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We derive a cohomological formula for the analytic index of the Dirac-Ramond operator and we exhibit its modular properties.

index theory | families index | Dirac-Ramond operator | elliptic genera | string theory | modular forms

The Atiyah-Singer index theorem is important in many different areas of mathematics and is also at the heart of many problems in physics. In quantum field theory it comes into play through the Dirac operator. Both the ordinary index [1], and the families index [2] of the Dirac operator reveal features of quantum field theories. The Dirac-Ramond operator is the extension to superstring theory of the ordinary Dirac operator in field theory; it is the Dirac operator on loop space and its ordinary index [3, 4, 5, 6] is given by the string genus, while the elliptic genus of Ochanine and Landweber and Stong [7, 8] corresponds to the index of one of its twisted version. Both the string genus and the elliptic genus have been extensively studied. They have given rise to an extension of the rigidity theorems of the elliptic genus to the case of families [9] and to different elliptic cohomology theories, most notably to the theory of topological modular forms (tmf) [10]. However, the analytic families index of the Dirac-Ramond operator has not yet been formulated and in this paper we derive for it a cohomological formula, with remarkable modular properties, using methods from field theory and string theory.

Consider a family of Dirac operators parametrized by a space X. The zero modes of the Dirac operator define a virtual vector bundle Ind over X, called the index bundle. One of the outcomes of the families index theorem is a cohomological expression for the Chern character of this bundle. The first two terms in the expansion of this Chern character in characteristic classes are the dimension of the vector bundle (i.e. the ordinary index of the Dirac operator) and its first Chern class respectively. In the Dirac-Ramond case, the analytic index is not an integer but a modular (or nearly modular) function whose Fourier coefficients are integers.

For simplicity, the gist of the argument and the exposition of the methods will be presented in the case of the Dirac operator. *Mutatis mutandis* they generalize straightforwardly to the Dirac-Ramond operator. It obtains, *en passant*, a novel presentation of the cohomological Dirac families index theorem that displays its intimate relation to Berry's phase [11]. When needed we will revert to a full discussion of the Dirac-Ramond case, which is our main concern.

Background. The treatment of the Atiyah-Singer index theorem with quantum field theory techniques has become well known [12, 13, 14]. It rests on a few general principles. Consider a supersymmetric quantum system. The spectrum of its hamiltonian consists of bosonic and fermionic states. The bosonic and fermionic eigenstates of non zero energy are paired with each other by supersymmetry. The generator of supersymmetry is a Dirac-like operator which commutes with the hamiltonian and anticommutes with fermion parity. The analytical index of this Dirac operator, *i.e.* the difference between the number of its bosonic and fermionic zeros modes, is then easily computed as the supertrace of the quantum evolution operator. The usual correspondence between the hamiltonian and the lagrangian formulation of quantum mechanics gives an equality between this trace and a supersymmetric path integral that is evaluated by the stationary phase approximation which is exact and shows that the path integral localizes. This gives the expression for the topological index. In this language, twisting the Dirac operator by some vector bundle simply amounts to coupling the quantum system to additional degrees of freedom. The canonical quantization of these degrees of freedom produces the required vector bundle. In this case, the A-roof genus is simply replaced by the product of the A-roof genus with the Chern character of the vector bundle. Similarly, the equivariant cases corresponds to having additional symmetries in the supersymmetric quantum system. Any special case of the Atiyah-Singer theorem can be treated in this manner.

In their paper on the families index theorem, Atiyah and Singer [2] consider a family of elliptic operators parametrized by a compact topological space. Consider a smooth family of metrics on a riemannian spin manifold Y parametrized by a space X. The manifold Y with metric parametrized by $x \in X$ will be denoted by Y_x. The idea is to put X and Y together into a fiber bundle $Z \to X$ where at each point $x \in X$ the fiber over x is a manifold isomorphic to Y with metric $g_Y(x,\cdot)$. It is well known that the index of $\not{D}^{Y_x}$ is independent of the metric $g_Y(x,\cdot)$. However the Dirac operator $\not{D}^{Y_x}$ and its zero modes change with the metric. Following Grothendieck, Atiyah and Singer asked how the space of zero modes changes as x varies over X. At each x, the space of zero modes is a finite dimensional vector space. Roughly speaking, this means that the space of zero modes of $\not{D}^{Y_x}$ is a finite dimensional vector bundle over X. This is not quite correct because the dimensionality of the vector space will jump if the number of solutions to the equation $\not{D}^{Y_x}\psi = 0$ changes with x. Only the index of the operator is protected from these jumps. If $\mathcal{Z}_{\pm}^{Y_x}$ are the vector spaces of respectively positive and negative chirality solutions to the Dirac equation $\not{D}^{Y_x}\psi = 0$ with metric $g_Y(x,\cdot)$ then $\mathrm{ind}(\not{D}^{Y_x}) = \dim \mathcal{Z}_+^{Y_x} - \dim \mathcal{Z}_-^{Y_x}$ is independent of x. Atiyah and Singer show that the virtual vector spaces $\mathcal{Z}_+^{Y_x} \ominus \mathcal{Z}_-^{Y_x}$ can be put together over X to make a virtual vector bundle $\mathrm{Ind}(\not{D}^{Y})$ over X, *i.e.* an element of $K-$theory, called the index bundle. Because it is well suited to the methods of quantum field theory and the study of the string genus, we consider here a more restrictive family given by a riemannian submersion. A riemannian submersion is a family of metrics with a special relationship between the geometries of Z and X. Pick a point $z \in Z$ that projects to $x \in X$. At z there is an orthogonal decomposition of the tangent space $T_zZ = H_z \oplus V_z$. Here $V_z \subset T_zZ$ is the "vertical subspace" consisting of vector that are parallel to the fiber. The "horizontal subspace" H_z is the orthogonal complement of V_z. A vector $v \in T_xX$ has a unique horizontal lift to

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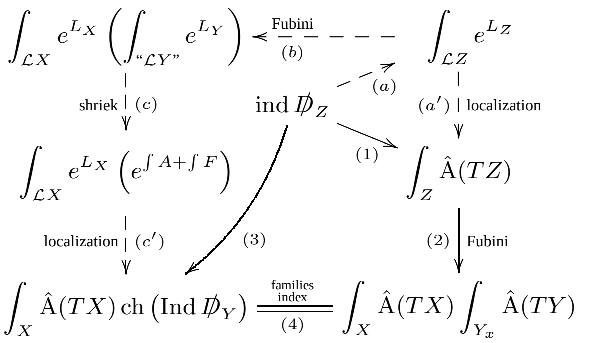
a vector $\tilde{v} \in H_z$. The condition for a riemannian submersion is that $\|v\|_X = \|\tilde{v}\|_Z$ for every $z \in \pi^{-1}(x)$. Note that the restriction of the metric on Z to a vertical subspaces gives a metric on $Z_x \approx Y_x$. If x^i are local coordinates on the base X and if y^a are local coordinates on the fiber Y then (x, y) are local coordinates on Z. The fibers are the submanifolds with x fixed. The metric of a submersion is locally of the "Kaluza-Klein" form

$$ds_Z^2 = g_{ij}(x)dx^i\,dx^i + g_{ab}(x,y)\left(dy^a + C^a{}_i(x,y)dx^i\right)\left(dy^b + C^b{}_j(x,y)dx^j\right) . \quad [1]$$

The metric above leads to a supersymmetric lagrangian on Z with the following schematic form $L_Z(x, y) = L_X(x) + L_Y(x, y)$ where L_X is the pullback of the supersymmetric lagrangian on the base and L_Y is roughly the lagrangian on Y depending parametrically on x.

Outline of the argument. Since the original paper of Atiyah and Singer there have been a number of different proofs of the families index theorem. In what follows we present yet another derivation which has the remarkable feature that its generalization to loop space obtains for the first time a families index theorem for the Dirac-Ramond operator. The multiplicative property of the index [2] is central to our argument. It states that for a submersion $Z \to X$, $\operatorname{ind} \not{D}_Z$ is the index of the Dirac operator on X twisted by the index bundle of $\not{D}_Y$. In our setup this multiplicative property is a reflection of Fubini's integration theorem. It is important to keep in mind that our approach is geometrical rather than topological. This leads to de Rham cohomology represented by differential forms and thus, in our final formula, we lose all torsion phenomena that might be of interest. From now on the expression "index theorem" (and variants thereof) is shorthand for "cohomological form of the index theorem" (and respective variants).

The following diagram displays clearly the architecture of our arguments:



[∗]

The dashed arrows refer to path integral operations while the solid arrows indicate procedures in the hamiltonian or operator description. To prove the families index theorem one usually performs the following steps in a straightforward manner.

(1) The Atiyah-Singer index theorem tells us that on Z $\operatorname{ind} \not{D}_Z = \int_Z \hat{A}(TZ)$.

(2) Any connection can be used to compute the integral but the computation simplifies when the family is described by a riemannian submersion. Using $T_z Z = H_z \oplus V_z$, the reduced structure group and the special connection [15], we see that $\hat{A}(T_z Z) = \hat{A}(H_z) \wedge \hat{A}(V_z)$. Moreover the properties of the submersion connection allows us to identify $\hat{A}(H_z)$ with $\hat{A}(T_x X)$ which is intrinsically defined on the base X. This implies by Fubini's theorem that

$$\operatorname{ind}(\not{D}_Z) = \int_X \hat{A}(TX) \wedge \int_{Y_x} \hat{A}\left(T_{(x,\cdot)}Y\right) . \quad [2]$$

(3) There is a second way of determining $\operatorname{ind} \not{D}_Z$. Using our riemannian submersion geometry on Z we can show that solving for the zero modes of $\not{D}_Z$ is the same as the following procedure. First determine the zero modes of $\not{D}_Y$ at fixed $x \in X$. Next solve a modified Dirac equation on X. The corresponding Dirac operator is coupled to a vector bundle with a connection constructed with the zero modes of $\not{D}_Y$. The Atiyah-Singer index theorem tells you that the index of this operator is given by the integral in the lower left hand corner.

(4) The cohomological formula for the families index theorem of Atiyah and Singer follows from the identification of these two ways of computing $\operatorname{ind} \not{D}_Z$.

This procedure is, *mutatis mutandis*, the one used by Bismut [16] and Bismut and Freed [17]. However, as we do not know how to generalize step (3) in the Dirac-Ramond case, we adopt a different strategy that bypasses step (3). This new proof relies on the path integral computations associated with the dashed arrows of the diagram. Every step now has a natural extension to the loop space, *i.e.*, string theory.

We begin with $\operatorname{ind} \not{D}_Z$ at the center of the diagram. The index is as usual given by the supertrace $\operatorname{Tr}(-1)^F$ in an appropriate field theory where $(-1)^F$ is fermionic parity.

(a) $\operatorname{Tr}(-1)^F = \operatorname{ind} \not{D}_Z$ is given by a supersymmetric path integral over the loop space $\mathcal{L}Z$ with appropriate boundary conditions [12, 13, 14]. This is a direct result of the equivalence between the hamiltonian and the lagrangian formulations of quantum mechanics.

(a') The path integral calculation localizes on the "constant loops" in $\mathcal{L}Z$, *i.e.*, the manifold Z. Steps (a) and (a') are equivalent to step (1) above and constitute the standard path integral derivation of the Atiyah-Singer index formula and we can then continue using Fubini's theorem (arrow (2)).

(b) As a result of the submersion geometry, see **[1]**, the lagrangian L_Z splits naturally into two pieces and one of them depends only on the base X. The Fubini theorem can be used to factorize the path integral into two factors. Here "$\mathcal{L}Y$" is the inverse image of the projection $\pi : Z \to X$ of a loop in X.

(c) This is the main and the only delicate step needed to derive the cohomological formula for the families index in the Dirac-Ramond case and constitutes one of the main results of this paper. The path integral over the fiber is the reflection in the path integral of the shriek map in K-Theory [2]. To compute the Y path integral we notice that it satisfies a time dependent super-Schrödinger equation. We then prove a new theorem in supersymmetric quantum mechanics that shows that this path integral over the fiber is exactly given by the supersymmetric parallel transport term in between the parentheses. The result is the standard path integral expression of the index for the Dirac operator coupled to a bundle with connection A and curvature F [13]. By construction this bundle is the index bundle. This is the argument that allows us to bypass step (3).

(c') The path integral calculation localizes on "constant loops" in X, *i.e.*, the manifold X, and the result of the computation is the A-roof genus times the Chern character of the $\operatorname{Ind} \not{D}_Y$. This step is standard and well known.

(4) *cf. ibidem*

We can now justify the approximations we will be performing. Our starting point for the derivation of the families index theorem is the computation of the index of the Dirac operator on the riemannian submersion Z. The index is an integer and therefore cannot change as we deform the space Z. If we fix a loop in the base X, the Y path integral (in step (c)) is $\operatorname{Tr}(-1)^F U_Y(T, 0)$ where $U_Y(t, \tau)$ is the time evolution operator on Y and F is fermion number. The time development is obtained from a study of the super-Schrödinger equation and is summarized in **[13]**. We will show that in the computation of the supertrace there is an exact cancellation and the only contribution comes from zero modes. Since the $\operatorname{ind} Z$ is an integer we can go to a parameter regime where the adiabatic approximation is valid and in this way we clarify the contribution of the zero modes. In the adiabatic approximation, the travel time T around a loop is taken to be very large, and the riemannian submersion metric is blown up in such a way that the evolution is very slow as one goes from $t = 0$ to $t = T$. The contribution from the zero modes in the adiabatic approximation is given by super-parallel transport in the index bundle around the loop in X. Our remarks imply that this is an exact result.

Details of step (c). The proof of step (c) in diagram **[$*$]** requires a discussion of the definition of the super-heat kernel in supersymmetric quantum mechanics. The standard framework is the following. On a $(1|1)$ super-manifold with coordinates (t, τ), where τ is a Grassmann variable, the supersymmetry transformation acts as $t \to t + i\epsilon\tau$ and $\tau \to \tau + \epsilon$. The generator of supersymmetry is $Q = \partial_\tau + i\tau\partial_t$ with $Q^2 = i\partial_t$. The superderivative is $D = \partial_\tau - i\tau\partial_t$ with $D^2 = -i\partial_t$ and anti-commutator $\{D, Q\} = 0$. After quantization Q becomes an operator on the Hilbert space and we will interpret Q this way from now on. The fundamental solution of the super-Schrödinger equation

$$D\Phi(t,\tau) = Q\Phi(t,\tau) \qquad \textbf{[3]}$$

is the super-heat kernel. This equation was introduced in [13] to study the index of the Dirac operator in an intrinsically supersymmetric covariant manner. We are interested in a generalization of the above that is analogous to going from a time independent hamiltonian to a time dependent one. We are interested in solving **[3]** where we have (t, τ) dependence, *i.e.*, $Q(t, \tau) = Q_0(t) + \tau Q_1(t)$. Note that in this case $\{D, Q\} \neq 0$. The fundamental solution to the super-Schrödinger equation with initial value $U_Y(0, 0) = I$ is the super-heat kernel $U_Y(t, \tau)$. The equivalence of the operator and the path integral formulations of quantum mechanics tells us that with supersymmetric boundary conditions we have

$$\operatorname{Tr}(-1)^F U_Y(T,0) = \int_{\pi^{-1}(\gamma)} e^{L_Y}\,. \qquad \textbf{[4]}$$

Here $\pi^{-1}(\gamma)$ is the inverse image under $\pi : Z \to X$ of a superloop γ on X. One of our key results is that the left hand side of the equation above is exactly given by the super-holonomy on the index bundle around the superloop γ, the generalization of the heat kernel expression for the index to the families case. We can expand the wavefunction as

$$\Phi(t,\tau) = \sum_n \phi_n(t,\tau) b_n(t,\tau)\,, \qquad \textbf{[5]}$$

where $\{\phi_n\}$ will be taken to be a complete orthonormal basis with $Q(t,\tau)\phi_n(t,\tau) = \lambda_n(t)\phi_n(t,\tau)$. The eigenfunctions of Q can be constructed if we know the eigenfunctions of Q_0. Let ϕ_0 be an eigenfunction of Q_0, $Q_0\phi_0 = \lambda\phi_0$ then $\phi = \phi_0 - (Q_0 - \lambda)^{-1}\tau Q_1\phi_0$ is an eigenfunction of Q with eigenvalue λ. The resolvent is defined to vanish on $\ker(Q_0 - \lambda)$. Inserting **[5]** into **[3]** and taking the inner product with ϕ_m gives the exact equation

$$Db_m + \sum_n \big(\phi_m, D\phi_n\big) b_n = \lambda_m b_m\,. \qquad \textbf{[6]}$$

Notice that the super-Schrödinger equation **[3]** is very similar to the equation which defines super-parallel transport. Assume we have a $(1|1)$ superparticle moving on a manifold M where there is a non-abelian connection A. The motion of the particle is described by the superfield $X^\mu(t,\tau) = x^\mu(t) + i\tau\xi^\mu(t)$. The manifold X is not to be confused with the superfield $X(t,\tau)$ that describes a superloop γ on X. The super-parallel transport equation,

$$D\Phi(t,\tau) + A_\mu(X) DX^\mu\ \Phi(t,\tau) = 0m \qquad \textbf{[7]}$$

is a multi-component super-Schrödinger equation with $Q(t,\tau) = -A_\mu(X)DX^\mu$. If we split the super-parallel transport into bosonic and fermionic components, with $\Phi(t,\tau) = \Phi_b(t) + i\tau\Phi_f(t)$, we obtain

$$\dot\Phi_b + \left(A_\mu \dot x^\mu - \tfrac{i}{2}F_{\mu\nu}\xi^\mu\xi^\nu\right)\Phi_b = 0\,, \qquad \textbf{[8]}$$

$$\Phi_f + A_\mu\xi^\mu\Phi_b = 0\,, \qquad \textbf{[9]}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Super-parallel transport is ordinary parallel transport with an extra "rotation" given by a Pauli $\vec\sigma \cdot \vec B$ type coupling. Note that $\Phi(t,\tau) = (1 - i\tau A_\mu(x)\xi^\mu)\,\Phi_b(t)$ and thus $\Phi(t, \tau = 0) = \Phi_b(t)$. We use this later when we apply the same methodology to **[6]**.

The key point is that we only have to compute the supertrace of $U_Y(t, 0))$ and not the full operator. The term $(\phi_m, D\phi_n)$ in **[6]** gives a supersymmetric Berry-Simon connection [18]. In the type of systems we are studying, Q depends on (t, τ) implicitly through a bosonic superfield $X(t,\tau)$, with $\tau X(t,\tau) = X(t,\tau)\tau$ which is apparent from the form of the lagrangian L_Z. We have $\phi_n\big(X(t,\tau)\big)$ and $D\phi_n = (\partial\phi_n/\partial x^\mu)(X)\ DX^\mu$. With this in mind we conclude that $(\phi_m, D\phi_n) = A_\mu^{mn}(X)\ DX^\mu$ where $A_\mu^{mn}(X) = (\phi_m(X), (\partial\phi_n/\partial x^\mu)(X))$. Thus equation **[6]** may be written as

$$Db_m + \sum_n A_\mu^{mn}(X)\, DX^\mu\, b_n = \lambda_m b_m\,. \qquad \textbf{[10]}$$

Let Λ be the matrix with the λ_n on the diagonal. Using **[8]** and **[9]** we find $b(t,\tau) = \big[1 - \tau\big(A_\nu(x(t))\xi^\nu(t) + i\Lambda(t)\big)\big]\, b^b(t)$ where the bosonic component b^b of $b(t,\tau) = b^b(t) + i\tau b^f(t)$ satisfies

$$\dot b^b + \big\{i\Lambda^2 + \big(A_\mu\dot x^\mu - \tfrac{i}{2}F_{\mu\nu}\xi^\mu\xi^\nu\big) + [A_\mu, \Lambda]\,\xi^\mu\big)\big\}b^b = 0\,, \qquad \textbf{[11]}$$

and contains all the information necessary to determine $U_Y(T, 0)$. We choose positive integers n to label the orthonormal eigenvectors $\phi_{0n} = \varphi_n$ of Q_0 with eigenvalue $\lambda_n > 0$. The eigenvector $\varphi_{-n} = (-1)^F\varphi_n$ has eigenvalue $-\lambda_n < 0$. The zero modes of Q_0 are indexed by z and denoted by φ_z. A detailed analysis of **[11]** shows that $b_n^b(t)$ and $b_{-n}^b(t)$ satisfy the same differential equation when the action of $(-1)^F$ is taken into account. Therefore we have an exact cancellation in the supertrace $\operatorname{Tr}(-1)^F U_Y(T,0)$ from the states orthogonal to $\ker Q_0$. Since the evolution is given by a first order differential equation this argument can be applied at each instant of time and can be adapted to the case when the kernel jumps. We still have to compute the contribution of the zero modes to the supertrace. This can be done exactly by the adiabatic approximation that we review shortly.

Applying these ideas to the zero modes we will obtain a refinement to the adiabatic theorem. Within the adiabatic approximation, the amplitudes b_z for the zero modes satisfy the equation

$$Db_z + \sum_{z'} A_\mu^{z,z'}(X)(DX^\mu)b_{z'} = 0 \qquad \textbf{[12]}$$

that is the super-parallel transport equation **[7]**. This gives the exact result

$$\operatorname{Tr}(-1)^F U_Y(T,0) = \operatorname{Tr}(-1)^F\Phi_b(T)\,, \qquad \textbf{[13]}$$

where Φ_b is the superholonomy given by the super-parallel transport equation **[8]**. This is the refinement of Berry's phase to supersymmetric quantum mechanics. This superparallel transport is now an additional term that has to be added to supersymmetric lagrangian on X. It reflects a coupling of the superparticle on X to a gauge field on the index bundle. Next, we can perform the Y path integral in step (c) and as usual the X path integrals in step (c') and express $\operatorname{ind} \not{D}_Z$ as

$$\int_{\mathcal{L}X} e^{L_X}\left(e^{\int A + \int F}\right) = \int_X \hat{\mathrm{A}}(TX)\,\mathrm{ch}\left(\mathrm{Ind}\,\not{D}_Y\right) \qquad \textbf{[14]}$$

If we compare **[14]** and **[2]**, we get

$$\int_X \hat{\mathrm{A}}(TX)\left(\mathrm{ch}(\mathrm{Ind}(\not{D}^Y)) - \int_{Y_x} \hat{\mathrm{A}}\left(T_{(x,\cdot)}Y\right)\right) = 0.$$

Thus we have that as cohomology classes on X

$$\left[\mathrm{ch}\left(\mathrm{Ind}(\not{D}^Y)\right)\right] = \left[\int_{Y_x} \hat{\mathrm{A}}\left(T_{(x,\cdot)}Y\right)\right], \qquad \textbf{[15]}$$

where $[\,\cdot\,]$ denotes the cohomology equivalence class. This is the cohomological families index formula for the Dirac operator which ends our discussion of the the Dirac operator case.

Adiabatic approximation with symmetries. We now turn our attention to the Dirac-Ramond operator. The new feature is that $P(t)$, the spatial translation operator along the loop, commutes with the hamiltonian $H(t)$. Our analysis requires the use of the adiabatic approximation in the presence of symmetries, a topic that is not addressed in textbooks. To simplify the discussion supersymmetry will be ignored. In the adiabatic approximation we scale the time so that the Schrödinger equation becomes $i\,\partial\psi/\partial t = TH(t)\psi(t)$. Let $\{\varphi_r(t)\}$ be an orthonormal basis of eigenvectors of $H(t)$ with eigenvalue $E_r(t)$. The orthonormal basis expansion for wavefunctions will be written as

$$\psi(t) = \sum_r e^{-iT\int_0^t E_r(t')\,dt'}\; a_r(t)\varphi_r(t)\,.$$

Inserting this into the Schrödinger equation and taking the inner product with φ_s gives the exact equation

$$\dot{a}_s(t) + \sum_r e^{iT\int_0^t (E_s(t')-E_r(t'))\,dt'}\;\langle\varphi_s|\,\dot{\varphi}_r\rangle\,a_r(t) = 0\,. \qquad \textbf{[16]}$$

As $T\to\infty$ an oscillating term contributes very little [19] and the states φ_s with $E_s(t) = E_r(t)$ for all $t>0$ are the only ones needed. In general there will only be one such state except in cases in which a symmetry enforces a multiplicity. In these cases one obtains the adiabatic approximation result $\dot{a}_s(t) + \sum_{r,E_r=E_s}\langle\varphi_s|\,\dot{\varphi}_r\rangle\,a_r(t) \approx 0$ within a degenerate energy level. On the eigenspace with eigenvalue E_s we get a connection along the family of hamiltonians given by $A_{sr}(t) = \langle\varphi_s|\,\dot{\varphi}_r\rangle$. The holonomy of this connection is the non-abelian Berry's phase [20]. This connection is not unitary because the volume element of the fiber Y_x can change as x varies [21]. The hermitian piece of the connection is associated with the varying volume element and takes the form: $\int_{Y_x} dy\,\sqrt{g_Y}\,\mathrm{Tr}(g_Y^{-1}\dot{g}_Y)\varphi_s^*\varphi_r$. Standard perturbation theory computations show that the connection has an "irreducible" part and a "perturbative" part. The topological information is contained in the "irreducible" part while the "perturbative" part, a differential form of type ad, corresponds to a translation in the affine space of connections. The perturbative parts can be ignored when discussing topological invariants. Thus we can ignore the fiber volume related part of the connection. The parts of the connection associated to transitions between different eigenvalues of $H(t)$ are purely "perturbative" because $\langle\varphi_s|\,\dot{\varphi}_r\rangle = \langle\varphi_s|\,\dot{H}\,|\varphi_r\rangle/(E_r - E_s)$. Moreover if we write an orthogonal direct sum for the Hilbert space $\mathcal{H} = \bigoplus_{E(t)}\mathcal{H}_{E(t)}$, in terms of the eigenspaces of $H(t)$ then the "irreducible" part of the connection is a direct sum $A = \bigoplus_{E(t)} A_{E(t)}$ where each piece $A_{E(t)}$ may be taken to be a unitary $\mathrm{U}(\dim\mathcal{H}_{E(t)})$ connection.

We can extend this analysis to a theory with additional symmetries because we have two commuting symmetries in the study of the Dirac-Ramond operator. Here we only consider the case of a maximally commuting algebra of self-adjoint operators $\mathcal{C}(t)$. Its basis will be written as $\{H(t) = H_0(t), H_1(t),\ldots,H_l(t)\}$. We assume that the spectrum of any operator in $\mathcal{C}(t)$ is discrete. Since $\mathcal{C}(t)$ is abelian, its irreducible representations are one dimensional. We can find simultaneously eigenvectors of all the operators in $\mathcal{C}(t)$. A state ψ has weight $\lambda(t)$ if $H_i(t)\psi = \lambda_i(t)\psi$ where $\lambda(t) = (E(t),\lambda_1(t),\ldots,\lambda_l(t))$. We will express the Hilbert space as an orthogonal direct sum $\mathcal{H} = \bigoplus_{\lambda(t)}\mathcal{H}_{\lambda(t)}$. As t varies, the subspaces $\mathcal{H}_{\lambda(t)}$ also vary. Assume that $\varphi_\lambda(t)$ and $\varphi_\mu(t)$ are normalized eigenvectors with respective weights $\lambda(t)$ and $\mu(t)$, and $\lambda(t)\neq\mu(t)$. It follows that there exists j such that $\lambda_j(t)\neq\mu_j(t)$. Applying the same analysis as above to the operator $H_i(t)$, one concludes on the one hand that the connection between the subspaces $\mathcal{H}_{\lambda(t)}$ and $\mathcal{H}_{\mu(t)}$ is perturbative and on the other hand that the "irreducible" part of the connection is a direct sum $A = \bigoplus_{\lambda(t)} A_{\lambda(t)}$ where each piece $A_{\lambda(t)}$ is a $\mathrm{U}(\dim\mathcal{H}_{\lambda(t)})$ connection on $\mathcal{H}_{\lambda(t)}$.

For simplicity consider a situation with only two commuting operators $H(t)$ and $P(t)$ and where the spectrum of $P(t)$ takes integer values. For the adiabatic time evolution by the operator $H(t)+\theta P(t)/T$, the contribution from a subspace of the Hilbert space with energy E and with P eigenvalue n is schematically given by $e^{-i\theta n}\,\mathrm{Tr}\,e^{-\int A}$, where A is the Berry-Simon connection on that subspace.

In the Dirac-Ramond model we have a $(0,1)$ supersymmetry. This means that there are operators L_0, $\bar{L}_0$ that commute with each other and that the Dirac-Ramond operator satisfies $\bar{G}_0^2 = \bar{L}_0 - \bar{c}/24$. Its index is given by the supertrace $\mathrm{Tr}(-1)^F q^{L_0-c/24}\bar{q}^{\bar{G}_0^2}$. To relate this to the previous discussion we note that the time development operator is $H = (L_0 - c/24) + (\bar{L}_0 - \bar{c}/24)$ and the spatial translation operator is $P = (L_0 - c/24) - (\bar{L}_0 - \bar{c}/24)$. Applying the previous analysis we conclude that the result of step (c) in the Dirac-Ramond scenario is an expression of the form $\sum_{n=0}^\infty q^{n-c/24}\;\Phi_n$ where Φ_n is the superholonomy of the index bundle at "n-th level eigenspace" of $L_0 - c/24$. Step (c') gives localization and will lead to formula **[18]**.

The families index for Dirac-Ramond. The existence of a Dirac-Ramond operator on Z requires $\mathrm{p}_1(Z) = 0$ and $\dim Z = n = 4\hat{n}$ while the submersion structure implies by restriction that $\mathrm{p}_1\,Y = 0$. The family defined by Z has base X with $\dim X = p$ and fiber isomorphic to Y with $\dim Y = m$. There will be two distinct cases to consider. The first case is $p = 4\hat{p}$ and $m = 4\widehat{m}$ where $\hat{p}\geq 0$ and $\widehat{m}\geq 1$. The second case is $p = 4\hat{p}+2$ and $m = 4\widehat{m}-2$ where $\hat{p}\geq 0$ and $\widehat{m}\geq 1$. Note that in both cases $n = 4\hat{n} = 4(\hat{p}+\widehat{m})$.

In the Dirac-Ramond case with (0,1) supersymmetry, the right hand side **[15]** becomes

$$\int_{Y_x}\hat{s}(\Omega_Y,\tau) = \frac{1}{\eta(\tau)^m}\int_{Y_x}\hat{a}(\Omega_Y,\tau)\,, \qquad \textbf{[17]}$$

where Y_x is the fiber of $Z\to X$ over $x\in X$ and Ω_Y is the curvature 2-form using the submersion connection of the vertical tangent bundle. The notations we use for the string genus $\hat{s}$ and for $\hat{a}$ are summarized in the appendix. From the path integral point of view, the Dirac and the Dirac-Ramond case differ by the essential presence of a full right Virasoro algebra commuting with supersymmetry. In particular since $[L_0,\overline{G}_0] = 0$, the index bundle for the family defined by $Z\to X$ has an orthogonal direct sum decomposition into representations of L_0 given by $\mathrm{Ind}(Z\to X) = \bigoplus_{n=0}^\infty \mathrm{Ind}_n(Z\to X)$, where n denotes the grading with respect to L_0. Our adiabatic invariance arguments in the presence of an abelian symmetry show that this leads to a connection $A = \bigoplus_{n=0}^\infty A^{(n)}$ where $A^{(n)}$ acts only on $\mathrm{Ind}_n(Z\to X)$. There are no off diagonal pieces that connect subspaces labeled by different values of n. We denote the curvature of $A^{(n)}$ by $F^{(n)}$. The left hand side of the formula for the family's index theorem is given by what we call the graded string Chern character

$$\mathrm{sch}(\tau,F) = \sum_{n=0}^\infty q^{n-m/24}\;\mathrm{ch}\left(iF^{(n)}/2\pi\right), \qquad \textbf{[18]}$$

where ch is the Chern character.

Given the foregoing, we are in a position to write down our main result:

$$\mathrm{sch}(\tau, F) = \left[\int_{Y_x} \hat{s}(\Omega_Y, \tau)\right] . \qquad [19]$$

This formula, the equivalent of **[15]** for loop space, gives a families index theorem for the Dirac-Ramond operator. Its right hand side can be rewritten in a more explicit way. Let $\hat{s} = \sum_{k=0}^{\infty} \hat{s}_{4k}(\Omega_Y, \tau)$, and $\hat{a} = \sum_{k=0}^{\infty} \hat{a}_{4k}(\Omega_Y, \tau)$ where $\hat{s}_{4k}$ and $\hat{a}_{4k}$ are $4k$-forms on Z. Note that $\hat{a}_{4k}(\Omega_Y, \tau) = \eta(\tau)^m \hat{s}_{4k}(\Omega_Y, \tau)$. It is convenient to define $\hat{a}_l = 0$ if $l \not\equiv 0 \bmod 4$ and likewise for $\hat{s}_l$. Integration along the fiber reduces the $4k$-form to a $4k - m$ form on the base X

$$\begin{aligned} \int_{Y_x} \hat{a}_{4k}(\Omega_Y, \tau) &= \alpha_{4k-m}(x, \tau) \\ \int_{Y_x} \hat{s}_{4k}(\Omega_Y, \tau) &= \sigma_{4k-m}(x, \tau). \end{aligned} \qquad [20]$$

Note that $\alpha_l(x, \tau) = \eta(\tau)^m \sigma_l(x, \tau)$, and if $l < 0$ then $\alpha_l = 0$, $\sigma_l = 0$. It then follows from the modular properties of $\hat{a}$ that α_{4k-m} transforms as a modular form of weight $2k$ while

$$\sigma_{4k-m}(\tau + 1) = e^{-2\pi i m/24} \sigma_{4k-m}(\tau) , \qquad [21]$$

$$\sigma_{4k-m}(-1/\tau) = e^{2\pi i m/8} \tau^{(4k-m)/2} \sigma_{4k-m}(\tau) . \qquad [22]$$

More precisely if we take $4k - m$ vectors $X_1, \ldots, X_{4k-m} \in T_x X$ then $\alpha_{4k-m}(x, \tau)(X_1, \ldots, X_{4k-m})$ will be a modular form in τ of weight $2k$. Observe that $0 \le 4k - m \le p = \dim X$ and therefore $\dim Y = m \le 4k \le p + m = \dim Z$. The range for the weight of the modular form is $\frac{1}{2} \dim Y \le 2k \le \frac{1}{2} \dim Z$. We can make the weight of α_{4k-m} as large as possible by making the parameter space X have high dimensionality. All this may be summarized in the formulae

$$\mathrm{sch}(\tau + 1, F) = e^{-2\pi i m/24} \, \mathrm{sch}(\tau, F) , \qquad [23]$$

$$\mathrm{sch}(-1/\tau, F/\tau) = e^{2\pi i m/8} \, \mathrm{sch}(\tau, F) . \qquad [24]$$

This extends the modular properties of the index **[35]** to the full index bundle. If we write $\mathrm{sch}(\tau, F) = \alpha(\tau, F)/\eta(\tau)^m$ then

$$\alpha(\tau + 1, F) = \alpha(\tau, F) , \qquad [25]$$

$$\alpha(-1/\tau, F/\tau) = \tau^{m/2} \alpha(\tau, F) . \qquad [26]$$

If $y_j^{(n)}$ are the formal eigenvalues of $iF^{(n)}/2\pi$ then $\mathrm{ch}(iF^{(n)}/2\pi) = \sum_j e^{y_j^{(n)}}$. The Chern character is formally invariant under the transformation $y_j^{(n)} \to y_j^{(n)} + 2\pi i m_j^{(n)}$ where $m_j^{(n)} \in \mathbb{Z}$ and therefore $\alpha(\tau, y_j^{(n)})$ should be periodic

$$\alpha(\tau, y_j^{(n)}) = \alpha(\tau, y_j^{(n)} + 2\pi i m_j^{(n)}) \qquad [27]$$

Combining this with **[25]** and **[26]** we see that there is an additional formal periodicity

$$\alpha(\tau, y_j^{(n)}) = \alpha(\tau, y_j^{(n)} + 2\pi i l_j^{(n)} \tau) \text{ where } l_j^{(n)} \in \mathbb{Z}. \qquad [28]$$

These formal transformation properties of the α remind one of the transformation rules of Jacobi forms. However this formal transformation cannot be a true invariance. One way to see this is to fix a positive integer r and consider $y_j^{(n)} \to y_j^{(n)} - 2\pi i r \delta_{nr}$. This transformation completely eliminates the $q^{r-m/24}$ term from **[18]**. We conclude that the $y_j^{(n)}$ are not all independent as can be seen clearly in the particular cases studied in **[32]** and **[33]** below. For example **[32]** implies that all the first Chern classes are related and determined by the single Chern class allowed in the right hand side.

The modular properties of σ_k severely constrains the non trivial cohomology classes of the index bundle of the Dirac-Ramond operator. We rewrite **[19]** as

$$\sum_{n=0}^{\infty} q^{n-m/24} \, \mathrm{ch}\left(iF^{(n)}/2\pi\right) = \sum_{k=0}^{\infty} \sigma_{4k-m}(\tau) \qquad [29]$$

and the classes are described by a universal formula. Let $\dim Y = m = 4\widehat{m} - 2\varepsilon$ where $\varepsilon = 0, 1$ then the $4j + 2\varepsilon$ cohomology class of the index bundle is given by

$$\frac{1}{(2j+\varepsilon)!} \sum_{n=0}^{\infty} q^{n-m/24} \, \mathrm{Tr}\left(\frac{iF^{(n)}}{2\pi}\right)^{2j+\varepsilon} = \sigma_{4j+2\varepsilon}(\tau) \qquad [30]$$

where

$$\sigma_{4j+2\varepsilon}(\tau) = \frac{\alpha_{4j+2\varepsilon}(\tau)}{\eta(\tau)^m} = \int_{Y_x} \hat{s}_{4j+4\widehat{m}}(\Omega_Y, \tau) \qquad [31]$$

Note that $\alpha_{4j+2\varepsilon}$ is of modular weight $2j + 2\widehat{m}$. An important reminder is that with our conventions $\widehat{m} \ge 1$ and therefore there is a bound on the modular weight of $\alpha_{4j+2\varepsilon}$ given by $2j + 2\widehat{m} \ge 2j + 2$.

Some applications. Two different approaches come to mind when trying to exploit the cohomological formula **[31]**. On the one hand we can use what we know about $\dim M_k$, the dimensionality of the space of modular forms of weight k, and in particular look at spaces of modular form of low dimensionality. This restricts $\alpha_{4j+2\epsilon}$ and can place strong constraints on the cohomological classes. Alternatively we can study cohomology of a specified degree.

As a first example we look at the case where $\alpha_{4j+2\varepsilon}$ has modular weight 4 . There are two possibilities for $(j, \widehat{m})$ given by $j = 0$, $\widehat{m} = 2$; and $j = 1$, $\widehat{m} = 1$. Because $\dim M_4 = 1$ we have that $\alpha_{4j+2\varepsilon}(x, \tau) = E_4(\tau)\tilde{\alpha}_{4j+2\varepsilon}(x)$ where $\tilde{\alpha}$ a closed $(4j + 2\varepsilon)$-form on X that is independent of τ and E_4 is the Eisenstein series. When $\epsilon = 1$, $\dim Y = 4\widehat{m} - 2$. If $j = 0$, $\widehat{m} = 2$ then we are looking at the second cohomology of the index bundle for the Dirac-Ramond operator on a manifold of dimension $\dim Y = 4\widehat{m} - 2 = 6$ and we can take X to be a two dimensional manifold. We see that

$$\sum_{n=0}^{\infty} q^{n-1/4} \, \mathrm{Tr}\left(\frac{iF^{(n)}}{2\pi}\right) = \frac{E_4(\tau)}{\eta(\tau)^6} \, \mathrm{Tr}\left(\frac{iF^{(0)}}{2\pi}\right) . \qquad [32]$$

The first Chern class of the determinant line bundle of Ind_n is proportional to the first Chern class of the determinant line bundle of the Dirac operator. If you eliminate the "anomaly" associated with the Dirac operator on Y then you eliminate the anomaly for the Dirac operator coupled to appropriate powers of TY. The other case has $j = 1$ with $\dim Y = 4\widehat{m} - 2 = 2$ where we reach the conclusion that the 6-cohomology of the index bundle (need $\dim X \ge 6$) is given by

$$\sum_{n=0}^{\infty} q^{n-1/12} \, \mathrm{Tr}\left(\frac{iF^{(n)}}{2\pi}\right)^3 = \frac{E_4(\tau)}{\eta(\tau)^2} \, \mathrm{Tr}\left(\frac{iF^{(0)}}{2\pi}\right)^3 . \qquad [33]$$

An interesting example of the second type from a physics point of view is given by two cohomology. This case corresponds to $j = 0$, $\varepsilon = 1$ and gives the first Chern class of the determinant line bundles associated with the various index bundles Ind_n. The manifold Y has dimension $m = 4\widehat{m} - 2$. We can take X to be a 2-manifold. The 2-form α_2 has weight $2\widehat{m} = \dim Y/2 + 1$ so that if $\dim Y = 6, 10, 14, 18, 26$ (respectively $2\widehat{m} = 4, 6, 8, 10, 14$) then $\dim M_{2\widehat{m}} = 1$ and we conclude that the first Chern classes of the index bundle are given by

$$\sum_{n=0}^{\infty} q^{n-m/24} \, \mathrm{Tr}\left(\frac{iF^{(n)}}{2\pi}\right) = \frac{E_{2\widehat{m}}(\tau)}{\eta(\tau)^m} \, \mathrm{Tr}\left(\frac{iF^{(0)}}{2\pi}\right) .$$

In this range if the determinant line bundle of the Dirac operator has vanishing first Chern class then so do all the determinant line bundles for the higher operators. In the especially interesting case with $\dim Y = 10$, *i.e.*, $2\widehat{m} = 6$, the result is

$$\sum_{n=0}^{\infty} q^{n-5/12}\,\mathrm{Tr}\left(\frac{iF^{(n)}}{2\pi}\right) = \frac{E_6(\tau)}{\eta(\tau)^{10}}\;\mathrm{Tr}\left(\frac{iF^{(0)}}{2\pi}\right).$$

Hence you see that $\mathrm{Tr}(iF^{(1)}/2\pi) = -494\,\mathrm{Tr}(iF^{(0)}/2\pi)$. $F^{(0)}$ is the curvature of the index bundle of the Dirac operator and $F^{(1)}$ is the curvature of the index bundle of the Dirac operator coupled to TY. If you compare this to the calculation of Alvarez-Gaumé and Witten [22] for the gravitational anomalies in type IIB supergravity in a manifold with $\mathrm{p}_1(Z) = 0$ you find that $\mathcal{A}_{3/2} = -495\mathcal{A}_{1/2}$ where $\mathcal{A}_{1/2}$ is the anomaly contribution from the chiral spinor and $\mathcal{A}_{3/2}$ is the contribution from the chiral gravitino. The difference of 1 corresponds to the longitudinal component of a Rarita-Schwinger field ψ_μ that must be accounted for correctly to get the physical gravitino. Note that $\dim Z = 12$ and $\mathrm{p}_1(Z) = 0$ tell us that things can only depend on p_3 so all the anomalies will be proportional.

Conclusions and outlook. We have shown that the characteristic classes of the index bundle of the Dirac-Ramond operator have remarkable modular properties. The discussion was here restricted to families described by a riemannian submersion. In a forthcoming longer publication we extend the analysis to the Dirac-Ramond operator coupled to various infinite dimensional vector bundles and show how symmetries constrain the structure by using the representation theory of Virasoro and chiral algebras. An open important question is the link of our geometrical methods to tmf.

Appendix. The index of the Dirac-Ramond operator for $(0,1)$ supersymmetry on a manifold M with $\mathrm{p}_1(M) = 0$ is given by $\mathrm{Tr}_{\ker \overline{G}_0}(-1)^F q^{L_0 - c/24} = \int_M \hat{s}(M,\tau)$, where $\overline{G}_0$ is the generator of the right handed supersymmetry. The integrand is the string genus $\hat{s}(M,\tau) = \hat{a}(M,\tau)/\eta(\tau)^d$ where $d = \dim M$ and

$$\hat{a}(M,\tau) = \prod_{j=1}^{d/2} \frac{ix_j/2\pi}{\sigma(ix_j/2\pi,\tau)}\,. \qquad [34]$$

Here η is the Dedekind eta function $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, where $q = e^{2\pi i\tau}$ and σ is the Weierstrass function. The modular transformation properties of the string genus follow from $\hat{a}(x,\tau+1) = \hat{a}(x,\tau)$ and $\hat{a}(x/\tau,-1/\tau) = \hat{a}(x,\tau)$ This implies for the string genus $\hat{s}(x/\tau,-1/\tau) = \hat{s}(x,\tau)/(-i\tau)^{d/2}$. Since integration over M picks out the differential form with degree $\dim M$, $\int_M \hat{a}(M,\tau)$ is a modular form of weight $d/2$ and

$$\begin{aligned} \int_M \hat{s}(M,-1/\tau) &= e^{2\pi i d/8} \int_M \hat{s}(M,\tau)\,, \\ \int_M \hat{s}(M,\tau+1) &= e^{-2\pi i d/24} \int_M \hat{s}(M,\tau)\,. \end{aligned} \qquad [35]$$

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