What happens after a default: the conditional density approach

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May 5, 2009

Abstract

We present a general model for default time, making precise the role of the intensity process, and showing that this process allows for a knowledge of the conditional distribution of the default only "before the default". This lack of information is crucial while working in a multi-default setting. In a single default case, the knowledge of the intensity process does not allow to compute the price of defaultable claims, except in the case where immersion property is satisfied. We propose in this paper the density approach for default time. The density process will give a full characterization of the links between the default time and the reference filtration, in particular "after the default time". We also investigate the description of martingales in the full filtration in terms of martingales in the reference filtration, and the impact of Girsanov transformation on the density and intensity processes, and also on the immersion property.

1 Introduction

Modelling default time for a single credit event has been largely studied in the literature, the main approaches being the structural, the reduced-form and the intensity ones. In this context, most works are concentrated (for pricing purpose) on the computation of conditional expectation of payoffs, given that the default has not occurred, in the case where immersion property is satisfied. In this paper, we are also interested in what happens after a default occurs: we find it

*This work has benefited from financial support by La Fondation du Risque et Fédération Bancaire Française.

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important to investigate the impact of a default event on the rest of the market and what goes on afterwards.

Furthermore, in a multi-default setting, it will be important to compute the prices of a portfolio derivative on the disjoint sets before the first default, after the first and before the second and so on. Our work will allow us to use a recurrence procedure to provide these computations, which will be presented in a companion paper [5].

We start with the knowledge of the conditional distribution of the default time τ, with respect to a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and we assume that this conditional distribution admits a density (see the first section below for a precise definition). We firstly reformulate the classical computation result of conditional expectations with respect to the observation σ-algebra $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau \wedge t)$ before the default time τ, i.e., on the set $\{t < \tau\}$. The main purpose is then to deduce what happens after the default occurs, i.e., on the set $\{\tau \leq t\}$. We shall emphasize that the density approach is suitable in this after-default study and explain why the intensity approach is inadequate for this case. We present computation results of $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ conditional expectations on the set $\{\tau \leq t\}$ by using the conditional density of τ and point out that the whole term structure of the density is needed. By establishing an explicit link between (part of) density and intensity, which correspond respectively to the additive and multiplicative decomposition related to the survival process (Azéma supermartingale of τ), we make clear that the intensity can be deduced from the density, but that the reverse does not hold, except when certain strong assumption, as the H-hypothesis, holds.

Note that, even if the "density" point of view is inspired by the enlargement of filtration theory, we shall not use classical results on the progressive enlargement of filtration. In fact, we take the opposite point of view: we are interested in $\mathbb{G}$-martingales and their characterization in terms of $\mathbb{F}$-(local) martingales. Moreover, these characterization results allow us to give a proof of a decomposition of $\mathbb{F}$-(local) martingales in terms of $\mathbb{G}$-semimartingales.

We study how the parameters of the default (i.e., the survival process, the intensity, the density) are modified by a change of probability in a general setting (we do not assume that we are working in a Brownian filtration, except for some examples), and we characterize changes of probability that do not affect the intensity process. We pay attention to the specific case where the dynamics of underlying default-free processes are changed only after the default.

The paper is organized as follows. We first introduce in Section 2 the different types of information we are dealing with and the key hypothesis of density. In Section 3, we establish results on computation of conditional expectations, on the "before default" and "after default" sets. The H-hypothesis is then discussed. The dynamic properties of the density process are presented in Section 4 where we make precise the links between this density process and the intensity process. In the last section, we present the characterization of $\mathbb{G}$-martingales in terms of $\mathbb{F}$-local martingales. We give a Girsanov type property and discuss the stability of immersion property and invariance of intensity.

2 The Different Sources of Information

In this section, we specify the link between the two filtrations $\mathbb{F}$ and $\mathbb{G}$, and make some hypotheses on the default time. Our aim is to measure the consequence of a default event in terms of pricing various contingent claims.

We start as usual with a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$. Before the default time τ, i.e., on the set $\{t < \tau\}$, the σ-algebra $\mathcal{F}_t$ represents the information accessible to the investors at time t. When the default occurs, the investors will add this new information (i.e., the knowledge of τ) to the σ-algebra $\mathcal{F}_t$.

More precisely, a strictly positive and finite random variable τ (the default time) is given on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. This space is supposed to be endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ which satisfies the usual conditions, that is, the filtration $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{A}$.

One of our goals is to show how the information contained in the reference filtration $\mathbb{F}$ can be used to obtain information on the distribution of τ. We assume that, for any t, the conditional distribution of τ with respect to $\mathcal{F}_t$ is smooth, i.e., that the $\mathcal{F}_t$-conditional distribution of τ admits a density with respect to some positive σ-finite measure η on $\mathbb{R}^+$. As an immediate consequence, the unconditional distribution of τ is absolutely continuous w.r.t. η. Another consequence is that τ can not be an $\mathbb{F}$-stopping time.

In other terms, we introduce the following hypothesis[1], that we call density hypothesis. This hypothesis will be in force in the paper.

Hypothesis 2.1 (**Density hypothesis.**) We assume that η is a non-negative non-atomic measure on $\mathbb{R}^+$ such that, for any time $t \geq 0$, there exists an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(\omega, \theta) \to \alpha_t(\omega, \theta)$ which satisfies

$$\mathbb{P}(\tau \in d\theta | \mathcal{F}_t) =: \alpha_t(\theta)\eta(d\theta), \quad \mathbb{P} - a.s. \tag{1}$$

[1]This hypothesis has been discussed by Jacod [9] in the initial enlargement of filtration framework. The same assumption also appears in the dynamic Bayesian framework [7]. We do not assume that η is finite, allowing for the specific case of the Lebesgue measure.

The family $\alpha_t(.)$ is called the *conditional density* of τ with respect to η given $\mathcal{F}_t$ (in short the *density* of τ if no ambiguity). Then, the distribution of τ is given by $\mathbb{P}(\tau > \theta) = \int_\theta^\infty \alpha_0(u)\eta(du)$. Note that, from the definition and the hypothesis that τ is finite, for any t, $\int_0^\infty \alpha_t(\theta)\eta(d\theta) = 1\,(a.s.)$. By definition of the conditional expectation, for any (bounded) Borel function f, $\mathbb{E}[f(\tau)|\mathcal{F}_t] = \int_0^\infty f(u)\alpha_t(u)\eta(du)(a.s.)$. The conditional distribution of τ is also characterized by the survival probability function

$$S_t(\theta) := \mathbb{P}(\tau > \theta|\mathcal{F}_t) = \int_\theta^\infty \alpha_t(u)\eta(du) \tag{2}$$

The family of random variables

$$S_t := S_t(t) = \mathbb{P}(\tau > t|\mathcal{F}_t) = \int_t^\infty \alpha_t(u)\eta(du)$$

plays a key role in what follows. Observe that one has

$$\{\tau > t\} \subset \{S_t > 0\} =: A_t \tag{3}$$

(where the inclusion is up to a negligible set) since $\mathbb{P}(A_t^c \cap \{\tau > t\}) = 0$. Note also that $S_t(\theta) = \mathbb{E}(S_\theta|\mathcal{F}_t)$ for any $\theta \geq t$.

More generally, if an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(\omega, \theta) \to Y_t(\omega, \theta)$ is given, the $\mathcal{F}_t$-conditional expectation of the r.v. $Y_t(\tau) := Y_t(\omega, \tau(\omega))$, assumed to be integrable, is given by

$$\mathbb{E}[Y_t(\tau)|\mathcal{F}_t] = \int_0^\infty Y_t(u)\alpha_t(u)\eta(du)\,. \tag{4}$$

Notation: In what follows, we shall simply say that $Y_t(\theta)$ is an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-random variable and even that $Y_t(\tau)$ is an $\mathcal{F}_t \otimes \sigma(\tau)$-random variable as a short cut for $Y_t(\theta)$ is an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function.

Corollary 2.2 *The default time τ avoids $\mathbb{F}$-stopping times, i.e., $\mathbb{P}(\tau = \xi) = 0$ for every $\mathbb{F}$-stopping time ξ.*

PROOF: Let ξ be an $\mathbb{F}$-stopping time bounded by a constant T. Then, the random variable $H_\xi(t) = 1_{\{\xi = t\}}$ is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^+)$-measurable, and, η being non-atomic

$$\mathbb{E}[H_\xi(\tau)|\mathcal{F}_t] = \mathbb{E}[\,\mathbb{E}[H_\xi(\tau)|\mathcal{F}_T]\,|\mathcal{F}_t] = \mathbb{E}[\int_0^\infty H_\xi(u)\alpha_T(u)\eta(du)|\mathcal{F}_t] = 0\,.$$

Hence, $\mathbb{E}[H_\xi(\tau)] = \mathbb{P}(\xi = \tau) = 0$. □

Remark 2.3 By using density, we adopt an additive point of view to represent the conditional probability of τ: the conditional survival function $S_t(\theta) = \mathbb{P}(\tau > \theta\,|\,\mathcal{F}_t)$ is written on the form $S_t(\theta) = \int_\theta^\infty \alpha_t(u)\eta(du)$. In the default framework, the "intensity" point of view is often preferred, and one uses the multiplicative representation $S_t(\theta) = \exp(-\int_0^\theta \lambda_t(u)\eta(du))$. The family of $\mathcal{F}_t$-measurable random variables $\lambda_t(u) = -\partial_u \ln S_t(u)$ is called the "forward hazard rate". We shall discuss and compare these two points of view further on.

3 Computation of conditional expectations in a default setting

The specific information related to the default time is the knowledge of this time when it occurs. It is defined in mathematical terms as follows: let $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$ be the smallest right-continuous filtration such that τ is a $\mathbb{D}$-stopping time; in other words, $\mathcal{D}_t$ is given by $\mathcal{D}_t = \mathcal{D}^0_{t+}$ where $\mathcal{D}^0_t = \sigma(\tau \wedge t)$. This filtration $\mathbb{D}$ represents the default information, that will be "added" to the reference filtration; the filtration $\mathbb{G} := \mathbb{F} \vee \mathbb{D}$ is the smallest filtration containing $\mathbb{F}$ and making τ a stopping time. Moreover, any $\mathcal{G}_t$-measurable r.v. $H^{\mathbb{G}}_t$ may be represented as $H^{\mathbb{G}}_t = H^{\mathbb{F}}_t 1_{\{\tau>t\}} + H_t(\tau)1_{\{\tau\leq t\}}$ where $H^{\mathbb{F}}_t$ is an $\mathcal{F}_t$-measurable random variable and $H_t(\tau)$ is $\mathcal{F}_t \otimes \sigma(\tau)$-measurable. In particular,

$$H^{\mathbb{G}}_t\, 1_{\{\tau>t\}} = H^{\mathbb{F}}_t 1_{\{\tau>t\}} \quad a.s., \tag{5}$$

where the random variable $H^{\mathbb{F}}_t$ is the $\mathcal{F}_t$-conditional expectation of $H^{\mathbb{G}}_t$ given the event $\{\tau > t\}$, i.e.,

$$H^{\mathbb{F}}_t = \frac{\mathbb{E}[H^{\mathbb{G}}_t 1_{\{\tau>t\}}|\mathcal{F}_t]}{\mathbb{P}(\tau > t|\mathcal{F}_t)} = \frac{\mathbb{E}[H^{\mathbb{G}}_t 1_{\{\tau>t\}}|\mathcal{F}_t]}{S_t} \quad a.s. \text{ on } A_t; \quad H^{\mathbb{F}}_t = 0 \quad \text{on the complementary set}. \tag{6}$$

3.1 Conditional expectations

The definition of $\mathbb{G}$ allows us to compute conditional expectations with respect to $\mathcal{G}_t$ in terms of conditional expectations with respect to $\mathcal{F}_t$. This will be done in two steps, depending whether or not the default has occurred: as we explained above, before the default, the only information contained in $\mathcal{G}_t$ is $\mathcal{F}_t$, after the default, the information contained in $\mathcal{G}_t$ is, roughly speaking, $\mathcal{F}_t \vee \sigma(\tau)$.

The $\mathcal{G}_t$-conditional expectation of an integrable $\sigma(\tau)$-measurable r.v. (of the form $f(\tau)$) is given by

$$\alpha^{\mathbb{G}}_t(f) := \mathbb{E}[f(\tau)|\mathcal{G}_t] = \alpha^{\text{bd}}_t(f)\, 1_{\{\tau>t\}} + f(\tau)\, 1_{\{\tau\leq t\}}$$

where α^{bd}_t is the value of the $\mathcal{G}_t$-conditional distribution **b**efore the **d**efault, given by

$$\alpha^{\text{bd}}_t(f) := \frac{\mathbb{E}[f(\tau)1_{\{\tau>t\}}|\mathcal{F}_t]}{\mathbb{P}(\tau > t|\mathcal{F}_t)} \quad a.s. \text{ on } A_t; \quad \alpha^{\text{bd}}_t(f) := 0 \quad \text{on the complementary set.}$$

Recall the notation $S_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$. On the set A_t, the "before the default" conditional distribution α_t^{bd} admits a density $\alpha_t^{\mathrm{bd}}(u)$ with respect to η, given by

$$\alpha_t^{\mathrm{bd}}(u) = \frac{1}{S_t}\mathbf{1}_{[t,\infty)}(u)\alpha_t(u)\eta(du) \quad a.s..$$

The same calculation as in (4) can be performed in this new framework and extended to the computation of $\mathcal{G}_t$-conditional expectations for a bounded $\mathcal{F}_T \otimes \sigma(\tau)$-r.v..

Theorem 3.1 *Let $Y_T(\tau)$ be a bounded $\mathcal{F}_T \otimes \sigma(\tau)$-random variable. Then, for $t \le T$*

$$E[Y_T(\tau)|\mathcal{G}_t] = Y_t^{\mathrm{bd}} 1_{\{t<\tau\}} + Y_t^{\mathrm{ad}}(T,\tau)1_{\{\tau\le t\}} \quad d\mathbb{P} - a.s.$$

where

$$\begin{aligned} Y_t^{\mathrm{bd}} &= \frac{\mathbb{E}\big[\int_t^\infty Y_T(u)\alpha_T(u)\eta(du)|\mathcal{F}_t\big]}{S_t} \quad d\mathbb{P} - a.s.\ on\ A_t, \\ Y_t^{\mathrm{ad}}(T,\theta) &= \frac{\mathbb{E}\big[Y_T(\theta)\alpha_T(\theta)\big|\mathcal{F}_t\big]}{\alpha_t(\theta)} 1_{\{\alpha_t(\theta)>0\}} \quad d\mathbb{P} - a.s.. \end{aligned} \tag{7}$$

Proof: The computation on the set $\{t < \tau\}$ (the pre-default case) is obtained following (5), (6) and using (4). For the after-default case, we note that, by definition of $\mathbb{G}$, any $\mathcal{G}_t$-measurable r.v. can be written on the set $\{\tau \le t\}$ as $H_t(\tau)1_{\{\tau\le t\}}$. Assuming that $H_t(\tau)$ is positive or bounded, and using the density $\alpha_t(\theta)$, we obtain

$$\begin{aligned} \mathbb{E}[H_t(\tau)1_{\{\tau\le t\}}Y_T(\tau)] &= \int_0^\infty d\theta\, \mathbb{E}[H_t(\theta)1_{\{\theta\le t\}}Y_T(\theta)\alpha_T(\theta)] = \int_0^\infty d\theta\, \mathbb{E}\big[H_t(\theta)1_{\{\theta\le t\}}\mathbb{E}[Y_T(\theta)\alpha_T(\theta)|\mathcal{F}_t]\big] \\ &= \int_0^\infty d\theta\, \mathbb{E}\Big[H_t(\theta)1_{\{\theta\le t\}}Y_t^{ad}(T,\theta)\alpha_t(\theta)\Big] = \mathbb{E}\Big[H_t(\tau)1_{\{\tau\le t\}}Y_t^{ad}(T,\tau)\Big], \end{aligned}$$

which implies immediately (7). □

3.2 Immersion property or H-hypothesis

In the form of the density $\alpha_t(\theta) = \mathbb{P}(\tau \in d\theta|\mathcal{F}_t)/\eta(d\theta)$, the parameter θ plays the role of the default time. It is hence natural to consider the particular case where

$$\alpha_t(\theta) = \alpha_\theta(\theta), \quad \forall\, \theta \le t\,, \tag{8}$$

i.e., the case when the information contained in the reference filtration after the default time does not give new information on the conditional distribution of the default. In that case

$$S_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = 1 - \int_0^t \alpha_t(u)\eta(du) = 1 - \int_0^t \alpha_u(u)\eta(du)$$

and, in particular S is decreasing [2]. Furthermore,

$$S_t = 1 - \int_0^t \alpha_T(u)\eta(du) = \mathbb{P}(\tau > t|\mathcal{F}_T) = S_T(t) \ a.s.$$

for any $T \geq t$ and it follows that $\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_\infty)$. This last equality is known to be equivalent to the immersion property ([3]), also known as the H-hypothesis, stated as: for any fixed t and any bounded $\mathcal{G}_t$-measurable r.v. $Y_t^{\mathbb{G}}$,

$$\mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_\infty] = \mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] \quad a.s.. \tag{9}$$

Conversely, if immersion property holds, then (8) holds. In that case, the conditional survival functions $S_t(\theta)$ are constant in time on $[\theta, \infty)$, i.e., $S_t(\theta) = S_\theta(\theta)$ for $t > \theta$. Then the previous result (7) takes a simpler form: $Y_t^{\mathrm{ad}}(T,\theta) = \mathbb{E}[Y_T(\theta)|\mathcal{F}_t]$, $a.s.$ for $\theta \leq t \leq T$, on the set $\{\alpha_\theta(\theta) > 0\}$.

Under immersion property, the knowledge of S implies that of the conditional distribution of τ for all positive t and θ: indeed, one has $S_t(\theta) = \mathbb{E}[S_\theta|\mathcal{F}_t]$ (note that, for $\theta < t$, this equality reduces to $S_t(\theta) = S_\theta(\theta) = S_\theta$).

Remarks 3.2

1) The decreasing property of S (equivalent to the fact that τ is a pseudo-stopping time (see [13])) does not imply the H-hypothesis, but only that $\mathbb{F}$-bounded martingales stopped at τ are $\mathbb{G}$-martingales (see also [6]). We shall revisit this property in Remarks 4.2 and 4.9 and Corollary 5.4.

2) The most important example where immersion holds is the widely studied Cox-process model introduced by Lando [12].

4 Dynamic point of view and density process

Our aim is here to give a dynamic study of the previous results. We shall call $(S_t, t \geq 0)$ the survival process, which is an $\mathbb{F}$-supermartingale. We have obtained equalities for fixed t, we would like to study the conditional expectations as processes. One of the goals is to recover the value of the intensity of the random time, and the decompositions of S. Another one is to study the link between $\mathbb{G}$ and $\mathbb{F}$ martingales: this is of main interest for pricing.

In this section, we present the dynamic version of the previous results in terms of $\mathbb{F}$ or $\mathbb{G}$ martingales or supermartingales. To be more precise, we need some "universal" regularity on the paths of the density process. We shall treat some technical problems in Subsection 4.1 which can be skipped for the first reading.

[2] and continuous, this last property will be useful later.

4.1 Regular Version of Martingales

One of the major difficulties is to prove the existence of a universal càdlàg martingale version of this family of densities. Fortunately, results of Jacod [9] or Stricker and Yor [15] help us to solve this technical problem.

Jacod ([9], Lemme 1.8) establishes the existence of a universal càdlàg version of the density process in the following sense: there exists a non negative function $\alpha_t(\omega,\theta)$ càdlàg in t, optional w.r.t. the filtration $\widehat{\mathbb{F}}$ on $\widehat{\Omega}=\Omega\times\mathbb{R}^+$, generated by $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R}^+)$, such that

- for any θ, $\alpha_.(\theta)$ is an $\mathbb{F}$-martingale; moreover, denoting $\zeta^\theta=\inf\{t:\alpha_{t-}(\theta)=0\}$, then $\alpha_.(\theta)>0$, and $\alpha_-(\theta)>0$ on $[0,\zeta^\theta)$, and $\alpha_.(\theta)=0$ on $[\zeta^\theta,\infty)$.
- For any bounded family $(Y_t(\omega,\theta),t\geq 0)$ measurable w.r.t. $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)$, (where $\mathcal{P}(\mathbb{F})$ is the $\mathbb{F}$-predictable σ-field), the $\mathbb{F}$-predictable projection of the process $Y_t(\omega,\tau)$ is the process $Y_t^{(p)}=\int\alpha_{t-}(\theta)Y_t(\theta)\eta(d\theta)$.
  In particular, for any t, $\mathbb{P}(\zeta^\tau<t)=\mathbb{E}[\int_0^\infty\alpha_{t-}(\theta)1_{\{\zeta^\theta<t\}}\eta(d\theta)]=0$. So, ζ^τ is infinite a.s.

We are also concerned with the càdlàg version of the martingale $(S_t(u),t\geq 0)$ for any $u\in\mathbb{R}^+$. By the previous result, we have a universal version of their predictable projections,

$$S_{t-}(u)=S_t^{(p)}(u)=\int_u^\infty\alpha_{t-}(\theta)\eta(d\theta).$$

It remains to define $S_t(u)=\lim_{q\in\mathbb{Q}^+,\,q\downarrow t}S_q^{(p)}(u)$ to obtain a universal càdlàg version of the martingales $S_.(u)$.
Remark that to show directly that $\int_u^\infty\alpha_t(\theta)\eta(d\theta)$ is a càdlàg process, we need stronger assumption on the process $\alpha_t(\theta)$ which allows us to apply the Lebesgue theorem w.r.t. $\eta(d\theta)$.

We say that the process $(Y_t(\omega,\theta),t\geq 0)$ is $\mathbb{F}$-optional if it is $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)$-measurable, where $\mathcal{O}(\mathbb{F})$ is the optional σ-field of $\mathbb{F}$. In particular, the process $(Y_t(\omega,t),t\geq 0)$ is optional.

4.2 Density and intensity processes

We are now interested in the relationship between the density and the intensity process of τ. As we shall see, this is closely related to the (additive and multiplicative) decompositions of the supermartingale S.

4.2.1 $\mathbb{F}$-decompositions of the survival process S

In this section, we characterize the martingale and the predictable increasing part of the additive and multiplicative Doob-Meyer decomposition of the supermartingale S in terms of the density.

Proposition 4.1
1) The Doob-Meyer decomposition of the survival process S is given by $S_t = 1 + M^{\mathbb{F}}_t - \int_0^t \alpha_u(u)\eta(du)$ where $M^{\mathbb{F}}$ is the càdlàg square-integrable $\mathbb{F}$-martingale defined by

$$M^{\mathbb{F}}_t = -\int_0^t \big(\alpha_t(u) - \alpha_u(u)\big)\eta(du) = \mathbb{E}[\int_0^\infty \alpha_u(u)\eta(du)|\mathcal{F}_t] - 1, \ a.s.$$

2) Let $\zeta^{\mathbb{F}} := \inf\{t : S_{t-} = 0\}$ and define $\lambda^{\mathbb{F}}_t := \frac{\alpha_t(t)}{S_{t-}}$ for any $t < \zeta^{\mathbb{F}}$ and $\lambda^{\mathbb{F}}_t := \lambda^{\mathbb{F}}_{t\wedge\zeta^{\mathbb{F}}}$ for any $t \geq \zeta^{\mathbb{F}}$. The multiplicative decomposition of S is given by

$$S_t = L^{\mathbb{F}}_t e^{-\int_0^t \lambda^{\mathbb{F}}_s \eta(ds)} \tag{10}$$

where $L^{\mathbb{F}}$ is the $\mathbb{F}$-local martingale solution of $dL^{\mathbb{F}}_t = e^{\int_0^t \lambda^{\mathbb{F}}_s \eta(ds)} dM^{\mathbb{F}}_t, \ L^{\mathbb{F}}_0 = 1.$

Proof: 1) First notice that $(\int_0^t \alpha_u(u)\eta(du), t \geq 0)$ is an $\mathbb{F}$-adapted continuous increasing process (the measure η does not have any atom). By the martingale property of $(\alpha_t(\theta), t \geq 0)$, for any fixed t,

$$S_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \int_t^\infty \alpha_t(u)\eta(du) = \mathbb{E}[\int_t^\infty \alpha_u(u)\eta(du)|\mathcal{F}_t], \ a.s..$$

Therefore, the non-negative process $S_t + \int_0^t \alpha_u(u)\eta(du) = \mathbb{E}[\int_0^\infty \alpha_u(u)\eta(du)|\mathcal{F}_t]$ is a square-integrable martingale since

$$\mathbb{E}\Big[\big(\int_0^\infty \alpha_u(u)\eta(du)\big)^2\Big] = 2\mathbb{E}\Big[\int_0^\infty \int_u^\infty \alpha_s(s)\eta(ds)\alpha_u(u)\eta(du)\Big] = 2\mathbb{E}\Big[\int_0^\infty S_u\alpha_u(u)\eta(du)\Big] \leq 2.$$

We shall choose its càdlàg version if needed. Using the fact that $\int_0^\infty \alpha_t(u)\eta(du) = 1$, we obtain

$$\forall t, \ \mathbb{E}[\int_0^\infty \alpha_u(u)\eta(du)|\mathcal{F}_t] = 1 - \int_0^t (\alpha_t(\theta) - \alpha_\theta(\theta))\eta(d\theta), \ a.s.$$

and the result follows.
2) Setting $L^{\mathbb{F}}_t = S_t e^{\int_0^t \lambda^{\mathbb{F}}_s \eta(ds)}$, integration by parts formula and 1) yield to

$$dL^{\mathbb{F}}_t = e^{\int_0^t \lambda^{\mathbb{F}}_s \eta(ds)} dS_t + e^{\int_0^t \lambda^{\mathbb{F}}_s \eta(ds)} \lambda^{\mathbb{F}}_t S_t \eta(dt) = e^{\int_0^t \lambda^{\mathbb{F}}_s \eta(ds)} dM^{\mathbb{F}}_t,$$

which implies the result. □

Remarks 4.2

1) Note that, from (3), $\mathbb{P}(\zeta^{\mathbb{F}} \geq \tau) = 1$.

2) The survival process S is a decreasing process if and only if the martingale $M^{\mathbb{F}}$ is constant ($M^{\mathbb{F}} \equiv 0$) or equivalently if and only if the martingale $L^{\mathbb{F}}$ is constant ($L^{\mathbb{F}} \equiv 1$). In that case, by Proposition 4.1, S is the continuous decreasing process $S_t = e^{-\int_0^t \lambda_s^{\mathbb{F}} \eta(ds)}$. Moreover, for any pair (t, θ), $t \leq \theta$, the conditional distribution is given by $S_t(\theta) = \mathbb{E}[e^{-\int_0^\theta \lambda_s^{\mathbb{F}} \eta(ds)} | \mathcal{F}_t]$.

3) The condition $M^{\mathbb{F}} \equiv 0$ can be written as $\int_0^t (\alpha_t(u) - \alpha_u(u))\eta(du) = 0$ and is satisfied if, for $t \geq u$, $\alpha_t(u) - \alpha_u(u) = 0$ (immersion property), but the converse is obviously not true (Remark 3.2).

4.2.2 Relationship with the $\mathbb{G}$-intensity

The intensity approach has been largely studied in the credit literature. We study now in more details the relationship between the density and the intensity, and notably between the $\mathbb{F}$-density process of τ and its intensity process with respect to $\mathbb{G}$. We first recall some definitions.

Definition 4.3 Let τ be a $\mathbb{G}$-stopping time. The $\mathbb{G}$-*compensator* of τ is the $\mathbb{G}$-predictable increasing process $\Lambda^{\mathbb{G}}$ such that the process $(N_t^{\mathbb{G}} = 1_{\{\tau \leq t\}} - \Lambda_t^{\mathbb{G}}, t \geq 0)$ is a $\mathbb{G}$-martingale. If $\Lambda^{\mathbb{G}}$ is absolutely continuous with respect to the measure η, the $\mathbb{G}$-adapted process $\lambda^{\mathbb{G}}$ such that $\Lambda_t^{\mathbb{G}} = \int_0^t \lambda_s^{\mathbb{G}} \eta(ds)$ is called the $(\mathbb{G}, \eta)$-*intensity process* or the $\mathbb{G}$-*intensity* if there is no ambiguity. The $\mathbb{G}$-compensator is stopped at τ, i.e., $\Lambda_t^{\mathbb{G}} = \Lambda_{t \wedge \tau}^{\mathbb{G}}$. Hence, $\lambda_t^{\mathbb{G}} = 0$ when $t > \tau$.

The following results give the $\mathbb{G}$-intensity of τ in terms of $\mathbb{F}$-density, and conversely the $\mathbb{F}$-density $\alpha_t(\theta)$ in terms of the $\mathbb{G}$-intensity, but only for $\theta \geq t$.

Proposition 4.4

1) The random time τ admits a $(\mathbb{G}, \eta)$-intensity given by

$$\lambda_t^{\mathbb{G}} = 1_{\{\tau > t\}} \lambda_t^{\mathbb{F}} = 1_{\{\tau > t\}} \frac{\alpha_t(t)}{S_t}, \quad \eta(dt) \text{ a.s..} \tag{11}$$

The processes $(N_t^{\mathbb{G}} := 1_{\{\tau \leq t\}} - \int_0^t \lambda_s^{\mathbb{G}} \eta(ds), t \geq 0)$, and $(L_t^{\mathbb{G}} := 1_{\{\tau > t\}} e^{\int_0^t \lambda_s^{\mathbb{G}} \eta(ds)}, t \geq 0)$ are $\mathbb{G}$-local martingales.

2) For any $t < \zeta^{\mathbb{F}}$ and $\theta \geq t$, we have: $\alpha_t(\theta) = \mathbb{E}[\lambda_\theta^{\mathbb{G}} | \mathcal{F}_t]$.

Then, the $\mathbb{F}$-optional projections of the local martingales $N^{\mathbb{G}}$ and $L^{\mathbb{G}}$ are the $\mathbb{F}$-local martingales $-M^{\mathbb{F}}$ and $L^{\mathbb{F}}$.

Proof: 1) The $\mathbb{G}$-local martingale property of $N^{\mathbb{G}}$ is equivalent to the $\mathbb{G}$-local martingale property of $L_t^{\mathbb{G}} = 1_{\{\tau>t\}} e^{\int_0^t \lambda_s^{\mathbb{G}} \eta(ds)} = 1_{\{\tau>t\}} e^{\int_0^t \lambda_s^{\mathbb{F}} \eta(ds)}$, since

$$dL_t^{\mathbb{G}} = -L_{t-}^{\mathbb{G}} d\mathbf{1}_{\{\tau\le t\}} + \mathbf{1}_{\{\tau>t\}} e^{\int_0^t \lambda_s^{\mathbb{F}} \eta(ds)} \lambda_t^{\mathbb{F}} \eta(dt) = -L_{t-}^{\mathbb{G}} dN_t^{\mathbb{G}}$$

Since the process $\int_0^t \lambda_s^{\mathbb{F}} \eta(ds)$ is continuous, we can proceed by localization introducing the $\mathbb{G}$-stopping times $\tau_n = \tau 1_{\{\tau\le T_n\}} + \infty 1_{\{\tau>T_n\}}$ where $T_n = \inf\{t : \int_0^t \lambda_s^{\mathbb{F}} \eta(ds) > n\}$. Then, the martingale property of the stopped process $L_{t\wedge\tau^n}^{\mathbb{G}} = L_t^{\mathbb{G},n}$ follows from the $\mathbb{F}$-martingale property of $L_{t\wedge T^n}^{\mathbb{F}} = L_t^{\mathbb{F},n}$, since for any $s \le t$,

$$\mathbb{E}[L_t^{\mathbb{G},n}|\mathcal{G}_s] = \mathbb{E}[1_{\{\tau>t\wedge T^n\}} e^{\int_0^{t\wedge T^n} \lambda_u^{\mathbb{F}} \eta(du)}|\mathcal{G}_s] = 1_{\{\tau>s\wedge T^n\}} \frac{\mathbb{E}[1_{\{\tau>t\wedge T^n\}} e^{\int_0^{t\wedge T^n} \lambda_u^{\mathbb{F}} \eta(du)}|\mathcal{F}_s]}{S_s}$$
$$= 1_{\{\tau>s\}} \frac{\mathbb{E}[S_{t\wedge T^n} e^{\int_0^{t\wedge T^n} \lambda_u^{\mathbb{F}} \eta(du)}|\mathcal{F}_s]}{S_s} = 1_{\{\tau>s\wedge T^n\}} \frac{L_s^{\mathbb{F},n}}{S_s}$$

where the last equality follows from the $\mathbb{F}$-martingale property of $L^{\mathbb{F},n}$.
Then, the form of the intensities follows from the definition.
2) By the martingale property of density, for any $\theta \ge t$, $\alpha_t(\theta) = \mathbb{E}[\alpha_\theta(\theta)|\mathcal{F}_t]$. Using the definition of S, and the value of $\lambda^{\mathbb{G}}$ given in 1), we obtain

$$\alpha_t(\theta) = \mathbb{E}\Big[\alpha_\theta(\theta)\frac{1_{\{\tau>\theta\}}}{S_\theta}|\mathcal{F}_t\Big] = \mathbb{E}[\lambda_\theta^{\mathbb{G}}|\mathcal{F}_t], \quad \forall t < \zeta^{\mathbb{F}},\ a.s.$$

Hence, the value of the density can be partially deduced from the intensity.
The $\mathbb{F}$-projection of the local martingale $L_t^{\mathbb{G}} = 1_{\{\tau>t\}} e^{\int_0^t \lambda_s^{\mathbb{G}} \eta(ds)}$ is the local martingale $S_t e^{\int_0^t \lambda_s^{\mathbb{F}} \eta(ds)} = L_t^{\mathbb{F}}$ by definition of the survival process S. Similarly, since $\alpha_t(\theta) = \mathbb{E}[\lambda_\theta^{\mathbb{G}}|\mathcal{F}_t]$, the $\mathbb{F}$-projection of the martingale $N_t^{\mathbb{G}} = 1_{\{\tau\le t\}} - \int_0^t \lambda_s^{\mathbb{G}} \eta(ds)$ is $1 - S_t - \int_0^t \alpha_s(s)\eta(ds) = -M_t^{\mathbb{F}}$.
□

Remarks 4.5

1) Since the intensity process is continuous, τ is a totally inaccessible $\mathbb{G}$-stopping time.
2) The density hypothesis, and the fact that η is non-atomic allow us to choose $\alpha_s(s)/S_s$ as an intensity, instead of $\alpha_s(s)/S_{s-}$ as it is usually done (see [6] in the case where the numerator $\alpha_s(s)$ represents the derivative of the compensator of S).
3) Proposition 4.1 shows that density and intensity approaches correspond respectively to the additive and the multiplicative decomposition point of view of the survival process S.

We now use the density-intensity relationship to characterize the pure jump $\mathbb{G}$-martingales having only one jump at τ.

Corollary 4.6

1) For any locally bounded $\mathbb{G}$-optional process $H^{\mathbb{G}}$, the process

$$N_t^{H,\mathbb{G}} := H_\tau^{\mathbb{G}} 1_{\{\tau \le t\}} - \int_0^{t\wedge\tau} \frac{\alpha_s(s)}{S_s} H_s^{\mathbb{G}} \eta(ds) = \int_0^t H_s^{\mathbb{G}} \, dN_s^{\mathbb{G}}, \quad t \ge 0 \tag{12}$$

is a $\mathbb{G}$-local martingale.

2) Conversely, any pure jump $\mathbb{G}$-martingale $M^{\mathbb{G}}$ which has only one locally bounded jump at τ can be written on the form (12), with $H_\tau^{\mathbb{G}} = M_\tau^{\mathbb{G}} - M_{\tau-}^{\mathbb{G}}$.

3) Any nonnegative pure jump $\mathbb{G}$-martingale $U^{\mathbb{G}}$ such that $U_0^{\mathbb{G}} = 1$, with only one jump at time τ has the following representation

$$U_t^{\mathbb{G}} = \big(u_\tau \mathbf{1}_{\{\tau \le t\}} + \mathbf{1}_{\{t<\tau\}}\big) e^{-\int_0^{t\wedge\tau}(u_s-1)\lambda_s^{\mathbb{F}}\eta(ds)}$$

where u is a positive $\mathbb{F}$-optional process associated with the relative jump such that $u_\tau = U_\tau^{\mathbb{G}}/U_{\tau-}^{\mathbb{G}}$.

Proof: 1) The $\mathbb{G}$-martingale property of $N^{\mathbb{G}}$ implies that $N^{H,\mathbb{G}}$ defined in (12) is a $\mathbb{G}$-martingale for any bounded predictable process $H^{\mathbb{G}}$. From a reinforcement of (5), if $H^{\mathbb{G}}$ is a $\mathbb{G}$-predictable process (typically $H_{t_0}^{\mathbb{G}} \mathbf{1}_{]t_0,\infty]}$), there exists an $\mathbb{F}$-predictable process $H^{\mathbb{F}}$ such that $H_\tau^{\mathbb{G}} = H_\tau^{\mathbb{F}}$, *a.s.*. Then the process $H^{\mathbb{G}}$ may be replaced by its representative $H^{\mathbb{F}}$ in the previous relations.

Let $Y_s^{\mathbb{G}}$ be a bounded $\mathcal{G}_s$-random variable, expressed in terms of $\mathbb{F}$-random variables as $Y_s^{\mathbb{G}} = Y_s^{\mathbb{F}} \mathbf{1}_{\{s<\tau\}} + Y_s^{\mathbb{F}}(\tau)\mathbf{1}_{\{\tau \le s\}}$ where $Y_s^{\mathbb{F}} \in \mathcal{F}_s$ and $Y_s^{\mathbb{F}}(\theta) \in \mathcal{F}_s \otimes \mathcal{B}([0,s])$, (typically $Y_s^{\mathbb{F}}(\theta) = Y_s^{\mathbb{F}} \times g(\theta)\mathbf{1}_{[0,s]}$). Then, the $\mathbb{G}$-martingale property still holds for the process $N^{H,\mathbb{G}}$ where $H^{\mathbb{G}}$ is the $\mathbb{G}$-optional process $H_s^{\mathbb{G}} = Y_s^{\mathbb{G}} \mathbf{1}_{[s,\infty)}$. The optional σ-field being generated by such processes, the assertion holds for any $\mathbb{G}$-optional process.

2) For the converse, observe that the locally bounded jump $H_\tau^{\mathbb{G}}$ of the martingale $M^{\mathbb{G}}$ at time τ is the value at time τ of some locally bounded $\mathbb{F}$-optional process $H^{\mathbb{F}}$. Then the difference $M^{\mathbb{G}} - N^{H,\mathbb{G}}$ is a finite variation local martingale without jump, that is a constant process.

3) It is easy to calculate the differential of the finite variation process $U^{\mathbb{G}}$ as

$$dU_t^{\mathbb{G}} = -U_t^{\mathbb{G}}(u_t - 1)\lambda_t^{\mathbb{G}}\eta(dt) + U_{t-}^{\mathbb{G}}((u_t-1)(dN_t^{\mathbb{G}} + \lambda_t^{\mathbb{G}}\eta(dt)) = U_{t-}^{\mathbb{G}}(u_t-1)dN_t^{\mathbb{G}}.$$

Then $U^{\mathbb{G}}$ is the exponential martingale of the purely jump martingale $(u_t - 1)dN_t^{\mathbb{G}}$. □

4.3 An example of HJM type

We now give some examples, where we point out similarities with Heath-Jarrow-Morton models. Here, our aim is not to present a general framework, therefore, we reduce our attention to the case where the reference filtration $\mathbb{F}$ is generated by a multidimensional standard Brownian motion W.

The following two propositions, which model the dynamics of the conditional probability $S(\theta)$, correspond respectively to the additive and multiplicative points of view. From the predictable representation theorem in the Brownian filtration, applied to the family of bounded martingales $(S_t(\theta), t \geq 0)$, $\theta \geq 0$, there exists a family of $\mathbb{F}$-predictable processes $(Z_t(\theta), t \geq 0)$ such that

$$dS_t(\theta) = Z_t(\theta) dW_t, \quad a.s. \tag{13}$$

Proposition 4.7 *Let $dS_t(\theta) = Z_t(\theta)dW_t$ be the martingale representation of $(S_t(\theta), t \geq 0)$ and assume that the processes $(Z_t(\theta); t \geq 0)$ are differentiable in the following sense: there exists a family of processes $(z_t(\theta), t \geq 0)$, bounded by an integrable process, such that $Z_t(\theta) = \int_0^\theta z_t(u)\eta(du)$. Then,*

1) The density martingales have the following dynamics $d\alpha_t(\theta) = -z_t(\theta)dW_t$.

2) The survival process S evolves as $dS_t = -\alpha_t(t)\eta(dt) + Z_t(t)dW_t$.

3) With more regularity assumptions, if $(\partial_\theta \alpha_t(\theta))_{\theta=t}$ is simply denoted by $\partial_\theta \alpha_t(t)$, then the process $\alpha_t(t)$ is driven by :

$$d\alpha_t(t) = \partial_\theta \alpha_t(t)\eta(dt) - z_t(t)dW_t$$

Proof: Observe that $Z(0) = 0$ since $S(0) = 1$, hence the existence of z is related with some smoothness conditions. Then

$$S_t(\theta) = S_0(\theta) + \int_0^t Z_u(\theta)dW_u = S_0(\theta) + \int_0^\theta \eta(dv) \int_0^t z_u(v)dW_u$$

and 1) follows. Furthermore, by using Proposition 4.1 and integration by parts,

$$M_t^{\mathbb{F}} = \int_0^t (\alpha_t(u) - \alpha_u(u))\eta(du) = \int_0^t \eta(du) \int_u^t z_s(u)dW_s = \int_0^t dW_s \Big(\int_0^s z_s(u)\eta(du) \Big)$$

which implies 2).

3) Let us use the short notation introduced above. We follow the same way as for the decomposition of S, by studying the process

$$\alpha_t(t) - \int_0^t (\partial_\theta \alpha_s)(s)\eta(d\theta) = \alpha_t(0) + \int_0^t (\partial_\theta \alpha_t)(s)\eta(ds) - \int_0^t (\partial_\theta \alpha_s)(s)\eta(ds)$$

Using martingale representation of $\alpha_t(\theta)$ and integration by parts, (assuming that smoothness hypothesis allows these operations) the integral in the RHS is a stochastic integral,

$$\int_0^t \Big((\partial_\theta \alpha_t)(s) - (\partial_\theta \alpha_s)(s) \Big) \eta(ds) = -\int_0^t \eta(ds) \partial_\theta (\int_s^t z_u(\theta) dW_u)$$
$$= -\int_0^t \eta(ds) \int_s^t \partial_\theta z_u(s) dW_u = -\int_0^t dW_u \int_0^u \eta(ds) \partial_\theta z_u(s) = -\int_0^t dW_u (z_u(u) - z_u(0))$$

The stochastic integral $\int_0^t dW_u z_u(0)$ is the stochastic part of the martingale $\alpha_t(0)$, and so the property 3) holds true. □

We now consider $(S_t(\theta), t \geq 0)$ in the classical HJM models (see [14]) where its dynamics is given in multiplicative form. By definition, the forward hazard rate $\lambda_t(\theta)$ of τ is given by $\lambda_t(\theta) = -\partial_\theta \ln S_t(\theta)$ and the density can then be calculated as $\alpha_t(\theta) = \lambda_t(\theta) S_t(\theta)$. As noted in Remark 2.3, $\lambda(\theta)$ plays the same role as the spot forward rate in the interest rate models. Classically, HJM framework is studied for time smaller than maturity, i.e. $t \leq T$. Here we consider all positive pairs (t, θ).

Proposition 4.8 *For any $t, \theta \geq 0$, let $\Psi_t(\theta) = \frac{Z_t(\theta)}{S_t(\theta)}$ with the notation of Proposition 4.7. We assume that $\psi_t(\theta)$ defined by $\Psi_t(\theta) = \int_0^\theta \psi_t(u)\eta(du)$ is bounded by some integrable process. Then*
1) $S_t(\theta) = S_0(\theta) \exp\left(\int_0^t \Psi_s(\theta) dW_s - \frac{1}{2}\int_0^t |\Psi_s(\theta)|^2 ds\right)$;
2) The forward hazard rate $\lambda(\theta)$ has the dynamics: $\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \psi_s(\theta) dW_s + \int_0^t \psi_s(\theta)\Psi_s(\theta)^* ds$;
3) $S_t = \exp\left(-\int_0^t \lambda_s^{\mathbb{F}} \eta(ds) + \int_0^t \Psi_s(s) dW_s - \frac{1}{2}\int_0^t |\Psi_s(s)|^2 ds\right)$;

Proof: By choice of notation, the process $S_t(\theta)$ is the solution of the equation

$$\frac{dS_t(\theta)}{S_t(\theta)} = \Psi_t(\theta) dW_t, \qquad \forall\, t, \theta \geq 0. \tag{14}$$

Hence 1), from which we deduce immediately 2) by differentiation w.r.t. θ.
3) This representation is the multiplicative version of the additive decomposition of S. There is not technical difficulties because S is continuous. □

Remarks 4.9 If $\Psi_s(s) = 0$, then $S_t = \exp(-\int_0^t \lambda_s^{\mathbb{F}} \eta(ds))$, which is decreasing. For the (H)-hypothesis to hold, it needs $\Psi_s(\theta) = 0$ for any $s \geq \theta$.
As a conditional survival probability, $S_t(\theta)$ is decreasing on θ, which is equivalent to that $\lambda_t(\theta)$ is positive. When $\theta > t$, this property is implied by the weaker condition $\lambda_t(t) \geq 0$. That is similar as for the zero coupon bond prices. But when $\theta < t$, additional assumption is necessary. We do not characterize this condition.

Remark 4.10 The above results are not restricted to the Brownian filtration and can be easily extended to more general filtrations under similar representation $dS_t(\theta) = Z_t(\theta) dM_t$ where M is a martingale which can include jumps. In this case, Proposition 4.7 can be generalized in a similar form; for Proposition 4.8, more attention should be payed to Doléans-Dade exponential martingales with jumps.

Example: We now give a particular example which provides a large class of forward rate processes. The non-negativity of λ is satisfied, by 2) of Proposition 4.8, if

• for any θ, the process $\psi(\theta)\Psi(\theta)$ is non negative, or if $\psi(\theta)$ is non negative;

• for any θ, the local martingale $\zeta_t(\theta) = \lambda_0(\theta) - \int_0^t \psi_s(\theta) dW_s$ is a Doléans-Dade exponential of some martingale, i.e., is solution of

$$\zeta_t(\theta) = \lambda_0(\theta) + \int_0^t \zeta_s(\theta) b_s(\theta) dW_s \,,$$

that is, if $-\int_0^t \psi_s(\theta) dW_s = \int_0^t b_s(\theta)\zeta_s(\theta) dW_s$. Here the initial condition is a positive constant $\lambda_0(\theta)$. Hence, we set

$$\psi_t(\theta) = -b_t(\theta)\zeta_t(\theta) = -b_t(\theta)\lambda_0(\theta) \exp\left(\int_0^t b_s(\theta) dW_s - \frac{1}{2}\int_0^t b_s^2(\theta) ds\right)$$

where λ_0 is a positive intensity function and $b(\theta)$ is a non-positive $\mathbb{F}$-adapted process. Then, the family

$$\alpha_t(\theta) = \lambda_t(\theta) \exp\left(-\int_0^\theta \lambda_t(v)\, dv\right),$$

where

$$\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \psi_s(\theta)\, dW_s + \int_0^t \psi_s(\theta)\Psi_s(\theta)\, ds$$

satisfies the required assumptions.

5 Characterization of $\mathbb{G}$-martingales in terms of $\mathbb{F}$-martingales

In the theory of pricing and hedging, martingale properties play a very important role. In this section, we study the martingale characterization when taking into account information of the default occurrence. The classical question in the enlargement of filtration theory is to give decomposition of $\mathbb{F}$-martingales in terms of $\mathbb{G}$-semimartingales. For the credit problems, we are concerned with the problem in a converse sense, that is, with the links between $\mathbb{G}$-martingales and $\mathbb{F}$-(local) martingales. In the literature, $\mathbb{G}$-martingales which are stopped at τ have been investigated, particularly in the credit context. For our analysis of after-default events, we are furthermore interested in the martingales which start at the default time τ and in martingales having one jump at τ, as the ones introduced in Corollary 4.6. We shall give characterization results for these types of $\mathbb{G}$-martingales in the following, by using a coherent formulation in the density framework.

5.1 $\mathbb{G}$-martingale characterization

Any $\mathbb{G}$-martingale may be split into two martingales, the first one stopped at time τ and the second one starting at time τ, that is

$$Y_t^{\mathbb{G}} = Y_t^{\text{bd},\mathbb{G}} + Y_t^{\text{ad},\mathbb{G}}$$

where $Y_t^{\text{bd},\mathbb{G}} := Y_{t\wedge\tau}^{\mathbb{G}}$ and $Y_t^{\text{ad},\mathbb{G}} := (Y_t^{\mathbb{G}} - Y_\tau^{\mathbb{G}})1_{\{\tau\le t\}}$. We now study the two types of martingales respectively.

The density hypothesis allows us to provide easily a characterization[3] of $\mathbb{G}$-martingales stopped at time τ.

Proposition 5.1 *A $\mathbb{G}$-adapted càdlàg process $Y^{\mathbb{G}}$ is a closed $\mathbb{G}$-martingale stopped at time τ if and only if there exist an $\mathbb{F}$-adapted càdlàg process Y defined on $[0,\zeta^{\mathbb{F}})$ and an $\mathbb{F}$-optional process Z such that $Y_t^{\mathbb{G}} = Y_t 1_{\{\tau>t\}} + Z_\tau 1_{\{\tau\le t\}}$ a.s. and that*

$$\Big(U_t := Y_t S_t + \int_0^t Z_s \alpha_s(s)\eta(ds),\ t\ge 0\Big) \quad \text{is an } \mathbb{F}\text{-martingale on } [0,\zeta^{\mathbb{F}}). \tag{15}$$

Equivalently, using the multiplicative decomposition of S as $S_t = L_t^{\mathbb{F}} e^{-\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)}$ on $[0,\zeta^{\mathbb{F}})$, the above condition (15) is equivalent to

$$\Big(L_t^{\mathbb{F}}[Y_t + \int_0^t (Z_s - Y_s)\lambda_s^{\mathbb{F}}\eta(ds)],\ t\ge 0\Big) \text{ is an } \mathbb{F}\text{-local martingale on } [0,\zeta^{\mathbb{F}}). \tag{16}$$

Proof: The conditional expectation of $Y_t^{\mathbb{G}}$ given $\mathcal{F}_t$ is the $\mathbb{F}$-martingale defined on $[0,\zeta^{\mathbb{F}})$ as $Y_t^{\mathbb{F}} = \mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] = Y_t S_t + \int_0^t Z_s \alpha_t(s)\eta(ds)$ by using the $\mathcal{F}_t$-density of τ. Notice that $Y_t^{\mathbb{F}}$ differs from U_t by $(\int_0^t Z_s(\alpha_t(s) - \alpha_s(s))\eta(ds), t\ge 0)$, which is an $\mathbb{F}$-local martingale (this can be easily checked using that Z is locally bounded and $(\alpha_t(s), t\ge 0)$ is $\mathbb{F}$-martingale). So U is also an $\mathbb{F}$-local martingale. Moreover, since $\mathbb{E}[|Y_t^{\mathbb{G}}|] < \infty$, for any $\mathbb{F}$-stopping time ϑ, the quantity $Y_\vartheta 1_{\{\tau>\vartheta\}}$ is integrable, hence $Y_\vartheta S_\vartheta$ is also integrable, and

$$\mathbb{E}[\int_0^{\zeta^{\mathbb{F}}} |Z_s|\alpha_s(s)\eta(ds)] = \mathbb{E}[|Y_\tau^{\mathbb{G}}|] < \infty,$$

which establishes that U is a martingale.

Conversely, if U is an $\mathbb{F}$-local martingale, it is easy to verify by Theorem 3.1 that $\mathbb{E}[Y_T^{\mathbb{G}} - Y_t^{\mathbb{G}}|\mathcal{G}_t] = 0$, a.s..

The second formulation is based on the multiplicative representation $S_t = L_t^{\mathbb{F}} e^{-\Lambda_t^{\mathbb{F}}}$ where $\Lambda_t^{\mathbb{F}} = \int_0^t \lambda_s^{\mathbb{F}}\eta(ds)$ is a continuous increasing process. Since $e^{\Lambda_t^{\mathbb{F}}} Y_t S_t = Y_t L_t^{\mathbb{F}}$ and $\alpha_t(t) = \lambda_t^{\mathbb{F}} S_t$, we have

$$d(Y_t L_t^{\mathbb{F}}) = e^{\Lambda_t^{\mathbb{F}}} d(Y_t S_t) + e^{\Lambda_t^{\mathbb{F}}} Y_t S_t \lambda_t^{\mathbb{F}}\eta(dt) = e^{\Lambda_t^{\mathbb{F}}} dU_t + (Y_t - Z_t)\lambda_t^{\mathbb{F}} L_t^{\mathbb{F}}\eta(dt).$$

[3] The following proposition was established in [2, Lemma 4.1.3] in a hazard process setting.

The local martingale property of the process U is then equivalent to that of $(Y_tL^{\mathbb{F}}_t - \int_0^t (Y_s - Z_s)\lambda^{\mathbb{F}}_s L^{\mathbb{F}}_s \eta(ds), t \geq 0)$, and then to the condition (16). □

Remark 5.2 A $\mathbb{G}$-martingale stopped at time τ and equal to 1 on $[0, \tau)$ is constant on $[0, \tau]$. Indeed, integration by parts formula proves that $(L^{\mathbb{F}}_t \int_0^t (1 - Z_s)\lambda^{\mathbb{F}}_s \eta(ds), t \geq 0)$ is a local martingale if and only if the continuous bounded variation process $(\int_0^t L^{\mathbb{F}}_s (1 - Z_s)\lambda^{\mathbb{F}}_s \eta(ds), t \geq 0)$ is a local-martingale, that is if $L^{\mathbb{F}}_s (1 - Z_s)\lambda^{\mathbb{F}}_s = 0$, which implies that $Z_s = 1$ on $[0, \zeta^{\mathbb{F}})$.

The before-default $\mathbb{G}$-martingale $Y^{\mathrm{bd},\mathbb{G}}$ can always be separated into two parts: a martingale which is stopped at τ and is continuous at τ; and a martingale which has a jump at τ.

Lemma 5.3 *Let $Y^{\mathrm{bd},\mathbb{G}}$ be a $\mathbb{G}$-martingale stopped at τ of the form $Y^{\mathrm{bd},\mathbb{G}}_t = Y_t 1_{\{\tau > t\}} + Z_\tau 1_{\{\tau \leq t\}}$. Then there exist two $\mathbb{G}$-martingales $Y^{\mathrm{c,bd}}$ and $Y^{\mathrm{d,bd}}$ such that $Y^{\mathrm{bd},\mathbb{G}} = Y^{\mathrm{c,bd}} + Y^{\mathrm{d,bd}}$ which satisfy the following conditions:*
1) $(Y^{\mathrm{d,bd}}_t = (Z_\tau - Y_\tau)1_{\{\tau \leq t\}} - \int_0^{t\wedge\tau} (Z_s - Y_s)\lambda^{\mathbb{F}}_s \eta(ds), t \geq 0)$ is a $\mathbb{G}$-martingale with a single jump at τ;
2) $(Y^{\mathrm{c,bd}}_t = \widetilde{Y}_{\tau\wedge t}, t \geq 0)$ is continuous at τ, where $\widetilde{Y}_t = Y_t + \int_0^t (Z_s - Y_s)\lambda^{\mathbb{F}}_s \eta(ds)$.

Proof: From Corollary 4.6, $Y^{\mathrm{d,bd}}$ is a martingale. The result follows. □

Corollary 5.4 *With the above notation, a martingale $Y^{\mathbb{G}}$ which is stopped and continuous at τ is characterized by: $(L^{\mathbb{F}}_t Y_t, t \geq 0)$ is an $\mathbb{F}$-local martingale. Furthermore, if $L^{\mathbb{F}}$ is a martingale, then this condition is equivalent to that Y is an $\mathbb{F}$-local martingale w.r.t. the probability measure $\mathbb{P}^L = L^{\mathbb{F}}_T \mathbb{P}$. In particular, under the immersion assumption, the $\mathbb{G}$-martingales stopped at time τ and continuous at τ are $\mathbb{F}$-martingales stopped at τ.*

Remark 5.5 Under immersion, $L^{\mathbb{F}}_t = 1$. So a process $Y^{\mathbb{G}}$ stopped at τ and continuous at time τ is a $\mathbb{G}$-martingale if and only if Y is an $\mathbb{F}$-local martingale.

We now concentrate on the $\mathbb{G}$-martingales starting at τ, which, as we can see below, are easier to characterize. The following proposition is a direct consequence of Theorem 3.1.

Proposition 5.6 *Any càdlàg integrable process $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale starting at τ with $Y_\tau = 0$ if and only if there exists an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-optional process $(Y_t(.), t \geq 0)$ such that $Y_t(t) = 0$ and $Y^{\mathbb{G}}_t = Y_t(\tau)1_{\{\tau \leq t\}}$ and that, for any $\theta > 0$, $(Y_t(\theta)\alpha_t(\theta), t \geq \theta \geq 0)$ are $\mathbb{F}$-martingales on $[\theta, \zeta^\theta)$, where ζ^θ is defined as in Section 4.1.*

Combining the previous results, we give the characterization of general $\mathbb{G}$-martingale.

Theorem 5.7 *A càdlàg process $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale if and only if there exist an $\mathbb{F}$-adapted càdlàg process Y and an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-optional process $Y_t(.)$ such that $Y^{\mathbb{G}}_t = Y_t 1_{\{\tau>t\}} + Y_t(\tau)1_{\{\tau\leq t\}}$ and*
1) the process $(Y_t S_t + \int_0^t Y_s(s)\alpha_s(s)\eta(ds), t \geq 0)$ or equivalently $(L^{\mathbb{F}}_t[Y_t + \int_0^t (Y_s(s) - Y_s)\lambda^{\mathbb{F}}_s \eta(ds)], t \geq 0)$ is an $\mathbb{F}$-local martingale;
2) for any $\theta \geq 0$, $(Y_t(\theta)\alpha_t(\theta), t \geq \theta)$ is an $\mathbb{F}$-martingale on $[\theta, \zeta^\theta)$.

PROOF: Notice that $Y^{\mathrm{ad},\mathbb{G}}_t = (Y_t(\tau) - Y_\tau(\tau))1_{\{\tau\leq t\}}$. Then the theorem follows directly by applying Propositions 5.1 and 5.6 on $Y^{\mathrm{bd},\mathbb{G}}$ and $Y^{\mathrm{ad},\mathbb{G}}$ respectively. □

Remark 5.8 We observe again the fact that to characterize what goes on before the default, it suffices to know the survival process S or the intensity $\lambda^{\mathbb{F}}$. However, for the after-default studies, we need the whole conditional distribution of τ, i.e., $\alpha_t(\theta)$ where $\theta \leq t$.

5.2 Decomposition of $\mathbb{F}$-(local) martingale

An important result in the enlargement of filtration theory is the decomposition of $\mathbb{F}$-(local) martingales as $\mathbb{G}$-semimartingales. Using the above results, we provide an alternative proof for a result established in [10], simplified by using the fact that any $\mathbb{F}$-martingale is continuous at time τ. Our method is interesting, since it gives the intuition of the decomposition without using any result on enlargement of filtrations.

Proposition 5.9 *Any $\mathbb{F}$-martingale $Y^{\mathbb{F}}$ is a $\mathbb{G}$-semimartingale can be written as $Y^{\mathbb{F}}_t = M^{Y,\mathbb{G}}_t + A^{Y,\mathbb{G}}_t$ where $M^{Y,\mathbb{G}}$ is a $\mathbb{G}$-martingale and $(A^{Y,\mathbb{G}}_t := A_t 1_{\{\tau>t\}} + A_t(\tau)1_{\{\tau\leq t\}}, t \geq 0)$ is an optional process with finite variation. Here*

$$A_t = \int_0^t \frac{d[Y^{\mathbb{F}}, S]_s}{S_s} \quad and \quad A_t(\theta) = \int_\theta^t \frac{d[Y^{\mathbb{F}}, \alpha(\theta)]_s}{\alpha_s(\theta)}. \tag{17}$$

PROOF: On the one hand, assuming that $Y^{\mathbb{F}}$ is a $\mathbb{G}$-semimartingale, it can be decomposed as the sum of a $\mathbb{G}$-(local)martingale and a $\mathbb{G}$-optional process $A^{Y,\mathbb{G}}$ with finite variation which can be written as $A_t 1_{\{\tau>t\}} + A_t(\tau)1_{\{\tau\leq t\}}$ where A and $A(\theta)$ are still unknown. Note that, since $Y^{\mathbb{F}}$ has no jump at τ (indeed, τ avoids $\mathbb{F}$-stopping times - see Corollary 2.2), we can choose $M^{Y,\mathbb{G}}$ such that $M^{Y,\mathbb{G}}$ and hence $A^{Y,\mathbb{G}}$ have no jump at τ. Applying the martingale characterization result obtained in Theorem 5.7 to the $\mathbb{G}$-local martingale

$$Y^{\mathbb{F}}_t - A^{Y,\mathbb{G}}_t = (Y^{\mathbb{F}}_t - A_t)1_{\{\tau>t\}} + (Y^{\mathbb{F}}_t - A_t(\tau))1_{\{\tau\leq t\}}$$

leads to the fact that the two processes

$$((Y_t^{\mathbb{F}} - A_t)L_t^{\mathbb{F}}, t \geq 0) \quad \text{and} \quad (\alpha_t(\theta)(Y_t^{\mathbb{F}} - A_t(\theta)), t \geq \theta) \tag{18}$$

are $\mathbb{F}$-(local) martingales. Since

$$d\big((Y_t^{\mathbb{F}} - A_t)L_t^{\mathbb{F}}\big) = (Y_{t-}^{\mathbb{F}} - A_{t-})dL_t^{\mathbb{F}} + L_{t-}^{\mathbb{F}} d(Y_t^{\mathbb{F}} - A_t) + d\langle Y^{\mathbb{F}}, L^{\mathbb{F}}\rangle_t^c + \Delta(Y_t^{\mathbb{F}} - A_t)\Delta L_t^{\mathbb{F}}$$

and

$$-L_{t-}^{\mathbb{F}} dA_t - \Delta A_t \Delta L_t^{\mathbb{F}} = -L_t^{\mathbb{F}} dA_t,$$

based on the intuition given by the Girsanov theorem, natural candidate for the finite variation processes A is $dA_t = d[Y^{\mathbb{F}}, L^{\mathbb{F}}]_t / L_t^{\mathbb{F}}$ where $[\ ,\ ]$ denotes the co-variation process. Similarly, $dA_t(\theta) = d[Y^{\mathbb{F}}, \alpha(\theta)]_t / \alpha_t(\theta)$. Then, using the fact that $Y^{\mathbb{F}}, L^{\mathbb{F}}, \alpha(\theta)$ are $\mathbb{F}$-local martingales, we obtain that $A = (1/L^{\mathbb{F}}) \star [Y^{\mathbb{F}}, L^{\mathbb{F}}]$ where $\star$ denotes the integration of $1/L^{\mathbb{F}}$ w.r.t. $[Y^{\mathbb{F}}, L^{\mathbb{F}}]$, and $A(\theta) = (1/\alpha(\theta)) \star [Y^{\mathbb{F}}, \alpha(\theta)]$ similarly. Then, since S is the product of the martingale L^F and an continuous increasing process $e^{\Lambda^{\mathbb{F}}}$, we have $d[Y^{\mathbb{F}}, L^{\mathbb{F}}]_t / L_t^{\mathbb{F}} = d[Y^{\mathbb{F}}, S]_t / S_t$ and obtain the first equality in (17).
On the other hand, define the optional process $A^{Y,\mathbb{G}}$ by using (17). It is not difficult to verify by Theorem 5.7 that $Y^{\mathbb{F}} - A^{Y,\mathbb{G}}$ is a $\mathbb{G}$-local martingale. It follows that $Y^{\mathbb{F}}$ is indeed a $\mathbb{G}$-semimartingale. □

Remark 5.10 Note that our decomposition differs from the usual one, since our process A is optional (and not predictable) and that we are using the co-variation process, instead of the predictable co-variation process. As a consequence our decomposition is not unique.

5.3 Girsanov theorem

Change of probability measure is a key tool in derivative pricing as in martingale theory. In credit risk framework, we are also able to calculate parameters of the conditional distribution of the default time w.r.t. a new probability measure. The links between change of probability measure and the initial enlargement have been established, in particular, in [9] and [1]. In statistics, it is motivated by the Bayesian approach [7].

We present a Girsanov type result, where the Radon-Nikodým density is given in an additive form instead of in a multiplicative one as in the classical literature. This makes the density of τ having simple form under the new probability measure.

Theorem 5.11 ***(Girsanov's theorem)*** *Let $Q_t^{\mathbb{G}} = q_t 1_{\{\tau > t\}} + q_t(\tau)1_{\{\tau \leq t\}}$ be a càdlàg positive $\mathbb{G}$-martingale with $Q_0^{\mathbb{G}} = q_0 = 1$. Let $\mathbb{Q}$ be the probability measure defined on $\mathcal{G}_t$ by $d\mathbb{Q} = Q_t^{\mathbb{G}} d\mathbb{P}$*

for any $t \in \mathbb{R}_+$ and $\mathbb{Q}^{\mathbb{F}}$ be the restriction of $\mathbb{Q}$ to $\mathbb{F}$, which has Radon-Nikodým density $Q^{\mathbb{F}}$, given by the projection of $Q^{\mathbb{G}}$ on $\mathbb{F}$, that is $Q^{\mathbb{F}}_t = q_t S_t + \int_0^t q_t(u)\alpha_t(u)\eta(du)$.
Then $(\Omega, \mathbb{Q}, \mathbb{G}, \mathbb{F}, \tau)$ satisfies the density hypothesis with the $(\mathbb{F}, \mathbb{Q})$-density of τ given in closed form after the default, that is for $\theta \le t$ by

$$\alpha^{\mathbb{Q}}_t(\theta) = \alpha_t(\theta)\frac{q_t(\theta)}{Q^{\mathbb{F}}_t}, \qquad \eta(d\theta)\text{- a.s.};$$

and, only via a conditional expectation before the default, that is for $t \le \theta$, by

$$\alpha^{\mathbb{Q}}_t(\theta) = \frac{1}{Q^{\mathbb{F}}_t}\,\mathbb{E}^{\mathbb{P}}[\alpha_\theta(\theta)q_\theta(\theta)|\mathcal{F}_t].$$

Furthermore:
1) the $\mathbb{Q}$-conditional survival process is defined on $[0, \zeta^{\mathbb{F}})$ by $S^{\mathbb{Q}}_t = S_t\frac{q_t}{Q^{\mathbb{F}}_t}$, and is null after $\zeta^{\mathbb{F}}$;
2) the $(\mathbb{F}, \mathbb{Q})$-local martingale $L^{\mathbb{F},\mathbb{Q}}$ is $(L^{\mathbb{F},\mathbb{Q}}_t = L^{\mathbb{F}}_t\frac{q_t}{Q^{\mathbb{F}}_t}\exp\int_0^t(\lambda^{\mathbb{F},\mathbb{Q}}_s - \lambda^{\mathbb{F}}_s)\eta(ds), t \in [0, \zeta^{\mathbb{F}}))$;
3) the $(\mathbb{F}, \mathbb{Q})$-intensity process is $\lambda^{\mathbb{F},\mathbb{Q}}_t = \lambda^{\mathbb{F}}_t\frac{q_t(t)}{q_t}, \eta(dt)$-a.s.

Proof: The expression of the density process after the default $(\alpha^{\mathbb{Q}}_t(\theta), \theta \le t)$ is an immediate consequence of definition. Before the default, the density may be only obtained via a conditional expectation form given by

$$\begin{aligned}\alpha^{\mathbb{Q}}_t(\theta) &= \mathbb{E}^{\mathbb{Q}}\big[\alpha_\theta(\theta)\frac{q_\theta(\theta)}{Q^{\mathbb{F}}_\theta}|\mathcal{F}_t\big] = \frac{1}{Q^{\mathbb{F}}_t}\,\mathbb{E}^{\mathbb{P}}\big[Q^{\mathbb{F}}_\theta\alpha_\theta(\theta)\frac{q_\theta(\theta)}{Q^{\mathbb{F}}_\theta}|\mathcal{F}_t\big] \\ &= \frac{1}{Q^{\mathbb{F}}_t}\,\mathbb{E}^{\mathbb{P}}[\alpha_\theta(\theta)q_\theta(\theta)|\mathcal{F}_t].\end{aligned}$$

For any $t \in [0, \zeta^{\mathbb{F}})$, the $\mathbb{Q}$-conditional survival probability can be calculated by

$$S^{\mathbb{Q}}_t = \mathbb{Q}(\tau > t|\mathcal{F}_t) = \frac{\mathbb{E}^{\mathbb{P}}[1_{\{\tau>t\}}Q^{\mathbb{G}}_t|\mathcal{F}_t]}{Q^{\mathbb{F}}_t} = q_t\frac{S_t}{Q^{\mathbb{F}}_t}$$

and finally, we use $\lambda^{\mathbb{F},\mathbb{Q}}_t = \alpha^{\mathbb{Q}}_t(t)/S^{\mathbb{Q}}_t$ and $L^{\mathbb{F},\mathbb{Q}}_t = S^{\mathbb{Q}}_t e^{\int_0^t \lambda^{\mathbb{F},\mathbb{Q}}_s \eta(ds)}$ to complete the proof. □

It is known, from [8], that under density hypothesis, there exists at least a change of probability, such that immersion property holds under this change of probability. Theorem 5.11 provides a full characterization of such changes of probability.

Corollary 5.12 *We keep the notation of Theorem 5.11. The change of probability measure generated by the two processes*

$$q_t = (L^{\mathbb{F}}_t)^{-1}, \qquad q_t(\theta) = \frac{\alpha_\theta(\theta)}{\alpha_t(\theta)} \tag{19}$$

provides a model where the immersion property holds true under $\mathbb{Q}$, and where the intensity processes does not change, i.e., remains $\lambda^{\mathbb{F}}$.
More generally, the only changes of probability measure for which the immersion property holds with the same intensity process are generated by a process q such that $(q_t L_t^{\mathbb{F}}, t \geq 0)$ is a uniformly integrable martingale.

PROOF: : Any change of probability measure with immersion property and the same intensity processes is characterized by the martingale property of the product $Q^{\mathbb{F}} = q.\, L^{\mathbb{F}}$. Moreover, given q, the immersion property determines in a unique way the process $(q_t(\theta); t \geq \theta)$ via the boundary condition $q_\theta(\theta) = q_\theta$ and the equalities

$$\alpha_t^{\mathbb{Q}}(\theta) = \alpha_t(\theta)\frac{q_t(\theta)}{q_t\, L_t^{\mathbb{F}}} = \alpha_\theta(\theta)\frac{q_\theta(\theta)}{q_\theta\, L_\theta^{\mathbb{F}}} = \alpha_\theta(\theta)(1/L_\theta^{\mathbb{F}}).$$

The martingale $Q^{\mathbb{F}} = q.\, L^{\mathbb{F}}$ has to satisfy the compatibility condition

$$\begin{aligned} Q_t^{\mathbb{F}} &= q_t\, L_t^{\mathbb{F}} = q_t S_t + Q_t^{\mathbb{F}} \int_0^t \alpha_u(u)(1/L_u^{\mathbb{F}})\eta(du) \\ &= Q_t^{\mathbb{F}}\big(e^{-\int_0^t \lambda_s^{\mathbb{F},\mathbb{P}}\eta(ds)} + \int_0^t e^{-\int_0^u \lambda_s^{\mathbb{F},\mathbb{P}}\eta(ds)}\lambda_u^{\mathbb{F},\mathbb{P}}\,\eta(du)\big) \end{aligned} \tag{20}$$

where the last equality comes from the identities (10) and (11). The term in the bracket in (20) is of finite variation and is hence equal to 1. Then the $Q^{\mathbb{F}}$-compatibility condition is always satisfied. So the only constraint on the process q is the martingale property of $q.\, L^{\mathbb{F}}$. □

It is well known, from Kusuoka [11], that immersion property is not stable by a change of probability. In the following, we shall in a first step characterize, under density hypothesis, changes of probability which preserve this immersion property, that is, H-hypothesis is satisfied under both $\mathbb{P}$ and $\mathbb{Q}$. (See also [4] for a different study of changes of probabilities preserving immersion property.) In a second step, we shall study changes of probability which preserve the information before the default, and give the impact of a change of probability after the default.

Corollary 5.13 *We keep the notation of Theorem 5.11, and assume immersion property under $\mathbb{P}$.*
1) Let the Radon-Nikodým density $(Q_t^{\mathbb{G}}, t \geq 0)$ be a pure jump martingale with only one jump at time τ. Then, the $(\mathbb{F},\mathbb{P})$-martingale $(Q_t^{\mathbb{F}}, t \geq 0)$ is the constant martingale equal to 1. Under $\mathbb{Q}$, the intensity process is $\lambda_t^{\mathbb{F},\mathbb{Q}} = \lambda_t^{\mathbb{F}}\dfrac{q_t(t)}{q_t}$, $\eta(dt)$-a.s., and the immersion property still holds.
2) Conversely, the only changes of probability measure compatible with immersion property have Radon-Nikodým densities that are the product of a pure jump positive martingale with only one jump at time τ, and a positive $\mathbb{F}$-martingale.

PROOF: : The previous Girsanov theorem 5.11 gives immediately the intensity characterization. From Lemma 5.3, the pure jump martingale $(Q_t^{\mathbb{G}}, t \geq 0)$ is a finite variation process and $(Q_t^{\mathbb{G}} = q_t, t < \tau)$ is a continuous process with bounded variation. Since immersion property holds, S is a continuous decreasing process (see footnote 2), and $Q_t^{\mathbb{F}} = q_t S_t + \int_0^t q_u(u)\alpha_t(u)\eta(du) = q_t S_t + \int_0^t q_u(u)\alpha_u(u)\eta(du)$ is a continuous martingale with finite variation. Since $Q_0^{\mathbb{F}} = 1$, then at any time, $Q_t^{\mathbb{F}} = 1$, *a.s.* By the other results established in Girsanov's theorem 5.11, this key point implies that the new density is constant after the default, so that the immersion property still holds.
2) Thanks to the first part of this corollary, we can restrict our attention to the case when in the both universe the intensity processes are the same. Then the Radon-Nikodým density is continuous at time τ and the two processes $(q_t, t \geq 0)$ and $(q_t(\theta), t \geq \theta)$ are $\mathbb{F}$-(local) martingales. Assume now that the immersion property holds also under the new probability measure $\mathbb{Q}$. Both martingales $L^{\mathbb{F},\mathbb{P}}$ and $L^{\mathbb{F},\mathbb{Q}}$ are constant, and $Q_t^{\mathbb{F}} = q_t$. Moreover the $\mathbb{Q}$-density process being constant after the default $(\theta < t)$, $q_t(\theta)/q_t = q_\theta(\theta)/q_\theta = 1$, *a.s.*. The processes $Q^{\mathbb{G}}$, $Q^{\mathbb{F}}$ and q are undistinguishable. □

As shown in this paper, the knowledge of the intensity does not allow to give full information on the law of the default, except if immersion property holds. Starting with a model under which immersion property holds, taking $q_t(t) = q_t$ in Theorem 5.11 will lead us to a model where the default time admits the same intensity whereas immersion property does not hold, and then the impact of the default changes the dynamics of the default-free assets. We present a specific case where, under the two probability measures, the dynamics of these assets are the same before the default but are changed after the default, a phenomenon that is observed in the actual crisis. We impose that the new probability $\mathbb{Q}$ coincide with $\mathbb{P}$ on the σ-algebra $\mathcal{G}_\tau$. In particular, if m is an $(\mathbb{F}, \mathbb{P})$-martingale, the process $(m_{t\wedge\tau}, t \geq 0)$ will be an $((\mathcal{G}_{t\wedge\tau}, t \geq 0),\mathbb{Q})$-martingale (but not necessarily an $(\mathbb{G}, \mathbb{Q})$-martingale). From Theorems 5.7 and 5.11 and Corollary 5.13, one gets the obvious proposition.

Proposition 5.14 *Let $(\Omega, \mathbb{P}, \mathbb{F}, \mathbb{G}, \tau)$ be a model satisfying the immersion property.*
Let $(q_t(\theta), t \geq \theta)$ be a family of positive $(\mathbb{F}, \mathbb{P})$-martingales such that $q_\theta(\theta) = 1$ and let $\mathbb{Q}$ be the probability measure with Radon-Nikodým density equal to the $(\mathbb{G}, \mathbb{P})$-martingale

$$Q_t^{\mathbb{G}} = 1_{\{\tau > t\}} + q_t(\tau)1_{\{\tau \leq t\}} . \tag{21}$$

Then, $\mathbb{Q}$ and $\mathbb{P}$ coincide on $\mathcal{G}_\tau$ and the $\mathbb{P}$ and $\mathbb{Q}$ intensities of τ are the same.
Furthermore, if $S^{\mathbb{Q}}$ is the $\mathbb{Q}$-survival process, the processes $(S_t/S_t^{\mathbb{Q}}, t \geq 0)$ and the family $(\alpha_t^{\mathbb{Q}}(\theta)\, S_t/S_t^{\mathbb{Q}},\, t \geq \theta)$ are $(\mathbb{F}, \mathbb{P})$-martingales.

Proof: The first part is a direct consequence of the previous results. It remains to note that $Q_t^{\mathbb{F}} = \frac{S_t}{S_t^{\mathbb{Q}}}$ and, for $t \geq \theta$, $\alpha_t^{\mathbb{Q}}(\theta) = \alpha_\theta(\theta)\frac{q_t(\theta)}{Q_t^{\mathbb{F}}} = \alpha_\theta(\theta)\frac{q_t(\theta)}{S_t}S_t^{\mathbb{Q}}$; hence the martingale properties follow from the ones of $Q_t^{\mathbb{F}}$ and $q_t(\theta)$. □

This result admits a converse. For the sake of simplicity we assume the condition $S_t^* > 0, \forall t \in \mathbb{R}_+$. This assumption can be removed, using the terminal time $\zeta^{*,\mathbb{F}}$.

Proposition 5.15 *Let $(\Omega, \mathbb{P}, \mathbb{F}, \mathbb{G}, \tau)$ be a model satisfying the H-hypothesis, with the decreasing survival process $S_t = \exp(-\int_0^t \lambda_s^{\mathbb{F}}\eta(ds))$.*
Let $(\alpha_t^*(\theta), t \geq \theta)$ be a given family, where, for all $\theta > 0$, $\alpha^*(\theta)$ is a non-negative process and define $S_t^* = 1 - \int_0^t \alpha_t^*(\theta)\eta(d\theta)$. Assume that $S_\infty^* = 0$ and $S_t^* > 0, \forall t \in \mathbb{R}_+$ and that

$$\begin{cases} \forall\theta,\ \alpha_\theta^*(\theta) = S_\theta^*\lambda_\theta^{\mathbb{F}} = \alpha_\theta(\theta)\frac{S_\theta^*}{S_\theta} \\ \textit{the processes } \left(\frac{S_t}{S_t^*}, t \geq 0\right) \textit{ and } \left(\alpha_t^*(\theta)\frac{S_t}{S_t^*}, t \geq \theta\right) \textit{ are } (\mathbb{F}, \mathbb{P})\textit{-martingales.} \end{cases} \tag{22}$$

Let

$$Q_t^{\mathbb{G}} := 1_{\{\tau > t\}} + \frac{\alpha_t^*(\tau)}{\alpha_\tau(\tau)}\frac{S_t}{S_t^*}1_{\{\tau \leq t\}}, \tag{23}$$

and $\mathbb{Q}$ be the probability measure with Radon-Nikodỳm density the $(\mathbb{G}, \mathbb{P})$-martingale $Q^{\mathbb{G}}$. Then, $\mathbb{Q}$ is equal to $\mathbb{P}$ on $\mathcal{G}_\tau$ and

$$\lambda^{\mathbb{Q},\mathbb{F}} = \lambda^{\mathbb{F}}, \quad \alpha_t^{\mathbb{Q}}(\theta) = \alpha_t^*(\theta)\,, \forall t \geq \theta \quad \textit{and} \quad S^{\mathbb{Q}} = S^* \tag{24}$$

Proof: We set

$$q_t(\theta) = \frac{\alpha_t^*(\theta)}{\alpha_\theta(\theta)}\frac{S_t}{S_t^*}.$$

Note that $q_t(t) = 1$ since $\alpha_s^*(s) = S_s^*\lambda_s^{\mathbb{F}} = S_s^*\alpha_s(s)/S_s$. For every θ, the processes $(q_t(\theta), t \geq \theta)$ are martingales since $(\alpha_t^*(\theta)S_t/S_t^*,\ t \geq \theta)$ are martingales. From Theorem 5.11, the $\mathbb{F}$-projection of the Radon-Nikodým density $Q^{\mathbb{G}}$ is

$$Q_t^{\mathbb{F}} = S_t + \int_0^t \frac{\alpha_t^*(s)}{\alpha_t(s)}\frac{S_t}{S_t^*}\alpha_t(s)\eta(ds) = S_t\left(1 + \int_0^t \alpha_t^*(s)\frac{1}{S_t^*}\eta(ds)\right),$$

and the survival probability is

$$\frac{S_t}{Q_t^{\mathbb{F}}} = \left(1 + \frac{1}{S_t^*}\int_0^t \alpha_t^*(s)\eta(ds)\right)^{-1} = S_t^*\left(S_t^* + \int_0^t \alpha_t^*(s)\eta(ds)\right)^{-1} = S_t^*.$$

It remains to note that the condition $\alpha_s^*(s) = S_s^*\lambda_s^{\mathbb{F}}$ is equivalent to the fact that the intensity of τ under $\mathbb{Q}$ is $\lambda^{\mathbb{F}}$. □

6 Conclusion

Our study relies on the impact of information related to the default time on the market.

Starting from a default-free model, where some assets are traded with the knowledge of a reference filtration $\mathbb{F}$, we consider the case where the participants of the market take into account the possibility of a default in view of trading default-sensitive asset. If we are only concerned by what happens up to the default time, the natural assumption is to assume immersion property with stochastic intensity process adapted to the default-free market evolution.
The final step is to anticipate that the default should have a large impact on the market, as now after the crisis. In particular, with the non constant "after default" density, we express how the default-free market is modified after the default. In addition, hedging strategies of default-free contingent claims are not the same in the both universes.

In a following paper [5], we shall apply this methodology to several default times, making this tool powerful for correlation of defaults. In another paper, we shall provide explicit examples of density processes, and give some general construction of these processes.

References

[1] Ankirchner, S., S. Dereich and P. Imkeller (2007): "Enlargement of filtrations and continuous Girsanov-type embeddings", in *Séminaire de Probabilités XL*, LNM1899, Springer.

[2] Bielecki, T.R. and M. Jeanblanc and M. Rutkowski (2006): "Credit risk", AMAMEF Side/Antalya.

[3] Brémaud, P. and M. Yor (1981): "Changes of filtrations and of probability measures", *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 45(4), 269-295.

[4] Coculescu, D., M. Jeanblanc and A. Nikeghbali (2009): "Default times, non arbitrage conditions and change of probability measures", working paper.

[5] El Karoui, N., M. Jeanblanc, and Y. Jiao (2009): "Successive default times", working Paper.

[6] Elliott, R.J., M. Jeanblanc and M. Yor (2000): "On models of default risk", *Mathematical Finance*, 10, 179-195.

[7] Gasbarra, D., E. Valkeila and L. Vostrikova (2006): "Enlargement of filtration and additional information in pricing models: Bayesian approach", in *From Stochastic Calculus to Mathematical Finance*, 257-285, Springer.

[8] Grorud and M. Pontier (2001): "Asymmetrical information and incomplete markets", *International Journal of Theoretical and Applied Finances*, 4, 285-302.

[9] Jacod, J. (1987): "Grossissement initial, Hypothèse (H') et théorème de Girsanov", in *Séminaire de Calcul Stochastique*, (1982/1983), vol. 1118, Lecture Notes in Mathematics, Springer.

[10] Jeanblanc, M. and Y. Le Cam (2008): "Progressive enlargement of filtration with initial times", to appear in *Stochastic Processes and their Applications.*

[11] Kusuoka, S. (1999): "A remark on default risk models", *Advances in Mathematical Economics*, 1, 69-82.

[12] Lando, D. (1998): "On Cox processes and credit risky securities", *Derivatives Research*, 2, 99-120.

[13] Nikeghbali, A. and M. Yor (2005): "A definition and some properties of pseudo-stopping times", *the Annals of Probability*, 33, 1804-1824.

[14] Schönbucher, P.J. (2003): Credit derivatives pricing models, models, pricing and implementation, Wiley.

[15] Stricker, C. and M. Yor (1978): "Calcul stochastique dépendant d'un paramètre", *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 45, 109-133.