Geometric Interpretation of Second Elliptic Integrable System

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Abstract.
In this paper we give a geometrical interpretation of all the second elliptic integrable systems associated to 4-symmetric spaces. We first show that a 4-symmetric space G/G_0 can be embedded into the twistor space of the corresponding symmetric space G/H. Then we prove that the second elliptic system is equivalent to the vertical harmonicity of an admissible twistor lift J taking values in $G/G_0 \hookrightarrow \Sigma(G/H)$. We begin the paper with an example: $G/H = \mathbb{R}^4$. We also study the structure of 4-symmetric bundles over Riemannian symmetric spaces.

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Introduction

The first example of second elliptic integrable system associated to a 4-symmetric space was given in [7]: the authors showed that the Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2$ are solutions of one such integrable system. Later they generalized their result to complex two-dimensional Hermitian symmetric spaces, [9]. In [12], we presented a new class of geometric problems for surfaces in the Euclidean space of dimension 8 by identifying $\mathbb{R}^8$ with the set of octonions $\mathbb{O}$, and we proved that these problems are solutions of a second elliptic integrable system. Using the left multiplication in $\mathbb{O}$ by the vectors of the canonical basis of $\operatorname{Im}\mathbb{O}$ we defined a family $\{\omega_i,\, 1 \leq i \leq 7\}$ of canonical symplectic forms in $\mathbb{O}$. This allowed us to define the notion of ω_I*-isotropic surfaces*, for $I \subsetneqq \{1, ..., 7\}$. Using the cross-product in $\mathbb{O}$ we defined a map $\rho\colon Gr_2(\mathbb{O}) \to S^6$ from the Grassmannian of planes in $\mathbb{O}$ to S^6. This allowed us to associate to each surface Σ in $\mathbb{O}$ a function $\rho_\Sigma\colon \Sigma \to S^6$. In the case of ω_I-isotropic surfaces, ρ_Σ takes values in a subsphere $S^I = S(\oplus_{i\notin I, i>0}\mathbb{R}e_i) \simeq S^{6-|I|}$. We showed that the surfaces in $\mathbb{O}$ such that ρ_Σ is harmonic (ρ*-harmonic surfaces*) are solutions of a completely integrable system $\mathcal{S}$. More generally we showed that the ω_I-isotropic ρ-harmonic surfaces are solutions of a completely integrable system $\mathcal{S}_I$. Hence we built a family $(\mathcal{S}_I)$ indexed by I, of set of surfaces solutions of an integrable system, all included in $\mathcal{S} = \mathcal{S}_\emptyset$, such that $I \subset J$ implies $\mathcal{S}_J \subset \mathcal{S}_I$. Each $\mathcal{S}_I$ is a second elliptic integrable system (in the sense of C.L. Terng). This means that the equations of this system are equivalent to the zero curvature equation :

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0,$$

for all $\lambda \in \mathbb{C}^*$, and where $\alpha_\lambda = \lambda^{-2}\alpha_2' + \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha_2''$.
By restriction to the quaternions $\mathbb{H} \subset \mathbb{O}$ of our theory we obtain a new class of surfaces: the ω_I-isotropic ρ-harmonic surfaces in $\mathbb{H}$. Then $\rho(Gr_2(\mathbb{H})) = S^2$ and $|I| = 0, 1$ or 2. For $|I| = 1$ we obtain the Hamiltonian Stationary Lagrangian surfaces in $\mathbb{R}^4$ and for $|I| = 2$, the special Lagrangian surfaces. By restriction to $\operatorname{Im}\mathbb{H}$, we obtain the CMC surfaces of $\mathbb{R}^3$.

Besides, in [13], we found a supersymmetric interpretation of all the second elliptic integrable systems associated to 4-symmetric spaces in terms of super harmonic maps into symmetric spaces. This led us to conjecture that this system has a geometric interpretation in terms of surfaces with values in a symmetric space, such that a certain associated map is harmonic as this is the case for Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric spaces or for ρ-harmonic surfaces of $\mathbb{O}$.
In this paper we give the answer to this conjecture. More precisely, we give a geometric interpretation – in terms of vertical harmonic twistor lifts – of all the second elliptic integrable systems associated to 4-symmetric spaces. Indeed given a 4-symmetric space G/G_0, and its order four automorphism $\tau\colon G \to G$, then the involution $\sigma = \tau^2$ gives rise to the symmetric space G/H, with $H = G^\sigma$. Then we prove that the second elliptic integrable system associated to the 4-symmetric space G/G_0 is exactly the equation of vertical harmonicity for an admissible twistor lift in G/H. More precisely, given a 4-symmetric space G/G_0, and its associated symmetric space G/H, then G/G_0 is a subbundle of the twistor space $\Sigma(G/H)$. We prove that the second elliptic integrable systems associated to G/G_0, is the system of equations for maps $J\colon \mathbb{C} \to G/G_0 \subset \Sigma(G/H)$ such that J is compatible with the Gauss map of $X\colon \mathbb{C} \to G/H$, the projection of J into G/H, i.e. X is J-holomorphic (admissible twistor lift), and such that J is vertically harmonic. We prove also that an admissible twistor lift $J\colon \mathbb{C} \to G/G_0$ is harmonic if and only if it is vertically harmonic and $X\colon \mathbb{C} \to G/H$ is harmonic.
We begin the paper with an example: $\mathbb{R}^4$. This case was just mentioned briefly at the end of [12] as a restriction of the difficult problem in $\mathbb{O}$. In this paper we study this problem independently and in detail. However, we also present a formulation of this problem in terms of twistor lifts which seems to be the appropriate formulation. Besides, in dimension 4 we have unicity of the twistor lift (in $\Sigma^+(G/H)$ and $\Sigma^-(G/H)$ respectively) so we are in this case in the presence of a theory of surfaces (and not, as in the general case, a theory of twistor lift). Hence we can speak about ρ-harmonic surfaces in this dimension (which are exactly the solutions of the second elliptic integrable system). In our work we are led to prove some theorems on the structure of 4-symmetric bundles. Indeed we

want to answer the following questions. Given a Riemannian symmetric space, do there exist 4-symmetric bundles over it? In other words, does its twistor bundle contain 4-symmetric subbundles, and if yes, how can we characterize these 4-symmetric components? are they isomorphic? How are they distributed in the twistor space ? Do they form a partition of the twistor space? The 4-symmetric spaces have been classified (at least in the compact case, see [11, 16]). However, our point of view is different: we want to keep an intrinsic point of view as long as possible, therefore we deal with the Riemannian symmetric space and a (locally) 4-symmetric bundle defined over it, and we try to forget as much as possible the order four automorphism of the Lie algebra. Our aim is to give a formulation of our problem which is as general and intrinsic as possible. For example, our definition of vertical harmonicity holds for any Riemannian manifold. Moreover we prove the following characterization: to define a (locally) 4-symmetric bundle over M is equivalent to give ourself $J_0 \in \Sigma(T_{p_0}M)$, an (orthogonal) almost complex structure in $T_{p_0}M$, which leaves invariant the curvature. We obtain the following picture: the submanifold of the twistor bundle leaving invariant the curvature is the disjoint union of all the maximal (locally) 4-symmetric subbundle, which are orbits (under the action of some subgroups of $\mathrm{Is}(M)$). Each isomorphism class of orbits defines a different second elliptic integrable system.
Our paper is organized as follows. In Section 1 we deal with the ρ-harmonic surfaces in $\mathbb{R}^4$. Section 2 contains our main result: the interpretation of the second elliptic integrable systems associated to a 4-symmetric space in terms of vertical harmonicity of an admissible twistor lift. Then Sections 3 and 4 are devoted to the study of the structure of 4-symmetric bundles over symmetric spaces. The last Section presents some examples of 4-symmetric bundles.

1 ρ-harmonic surfaces in $\mathbb{H}$

1.1 Cross product, complex structure and Grassmannian of planes in $\mathbb{H}$

We consider the space $\mathbb{R}^4 = \mathbb{H}$ with its canonical basis $(1, i, j, k)$ (which we denote also by $(e_i)_{0\leq i\leq 3}$). Let $P = q \wedge q'$ be an oriented plane of $\mathbb{H}$ (itself oriented by its canonical basis) then there exists an unique positive complex structure[1] $I_P \in \Sigma^+(P)$ on the plane P. It is defined by $I_P(q) = q'$, $I_P(q') = -q$ if (q, q') is orthogonal. Next, we can extend it in an unique way to a positive (resp. negative) complex structure in $\mathbb{H} = P \oplus P^\perp$, J_P^+ (resp. J_P^-) given by

$$\begin{aligned} J_P^+ &= I_P \oplus I_{P^\perp} \\ J_P^- &= I_P \oplus -I_{P^\perp} \end{aligned} \tag{1}$$

[1] In all the paper, for any oriented Euclidean space E, $\Sigma(E) = \{J \in SO(E) | J^2 = -\mathrm{Id}\}$, and $\Sigma(M)$ denotes the twistor bundle of the Riemannian manifold M.

($P^\perp$ is oriented so that $= P \oplus P^\perp$ is positively oriented). Hence we obtain a surjective map:

$$\begin{array}{rccc} J^+\colon & Gr_2(\mathbb{H}) & \to & \Sigma^+(\mathbb{H}) \\ & q\wedge q' & \mapsto & J^+_{q\wedge q'} \end{array} \tag{2}$$

$Gr_2(\mathbb{H})$ being the Grassmannian of oriented planes in $\mathbb{H}$, and in the same way a surjective map $J^-\colon Gr_2(\mathbb{H}) \to \Sigma^-(\mathbb{H})$.
Besides, we have

$$J^+_{q\wedge q'} = L_{q\times_L q'} = \frac{1}{2}(L_{q'}L_{\overline{q}} - L_q L_{\overline{q'}}),$$

where $q\times_L q' = -\mathrm{Im}\,(q\cdot \overline{q'}) = \mathrm{Im}\,(q'\cdot \overline{q})$ is the left cross product (it is a bilinear skew map from $\mathbb{H}\times\mathbb{H}$ to $\mathrm{Im}\,\mathbb{H}$). Indeed, if (q,q') is orthonormal then $q\times_L q' = -q\cdot\overline{q'} \in S(\mathrm{Im}\,\mathbb{H})$ so $L_{q\times_L q'}$ is a complex structure in $\mathbb{H}$ and it is positive (because $\{L_u, u\in S^2\}$ is connected and $L_i\in\Sigma^+(\mathbb{H})$ because $(1, L_i(1), j, L_i(j))$ is positively oriented). Moreover if (q,q') is orthonormal then $L_{q\times_L q'}(q) = (q'\overline{q})q = q'$. Hence $L_{q\times_L q'} = J^+_{q\wedge q'}$. Thus we obtain a diffeomorphism:

$$\begin{array}{ccc} \Sigma^+(\mathbb{H}) & \xrightarrow{\sim} & S^2 \\ J & \longmapsto & J(1) \end{array}. \tag{3}$$

Under this identification, the map (2) becomes

$$\begin{array}{rccc} \rho_+\colon & Gr_2(\mathbb{H}) & \to & S^2 \\ & q\wedge q' & \mapsto & q\times_L q' \end{array}.$$

We can do the same for $\Sigma^-(\mathbb{H})$. We obtain that $J^-_{q\wedge q'} = R_{\overline{q\times_R q'}} = -R_{q\times_R q'} = \frac{1}{2}(R_{q'}R_{\overline{q}} - R_q R_{\overline{q'}})$, where $q\times_R q' = -\mathrm{Im}\,(\overline{q}\cdot q') = \mathrm{Im}\,(\overline{q'}\cdot q)$ is the right cross product (it is a bilinear skew map from $\mathbb{H}\times\mathbb{H}$ to $\mathrm{Im}\,\mathbb{H}$). Then we have the same identification between $\Sigma^-(\mathbb{H})$ and S^2, as in (3). Under this identification J^- becomes

$$\begin{array}{rccc} \rho_-\colon & Gr_2(\mathbb{H}) & \to & S^2 \\ & q\wedge q' & \mapsto & q\times_R q' \end{array}.$$

1.2 Action of $SO(4)$

Recall the following 2-sheeted covering of $SO(4)$:

$$\begin{array}{rccc} \chi\colon & S^3\times S^3 & \to & SO(4) \\ & (a,b) & \mapsto & L_a R_{\overline{b}} \end{array}$$

and set $Spin(3)_+ = \{L_a, a\in S^3\}$, $Spin(3)_- = \{R_{\overline{b}}, b\in S^3\}$, then $SO(4) = Spin(3)_+Spin(3)_- = Spin(3)_-Spin(3)_+$. We have the two following representations of $Spin(3)_\varepsilon$:

$$\chi^+\colon L_a \mapsto \mathrm{int}_a = L_a R_{\overline{a}} \in SO(\mathrm{Im}\,\mathbb{H}), \quad \chi^-\colon R_{\overline{b}} \mapsto \mathrm{int}_b = L_b R_{\overline{b}} \in SO(\mathrm{Im}\,\mathbb{H}).$$

Then the map ρ_ε is $Spin(3)$-equivariant: for all $q,q'\in\mathbb{H}$, $g = L_a R_{\overline{b}} \in SO(4)$,

$$\begin{array}{rcccl} (gq)\times_L(gq') & = & a(q\times_L q')\overline{a} & = & \mathrm{int}_a(q\times_L q') \\ (gq)\times_R(gq') & = & b(q\times_R q')\overline{b} & = & \mathrm{int}_b(q\times_R q'). \end{array}$$

Hence we have $\forall g \in SO(4)$,

$$\rho_\varepsilon(g(q \wedge q')) = \chi_g^\varepsilon(\rho_\varepsilon(q \wedge q'))$$

(where we have extended χ^ε to $SO(4)$ in an obvious way: $\chi^+(L_a R_{\overline{b}}) = \chi^+(L_a)$, $\chi^-(L_a R_{\overline{b}}) = \chi^-(R_{\overline{b}})$). Besides the map J^ε is also $Spin(3)$-equivariant, in other words the identification (3) is $Spin(3)$-equivariant:

$$\forall g \in SO(4), \quad g J^+_{q\wedge q'} g^{-1} = L_a R_{\overline{b}} \, L_{q \times_L q'} \, R_b L_{\overline{a}} = L_{a(q\times_L q')a^{-1}} = J^+_{g(q\wedge q')} \ .$$

The action of $Spin(3)_+ = SU(\mathbb{R}^4, R_e)$ (resp. $Spin(3)_- = SU(\mathbb{R}^4, L_e)$) on $\Sigma^-(\mathbb{H})$ (resp. $\Sigma^+(\mathbb{H})$) is trivial. Hence $SO(4)$ acts on $\Sigma^\varepsilon(\mathbb{H})$ only by its component $Spin(3)_\varepsilon$ (in the same way it acts on S^2_ε only by its component $Spin(3)_\varepsilon$ via χ^ε). In fact, the equality $gJ^+_{q\wedge q'}g^{-1} = J^+_{g(q\wedge q')}$ results immediately from the definition of $J^+_{q\wedge q'}$ and the fact that g is a positive isometry. This natural equality which is equivalent to what we called the fundamental property in [12]: $(gq) \times (gq') = \chi_g(q \times q')$, is characteristic of dimension 4: in this case it is possible to associate in a natural way (which depends only on the metric and the orientation) to each plane a complex structure, which is not possible in higher dimension. In dimension 8, we must choose an octonionic structure in $\mathbb{R}^8$ to do that (see [12]).

1.3 The Grassmannian $Gr_2(\mathbb{H})$ is a product of spheres

Theorem 1 *The map*

$$\begin{array}{rccc} \rho_+ \times \rho_- : & Gr_2(\mathbb{H}) & \to & S^2 \times S^2 \\ & q \wedge q' & \mapsto & (q \times_L q', q \times_R q') \end{array}$$

is a diffeomorphism.

Proof. $SO(3) \times SO(3)$ acts transitively on $S^2 \times S^2$ so $SO(4)$ acts transitively on $S^2 \times S^2$ via $\chi^+ \times \chi^-$, thus $\rho_+ \times \rho_-$ is surjective.
Let $e \in S(\operatorname{Im}\mathbb{H})$, $g = L_a R_{\overline{b}}$, $g' = L_{a'} R_{\overline{b'}} \in SO(4)$ then we have[2]

$$\begin{array}{rcl} \rho_+ \times \rho_-(g(1\wedge e)) = \rho_+ \times \rho_-(g'(1\wedge e)) & \Longleftrightarrow & (aea^{-1}, -beb^{-1}) = (a'ea'^{-1}, -b'eb'^{-1}) \\ & \Longleftrightarrow & a'^{-1}a, b'^{-1}b \in S^1(e) \\ & \Longrightarrow & (L_{a'}R_{\overline{b'}})^{-1}(L_a R_{\overline{b}})(1\wedge e) = 1 \wedge e \\ & \Longrightarrow & g(1\wedge e) = g'(1 \wedge e). \end{array}$$

Hence, since $SO(4)$ acts transitively on $Gr_2(\mathbb{H})$, we have proved that $\rho_+ \times \rho_-$ is injective and that

$$\rho_+ \times \rho_-(g(1\wedge e)) = \rho_+ \times \rho_-(g'(1\wedge e)) \Longleftrightarrow (a'^{-1}a, b'^{-1}b) \in S^1(e) \times S^1(e)$$

[2]setting $S^1(e) = \{\cos\theta + \sin\theta \, e, \, \theta \in \mathbb{R}\}$,

(in the previous sequence of implications, the last proposition implies the first one so all the propositions are equivalent). This completes the proof. ■

As it is the case in [12], it is useful here to introduce a function $\tilde{\rho}_\varepsilon$ on $Spin(3)_\varepsilon$ corresponding to ρ_ε: we define $\tilde{\rho}_{\varepsilon e}\colon Spin(3)_\varepsilon \to S^2$ by $\tilde{\rho}_{\varepsilon e}(g) = \chi_g^\varepsilon(e)$ (where $e \in S(\operatorname{Im}\mathbb{H}) = S^2$), i.e. under the identification $Spin(3)_\varepsilon = S^3$ we have $\tilde{\rho}_{\varepsilon e}(a) = \operatorname{int}_a(e) = aea^{-1}$, which is nothing but the Hopf fibration $S^3 \to S^3/S^1(e)$. If $\rho_\varepsilon(e_1 \wedge e_2) = e$ then $\tilde{\rho}_{\varepsilon e}(g) = \rho_\varepsilon(g(e_1 \wedge e_2))$. In the following, we will forget the index e. Hence, if we take $e_1 \wedge e_2$ such that $\rho_\varepsilon(e_1 \wedge e_2) = e$ for $\varepsilon = \pm 1$ (i.e. $e_1 \wedge e_2 = (1 \wedge e)^\perp$ which means also that (e, e_1, e_2) is a direct orthonormal basis of $\operatorname{Im}\mathbb{H}$) then we have the following commutative diagram:

$$
\begin{array}{ccc}
S^3 \times S^3 & \xrightarrow{\ \chi\ } & SO(4) \\
{\scriptstyle \tilde{\rho}_+ \times \tilde{\rho}_-}\Big\downarrow & & \Big\downarrow {\scriptstyle g \mapsto g(e_1 \wedge e_2)} \\
S^2 \times S^2 & \xleftarrow[\rho_+ \times \rho_-]{\simeq} & Gr_2(\mathbb{H})
\end{array}
$$

Let us now consider the restriction to $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$ of this diagram. First the universal covering $Spin(3) \to SO(3)$ is obtained by restriction to $\Delta_3 = \{(a,a),\ a \in S^3\} \simeq S^3$ of $\chi\colon S^3 \times S^3 \to SO(4)$, which gives the covering $(a,a) \mapsto \operatorname{int}_a$. Then supposing in addition that $e_1, e_2 \in \operatorname{Im}\mathbb{H}$, the restriction to $SO(3)$ of $SO(4) \to Gr_2(\mathbb{H})$ is only the surjective map $g \in SO(3) \mapsto g(e_1 \wedge e_2) \in Gr_3(\mathbb{R}^3)$. And the restriction to $Gr_2(\mathbb{R}^3)$ of $\rho_+ \times \rho_-$ gives the diffeomorphism $\rho\colon u \wedge v \in Gr_2(\mathbb{R}^3) \to u \times v \in S^2$. Finally the restriction to Δ_3 of $\tilde{\rho}_+ \times \tilde{\rho}_-$ gives the Hopf fibration $\tilde{\rho}\colon a \in S^3 \mapsto aea^{-1} \in S^2$. So by restriction to $\mathbb{R}^3$, we obtain the classical commutative diagram:

$$
\begin{array}{ccc}
S^3 & \xrightarrow{\ \chi_3\ } & SO(3) \\
{\scriptstyle \mathrm{Hopf}}\Big\downarrow & & \Big\downarrow \\
S^2 & \xleftarrow{\ \simeq\ } & Gr_2(\mathbb{R}^3)
\end{array}
$$

Remark 1 Besides if we use $\Sigma^\varepsilon(\mathbb{H})$ instead of the sphere S^2 the Hopf fibration $\tilde{\rho}_\varepsilon$ becomes $SU(2, J_{1\wedge e}^{-\varepsilon}) \to \Sigma^\varepsilon(\mathbb{H}) = SU(2, J_{1\wedge e}^{-\varepsilon})/U(1)_\varepsilon = SO(4)/U(2, J_{1\wedge e}^{\varepsilon})$ where $U(1)_+ = R_{S^1(e)} = \exp(\mathbb{R}.R_e)$, $U(1)_- = L_{S^1(e)} = \exp(\mathbb{R}.L_e)$.

1.4 The ρ-harmonic ω_I-isotropic surfaces

We recall here in the particular case of $\mathbb{H} = \mathbb{R}^4$ our result obtained in [12] about ρ-harmonic surfaces. To do that, we need to introduce some notations and definitions. We have

$$\rho_\varepsilon(q \wedge q') = -\varepsilon \sum_{i=1}^{3} \omega_i^\varepsilon(q, q') e_i$$

where $(e_i)_{1\leq i\leq 3} = (i,j,k)$ and $\omega_i^\varepsilon = \langle \cdot, J^\varepsilon_{1\wedge e_i}\cdot\rangle$ (i.e. $\omega_i^+ = \langle \cdot, L_{e_i}\cdot\rangle$, $\omega_i^- = \langle \cdot, R_{e_i}\cdot\rangle$). Let us set, for $I \subsetneq \{1,2,3\}$,

$$Q_I^\varepsilon = \{P \in Gr_2(\mathbb{H}) |\, \omega_i^\varepsilon(P) = 0,\, i \in I\},$$

then $Q_\varnothing = Gr_2(\mathbb{H})$, $Q_{\{k\}} = \{P \in Gr_2(\mathbb{H})$, Lagrangian for $\omega_k^\varepsilon\}$, and $Q^\varepsilon_{\{k,l\}}$ is the set of special Lagrangian planes (more precisely the ω_k^ε-Lagrangian planes P such that $\det_{\mathbb{C}^2}(P) = \pm i$ under the identification: $x \in \mathbb{R}^4 \mapsto (x_0 + ix_k, x_l + i\varepsilon x_{k\wedge l}) \in \mathbb{C}^2$, with $(k,l,k\wedge l)$ cyclic permutation of $(1,2,3)$; for example, if $(k,l) = (1,2)$, it is the identification $(z_1,z_2) \in \mathbb{C}^2 \mapsto z_1 + z_2 j \in \mathbb{H}$ for $\varepsilon = 1$ and $(z_1,z_2) \mapsto z_1 + jz_2$ for $\varepsilon = -1$). We have also $\rho_\varepsilon(Q_I) = S^I = S(\bigoplus_{i\notin I}\mathbb{R}e_i) = S^2, S^1, \{\pm e_k\}$ for $|I| = 0,1,2$ respectively. Besides we have for $I = \{i\} \subset \{1,2,3\}$, that $J^+(Q_I) = L_{S^I} = S^1(\mathbb{R}L_{e_j} \oplus \mathbb{R}L_{e_k})$ is the circle of positive complex structures which anticommute with L_{e_i}; and for $I = \{i,j\} \subset \{1,2,3\}$, $J^+(Q_I) = L_{S^I} = \{\pm L_{e_k}\}$.
We denote by G_I^ε the subgroup of $Spin(3)_\varepsilon$ which conserves ω_i^ε, for all $i \in I$; this is the subgroup of $Spin(3)_\varepsilon$ which commutes with L_{e_i}, for all $i \in I$. Then $G_I^\varepsilon = S^3, S^1, \{\pm 1\}$ for $|I| = 0,1,2$ respectively. We can also consider instead of $Spin(3)_\varepsilon$ the group $SO(4)$ (which is equivalent to add the component $Spin(3)_{-\varepsilon}$ which is useless), then we have $G_I^\varepsilon = SO(4), U(2, J^\varepsilon_{1\wedge e_i}), SU(2, J^\varepsilon_{1\wedge e_i})$ for $|I| = 0,1,2$ respectively. Let $e \in S(\bigoplus_{i\notin I}\mathbb{R}e_i)$. The inner automorphism, $\mathrm{Int}J^\varepsilon_{1\wedge e}$, defines on G_I^ε an involution which gives rise to the symmetric space $S^I = G_I^\varepsilon / G^\varepsilon_{I\cup\{k\}}$ and in the Lie algebra of G_I^ε, $\mathfrak{g}_I^\varepsilon$, to the eigenspace decomposition of $\mathrm{Ad}J^\varepsilon_{1\wedge e}$:

$$\mathfrak{g}_I^\varepsilon = \mathfrak{g}_0^\varepsilon(I) \oplus \mathfrak{g}_2^\varepsilon(I)$$

with $\mathfrak{g}_0^\varepsilon(I) = \ker(\mathrm{Ad}J^+_{1\wedge e} - \mathrm{Id})$, $\mathfrak{g}_2^\varepsilon(I) = \ker(\mathrm{Ad}J^\varepsilon_{1\wedge e} + \mathrm{Id})$.
Let us introduce $\mathcal{G}_I^\varepsilon = G_I^\varepsilon \ltimes \mathbb{R}^4$ the group of affine isometries of which the linear part is in G_I^ε, and its Lie algebra: $\mathfrak{g}^\varepsilon(I) = \mathfrak{g}_I^\varepsilon \oplus \mathbb{R}^4$. Consider the automorphism of the group $\mathcal{G}_I^\varepsilon$: $\tau_e^\varepsilon = \mathrm{Int}(-\varepsilon J^\varepsilon_{1\wedge e}, 0)$ with $e \in S(\bigoplus_{i\notin I}\mathbb{R}e_i)$. This is an order four automorphism which gives us an eigenspace decomposition of $\mathfrak{g}^\varepsilon(I)^{\mathbb{C}}$:

$$\mathfrak{g}^\varepsilon(I)^{\mathbb{C}} = \bigoplus_{k\in\mathbb{Z}_4} \tilde{\mathfrak{g}}_k^\varepsilon(I)$$

with $\tilde{\mathfrak{g}}^\varepsilon_{\pm1}(I) = \mathfrak{g}^\varepsilon_{\pm1} = \ker(J^\varepsilon_{1\wedge e} \pm i\mathrm{Id})$, $\tilde{\mathfrak{g}}_0^\varepsilon(I) = \mathfrak{g}_0^\varepsilon(I)^{\mathbb{C}}$, $\tilde{\mathfrak{g}}_2^\varepsilon(I) = \mathfrak{g}_2^\varepsilon(I)^{\mathbb{C}}$. Moreover we have $[\tilde{\mathfrak{g}}_k^\varepsilon(I), \tilde{\mathfrak{g}}_l^\varepsilon(I)] \subset \tilde{\mathfrak{g}}^\varepsilon_{k+l}(I)$.
We fix a value of $\varepsilon = \pm 1$. Then let us define as in [12]:

Definition 1 *Let L be an immersed surface in $\mathbb{H}$, then a map $\rho_L\colon L \to S^2$ is associated to it, defined by $\rho_L(z) = \rho_\varepsilon(T_zL)$ i.e. if $X\colon L \to \mathbb{H}$ is the immersion then $\rho_L = X^*\rho_\varepsilon$. We will say that L is ρ-harmonic if ρ_L is harmonic[3].*
Let $I \subsetneq \{1,2,3\}$, we will say that L is ω_I-isotropic if $\forall z \in L, T_zL \in Q_I^\varepsilon$ (i.e. L is ω_i^ε-isotropic for all $i \in I$) which is equivalent to: ρ_L takes values in $S^I = S(\oplus_{i\notin I}\mathbb{R}e_i) \subset S^2$. Hence for $|I| = 1$, the ρ-harmonic ω_I^ε-isotropic

[3]with respect to the induced metric on S^2.

*surfaces are the **Hamiltonian stationary Lagrangian surfaces** in $\mathbb{C}^2$, and for $|I| = 2$, these are the **special Lagrangian surfaces** in $\mathbb{C}^2$ (see above for the identification $\mathbb{R}^4 \simeq \mathbb{C}^2$).*
If it could be an ambiguity as concerned the value of $\varepsilon = \pm 1$, we will use the qualificatifs "left" and "right" respectively to design these two values. A lifted conformal left (resp. right) ω_I-isotropic immersion - $LC\omega_I$ - (if $I = \varnothing$ we will say a lifted conformal immersion or simply a lift) is a map $U = (F, X)\colon L \to \mathcal{G}_I^\varepsilon$ such that X is a conformal ω_I^ε-isotropic immersion and $\tilde{\rho}_e \circ F = \rho_L$.

We have obtained the following result in [12]:

Theorem 2 *Let Ω be a simply connected open domain in $\mathbb{C}$, and α an 1-form on Ω with values in $\mathfrak{g}(I)$, then*

- *α is the Maurer-Cartan form of a $LC\omega_I$ if and only if*

$$d\alpha + \alpha \wedge \alpha = 0, \quad \alpha''_{-1} = 0 \quad \text{and} \quad \alpha'_{-1} \text{ does not vanish}$$

- *furthermore, α corresponds to a ρ-harmonic ω_I-isotropic conformal immersion if and only if the extended Maurer-Cartan form $\alpha_\lambda = \lambda^{-2}\alpha'_2 + \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha''_2$ satisfies*

$$d\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda = 0, \ \forall \lambda \in \mathbb{C}^*.$$

Let us recall the proof given in [12].

Proof. To fix ideas, we take $\varepsilon = 1$. α is a Maurer-Cartan form if and only if it satisfies the Maurer-Cartan equation. In this case, it can be integrated by $U = (F, X)\colon \Omega \to \mathcal{G}_I$, $\alpha = U^{-1}.dU, U(z_0) = 1$. Hence $\alpha = U^{-1}.dU = (F^{-1}.dF, F^{-1}.dX)$. Moreover, $F^{-1}.dX = \alpha_{-1} + \alpha_1$ is real and $\mathfrak{g}_{\pm 1} = \{V \pm iL_eV, V \in \mathbb{H}\}$ so $\alpha_{-1} = \overline{\alpha_1}$. Hence $\alpha''_{-1} = 0 \Longleftrightarrow \alpha''_{-1} = \overline{\alpha'_1} = 0 \Longleftrightarrow \alpha_{-1} = (F^{-1}\frac{\partial X}{\partial z})dz \Longleftrightarrow F^{-1}\frac{\partial X}{\partial y} = L_e(F^{-1}\frac{\partial X}{\partial x}) \Longleftrightarrow F^{-1}dX = h(q_0 du + q'_0 dv)$ with $h \in C^\infty(\Omega, \mathbb{R})$, $q_0, q'_0 \in C^\infty(\Omega, S^3)$, $\langle q_0, q'_0\rangle = 0$ and $\rho(q_0, q'_0) = e$. Thus we have $(\alpha''_{-1} = 0$ and $\alpha'_{-1} \neq 0) \Longleftrightarrow dX = e^f(q du + q' dv)$ with $f \in C^\infty(\Omega, \mathbb{R})$, (q, q') orthonormal and $\rho(q, q') = \tilde{\rho}_e(F)$ i.e. $\rho_X = \tilde{\rho}_e(F)$. This proves the first point.
Hence we have the decomposition $\alpha = \alpha_2 + \alpha_{-1} + \alpha_0 + \alpha_1 = \alpha'_2 + \alpha'_{-1} + \alpha_0 + \alpha''_1 + \alpha''_2$. Furthermore, using the commutation relations $[\tilde{\mathfrak{g}}_k(I), \tilde{\mathfrak{g}}_l(I)] \subset \tilde{\mathfrak{g}}_{k+l}(I)$, $[\mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 1}] = \{0\}$, we obtain

$$\begin{aligned} d\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda \ = \ & \lambda^{-2}(d\alpha'_2 + [\alpha_0 \wedge \alpha'_2)]) \\ & + \lambda^{-1}(d\alpha'_{-1} + [\alpha'_{-1} \wedge \alpha_0] + [\alpha''_1 \wedge \alpha'_2]) \\ & + (d\alpha_0 + \frac{1}{2}[\alpha_0 \wedge \alpha_0] + \frac{1}{2}[\alpha'_2 \wedge \alpha''_2]) \\ & + \lambda(d\alpha''_1 + [\alpha''_1 \wedge \alpha_0] + [\alpha'_{-1} \wedge \alpha''_2]) \\ & + \lambda^2(d\alpha''_2 + [\alpha_0 \wedge \alpha''_2]), \end{aligned}$$

the coefficients of $\lambda^{-1}, \lambda^0, \lambda$ are respectively the projections of $d\alpha + \alpha \wedge \alpha$ on $\mathfrak{g}_{-1}, \mathfrak{g}_0, \mathfrak{g}_1$ respectively so they vanish and hence

$$d\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda = d\beta_{\lambda^2} + \beta_{\lambda^2} \wedge \beta_{\lambda^2}$$

where $\beta_\lambda = \lambda^{-1}\alpha_2' + \alpha_0 + \lambda\alpha_2''$ is the extended Maurer-Cartan form of $\beta = F^{-1}.dF$, the Maurer-Cartan form of the lift $F \in G_I$ of $\rho_X \in S^I$. According to [6], we know that ρ_X is harmonic if and only if $d\beta_\lambda + \beta_\lambda \wedge \beta_\lambda = 0$, $\forall \lambda \in \mathbb{C}^*$. This proves the second point and completes the proof. ■

Remark 2 We have $\rho_-(x,y) = -\mathrm{Im}\,(\overline{x}.y) = \rho_+(\overline{x},\overline{y})$. Hence $X\colon \Omega \to \mathbb{H}$ is ρ_--harmonic if and only if $\overline{X}$ is ρ_+-harmonic, and X is ω_I^--isotropic if and only if $\overline{X}$ is ω_I^+-isotropic. Besides if $U = (F,X)\colon \Omega \to G_I \ltimes \mathbb{H}$ is a left LCω_I then we have $F = L_a$ and $aea^{-1} = \rho_X = \rho_+(q,q')$ with $dX = e^\omega(qdu + q'dv)$, (q,q') orthonormal. Thus $\rho_-(\overline{q},\overline{q}') = aea^{-1}$ and hence $\underline{U} = (R_{\overline{a}}, \overline{X})$ is a right LCω_I.

Remark 3 The restriction to $\mathrm{Im}\,\mathbb{H} = \mathbb{R}^3$ of the left (or right) cross product gives us the usual cross product in $\mathbb{R}^3$. Hence a surface in $\mathrm{Im}\,\mathbb{H}$ is left (resp. right) ρ-harmonic if and only if it is a constant mean curvature surface.
In the same way, it is easy to see that a surface in S^3 is left (resp. right) ρ-harmonic if and only if it is a constant mean curvature surface.

Remark 4 We can apply now the Dorfmeister-Pedit-Wu method (DPW) to obtain a Weierstrass representation of ρ-harmonic surfaces (see [6, 7, 9, 12, 13]). There are non-trivial technical difficulties in establishing DPW, such as proving loop group splittings ([6, 14]).

2 Second Elliptic Integrable Systems

2.1 4-symmetric spaces and twistor spaces

Definition 2 *Let M be a Riemannian symmetric space. We will say that a Lie group G acts symmetrically on M or that M is a G-symmetric space if G acts transitively and isometrically on M and if there exists an involutive automorphism of G, σ, such that H the isotropy subgroup at a fixed point $p_0 \in M$, satisfies $(G^\sigma)^0 \subset H \subset G^\sigma$. We will say also that G/H is a symmetric realisation of M.*
We will say that a G-homogeneous space $N = G/G_0$ is a 4-symmetric bundle over the G-symmetric space M if there exists an order four automorphism τ of G, such that $(G^\tau)^0 \subset G_0 \subset G^\tau$, and (G,τ) gives rise to the symmetric space M, i.e. $\sigma = \tau^2$ and $G_0 \subset H$.
A G-homogeneous space $N = G/G_0$ is a locally 4-symmetric space if there exists an order four automorphism of the Lie algebra $\mathfrak{g} = \mathrm{Lie}\,G$, $\tau\colon \mathfrak{g} \to \mathfrak{g}$ such that $\mathfrak{g}^\tau = \mathrm{Lie}\,G_0$. We will say that G/G_0 is a locally 4-symmetric bundle over the G-symmetric space M if $\tau^2 = \sigma$ (and $G_0 \subset H$).

Let us consider M a G-symmetric space with $\tau\colon \mathfrak{g} \to \mathfrak{g}$ an order four automorphism such that $\tau^2 = \sigma$. The automorphism τ gives us an eigenspace decomposition of $\mathfrak{g}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{k\in\mathbb{Z}_4} \tilde{\mathfrak{g}}_k$$

where $\tilde{\mathfrak{g}}_k$ is the $e^{ik\pi/2}$-eigenspace of τ. We have clearly $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0^{\mathbb{C}}$, $\overline{\tilde{\mathfrak{g}}_k} = \tilde{\mathfrak{g}}_{-k}$ and $[\tilde{\mathfrak{g}}_k, \tilde{\mathfrak{g}}_l] \subset \tilde{\mathfrak{g}}_{k+l}$. We define $\mathfrak{g}_2$, $\mathfrak{m}$ and $\underline{\mathfrak{g}}_1$ by

$$\tilde{\mathfrak{g}}_2 = \mathfrak{g}_2^{\mathbb{C}},\ \mathfrak{m}^{\mathbb{C}} = \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_1 \ \text{ and }\ \underline{\mathfrak{g}}_1^{\mathbb{C}} = \bigoplus_{k\in\mathbb{Z}_4\setminus\{0\}} \tilde{\mathfrak{g}}_k,$$

it is possible because $\overline{\tilde{\mathfrak{g}}_2} = \tilde{\mathfrak{g}}_2$ and $\overline{\tilde{\mathfrak{g}}_{-1}} = \tilde{\mathfrak{g}}_1$. Let us set $\mathfrak{g}_{-1} = \tilde{\mathfrak{g}}_{-1}, \mathfrak{g}_1 = \tilde{\mathfrak{g}}_1$ (i.e. we forget the " $\tilde{}$ "), $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_2$. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

is the eigenspace decomposition of the involutive automorphism σ, $\mathfrak{h}$ is the Lie algebra of H, the isotropy subgroup of G at a reference point p_0, and $\mathfrak{m}$ is identified to the tangent space $T_{p_0}M$. Besides we remark that $\tau_{|\mathfrak{m}} \in \Sigma(\mathfrak{m})$ (since $\tau_{|\mathfrak{m}^{\mathbb{C}}} = -i\mathrm{Id}_{\mathfrak{g}_{-1}} \oplus i\mathrm{Id}_{\mathfrak{g}_1}$)[4], which gives us the following theorem (proved in section 3.2).

Theorem 3 *Let us consider M a Riemannian G-symmetric space and $\tau\colon \mathfrak{g} \to \mathfrak{g}$ an order four automorphism such that $\tau^2 = \sigma$. Let us make G acting on $\Sigma(M)$: $g \cdot J = gJg^{-1}$. Let $J_0 \in \Sigma(T_{p_0}M)$ be the complex structure corresponding[5] to $-\tau_{|\mathfrak{m}} \in \Sigma(\mathfrak{m})$, under the identification $T_{p_0}M = \mathfrak{m}$. Then the orbit of J_0 under the action of G is an immersed submanifold of $\Sigma(M)$. Denoting by G_0 the stabilizer of J_0, then $\mathrm{Lie}\, G_0 = \mathfrak{g}^\tau$ and thus G/G_0 is a locally 4-symmetric bundle over M, and the natural map*

$$\begin{array}{rccc} i\colon & G/G_0 & \longrightarrow & \Sigma(M) \\ & g.G_0 & \longmapsto & gJ_0g^{-1} \end{array}$$

is an injective immersion and a morphism of bundle. Moreover, if the image of G in $\mathrm{Is}(M)$ (the group of isometry of M) is closed, then i is an embedding.

2.2 The second elliptic integrable system associated to a 4-symmetric space

We give ourself M a Riemannian G-symmetric space with $\tau\colon \mathfrak{g} \to \mathfrak{g}$ an order four automorphism such that $\tau^2 = \sigma$, and $N = G/G_0$ the associated locally 4-symmetric space given by theorem 3. We use the same notations as in Section 2.1. Then let us recall what is a second elliptic system according to C.L. Terng (see [15]).

[4] We choose a metric in $\mathfrak{m}$ invariant by $\tau_{|\mathfrak{m}}$ (and of course by $\mathrm{Ad}H$), see section 3.1

[5] About the choice of $-\tau_{|\mathfrak{m}}$ (instead of $\tau_{|\mathfrak{m}}$) and its link to the $(1,0)$-splitting, see theorem 4 and remark 13, for later explanation.

Definition 3 *The second $(\mathfrak{g},\tau)$-system is the equation for $(u_0,u_1,u_2)\colon \mathbb{C}\to \oplus_{j=0}^{2}\tilde{\mathfrak{g}}_{-j}$,*

$$\begin{cases} \partial_{\bar z}u_2 + [\bar u_0, u_2] = 0 & (a)\\ \partial_{\bar z}u_1 + [\bar u_0, u_1] + [\bar u_1, u_2] = 0 & (b)\\ -\partial_{\bar z}u_0 + \partial_z \bar u_0 + [u_0,\bar u_0] + [u_1,\bar u_1] + [u_2,\bar u_2] = 0. & (c)\end{cases} \tag{4}$$

It is equivalent to say that the 1-form

$$\alpha_\lambda = \sum_{i=0}^{2} \lambda^{-i}u_i dz + \lambda^i \bar u_i d\bar z = \lambda^{-2}\alpha_2' + \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1'' + \lambda^2\alpha_2'' \tag{5}$$

satisfies the zero curvature equation:

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0, \tag{6}$$

for all $\lambda\in\mathbb{C}^*$. We will speak about the (G,τ)-system (τ is an automorphism of $\mathrm{Lie}\,G=\mathfrak{g}$*) when we will look for solutions of the $(\mathfrak{g},\tau)$-system in G, i.e. maps $U\colon\mathbb{C}\to G$ such that their Maurer-Cartan form is solution of the $(\mathfrak{g},\tau)$-system, in other words when we integrate the zero curvature equation (6) in G. We will call (geometric) solution of the second elliptic integrable system associated to the locally 4-symmetric space G/G_0 a map $J\colon\mathbb{C}\to G/G_0$ which can be lifted into a solution $U\colon\mathbb{C}\to G$ of (4).*

Remark 5 In (4), $\{\mathrm{Im}((a)), (b), (c)\}$ is equivalent to $d\alpha + \frac{1}{2}[\alpha\wedge\alpha] = 0$. Hence the additional condition added to the Maurer-Cartan equation by the zero curvature equation (6) is $\mathrm{Re}\left(\partial_{\bar z}\alpha_2'(\frac{\partial}{\partial z}) + \left[\alpha_0''(\frac{\partial}{\partial \bar z}), \alpha_2'(\frac{\partial}{\partial z})\right]\right) = 0$ or equivalently

$$d(\star\alpha_2) + [\alpha_0\wedge(\star\alpha_2)] = 0.$$

The first example of second elliptic system was given by F. Hélein and P. Romon (see [7, 9]): they showed that the equations for Hamiltonian stationary Lagrangian surfaces in 4-dimension Hermitian symmetric spaces are exactly the second elliptic system associated to certain 4-symmetric spaces. Then in [12], we found another example in $\mathbb{O}$: the ρ-harmonic surfaces in $\mathbb{O}$, which by restriction to $\mathbb{H}$ gave us the ρ-harmonic surfaces in $\mathbb{H}$ (studied in section 1) which generalize the Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2$.

Definition 4 *Let M be a Riemannian manifold and ∇ its Levi-Civita connection which induces a connection on* $\mathrm{End}(TM)$*. Let us define for each $(p,J_p)\in\Sigma(M)$ the orthogonal projection*

$$\mathrm{pr}^\perp(p,J_p)\colon \mathrm{End}(T_pM)\to T_{J_p}(\Sigma(T_pM))$$

(T_pM is an Euclidean vector space so $\Sigma(T_pM)$ is a submanifold of the Euclidean space $\mathrm{End}(T_pM)$ *and so $T_{J_p}\Sigma(T_pM)$ is a vector subspace of* $\mathrm{End}(T_pM)$ *and we*

can consider the orthogonal projection on this subspace). Given L a Riemannian surface and $J\colon L\to\Sigma(M)$ we set

$$\Delta J = \mathrm{pr}^{\perp}(J).\mathrm{Tr}(\nabla^2 J)$$

where Tr *is the trace with respect to the metric on L (in fact, we take the vertical part of the rough Laplacian) . We will say that J is vertically harmonic if $\Delta J = 0$. This notion depends only on the conformal structure on L.*

Definition 5 *Let (L, j) be a Riemann surface, M an oriented manifold and $X\colon L\to M$ an immersion. Let $J\colon L\to X^*(\Sigma(M))$ be an almost complex structure on the vector bundle $X^*(TM)$. Then we will say that J is an admissible twistor lift of X if one of the following equivalent statements holds:*

(i) *X is J-holomorphic: $\star dX := dX\circ j = J.dX$*

(ii) *J is an extension of the complex structure on the oriented tangent plane $P = X_*(TL)$ induced by j, the complex structure of L, or equivalently J induces the complex structure j in L.*

(iii) *X is a conformal immersion and J stabilizes the tangent plane $X_*(TL)$, i.e. for all $z\in L$, J_z stabilizes $X_*(T_zL)$ and induces on it the same orientation, which we will denote by $J \circlearrowleft X_*(TL)$*

(iv) *X is a conformal immersion and J is an extension of the unique positive complex structure I_P of the tangent plan $P = X_*(TL)$.*

Finally, we will say that a map $J\colon L\to\Sigma(M)$ is an admissible twistor lift if its projection $X = \mathrm{pr}_M\circ J\colon L\to M$ is an immersion and J is an admissible twistor lift of it.

Theorem 4 *Let L be a simply connected Riemann surface and (G,τ) a locally 4-symmetric bundle over a symmetric space $M = G/H$. Let $J_0\in\Sigma(T_{p_0}M)$ be the complex structure corresponding to $-\tau_{|\mathfrak{m}}$ (see Section 2.1). Let be $J_X\colon L\to i(G/G_0)\subset\Sigma(G/H)$. Then the two following statements are equivalent:*

- *J_X is an admissible twistor lift.*
- *Any lift $F\colon L\to G$ of J_X ($FJ_0F^{-1} = J_X$) gives rise to a Maurer-Cartan form $\alpha = F^{-1}.dF$ which satisfies: $\alpha''_{-1} = \alpha'_1 = 0$ and α'_{-1} does not vanish.*

Furthermore, under these statements, $J_X\colon L\to\Sigma(G/H)$ is vertically harmonic if and only if $J_X\colon L\to G/G_0$ is solution of the second elliptic integrable system associated to the locally 4-symmetric space (G,τ), i.e.

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda\wedge\alpha_\lambda] = 0,\quad \forall\lambda\in\mathbb{C}^*,$$

where $\alpha_\lambda = \lambda^{-2}\alpha'_2 + \lambda^{-1}\alpha'_{-1} + \alpha_0 + \lambda\alpha''_1 + \lambda^2\alpha''_2$ is the extended Maurer-Cartan form of α.

Proof. For the first point, let us make F^{-1} acting on the equation $dX \circ j = J_X.dX$, we obtain $\alpha_{\mathfrak{m}} \circ j = -\tau_{|\mathfrak{m}}(\alpha_{\mathfrak{m}})$ which is equivalent to $\alpha''_{-1} = \alpha'_1 = 0$. Thus $\alpha_{-1}(\frac{\partial}{\partial z}) = \alpha_{\mathfrak{m}}(\frac{\partial}{\partial z}) = F^{-1}.\frac{\partial X}{\partial z}$, and X is an immersion if and only if α'_{-1} does not vanish.
For the second point, let us recall that $\mathrm{End}(T_pM) = \mathrm{sym}(T_pM) \overset{\perp}{\oplus} \mathfrak{so}(T_pM)$ and given $J \in \Sigma(T_pM)$, we have $T_J\Sigma(T_pM) = \mathrm{Ant}(J) = \{A \in \mathfrak{so}(T_pM) | AJ + JA = 0\}$ and $(T_J\Sigma(T_pM))^{\perp} \cap \mathfrak{so}(T_pM) = \mathrm{Com}(J) = \{A \in \mathfrak{so}(T_pM) | [A, J] = 0\}$.
Now, let us compute the connection $X^*\nabla$ on $X^*(\mathrm{End}(TM))$, in terms of the Lie algebra setting. Let A be a section of $X^*(\mathrm{End}(TM))$ and Y a section of $X^*(TM)$. Let $A_0 \in C^{\infty}(L, \mathrm{End}(T_{p_0}M))$ be defined by $A_{F.p_0} = FA_0F^{-1}$ and $A_{\mathfrak{m}} \in C^{\infty}(L, \mathrm{End}(\mathfrak{m}))$ its image under the identification $T_{p_0}M = \mathfrak{m}$. Then $A_{F.p_0}$ corresponds to $\mathrm{Ad}F \circ A_{\mathfrak{m}} \circ \mathrm{Ad}F^{-1}$ (under the identification $TM = [\mathfrak{m}] := \{(g.p_0, \mathrm{Ad}g(\xi)),\ \xi \in \mathfrak{m}, g \in G\}$, see section 3.1). In particular $(J_X)_{\mathfrak{m}} = -\tau_{|\mathfrak{m}}$ (we suppose $F(p_0) = 1$). We set also $Y = \mathrm{Ad}F(\xi).p_0$, $\xi \in C^{\infty}(L, \mathfrak{m})$. From now, we do the identification $TM = [\mathfrak{m}]$ without precising it. Then, denoting by $[\ ,\ ]_{\mathfrak{m}}$ the $\mathfrak{m}$-component of the Lie bracket, we have

$$\begin{aligned}(\nabla A)(Y) &= \nabla(AY) - A(\nabla Y)\\ &= \mathrm{Ad}F\left([d(A_{\mathfrak{m}}\xi) + [\alpha, A_{\mathfrak{m}}.\xi]]_{\mathfrak{m}} - A_{\mathfrak{m}}(d\xi + [\alpha, \xi]_{\mathfrak{m}})\right)\\ &= \mathrm{Ad}F\left((dA_{\mathfrak{m}})\xi + (\mathrm{ad}\alpha_{\mathfrak{h}} \circ A_{\mathfrak{m}} - A_{\mathfrak{m}} \circ \mathrm{ad}\alpha_{\mathfrak{h}})\xi\right).\end{aligned}$$

Hence

$$\nabla A = \mathrm{Ad}F(dA_{\mathfrak{m}} + [\mathrm{ad}_{\mathfrak{m}}\alpha_{\mathfrak{h}}, A_{\mathfrak{m}}]).$$

In particular,[6]

$$\nabla_{\frac{\partial}{\partial z}} J_X = -2\mathrm{Ad}F(\mathrm{ad}_{\mathfrak{m}}\alpha'_2 \circ \tau_{|\mathfrak{m}})$$

(because $\mathrm{ad}_{\mathfrak{m}}\mathfrak{g}_0$ commutes with $\tau_{|\mathfrak{m}}$ whereas $\mathrm{ad}_{\mathfrak{m}}\mathfrak{g}_2$ anticommutes with it) and thus

$$\begin{aligned}\nabla_{\frac{\partial}{\partial \bar z}}(\nabla_{\frac{\partial}{\partial z}} J_X) &= -2\mathrm{Ad}F\left(\mathrm{ad}_{\mathfrak{m}}(\partial_{\bar z}\alpha'_2) \circ \tau_{|\mathfrak{m}} + [\mathrm{ad}_{\mathfrak{m}}(\alpha''_{\mathfrak{h}}), \mathrm{ad}_{\mathfrak{m}}(\alpha'_2) \circ \tau_{|\mathfrak{m}}]\right)\\ &= -2\mathrm{Ad}F\left(\mathrm{ad}_{\mathfrak{m}}(\partial_{\bar z}\alpha'_2) \circ \tau_{|\mathfrak{m}} + \mathrm{ad}_{\mathfrak{m}}([\alpha''_0, \alpha'_2]) \circ \tau_{|\mathfrak{m}}\right.\\ &\qquad \left. + [\mathrm{ad}_{\mathfrak{m}}\alpha''_2, \mathrm{ad}_{\mathfrak{m}}(\alpha'_2) \circ \tau_{|\mathfrak{m}}]\right)\\ &= -2\mathrm{Ad}F\left(\mathrm{ad}_{\mathfrak{m}}(\partial_{\bar z}\alpha'_2 + [\alpha''_0, \alpha'_2]) \circ \tau_{|\mathfrak{m}} + [\mathrm{ad}_{\mathfrak{m}}\alpha''_2, \mathrm{ad}_{\mathfrak{m}}(\alpha'_2) \circ \tau_{|\mathfrak{m}}]\right)\end{aligned}$$

but $-\mathrm{Ad}F\left([\mathrm{ad}_{\mathfrak{m}}\alpha''_2, \mathrm{ad}_{\mathfrak{m}}(\alpha'_2) \circ \tau_{|\mathfrak{m}}]\right)$ commutes with $-\mathrm{Ad}F(\tau_{|\mathfrak{m}}) = J_X$ so it is orthogonal to $T_J\Sigma(T_pM)$ thus

$$\mathrm{pr}^{\perp}(J_X).\nabla_{\frac{\partial}{\partial \bar z}}(\nabla_{\frac{\partial}{\partial z}} J_X) = -2\mathrm{Ad}F\left(\mathrm{ad}_{\mathfrak{m}}(\partial_{\bar z}\alpha'_2 + [\alpha''_0, \alpha'_2]) \circ \tau_{|\mathfrak{m}}\right).$$

Hence, since $\mathrm{ad}_{\mathfrak{m}}$ is injective[7]

$$\Delta J_X = 0 \Longleftrightarrow \mathrm{Re}\left(\partial_{\bar z}\alpha'_2 + [\alpha''_0, \alpha'_2]\right) = 0. \tag{7}$$

This completes the proof. ∎

[6] In all the proof, we will merge α'_k (resp. α''_k) with $\alpha'_k(\frac{\partial}{\partial z})$ (resp. $\alpha''_k(\frac{\partial}{\partial \bar z})$), and in particular write '$[\alpha''_k, \alpha'_l]$' instead of '$[\alpha''_k(\frac{\partial}{\partial \bar z}), \alpha'_l(\frac{\partial}{\partial z})]$'. z is a local holomorphic coordinate in L.

[7] We can do this hypothesis without loss of generality, see section 3.1.

Remark 6 The equivalence (7) holds for any map $J_X\colon L \to i(G/G_0)$. Indeed, we have not used the fact that J_X is an admissible twistor lift to prove this equivalence.

Theorem 5 *Let $J_X\colon L \to G/G_0 \hookrightarrow \Sigma(G/H)$ be an admissible twistor lift. Then $J_X\colon L \to G/G_0$ is harmonic*[8] *if and only if $X\colon L \to G/H$ is harmonic and J_X is vertically harmonic.*

Proof. $J_X\colon L \to G/G_0$ is harmonic if and only if the Maurer-Cartan form $\alpha = F^{-1}.dF$ of the lift $F\colon L \to G$ of J_X ($FJ_0F^{-1} = J_X$) satisfies (see [3])

$$\partial_{\bar z}\underline{\alpha}_1' + [\alpha_0'', \underline{\alpha}_1'] + \frac{1}{2}[\underline{\alpha}_1'', \underline{\alpha}_1']_{\underline{\mathfrak{g}}_1} = 0$$

(where $\mathfrak{g} = \mathfrak{g}_0 \oplus \underline{\mathfrak{g}}_1$ is the reductive decomposition corresponding to the homogeneous space G/G_0, see Section 2.1) which splits into

$$\left\{\begin{array}{l} \partial_{\bar z}\alpha_2' + [\alpha_0'', \alpha_2'] + \frac{1}{2}[\alpha_1'', \alpha_1'] + \frac{1}{2}[\alpha_{-1}'', \alpha_{-1}'] = 0 \\ \partial_{\bar z}\alpha_{-1}' + [\alpha_0'', \alpha_{-1}'] + \frac{1}{2}[\alpha_2'', \alpha_1'] + \frac{1}{2}[\alpha_1'', \alpha_2'] = 0 \\ \partial_{\bar z}\alpha_1' + [\alpha_0'', \alpha_1'] + \frac{1}{2}[\alpha_2'', \alpha_{-1}'] + \frac{1}{2}[\alpha_{-1}'', \alpha_2'] = 0. \end{array}\right. \tag{8}$$

then, using $\alpha_{-1}'' = \alpha_1' = 0$, we obtain

$$\left\{\begin{array}{l} \partial_{\bar z}\alpha_2' + [\alpha_0'', \alpha_2'] = 0 \\ \partial_{\bar z}\alpha_{-1}' + [\alpha_0'', \alpha_{-1}'] = 0 \\ [\alpha_2'', \alpha_{-1}'] = 0 \end{array}\right.$$

(in the second equation, we have used $[\alpha_1'', \alpha_2'] = -\overline{[\alpha_2'', \alpha_{-1}']} = 0$).
Besides $X\colon L \to G/H$ is harmonic if and only if we have

$$\partial_{\bar z}\alpha_{\mathfrak{m}}' + [\alpha_{\mathfrak{h}}'', \alpha_{\mathfrak{m}}'] = 0$$

which splits into

$$\left\{\begin{array}{l} \partial_{\bar z}\alpha_{-1}' + [\alpha_0'', \alpha_{-1}'] + [\alpha_2'', \alpha_1'] = 0 \\ \partial_{\bar z}\alpha_1' + [\alpha_0'', \alpha_1'] + [\alpha_2'', \alpha_{-1}'] = 0. \end{array}\right. \tag{9}$$

and using $\alpha_{-1}'' = \alpha_1' = 0$, we obtain

$$\left\{\begin{array}{l} \partial_{\bar z}\alpha_{-1}' + [\alpha_0'', \alpha_{-1}'] = 0 \\ [\alpha_2'', \alpha_{-1}'] = 0. \end{array}\right.$$

This completes the proof. ■

[8] with respect to any metric induced by an AdG-invariant metric in $\mathfrak{g}$.

3 Structure of 4-symmetric bundles over symmetric spaces

3.1 4-symmetric spaces

Let G be a Lie group with Lie algebra $\mathfrak{g}$, $\tau\colon G \to G$ an order four automorphism with the fixed point subgroup G^τ, and the corresponding Lie algebra $\mathfrak{g}_0 = \mathfrak{g}^\tau$. Let G_0 be a subgroup of G such that $(G^\tau)^0 \subset G_0 \subset G^\tau$, then $\mathrm{Lie}\, G_0 = \mathfrak{g}_0$ and G/G_0 is a 4-symmetric space. The automorphism τ gives us an eigenspace decomposition of $\mathfrak{g}^{\mathbb{C}}$ for which we use the notation of section 2.1. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the eigenspace decomposition of the involutive automorphism $\sigma = \tau^2$. Let H be a subgroup of G such that $(G^\sigma)^0 \subset H \subset G^\sigma$ then $\mathrm{Lie}\, H = \mathfrak{h}$ and G/H is a symmetric space. We will often suppose that G_0 and H are chosen such that $G_0 = G^\tau \cap H$. With this condition, $G_0 \subset H$ so that G/G_0 is a bundle over G/H. Recall that the tangent bundle TM is canonically isomorphic to the subbundle $[\mathfrak{m}]$ of the trivial bundle $M \times \mathfrak{g}$, with fibre $\mathrm{Ad}g(\mathfrak{m})$ over the point $x = g.H \in M$. Under this identification the canonical G-invariant connection of M is just the flat differentiation in $M \times \mathfrak{g}$ followed by the projection on $[\mathfrak{m}]$ along $[\mathfrak{h}]$ (which is defined in the same way as $\mathfrak{m}$) (see [4]). For the homogeneous space $N = G/G_0$ we have the following reductive decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \underline{\mathfrak{g}}_1 \tag{10}$$

($\underline{\mathfrak{g}}_1$ can be written $\underline{\mathfrak{g}}_1 = \mathfrak{m} \oplus \mathfrak{g}_2$) with $[\mathfrak{g}_0, \underline{\mathfrak{g}}_1] \subset \underline{\mathfrak{g}}_1$. As for the symmetric space G/H, we can identify the tangent bundle TN with the subbundle $[\underline{\mathfrak{g}}_1]$ of the trivial bundle $N \times \mathfrak{g}$, with fibre $\mathrm{Ad}g(\underline{\mathfrak{g}}_1)$ over the point $y = g.G_0 \in N$.
The symmetric space $M = G/H$ is Riemannian if it admits a G-invariant metric, which is equivalent to say that $\mathfrak{m}$ admits an $\mathrm{Ad}(H)$-invariant inner product or equivalently, that $\mathrm{Ad}_{\mathfrak{m}}(H)$ be relatively compact[9]. We remark that the Levi-Civita connection coincides with the previous canonical G-invariant connection and in particular is independent of the G-invariant metric chosen. We will always suppose that the symmetric spaces M which we consider are Riemannian. We will in addition to that suppose that the $\mathrm{Ad}(H)$-invariant inner product in $\mathfrak{m}$ is also invariant by $\tau_{|\mathfrak{m}}$ (such an inner product always exists when $\mathrm{Ad}_{\mathfrak{m}}(H)$ is relatively compact, see the appendix). We will also suppose that M is connected, then G^0 acts transitively on M and so we can suppose that G is connected.
We want to study the Riemannian symmetric spaces M such that there exists a 4-symmetric space (G, τ) which gives rise to M in the same way as above. For that, let us recall the following theorem:

Theorem 6 *[2, 10] Let M be a Riemannian manifold.*

(a) *The group $\mathrm{Is}(M)$ of all the isometries of M is a Lie group and acts differentiably on M.*

[9] In the literature, it is often supposed that $\mathrm{Ad}_{\mathfrak{m}}(H)$ is compact. We will see that these two hypothesis are in fact equivalent.

(b) *Let $p_0 \in M$, then an isometry f of M is determined by the image $f(p_0)$ of the point p_0 and the corresponding tangent map $T_{p_0}f$ (i.e. if $f(p_0) = g(p_0)$ and $T_{p_0}f = T_{p_0}g$ then $f = g$).*

(c) *The isotropy subgroup $\mathrm{Is}_{p_0}(M) = \{f \in \mathrm{Is}(M); f(p_0) = p_0\}$ is a closed subgroup of $\mathrm{Is}(M)$ and the linear isotropy representation $\rho_{p_0} \colon f \in \mathrm{Is}_{p_0}(M) \mapsto T_{p_0}f \in O(T_{p_0}M)$ is an isomorphism from $\mathrm{Is}_{p_0}(M)$ onto a closed subgroup of $O(T_{p_0}M)$. Hence $\mathrm{Is}_{p_0}(M)$ is a compact subgroup of $\mathrm{Is}(M)$.*

(d) *If M is a Riemannian homogeneous space, $M = G/H$ with $G = \mathrm{Is}(M)$, $H = \mathrm{Is}_{p_0}(M)$ and $\mathfrak{m}$ an $\mathrm{Ad}H$-invariant space such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, then the previous closed subgroup, image of H by the preceding isomorphism ρ_{p_0}, i.e. the linear isotropy subgroup H^* can be identified to $\mathrm{Ad}_{\mathfrak{m}}H$. More precisely the linear isometry $\xi \in \mathfrak{m} \mapsto \xi.p_0 \in T_{p_0}M$ gives rise to an isomorphism from $O(\mathfrak{m})$ onto $O(T_{p_0}M)$ which sends $\mathrm{Ad}_{\mathfrak{m}}H$ onto H^*. Hence the linear adjoint representation of H on $\mathfrak{m}$: $g \in H \mapsto \mathrm{Ad}_{\mathfrak{m}}g \in \mathrm{Ad}_{\mathfrak{m}}H$ is an isomorphism (of Lie groups). $H \cong H^* \cong \mathrm{Ad}_{\mathfrak{m}}H$.*

3.1.1 First convenient hypothesis.

There may be more than one Lie group G acting symmetrically on a Riemannian symmetric space M. Besides, we have a convenient way to work on Riemannian symmetric spaces: it is to consider that G is a subgroup of the group of isometries of M, $\mathrm{Is}(M)$, which is equivalent to suppose that G acts effectively on M, i.e. H, the isotropy subgroup at a fixed point p_0 does not contain non-trivial normal subgroup of G (see [2]). It is always possible because the kernel K of the natural morphism $\phi_H \colon G \to \mathrm{Is}(M)$ is the maximal normal subgroup of G contained in H [10], and $G' = G/K$ acts transitively and effectively on $M = G/H$ with isotropy subgroup $H' = H/K$. Thus $M = G'/H'$ and since $K \subset H \subset G^\sigma$, then σ gives rise to an involutive morphism $\sigma' \colon G' \to G'$ such that $(G'^{\sigma'})^0 \subset H' \subset G'^{\sigma'}$. Now, let us suppose that there exists an order four automorphism $\tau \colon G \to G$ such that $\sigma = \tau^2$. Then it gives rise to an isomorphism $\tau' \colon G/K \to G/\tau(K)$. We would like that $\tau(K) = K$. It is the case if $\tau(H) = H$: K and $\tau(K)$ are respectively the maximal normal subgroups of G contained in H and $\tau(H)$ respectively, and so if $\tau(H) = H$ then $K = \tau(K)$.

Let us suppose that $\tau(K) = K$, then τ gives rise to an order four automorphism $\tau' \colon G/K \to G/K$ such that $\sigma' = \tau'^2$. With our convention we have $G'_0 = G'^{\tau'} \cap H'$, then we obtain a 4-symmetric bundle $N'_{min} = G'/G'_0$ over M. Hence, when G'_0 describes all the possible choices: $(G'^{\tau'})^0 \subset G'_0 \subset G'^{\tau'} \cap H'$, we obtain a family of 4-symmetric bundles $N' = G'/G'_0$ over M which are discrete coverings of $N'_{min} = G'/(G'^{\tau'} \cap H')$ and of which $N'_{max} = G'/(G'^{\tau'})^0$ is a discrete covering. For example, if we choose $G'_0 = \pi_K(G_0 K)$, we obtain the 4-symmetric bundle

[10] $K = \ker \phi_H = \ker \rho_{p_0} = \ker \mathrm{Ad}_{\mathfrak{m}}$

over M, $N' = (G/K)/\pi_K(G_0K) = G/G_0K = N/K$ [11].

Let us come back to the general case (i.e. we do not suppose that $\tau(K) = K$). Since $\tau(\mathfrak{h}) = \mathfrak{h}$, we have $\tau(H^0) = H^0$ and thus denoting by K_0 the maximal normal subgroup of G contained in H^0 (we have $K^0 \subset K_0 \subset K \cap H^0$), then $\tau(K_0) = K_0$ for the same reason as above (in particular, if $K_0 = K$ i.e. $K \subset H^0$, then we are in the previous case: $\tau(K) = K$). Hence τ gives rise to an order four automorphism $\tilde{\tau}\colon G/K_0 \to G/K_0$ and we are in the case considered above if we consider the symmetric space $\tilde{M} = G/H^0$ (instead of M). Let us precise this point. Indeed $\tilde{M}$ is a (G/K_0)-symmetric space and $\tilde{G} = G/K_0$ acts effectively on it (the isotropy subgroup $\tilde{H} = H^0/K_0$ does not contain non-trivial normal subgroup of G/K_0): as above σ gives rise to an involutive automorphism $\tilde{\sigma}$ of $\tilde{G} = G/K_0$ such that $\tilde{H} = (\tilde{G}^{\tilde{\sigma}})^0$ and $\tilde{\tau}$ is an order four automorphism of G/K_0 such that $\tilde{\tau}^2 = \tilde{\sigma}$. Finally, as above we obtain a family of 4-symmetric bundles $\tilde{N} = \tilde{G}/\tilde{G}_0$ over $\tilde{M}$ when $\tilde{G}_0$ describes the set of all possible choices: $(\tilde{G}^{\tilde{\tau}})^0 \subset \tilde{G}_0 \subset \tilde{G}^{\tilde{\tau}} \cap \tilde{H}$.
Moreover, the involution $\tilde{\sigma}$ of G/K_0 gives rise also to the G/K_0-symmetric space M (i.e. $(\tilde{G}^{\tilde{\sigma}})^0 \subset H/K_0 \subset \tilde{G}^{\tilde{\sigma}}$ or equivalently M belongs to the family of G/K_0-symmetric spaces defined by $\tilde{\sigma}$ (of which $\tilde{M}$ is a discrete covering)).

In the same way, we have $\tau(G^\sigma) = G^\sigma$ and thus we can do the same as above for the symmetric space $M_{min} = G/G^\sigma$.

Nevertheless, in general, it is possible that $\tau(K) \neq K$ and then τ does not give rise to an order four automorphism of $G' = G/K$ but only to the isomorphism $\tau'\colon G/K \to G/\tau(K)$. However, the tangent map $T_e\tau' = T_e\tilde{\tau}$ is an order four automorphism of the Lie algebra $\mathrm{Lie}(G/K) = \mathrm{Lie}(G/\tau(K)) = \mathrm{Lie}(G/K_0) = \mathfrak{g}/\mathfrak{k}$, and we have $(T_e\tau')^2 = T_e\sigma'$, thus $N/K = (G/K)/\pi_K(G_0K)$ is a locally 4-symmetric bundle over M ($\mathrm{Lie}\,\pi_K(G_0K) = \mathfrak{g}^{T_e\tau'}$).

Hence we have two good settings to study the Riemannian symmetric spaces M over which a 4-symmetric bundle can be defined, if we want to work only with subgroups of $\mathrm{Is}(M)$.

The first possibility is to consider that we begin by giving ourself an order four automorphism $\tau\colon G \to G$ and that we always choose the Riemannian symmetric space $\tilde{M} = G/H$ with $H = (G^{\tau^2})^0$ (respectively $M_{min} = G/H$ with $H = G^{\tau^2}$). In other words, in the family of G-symmetric space corresponding to $\sigma = \tau^2$ (i.e. $(G^\sigma)^0 \subset H \subset G^\sigma$), we choose the "maximal" one $\tilde{M} = G/(G^\sigma)^0$, which is a discrete covering of all the others (respectively the "minimal" one $M_{min} = G/G^\sigma$, of which all the others are discrete coverings). Then according to what precedes, we can always suppose that G is a subgroup of $\mathrm{Is}(\tilde{M})$ (respectively of $\mathrm{Is}(M_{min})$).

The second possibility is to work with locally 4-symmetric spaces. In other words we begin by a Riemannian symmetric space over which there exists a locally 4-symmetric bundle. It means that we work with the following setting:

[11] In the writing $N' = N/K$, K does not act freely on N in general: it is $K' = K/K \cap G_0$ which acts freely on N and we have $N' = N/K = N/K'$. In particular it is possible that $N/K = N$ for a non-trivial K (see section 5.3).

a Riemannian symmetric spaces M with G a subgroup of $\mathrm{Is}(M)$ acting symmetrically on M and an order four automorphism $\tau\colon \mathfrak{g} \to \mathfrak{g}$, such that $\tau^2 = \sigma$. To define the locally 4-symmetric space N in this setting, we must tell how we define G_0. We will set

$$G_0 = \{g \in H |\, \mathrm{Ad}_{\mathfrak{m}} g \circ \tau_{|\mathfrak{m}} \circ \mathrm{Ad}_{\mathfrak{m}} g^{-1} = \tau_{|\mathfrak{m}}\}. \tag{11}$$

First, we have to verify that if τ can be integrated by an automorphism of G, also denoted by τ, then we have $G_0 = G^\tau \cap H$. Indeed, if $g \in G^\tau \cap H$, then $\mathrm{Ad} g \circ \tau \circ \mathrm{Ad} g^{-1} = \mathrm{Ad}(g.\tau(g)^{-1}) \circ \tau = \tau$ and since $\mathrm{Ad} H$ stabilizes $\mathfrak{m}$, we have $\mathrm{Ad}_{\mathfrak{m}} g \circ \tau_{|\mathfrak{m}} \circ \mathrm{Ad}_{\mathfrak{m}} g^{-1} = \tau_{|\mathfrak{m}}$ by taking the restriction to $\mathfrak{m}$ of the preceding equation. Conversely, suppose that $g \in H$ and $\mathrm{Ad}_{\mathfrak{m}} g \circ \tau_{|\mathfrak{m}} \circ \mathrm{Ad}_{\mathfrak{m}} g^{-1} = \tau_{|\mathfrak{m}}$, then $\mathrm{Ad}(g.\tau(g)^{-1}) \circ \tau_{|\mathfrak{m}} = \tau_{|\mathfrak{m}}$ so since $\tau_{|\mathfrak{m}}$ is surjective, $\mathrm{Ad}(g.\tau(g)^{-1})_{|\mathfrak{m}} = \mathrm{Id}_{\mathfrak{m}}$ and since the adjoint representation of H on $\mathfrak{m}$ is injective (because we suppose that G is a subgroup of $\mathrm{Is}(M)$, and thus H is a subgroup of $\mathrm{Is}_{p_0}(M)$) it follows that $g.\tau(g)^{-1} = 1$. Finally, $g \in G^\tau \cap H$. Thus our definition (11) is coherent with our convention which holds when τ can be integrated by an automorphism of G.
Besides, it is easy to see that $\mathrm{Lie}\, G_0 = \{a \in \mathfrak{h} |\, \mathrm{ad}_{\mathfrak{m}} a \circ \tau_{|\mathfrak{m}} = \tau_{|\mathfrak{m}} \circ \mathrm{ad}_{\mathfrak{m}} a\} = \mathfrak{g}_0$. (Indeed, $\forall a \in \mathfrak{g}_0$, $\mathrm{ad} a \circ \tau = \tau \circ \mathrm{ad} a$, and $\forall a \in \mathfrak{g}_2$, $\mathrm{ad} a \circ \tau = -\tau \circ \mathrm{ad} a$, moreover $\tau_{|\mathfrak{m}} \circ \mathrm{ad}_{\mathfrak{m}} a = 0 \Rightarrow \mathrm{ad}_{\mathfrak{m}} a = 0 \Rightarrow a = 0$ because $a \in \mathfrak{h} \mapsto \mathrm{ad}_{\mathfrak{m}} a$ is the tangent map of $h \in H \mapsto \mathrm{Ad}_{\mathfrak{m}} h$ which is an injective morphism). Hence $N = G/G_0$ is a locally 4-symmetric bundle over M.
Further, let $\pi\colon \tilde{G} \to G$ be the universal covering of G, and $D = \ker \pi$. Then τ can be integrated by $\tilde{\tau}\colon \tilde{G} \to G$. Set $\tilde{\sigma} = \tilde{\tau}^2$, then $\sigma \circ \pi = \pi \circ \tilde{\sigma}$ and $T_1\sigma = T_1\tilde{\sigma} = (T_1\tilde{\tau})^2$. $\tilde{G}$ acts almost effectively on M with isotropy subgroup $\tilde{H} = \pi^{-1}(H)$ and almost effectively on $\tilde{M} = \tilde{G}/\tilde{H}^0$ which is the universal covering of M (see [10]). Besides, if $\tilde{G}$ does not act effectively on $\tilde{M}$, then we take D_0 the maximal normal subgroup of $\tilde{G}$ included in $\tilde{H}^0$, and then we quotient by it, so that we obtain an effective action of $\tilde{G}/D_0$ on $\tilde{M}$ and $\tilde{\tau}$ gives rise to an automorphism of $\tilde{G}/D_0$, according to above. Thus we are in the first possibility. Besides it is easy to see that $\forall g \in \tilde{G}$, $\mathrm{Ad} g = \mathrm{Ad}\pi(g)$ (more precisely $T_1\pi \circ \mathrm{Ad} g = \mathrm{Ad}\pi(g) \circ T_1\pi$ and we identify $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ so that $T_1\pi = \mathrm{Id}$). Thus $\tilde{G}_0 = \tilde{G}^{\tilde{\tau}} \cap \tilde{H}^0 = \{g \in \tilde{H}^0 |\, \mathrm{Ad} g \circ \tau_{|\mathfrak{m}} \circ \mathrm{Ad} g^{-1} = \tau_{|\mathfrak{m}}\} \subset \pi^{-1}(G_0)$. Hence the 4-symmetric space $\tilde{G}/\tilde{G}_0$ is a discrete covering of the locally 4-symmetric space G/G_0 and we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{G}/\tilde{G}_0 & \longrightarrow & G/G_0 \\ \big\downarrow & & \big\downarrow \\ \tilde{M} & \longrightarrow & M\ . \end{array} \tag{12}$$

In conclusion, the two possibilities are equivalent, but we will use the second one because it works with any symmetric space M, whereas the first one needs that we choose a certain covering of M (for example its universal covering).

Remark 7 We see that in the preceding reasoning (this using the universal covering $\tilde{G}$) we need only the automorphism of Lie algebra τ (and not the symmetric space M). Hence, we can consider that we work in the Lie algebra setting and give ourself an order four automorphism τ of $\mathfrak{g}$. Then we consider the family of associated pairs (G, H) where G is a connected Lie group with Lie algebra $\mathfrak{g}$ and H a closed Lie subgroup with Lie algebra $\mathfrak{h} = \mathfrak{g}^{\sigma}$. To each such pair corresponds the locally symmetric space $M = G/H$ and defining G_0 by (11), the locally 4-symmetric bundle $N = G/G_0$ over M. Let $\tilde{G}$ be a simply connected Lie group with Lie algebra $\mathfrak{g}$, then τ and σ integrate in $\tilde{G}$ and thus for $\tilde{H}$ the closed subgroup with Lie algebra $\mathfrak{h}$, we can take any subgroup such that $(\tilde{G}^{\tilde{\sigma}})^0 \subset \tilde{H} \subset \tilde{G}^{\tilde{\sigma}}$ (which implies that $\tilde{H}$ is closed). If we suppose $\tilde{H}$ connected, i.e. $\tilde{H} = (\tilde{G}^{\tilde{\sigma}})^0$, then $\tilde{M} = \tilde{G}/\tilde{H}$ is a symmetric space and is also the universal covering of all the locally symmetric spaces $M = G/H$ when (G, H) describes all the associated pairs (see [10]), and we have the above commutative diagram between the 4-symmetric bundle $\tilde{N} = \tilde{G}/\tilde{G}_0$ over $\tilde{M}$ and the locally 4-symmetric bundle $N = G/G_0$ over M. Moreover if $\tilde{M}$ is Riemannian then all the symmetric spaces $M = G/H$ when (G, H) describes all the symmetric associated pairs are Riemannian (see appendix, corollary 3).

Remark 8 Let us consider M a G-symmetric space, $G \subset \mathrm{Is}(M)$, and $\tau \colon \mathfrak{g} \to \mathfrak{g}$ an order four automorphism such that $\tau^2 = \sigma$. Then we have $\tau_{|\mathfrak{m}} \in \Sigma(\mathfrak{m})$ ($\tau_{|\mathfrak{m}^{\mathbb{C}}} = -i\mathrm{Id}_{\mathfrak{g}_{-1}} \oplus i\mathrm{Id}_{\mathfrak{g}_1}$) and it is easy to see that

$$\forall a \in \mathfrak{h},\ \tau_{|\mathfrak{h}}(a) = \mathrm{ad}_{\mathfrak{m}}^{-1}(\tau_{|\mathfrak{m}} \circ \mathrm{ad}_{\mathfrak{m}} a \circ \tau_{|\mathfrak{m}}^{-1}).$$

In other words, under the identification $\mathfrak{h} \simeq \mathrm{ad}_{\mathfrak{m}}\mathfrak{h} \subset \mathfrak{so}(\mathfrak{m})$, $\tau_{|\mathfrak{h}}$ is the restriction to $\mathfrak{h}$ of $\mathrm{Ad}(\tau_{|\mathfrak{m}}) \colon \mathfrak{so}(\mathfrak{m}) \to \mathfrak{so}(\mathfrak{m})$. Hence τ is determined by $\tau_{|\mathfrak{m}}$. Besides $\tau_{|\mathfrak{h}}$ is the tangent map of the isomorphism τ_H:

$$\tau_H(g) = \mathrm{Ad}_{\mathfrak{m}}^{-1}(\tau_{|\mathfrak{m}} \circ \mathrm{Ad}_{\mathfrak{m}} g \circ \tau_{|\mathfrak{m}}^{-1}),$$

for $g \in H^0$ (and more generally for $g \in \mathrm{Ad}_{\mathfrak{m}}^{-1} \circ (\mathrm{Int}\tau_{|\mathfrak{m}})^{-1} \circ \mathrm{Ad}_{\mathfrak{m}}(H)$). Under the identification $H \simeq \mathrm{Ad}_{\mathfrak{m}} H \subset O(\mathfrak{m})$ it is the restriction to H^0 of the involution $\mathrm{Int}\tau_{|\mathfrak{m}} \colon O(\mathfrak{m}) \to O(\mathfrak{m})$. According to the definition (11) of G_0, we have $G_0 = H^{\tau_H}$. Besides $\tau_H(H^0) = H^0$, thus H^0/G_0^0 is an H^0-symmetric space. If $\mathrm{Int}\tau_{|\mathfrak{m}}(\mathrm{Ad}_{\mathfrak{m}} H) = (\mathrm{Ad}_{\mathfrak{m}} H)$, then τ_H is defined in H and $\tau_H(H) = H$, then H/G_0 is an H-symmetric space (if $\tau_H(H) \neq H$ it is only a locally symmetric space). Obviously, if τ can be integrated in G then $\tau_H = \tau_{|H}$.

Definition 6 *Let M be a G-symmetric space. Let* $\mathrm{Aut}(\mathfrak{m})$ *be the subgroup of* $O(\mathfrak{m})$ *defined by:*

$$\mathrm{Aut}(\mathfrak{m}) = \{F \in O(\mathfrak{m}) \mid F(\mathrm{ad}_{\mathfrak{m}}[v, v'])F^{-1} = \mathrm{ad}_{\mathfrak{m}}[Fv, Fv'],\ \forall v, v' \in \mathfrak{m}\},$$

it is the subgroup of $O(\mathfrak{m})$ *which leaves invariant* $\mathrm{ad}_{\mathfrak{m}}([\cdot,\cdot]_{|\mathfrak{m}\times\mathfrak{m}}) \in (\Lambda^2\mathfrak{m}^*) \otimes \mathfrak{so}(\mathfrak{m})$. *Its Lie Algebra*

$$\mathrm{Der}(\mathfrak{m}) = \{A \in \mathfrak{so}(\mathfrak{m}) \mid [A, \mathrm{ad}_{\mathfrak{m}}[v, v']] = \mathrm{ad}_{\mathfrak{m}}[Av, v'] + \mathrm{ad}_{\mathfrak{m}}[v, Av'],\ \forall v, v' \in \mathfrak{m}\}$$

is the Lie subalgebra of $\mathfrak{so}(\mathfrak{m})$ *which (acting by derivation) leaves invariant* $\mathrm{ad}_{\mathfrak{m}}([\cdot,\cdot]_{|\mathfrak{m}\times\mathfrak{m}}) \in (\Lambda^2\mathfrak{m}^*)\otimes\mathfrak{so}(\mathfrak{m})$.

Theorem 7 *Let M be a G-symmetric space, $G \subset \mathrm{Is}(M)$, and $\tau\colon \mathfrak{g} \to \mathfrak{g}$ an order four automorphism such that $\tau^2 = \sigma$. Then $\tau_{|\mathfrak{m}} \in \mathrm{Aut}(\mathfrak{m})$ and τ can be extended in an unique way to the Lie algebra $\mathrm{Der}(\mathfrak{m})\oplus\mathfrak{m}$ endowed with the Lie bracket*

$$[(A,v),(A',v')] = ([A,A'] + \mathrm{ad}_{\mathfrak{m}}[v,v'],\ A.v' - A'.v)$$

and of which $\mathfrak{g}$ is a Lie subalgebra, under the inclusion $a + v \in \mathfrak{h}\oplus\mathfrak{m} \mapsto (\mathrm{ad}_{\mathfrak{m}}a, v) \in \mathrm{Der}(\mathfrak{m})\oplus\mathfrak{m}$, by setting

$$\underline{\tau}_{|\mathfrak{m}} = \tau_{|\mathfrak{m}} \quad \textit{and} \quad \underline{\tau}_{|\mathrm{Der}(\mathfrak{m})} = \mathrm{Ad}\tau_{|\mathfrak{m}}\,. \tag{13}$$

Conversely, given $\tau_{\mathfrak{m}} \in O(\mathfrak{m})$, the linear map $\underline{\tau}$ defined by (13) is an automorphism of the Lie algebra $\mathrm{Der}(\mathfrak{m})\oplus\mathfrak{m}$ if and only if $\tau_{\mathfrak{m}} \in \mathrm{Aut}(\mathfrak{m})$. Besides it satisfies $\underline{\tau}^2 = \mathrm{Id}_{\mathrm{Der}(\mathfrak{m})}\oplus -\mathrm{Id}_{\mathfrak{m}}$ (and in particular is of order four) if and only if $\tau_{\mathfrak{m}} \in \Sigma(\mathfrak{m})$.
Hence, to define a locally 4-symmetric bundle over the Riemannian symmetric space M (with the realisation $M = G/H$, i.e. τ is an automorphism of $\mathfrak{g}$ such that $\tau^2 = \sigma$) is equivalent to give ourself $\tau_{\mathfrak{m}} \in \Sigma(\mathfrak{m})\cap\mathrm{Aut}(\mathfrak{m})$ such that the order four automorphism $\underline{\tau}$ of $\mathrm{Der}(\mathfrak{m})\oplus\mathfrak{m}$ stabilizes $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$, i.e. such that $\tau_{\mathfrak{m}}(\mathrm{ad}_{\mathfrak{m}}\mathfrak{h})\tau_{\mathfrak{m}}^{-1} = \mathrm{ad}_{\mathfrak{m}}\mathfrak{h}$ (i.e. $\mathrm{ad}_{\mathfrak{m}}\mathfrak{h}$ is a subalgebra of $\mathrm{Der}(\mathfrak{m})$ invariant by $\mathrm{Ad}\tau_{\mathfrak{m}}$). Then $\tau = \underline{\tau}_{|\mathfrak{g}}$ is an order four automorphism of $\mathfrak{g}$ such that $\tau^2 = \mathrm{Id}_{\mathfrak{h}}\oplus -\mathrm{Id}_{\mathfrak{m}} = \sigma$.

Proof. First $\tau_{|\mathfrak{m}} \in \mathrm{Aut}(\mathfrak{m})$: that follows from the fact that τ is an automorphism, so $\tau\circ\mathrm{ad}a\circ\tau^{-1} = \mathrm{ad}\tau(a)\,,\forall a\in\mathfrak{g}$.
Second, $\mathrm{Der}(\mathfrak{m})\oplus\mathfrak{m}$ is a Lie subalgebra . We have to check that the Jacobi identity is satisfied. It is a straightforward computation (see [10]). Then we have to check that $\underline{\tau}$ is an automorphism if and only if $\tau_{\mathfrak{m}} \in \mathrm{Aut}(\mathfrak{m})$.
If $\tau_{\mathfrak{m}} \in \mathrm{Aut}(\mathfrak{m})$ then

- if $A, A' \in \mathrm{Der}(\mathfrak{m})$, $\underline{\tau}([A,A']) = [\underline{\tau}(A),\underline{\tau}(A')]$ because $\underline{\tau}_{\mathrm{Der}(\mathfrak{m})} = \mathrm{Ad}\tau_{\mathfrak{m}}$ is an automorphism of $\mathrm{Der}(\mathfrak{m})$.
- if $A \in \mathrm{Der}(\mathfrak{m})$, $v\in\mathfrak{m}$, $\underline{\tau}([A,v]) = \tau_{\mathfrak{m}}(A.v) = \tau_{\mathfrak{m}}A\tau_{\mathfrak{m}}^{-1}(\tau_{\mathfrak{m}}.v) = [\underline{\tau}(A),\underline{\tau}(v)]$
- if $v, v' \in \mathfrak{m}$, $\underline{\tau}([v,v']) = \mathrm{Ad}\tau_{\mathfrak{m}}(\mathrm{ad}_{\mathfrak{m}}[v,v']) = \mathrm{ad}_{\mathfrak{m}}([\tau_{\mathfrak{m}}v,\tau_{\mathfrak{m}}v']) = [\underline{\tau}(v),\underline{\tau}(v')]$ because $\tau_{\mathfrak{m}} \in \mathrm{Aut}(\mathfrak{m})$.

Finally $\underline{\tau}$ is an automorphism and the unique extension of τ (because it is determined by $\tau_{|\mathfrak{m}}$, see remark 8).
Conversely if $\underline{\tau}$ is an automorphism of Lie algebra then

$$\tau_{\mathfrak{m}}\mathrm{ad}_{\mathfrak{m}}([v,v'])\tau_{\mathfrak{m}}^{-1} = (\underline{\tau}\,\mathrm{ad}([v,v'])\underline{\tau}^{-1})_{|\mathfrak{m}} = (\mathrm{ad}\underline{\tau}([v,v']))_{|\mathfrak{m}} = \mathrm{ad}_{\mathfrak{m}}([\underline{\tau}(v),\underline{\tau}(v')]) = \mathrm{ad}_{\mathfrak{m}}([\tau_{\mathfrak{m}}v,\tau_{\mathfrak{m}}v']).$$

Thus $\tau_{\mathfrak{m}} \in \mathrm{Aut}(\mathfrak{m})$.
The last assertion of the theorem follows from what precedes. This completes the proof. ■

Remark 9 Let $\tau_{\mathfrak{m}} \in \Sigma(\mathfrak{m})$ then the condition $\mathrm{Ad}\tau_{\mathfrak{m}}(\mathrm{ad}_{\mathfrak{m}}\mathfrak{h}) = \mathrm{ad}_{\mathfrak{m}}\mathfrak{h}$ implies that there exists an automorphism $\tau_{\mathfrak{h}}$ of $\mathfrak{h}$ defined by $\forall a \in \mathfrak{h}$, $\mathrm{Ad}\tau_{\mathfrak{m}}(\mathrm{ad}_{\mathfrak{m}}a) = \mathrm{ad}_{\mathfrak{m}}\tau_{\mathfrak{h}}(a)$, i.e. $\tau_{\mathfrak{h}} = \mathrm{ad}_{\mathfrak{m}}^{-1} \circ \mathrm{Ad}\tau_{\mathfrak{m}} \circ \mathrm{ad}_{\mathfrak{m}}$. Then the condition $\tau_{\mathfrak{m}} \in \mathrm{Aut}(\mathfrak{m})$ is equivalent to

$$\tau_{\mathfrak{h}}([v, v']) = [\tau_{\mathfrak{m}}v, \tau_{\mathfrak{m}}v'],\ \forall v, v' \in \mathfrak{m}.$$

And obviously, if these two conditions are satisfied then we have $\tau_{\mathfrak{h}} = \tau_{|\mathfrak{h}}$ (where $\tau = \underline{\tau}_{\mathfrak{g}}$ is given by the theorem 7).

Remark 10 Let us consider the map

$$s\colon g \in \mathrm{Is}_{p_0}(M) \mapsto \mathrm{Ad}_{\mathfrak{m}}g \circ \tau_{|\mathfrak{m}} \circ \mathrm{Ad}_{\mathfrak{m}}g^{-1} \in \Sigma(\mathfrak{m})$$

and set $\underline{G_0} = \{g \in \mathrm{Is}_{p_0}(M) | s(g) = \tau_{|\mathfrak{m}}\}$. Then $\mathrm{Is}_{p_0}(M)$ acts on $\Sigma(\mathfrak{m})$ by $g.J = \mathrm{Ad}_{\mathfrak{m}}g \circ J \circ \mathrm{Ad}_{\mathfrak{m}}g^{-1}$ and $s(g) = g.\tau_{|\mathfrak{m}}$, and $\underline{G_0} = \mathrm{Stab}_{\mathrm{Is}_{p_0}(M)}(\tau_{|\mathfrak{m}})$. In the same way, the subgroup $H = \mathrm{Is}_{p_0}(M) \cap G$ acts on $\Sigma(M)$ and $G_0 = \mathrm{Stab}_H(\tau_{|\mathfrak{m}})$. Then $s(\mathrm{Is}_{p_0}(M)) = \mathrm{Is}_{p_0}(M)/\underline{G_0}$ is a compact submanifold of $\Sigma(\mathfrak{m})$, and $s(H) = H/G_0$ is a relatively compact (immersed) submanifolds of $\Sigma(\mathfrak{m})$.

3.1.2 Second convenient hypothesis.

An other convenient hypothesis on G is to consider that it is a closed subgroup of $\mathrm{Is}(M)$ (and not only an immersed subgroup). It is always possible to work with this hypothesis. Let us make precise this point. Let σ_{p_0} be the symmetry of M around p_0 (defined by σ): $\sigma_{p_0} \in \mathrm{Is}(M)$, $\sigma_{p_0}(p_0) = p_0$ and $T_{p_0}\sigma_{p_0} = -\mathrm{Id}$. Then σ_{p_0} belongs to the isotropy subgroup $\mathrm{Is}_{p_0}(M) = \{f \in \mathrm{Is}(M); f(p_0) = p_0\}$, and we can define the involution of $\mathrm{Is}(M)$:

$$\sigma_{\mathrm{Is}(M)} = \mathrm{Int}(\sigma_{p_0})\colon g \in \mathrm{Is}(M) \mapsto \sigma_{p_0} \circ g \circ \sigma_{p_0}^{-1} \in \mathrm{Is}(M).$$

It is easy to see that we have

$$(\mathrm{Is}(M)^{\sigma_{\mathrm{Is}(M)}})^0 \subset \mathrm{Is}_{p_0}(M) \subset \mathrm{Is}(M)^{\sigma_{\mathrm{Is}(M)}} \tag{14}$$

(see [10, 2]). The result of this is that $\sigma\colon G \to G$ is the restriction of $\sigma_{\mathrm{Is}(M)}$ to $G \subset \mathrm{Is}(M)$ (they induce σ_{p_0} on $M = G/H$ and the identity on H, thus, since G is locally isomorphic to $M \times H$, they are identical, see also [10]). Moreover there exists an unique subgroup $\bar{G}$ of $\mathrm{Diff}(M)$ such that for any G-invariant Riemannian metric b on M, the group $\bar{G}$ is the closure of G in $\mathrm{Is}(M, b)$: $\mathrm{Is}(M, b)$ is closed in $\mathrm{Diff}(M)$ and so the closure of G in $\mathrm{Is}(M, b)$ is its closure in $\mathrm{Diff}(M)$ and thus it does not depend on b (see [2, 10]). Then σ extends in an unique way to an involutive morphism $\bar{\sigma}\colon \bar{G} \to \bar{G}$, which is the restriction of $\sigma_{\mathrm{Is}(M)}$ to $\bar{G}$. Hence denoting by $\hat{H}$ the isotropy subgroup of $\bar{G}$ at p_0, $\hat{H} = \mathrm{Is}_{p_0}(M) \cap \bar{G}$, we have according to (14), $(\bar{G}^{\bar{\sigma}})^0 \subset \hat{H} \subset \bar{G}^{\bar{\sigma}}$. Besides $\bar{\sigma}$ gives rise to the symmetric decomposition $\mathrm{Lie}\,\bar{G} = \mathrm{Lie}\,\hat{H} \oplus \mathfrak{m}$.

In addition to that, we have $\hat{H} = \bar{H}$. Indeed, let $\Phi\colon U \times \mathrm{Is}_{p_0}(M) \to \mathrm{Is}(M)$ be a local trivialisation of $\mathrm{Is}(M) \to M$, such that $\Phi(p_0, h) = h$, and $\Phi(U \times H) = \Phi(U \times \mathrm{Is}_{p_0}(M)) \cap G$ (take $\Phi(p, h) = \phi(p).h$, with $\phi\colon U \to G$ a local section

such that $\phi(p_0) = 1$). Further, if $g \in \mathrm{Is}_{p_0}(M) \cap \bar{G}$ and (g_n) is a sequence of $G \cap \Phi(U \times \mathrm{Is}_{p_0}(M))$ such that $g_n \to g$, then $\Phi^{-1}(g_n) = (u_n, h_n) \in U \times H$ converges to $\Phi^{-1}(g) = (p_0, g)$, thus $h_n \to g$ so $g \in \bar{H}$.
Moreover, $\bar{H}$ is a closed subgroup of $\mathrm{Is}_{p_0}(M)$, thus it is compact. Hence, we have the symmetric realisation $M = \bar{G}/\bar{H}$ and $\mathrm{Ad}_{\mathfrak{m}}(\bar{H})$ is compact: we have showed that the hypothesis "$\mathrm{Ad}_{\mathfrak{m}}(H)$ relatively compact" and "$\mathrm{Ad}_{\mathfrak{m}}(H)$ compact" give the same symmetric spaces. Moreover, by using the preceding reasoning (to prove $\hat{H} = \bar{H}$) it is easy to see that if $\mathrm{Ad}_{\mathfrak{m}}(H)$ is compact then G is closed in $\mathrm{Is}(M)$ (see also [10]) so that the hypothesis "$\mathrm{Ad}_{\mathfrak{m}}(H)$ is compact" and "G is closed in $\mathrm{Is}(M)$" are in fact equivalent.
Besides, the closure of G is the same in $\mathrm{Is}(M)$ and in $\mathrm{Is}(\tilde{M})$ with $\tilde{M} = G/H^0$: since M and $\tilde{M}$ are complete (a Riemannian homogeneous space is complete) then $\mathrm{Is}(M)$ and $\mathrm{Is}(\tilde{M})$ are complete (see [10]), and thus the closure of G in one of this group is the completed of G.
Now, let us suppose that we have a locally 4-symmetric bundle over M.

Theorem 8 *Let us consider M a G-symmetric space with $G \subset \mathrm{Is}(M)$ and $\tau\colon \mathfrak{g} \to \mathfrak{g}$ an order four automorphism such that $\tau^2 = \sigma$. Then the extension $\underline{\tau}$ of τ, given by theorem 7 stabilizes the Lie algebra,* $\mathrm{Lie}\,\bar{G}$*, of the closure of G in* $\mathrm{Is}(M)$*:*

$$\underline{\tau}(\mathrm{Lie}\,\bar{G}) = \mathrm{Lie}\,\bar{G}.$$

Then denoting by $\bar{\tau} := \underline{\tau}_{|\mathrm{Lie}\,\bar{G}}$ the extension of τ to $\mathrm{Lie}\,\bar{G}$ *(given by theorem 7), the subgroup fixed by $\bar{\tau}$ (defined by (11)) is the closure of G_0:*

$$\hat{G}_0 := \{g \in \bar{H} \mid \tau_{|\mathfrak{m}} \circ \mathrm{Ad}_{\mathfrak{m}} g \circ \tau_{|\mathfrak{m}}^{-1} = \mathrm{Ad}_{\mathfrak{m}} g\} = \bar{G}_0.$$

Finally the new locally 4-symmetric bundle over M defined by $\bar{\tau}$ is $\bar{G}/\bar{G}_0$, and using the notation of remark 10, the fibre of $\bar{G}/\bar{G}_0$, $\hat{S}_0 := s(\bar{H}) = \bar{H}/\bar{G}_0$, is the closure of the fibre S_0 of G/G_0, $\overline{S_0} = \overline{s(H)} = \overline{(H/G_0)}$, in the maximal fibre over M: $\underline{S_0} := s(\mathrm{Is}_{p_0}(M)) = \mathrm{Is}_{p_0}(M)/\underline{G_0}$.

Remark 11 In particular, if we suppose that we have an order four automorphism τ of G, such that $\tau^2 = \sigma$, then since τ is uniformly continuous, it extends into an order four automorphism $\bar{\tau}\colon \bar{G} \to \bar{G}$ (because $\mathrm{Is}(M)$ is complete) and obviously $\bar{\tau}^2 = \bar{\sigma}$.

The following theorem precises the link between the Lie algebra setting and the one of the Riemannian symmetric space M (first point of theorem 9), which will allow us (in theorem 10) to translate the theorem 7 in terms of the setting of M. The two last points (of theorem 9) characterize the "satisfying cases": any element in $\mathrm{Aut}(\mathfrak{m})$ defines an automorphism in $\mathfrak{Is}(M)$ (an example of "unsatisfactory" case is given by $M = \mathbb{R}^{2n-r} \times \mathbb{T}^r$, see section 4.2).

Theorem 9 *Let us consider M a Riemannian symmetric space and $\tilde{M}$ its universal covering.*

• *The curvature operator (in M) is given by* $R_{p_0}(\cdot,\cdot) = -\mathrm{ad}_{\mathfrak{m}}([\cdot,\cdot]_{|\mathfrak{m}\times\mathfrak{m}})$ *and thus*[12]

$$\begin{aligned} &\mathrm{Der}(\mathfrak{m}) = \mathfrak{Is}_{p_0}(\tilde{M}) \supset \mathfrak{Is}_{p_0}(M) \supset \mathrm{Lie}\,\mathrm{Hol}(M) \\ &\mathrm{Aut}(\mathfrak{m}) \supset \mathrm{Is}_{p_0}(\tilde{M}) \supset \mathrm{Is}_{p_0}(M) \supset \mathrm{Hol}(M) \end{aligned} \tag{15}$$

(using the identification $T_{p_0}M = \mathfrak{m}$*) and* $\mathrm{Der}(\mathfrak{m}) \oplus \mathfrak{m} = \mathfrak{Is}(\tilde{M})$.
• *Moreover the following statements are equivalent:*
(i) $\mathfrak{Is}_{p_0}(\tilde{M}) = \mathfrak{Is}_{p_0}(M)$ *(i.e.* $\mathfrak{Is}(\tilde{M}) = \mathfrak{Is}(M)$*)*
(ii) $M = M' \times M_0$*, with* M' *of the semisimple type (i.e.* $\mathrm{Is}(M')$ *is semisimple) and* M_0 *is Euclidean.*
(iii) $\mathfrak{h}_0 = \mathfrak{so}(\mathfrak{m}_0)$*, where* $\mathfrak{h}_0$ *and* $\mathfrak{m}_0$ *are respectively the Euclidean part of* $\mathfrak{Is}_{p_0}(M)$ *and* $\mathfrak{m}$ *respectively, in the decomposition* $\mathfrak{Is}(M) = \mathfrak{g}' \oplus \mathfrak{g}_0$*, with* $\mathfrak{g}'$ *semisimple and* $\mathfrak{g}_0$ *of the Euclidean type.*

• *Moreover the following statements are also equivalent:*
(i) $\mathfrak{Is}_{p_0}(\tilde{M}) = \mathfrak{Is}_{p_0}(M) \oplus \mathfrak{so}(\mathfrak{m}_0)$
(ii) $\mathfrak{h}_0 = 0$
(iii) Let $\tilde{M} = \tilde{M}' \times \tilde{M}_0$ *be the decomposition of* $\tilde{M}$ *into the semisimple and Euclidean type,* Γ *the group of deck transformations of the covering* $\pi\colon \tilde{M} \to M$*. Then the projection on the Euclidean factor (in* $\mathrm{Is}(\tilde{M}) = \mathrm{Is}(\tilde{M}') \times \mathrm{Is}(\tilde{M}_0)$*) of* Γ *satisfies* $\Gamma_0 \cong \mathbb{Z}^r$ *with* $r = \dim \tilde{M}_0$ *so that* $\tilde{M}_0/\Gamma_0 = \mathbb{T}^r$.

Further $\mathrm{Aut}(\mathfrak{m})$ *stabilizes* $\mathfrak{Is}_{p_0}(M)$ *if and only if one of the 6 previous statements holds i.e. if and only if* $\mathfrak{Is}(\tilde{M})/\mathfrak{Is}(M) = \{0\}$ *or* $\mathfrak{so}(\mathfrak{m}_0)$*. Denoting by* $\mathrm{Aut}(\mathfrak{m})^*$ *the subgroup of* $\mathrm{Aut}(\mathfrak{m})$ *which stabilizes* $\mathfrak{Is}_{p_0}(M)$*, then the maximal subalgebra of* $\mathfrak{Is}_{p_0}(M)$ *invariant by* $\mathrm{Aut}(\mathfrak{m})$ *is* $\mathfrak{Is}_{p_0}(M)$ *if* $\mathrm{Aut}(\mathfrak{m}) = \mathrm{Aut}(\mathfrak{m})^*$ *and* $\mathfrak{h}' = \mathfrak{Is}_{p_0}(\tilde{M}')$ *if not.*

Let us consider M a G-symmetric space with $G \subset \mathrm{Is}(M)$ and $\tau\colon \mathfrak{g} \to \mathfrak{g}$ an order four automorphism such that $\tau^2 = \sigma$.
Then the extension $\underline{\tau}$ of τ, given by theorem 7 defines a maximal locally 4-symmetric bundle over M. Indeed let $\underline{\mathfrak{g}}$ be the maximal subalgebra of $\mathfrak{Is}(M)$ invariant by $\underline{\tau}$, and $\underline{G}$ the subgroup of $\mathrm{Is}(M)$ generated by it. Then $\underline{G}$ is a closed subgroup of $\mathrm{Is}(M)$ acting symmetrically on M: since $\underline{G}$ is connected, it is invariant by $\sigma_{\mathrm{Is}(M)}$ (and it contains G) thus it acts symmetrically on M, then $\underline{G}$ is closed as an immediate consequence of the maximality and theorem 8. Therefore, $\underline{\tau}_{|\underline{\mathfrak{g}}}$ defines a maximal locally 4-symmetric bundle over M, with the realisation $M = \underline{G}/\underline{H}$.
We can also define a minimal locally 4-symmetric bundle over M, by considering the subalgebra $\mathfrak{g}' \oplus \mathfrak{m}_0$ (where $\mathfrak{g}'$ is the semisimple part of $\mathfrak{Is}(M)$ and $\mathfrak{m}_0$ the Euclidean part of $\mathfrak{m}$).

Theorem 10 *In conclusion, given any (even-dimensional) Riemannian symmetric space M, to define over it a locally 4-symmetric bundle is equivalent to give ourself* $J_0 \in \Sigma(T_{p_0}M)$ *which leaves invariant the curvature* R_{p_0}*:*

$$R_{p_0}(J_0X, J_0Y) = J_0 R_{p_0}(X,Y) J_0^{-1}\ \forall X, Y \in T_{p_0}M.$$

[12] $\mathrm{Hol}(M)$ is the holonomy group of M

Then the order four automorphism of $\mathfrak{Is}(\tilde{M})$, $\underline{\tau}$, defined by J_0, defines the maximal locally 4-symmetric bundle over M, $\underline{N} = \underline{G}/\underline{G_0}$ with $\underline{G_0} = \{g \in \underline{H} \mid J_0 g J_0^{-1} = g\}$. Moreover, any locally 4-symmetric bundle over M is a subbundle of one such maximal bundle and to define such a subbundle N is equivalent to give ourself a Lie subgroup $G \subset \mathrm{Is}(M)$ acting symmetrically on M such that $\underline{\tau}(\mathfrak{g}) = \mathfrak{g}$ i.e. $\mathrm{Ad}J_0(\mathfrak{h}) = \mathfrak{h}$. In this case, the closure $\bar{N}$ of $N = G/G_0$ in the (unique) maximal locally 4-symmetric bundle over M containing N, $\underline{N}$, is also a locally 4-symmetric bundle over M and we have $\bar{N} = \bar{G}/\bar{G}_0$, $M = \bar{G}/\bar{H}$.

Proof of theorem 8 We have to check that $\underline{\tau}(\mathrm{Lie}\,\bar{G}) = \mathrm{Lie}\,\bar{G}$, i.e. according to the theorem 7, $\mathrm{Int}\tau_{\mathfrak{m}}(\mathrm{Lie}\,\bar{H}) = \mathrm{Lie}\,\bar{H}$. We still have $\mathrm{Int}\tau_{\mathfrak{m}}(H^0) = H^0$, thus $\mathrm{Int}\tau_{\mathfrak{m}}(\overline{H^0}) = \overline{H^0}$. It remains to verify that $(\bar{H})^0 = \overline{H^0}$. But this is simply the result of the fact that $\tilde{M} := G/H^0 = \bar{G}/\overline{H^0}$ (the closures in $\mathrm{Is}(\tilde{M})$ and $\mathrm{Is}(M)$ are the same) is a discrete covering of $M = \bar{G}/\bar{H}$. Indeed $(\bar{H})^0$ is closed thus $(\bar{H})^0 \supset \overline{H^0}$ and then we have

$$\tilde{M} = \bar{G}/\overline{H^0} \xrightarrow{\text{fibration}} \bar{G}/(\bar{H})^0 \xrightarrow{\text{covering}} \bar{G}/\bar{H} = M$$

and $\tilde{M} \xrightarrow{\text{covering}} M$, hence $(\bar{H})^0/\overline{H^0}$ is discrete but the two groups are connected ($(\bar{H})^0$ suffices) thus $(\bar{H})^0 = \overline{H^0}$. We have proved that $\underline{\tau}(\mathrm{Lie}\,\bar{G}) = \mathrm{Lie}\,\bar{G}$.
Using the notation of remark 10, we have, since $\bar{H}$ is compact, $s(\bar{H}) = \overline{s(H)}$, hence using the same method as for $\hat{H}$, we can easily show that $\hat{G}_0 := s^{-1}(\tau_{|\mathfrak{m}}) \cap \bar{H} = \bar{G}_0$ and thus $s(\bar{H}) = \bar{H}/\bar{G}_0$. Finally, the new locally 4-symmetric space is $\bar{G}/\bar{G}_0$. This completes the proof. ■
Proof of theorem 9 For the first point see [10]. For the following points, see sections 4.1 and 4.2. ■
Proof of theorem 10 The first assertions are nothing but the translation of theorem 7, using theorem 9. Then, we have to prove that $\bar{G}/\bar{G}_0$ is the closure $\bar{N}$ of $N = G/G_0$ in $\underline{G}/\underline{G_0}$. Let $\pi_{J_0} \colon \underline{G} \to \underline{G}/\underline{G_0}$ be the projection map, then we have $\pi_{J_0}(G) = G/\underline{G_0} \cap G = G/G_0$ (according to definition (11)) and thus $\pi_{J_0}(\bar{G}) \subset \overline{\pi_{J_0}(G)} = \bar{N}$ but $\pi_{J_0}(\bar{G}) = \bar{G}/\underline{G_0} \cap \bar{G} = \bar{G}/\bar{G}_0$ (according to definition (11) and $\hat{G}_0 = \bar{G}_0$). Hence $\bar{G}/\bar{G}_0 \subset \bar{N}$. These are together subbundle (over M) of $\underline{N}$ and using a trivialisation of $\underline{N} = \underline{G}/\bar{G}_0 \to M$ (same reasoning as for $\hat{H}$) it is easy to see that the fibre of $\bar{N}$ (over p_0) is $\bar{H}/\bar{G}_0$ which implies that $\bar{G}/\bar{G}_0 = \bar{N}$. This completes the proof. ■

Remark 12 According to the definition (11), $\tau_{|\mathfrak{m}}$ and $-\tau_{|\mathfrak{m}}$ give rise to the same group G_0. Moreover $\tau_{|\mathfrak{m}} = (\tau^{-1})_{|\mathfrak{m}}$ and in particular if τ integrates in G then $G^{\tau} = G^{\tau^{-1}}$. Besides $(\tau^{-1})^2 = \sigma^{-1} = \sigma$, hence τ^{-1} defines the same locally 4-symmetric bundle over M as τ. Moreover, given any $\tau_{\mathfrak{m}} \in \Sigma(\mathfrak{m}) \cap \mathrm{Aut}(\mathfrak{m})$, then $-\tau_{\mathfrak{m}} \in \Sigma(\mathfrak{m}) \cap \mathrm{Aut}(\mathfrak{m})$ and gives rise (according to theorem 7) to the automorphism $\underline{\tau}^{-1}$ which gives rise to the same maximal locally 4-symmetric bundle over M and the same family of 4-symmetric subbundle over M.

From now, we will always suppose that G is a closed subgroup of $\mathrm{Is}(M)^0$. The result of this is that the isotropy subgroup of G at the point p_0, $H = \mathrm{Stab}_G(p_0)$

is compact and can be identified (via the adjoint representation on $\mathfrak{m}$, resp. via the linear isotropy representation) to a closed subgroup of $O(\mathfrak{m})$ (resp. of $O(T_{p_0}M)$). Then according to theorem 10, to study the case of non-closed subgroup of $\mathrm{Is}(M)^0$ (or equivalently the non-closed locally 4-symmetric bundle over M), we have just to consider the non-closed subgroups of our closed group G, acting symmetrically on M, and whose Lie algebra is invariant by τ.

3.2 Twistor subbundle

We give ourself a locally 4-symmetric bundle $N = G/G_0$ (defined by an order four automorphism τ and by (11)) over a symmetric space $M = G/H$. We will show that G/G_0 is a subbundle of the twistor bundle $\Sigma(G/H)$. Under the isomorphism between TM and $[\mathfrak{m}] = \{(g.p_0, \mathrm{Ad}g(\xi)), \xi \in \mathfrak{m}, g \in G\}$, $T_{p_0}M$ is identified to $\mathfrak{m}$: $\xi \in \mathfrak{m} \mapsto \xi.p_0 \in T_{p_0}M$ is an isomorphism of vector spaces. Then to $\tau_{|\mathfrak{m}} \in \Sigma(\mathfrak{m})$ (resp. to $-\tau_{|\mathfrak{m}} = \tau_{|\mathfrak{m}}^{-1} \in \Sigma(\mathfrak{m})$) corresponds $J_0 \in \Sigma(T_{p_0}M)$, and more generally to $\mathrm{Ad}g \circ \tau_{|\mathfrak{m}} \circ \mathrm{Ad}g^{-1} \in \Sigma(\mathrm{Ad}g(\mathfrak{m}))$ (resp. $\mathrm{Ad}g \circ \tau_{|\mathfrak{m}}^{-1} \circ \mathrm{Ad}g^{-1}$) corresponds $gJ_0g^{-1} \in \Sigma(T_{g.p_0}M)$. Thus we have defined a map

$$\begin{array}{rccc} \rho_{J_0}\colon & G & \longrightarrow & \Sigma(M) \\ & g & \longmapsto & gJ_0g^{-1} \end{array}$$

which according to the definition (11) of G_0 gives rise under quotient to the injective map:

$$\begin{array}{rccc} i\colon & G/G_0 & \longrightarrow & \Sigma(M) \\ & g.G_0 & \longmapsto & gJ_0g^{-1}\ . \end{array}$$

Moreover, i is an embedding. Indeed, G acts smoothly on the manifolds $\Sigma(M)$ and so the map $g \in G \mapsto gJ_0g^{-1} \in \Sigma(M)$ is of constant rank. Thus $i\colon G/G_0 \to \Sigma(M)$ is an injective map of constant rank and so it is an injective immersion. We can add that $i\colon G/\mathrm{Stab}_G(J_0) \to G.J_0$ is an homeomorphism if the orbit $G.J_0$ is locally closed in $\Sigma(M)$ (see [5]). We will show directly that $i(G/G_0) = G.J_0$ is a subbundle of $\Sigma(M)$.

First, let us precise the fibration $G/G_0 \to G/H$. We have the isomorphism of bundle: $G/G_0 \cong G \times_H H/G_0$. In particular, the fibre type of G/G_0 is H/G_0. Besides i is a morphism of bundle (over M). Since i is also an injective immersion, we can identify the fibres of G/G_0 and $i(G/G_0)$ respectively over the point $g.p_0 \in M$. The fibre of $i(G/G_0)$ over $p = g.p_0$ is gS_0g^{-1} where $S_0 = \mathrm{Int}(H)(J_0) \subset \Sigma(T_{p_0}M)$ is the fibre over p_0.[13]

Now let us show that $i(G/G_0)$ is a subbundle of $\Sigma(M)$. Let $\sigma\colon U \subset G/H \to G$ be a local section of the fibration $\pi_H\colon G \to G/H$. Then we have the following trivialisation of $\Sigma(U)$:

$$\Phi\colon (p, J) \in U \times \Sigma(T_{p_0}M) \longmapsto (p, \sigma(p)J\sigma(p)^{-1}) \in \Sigma(U)$$

[13] we remark that $H \subset O(T_{p_0}M)$, $G_0 \subset U(T_{p_0}M, J_0)$ and $S_0 = H/G_0$ is a compact submanifold of $\Sigma(T_{p_0}M)$.

and we have $\Phi(U\times S_0)=\bigsqcup_{p\in U}\{p\}\times(\sigma(p)S_0\sigma(p)^{-1})=i(G/G_0)\cap\Sigma(U)$. Thus $i(G/G_0)$ is a subbundle of $\Sigma(M)$, hence i is an embedding.

Let us recapitulate what precedes:

Theorem 11 *The map*

$$\begin{array}{rccc} i\colon & G/G_0 & \longrightarrow & \Sigma(M) \\ & g.G_0 & \longmapsto & gJ_0g^{-1}\ . \end{array}$$

is an embedding and a morphism of bundle from G/G_0 into $\Sigma(M)$. Besides the fibre of $i(G/G_0)$ over the point $p=g.p_0$ is gS_0g^{-1}, with $S_0=\mathrm{Int}(H)(J_0)$ and $J_0\in\Sigma(T_{p_0}M)$ corresponding to $\tau_{|\mathfrak{m}}\in\Sigma(\mathfrak{m})$ (resp. to $\tau_{|\mathfrak{m}}^{-1}$) .

Remark 13 Given one order four automorphism, we have two different ways to embed G/G_0 into $\Sigma(M)$ by $J_0=\pm\tau_{|\mathfrak{m}}$. The two submanifolds $i_{J_0}(G/G_0)$ and $i_{-J_0}(G/G_0)$ are isomorphic by $J\mapsto -J$. These are identical if and only if H contains an element which anticommutes with J_0. If $\dim M=2\mod 4$ then they lie in different connected components of the twistor space (one in $\Sigma^+(M)$ and the other one in $\Sigma^-(M)$). In theorem 4 we use $-\tau_{|\mathfrak{m}}$ to respect the convention: α_{-1} is an $(1,0)$-form.

Remark 14 If we consider a locally 4-symmetric bundle $N=G/G_0$ over M, with G_0 not defined by (11), then i is not injective in general: to obtain an injective map i, we must consider the locally 4-symmetric space $G/\pi_K^{-1}(G_0')=(G/K)/G_0'$ where $K=\ker\mathrm{Ad}_{\mathfrak{m}}$ and G_0' is the subgroup of $H'=H/K$ defined by (11). In particular, we see that in general a 4-symmetric space G/G_0 is not a submanifold of a twistor space (see section 5). Moreover, we can see the aim of our definition (11) (and in particular of our convention $G_0=G^\tau\cap H$): it is to obtain an injective map i.

Remark 15 For any covering $\pi\colon\tilde G\to G$, $\tilde G$ acting symmetrically on M, we have $\tilde\rho_{J_0}(\tilde G)=\rho_{J_0}(G)=i_{J_0}(G/G_0)$: the locally 4-symmetric subbundle of $\Sigma(M)$, $i_{J_0}(G/G_0)$ does not depend on the choice of the group G (we have chosen for G, the subgroup of $\mathrm{Is}(M)$ generated by $\mathfrak{g}$).

Moreover, $\rho_{J_0}(\underline{G})=i_{J_0}(\underline{N})$ is a maximal locally 4-symmetric subbundle in $\Sigma(M)$. Now, suppose that we work with a non-closed subgroup $G'\subset\mathrm{Is}(M)$, then $\rho_{J_0}(G')=i_{J_0}(G'/G_0')$ is an immersed subbundle in $\Sigma(M)$: $\Phi(U\times S_0')=\bigsqcup_{p\in U}\{p\}\times(\sigma(p)S_0'\sigma(p)^{-1})=i(G'/G_0')\cap\Sigma(U)$ but the fibre $S_p'=\sigma(p)S_0'\sigma(p)^{-1}$ is only a (non-closed relatively compact) immersed submanifold in $\Sigma(T_pM)$. And since i_{J_0} is an embedding (from $\underline{N}$ into $\Sigma(M)$) we have $i_{J_0}(\bar G'/\bar G_0')=i_{J_0}(\bar N')=\overline{i_{J_0}(N')}$. In others words, taking the closure of G' in $\mathrm{Is}(M)$ is equivalent to take the closure of N' in $\underline{N}$ according to theorem 10 which is equivalent to take the closure of $i_{J_0}(N')$ in $i_{J_0}(\underline{N})$.

Remark 16 The maximal locally 4-symmetric bundles $\underline{N}$ are disjoint : these are orbits in $\Sigma(M)$. More precisely these are suborbits of $\mathrm{Is}(M)^0$-orbits in the form $\underline{G} \cdot J_0$ in $\Sigma(M) \cap \mathrm{Aut}(M)$ with $\mathrm{Aut}(M) = \bigsqcup_{p\in M} \{p\} \times \mathrm{Aut}(T_pM)$ (see Sections 4.1 and 4.2). In particular, $\Sigma(M) \cap \mathrm{Aut}(M)$ is the disjoint union of all the maximal locally 4-symmetric bundles over M. Moreover, the set of maximal locally 4-symmetric bundles over M, $\mathcal{N}(M)$, contains the subset $\mathcal{N}^*(M)$ of elements which are $\mathrm{Is}(M)^0$-orbits, i.e. elements $\rho_{J_0}(\mathrm{Is}(M)^0)$ with $J_0 \in \Sigma(\mathfrak{m}) \cap \mathrm{Aut}^*(\mathfrak{m})$: $\mathcal{N}^*(M) = (\Sigma(M) \cap \mathrm{Aut}^*(M))/\mathrm{Is}(M)^0 \subset \Sigma(M)/\mathrm{Is}(M)^0$.

Remark 17 The Riemannian manifold $M = G/H$ is orientable if and only if $\mathrm{Ad}_{\mathfrak{m}} H \subset SO(\mathfrak{m})$ (or equivalently $H \subset SO(T_{p_0}M)$). Besides $\tau_{|\mathfrak{m}} \in \Sigma^{\varepsilon}(\mathfrak{m})$, and to fix ideas, let us suppose that $\varepsilon = 1$. Then, if M is orientable, i is an embedding from G/G_0 into $\Sigma^+(M)$. Moreover, if we work with $\tilde{M} = G/H^0$, we are sure that $H^0 \subset SO(T_{p_0}\tilde{M})$. Hence, if we work with what we called the first possibility (see section 3.1.1), i takes values in $\Sigma^+(\tilde{M})$. In other words, given a locally 4-symmetric bundle over M, the corresponding 4-symmetric bundle over its universal covering $\tilde{M}$ (see remark 7) is embedded in $\Sigma^+(\tilde{M})$.
Let us consider more generally any covering $\pi\colon \tilde{M} \to M$ then it induces the covering $\pi_\Sigma\colon \Sigma(\tilde{M}) \to \Sigma(M)$ which is also a morphism of bundle over $\pi\colon \tilde{M} \to M$. It is defined by

$$\pi_\Sigma\colon J_{\tilde{x}} \in \Sigma(T_{\tilde{x}}\tilde{M}) \mapsto T_{\tilde{x}}\pi \circ J_{\tilde{x}} \circ (T_{\tilde{x}}\pi)^{-1} \in \Sigma(T_xM).$$

Now, let us suppose that π comes from a covering $\tilde{\pi}\colon \tilde{G} \to G$ and that we have $M = G/H$, $\tilde{M} = \tilde{G}/\tilde{H}^0$ (symmetric realisation) with $\tilde{H} = \tilde{\pi}^{-1}(H)$ and $G \subset \mathrm{Is}(M)$, $\tilde{G} \subset \mathrm{Is}(\tilde{M})$ (see section 3.1.1). Then we have

$$T_{\tilde{x}}\pi \circ (\tilde{g}J_{\tilde{p}_0}\tilde{g}^{-1}) \circ (T_{\tilde{x}}\pi)^{-1} = gJ_{p_0}g^{-1}$$

with $\tilde{x} = \tilde{g}.\tilde{p}_0$, $g = \tilde{\pi}(\tilde{g})$. Hence the restriction of π_Σ to $\tilde{G}/\tilde{G}_0$ gives rise to the morphism of bundle (12)[14]. Moreover[15]

$$S_0 = \mathrm{Int}(H)(J_0) = \bigcup_{h\in\tilde{H}/\tilde{H}^0} h\tilde{S}_0h^{-1}$$

with, since $\tilde{H}^0 \subset SO(T_{p_0}\tilde{M})$, $\tilde{S}_0 \subset \Sigma^+(T_{p_0}\tilde{M})$. Further if $H \subset O(T_{p_0}M)$ is not included in $SO(T_{p_0}M)$ (i.e. M is not orientable), then we have

$$\pi_\Sigma(\Sigma^+(\tilde{M})) = \Sigma(M).$$

Remark 18 Let us see what happens when we change M, in theorem 4. Let $\tilde{G}$ be a covering of G, acting symmetrically on a covering $\tilde{M}$ of M, $\pi\colon \tilde{M} \to M$, with $\tilde{G} \subset \mathrm{Is}(\tilde{M})$. Then according to remark 17, we have $\pi_\Sigma \circ i_{J_{\tilde{p}_0}} = i_{J_{p_0}} \circ \pi_0$,

[14] i.e. $\pi_\Sigma \circ i_{J_{\tilde{p}_0}} = i_{J_{p_0}} \circ \pi_0$, where $\pi_0\colon \tilde{G}/\tilde{G}_0 \to G/G_0$ is given by (12).

[15] In fact, $h\tilde{S}_0h^{-1}$ means obviously $T_{h.p_0}\pi \circ (h\tilde{S}_0h^{-1}) \circ T_{h.p_0}\pi^{-1}$. π_Σ allows to consider the fibres $\Sigma(T_{\tilde{x}}\tilde{M})$ as included in the fibre $\Sigma(T_xM)$, with $x = \pi(\tilde{x})$.

with $\pi_0\colon \tilde{G}/\tilde{G}_0 \to G/G_0$ the morphism of bundle (over $\pi\colon \tilde{M} \to M$) given by (12). Then given any solution α of the $(\mathfrak{g},\tau)$-system (6), let us integrate it in $\tilde{G}$ and G respectively, $\tilde{U}\colon L \to \tilde{G}$, $U\colon L \to G$ with $\tilde{U}(0) = 1$, $U(0) = 1$ (0 is a reference point in L), we have $\tilde{\pi} \circ \tilde{U} = U$. Then let us project these lifts in $\tilde{G}/\tilde{G}_0$ and G/G_0 respectively: we obtain the geometric solutions $\tilde{J}\colon L \to \tilde{G}/\tilde{G}_0$ and $J\colon L \to G/G_0$ respectively and we have $\pi_0 \circ \tilde{J} = J$. Then let us embed these into the twistor spaces $\Sigma(\tilde{M})$ and $\Sigma(M)$ to obtain the admissible twistor lifts $\tilde{J}_{\tilde{X}}\colon L \to i_{\tilde{J}_0}(\tilde{G}/\tilde{G}_0)$ and $J_X\colon L \to i_{J_0}(G/G_0)$ respectively which are related by $\pi_\Sigma \circ \tilde{J}_{\tilde{X}} = J_X$, and in particular $\pi \circ \tilde{X} = X$.

4 Splitting of M into the 3 types of symmetric spaces

In the following theorems and corollaries, we study the behaviour of the automorphism τ with respect to the de Rham decomposition of M.

Theorem 12 [10, 2] *Let M be a simply connected Riemannian symmetric space. Then M is a product*

$$M = M_0 \times M_- \times M_+$$

where M_0 is an Euclidean space, M_- and M_+ are Riemannian symmetric spaces of the compact and non-compact types respectively. In particular

$$M = M_0 \times M'$$

where M' has a group of isometries $G = \mathrm{Is}(M')$ semisimple and its isotropy subgroup at $p_0 \in M'$, H, (which is connected because M' is simply connected) is equal to the holonomy group of M'. Hence a Riemannian symmetric space M of which the isometry group is semisimple (which is equivalent to say that its universal covering has not Euclidean factor, or equivalently the Lie algebra of G does not contain non-trivial abelian ideal, i.e. its Killing form is non-degenerated) has a unique symmetric realisation G/H, with G acting effectively. In this unique realisation, we have necessarily $G = \mathrm{Is}(M)^0$ [16] *and $H = \mathrm{Is}^0_{p_0}(M) := \mathrm{Is}_{p_0}(M) \cap \mathrm{Is}(M)^0 (\supset \mathrm{Is}_{p_0}(M)^0)$. Further the Lie algebra $\mathfrak{Is}_{p_0}(M) = \mathrm{Der}(\mathfrak{m}) = \mathfrak{Hol}(M)$ is spanned by $[\mathfrak{m},\mathfrak{m}] = \{R_{p_0}(X,Y),\ X,Y \in T_{p_0}(M)\}$.*
Moreover the universal covering of such a Riemannian symmetric space M, admits a decomposition into a product of irreducible Riemannian symmetric spaces (i.e with linear isotropy representations which are irreducible)

$$\tilde{M} = M_1 \times \cdots \times M_r.$$

[16] as usual, we suppose that G is connected

Theorem 13 *Let us consider the decomposition of* $(\mathfrak{g},\sigma)$ *into the sum of orthogonal (for the Killing form) ideals of the compact, non-compact and Euclidean types respectively:*

$$\mathfrak{g}=\mathfrak{l}_0\oplus\mathfrak{l}_-\oplus\mathfrak{l}_+$$

and let $\mathfrak{l}_\alpha=\mathfrak{h}_\alpha\oplus\mathfrak{m}_\alpha$ *be the eigenspace decomposition of the involution* $\sigma_{|\mathfrak{l}_\alpha}$*. Suppose now that we have an order four automorphism* $\tau\colon\mathfrak{g}\to\mathfrak{g}$ *with* $\tau^2=\sigma$*. Then* $\tau(\mathfrak{l}_\alpha)=\mathfrak{l}_\alpha$*,* $\tau(\mathfrak{h}_\alpha)=\mathfrak{h}_\alpha$*,* $\tau(\mathfrak{m}_\alpha)=\mathfrak{m}_\alpha$ *for* $\alpha=0,-,+$*. Hence* $\tau_{\mathfrak{m}}=\oplus_\alpha\tau_{\mathfrak{m}_\alpha}$*, with* $\tau_{\mathfrak{m}_\alpha}\in\Sigma(\mathfrak{m}_\alpha)$*, and* $\tau_{|\mathfrak{l}_\alpha}$ *is the automorphism of* $\mathfrak{l}_\alpha$ *defined by* $\tau_{\mathfrak{m}_\alpha}$ *according to theorem 7 and we have* $\tau^2_{|\mathfrak{l}_\alpha}=\sigma_{|\mathfrak{l}_\alpha}$*. Moreover, we have* $\mathrm{Aut}(\mathfrak{m})=\prod_\alpha\mathrm{Aut}(\mathfrak{m}_\alpha)$*.*

Corollary 1 *Let* M *be a* G*-symmetric space,* $G\subset\mathrm{Is}(M)$ *and* $\tau\colon\mathfrak{g}\to\mathfrak{g}$ *an order four automorphism with* $\tau^2=\sigma$*. Let* $\tilde{M}$ *be its universal covering, which has a symmetric realisation* $\tilde{M}=\tilde{G}/\tilde{H}^0$*, with* $\pi\colon\tilde{G}\to G$ *a covering of* G*,* $\tilde{H}=\pi^{-1}(H)$ *and* $\tilde{G}\subset\mathrm{Is}(\tilde{M})$*, such that* τ *integrates into* $\tilde{\tau}\colon\tilde{G}\to\tilde{G}$*. Then the decomposition of* $\mathfrak{g}$ *into 3 ideals of different types gives rise to the following decomposition of* $\tilde{G}$*:*

$$\tilde{G}=L_0\times L_-\times L_+$$

which induces the following decompositions of $\tilde{H}^0$ *and* $\tilde{G}_0=\tilde{H}^0\cap\tilde{G}^{\tilde{\tau}}$*, corresponding also to the decompositions* $\mathfrak{h}=\oplus_\alpha\mathfrak{h}_\alpha$ *and* $\mathfrak{g}_0=\oplus_\alpha(\mathfrak{g}_0)_\alpha$*:*

$$\tilde{H}^0 = H_0\times H_-\times H_+ \qquad (16)$$

$$\tilde{G}_0 = (G_0)_0\times(G_0)_-\times(G_0)_+\,. \qquad (17)$$

Hence $\tilde{M}=M_0\times M_-\times M_+$ *and* $\tilde{N}=N_0\times N_-\times N_+$ *with* $M_\alpha=L_\alpha/H_\alpha$*,* $N_\alpha=L_\alpha/(G_0)_\alpha$*. Besides* $\tilde{\sigma}$ *and* $\tilde{\tau}$ *admit the decompositions* $\tilde{\sigma}=\prod_\alpha\tilde{\sigma}_\alpha$ *and* $\tilde{\tau}=\prod_\alpha\tilde{\tau}_\alpha$*, and* $H_\alpha=(L_\alpha^{\tilde{\sigma}_\alpha})^0$*,* $(G_0)_\alpha=H_\alpha\cap L_\alpha^{\tau_\alpha}=(L_\alpha)_0$*. Moreover* N_α *is a 4- symmetric bundle over* M_α*.*

Theorem 14 *Let us consider the decomposition of* $(\mathfrak{g},\sigma)$ *into the sum of orthogonal (for the Killing form) ideals:*

$$\mathfrak{g}=\oplus_{i=0}^r\mathfrak{g}_i \qquad (18)$$

with $\mathfrak{g}_0$ *abelian and* $(\mathfrak{g}_i,\sigma_{|\mathfrak{g}_i})$ *irreducible, and let* $\mathfrak{g}_i=\mathfrak{h}_i\oplus\mathfrak{m}_i$ *be the eigenspace decomposition of* $\sigma_{|\mathfrak{g}_i}$*. Suppose now that we have an order four automorphism* $\tau\colon\mathfrak{g}\to\mathfrak{g}$ *such that* $\tau^2=\sigma$*.*
There exists an unique decomposition of $\mathfrak{g}$*:*

$$\mathfrak{g}=\mathfrak{g}_0\oplus(\oplus_{i=1}^{r'}\mathfrak{g}'_i) \qquad (19)$$

where $\mathfrak{g}'_i=\mathfrak{g}_i$ *or* $\mathfrak{g}'_i=\mathfrak{g}_i\oplus\mathfrak{g}_j$ *with* $(\mathfrak{g}_i,\sigma_{|\mathfrak{g}_i})$ *and* $(\mathfrak{g}_j,\sigma_{|\mathfrak{g}_j})$ *isomorphic, such that* $\tau(\mathfrak{g}'_i)=\mathfrak{g}'_i$*. Besides if* $\mathfrak{g}'_i=\mathfrak{h}'_i\oplus\mathfrak{m}'_i$ *is the eigenspace decomposition of* $\sigma_{|\mathfrak{g}'_i}$*, then* $\tau(\mathfrak{h}'_i)=\mathfrak{h}'_i$*,* $\tau(\mathfrak{m}'_i)=\mathfrak{m}'_i$*. Moreover if* $\mathfrak{g}'_i=\mathfrak{g}_i\oplus\mathfrak{g}_j$ *then* $\tau(\mathfrak{g}_i)=\mathfrak{g}_j$*,* $\tau(\mathfrak{h}_i)=\mathfrak{h}_j$*,* $\tau(\mathfrak{m}_i)=\mathfrak{m}_j$*. Hence* $\tau_{\mathfrak{m}}=\oplus_{i=0}^{r'}\tau_{\mathfrak{m}'_i}$ *with* $\tau_{\mathfrak{m}'_i}\in\Sigma(\tau_{\mathfrak{m}'_i})$*, and* $\tau_{|\mathfrak{g}'_i}$ *is the automorphism of* $\mathfrak{g}'_i$ *defined by* $\tau_{\mathfrak{m}'_i}$ *according to theorem 7 and we have* $\tau^2_{|\mathfrak{g}'_i}=\sigma_{|\mathfrak{g}'_i}$*.*

Corollary 2 *Let M be a G-symmetric space, $G \subset \mathrm{Is}(M)$ and $\tau\colon \mathfrak{g} \to \mathfrak{g}$ an order four automorphism with $\tau^2 = \sigma$. Let M be its universal covering, which has a symmetric realisation $\tilde{M} = \tilde{G}/\tilde{H}^0$, with $\pi\colon \tilde{G} \to G$ a covering of G, $\tilde{H} = \pi^{-1}(H)$ and $\tilde{G} \subset \mathrm{Is}(\tilde{M})$, such that τ integrates into $\tilde{\tau}\colon \tilde{G} \to \tilde{G}$. Then the decomposition of $\mathfrak{g}$,(18), gives rise to the following decomposition of $\tilde{G}$:*

$$\tilde{G} = L_0 \times L_1 \times \cdots \times L_r$$

which induces the following decomposition of $\tilde{H}^0$, corresponding also to the decomposition $\mathfrak{h} = \oplus_{i=0}^r \mathfrak{h}_i$:

$$\tilde{H}^0 = H_0 \times H_1 \times \cdots \times H_r.$$

Then $\tilde{\sigma}$ admits the decomposition $\tilde{\sigma} = \prod_{i=0}^r \tilde{\sigma}_i$ (with $\tilde{\sigma}_i$ involution of L_i) and $H_i = (L_i^{\tilde{\sigma}_i})^0$. Moreover there exists an unique decomposition of $\tilde{G}$:

$$\tilde{G} = L'_0 \times L'_1 \times \cdots \times L'_{r'} \tag{20}$$

where $L'_i = L_i$ or $L'_i = L_i \times L_j$ with $(L_i, \tilde{\sigma}_i)$ and $(L_j, \tilde{\sigma}_j)$ isomorphic. Then τ admits the decomposition $\tilde{\tau} = \prod_{i=0}^{r'} \tilde{\tau}'_i$ with $\tilde{\tau}'_i$ order four automorphism of L'_i. Further, by identifying $(L_i, \tilde{\sigma}_i)$ and $(L_j, \tilde{\sigma}_j)$ (when $L'_i = L_i \times L_j$), then in (20), we have either $L'_i = L_i$ and then $\tilde{\tau}'_i = \tilde{\tau}_i$ is an order four automorphism of L_i so that $(L'_i)^{\tilde{\tau}'_i} = (L_i)^{\tilde{\tau}_i}$, or $L'_i = L_i \times L_i$ and then

$$\tilde{\tau}'_i \colon (a, b) \in L_i \times L_i \mapsto (\sigma_i(b), a) \in L_i \times L_i$$

so that $(L'_i)^{\tilde{\tau}'_i} = \Delta(H_i) \subset H_i \times H_i$. Hence $\tilde{M} = M_0 \times M_1 \times \cdots \times M_r$ with $M_i = L_i/H_i$, and $\tilde{N} = N'_0 \times N'_1 \times \cdots \times N'_{r'}$ where either $N'_i = N_i = L_i/(L_i)_0$ is a 4-symmetric bundle over M_i, or $N'_i = L_i \times L_i/\Delta(H_i)$ is a 4-symmetric bundle over $M_i \times M_i = L_i \times L_i/H_i \times H_i$ (and the fibre $H_i \times H_i/\Delta(H_i) \simeq H_i$ is a group).

Proofs of theorems 13,14 and corollaries 1,2 Use the fact that $\tau_{\mathfrak{m}}$ leaves invariant the metric in $\mathfrak{m}$ and the restriction to $\mathfrak{m}$ of the Killing form. ■

4.1 The semisimple case

Definition 7 *We will say that the Riemannian symmetric space M is of semisimple type if* $\mathrm{Is}(M)$ *is semisimple.*

Theorem 15 *If M is of semisimple type then each (connected) locally 4-symmetric bundle over M is maximal and in the form $\underline{N} = \mathrm{Is}(M)^0/\underline{G_0}$, i.e. is an $\mathrm{Is}(M)^0$-orbit in $\Sigma(M) \cap \mathrm{Aut}(M)$. In other words the set of locally 4-symmetric bundles over M is $\mathcal{N}(M) = (\Sigma(M) \cap \mathrm{Aut}(M))/\mathrm{Is}(M)^0 \subset \Sigma(M)/\mathrm{Is}(M)^0$.*

Remark 19 The "size" of a maximal (locally) 4-symmetric bundle over M in the twistor bundle $\Sigma(M)$ depends on the "size" of the isotropy subgroup $\mathrm{Is}_{p_0}(M)$ and on $J_0 \in \Sigma(T_{p_0}M)$. In other words, if we want a fibre $S_0 \subset \Sigma(T_{p_0}M)$ of maximal dimension, we must find $J_0 \in \Sigma(T_{p_0}M) \cap \mathrm{Aut}(T_{p_0}M) \supset \Sigma(T_{p_0}M) \cap \mathrm{Is}_{p_0}(M)$ such that $T_{J_0}\underline{S_0} = \underline{\mathfrak{g}_2}(J_0) := \{A \in \mathfrak{Is}_{p_0}(M) \mid AJ_0 + J_0A = 0\}$ is of maximal dimension, or equivalently such that $\underline{\mathfrak{g}_0}(J_0) = \{A \in \mathfrak{Is}_{p_0}(M) \mid AJ_0 - J_0A = 0\}$ is of minimal dimension.

Remark 20 It is possible that there exist different non-isomorphic locally 4-symmetric bundles over M (see section 5.3). And it is also possible that there does not exist any locally 4-symmetric bundle over M. For example: $M = S^1 \times S^3$, then $\mathrm{Is}(M) = SO(2) \times SO(4)$ and $\mathrm{Is}_{p_0}(M) = SO(3)$, and there does not exist $J_0 \in \Sigma(\mathbb{R}^4)$ such that $J_0 SO(3) J_0^{-1} = SO(3)$.

Moreover we have the following obvious theorem (see also [11]):

Theorem 16 *Let $(\mathfrak{g}, \sigma)$ be an orthogonal symmetric Lie algebra. Then set $\mathfrak{g}^* = \mathfrak{h} \oplus i\mathfrak{m}$ and $\sigma^* = \mathrm{Id}_{\mathfrak{h}} \oplus -\mathrm{Id}_{i\mathfrak{m}}$. Then $(\mathfrak{g}^*, \sigma^*)$ is an orthogonal symmetric Lie algebra. If $(\mathfrak{g}, \sigma)$ is of the compact type then $(\mathfrak{g}^*, \sigma^*)$ is of the non-compact type and conversely. Now, for $\tau_{\mathfrak{m}} \in \mathrm{End}(\mathfrak{m})$, set $\tau_{\mathfrak{m}}^* \colon iv \in i\mathfrak{m} \mapsto i\tau_{\mathfrak{m}}(v)$. Then*

$$\tau_{\mathfrak{m}} \in \mathrm{Aut}(\mathfrak{m}) \Longleftrightarrow \tau_{\mathfrak{m}}^* \in \mathrm{Aut}(i\mathfrak{m})$$

and $\tau_{\mathfrak{m}} \in \Sigma(\mathfrak{m})$ if and only if $\tau_{\mathfrak{m}}^* \in \Sigma(i\mathfrak{m})$. In this case ($\tau_{\mathfrak{m}} \in \mathrm{Aut}(\mathfrak{m}) \cap \Sigma(\mathfrak{m})$) let τ (resp. τ^*) be the automorphism of $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) defined by $\tau_{\mathfrak{m}}$ (resp. $\tau_{\mathfrak{m}}^*$) and denoting by $A^{\mathbb{C}} \in \mathrm{End}(V^{\mathbb{C}})$ the extension to $V^{\mathbb{C}}$ of $A \in \mathrm{End}(V)$ (V real vector space) then we have

$$\tau^{\mathbb{C}} = \tau^{*\mathbb{C}} \qquad i.e.\ \tau^* = \tau^{\mathbb{C}}_{|\mathfrak{g}^*}$$

Theorem 17 *Let M be an irreducible symmetric spaces of type II (compact type) or type IV (non-compact type) then there does not exist any (non-trivial) locally 4-symmetric bundle over M. Equivalently $\mathrm{Aut}(M) \cap \Sigma(M) = \varnothing$, in other words, there does not exist any automorphism τ of $\mathfrak{Is}(M)$ such that $\tau^2 = \sigma$.*

Proof. By duality, it is enough to prove the assertion for the compact type. In this case let $\tilde{M}$ be the universal covering of M, we have $\tilde{M} = H \times H/\Delta(H)$ and $\tilde{\sigma} \colon (a,b) \in G \times G \mapsto (b,a)$. Then an automorphism $\tau \colon \mathfrak{g} \to \mathfrak{g}$ must send $\mathfrak{g}_1 = \mathfrak{h} \oplus \{0\}$ either on $\mathfrak{g}_1$ or on $\mathfrak{g}_2 = \{0\} \oplus \mathfrak{h}$ and idem for $\mathfrak{g}_2$, and thus for any automorphism we have $\tau^2(\mathfrak{g}_i) = \mathfrak{g}_i$ and hence we cannot have $\tau^2 = \sigma$. This completes the proof. ∎

4.2 The Euclidean case

Theorem 18 *Let $M = \mathbb{R}^{2n}$ with its canonical inner product. Then $\mathrm{Is}(M) = O(2n) \ltimes \mathbb{R}^{2n}$ the group of affine isometries in $\mathbb{R}^{2n}$. Hence for any $p_0 \in \mathbb{R}^{2n}$, we have $\mathrm{Is}_{p_0}(M) = \{(F, (\mathrm{Id} - F)p_0), F \in O(2n)\} \simeq O(2n)$. In particular for $p_0 = 0$, $\mathrm{Is}_{p_0}(M) = O(2n)$. Thus we have $\forall p_0 \in \mathbb{R}^{2n}$, $\mathrm{Is}(M) = \mathrm{Is}_{p_0}(M) \ltimes \mathbb{R}^{2n}$. Further $M = G/H$ is a symmetric realisation with G acting effectively if and* *only if* $G = H \ltimes \mathbb{R}^{2n}$ *with* $H \subset \mathrm{Is}_{p_0}(\mathbb{R}^{2n})$ *for some* $p_0 \in \mathbb{R}^{2n}$*. Then we have* $G = H_0 \ltimes \mathbb{R}^{2n}$ *with* $H_0 = \mathrm{pr}_{O(2n)}(H) \subset O(2n)$*. The involution for this realisation is*

$$\sigma = \mathrm{Int}(-\mathrm{Id}, 2p_0)\colon (h,x) \in G \mapsto (h, 2(\mathrm{Id}-h)p_0 - x)$$

giving rise to the symmetry around p_0*:* $\sigma_0\colon x \in \mathbb{R}^{2n} \mapsto -(x-p_0)+p_0 \in \mathbb{R}^{2n}$*. Let us fix* $p_0 = 0$*, so that for any symmetric realisation we have* $H \subset \mathrm{Is}_{p_0}(M) = O(2n)$ *and* $\sigma = \mathrm{Int}(-\mathrm{Id}, 0)$*.*
All (connected) locally 4-symmetric bundles over M *are globally 4-symmetric bundles over* M*. The twistor bundle,* $\Sigma(\mathbb{R}^{2n}) \times \mathbb{R}^{2n}$*, is a globally 4-symmetric bundle over* M*. All the (connected) 4-symmetric bundles over* $\mathbb{R}^{2n}$ *are in the form:* $S_0 \times \mathbb{R}^{2n}$ *where* S_0 *is a compact Riemannian symmetric space embedded*[17] *in* $\Sigma^\varepsilon(\mathbb{R}^{2n})$*. Besides* $\mathrm{Aut}(T_{p_0}M) = \mathrm{Is}_{p_0}(M) = O(2n)$ *so that any* $J_0 \in \Sigma(\mathbb{R}^{2n})$ *defines the maximal 4-symmetric bundle* $\Sigma(\mathbb{R}^{2n}) \times \mathbb{R}^{2n} = (O(2n) \ltimes \mathbb{R}^{2n})/U(\mathbb{R}^{2n}, J_0)$*.*

Theorem 19 *Let* M *be an Euclidean Riemannian symmetric space (i.e. its universal covering is an Euclidean space* $\mathbb{R}^{2n}$*). Then* $M = \mathbb{R}^{2p} \oplus \mathbb{T}^{2q}$, $\mathrm{Is}(M) = O(2p) \times (\mathfrak{S}_{2q} \otimes \{\pm 1\}) \ltimes M$ *(*$\mathfrak{S}_{2q}$ *is the group of permutations) and denoting by* $\pi\colon \mathbb{R}^{2n} \to M$ *the universal covering, and* $p_0 = \pi(0)$*, then* $\mathrm{Is}_{p_0}(M) = O(2p) \times (\mathfrak{S}_{2q} \otimes \{\pm 1\})$*. Moreover* $\mathrm{Aut}(\mathbb{R}^{2n}) = O(2n)$*, and* $J_0 \in \Sigma(\mathbb{R}^{2n})$ *defines the (connected) maximal 4-symmetric bundle over* M*:* $(\Sigma(E^{2l}) \times \{J_{0|E^{2l\perp}}\}) \times M$*, where* E^{2l} *is the (unique) maximal subspace in* $\mathbb{R}^{2p}$ *invariant by* J_0*. In particular,* $\mathrm{Aut}^*(M) \cap \Sigma(M) = \Sigma(\mathbb{R}^{2p}) \times \Sigma(\mathbb{R}^{2q}) \times M$*.*

Proof. Let $\tilde\pi\colon \tilde G \to G$ be a covering of $G = \mathrm{Is}(M)^0$ acting symmetrically and effectively on $\tilde M = \mathbb{R}^{2n}$ and $\tilde\sigma\colon \tilde G \to \tilde G$ the corresponding involution. Then setting $\tilde H = (\tilde G^{\tilde\sigma})^0$, we have according to the previous theorem $\tilde G = \tilde H \ltimes \mathbb{R}^{2n}$ and $\tilde H \subset SO(2n)$. Then setting $D = \ker\pi$, D is a discrete central subgroup of $\tilde G$. Besides it is easy to see that $\mathrm{Cent}(\tilde G) = \mathrm{Cent}(\tilde H \times \mathbb{R}^{2n}) = \mathbb{R}^{2q}$ where $\mathbb{R}^{2q}$ is the maximal subspace of $\mathbb{R}^{2n}$ fixed by $\tilde H$, i.e. $\tilde H \subset SO(2p) \times \{\mathrm{Id}_{2q}\}$ $(2p+2q = 2n)$. Hence $D = \oplus_{i=1}^r \mathbb{Z}e_i$ with $(e_i)_{1\le i\le r}$ $\mathbb{R}$-free so that $G = \mathrm{Is}(M)^0 = \tilde G/D = \tilde H \ltimes M'$ with $M' = \mathbb{R}^{2p} \oplus \mathbb{R}^{2q-r} \oplus \mathbb{R}^r/\mathbb{Z}^r$. Moreover we have $\sigma\colon (h,x) \in \tilde H \times M' \to (h,-x)$ because $\tilde\sigma = \mathrm{Int}(-\mathrm{Id},0)$ (see the previous theorem) and thus $G^\sigma = \tilde H$ but the isotropy subgroup of G at p_0 satisfies $H \supset \tilde\pi(\tilde H)$ (because $\tilde H$ is connected), but $\tilde\pi(\tilde H) = \tilde H$ $(D \cap \tilde H = \{1\})$ and thus $H = \tilde H$. Thus $M = G/H = M'$. Now, we have to compute $\mathrm{Is}(M)$, we know that $\mathrm{Is}(M)^0 = H \ltimes M \subset SO(2p) \ltimes M$. In the other hand, any $g \in \mathrm{Is}(M)$ can be lifted into $\tilde g \in O(2n) \ltimes \mathbb{R}^{2n}$, and conversely $\tilde g \in O(2n) \ltimes \mathbb{R}^{2n}$ corresponds to some $g \in \mathrm{Is}(M)$ if and only if $\tilde g(D) = D$ which is equivalent to $\tilde g \in [O(2p+2q-r) \times (GL_r(\mathbb{Z}) \cap O(\mathbb{R}^r))] \ltimes \mathbb{R}^{2n} = [O(2p+2q-r) \times (\mathfrak{S}_r \ltimes \{\pm\mathrm{Id}\})] \ltimes \mathbb{R}^{2n}$. Hence $\mathrm{Is}_{p_0}(M)^0 = SO(2p+2q-r)$ and thus $r = 2q$. Finally $M = \mathbb{R}^{2p} \oplus \mathbb{T}^{2q}$, $\mathrm{Is}(M) = O(2p) \times (\mathfrak{S}_r \ltimes \{\pm\mathrm{Id}\}) \ltimes M$, $\mathrm{Is}_{p_0}(M) = O(2p) \times (\mathfrak{S}_r \ltimes \{\pm\mathrm{Id}\})$, and $\mathrm{Is}_{p_0}(M)^0 = H = SO(2p)$. We conclude by remarking that $J_0 \in \Sigma(\mathbb{R}^{2n})$ satisfies $J_0 H J_0^{-1} = H$ for $H \subset SO(2p)$ connected

[17] only immersed if H is not closed in $O(2n)$

and maximal if and only if $H = SO(E^{2l})$ and $J_0 \in \Sigma(E^{2l}) \times \Sigma(E^{2l\perp})$. This completes the proof. ∎

Remark 21 We can use the second elliptic integrable system in the Euclidean case to "modelize" this system in the general case. Indeed, let us consider M a Riemannian symmetric space of the semisimple type (then its isotropy subgroup $H = \mathrm{Is}_{p_0}(M)$ is essentially its holonomy group, i.e. they have the same identity component) with $\tau\colon \mathfrak{g} \to \mathfrak{g}$ an order four automorphism such that $\tau^2 = \sigma$. Then we can associate to the corresponding locally 4-symmetric bundle N over M, the 4-symmetric bundle over $M_0 = \mathfrak{m} = H \ltimes \mathfrak{m}/H$: $N_0 = H \ltimes \mathfrak{m}/G_0 = S_0 \times \mathfrak{m} \subset \Sigma(\mathfrak{m}) \times \mathfrak{m}$, and to the second elliptic integrable system in N, its "linearized" in N_0. We conjecture that the "concrete" geometrical interpretation (i.e. in terms of the second fundamental form of the surface X etc...) is the same for the linearized and the initial system. This is what happens in dimension 4.

Remark 22 The second elliptic integrable system can be viewed as "a couplage" between the harmonic map equation in $S_0 = H/G_0$ and a kind of Dirac equation in $\mathfrak{g}_{-1}$: $\partial_{\bar{z}} u_1 + [\bar{u}_0, u_1] + [\bar{u}_1, u_2] = 0$. In the Euclidean case, the projection on the "group part", $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m} \to \mathfrak{h}$, of the second elliptic system is only the harmonic map equation in H/G_0. In other words, the second elliptic integrable system is only the harmonic map equation in H/G_0 and a kind of Dirac equation in $\mathbb{C}^n$ ($\cong (\mathfrak{g}_{-1}, J_0)$). In particular, if we apply any method of integrable systems theory using loop groups (DPW, Dressing action etc..) or something else (spectral curves) to the second elliptic system in G/G_0 and then project in the group part ($\mathrm{pr}\colon H \ltimes \mathfrak{m} \to H$), we obtain the same method applied to the first elliptic integrable system in H/G_0 i.e. the harmonic map equation in H/G_0. For example, if we apply the DPW method: given $\mu = (\mu_{\mathfrak{h}}, \mu_{\mathfrak{m}})$ a holomorphic potential, we have $\mathrm{pr}(\mathcal{W}_{G/G_0}(\mu)) = \mathcal{W}_{H/G_0}(\mu_{\mathfrak{h}})$ where $\mathcal{W}_{G/G_0}, \mathcal{W}_{H/G_0}$ are the Weierstrass representations for each elliptic system. Hence to solve the second elliptic system, we can first solve the harmonic map equation in H/G_0, by using any method of integrable systems theory which gives us a lift h in H of a harmonic map in H/G_0, and then we have to solve the Dirac equation with parameters u_0, u_2 given by the lift : $h^{-1}\partial_z h = u_0 + u_2$ following $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_2$ (see [12]). However, the Dirac equation is not intrinsic since it depends on the lift h of the harmonic map (see [12]).
In the particular case where S_0 is a group and $H = G_0 \rtimes S_0$, (for example $S_0 = G_0 \times G_0/G_0$), then we have a canonical lift and then the Dirac equation becomes intrinsic (see [12]). It is in particular what happens for Hamiltonian stationary Lagrangian surfaces : in $\mathbb{C}^2$ we have an intrinsic Dirac equation whereas in the others Hermitian symmetric spaces this equation does not exist (see [7, 8, 9]). It is also what happens in [12] when we take for S_0 the subsphere $S^3 \subset S^6$ (S^6 embeds in $\Sigma^+(\mathbb{R}^8)$ by the left multiplication in $\mathbb{O}$).

5 Examples of 4-symmetric bundles

We use the notations of section 3.1.

5.1 The sphere

Let us consider $M = S^{2n} = SO(2n+1)/SO(2n)$ with $G = SO(2n+1)$, $H = SO(2n)$ and the involution $\sigma = \mathrm{Int}(\mathrm{diag}(\mathrm{Id}_{2n}, -1))$. Then $G^\sigma = SO(2n) \bigsqcup O^-(2n) \times \{-1\}$. Hence $H = (H^\sigma)^0$, $M_{min} = \mathbb{RP}^{2n}$ and $M_{max} = S^{2n}$.[18] We have also

$$\mathfrak{h} = \mathfrak{so}(2n), \qquad \mathfrak{m} = \left\{ \begin{pmatrix} 0 & v \\ -v^t & 0 \end{pmatrix}, v \in \mathbb{R}^{2n} \right\} = \{ i_\mathfrak{m}(v), v \in \mathbb{R}^{2n} \}$$

where $i_\mathfrak{m} \colon \mathbb{R}^{2n} \to \mathfrak{m}$ is defined in an obvious way. Now, let us consider the action of H on $\mathfrak{m}$: for $h \in SO(2n)$, $\xi = i_\mathfrak{m}(v) \in \mathfrak{m}$, we have

$$\mathrm{Ad}_\mathfrak{m} h(\xi) = i_\mathfrak{m}(h.v)$$

hence $K = \ker \mathrm{Ad}_\mathfrak{m} = \{\mathrm{Id}\}$ and the action of G is effective (in fact $SO(2n+1)$ is simple because $2n+1$ is odd). Identifying $\mathfrak{m}$ with $\mathbb{R}^{2n}$ via $i_\mathfrak{m}$ we have: $\forall h \in SO(2n), \mathrm{Ad}_\mathfrak{m} h = h$ i.e. $\mathrm{Ad}_\mathfrak{m} = \mathrm{Id}$. Moreover $SO(2n+1)$ is the connected isometry group of S^{2n}. Now, according to theorem 7, define a locally 4-symmetric bundle over $M = S^{2n}$ is equivalent to give ourself $\tau_\mathfrak{m} \in \Sigma(\mathfrak{m}) \cap \mathrm{Aut}(\mathfrak{m}) = \Sigma(\mathfrak{m})$. Further, given $J_0 \in \Sigma^\varepsilon(\mathbb{R}^{2n})$, let us define the order four automorphism of G: $\tau = \mathrm{Int}(\mathrm{diag}(-J_0, 1))$. Then $\tau^2 = \sigma$ and since $\tau_H = \mathrm{Int} J_0$ and $\tau_{|\mathfrak{m}} = J_0$, we obtain all the locally 4-symmetric bundles over M which are all globally 4-symmetric bundles over M.
Moreover, we have $G^\tau = \mathrm{Com}(J_0) \cap SO(2n) = U(\mathbb{R}^{2n}, J_0)$. Hence $G^\tau = (G^\tau)^0 = G_0$ thus $S_0 = H/G_0 = \mathrm{Int}(SO(2n))(J_0) = \Sigma^\varepsilon(\mathbb{R}^{2n})$ and thus $N = G/G_0 = \Sigma^\varepsilon(S^{2n})$.

5.2 Real Grassmannian

More generally, let $p, q \in \mathbb{N}^*$ such that pq is even and let us consider $M = SO(p+q)/SO(p) \times SO(q) = Gr_p(\mathbb{R}^{p+q})$ (oriented p-planes in $\mathbb{R}^{p+q}$). Since p and q play symmetric roles, we will suppose that p is even and that it has the biggest divisor in the form 2^r. We have $\dim M = pq$ and the following setting

$G = SO(p+q)$, $H = SO(p) \times SO(q)$; $\sigma = \mathrm{Int}(\mathrm{diag}(\mathrm{Id}_p, -\mathrm{Id}_q))$ and
$G^\sigma = SO(p) \times SO(q) \bigsqcup O^-(p) \times O^-(q)$.

Then $H = (G^\sigma)^0$ so that $M_{min} = Gr_p^*(\mathbb{R}^{p+q})$ (non-oriented p-planes in $\mathbb{R}^{p+q}$) and $M_{max} = Gr_p(\mathbb{R}^{p+q}) = M$. Besides $\mathfrak{h} = \mathfrak{so}(p) \oplus \mathfrak{so}(q)$, and $\mathfrak{m} = \left\{ \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix}, B \in \mathfrak{gl}_{p,q}(\mathbb{R}) \right\} = i_\mathfrak{m}(\mathfrak{gl}_{p,q}(\mathbb{R}))$ ($i_\mathfrak{m}$ defined in an obvious way).
Now let us compute $\mathrm{Ad}_\mathfrak{m}$. For $h = \mathrm{diag}(A, C)$ and $\xi = i_\mathfrak{m}(B)$, we have:

$$\mathrm{Ad}_\mathfrak{m} h(\xi) = i_\mathfrak{m}(ABC^{-1}).$$

[18] M_{max} is simply connected and M_{min} is the adjoint space.

Under the identification $i_{\mathfrak{m}}$ we have $\mathrm{Ad}_{\mathfrak{m}}(A,C) = L_A R_{C^{-1}} = \chi(A,C)$, by introducing the morphism $\chi\colon (A,C) \in GL_p(\mathbb{R}) \times GL_q(\mathbb{R}) \mapsto L(A)R(C^{-1}) \in GL(\mathfrak{gl}_{p,q}(\mathbb{R}))$. Hence $K = \ker \mathrm{Ad}_{\mathfrak{m}} = \{\pm\mathrm{Id}\}$ if q is even and $K = \{\mathrm{Id}\}$ if not. Thus the connected isometry group of M, $\mathrm{Is}(M)^0$, is $G' = G/K = PSO(p+q)$ if q is even and $G' = G = SO(p+q)$ if not. Let us compute $\mathrm{Aut}(\mathfrak{m})$: we already know that $\mathrm{Aut}(\mathfrak{m}) \supset H \supset \mathrm{Aut}(\mathfrak{m})^0$. But, it is well known that the automorphisms of $\mathfrak{so}(n+1)$ are all inner automorphisms by $O(n+1)$ so we have $\mathrm{Aut}(\mathfrak{m}) = \{L_A R_{C^{-1}}, (A,C) \in O(p) \times O(q)\}$. Thus $J_0 = L(J_1)R(J_2^{-1}) \in \mathrm{Aut}(\mathfrak{m})$ is in $\Sigma(\mathfrak{m})$ if and only if :

$$\begin{cases} (J_1^2, J_2^2) = \pm(-\mathrm{Id}_p, \mathrm{Id}_q) & \text{if q is even,} \\ (J_1^2, J_2^2) = (-\mathrm{Id}_p, \mathrm{Id}_q) & \text{if q is odd.} \end{cases}$$

Then the associated order four automorphism is $\tau = \mathrm{Int}(\mathrm{diag}(J_1, J_2))$. In particular, $\tau(H) = H$ and $\tau_H = \mathrm{Int}J_1 \times \mathrm{Int}J_2$. Besides, $\mathrm{Aut}(\mathfrak{m}) \cap \Sigma(\mathfrak{m})$ has respectively $2(p+q+2)$ or $2(q+1)$ connected components if q is even or q is odd respectively. Each connected component is an $\mathrm{Ad}_{\mathfrak{m}}H$-orbit and corresponds to the fibre of a different maximal 4-symmetric bundle over M.
Moreover to fix ideas let us suppose that we have $J_1 \in \Sigma(\mathbb{R}^p), J_2 \in OS(\mathbb{R}^q)$, the set of orthogonal symmetries in $\mathbb{R}^q$, then $G^\tau = U(\mathbb{R}^p, J_1) \times S(O(E_1) \times O(E_2))$ with $E_1 = \ker(J_2 - \mathrm{Id})$, $E_2 = \ker(J_2 + \mathrm{Id})$. We have $G^\tau \subset H$. Let $OS_r(\mathbb{R}^q) = \mathrm{Int}(SO(q))(\mathrm{Id}_r, -\mathrm{Id}_{q-r})$ be the set of orthogonal symmetries in $\mathbb{R}^q$ with $\dim E_1 = r$. Then $H/G^\tau = \mathrm{Int}(H)(J_1, J_2) = \Sigma^\varepsilon(\mathbb{R}^p) \times OS_r(\mathbb{R}^q)$ (ε being determined by J_1) and

$$G/G^\tau = \{(x,J), x \in M, J \in \Sigma^\varepsilon(x) \times OS_r(x^\perp)\}. \tag{21}$$

Now let us compute G_0 according to (11): $h = (A,C) \in H$ is in G_0 if and only if $\mathrm{Ad}_{\mathfrak{m}}\tau(h) = \mathrm{Ad}_{\mathfrak{m}}h$ i.e.: if q is odd, $\tau(h) = h$, and $G_0 = G^\tau \cap H = G^\tau$; if q is even, $\tau(h) = \pm h$ (and $G_0 = \pi_K^{-1}(G_0')$ with $G_0' = {G'}^{\tau'} \cap H'$), i.e. $h \in G^\tau$ or $\tau(h) = -h$. The existence of solutions of this last equation depends on p, q and r (we remark that if h_1 is a solution then the set of solutions is $h_1 G^\tau$). One finds that the equation $\tau(h) = -h$ (q is even) has a solution in G^σ if and only if $\dim E_1 = \dim E_2 = q/2$ and that this solution is in H if $p/2$ is even and in $O^-(p) \times O^-(q)$ (the other component of G^σ) if $p/2$ is odd. Hence, if p is divisible by 4, q is even and $r = q/2$ (i.e. $J_0 \in \chi(\Sigma(\mathbb{R}^p) \times OS_{q/2}(\mathbb{R}^q))$), we have $G_0 = G^\tau \bigsqcup h_1 G^\tau$. In all the other cases we have $G_0 = G^\tau$.
In conclusion, let us denote by $N^L(r,\varepsilon) := N(J_0)$ (resp. $N^R(r,\varepsilon)$) the maximal 4-symmetric bundle over M corresponding to $J_0 \in \chi(\Sigma^\varepsilon(\mathbb{R}^p) \times OS_r(\mathbb{R}^q))$ (resp. $\chi(OS_r(\mathbb{R}^p) \times \Sigma^\varepsilon(\mathbb{R}^q))$. Then:

if p is not divisible by 4 or q is odd, $N^\alpha(r,\varepsilon)$ is given by (21), for all (α, r, ε),

if p is divisible by 4, q even not divisible by 4 then for $(\alpha, r) \neq (L, q/2)$, $N^\alpha(r,\varepsilon)$ is given by (21) and for $(\alpha, r) = (L, q/2)$ it is given by (22), below,

if p and q are divisible by 4, then for $(\alpha, r) \in \{(L, q/2), (R, p/2)\}$, $N^\alpha(r,\varepsilon)$ is given by (22), and for the other choices it is given by (21),

$$\begin{array}{rcl} N^L(r,\varepsilon) & = & \{(x,J), x\in M, J\in P(\Sigma^\varepsilon(x)\times OS_r(x^\perp))\} \\ N^R(r,\varepsilon) & = & \{(x,J), x\in M, J\in P(OS_r(x)\times \Sigma^\varepsilon(x^\perp))\} \end{array} \quad (22)$$

where $P(\Sigma^\varepsilon(x)\times OS_r(x^\perp)) = \Sigma^\varepsilon(x)\times OS_r(x^\perp)/\{\pm\mathrm{Id}\}$. In the cases described by (22), G/G^τ is not a submanifold of $\Sigma(M)$.

5.3 Complex Grassmannian

Let us consider $M = SU(p+q)/S(U(p)\times U(q)) = Gr_{p,\mathbb{C}}(\mathbb{C}^{p+q})$. We have $\dim M = 2pq$ and the following setting

$G = SU(p+q)$, $H = S(U(p)\times U(q))$; $\sigma = \mathrm{Int}(\mathrm{diag}(\mathrm{Id}_p, -\mathrm{Id}_q))$ and $G^\sigma = H = (G^\sigma)^0$.

Besides $\mathfrak{h} = \mathfrak{s}(\mathfrak{u}(p)\oplus\mathfrak{u}(q))$ and $\mathfrak{m} = \left\{\begin{pmatrix} 0 & B \\ -B^* & 0\end{pmatrix}, B\in\mathfrak{gl}_{p,q}(\mathbb{C})\right\} = i_\mathfrak{m}(\mathfrak{gl}_{p,q}(\mathbb{C}))$. Let us compute $\mathrm{Ad}_\mathfrak{m}$. For $h = \mathrm{diag}(A,C)$ and $\xi = i_\mathfrak{m}(B)$, we have:

$$\mathrm{Ad}_\mathfrak{m} h(\xi) = i_\mathfrak{m}(ABC^{-1}).$$

Under the identification $i_\mathfrak{m}$ we have $\mathrm{Ad}_\mathfrak{m}(A,C) = L_A R_{C^{-1}} = \chi(A,C)$, by introducing the morphism $\chi\colon (A,C)\in GL_p(\mathbb{C})\times GL_q(\mathbb{C}) \mapsto L(A)R(C^{-1}) \in GL(\mathfrak{gl}_{p,q}(\mathbb{C}))$[19]. Hence $K = \ker\mathrm{Ad}_\mathfrak{m} = \{(\lambda\mathrm{Id}_p,\lambda\mathrm{Id}_q), \lambda\in\mathbb{C}, \lambda^{p+q}=1\} = \hat{U}_{p+q}\mathrm{Id} \simeq \mathbb{Z}_{p+q}$ (with $\hat{U}_{p+q} = \exp(\frac{2i\pi}{p+q}\mathbb{Z})$). Thus $G' = G/K = PSU(p+q)$ and $H' = S(U(p)\times U(q))/\hat{U}_{p+q} \simeq S(U(p)\times U(q))$. The connected isometry group is the unitary group of M: $\mathrm{Is}(M)^0 = U(M) = G' = PSU(p+q)$.
It is well known that the group of automorphisms of $SU(p+q)$ has two components (the $\mathbb{C}$-linear one and the anti-$\mathbb{C}$-linear one) and is generated by the inner automorphisms and the complex conjugation: $g\in SU(p+q)\mapsto \bar{g}\in SU(p+q)$. In particular, $\mathrm{Aut}(\mathfrak{m}) = \mathrm{Ad}_\mathfrak{m} H \rtimes \{\mathrm{Id}, c\} = \chi(S(U(p)\times U(q))\cdot\{(\mathrm{Id},\mathrm{Id}),(b_p,b_q)\})$ with $c = L(b_p)R(b_q^{-1}) : B\in\mathfrak{gl}_{p,q}(\mathbb{C})\mapsto \bar{B}\in\mathfrak{gl}_{p,q}(\mathbb{C})$, $b_n\colon v\in\mathbb{C}^n\mapsto\bar{v}\in\mathbb{C}^n$.
The complex structure in $\mathfrak{m} = \mathfrak{gl}_{p,q}(\mathbb{C})$ is defined by $L(I_p) = R(I_q)$ where $I_n = i\mathrm{Id}_n$ is the canonical complex structure in $\mathbb{C}^n$, and the two connected components of $\mathrm{Aut}(\mathfrak{m})$ are respectively the elements in $\mathrm{Aut}(\mathfrak{m})$ which commute and those which anticommute with this complex structure.
Moreover, $J_0 = L(J_1)R(J_2^{-1}) \in \mathrm{Aut}(\mathfrak{m})^0 = \mathrm{Ad}_\mathfrak{m} H$ is in $\Sigma(\mathfrak{m})$ if and only if $(J_1^2, J_2^2)\in(-\mathrm{Id}_p,\mathrm{Id}_q)U(1)$. Then let us set $\Sigma_\lambda = \{(J_1,J_2)\in U(p)\times U(q) \mid (J_1^2,J_2^2) = \lambda(-\mathrm{Id}_p,\mathrm{Id}_q)\}$. Then we have $\chi(\Sigma_\lambda) = \chi(\Sigma_0)$ for all $\lambda\in U(1)$ since $\Sigma_\lambda = \lambda^{\frac{1}{2}}\Sigma_0$ with $\lambda^{\frac{1}{2}}$ a root of λ. Thus according to the following lemma, $\mathrm{Aut}(\mathfrak{m})^0\cap\Sigma(\mathfrak{m})$ has $(p+1)(q+1)$ connected components, which are $\mathrm{Ad}_\mathfrak{m} H$-orbits and correspond to the fibres of different maximal 4-symmetric bundles over M.

Lemma 1 *Let $J\in U(n)$, then $J^2 = -\mathrm{Id}$ (resp. $J^2 = \mathrm{Id}$) if and only if there exists $h\in U(n)$ such that $hJh^{-1} = \mathrm{diag}(i\mathrm{Id}_l, -i\mathrm{Id}_{n-l})$ for some $l\in\{0,\dots,n\}$ (resp. $hJh^{-1} = \mathrm{diag}(\mathrm{Id}_r, -\mathrm{Id}_{n-r})$ for some $r\in\{0,\dots,n\}$).*

[19]For the following it useful to keep in mind that we have $\mathrm{Ad}_\mathfrak{m} H = \chi(S(U(p)\times U(q))) = \chi(U(p)\times U(q))$ and $\ker\chi = \mathbb{C}^*\mathrm{Id}$.

Then the order four automorphism corresponding to J_0 is $\tau = \mathrm{Int}(\mathrm{diag}(J_1, J_2))$, with[20] $J_1 \in \mathrm{Ad}U(p)(i\mathrm{I}_{l,p-l}) \cong iGr_{l,\mathbb{C}}(\mathbb{C}^p)$, $J_2 \in \mathrm{Ad}U(q)(\mathrm{I}_{r,q-r}) \cong Gr_{r,\mathbb{C}}(\mathbb{C}^q)$. Hence $G^\tau = S(U(l) \times U(p-l) \times U(r) \times U(q-r))$; the fibre of the 4-symmetric space G/G^τ is $H/G^\tau = Gr_{l,\mathbb{C}}(\mathbb{C}^p) \times Gr_{r,\mathbb{C}}(\mathbb{C}^q)$, and

$$G/G^\tau = \{(x, J), x \in Gr_{p,\mathbb{C}}(\mathbb{C}^{p+q}), J \in Gr_{l,\mathbb{C}}(x) \times Gr_{r,\mathbb{C}}(x^\perp)\}. \qquad (23)$$

Further, G_0 is defined by: $\mathrm{Ad}_{\mathfrak{m}}\tau(h) = h, h \in H$, i.e. $(J_1 A J_1^{-1}, J_2 C J_2^{-1}) = \lambda(A, C)$ for some $\lambda \in K$. But it is easy to see that we must have $\lambda^2 = 1$ and thus $\tau(h) = \pm h$. One finds that $\tau(h) = -h$ has solutions if and only if p, q are even and $l = p/2, r = q/2$. Finally, in the $\mathbb{C}$-linear case, the maximal 4-symmetric bundle $N = G/G_0$ is given by

$$G/G_0 = \{(x, J), x \in Gr_{p,\mathbb{C}}(\mathbb{C}^{p+q}), J \in Gr_{l,\mathbb{C}}(x) \times Gr_{r,\mathbb{C}}(x^\perp)/\mathbb{Z}_2\} \qquad (24)$$

if p, q are even and $l = p/2, r = q/2$, and by (23) in all the other cases.
In the antilinear case, $J_0 = L(J_1)R(J_2^{-1}) \in \mathrm{Aut}(\mathfrak{m})^0.c$, with $(J_1, J_2) = (J_1' b_p, J_2' b_q)$, is in $\Sigma(\mathfrak{m})$ if and only if $(J_1^2, J_2^2) = (J_1'\overline{J_1'}, J_2'\overline{J_2'}) \in (-\mathrm{Id}_p, \mathrm{Id}_q).U(1)$. It is easy to see that we can only have

$$(J_1^2, J_2^2) = \pm(-\mathrm{Id}_p, \mathrm{Id}_q). \qquad (25)$$

Hence according to the following lemma:
– if p, q are odd then $\Sigma(\mathfrak{m}) \cap (\mathrm{Aut}(\mathfrak{m})^0.c) = \varnothing$,
– if p, q are even then the two signs $\pm$ are realized in (25) and thus $\Sigma(\mathfrak{m}) \cap (\mathrm{Aut}(\mathfrak{m})^0.c)$ has 2 connected components,
– if p, q have opposite parities, then only one sign is realized in (25) and $\Sigma(\mathfrak{m}) \cap (\mathrm{Aut}(\mathfrak{m})^0.c)$ has one component.

Lemma 2 *Let $E \subset \mathbb{C}^n$ be a Lagrangian n-plan, i.e. $E \overset{\perp}{\oplus} iE = \mathbb{C}^n$ and let b_E be the associated conjugation: $v + iw \mapsto v - iw$ for $v, w \in E$. Then $U(n).b_E = b_E.U(n)$ does not depend on E and is the set of anti-$\mathbb{C}$-linear isometries in $\mathbb{C}^n$ (the elements in $O(\mathbb{R}^{2n})$ which anticommute with the complex structure $I = i\mathrm{Id}$). Moreover for any J in this set there exists a Lagrangian n-plane E such that $J = J_E.b_E = b_E.J_E$ with $J_E \in O(E)$. Besides $J \in \Sigma(\mathbb{R}^{2n})$ (resp. $OS(\mathbb{R}^{2n})$) if and only if $J_E \in \Sigma(E)$ (resp. $OS(E)$). In particular $\Sigma(\mathbb{R}^{2n}) \cap (U(n).b_E) \neq \varnothing$ only if n is even, moreover $\Sigma(\mathbb{R}^{2n}) \cap (U(n).b_E) \subset \Sigma^+(\mathbb{R}^{2n})$. Then given any $J_1 \in \Sigma(\mathbb{R}^n)$ (resp. $OS(\mathbb{R}^n)$) there exists $h \in U(n)$ such that $h.E = \mathbb{R}^n$, $hJ_Eh^{-1} = J_1$ and thus $hJh^{-1} = J_1.b_{\mathbb{R}^n}$.*

Then the order four automorphism corresponding to J_0 is $\tau = \mathrm{Int}(\mathrm{diag}(J_1, J_2))$ with $J_1 \in \mathrm{Ad}U(p)(J_{\frac{p}{2}}.b_p)$, $J_2 \in \mathrm{Ad}U(q)(b_q)$ and $J_{\frac{p}{2}} = \begin{pmatrix} 0 & \mathrm{Id}_{\frac{p}{2}} \\ -\mathrm{Id}_{\frac{p}{2}} & 0 \end{pmatrix}$. In other words J_1 is any complex structure in $\mathbb{R}^{2p}$ anticommuting with I_p and J_2 is any orthogonal conjugation in $\mathbb{C}^q$. Hence, we have $G^\tau = Sp(p/2) \times SO(q)$. Hence

[20] $\mathrm{I}_{l,p-l} = \mathrm{diag}(\mathrm{Id}_l, -\mathrm{Id}_{p-l})$

$U(p) \times U(q)/G^\tau = \Sigma^+(\mathbb{C}^p)_- \times \mathrm{Lag}(\mathbb{C}^q)$ where $\Sigma^+(\mathbb{C}^p)_- = \Sigma(\mathbb{R}^{2p}) \cap \mathrm{Ant}(I_p)$ are the complex structures in $\mathbb{R}^{2p}$ anticommuting with I_p and $\mathrm{Lag}(\mathbb{C}^q)$ are the oriented Lagrangian planes in $\mathbb{C}^q$. Thus we have:

$$H/G^\tau = S(\Sigma^+(\mathbb{C}^p)_- \times \mathrm{Lag}(\mathbb{C}^q)) := \{(J,P) \in \Sigma^+(\mathbb{C}^p)_- \times \mathrm{Lag}(\mathbb{C}^q) \mid \det_{\mathbb{C}}(J)\det_{\mathbb{C}}(P) = 1\}.$$

It is easy to define $\det_{\mathbb{C}}$ on $\mathrm{Lag}(\mathbb{C}^q)$; and for $\Sigma^+(\mathbb{C}^p)_-$, we set $\det_{\mathbb{C}}(J) = \det_{\mathbb{C}}(E)$ for E any Lagrangian n-plane invariant by J (definition independent on the choice of E). Then

$$G/G^\tau = \{(x,J,P), x \in Gr_{p,\mathbb{C}}(\mathbb{C}^{p+q}), (J,P) \in \Sigma^+(x)_- \times \mathrm{Lag}(x^\perp)\}.$$

Let us compute G/G_0. We have to solve for $(A,C) \in U(p)\times U(q)$: $(J_{\frac{p}{2}}\bar{A}J_{\frac{p}{2}}^{-1}, \bar{C}) = \lambda(A,C)$ for $\lambda \in U(1)$ whose the solutions are $\pm\lambda^{\frac{1}{2}}(Sp(p/2) \times O(q))$. Hence we have $G_0' = G_0/K = \chi(U(1)(Sp(p/2) \times O(q))) = \chi(Sp(p/2) \times O(q)) =$

$$\begin{cases} \chi(Sp(p/2) \times SO(q)) & \text{if } q \text{ is odd} \\ \chi(G^\tau) \bigsqcup h_1\chi(G^\tau) & \text{if } q \text{ is even.} \end{cases}$$

Then $G'/G_0' = G/G_0 = U(p+q)/(U(1)(Sp(p/2)\times O(q))) = PSU(p+q)/P(Sp(p/2)\times O(q))$ hence $N = G/G_0$ is equal to $(G/G^\tau)/\mathbb{Z}_{p+q}$ if q is odd and to $(G/G^\tau)/\mathbb{Z}_{2(p+q)}$ if q is even.

6 Appendix

Theorem 20 *Let G be a connected Lie group with an involution σ. If $\mathrm{Ad}_{\mathfrak{m}}(G^\sigma)^0$ is compact (resp. relatively compact) then $\mathrm{Ad}_{\mathfrak{m}}H$ is compact (resp. relatively compact) for any H such that $(G^\sigma)^0 \subset H \subset G^\sigma$.*

Proof. According to [1] (lemma 2.7), $(G^\sigma)/(G^\sigma)^0$ is finite hence $H/(G^\sigma)^0$ is finite and the theorem follows. ■

Corollary 3 *We give ourself the same setting and notations as in remark 7. If $\tilde{H} = (\tilde{G}^{\tilde{\sigma}})^0$ satisfies: $\mathrm{Ad}_{\mathfrak{m}}\tilde{H}$ is compact (resp. relatively compact), then for any symmetric pair (G,H), $\mathrm{Ad}_{\mathfrak{m}}H$ is compact (resp. relatively compact). In other words if one symmetric pair (associated to $(\mathfrak{g},\sigma)$) is Riemannian then all the others are also.*

Proof. Since $\tilde{G}$ is simply connected, it is the universal covering of G and we have a covering $\pi\colon \tilde{G} \to G$. Then $\mathrm{Ad}_{\mathfrak{m}}\tilde{H} = \mathrm{Ad}_{\mathfrak{m}}H^0$ (there are connected with the same Lie algebra) hence $\mathrm{Ad}_{\mathfrak{m}}H^0$ is compact and then according to the previous theorem, $\mathrm{Ad}_{\mathfrak{m}}H$ is compact. ■

Corollary 4 *Let (G,H) be a symmetric pair with involution σ and $\tau\colon G \to G$ an order four automorphism such that $\tau^2 = \sigma$. Then if $\mathrm{Ad}_{\mathfrak{m}}H$ is compact (resp. relatively compact) then the subgroup generated by $\mathrm{Ad}_{\mathfrak{m}}H$ and $\tau_{|\mathfrak{m}}$, $\mathrm{Gr}(\mathrm{Ad}_{\mathfrak{m}}H, \tau_{|\mathfrak{m}})$ is compact (resp. relatively compact).*

Proof. We have $\tau_{|\mathfrak{m}}(\mathrm{Ad}_{\mathfrak{m}}G^\sigma)\tau_{|\mathfrak{m}}^{-1} = \mathrm{Ad}_{\mathfrak{m}}\tau(G^\sigma) = \mathrm{Ad}_{\mathfrak{m}}G^\sigma$. Hence $\mathrm{Gr}(\mathrm{Ad}_{\mathfrak{m}}G^\sigma, \tau_{|\mathfrak{m}}) = (\mathrm{Ad}_{\mathfrak{m}}G^\sigma)\mathrm{Gr}(\tau_{|\mathfrak{m}})$ which is (relatively) compact because so is $\mathrm{Ad}_{\mathfrak{m}}G^\sigma$, according to theorem 20, and then $\mathrm{Gr}(\mathrm{Ad}_{\mathfrak{m}}G^\sigma, \tau_{|\mathfrak{m}})$ is (relatively) compact because since $\mathrm{Gr}(\mathrm{Ad}_{\mathfrak{m}}H, \tau_{|\mathfrak{m}}) \supset (\mathrm{Ad}_{\mathfrak{m}}H)\mathrm{Gr}(\tau_{|\mathfrak{m}})$ then $\mathrm{Ad}_{\mathfrak{m}}G^\sigma/\mathrm{Ad}_{\mathfrak{m}}H$ is a covering of $\mathrm{Gr}(\mathrm{Ad}_{\mathfrak{m}}G^\sigma, \tau_{|\mathfrak{m}})/\mathrm{Gr}(\mathrm{Ad}_{\mathfrak{m}}H, \tau_{|\mathfrak{m}})$ which is consequently finite. ∎

Theorem 21 *Let (G, H) be a symmetric pair with involution $\sigma\colon G \to G$ and $\tau\colon \mathfrak{g} \to \mathfrak{g}$ an order four automorphism such that $\tau^2 = \sigma$. Then if $\mathrm{Ad}_{\mathfrak{m}}H$ is relatively compact then the subgroup generated by $\mathrm{Ad}_{\mathfrak{m}}H$ and $\tau_{|\mathfrak{m}}$, $\mathrm{Gr}(\mathrm{Ad}_{\mathfrak{m}}H, \tau_{|\mathfrak{m}})$ is relatively compact.*

Proof. Let $G' = \mathrm{Ad}G$, then $C := \ker \mathrm{Ad} =$ center of G and we can identify Ad to the covering $\pi\colon G \to G/C$ and G' to G/C. The automorphism σ gives rise to $\sigma'\colon G' \to G'$ such that $\sigma' \circ \pi = \pi \circ \sigma$. Besides the automorphism τ integrates in G' into τ' defined by $\tau' = \mathrm{Int}\tau\colon \mathrm{Ad}g \in G' \mapsto \tau \circ \mathrm{Ad}g \circ \tau^{-1}$ and we have $\tau' \circ \pi = \pi \circ \tau$ and $\tau'^2 = \sigma'$. Then according to corollary 4, $\mathrm{Gr}(\mathrm{Ad}_{\mathfrak{m}}G'^{\sigma'}, \tau_{|\mathfrak{m}})$ is relatively compact since according to corollary 3, $\mathrm{Ad}_{\mathfrak{m}}G'^{\sigma'}$ is relatively compact because $\mathrm{Ad}_{\mathfrak{m}}H$ is so. Moreover we have $G'^{\sigma'} \supset \pi(G^\sigma)$ then (since $\mathrm{Ad}\pi(g) = \mathrm{Ad}g\ \forall g \in G$) $\mathrm{Ad}_{\mathfrak{m}}G'^{\sigma'} \supset \mathrm{Ad}_{\mathfrak{m}}G^\sigma \supset \mathrm{Ad}_{\mathfrak{m}}H$ thus $\mathrm{Gr}(\mathrm{Ad}_{\mathfrak{m}}H, \tau_{|\mathfrak{m}})$ is relatively compact. ∎

Theorem 22 *Let $(\mathfrak{g}, \sigma)$ be an orthogonal symmetric Lie algebra*[21] *such that $\mathfrak{h} = \mathfrak{g}^\sigma$ contains no ideal $\neq 0$ in $\mathfrak{g}$. Then for any symmetric pair (G, H) associated with $(\mathfrak{g}, \mathfrak{h})$, the associated symmetric space $M = G/H$ is Riemannian. Moreover let $\tilde{G}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$, $\tilde{\sigma}$ integrating σ, $\tilde{H} = (\tilde{G}^{\tilde{\sigma}})^0$ and $\tilde{C}$ the center of $\tilde{G}$. Then we have $\tilde{H} = \tilde{G}^{\tilde{\sigma}}$. Further, for any subgroup S of $\tilde{C}$ put*

$$H_S = \{g \in \tilde{G} \mid \tilde{\sigma}(g) \in g.S\}.$$

The symmetric spaces M associated with $(\mathfrak{g}, \sigma)$ (i.e. (G, H) is associated with $(\mathfrak{g}, \mathfrak{h})$) are exactly the spaces $M = G/H$ with

$$G = \tilde{G}/S \quad \text{and} \quad H = H^*/S \tag{26}$$

where S varies through all $\tilde{\sigma}$-invariant subgroups of $\tilde{C}$ and H^* varies through all $\tilde{\sigma}$-invariant subgroups of $\tilde{G}$ such that $\tilde{H}S \subset H^* \subset H_S$. Hence, all the symmetric spaces $M = G/H = \tilde{G}/H^*$ associated with $(\mathfrak{g}, \sigma)$ cover the adjoint space of $(\mathfrak{g}, \sigma)$: $M' = G'/G'^{\sigma'} = \tilde{G}/H_{\tilde{C}}$[22] *and are covered by $\tilde{M} = \tilde{G}/\tilde{H}$ (the universal covering):*

$$\tilde{M} \to M \to M'. \tag{27}$$

Besides if $\langle\cdot,\cdot\rangle$ is an $\mathrm{Ad}_{\mathfrak{m}}G'^{\sigma'}$-invariant inner product then it is invariant by $\mathrm{ad}_{\mathfrak{m}}H = \mathrm{Ad}_{\mathfrak{m}}H^*$ for any H described above, and the coverings (27) are Riemannian, when $M, \tilde{M}, M'$ are endowed with the corresponding metrics.

[21] i.e. σ is an involutive automorphism and $\mathfrak{h} = \mathfrak{g}^\sigma$ is compactly embedded in $\mathfrak{g}$ (see [10])

[22] with the notation of the proof of theorem 21. For any (G, H) symmetric pair associated with $(\mathfrak{g}, \sigma)$, we have $G' = \mathrm{Ad}G = \mathrm{Int}(\mathfrak{g})$ the group of inner automorphism of $\mathfrak{g}$ (see [10]) and σ induces an automorphism σ' of $G' = \mathrm{Int}(\mathfrak{g})$.

Proof. We have only to prove $\tilde{H} = \tilde{G}^{\tilde{\sigma}}$, which follows from [1] (lemma 2.7). All the rest is an adaptation of [10] (Ch. VII, thm 9.1) using what precedes. This completes the proof. ■

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