbf{F}(\sigma_b,\mathbb{A},t)$ of the super-connection $\mathbb{A}^{\sigma_b}(t) = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} + \begin{pmatrix} 0 & it\overline{z} \\ itz & 0 \end{pmatrix}$ has the matrix form (see (31))

$$\mathbf{F}(\sigma_b,\mathbb{A},t) = \begin{pmatrix} -t^2|z|^2 & 0 \\ 0 & -t^2|z|^2 \end{pmatrix} - it\begin{pmatrix} 0 & d\overline{z} \\ dz & 0 \end{pmatrix}.$$

Thus

$$\mathrm{e}^{\mathbf{F}(\sigma_b,\mathbb{A},t)} = \mathrm{e}^{-t^2|z|^2}\begin{pmatrix} 1-\frac{t^2}{2}dzd\overline{z} & -itd\overline{z} \\ -itdz & 1+\frac{t^2}{2}dzd\overline{z} \end{pmatrix}$$

and $\eta(\sigma_b,\mathbb{A},t) = -\operatorname{Str}\left(iv_{\sigma_b}\ \mathrm{e}^{\mathbf{F}(\sigma_b,\mathbb{A},t)}\right)$ is equal to

$$\eta(\sigma_b,\mathbb{A},t) = -t(\overline{z}dz - zd\overline{z})\,\mathrm{e}^{-t^2|z|^2}. \tag{33}$$

When $z \neq 0$, we obtain that $\beta(\sigma_b, \mathbb{A})(z) = \int_0^\infty \eta(\sigma_b, \mathbb{A}, t)dt$ is equal to $\frac{1}{2|z|^2}(zd\overline{z} - \overline{z}dz) = -i\, d(\arg z)$. Thus we have

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_b) = [0, -i\, d(\arg z)]. \tag{34}$$

It is easy to see that $\mathrm{Ch}_{\mathrm{rel}}(\sigma_b)$ is a basis of the vector space $\mathcal{H}^*(\mathbb{C}, \mathbb{C} \setminus \{0\})$.

Take $f \in \mathcal{C}^\infty(\mathbb{R})$ with compact support and equal to 1 in a neighborhood of 0. Let $\chi(z) := f(|z|^2)$. Then the class $\mathrm{Ch}_c(\sigma_b) \in \mathcal{H}_c^*(\mathbb{R}^2)$ is represented by the differential form $c(\sigma_b, \mathbb{A}, \chi) = \chi\,\mathrm{Ch}(\mathbb{A}) + d\chi\beta(\sigma_b, \mathbb{A})$. Here the differential form $\mathrm{Ch}(\mathbb{A})$ is identically equal to 0. We obtain

$$\begin{aligned} c(\sigma_b, \mathbb{A}, \chi) &= d(f(|z|^2)) \wedge \beta(\sigma_b, \mathbb{A}) \\ &= -f'(|z|^2)d\overline{z} \wedge dz. \end{aligned}$$

Remark that $c(\sigma_b, \mathbb{A}, \chi)$ is compactly supported and of integral equal to $2i\pi$ on $\mathbb{R}^2$ (with orientation $dx \wedge dy$). Thus $\frac{1}{2i\pi}c(\sigma_b, \mathbb{A}, \chi)$ is a representative of the Thom form of $\mathbb{R}^2$.

Remark 3. *For $t > 0$, the Chern character of the super-connection $\mathbb{A}^{\sigma_b}(t)$ is the degree 2 differential form with "Gaussian look"*

$$\mathrm{Ch}(\sigma_b, \mathbb{A}, t) = -\,\mathrm{e}^{-t^2|z|^2}\, t^2 dz d\overline{z}.$$

For any $t > 0$, $\mathrm{Ch}(\sigma_b, \mathbb{A}, t)$ and $c(\sigma_b, \mathbb{A}, \chi)$ coincide in the cohomology $\mathcal{H}^*_{\text{dec-rap}}(\mathbb{R}^2)$, as follows from Proposition 7. In particular they have the same integral.

5.3.3. *The multiplicativity property on $\mathbb{C}^2$.* Following the notations of preceding example, we consider $\mathbb{C}^2$ with coordinates $z = (z_1, z_2)$ and morphisms $\sigma_1 = z_1$ and $\sigma_2 = z_2$. Then the tensor product morphism is

$$\sigma_1 \odot \sigma_2 = \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix}.$$

The morphism $\sigma_1 \odot \sigma_2$ has support $z_1 = z_2 = 0$. A calculation similar to the calculation done in the preceding section gives the following

Proposition 9. *The relative chern class $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2) \in \mathcal{H}^*(\mathbb{C}^2, \mathbb{C}^2 \setminus (0,0))$ is represented by $(0, \beta_{12})$, where*

$$\beta_{12} = \frac{-1}{2|z|^4}\Big((\overline{z_1}dz_1 - z_1 d\overline{z_1}) \wedge d\overline{z_2} \wedge dz_2 + (\overline{z_2}dz_2 - z_2 d\overline{z_2}) \wedge d\overline{z_1} \wedge dz_1\Big)$$

is a closed form on $\mathbb{C}^2 \setminus (0,0)$.

Remark that β_{12} is invariant under the symmetry group $U(2)$ of $\mathbb{C}^2$.

Recall that $\mathrm{Ch}_{\mathrm{rel}}(\sigma_k) = [0, \beta_k]$, with $\beta_k = -\frac{\overline{z_k}dz_k - z_k d\overline{z_k}}{2|z_k|^2}$. The wedge product $\beta_1 \wedge \beta_2$ is not defined on $\mathbb{C}^2 \setminus (0,0)$. Introduce a partition of unity Φ_1, Φ_2 with respect to the covering $U_1 \cup U_2$ of $\mathbb{C}^2 \setminus (0,0)$, with $U_k = \{z, z_k \neq 0\}$. Then the relative product $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma_2)$ has representative $(0, \beta)$, with $\beta = -d\Phi_1 \wedge \beta_1 \wedge \beta_2$.

We now compute the forms B_1, B_2 of the equation (5). The form $\eta_k(t)$ have been computed (Equation 33). From this it is easy to compute $B_1 = \Phi_1 \int_{0 \leq s \leq t} \eta_1(t)\eta_2(s)ds\, dt$ and B_2. We obtain

$$\begin{aligned} B_1 &= \Phi_1(z_1, z_2)\frac{(\overline{z_1}dz_1 - z_1 d\overline{z_1}) \wedge (\overline{z_2}dz_2 - z_2 d\overline{z_2})}{4|z_1|^2(|z_1|^2 + |z_2|^2)}, \\ B_2 &= \Phi_2(z_1, z_2)\frac{(\overline{z_1}dz_1 - z_1 d\overline{z_1}) \wedge (\overline{z_2}dz_2 - z_2 d\overline{z_2})}{4|z_2|^2(|z_1|^2 + |z_2|^2)}. \end{aligned}$$

Here $B_1 - B_2$ is a two form which is well defined on $\mathbb{C}^2 \setminus (0,0)$ and the relation $\Phi_1 + \Phi_2 = 1$ imply

$$\beta_{12} - \beta = d(B_1 - B_2).$$

This shows that the class $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2)$ is the product $[0, \beta_1] \diamond [0, \beta_2]$.

We can now look at the different representatives of the Chern class with compact support $\mathrm{Ch}_c(\sigma_1 \odot \sigma_2) \in \mathcal{H}_c^*(\mathbb{C}^2)$. Let $f \in \mathcal{C}^\infty(\mathbb{R})$ with compact support and equal to 1 in a neighborhood of 0. We consider the functions $\chi(z) = f(|z_1|^2 + |z_2|^2)$ and $\chi_k(z_k) = f(|z_k|^2)$. Let $\Omega = d\overline{z_1} \wedge dz_1 \wedge d\overline{z_2} \wedge dz_2$.

Proposition 10. *The Chern class* $\mathrm{Ch}_c(\sigma_1 \odot \sigma_2) \in \mathcal{H}_c^*(\mathbb{C}^2)$ *is represented by any of the following differential forms*

$$\begin{aligned} c(\sigma_1 \odot \sigma_2, \mathbb{A}, \chi) &= -\frac{f'\left(|z_1|^2 + |z_2|^2\right)}{|z_1|^2 + |z_2|^2}\,\Omega \\ c(\sigma_1, \mathbb{A}_1, \chi_1) \wedge c(\sigma_2, \mathbb{A}_2, \chi_2) &= f'\left(|z_1|^2\right) f'\left(|z_2|^2\right)\,\Omega. \end{aligned}$$

Clearly the first representative is "better", as it is invariant by the full symmetry group $SO(4)$ on $\mathbb{C}^2 = \mathbb{R}^4$.

6. Riemann-Roch formula in relative cohomology

In this section, we work with differential forms with *real* coefficients until Subsection 6.4.

6.1. **Some notations.** Let V be an Euclidean vector space of dimension d, with oriented orthonormal basis $e_1, e_2, \ldots, e_d$. We identify the Lie algebra $\mathfrak{so}(V)$ of $SO(V)$ with $\Lambda^2 V$ as follows: to an antisymmetric matrix A in $\mathfrak{so}(V)$, we associate the element $\sum_{i<j}(Ae_i, e_j) e_i \wedge e_j$ of $\Lambda^2 V$. This identification will be in place throughout this section.

The Berezin integral $\mathrm{T} : \Lambda V \to \mathbb{R}$ is the $\mathbb{R}$-linear map which vanishes on $\Lambda^i V$ for $i < d$ and is such that $\mathrm{T}(e_1 \wedge e_2 \wedge \cdots \wedge e_d) = 1$.

6.2. **Thom class in relative cohomology.** Let M be a manifold. Let $p : \mathcal{V} \to M$ be a real oriented Euclidean vector bundle over M of rank d. In this section, we give a construction for the relative Thom form, analogous to Quillen's construction of the Chern character. Here, we use the Berezin integral which is the "super-commutative" analog of the super-trace for endomorphisms of a super-space.

Recall the sub-space $\mathcal{A}^*_{\text{fiber cpt}}(\mathcal{V}) \subset \mathcal{A}^*(\mathcal{V})$ of (real) differential forms on $\mathcal{V}$ which have a compact support in the fibers of $p : \mathcal{V} \to M$. We have also defined the sub-space $\mathcal{A}^*_{\text{dec-rap}}(\mathcal{V})$. The integration over the fiber, that we denote by p_*, is well defined on the three spaces $\mathcal{A}^*(\mathcal{V}, \mathcal{V} \setminus M)$, $\mathcal{A}^*_{\text{fiber cpt}}(\mathcal{V})$ and $\mathcal{A}^*_{\text{dec-rap}}(\mathcal{V})$ and take values in $\mathcal{A}^*(M)$. A Thom form on $\mathcal{V}$ will be a (real) closed element which integrates to the constant function 1 on M.

Let ∇ be an Euclidean connection on $\mathcal{V}$. As the structure group of $\mathcal{V}$ is the Lie group $SO(V)$ with Lie algebra $\mathfrak{so}(V)$, the curvature $\mathbf{F}$ of ∇ is a two-form with values antisymmetric transformations of $\mathcal{V}$. We will identify the curvature to an element $\mathbf{F} \in \mathcal{A}^2(M, \Lambda^2 \mathcal{V})$ according to the isomorphism $\mathfrak{so}(V) \sim \Lambda^2 V$ described above. Let $\mathrm{T} : \Gamma(M, \Lambda \mathcal{V}) \to \mathcal{C}^\infty(M)$ be the Berezin integral that we extend to a $\mathbb{R}$-linear map $\mathrm{T} : \mathcal{A}^*(M, \Lambda \mathcal{V}) \to \mathcal{A}^*(M)$. The pfaffian of an element $L \in \mathcal{A}^*(M, \Lambda^2 \mathcal{V})$ is defined by: $\mathrm{Pf}(L) := \mathrm{T}(\mathrm{e}^L)$.

Definition 6.1. *Let ∇ be an Euclidean connection on $\mathcal{V}$, with curvature form* $\mathbf{F}$. *The Euler form* $\mathrm{Eul}(\mathcal{V},\nabla)\in\mathcal{A}^*(M)$ *of the bundle $\mathcal{V}\to M$ is the closed real differential form on M defined by* $\mathrm{Eul}(\mathcal{V},\nabla):=\mathrm{Pf}\left(-\frac{\mathbf{F}}{2\pi}\right)$. *The class of* $\mathrm{Eul}(\mathcal{V},\nabla)$, *which does not depend on ∇, is denoted by* $\mathrm{Eul}(\mathcal{V})\in\mathcal{H}^d(M)$.

Remark 4. *Since the pfaffian vanishes when the rank of $\mathcal{V}$ is odd, the Euler class* $\mathrm{Eul}(\mathcal{V})\in\mathcal{H}^d(M)$ *is identically equal to* 0 *when the rank of $\mathcal{V}$ is odd.*

Let us consider the vector bundle $p^*\mathcal{V}\to\mathcal{V}$ equipped with the pull-back connection $p^*\nabla$. Let $\mathbf{x}$ be the canonical section of the bundle $p^*\mathcal{V}$. We consider the $\mathbb{Z}/2\mathbb{Z}$ graded algebra $\mathcal{A}^*(\mathcal{V},\Lambda p^*\mathcal{V})$ which is equipped with the Berezin integral $\mathrm{T}:\mathcal{A}^*(\mathcal{V},\Lambda p^*\mathcal{V})\to\mathcal{A}^*(\mathcal{V})$.

Let $f_t^\nabla\in\mathcal{A}^*(\mathcal{V},\Lambda p^*\mathcal{V})$ be the element defined by the equation

$$f_t^\nabla=-t^2\|\mathbf{x}\|^2+t\,p^*\nabla\mathbf{x}+\frac{1}{2}p^*\mathbf{F}. \tag{35}$$

We consider the real differential forms on $\mathcal{V}$ defined by

$$\mathrm{C}^t_\wedge := \mathrm{T}\left(\mathrm{e}^{f_t^\nabla}\right), \tag{36}$$

$$\eta^t_\wedge := -\mathrm{T}\left(\mathbf{x}\,\mathrm{e}^{f_t^\nabla}\right). \tag{37}$$

Here the exponentials are computed in the super-algebra $\mathcal{A}^*(\mathcal{V},\Lambda p^*\mathcal{V})$. To be more concrete, this calculation is performed explicitly for a rank two bundle in Example 6.2 at the end of this subsection.

Lemma 7. *The differential form $\mathrm{C}^t_\wedge$ is closed. Furthermore,*

$$\frac{d}{dt}\mathrm{C}^t_\wedge=-d(\eta^t_\wedge). \tag{38}$$

Proof. The proof of the first point is given in [7] (Chapter 7, Theorem 7.41). We recall the proof. We denote by $\iota_\wedge(\mathbf{x})$ the derivation of the super-algebra $\mathcal{A}^*(\mathcal{V},\Lambda p^*\mathcal{V})$ such that $\iota_\wedge(\mathbf{x})s=\langle\mathbf{x},s\rangle$ when $s\in\mathcal{A}^0(\mathcal{V},\Lambda^1p^*\mathcal{V})$. We extend the connection $p^*\nabla$ to a derivation $\nabla^\wedge$ of $\mathcal{A}^*(\mathcal{V},\Lambda p^*\mathcal{V})$. We consider the derivation $\nabla^\wedge-2t\iota_\wedge(\mathbf{x})$ on $\mathcal{A}^*(\mathcal{V},\Lambda p^*\mathcal{V})$. It is easy to verify that

$$(\nabla^\wedge-2t\iota_\wedge(\mathbf{x}))f_t^\nabla=0. \tag{39}$$

Then, the exponential $\mathrm{e}^{f_t^\nabla}$ satisfies also $(\nabla^\wedge-2t\iota_\wedge(\mathbf{x}))(\mathrm{e}^{f_t^\nabla})=0$. The Berezin integral is such that $\mathrm{T}(\iota_\wedge(\mathbf{x})\alpha)=0$ and $\mathrm{T}(\nabla^\wedge\alpha)=d(\mathrm{T}(\alpha))$ for any $\alpha\in\mathcal{A}^*(\mathcal{V},\Lambda p^*\mathcal{V})$. This shows that $d\left(\mathrm{T}(\mathrm{e}^{f_t^\nabla})\right)=0$.

Let us prove the second point. We have $d\circ\mathrm{T}\left(\mathbf{x}\,\mathrm{e}^{f_t^\nabla}\right)=\mathrm{T}\circ(\nabla^\wedge-2t\iota_\wedge(\mathbf{x}))\left(\mathbf{x}\,\mathrm{e}^{f_t^\nabla}\right)$, and since $(\nabla^\wedge-2t\iota_\wedge(\mathbf{x}))\,\mathrm{e}^{f_t^\nabla}=0$, we get

$$\begin{aligned}(\nabla^\wedge-2t\iota_\wedge(\mathbf{x}))\left(\mathbf{x}\,\mathrm{e}^{f_t^\nabla}\right) &= \left((\nabla^\wedge-2t\iota_\wedge(\mathbf{x}))\cdot\mathbf{x}\right)\mathrm{e}^{f_t^\nabla}\\ &= (\nabla^\wedge\mathbf{x}-2t\|\mathbf{x}\|^2)\,\mathrm{e}^{f_t^\nabla}\\ &= \frac{d}{dt}\,\mathrm{e}^{f_t^\nabla}.\end{aligned}$$

□

When $t=0$, then $\mathrm{C}^0_\wedge$ is just equal to $\mathrm{Pf}(\frac{\mathbf{F}}{2})=(-\pi)^{d/2}\,\mathrm{Eul}(\mathcal{V},\nabla)$. When $t=1$, then $\mathrm{C}^1_\wedge=\mathrm{T}(\mathrm{e}^{f_1^\wedge})=\mathrm{e}^{-\|\mathbf{x}\|^2}Q$ is a closed form with a Gaussian look on $\mathcal{V}$: Q

a differential form on $\mathcal{V}$ with a polynomial growth on the fiber of $\mathcal{V} \to M$ (we will be more explicit in a short while). This differential form was considered by Mathai-Quillen in [17].

We have $\eta^t_\wedge = \mathrm{e}^{-t^2\|\mathbf{x}\|^2} Q(t)$ where $Q(t)$ is a differential form on $\mathcal{V}$ with a polynomial growth on the fiber of $\mathcal{V}$ and which depends polynomially on $t \in \mathbb{R}$. Thus, if $\mathbf{x} \neq 0$, when t goes to infinity, $\eta^t_\wedge$ is an exponentially decreasing function of t. We can thus define the following differential form on $\mathcal{V} \setminus M$:

$$\beta_\wedge = \int_0^\infty \eta^t_\wedge \, dt. \tag{40}$$

If we integrate (38) between 0 and ∞, we get $\mathrm{C}^0_\wedge = d(\beta_\wedge)$ on $\mathcal{V} \setminus M$. Thus the couple $(\mathrm{C}^0_\wedge, \beta_\wedge)$ defines a canonical relative class

$$\left[\mathrm{Pf}(\tfrac{\mathbf{F}}{2}), \beta_\wedge\right] \in \mathcal{H}^*(\mathcal{V}, \mathcal{V} \setminus M) \tag{41}$$

of degree equal to the rank of $\mathcal{V}$.

We give the explicit formula for this relative class in the case of a rank two Euclidean bundle in Example 6.2.

Consider now the cohomology with compact support in the fiber of $\mathcal{V}$. Let $\mathrm{C}_\mathcal{V}$ be the image of $\left[\mathrm{Pf}(\frac{\mathbf{F}}{2}), \beta_\wedge\right]$ through the map $\mathrm{p}_{\text{fiber cpt}} : \mathcal{H}^*(\mathcal{V}, \mathcal{V} \setminus M) \to \mathcal{H}^*_{\text{fiber cpt}}(\mathcal{V})$.

Proposition 11. *Let $\chi \in \mathcal{C}^\infty(\mathcal{V})$ be a function with compact support in the fibers and equal to 1 in a neighborhood of M. The form*

$$\mathrm{C}^\chi_\mathcal{V} = \chi \,\mathrm{Pf}(\tfrac{\mathbf{F}}{2}) + d\chi \beta_\wedge$$

is a closed differential form with compact support in the fibers on $\mathcal{V}$. Its cohomology class in $\mathcal{H}^*_{\text{fiber cpt}}(\mathcal{V})$ coincides with $\mathrm{C}_\mathcal{V}$: in particular, it does not depend on the choice of χ. We have $\frac{1}{\epsilon_d} p_*(\mathrm{C}^\chi_\mathcal{V}) = 1$, with $\epsilon_d = (-1)^{\frac{d(d-1)}{2}} \pi^{d/2}$. Thus $\frac{1}{\epsilon_d}\mathrm{C}^\chi_\mathcal{V}$ is a Thom form in $\mathcal{A}^*_{\text{fiber cpt}}(\mathcal{V})$.

Proof. The first assertions are consequence of the definition of $\mathrm{C}_\mathcal{V}$. To compute $p_*(\mathrm{C}^\chi_\mathcal{V})$, we may choose $\chi = f(\|\mathbf{x}\|^2)$ where $f \in \mathcal{C}^\infty(\mathbb{R})$ has a compact support and is equal to 1 in a neighborhood of 0. We work with a local oriented orthonormal frame $(e_1, \ldots, e_d)$ of $\mathcal{V}$: we have $\mathbf{x} = \sum_i x_i e_i$, and $p^*\nabla \mathbf{x} = \sum_i dx_i e_i + x_i p^* \nabla e_i$.

The component of maximal degree in the fibers of the differential form $\eta^t_\wedge$ is $(-1)^{\frac{d(d-1)}{2}} t^{d-1} \mathrm{e}^{-t^2\|x\|^2} \sum_k (-1)^k x_k dx_1 \cdots \widehat{dx_k} \cdots dx_d$ (see Proposition 13). Then, the component of maximal degree in the fibers of the differential form $d\chi \wedge \eta^t_\wedge$ is

$$-2(-1)^{\frac{d(d-1)}{2}} t^{d-1} f'(\|x\|^2)\|x\|^2 \,\mathrm{e}^{-t^2\|x\|^2}\, dx_1 \cdots dx_d.$$

Hence, using the change of variables $x \to \frac{1}{t}x$,

$$\begin{aligned} p_*(\mathrm{C}^\chi_\mathcal{V}) &= -2(-1)^{\frac{d(d-1)}{2}} \int_0^\infty t^{d-1} \left(\int_{\mathbb{R}^d} f'(\|x\|^2)\|x\|^2 \,\mathrm{e}^{-t^2\|x\|^2}\, dx \right) dt \\ &= (-1)^{\frac{d(d-1)}{2}} \int_0^\infty \underbrace{\left(\int_{\mathbb{R}^d} f'(\frac{\|x\|^2}{t^2})(\frac{-2\|x\|^2}{t^3}) \,\mathrm{e}^{-\|x\|^2}\, dx \right)}_{I(t)} dt. \end{aligned}$$

Since for $t > 0$, $I(t) = \frac{d}{dt}\left(\int_{\mathbb{R}^d} f(\frac{\|x\|^2}{t^2}) \,\mathrm{e}^{-\|x\|^2}\, dx\right)$, we have $p_*(\mathrm{C}^\chi_\mathcal{V}) = (-1)^{\frac{d(d-1)}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\|x\|^2}\, dx = (-1)^{\frac{d(d-1)}{2}} \pi^{d/2}$. □

Using the differential form $\mathrm{C}^1_\wedge$, it is possible to construct representatives of a Thom form with Gaussian look.

Proposition 12 (Mathai-Quillen). *The differential form* $\mathrm{C}^1_\wedge$ *is a closed form which belongs to* $\mathcal{A}^*_{\text{dec-rap}}(\mathcal{V})$. *We have* $\frac{1}{\epsilon_d} p_* \left(\mathrm{C}^1_\wedge\right) = 1$, *with* $\epsilon_d = (-1)^{\frac{d(d-1)}{2}} \pi^{d/2}$. *Thus* $\frac{1}{\epsilon_d}\mathrm{C}^1_\wedge$ *is a Thom form in* $\mathcal{A}^*_{\text{dec-rap}}(\mathcal{V})$.

Proof. By Proposition 13, the component of maximal degree in the fibers of the differential form $\mathrm{C}^1_\wedge$ is the term $(-1)^{\frac{d(d-1)}{2}} \mathrm{e}^{-\|x\|^2} dx_1 \cdots dx_d$. □

We summarize Propositions 11 and 12 in the following theorem.

Theorem 6.1. *Let* $p : \mathcal{V} \to M$ *be an oriented Euclidean vector bundle of rank* d *equipped with an Euclidean connection* ∇, *with curvature* $\mathbf{F}$. *Let* $\epsilon_d := (-1)^{\frac{d(d-1)}{2}} \pi^{d/2}$. *Let* $\mathrm{T} : \mathcal{A}^*(\mathcal{V}, \Lambda p^*\mathcal{V}) \to \mathcal{A}^*(\mathcal{V})$ *be the Berezin integral. Let*

$$\begin{aligned} f^\nabla_t &= -t^2\|\mathbf{x}\|^2 + tp^*\nabla\mathbf{x} + \frac{1}{2}p^*\mathbf{F}, \\ \eta^t_\wedge &= -\mathrm{T}\left(\mathbf{x}\,\mathrm{e}^{f^\nabla_t}\right), \\ \beta_\wedge &= \int_0^\infty \eta^t_\wedge\, dt. \end{aligned}$$

• $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V}, \nabla) = \frac{1}{\epsilon_d}\left(\mathrm{Pf}(\frac{\mathbf{F}}{2}), \beta_\wedge\right)$ *is a Thom form in* $\mathcal{A}^*(\mathcal{V}, \mathcal{V} \setminus M)$. *It defines a Thom class*

$$\mathrm{Th}_{\mathrm{rel}}(\mathcal{V}) \in \mathcal{H}^*(\mathcal{V}, \mathcal{V} \setminus M).$$

• $\mathrm{Th}_{\mathrm{c}}(\mathcal{V}, \nabla, \chi) = \frac{1}{\epsilon_d}\mathrm{C}^\chi_\mathcal{V} = \frac{1}{\epsilon_d}\left(\chi\,\mathrm{Pf}(\frac{\mathbf{F}}{2}) + d\chi\beta_\wedge\right)$ *is a Thom form in* $\mathcal{A}^*_c(\mathcal{V})$. *Here* $\chi \in \mathcal{C}^\infty(\mathcal{V})$ *is a function with compact support in the fibers of* $\mathcal{V}$ *and equal to* 1 *in a neighborhood of* M. *It defines a Thom class*

$$\mathrm{Th}_{\mathrm{c}}(\mathcal{V}) \in \mathcal{H}^*_c(\mathcal{V}).$$

• *The Mathai-Quillen form* $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V}, \nabla) = \frac{1}{\epsilon_d}\mathrm{C}^1_\wedge = \frac{1}{\epsilon_d}\mathrm{T}\left(\mathrm{e}^{f^\nabla_1}\right)$ *is a Thom form in* $\mathcal{A}^*_{\text{dec-rap}}(\mathcal{V})$. *It defines a Thom class*

$$\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V}) \in \mathcal{H}^*_{\text{dec-rap}}(\mathcal{V}).$$

Thus the use of the Berezin integral allowed us to give slim formulae for Thom forms in relative cohomology, as well as in compactly supported cohomology or in rapidly decreasing cohomology.

Example 6.2. Vector bundle of rank 2.

We write explicitly the formulae of this subsection in the case of an Euclidean bundle $\mathcal{V} \to M$ *of rank 2 in a local frame. Let* (e_1, e_2) *be a local oriented orthonormal frame. Let* ∇ *be an Euclidean connection on* $\mathcal{V}$, *so that* $\nabla e_1 = \eta e_2$, $\nabla e_2 = -\eta e_1$, *where* η *is a real valued one form on* M. *Then*

$$\mathbf{F} = d\eta(e_1 \wedge e_2), \qquad p^*\nabla\mathbf{x} = \eta_1 e_1 + \eta_2 e_2$$

with $\eta_1 = dx_1 - x_2\eta, \eta_2 = dx_2 + x_1\eta$, *and*

$$f^\nabla_t = -t^2\|x\|^2 + t(\eta_1 e_1 + \eta_2 e_2) + \frac{1}{2}d\eta(e_1 \wedge e_2).$$

The exponential of f^∇_t *in the super-algebra* $\mathcal{A}^*(\mathcal{V}, \Lambda p^*\mathcal{V})$ *is*

$$\mathrm{e}^{f_t^{\nabla}} = \mathrm{e}^{-t^2\|x\|^2}\left(1+\frac{d\eta}{2}e_1\wedge e_2 + t(\eta_1 e_1+\eta_2 e_2) - t^2(\eta_1\wedge\eta_2)e_1\wedge e_2\right).$$

Thus we have the formulae:

$$\begin{aligned}
\mathrm{C}^t_\wedge &= \mathrm{e}^{-t^2\|x\|^2}(\frac{d\eta}{2} - t^2\eta_1\wedge\eta_2),\\
\eta^t_\wedge &= t\,\mathrm{e}^{-t^2\|x\|^2}(x_1\eta_2 - x_2\eta_1),\\
\beta_\wedge &= \frac{x_1\eta_2 - x_2\eta_1}{2\|x\|^2}.
\end{aligned}$$

So Thom forms are given by

- $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V}) = \dfrac{-1}{2\pi}\left[d\eta, \eta + \dfrac{x_1dx_2 - x_2dx_1}{\|x\|^2}\right]$,
- $\mathrm{Th}_{\mathrm{c}}(\mathcal{V}) = \dfrac{-1}{2\pi}\Big(2f'(\|x\|^2)dx_1\wedge dx_2 + d\left(f(\|x\|^2)\wedge\eta\right)\Big)$,

where f is a compactly supported function on $\mathbb{R}$, identically equal to 1 in a neighborhood of 0,

- $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V}) = \dfrac{1}{2\pi}\,\mathrm{e}^{-\|x\|^2}\Big(2dx_1\wedge dx_2 - d\eta + d\left(\|x\|^2\right)\wedge\eta\Big)$.

6.3. **Explicit formulae for the Thom forms of a vector bundle.** Let us give explicit local formulae for a general Euclidean vector bundle.

Given a local oriented orthonormal frame $(e_1,\dots,e_d)$ of the vector bundle $\mathcal{V}$, we work with the identification $(m,x)\mapsto\sum_i x_ie_i(m)$ from $M\times\mathbb{R}^d$ into $\mathcal{V}$. The element $p^*\nabla\mathbf{x}$ is then equal to $\sum_i\eta_ie_i$ with $\eta_i = dx_i+\sum_k x_k(\nabla e_k, e_i)$.

If $I=[i_1,i_2,\dots,i_p]$ (with $i_1<i_2<\cdots<i_p$) is a subset of $[1,2,\dots,d]$, we use the notations $e_I = e_{i_1}\wedge\cdots\wedge e_{i_p}$ and $\eta_I = \eta_{i_1}\wedge\cdots\wedge\eta_{i_p}$. The curvature $\mathbf{F}$ decomposes as $\mathbf{F} := \sum_{i<j}\mathbf{F}_{ij}e_i\wedge e_j$. For any subset I of $[1,2,\dots,d]$, we consider the two form $\mathbf{F}_I := \sum_{i<j, i\in I, j\in I}\mathbf{F}_{ij}e_i\wedge e_j$ with values in $\Lambda\mathcal{V}_I$, where $\mathcal{V}_I$ is the sub-bundle generated by the $e_i, i\in I$. Let $\mathrm{Pf}(\mathbf{F}_I)$ be its pfaffian. One sees easily that

$$\mathrm{e}^{\frac{\mathbf{F}}{2}} = \sum_I \mathrm{Pf}(\tfrac{\mathbf{F}_I}{2})\,e_I \quad\text{in}\quad \mathcal{A}^*(M,\Lambda\mathcal{V}). \tag{42}$$

Only those I with $|I|$ even will contribute to the sum (42), as otherwise the pfaffian of $\mathbf{F}_I$ vanishes.

If I and J are two disjoint subsets of $\{1,2,\dots,d\}$, we denote by $\epsilon(I,J)$ the sign such that $e_I\wedge e_J = \epsilon(I,J)\,e_{I\cup J}$.

Proposition 13. • *We have* $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V}) = \frac{1}{\epsilon_d}\left[\mathrm{Pf}(\frac{\mathbf{F}}{2}),\beta_\wedge\right]$ *with*

$$\beta_\wedge = \sum_{k,I,J}\gamma_{(k,I,J)}\,\mathrm{Pf}\left(\tfrac{\mathbf{F}_I}{2}\right)\frac{x_k\eta_J}{\|x\|^{|J|+1}},$$

with

$$\gamma_{(k,I,J)} = -\frac{1}{2}(-1)^{\frac{|J|(|J|+1)}{2}}\Gamma\left(\tfrac{|J|+1}{2}\right)\epsilon(I,J)\epsilon(\{k\},I\cup J).$$

Here for $1 \leq k \leq d$, the sets I, J vary over the subsets of $\{1, 2, \dots, d\}$ such that $\{k\} \cup I \cup J$ is a partition of $\{1, 2, \dots, d\}$. Only those I with $|I|$ even will contribute to the sum.

- *The class $\mathrm{Th_c}(\mathcal{V})$ is represented by the closed differential form*

$$\frac{1}{\epsilon_d}\Big(f(\|x\|^2)\,\mathrm{Pf}(\tfrac{\mathbf{F}}{2}) + 2f'(\|x\|^2)(\sum x_i dx_i)\beta_\wedge\Big)$$

where f is a compactly supported function on $\mathbb{R}$, identically equal to 1 in a neighborhood of 0.

- *We have*

$$\mathrm{Th_{MQ}}(\mathcal{V}) = \frac{1}{(\pi)^{d/2}}\,\mathrm{e}^{-\|x\|^2}\sum_I (-1)^{\frac{|I|}{2}}\epsilon(I, I')\,\mathrm{Pf}(\tfrac{\mathbf{F}_I}{2})\eta_{I'}.$$

Here I runs over the subset of $\{1, 2, \dots, d\}$ with an even number of elements, and I' denotes the complement of I.

Proof. It follows from the explicit description of our forms and from the formula $\int_0^\infty \mathrm{e}^{-t^2} t^a dt = \frac{1}{2}\Gamma(\frac{a+1}{2})$. □

6.4. **More notations.** We recall notations from [7]. Let V be an Euclidean vector space of even dimension $d = 2n$ with oriented orthonormal basis $e_1, e_2, \dots, e_d$. Let $C(V)$ be the Clifford algebra of V. Then $C(V)$ is generated by elements c_i with relations $c_i c_j + c_j c_i = 0$, for $i \neq j$, and $c_i^2 = -1$. We denote by $\Sigma : C(V) \to \Lambda V$ the symbol isomorphism. Thus, for $1 \leq i_1 < i_2 < \cdots < i_k \leq d$, we have $\Sigma(c_{i_1} c_{i_2} \cdots c_{i_k}) = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$. Let $C^{[i]}(V) = \Sigma^{-1}(\Lambda^i V)$. We denote by $\tau : C^{[2]}(V) \to \mathfrak{so}(V)$ the map such that $\tau(c)v = cv - vc$, for $c \in C^{[2]}(V)$ and $v \in V$. Then $\tau(c_i c_j)(e_i) = 2e_j$, for $i \neq j$. We denote by $S = S^+ \oplus S^-$ the complex spinor space. We denote by $\mathbf{c}$ the Clifford action of $C(V)$ on S. If $v \in V$, then $\mathbf{c}(v)$ on S interchanges S^+ and S^- and satisfies $\mathbf{c}(v)^2 = -\|v\|^2 \mathrm{Id}_S$. The supertrace of the action of the even element $c_1 c_2 \dots c_d$ on S is $(-2i)^n$.

Let $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ be of dimension 2 . We consider the super-algebra $A \otimes C(V)$ where A is a super-commutative algebra. Then for a_1, a_2 odd elements in A, and b an even element of A, we have

$$\begin{aligned}\exp(a_1c_1 + a_2c_2 + bc_1c_2) &= \cos(b) + \sin(b)c_1c_2\\ &+ \frac{\sin(b)}{b}(a_1c_1 + a_2c_2) + \frac{\sin(b) - b\cos(b)}{b^2}a_1a_2 - \frac{\sin(b)}{b}a_1a_2c_1c_2.\end{aligned}$$

This formula can be verified using, for example, the differential equation $\frac{d}{dt}\exp(tX) = X\exp(tX)$ for the exponential (see also [7], proof of Proposition 7.43).

6.5. **Riemann-Roch relation.** Let $p : \mathcal{V} \to M$ be an oriented Euclidean vector bundle of even rank $d = 2n$ with spin structure, and let $\mathcal{S} \to M$ be the corresponding spin super-bundle. Let $C(\mathcal{V}) \to M$ be the Clifford bundle. We denote by

$$\mathbf{c} : C(\mathcal{V}) \to \mathrm{End}_{\mathbb{C}}(\mathcal{S})$$

the bundle map defined by the spinor representation.

The vector bundle $\mathcal{S}$ is provided with an Hermitian metric such that $\mathbf{c}(v)^* = -\mathbf{c}(v)$ for $v \in \mathcal{V}$. Consider the morphism $\sigma_{\mathcal{V}} : p^*\mathcal{S}^+ \to p^*\mathcal{S}^-$ defined by

$$\sigma_{\mathcal{V}} := -i\mathbf{c}(\mathbf{x})$$

where $\mathbf{x} : \mathcal{V} \to p^*\mathcal{V}$ is the canonical section. Then the odd linear map $v_{\sigma_\mathcal{V}} : p^*\mathcal{S} \to p^*\mathcal{S}$ is equal to $-i\mathbf{c}(\mathbf{x})$.

Let ∇ be an Euclidean connection on $\mathcal{V}$. Then as explained in Subsection 6.2, the curvature $\mathbf{F}$ of ∇ may be identified to an element of $\mathcal{A}(M, \Lambda^2\mathcal{V})$. Thus $\Sigma^{-1}\mathbf{F}$ is an element of $\mathcal{A}(M, C(\mathcal{V}))$, where $\Sigma : C(\mathcal{V}) \to \Lambda\mathcal{V}$ is the symbol bundle map. The connection ∇ induces a connection $\nabla^\mathcal{S}$ on $\mathcal{S}$ with curvature $\mathbf{F}^\mathcal{S}$ (see Lemma 8). We work with the family of super-connections on $p^*\mathcal{S}$:

$$\mathbb{A}^\sigma_t := p^*\nabla^\mathcal{S} + t\mathbf{c}(\mathbf{x})$$

We see that the curvature of the super-connection $\mathbb{A}^\sigma_t$ is the even element $\mathbf{F}^\mathbf{c}_t \in \mathcal{A}^*(\mathcal{V}, p^*\mathrm{End}_\mathbb{C}(\mathcal{S}))$, given by

$$\mathbf{F}^\mathbf{c}_t = -t^2\|\mathbf{x}\|^2 + t\mathbf{c}(p^*\nabla\mathbf{x}) + p^*\mathbf{F}^\mathcal{S}$$

where $\mathbf{F}^\mathcal{S} \in \mathcal{A}^2(M, \mathrm{End}_\mathbb{C}(\mathcal{S}))$ is the curvature of $\nabla^\mathcal{S}$.

Lemma 8. • *The following relation holds in $\mathcal{A}^2(M, \mathrm{End}_\mathbb{C}(\mathcal{S}))$:*

$$\mathbf{F}^\mathcal{S} = \frac{1}{2}\mathbf{c}(\Sigma^{-1}\mathbf{F}).$$

• *We have $\mathbf{F}^\mathbf{c}_t = \mathbf{c}(\tilde{f}_t)$, where $\tilde{f}_t \in \mathcal{A}^*(\mathcal{V}, p^*C(\mathcal{V}))$ is given by*

$$\tilde{f}_t = -t^2\|\mathbf{x}\|^2 + tp^*\nabla\mathbf{x} + \frac{1}{2}p^*\Sigma^{-1}\mathbf{F}.$$

• *The image of $\tilde{f}_t$ by the bundle map Σ is equal to the map $f^\nabla_t \in \mathcal{A}^*(\mathcal{V}, p^*\Lambda\mathcal{V})$ defined in (35).*

We consider now in parallel the closed differential forms

$$\mathrm{Ch}(\sigma_\mathcal{V}, \nabla^\mathcal{V}, t) := \mathrm{Str}\left(\exp(\mathbf{F}^\mathbf{c}_t)\right), \quad \mathrm{C}^t_\wedge := \mathrm{T}\left(\exp_\wedge(f^\nabla_t)\right).$$

In the first case the exponential is computed in the super-algebra $\mathcal{A}^*(\mathcal{V}, p^*\mathrm{End}_\mathbb{C}(\mathcal{S}))$, and the forms $\mathrm{Ch}(\sigma_\mathcal{V}, \nabla^\mathcal{V}, t)$ have *complex* coefficients. In the second case, the exponential is computed in the super-algebra $\mathcal{A}^*(\mathcal{V}, p^*\Lambda\mathcal{V})$, and the forms $\mathrm{C}^t_\wedge$ have *real* coefficients.

In Example 6.4, we will perform the explicit calculation of $\exp(\mathbf{F}^\mathbf{c}_t)$ for a bundle of rank two.

We also consider in parallel the differential forms

$$\eta^t_\mathbf{c} := -\,\mathrm{Str}\left(\mathbf{c}(\mathbf{x})\exp(\mathbf{F}^\mathbf{c}_t)\right), \quad \eta^t_\wedge := -\,\mathrm{T}\left(\mathbf{x}\cdot\exp_\wedge(f^\nabla_t)\right).$$

Note that the forms $\eta^t_\mathbf{c}$ have *complex* coefficients, and that the forms $\eta^t_\wedge$ have *real* coefficients.

In the next definition, we return to the original definition of the curvature $\mathbf{F}$ of the Euclidean connection ∇, that is we consider $\mathbf{F}$ as a 2-form with values antisymmetric transformations of $\mathcal{V}$.

Definition 6.2. *We associate to the vector bundle $\mathcal{V}$, equipped with the connection ∇, the closed real differential form on M defined by*

$$\widehat{A}(\nabla) := \det{}^{1/2}\left(\frac{\mathbf{F}}{\mathrm{e}^{\frac{\mathbf{F}}{2}} - \mathrm{e}^{-\frac{\mathbf{F}}{2}}}\right).$$

Its cohomology class $\widehat{A}(\mathcal{V}) \in \mathcal{H}^*(M)$ is the $\widehat{A}$-genus of $\mathcal{V}$.

Proposition 14. *We have the following equalities of differential forms on $\mathcal{V}$:*

$$\mathrm{Ch}(\sigma_{\mathcal{V}}, \nabla^{\mathcal{V}}, t) = (-2i)^n \ \widehat{A}(\nabla)^{-1} \ \mathrm{C}^t_{\wedge}$$

and

$$\eta^t_{\mathbf{c}} = (-2i)^n \ \widehat{A}(\nabla)^{-1} \ \eta^t_{\wedge}.$$

Proof. The proof of the first relation is done in [7], Section 7. The same proof works for the second equality. Let us give here a brief idea of the proof. Let Str_C be the super-trace on $\mathcal{A}^*(\mathcal{V}, p^*C(\mathcal{V}))$ such that $\mathrm{Str}_C(a) = \mathrm{Str}(\mathbf{c}(a))$ for any element $a \in \mathcal{A}^*(\mathcal{V}, p^*C(\mathcal{V}))$. We have then to show that

$$\mathrm{Str}_C\left(\exp_C(\tilde{f}_t)\right) = (-2i)^n \ \widehat{A}(\nabla)^{-1} \ \mathrm{T}\left(\exp_{\wedge}(\Sigma \tilde{f}_t)\right) \tag{43}$$

and

$$\mathrm{Str}_C\left(\mathbf{x} \cdot \exp_C(\tilde{f}_t)\right) = (-2i)^n \ \widehat{A}(\nabla)^{-1} \ \mathrm{T}\left(\mathbf{x} \cdot \exp_{\wedge}(\Sigma \tilde{f}_t)\right). \tag{44}$$

If V is an oriented Euclidean vector space of even dimension $2n$, we have the following fundamental relation between $\mathrm{Str}_C(\exp_C(a))$ and $\mathrm{T}(\exp_{\wedge}(\Sigma a))$ for $a \in C^2(V)$:

$$\mathrm{Str}_C(\exp_C(a)) = (-2i)^n \det{}^{1/2}\left(\frac{\mathrm{e}^{\frac{\tau(a)}{2}} - \mathrm{e}^{-\frac{\tau(a)}{2}}}{\tau(a)}\right) \mathrm{T}(\exp_{\wedge}(\Sigma a)) \tag{45}$$

(see [7], Section 3). We see then that (43) is an extension of (45) to the case where $a \in A^- \otimes C^1(V) + A^+ \otimes C^2(V)$ (here A is a super-commutative super-algebra). This is verified by an explicit computation when V is of dimension 2, using the formula for the exponential that we recalled in Subsection 6.4.

□

We can now conclude with the

Theorem 6.3. • *We have the following equality in $\mathcal{H}^*(\mathcal{V}, \mathcal{V} \setminus M)$:*

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_{\mathcal{V}}) = (2i\pi)^n \ \widehat{A}(\mathcal{V})^{-1} \ \mathrm{Th}_{\mathrm{rel}}(\mathcal{V}). \tag{46}$$

- *We have the following equality in $\mathcal{H}^*_{\mathrm{c}}(\mathcal{V})$:*

$$\mathrm{Ch}_{\mathrm{c}}(\sigma_{\mathcal{V}}) = (2i\pi)^n \ \widehat{A}(\mathcal{V})^{-1} \ \mathrm{Th}_{\mathrm{c}}(\mathcal{V}). \tag{47}$$

- *We have the following equality in $\mathcal{H}^*_{\text{dec-rap}}(\mathcal{V})$:*

$$\mathrm{Ch}_Q(\sigma_{\mathcal{V}}) = (2i\pi)^n \ \widehat{A}(\mathcal{V})^{-1} \ \mathrm{Th}_{\mathrm{MQ}}(\mathcal{V}). \tag{48}$$

Remark that these identities holds at the level of the representatives.

Example 6.4. Vector bundle of rank two.
We return to Example 6.2, and keep the same notations. Then we have

$$\tilde{f}_t = -t^2\|x\|^2 + t(\eta_1 c_1 + \eta_2 c_2) + \frac{1}{2}(d\eta)c_1 c_2.$$

We use the formula for the exponential recalled in Subsection 6.4, and we obtain

$$\mathrm{e}^{\tilde{f}_t} = \mathrm{e}^{-t^2\|x\|^2}\Big(\cos(\frac{d\eta}{2}) + \sin(\frac{d\eta}{2})c_1c_2 + t\frac{\sin(\frac{d\eta}{2})}{(\frac{d\eta}{2})}(\eta_1 c_1 + \eta_2 c_2)$$
$$+ t^2\frac{\sin(\frac{d\eta}{2}) - (\frac{d\eta}{2})\cos(\frac{d\eta}{2})}{(\frac{d\eta}{2})^2}\eta_1\eta_2 - t^2\frac{\sin(\frac{d\eta}{2})}{(\frac{d\eta}{2})}\eta_1\eta_2 c_1 c_2\Big).$$

The supertrace of the action of c_1c_2 on S is $-2i$. Thus we obtain

$$\begin{aligned}
\mathrm{Ch}(\sigma_{\mathcal{V}}, \nabla^{\mathcal{V}}, t) &= (-2i)\,\mathrm{e}^{-t^2\|x\|^2}\Big(\sin(\frac{d\eta}{2}) - t^2\frac{\sin(\frac{d\eta}{2})}{(\frac{d\eta}{2})}\eta_1\eta_2\Big),\\
\eta^t_{\mathbf{c}} &= (-2i)\; t\,\mathrm{e}^{-t^2\|x\|^2}\,\frac{\sin(\frac{d\eta}{2})}{(\frac{d\eta}{2})}\Big(x_1\eta_2 - x_2\eta_1\Big),\\
\beta_{\mathbf{c}} &= (-2i)\frac{\sin(\frac{d\eta}{2})}{(\frac{d\eta}{2})}\frac{(x_1\eta_2 - x_2\eta_1)}{2\|x\|^2}
\end{aligned}$$

Finally, the relative Chern character form associated to $\sigma_{\mathcal{V}}$ is

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_{\mathcal{V}}) = (-i)\frac{\sin(\frac{d\eta}{2})}{(\frac{d\eta}{2})}\left[d\eta, \eta + \frac{(x_1dx_2 - x_2dx_1)}{\|x\|^2}\right]. \tag{49}$$

We have

$$\det{}^{1/2}\left(\frac{\mathbf{F}}{\mathrm{e}^{\frac{\mathbf{F}}{2}} - \mathrm{e}^{-\frac{\mathbf{F}}{2}}}\right) = \frac{(\frac{d\eta}{2})}{\sin(\frac{d\eta}{2})}.$$

Thus we see that we have the relations

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_{\mathcal{V}}) = (2i\pi)\ \widehat{A}(\mathcal{V})^{-1}\ \mathrm{Th}_{\mathrm{rel}}(\mathcal{V}), \tag{50}$$

$$\mathrm{Ch}_{\mathrm{c}}(\sigma_{\mathcal{V}}) = (2i\pi)\ \widehat{A}(\mathcal{V})^{-1}\ \mathrm{Th}_{\mathrm{c}}(\mathcal{V}), \tag{51}$$

and

$$\mathrm{Ch}_{Q}(\sigma_{\mathcal{V}}) = (2i\pi)\ \widehat{A}(\mathcal{V})^{-1}\ \mathrm{Th}_{\mathrm{MQ}}(\mathcal{V}) \tag{52}$$

at the level of differential forms.

7. Appendix

We give a proof of the estimate used in this article. It is based on Volterra's expansion formula: if H and R are elements in a finite dimensional associative algebra, then $\mathrm{e}^{(H+R)} = \mathrm{e}^H + \sum_{k=1}^{\infty} I_k(H,R)$ where

$$I_k(H,R) = \int_{\Delta_k} \mathrm{e}^{s_1H}\, R\,\mathrm{e}^{s_2H}\, R\cdots R\,\mathrm{e}^{s_kH}\, R\,\mathrm{e}^{s_{k+1}H}\, ds_1\cdots ds_k \tag{53}$$

Here Δ_k is the simplex $\{s_i \geq 0\,;\, s_1 + s_2 + \cdots + s_k + s_{k+1} = 1\}$ which has the volume $\frac{1}{k!}$ for the measure $ds_1\cdots ds_k$.

Now, let $A = \oplus_{i=0}^R A_i$ be a complex finite dimensional graded commutative algebra with a norm $\|\cdot\|$ such that $\|ab\| \leq \|a\|\|b\|$. We denote by $A_+ = \oplus_{i=1}^R A_i$. Thus $\omega^{R+1} = 0$ for any $\omega \in A_+$. Let E be a finite dimensional Hermitian vector space. Then $\mathrm{End}(E)\otimes A$ is an algebra with a norm still denoted by $\|\cdot\|$. If $H \in \mathrm{End}(E)$, we denote also by H the element $H\otimes 1$ in $\mathrm{End}(E)\otimes A$.

We denote by $\mathrm{Herm}(E) \subset \mathrm{End}(E)$ the subspace formed by the Hermitian endomorphisms. When $H \in \mathrm{Herm}(E)$, we denote by $\mathrm{sm(H)} \in \mathbb{R}$ the smallest eigenvalue of H : we have
$$\left\| \mathrm{e}^{-H} \right\| = \mathrm{e}^{-\mathrm{sm(H)}}, \quad \text{for all} \quad H \in \mathrm{Herm}(E).$$

Lemma 9. *Let* $\mathcal{P}(t) = \sum_{k=0}^{R} \frac{t^k}{k!}$. *Then, for any* $R \in \mathrm{End}(E) \otimes A_+$, *and* $H \in \mathrm{Herm}(E)$, *we have*
$$\left\| \mathrm{e}^{-(H+R)} \right\| \leq \mathrm{e}^{-\mathrm{sm(H)}} \mathcal{P}(\|R\|).$$

Proof. Let $c = \mathrm{sm(H)}$. Then $\| \mathrm{e}^{-uH} \| = \mathrm{e}^{-uc}$ for all $u \geq 0$. The term $I_k(H,R)$ of the Volterra expansion vanishes for $k > R$ since the term $\mathrm{e}^{s_1 H} R \cdots R \,\mathrm{e}^{s_{k+1} H}$ belongs to $\mathrm{End}(E) \otimes A_k$. The norm of the term $I_k(H,R)$ is bounded by $\frac{1}{k!} \mathrm{e}^{-c} \|R\|^k$. Summing up in k, we obtain our estimate. □

The preceding estimates hold if we work in the algebra $\mathrm{End}(E) \otimes A$, where E is a super-vector space and A a super-commutative algebra.

References

[1] M.F. Atiyah, *K-Theory*, Benjamin, New-York, 1967.

[2] M.F. Atiyah, F. Hirzebruch, Vector bundles and homogeneous spaces, *Proceedings of Symposium in Pure Mathematics Vol. 3*, Amer. Math. Soc., 1961.

[3] M.F. Atiyah, I.M. Singer, The index of elliptic operators I, *Ann. Math.* **87**, 1968, p. 484-530.

[4] M.F. Atiyah, G.B. Segal, The index of elliptic operators II, *Ann. Math.* **87**, 1968, p. 531-545.

[5] M.F. Atiyah, I.M. Singer, The index of elliptic operators III, *Ann. Math.* **87**, 1968, p. 546-604.

[6] M.F. Atiyah, I.M. Singer, The index of elliptic operators IV, *Ann. Math.* **93**, 1971, p. 139-141.

[7] N. Berline, E. Getzler and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren, vol. 298, Springer, Berlin, 1991.

[8] N. Berline and M. Vergne, The equivariant Chern character and index of G-invariant operators. *In* D-Modules, representation theory and quantum groups, Venezia 1992. *Springer Lectures Notes in Math.* **1565** (1992).

[9] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, GTM, Springer.

[10] L. Illusie, Caractère de Chern. Classe de Todd, *Séminaire Henri Cartan*, **16 no. 1**, 1963-1964, Exposé No. 6, 9 p. (*At www.numdam.org*)

[11] L. Illusie, Compléments de K-théorie, *Séminaire Henri Cartan*, **16 no. 1**, 1963-1964, Exposé No. 15, 10 p. (*At www.numdam.org*)

[12] L. Illusie, Symboles elliptiques, *Séminaire Henri Cartan*, **16 no. 2**, 1963-1964, Exposé No. 19, 13 p. (*At www.numdam.org*)

[13] L. Illusie, *Complexe cotangent et déformations. I.* Lecture Notes in Math., No 239, Springer, Berlin, 1971.

[14] B. Iversen, Local Chern classes, *Annales Scientifiques de l'E.N.S.*, **9**, 1976, p. 155-169.

[15] M. Karoubi, Théorie générale des classes caractéristiques secondaires, *K-theory*, **4**, 1990, p. 55-87.

[16] M. Karoubi, Classes caractéristiques de fibrés feuilletés, holomorphes ou algébriques, *K-theory*, **8**, 1994, p. 153-211.

[17] V. Mathai and D. Quillen, Superconnections, Thom classes and equivariant differential forms, *Topology* **25**, 1986, p. 85-110.

[18] P-E. Paradan and M. Vergne, Equivariant Chern character with generalized coefficients. *Arxiv 0801.2822.*

[19] D. Quillen, Superconnections and the Chern Character, *Topology* **24**, 1985, p. 37-41.

[20] J.P. Schneiders, Introduction to characteristic classes and index theory, *Textos de Matemática*, Universidade de Lisboa, 2000.

Institut de Mathématiques et Modélisation de Montpellier (I3M), Université Montpellier 2, 34095 Montpellier
E-mail address: Paul-Emile.Paradan@math.univ-montp2.fr

Institut de Mathématiques de Jussieu, Théorie des Groupes, 75005 Paris – Ecole Polytechnique, Centre de mathématiques Laurent Schwartz, 91128 Palaiseau
E-mail address: vergne@math.polytechnique.fr