A dichotomy characterizing analytic digraphs of uncountable Borel chromatic number in any dimension.

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Abstract. We study the extension of the Kechris-Solecki-Todorčević dichotomy on analytic graphs to dimensions higher than 2. We prove that the extension is possible in any dimension, finite or infinite. The original proof works in the case of the finite dimension. We first prove that the natural extension does not work in the case of the infinite dimension, for the notion of continuous homomorphism used in the original theorem. Then we solve the problem in the case of the infinite dimension. Finally, we prove that the natural extension works in the case of the infinite dimension, but for the notion of Baire-measurable homomorphism.

1 Introduction.

The reader should see [K] for the standard descriptive set theoretic notation used in this paper. We study a definable coloring problem, in any dimension. We will need some more notation:

Notation. In this paper, $2\leq d\leq\omega$ will be a cardinal, i.e., any dimension of an actual product making sense in the context of descriptive set theory. The letters X, Y will refer to some sets. We set

$$\Delta^d(X):=\{(x_i)_{i\in d}\in X^d\mid\forall i\in d\ \ x_i=x_0\}.$$

Definition 1.1 *Let $A\subseteq X^d$. We say that A is a $\underline{digraph}$ if $A\cap\Delta^d(X)=\emptyset$.*

Notation. Let $u:X\rightarrow Y$ be a map. We define a map $u^d:X^d\rightarrow Y^d$ by

$$u^d[(x_i)_{i\in d}]:=[u(x_i)]_{i\in d}.$$

Definition 1.2 *Let $A\subseteq X^d$ be a digraph.*
(a) A $\underline{coloring}$ of $[X,A]$ is a map $c:X\rightarrow Y$ such that $A\cap(c^d)^{-1}[\Delta^d(Y)]=\emptyset$.
(b) Assume that X is a Polish space. The $\underline{Borel\ chromatic\ number}$ of $[X,A]$ is

$$\chi_B(A):=\text{min}\{\ \text{Card}(Y)\mid Y\text{ is a Polish space and there is a Borel coloring }c:X\rightarrow Y\text{ of }[X,A]\ \}.$$

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The goal of this paper is to characterize the analytic digraphs of uncountable Borel chromatic number. This has been done in [K-S-T] for graphs, i.e., for symmetric digraphs, when $d=2$. We will give such a characterization in terms of the following notion of comparison between relations.

Notation. Assume that X, Y are Polish spaces, and let A (resp., B) be a subset of X^d (resp., Y^d). We set

$$[X,A]\preceq_B [Y,B] \ \Leftrightarrow\ \exists u\!:\!X\to Y \text{ Borel with } A\subseteq (u^d)^{-1}(B).$$

In this case, we say that u is a Borel *homomorphism* from $[X,A]$ into $[Y,B]$. This notion essentially makes sense for digraphs (we can take u to be constant if B is not a digraph). If u is continuous (resp., Baire-measurable, arbitrary), then we write $\preceq_c$ (resp., $\preceq_{Bm}$, $\preceq$) instead of $\preceq_B$. Note that $\chi_B(A)\leq\omega$ is equivalent to $[X,A]\preceq_B [\omega,\neg\Delta^d(\omega)]$.

We also have to introduce minimum digraphs of uncountable Borel chromatic number:

• Let $\psi_d\!:\!\omega\to d^{<\omega}$ be the natural bijection, for $d\leq\omega$. More specifically,

- If $d<\omega$, then $\psi_d(0):=\emptyset$ is the sequence of length 0, $\psi_d(1):=0$, ..., $\psi_d(d):=d-1$ are the sequences of length 1, and so on.

- If $d=\omega$, then let $(p_n)_{n\in\omega}$ be the sequence of prime numbers, and $I:\omega^{<\omega}\to\omega$ defined by $I(\emptyset):=1$, and $I(s):=p_0^{s(0)+1}...p_{|s|-1}^{s(|s|-1)+1}$ if $s\neq\emptyset$. Note that I is one-to-one, so that there is an increasing bijection $\varphi\!:\!\text{Seq}:=I[\omega^{<\omega}]\to\omega$. If $t\in\text{Seq}\subseteq\omega$, then we will denote $\overline{t}:=I^{-1}(t)\in\omega^{<\omega}$. We set $\psi_\omega:=(\varphi\circ I)^{-1}\!:\!\omega\to\omega^{<\omega}$. Note that ψ_ω is a bijection.

• Note also that $|\psi_d(n)|\leq n$ if $n\in\omega$. Indeed, this is clear if $d<\omega$. If $d=\omega$, then

$$I[\psi_\omega(n)|0]<I[\psi_\omega(n)|1]<...<I[\psi_\omega(n)],$$

so that $(\varphi\circ I)[\psi_\omega(n)|0]<(\varphi\circ I)[\psi_\omega(n)|1]<...<(\varphi\circ I)[\psi_\omega(n)]=n$. This implies that $|\psi_\omega(n)|\leq n$.

• Let $n\in\omega$. As $|\psi_d(n)|\leq n$, we can define $s^d_n:=\psi_d(n)0^{n-|\psi_d(n)|}$. The crucial properties of the sequence $(s^d_n)_{n\in\omega}$ are the following:

- For each $s\in d^{<\omega}$, there is $n\in\omega$ such that $s\subseteq s^d_n$ (we say that $(s^d_n)_{n\in\omega}$ is *dense* in $d^{<\omega}$).

- $|s^d_n|=n$.

• We put

$$\mathbb{A}_d:=\{(s^d_n i\gamma)_{i\in d}\mid n\in\omega \text{ and } \gamma\in d^\omega\}\subseteq (d^\omega)^d.$$

Note that $\mathbb{A}_d\in\mathbf{\Sigma}^1_1$ since the map $(n,\gamma)\mapsto(s^d_n i\gamma)_{i\in d}$ is continuous.

The previous definitions were given, when $d=2$, in [K-S-T], where the following is proved:

Theorem 1.3 *(Kechris, Solecki, Todorčević) Let X be a Polish space, and $A\in\mathbf{\Sigma}^1_1(X^2)$. Then exactly one of the following holds:*
(a) $[X,A]\preceq_B [\omega,\neg\Delta^2(\omega)]$.
(b) $[2^\omega,\mathbb{A}_2]\preceq_c [X,A]$.

This result can be extended to any finite dimension d, with the same proof.

Theorem 1.4 *Let $d\geq 2$ be an integer, X a Polish space, and $A\in\mathbf{\Sigma}^1_1(X^d)$. Then exactly one of the following holds:*
(a) $[X,A]\preceq_B[\omega,\neg\Delta^d(\omega)]$.
(b) $[d^\omega,\mathbb{A}_d]\preceq_c[X,A]$.

We want to study the case of the infinite dimension.

Theorem 1.5 *We cannot extend Theorem 1.4 to the case where $d=\omega$.*

Notation. In order to get a positive result in the case of the infinite dimension, we put

$$\mathbb{G}:=\{\alpha\in\omega^\omega\mid\forall m\in\omega\ \exists n\geq m\ \ s^\omega_n0\subseteq\alpha\}.$$

Note that $\mathbb{G}$ is a dense G_δ subset of ω^ω.

The main result of this paper is the following:

Theorem 1.6 *Let X be a Polish space, and $A\in\mathbf{\Sigma}^1_1(X^\omega)$. Then exactly one of the following holds:*
(a) $[X,A]\preceq_B[\omega,\neg\Delta^\omega(\omega)]$.
(b) $[\mathbb{G},\mathbb{A}_\omega\cap\mathbb{G}^\omega]\preceq_c[X,A]$.

So we have a general characterization, in any dimension d, of analytic relations $A\subseteq X^d$ for which $[X,A]\not\preceq_B[\omega,\neg\Delta^d(\omega)]$. In particular, we have a characterization of analytic digraphs of uncountable Borel chromatic number.

Theorem 1.5 says that we cannot extend Theorem 1.4 to the case where $d=\omega$ for the notion of continuous homomorphism in (b). However, the extension of Theorem 1.4 to the case where $d=\omega$ is possible for the notion of Baire-measurable homomorphism:

Theorem 1.7 *Let X be a Polish space, and $A\in\mathbf{\Sigma}^1_1(X^\omega)$. Then exactly one of the following holds:*
(a) $[X,A]\preceq_B[\omega,\neg\Delta^\omega(\omega)]$.
(b) $[\omega^\omega,\mathbb{A}_\omega]\preceq_{Bm}[X,A]$.

2 The proof in finite dimension.

Let us start with two general lemmas:

Lemma 2.1 *Let G be a dense G_δ subset of d^ω. Then $[G,\mathbb{A}_d\cap G^d]\not\preceq_{Bm}[\omega,\neg\Delta^d(\omega)]$.*

Proof. We argue by contradiction. This gives a Baire-measurable function $u:G\rightarrow\omega$ such that $\mathbb{A}_d\cap G^d\subseteq(u^d)^{-1}[\neg\Delta^d(\omega)]$. As $G=\bigcup_{i\in\omega}\ u^{-1}(\{i\})$, there is an integer i_0 such that $u^{-1}(\{i_0\})$ is not meager and has the Baire property in G. This implies the existence of $s\in d^{<\omega}$ such that $(G\cap N_s)\setminus u^{-1}(\{i_0\})$ is meager. Let H be a dense G_δ subset of G such that $H\cap N_s\subseteq u^{-1}(\{i_0\})$. We choose $n\in\omega$ with $s\subseteq s^d_n$. Note that $f^i_n:N_{s^d_n0}\rightarrow N_{s^d_ni}$ defined by $f^i_n(s^d_n0\gamma):=s^d_ni\gamma$ is an homeomorphism. This implies that $\bigcap_{i\in\omega}\ (f^i_n)^{-1}(H)$ is a dense G_δ subset of $N_{s^d_n0}$. We choose $s^d_n0\gamma\in\bigcap_{i\in\omega}\ (f^i_n)^{-1}(H)$. We get $(s^d_ni\gamma)_{i\in d}\in\mathbb{A}_d\cap(H\cap N_s)^d\subseteq[u^{-1}(\{i_0\})]^d$, which contradicts the fact that $\mathbb{A}_d\cap G^d\subseteq(u^d)^{-1}[\neg\Delta^d(\omega)]$. □

Definition 2.2 *Let $A\subseteq X^d$. We say that $C\subseteq X$ is $\underline{A-discrete}$ if $A\cap C^d=\emptyset$.*

Notation. The reader should see [M] for the basic notions of effective descriptive set theory. Assume that X and X^d are recursively presented Polish spaces, and that $A\in\Sigma^1_1(X^d)$. We put

$$U:=\bigcup\ \{D\in\Delta^1_1(X)\mid D \text{ is } A\text{-discrete}\}.$$

Note that $U\in\Pi^1_1(X)$ if the projections are recursive.

Lemma 2.3 *Assume that X and X^d are recursively presented Polish spaces, $A\in\Sigma^1_1(X^d)$, and $U=X$. Then $[X,A]\preceq_B[\omega,\neg\Delta^d(\omega)]$.*

Proof. As $U=X$, there is a partition $(D_n)_{n\in\omega}$ of X into A-discrete Δ^1_1 sets. We define a Borel map $u:X\rightarrow\omega$ by $u(x)=n\Leftrightarrow x\in D_n$. If $(x_i)_{i\in d}\in A$, then we cannot have $[u(x_i)]_{i\in d}\in\Delta^d(\omega)$, since the D_n's are A-discrete. $\square$

We will recall the proof of Theorem 1.4, to show the problem appearing in the case of the infinite dimension. It is essentially identical to the one in [K-S-T], except that we do not use Choquet games.

Notation. Let Z be a recursively presented Polish space. The $\underline{Gandy-Harrington\ topology}$ on Z is generated by $\Sigma^1_1(Z)$ and denoted Σ_Z. It is finer than the initial topology of Z, so that $[Z,\Sigma_Z]$ is T_1. As $\Sigma^1_1(Z)$ is countable (see 3F.6 in [M]), $[Z,\Sigma_Z]$ is second countable. We set

$$\Omega_Z:=\{z\in Z\mid\omega_1^z=\omega_1^{\text{CK}}\}.$$

Recall that Ω_Z is $\Sigma^1_1(Z)$ and dense in $[Z,\Sigma_Z]$ (see III.1.5 in [S]; the second assertion is Gandy's basis theorem). Recall also that $W\cap\Omega_Z$ is a clopen subset of $[\Omega_Z,\Sigma_Z]$ for each $W\in\Sigma^1_1(Z)$. Indeed, it is obviously open. Let $f:Z\rightarrow\omega^\omega$ be Δ^1_1 such that $Z\setminus(W\cap\Omega_Z)=f^{-1}(WO)$ (see 4A.3 in [M]). We get

$$z\in\Omega_Z\setminus(W\cap\Omega_Z)\ \Leftrightarrow\ z\in\Omega_Z\ \text{ and }\ \exists\xi<\omega_1^{\text{CK}}\ (f(z)\in WO\ \text{ and }\ |f(z)|\leq\xi).$$

This proves that $W\cap\Omega_Z$ is closed (see 4A.2 in [M]). In particular, $[\Omega_Z,\Sigma_Z]$ is zero-dimensional, and regular. By Theorem 4.2 in [H-K-L] and 8.16.(iii) in [K], $[\Omega_Z,\Sigma_Z]$ is strong Choquet. By 8.18 in [K], $[\Omega_Z,\Sigma_Z]$ is a Polish space. So we fix a complete compatible metric d_Z on $[\Omega_Z,\Sigma_Z]$.

Proof of Theorem 1.4. Note first that we cannot have (a) and (b) simultaneously, by Lemma 2.1.

• We may assume that X is a recursively presented Polish space and that $A\in\Sigma^1_1(X^d)$. We set $\Phi:=\{C\subseteq X\mid C \text{ is } A\text{-discrete}\}$. As Φ is Π^1_1 on Σ^1_1, the first reflection theorem ensures that if $C\in\Sigma^1_1(X)$ is A-discrete, then there is $D\in\Delta^1_1(X)$ which is A-discrete and contains C (see 35.C in [K]).

• By Lemma 2.3 we may assume that $U\neq X$, so that $Y:=X\setminus U$ is a nonempty Σ^1_1 subset of X. The previous point gives the following key property:

$$\forall C\in\Sigma^1_1(X)\quad(\emptyset\neq C\subseteq Y\ \Rightarrow\ A\cap C^d\neq\emptyset).$$

• We construct $(x_s)_{s\in d^{<\omega}}\subseteq Y$, $(V_s)_{s\in d^{<\omega}}\subseteq \Sigma^1_1(X)$ and $(U_{n,t})_{(n,t)\in\omega\times d^{<\omega}}\subseteq \Sigma^1_1(X^d)$ satisfying the following conditions:

$$(1)\ x_s\in V_s\subseteq Y\cap\Omega_X \ \text{ and } \ (x_{s^d_n it})_{i\in d}\in U_{n,t}\subseteq A\cap Y^d\cap\Omega_{X^d},$$

$$(2)\ V_{sm}\subseteq V_s \ \text{ and } \ U_{n,tm}\subseteq U_{n,t},$$

$$(3)\ \text{diam}_{d_X}(V_s)\leq 2^{-|s|} \ \text{ and } \ \text{diam}_{d_{X^d}}(U_{n,t})\leq 2^{-n-1-|t|}.$$

• Assume that this is done. Fix $\alpha\in d^\omega$. Then $(V_{\alpha|p})_{p\in\omega}$ is a decreasing sequence of nonempty clopen subsets of $[\Omega_X,\Sigma_X]$ whose d_X-diameters tend to zero, so there is $u(\alpha)$ in their intersection. This defines $u : d^\omega\rightarrow X$. Note that $d_X[x_{\alpha|p},u(\alpha)]\leq \text{diam}_{d_X}(V_{\alpha|p})\leq 2^{-p}$, so that u is continuous and $(x_{\alpha|p})_{p\in\omega}$ tends to $u(\alpha)$ in $[X,\Sigma_X]$.

If $(s^d_n i\gamma)_{i\in d}\in\mathbb{A}_d$, then $(U_{n,\gamma|p})_{p\in\omega}$ is a decreasing sequence of nonempty clopen subsets of $[\Omega_{X^d},\Sigma_{X^d}]$ whose d_{X^d}-diameters tend to zero, so there is $(\alpha_i)_{i\in d}$ in their intersection. Note that $(\alpha_i)_{i\in d}\in A$. Moreover, the sequence $([x_{s^d_n i(\gamma|p)}]_{i\in d})_{p\in\omega}$ tends to $(\alpha_i)_{i\in d}$ in $[X^d,\Sigma_{X^d}]$, and in $[X^d,\Sigma^d_X]$ too. As $(x_{s^d_n i(\gamma|p)})_{p\in\omega}$ tends to $u(s^d_n i\gamma)$, we get $u(s^d_n i\gamma)=\alpha_i$, for each $i\in d$. Thus $[u(s^d_n i\gamma)]_{i\in d}\in A$.

• So it is enough to see that the construction is possible. As Y is a nonempty Σ^1_1 subset of X, we can choose $x_\emptyset\in Y\cap\Omega_X$, and $V_\emptyset\in\Sigma^1_1(X)$ such that $x_\emptyset\in V_\emptyset\subseteq Y\cap\Omega_X$ and $\text{diam}_{d_X}(V_\emptyset)\leq 1$. Assume that $(x_s)_{|s|\leq l}$, $(V_s)_{|s|\leq l}$ and $(U_{n,t})_{n+1+|t|\leq l}$ satisfying (1)-(3) have been constructed, which is the case for $l=0$. Let C be the following set:

$$\{x\in X\mid \exists (y_s)_{s\in d^l}\in X^{d^l}\ \ y_{s^d_l}=x \ \text{ and } \ \forall s\in d^l\ y_s\in V_s \ \text{ and } \ \forall n<l\ \ \forall t\in d^{l-n-1}\ (y_{s^d_n it})_{i\in d}\in U_{n,t}\}.$$

Then $C\in\Sigma^1_1(X)$ **since** d **is an integer**, $x_{s^d_l}\in C\subseteq Y$ by induction assumption. So there is $(x_{s^d_l i})_{i\in d}$ in $A\cap C^d\cap\Omega_{X^d}$, by the key property. As $x_{s^d_l m}\in C$, we get $(x_{sm})_{s\in d^l\setminus\{s^d_l\}}$. It remains to choose

- $V_{sm}\in\Sigma^1_1(X)$ with $x_{sm}\in V_{sm}\subseteq V_s$ and $\text{diam}_{d_X}(V_{sm})\leq 2^{-l-1}$, for $s\in d^l$ and $m\in d$.

- $U_{l,\emptyset}\in\Sigma^1_1(X^d)$ with $(x_{s^d_l i})_{i\in d}\in U_{l,\emptyset}\subseteq A\cap Y^d\cap\Omega_{X^d}$ and $\text{diam}_{d_{X^d}}(U_{l,\emptyset})\leq 2^{-l-1}$.

- $U_{n,tm}\in\Sigma^1_1(X^d)$ with $(x_{s^d_n itm})_{i\in d}\in U_{n,tm}\subseteq U_{n,t}$ and $\text{diam}_{d_{X^d}}(U_{n,tm})\leq 2^{-l-1}$, for (n,t) in $\omega\times d^{<\omega}$ with $n+1+|t|=l$ and $m\in d$. □

3 The natural extension in infinite dimension does not work.

Theorem 1.5 is a consequence of Lemma 2.1 and of the following result:

Theorem 3 $[\omega^\omega,\mathbb{A}_\omega]\not\leq_c[\mathbb{G},\mathbb{A}_\omega\cap\mathbb{G}^\omega]$.

Proof. We argue by contradiction. This gives a continuous map $u:\omega^\omega\rightarrow\mathbb{G}$ with $\mathbb{A}_\omega\subseteq (u^\omega)^{-1}(\mathbb{A}_\omega)$.

• Let us prove that there is $\alpha\in\omega^\omega$ and $(s_n)_{n\in\omega}\in(\omega^{<\omega})^\omega$ such that

$$u[\beta(0)0^{\alpha(0)}\beta(1)0^{\alpha(1)}...]=s_0\beta(0)s_1\beta(1)...$$

for each $\beta\in\omega^\omega$. We construct $\alpha(n)$ and s_n by induction on n. Assume that $\alpha|n$ and $(s_p)_{p<n}$ are constructed satisfying

$$s^\omega_{\Sigma_{j\leq p}[1+\alpha(j)]}\subseteq 0^\infty \text{ and } [t(0)0^{\alpha(0)}...t(p)0^{\alpha(p)}\subseteq\gamma\ \Rightarrow\ s_0t(0)...s_pt(p)\subseteq u(\gamma)]$$

for each $p<n$ and $t\in\omega^{p+1}$. We will construct $\alpha(n)$ and s_n satisfying

$$s^\omega_{\Sigma_{j\leq n}[1+\alpha(j)]}\subseteq 0^\infty \text{ and } [t(0)0^{\alpha(0)}...t(n)0^{\alpha(n)}\subseteq\gamma\ \Rightarrow\ s_0t(0)...s_nt(n)\subseteq u(\gamma)]$$

for each $t\in\omega^{n+1}$, which will be enough. Note first that there are $m\in\omega$ and $\delta\in\omega^\omega$ with $[u(s^\omega_{\Sigma_{j<n}[1+\alpha(j)]}i0^\infty)]_{i\in\omega}=(s^\omega_m i\delta)_{i\in\omega}$. As u is continuous, there is $p\in\omega$ such that

$$s^\omega_{\Sigma_{j<n}[1+\alpha(j)]}0^{p+1}\subseteq\gamma\ \Rightarrow\ s^\omega_m 0\subseteq u(\gamma),$$

$$s^\omega_{\Sigma_{j<n}[1+\alpha(j)]}10^{p}\ \subseteq\gamma\ \Rightarrow\ s^\omega_m 1\subseteq u(\gamma).$$

Note that $s^\omega_{\Sigma_{j<n}[1+\alpha(j)]}i0^p\subseteq\gamma\ \Rightarrow\ s^\omega_m i\subseteq u(\gamma)$, for each $i\in\omega$. Indeed, let $\varepsilon\in\omega^\omega$. Then $[u(s^\omega_{\Sigma_{j<n}[1+\alpha(j)]}i0^p\varepsilon)]_{i\in\omega}\in\mathbb{A}_\omega\cap[N_{s^\omega_m 0}\times N_{s^\omega_m 1}\times(\omega^\omega)^\omega]\subseteq\Pi_{i\in\omega}\ N_{s^\omega_m i}$. In particular, this implies that $s_00...s_{n-1}0\subseteq s^\omega_m$ since $s_00...s_{n-1}0\subseteq u(s^\omega_{\Sigma_{j<n}[1+\alpha(j)]}i0^p\varepsilon)$.

- If $n=0$, then we choose $\alpha(0)\geq p$ such that $0^{1+\alpha(0)}=s^\omega_{1+\alpha(0)}$, we set $s_0:=s^\omega_m$, and we are done.

- If $n>0$, then we set $s_n:=s^\omega_m-(s_00...s_{n-1}0)$. We will prove, by induction on $l\leq n$, that

$$\forall t\in\omega^{n+1}\ 0^{n-l}\subseteq t\ \Rightarrow\ [t(0)0^{\alpha(0)}...t(n-1)0^{\alpha(n-1)}t(n)0^p\subseteq\gamma\ \Rightarrow\ s_0t(0)...s_nt(n)\subseteq u(\gamma)].$$

We already proved it for $l=0$. Assume that it is true for $l<n$, let $t\in\omega^{n+1}$ with $0^{n-l-1}\subseteq t$, and assume that $t(0)0^{\alpha(0)}...t(n-1)0^{\alpha(n-1)}t(n)0^p\subseteq\gamma$. We set $\varepsilon:=\gamma-[t(0)0^{\alpha(0)}...t(n-1)0^{\alpha(n-1)}t(n)0^p]$. Then by induction assumption on l we get

$$s_00...s_{n-l-1}0s_{n-l}t(n-l)...s_nt(n)\subseteq u[s^\omega_{\Sigma_{j<n-l}[1+\alpha(j)]}t(n-l)0^{\alpha(n-l)}...t(n-1)0^{\alpha(n-1)}t(n)0^p\varepsilon].$$

But by induction assumption on n we get, for each $i\in\omega$,

$$s_00...s_{n-l-2}0s_{n-l-1}i\subseteq u[s^\omega_{\Sigma_{j<n-l-1}[1+\alpha(j)]}i0^{\alpha(n-l-1)}t(n-l)0^{\alpha(n-l)}...t(n-1)0^{\alpha(n-1)}t(n)0^p\varepsilon].$$

But $(u[s^\omega_{\Sigma_{j<n-l-1}[1+\alpha(j)]}i0^{\alpha(n-l-1)}t(n-l)0^{\alpha(n-l)}...t(n-1)0^{\alpha(n-1)}t(n)0^p\varepsilon])_{i\in\omega}\in\mathbb{A}_\omega$. This implies, for each $i\in\omega$, that $u[s^\omega_{\Sigma_{j<n-l-1}[1+\alpha(j)]}i0^{\alpha(n-l-1)}t(n-l)0^{\alpha(n-l)}...t(n-1)0^{\alpha(n-1)}t(n)0^p\varepsilon]$ begins with $s_00...s_{n-l-2}0s_{n-l-1}is_{n-l}t(n-l)...s_nt(n)$. In particular, this holds for $i=t(n-l-1)$, and we are done.

It remains to choose $\alpha(n)\geq p$ such that $0^{\Sigma_{j\leq n}[1+\alpha(j)]}=s^\omega_{\Sigma_{j\leq n}[1+\alpha(j)]}$.

• If $s \in \omega^{\leq\omega}$, then we set $N[s] := \text{Card}\{n \in \omega \mid s_n^\omega 0 \subseteq s\}$. Note that $N[\alpha] = \omega$ if $\alpha \in \mathbb{G}$. By induction on p, we can construct $\beta(p) \in \omega$ such that $N[s_0\beta(0)...s_p\beta(p)s_{p+1}] = N[s_0]$. This implies that $N[s_0\beta(0)s_1\beta(1)...] = N[s_0] < \omega$, and $u[\beta(0)0^{\alpha(0)}\beta(1)0^{\alpha(1)}...] \notin \mathbb{G}$ by the previous point, which is absurd. $\square$

4 The proof in infinite dimension.

Before proving Theorem 1.6, note first the following result:

Theorem 4.1 *There is no $(X_0, \mathbb{A}_0)$, where X_0 is a metrizable compact space and $\mathbb{A}_0 \in \mathbf{\Sigma}^1_1(X_0^\omega)$, such that for any Polish space X, and for any $A \in \mathbf{\Sigma}^1_1(X^\omega)$, exactly one of the following holds:*
(a) $[X, A] \preceq_B [\omega, \neg\Delta^\omega(\omega)]$.
(b) $[X_0, \mathbb{A}_0] \preceq_c [X, A]$.

Proof. Suppose towards a contradiction that such $(X_0, \mathbb{A}_0)$ exists. Note that $\mathbb{A}_0 \neq \emptyset$, since otherwise we would have $[X_0, \mathbb{A}_0] \preceq_B [\omega, \neg\Delta^\omega(\omega)]$. By Lemma 2.1, we now get some continuous $u : X_0 \to \omega^\omega$ such that $\mathbb{A}_0 \subseteq (u^\omega)^{-1}(\mathbb{A}_\omega)$. Then $u[X_0]$ will be a compact subset of ω^ω and hence contained in some product $k_0 \times k_1 \times ... \subseteq \omega^\omega$, where the k_i's are finite. Notice however that $(k_0 \times k_1 \times ...)^\omega \cap \mathbb{A}_\omega = \emptyset$, and thus $\mathbb{A}_0 \subseteq (u^\omega)^{-1}[(k_0 \times k_1 \times ...)^\omega \cap \mathbb{A}_\omega] = \emptyset$, which is a contradiction. $\square$

Assume temporarily that there is a Polish space X_0 and $\mathbb{A}_0$ such that the end of the statement of Theorem 4.1 holds. By Theorem 4.1, X_0 cannot be compact. Note that we may assume that X_0 is zero-dimensional, since there is a finer zero-dimensional Polish topology on X_0 (see 13.5 in [K]). This means that we can view X_0 as a closed subspace of ω^ω (see 7.8 in [K]). As X_0 is not compact, the tree associated with this closed set (see 2.4 in [K]) is not finite splitting (see 4.11 in [K]). The proof of Theorem 1.6 will have the same scheme as the proof of Theorem 1.4. We have to build infinitely many V_s's at some levels of the construction, since the tree associated with X_0 is not finite splitting. The only place where the proof of Theorem 1.4 does not work in infinite dimension is when we write "$C \in \Sigma^1_1(X)$".

The main modifications to make are the following:

- As we have to build infinitely many V_s's at some levels of the construction, it is not clear at all that C remains Σ^1_1, since Σ^1_1 is not closed under infinite intersections. However, Σ^1_1 is closed under $\forall^\omega$, and this will be enough. We will have to build the V_s's uniformly in s at each level of the construction to ensure that C is Σ^1_1, and it is possible. We will also ensure that there are only finitely many $U_{n,t}$'s at each level of the construction, to ensure that C is Σ^1_1.

- The reason why Theorem 3 is true is that we cannot control all the diameters in $\mathbb{G}$ at each level of a construction that would give a map $u : \omega^\omega \to \mathbb{G}$. We will only control finitely many diameters, since we want C to be Σ^1_1. This is the reason why we will work in $\mathbb{G}$ instead of ω^ω. This gives the possibility to control only one diameter at each level of the construction among the V_s's (and finitely many among the $U_{n,t}$'s). So the point in the proof of Theorem 1.6 is that we cannot build the Σ^1_1 sets uniformly at each level of the construction and control all the diameters at the same time.

Proof of Theorem 1.6. Note first that we cannot have (a) and (b) simultaneously, by Lemma 2.1.

• Note that there is a recursive map $\tilde{s}:\omega\rightarrow\omega$ such that $\tilde{s}(l)$ codes s^ω_l, i.e., $\tilde{s}(l)=I(s^\omega_l)$ (see the notation in the introduction). Indeed, there is a recursive map $\tilde{\varphi}:\omega\rightarrow\omega$ whose restriction to Seq is an increasing bijection from Seq onto ω. Now $(\tilde{\varphi}|_{\text{Seq}})^{-1}$ defines a recursive map $\tilde{\psi}_\omega:\omega\rightarrow\omega$. It remains to note that $\tilde{s}(l)=t$ is equivalent to

$t\in\text{Seq}$ and $\text{lh}(t)=l$ and $\forall i<l$ $[i<\text{lh}[\tilde{\psi}_\omega(l)]$ and $(t)_i=(\tilde{\psi}_\omega(l))_i]$ or $[i\geq\text{lh}[\tilde{\psi}_\omega(l)]$ and $(t)_i=0]$.

• We may assume that

- The X^{ω^l}'s are recursively presented Polish spaces, for $l\in\omega$.

- The projections are recursive.

- The maps $\Pi_l:\omega\times X^{\omega^l}\rightarrow X$ defined by

$$\Pi_l[t,(x_s)_{s\in\omega^l}]=x \;\;\Leftrightarrow\;\; t\in\text{Seq}\text{ and }\text{lh}(t)=l\text{ and }x=x_{\overline{t}}$$

are partial recursive functions on $\{t\in\omega\mid t\in\text{Seq}\text{ and }\text{lh}(t)=l\}\times X^{\omega^l}$, for $l\in\omega$.

- The maps $\Pi'_l:\omega^2\times X^{\omega^l}\rightarrow X^\omega$ defined by

$$\Pi'_l[n,t,(x_s)_{s\in\omega^l}]=(y_i)_{i\in\omega} \;\;\Leftrightarrow\;\; t\in\text{Seq}\text{ and }n+1+\text{lh}(t)=l\text{ and }\forall i\in\omega\;\; y_i=x_{s^\omega_n i\overline{t}}$$

are partial recursive functions on $\{(n,t)\in\omega^2\mid t\in\text{Seq}\text{ and }n+1+\text{lh}(t)=l\}\times X^{\omega^l}$, for $l\in\omega$.

- $A\in\varSigma^1_1(X^\omega)$.

• We set $\Phi:=\{C\subseteq X\mid C\text{ is }A\text{-discrete}\}$. As Φ is $\varPi^1_1$ on $\varSigma^1_1$, the first reflection theorem ensures that if $C\in\varSigma^1_1(X)$ is A-discrete, then there is $D\in\varDelta^1_1(X)$ which is A-discrete and contains C.

• By Lemma 2.3 we may assume that $U\neq X$, so that $Y:=X\setminus U$ is a nonempty $\varSigma^1_1$ subset of X. The previous point gives the following key property:

$$\forall C\in\varSigma^1_1(X)\;\;\;(\emptyset\neq C\subseteq Y\;\Rightarrow\; A\cap C^\omega\neq\emptyset).$$

• We construct $(x_s)_{s\in\omega^{<\omega}}\subseteq Y$, $(V_s)_{s\in\omega^{<\omega}}\subseteq\varSigma^1_1(X)$, and $(U_{n,t})_{(n,t)\in\omega\times\omega^{<\omega}}\subseteq\varSigma^1_1(X^\omega)$ satisfying the following conditions:

(1) $x_s\in V_s\subseteq Y\cap\Omega_X$ and $(x_{s^\omega_n it})_{i\in\omega}\in U_{n,t}\subseteq A\cap Y^\omega\cap\Omega_{X^\omega}$,

(2) $V_{sm}\subseteq V_s$ and $U_{n,tm}\subseteq U_{n,t}$,

(3) $\text{diam}_{d_X}(V_{s^\omega_l 0})\leq 2^{-l}$ and $[s^\omega_n 0t=s^\omega_l 0\;\Rightarrow\;\text{diam}_{d_{X^\omega}}(U_{n,t})\leq 2^{-l}]$,

(4) For any fixed $|s|$, the relation "$x\in V_s$" is a $\varSigma^1_1$ condition on (x,s),

(5) For any fixed n and fixed $|t|$, the relation "$(x_i)_{i\in\omega}\in U_{n,t}$" is a $\varSigma^1_1$ condition on $[(x_i)_{i\in\omega},t]$.

• Assume that this is done. Fix $\alpha\in\mathbb{G}$. Then $(V_{\alpha|p})_{p\in\omega}$ is a decreasing sequence of nonempty clopen subsets of $[\Omega_X,\Sigma_X]$ whose d_X-diameters tend to zero, so there is $u(\alpha)$ in their intersection. This defines $u:\mathbb{G}\rightarrow X$. Note that $d_X[x_{\alpha|p},u(\alpha)]\leq\mathrm{diam}_{d_X}(V_{\alpha|p})$, so that u is continuous and $(x_{\alpha|p})_{p\in\omega}$ tends to $u(\alpha)$ in $[X,\Sigma_X]$.

If $(s^\omega_n i\gamma)_{i\in\omega}\in\mathbb{A}_\omega\cap\mathbb{G}^\omega$, then $(U_{n,\gamma|p})_{p\in\omega}$ is a decreasing sequence of nonempty clopen subsets of $[\Omega_{X^\omega},\Sigma_{X^\omega}]$ whose d_{X^ω}-diameters tend to zero, so there is $(\alpha_i)_{i\in\omega}$ in their intersection. Note that $(\alpha_i)_{i\in\omega}\in A$. Moreover, the sequence $([x_{s^\omega_n i(\gamma|p)}]_{i\in\omega})_{p\in\omega}$ tends to $(\alpha_i)_{i\in\omega}$ in $[X^\omega,\Sigma_{X^\omega}]$, and in $[X^\omega,\Sigma^\omega_X]$ too. As $(x_{s^\omega_n i(\gamma|p)})_{p\in\omega}$ tends to $u(s^\omega_n i\gamma)$ in $[X,\Sigma_X]$, we get $u(s^\omega_n i\gamma)=\alpha_i$, for each $i\in\omega$. Thus $[u(s^\omega_n i\gamma)]_{i\in\omega}\in A$.

• So it is enough to see that the construction is possible. If $V_\emptyset$ is any Σ^1_1 set, then clearly (4) holds for s of length 0. Now suppose that V_s has been defined for all $s\in\omega^{\leq l}$ and that (4) holds. Then in order to define V_r for $r\in\omega^{l+1}$, while ensuring (4), we will let $V_{s^\omega_l 0}\subseteq V_{s^\omega_l}$ be some chosen Σ^1_1 set of diameter at most 2^{-l} (to be determined later on) and $V_{sm}:=V_s$ for all $sm\neq s^\omega_l 0$. Then for $r\in\omega^{l+1}$

$$x\in V_r\Leftrightarrow (r=s^\omega_l 0\ \text{ and }\ x\in V_{s^\omega_l 0})\ \text{ or }\ (r=sm\neq s^\omega_l 0\ \text{ and }\ x\in V_s),$$

which is Σ^1_1 in (x,r) by the induction hypothesis.

Similarly, if $U_{n,\emptyset}$ is any Σ^1_1 set, then clearly (5) holds for t of length 0. Now suppose that $U_{n,t}$ has been defined for all $t\in\omega^{\leq k}$ and that (5) holds. Then in order to define $U_{n,r}$ for $r\in\omega^{k+1}$, while ensuring (5), we again split into two cases. If $s^\omega_n 0r=s^\omega_n 0t0=s^\omega_l 0$, then $U_{n,r}\subseteq U_{n,t}$ will be some chosen Σ^1_1 set of diameter at most 2^{-l} (to be determined later on). On the other hand, if $s^\omega_n 0r=s^\omega_n 0tm\neq s^\omega_l 0$, then we set $U_{n,r}:=U_{n,t}$. Then for $r\in\omega^{k+1}$

$$(x_i)_{i\in\omega}\in U_{n,r}\Leftrightarrow\begin{cases}(s^\omega_n 0r=s^\omega_n 0t0=s^\omega_l 0\ \text{ and }\ (x_i)_{i\in\omega}\in U_{n,r})\\ \text{or}\\ (s^\omega_n 0r=s^\omega_n 0tm\neq s^\omega_l 0\ \text{ and }\ (x_i)_{i\in\omega}\in U_{n,t}),\end{cases}$$

which is Σ^1_1 in $[(x_i)_{i\in\omega},r]$ by the induction hypothesis, since $s^\omega_n 0r=s^\omega_l 0$ can hold for only finitely many $(n,r)\in\omega\times\omega^{<\omega}$.

Notice that in this way (2) and (3) are also satisfied, so it remains to define $V_{s^\omega_l 0}$, $U_{n,\emptyset}$ and $U_{n,r}$ for $s^\omega_n 0r=s^\omega_l 0$ of diameter small enough such that (1) also holds.

- As Y is a nonempty Σ^1_1 subset of X, we can choose $x_\emptyset\in Y\cap\Omega_X$, and set $V_\emptyset:=Y\cap\Omega_X$.

- The key property applied to $V_\emptyset$ gives $(x_i)_{i\in\omega}\in A\cap V^\omega_\emptyset\cap\Omega_{X^\omega}$. We choose $U_{0,\emptyset}\in\Sigma^1_1(X^\omega)$ such that $(x_i)_{i\in\omega}\in U_{0,\emptyset}\subseteq A\cap V^\omega_\emptyset\cap\Omega_{X^\omega}$ and $\mathrm{diam}_{d_{X^\omega}}(U_{0,\emptyset})\leq 1$. Then we choose $V_0\in\Sigma^1_1(X)$ such that $x_0\in V_0\subseteq V_\emptyset$ and $\mathrm{diam}_{d_X}(V_0)\leq 1$. Assume that $(x_s)_{|s|\leq l}$, $(V_s)_{|s|\leq l}$, and $(U_{n,t})_{n+1+|t|\leq l}$ satisfying (1)-(5) have been constructed, which is the case for $l\leq 1$.

- We put

$$C:=\Big\{\ x\in X\mid\exists(y_s)_{s\in\omega^l}\in X^{\omega^l}\ \ y_{s^\omega_l}=x\ \text{ and }\ \forall s\in\omega^l\ \ y_s\in V_s\ \text{ and }\ \forall n<l\ \ \forall t\in\omega^{l-n-1}\ \ (y_{s^\omega_n it})_{i\in\omega}\in U_{n,t}\ \Big\}.$$

Then $x_{s_l^\omega}\in C$, by induction assumption. Moreover, $C\in\Sigma^1_1$, by conditions (4) and (5) since Σ^1_1 is closed under $\forall^\omega$. The key property applied to C gives $(x_{s_l^\omega i})_{i\in\omega}\in A\cap C^\omega\cap\Omega_{X^\omega}$. As $x_{s_l^\omega m}\in C$, there is $(x_{sm})_{s\in\omega^l\setminus\{s_l^\omega\}}\subseteq X$ such that $x_{sm}\in V_s$ for each $s\in\omega^l$ and $(x_{s_n^\omega itm})_{i\in\omega}\in U_{n,t}$ for each $n<l$ and each $t\in\omega^{l-n-1}$. This defines $(x_s)_{s\in\omega^{l+1}}$.

We choose $U_{l,\emptyset}\in\Sigma^1_1(X^\omega)$ such that $(x_{s_l^\omega i})_{i\in\omega}\in U_{l,\emptyset}\subseteq A\cap V^\omega_{s_l^\omega}\cap\Omega_{X^\omega}$ and $\text{diam}_{d_{X^\omega}}(U_{l,\emptyset})\leq 2^{-l}$, and $V_{s_l^\omega 0}\in\Sigma^1_1(X)$ such that $x_{s_l^\omega 0}\in V_{s_l^\omega 0}\subseteq V_{s_l^\omega}$ and $\text{diam}_{d_X}(V_{s_l^\omega 0})\leq 2^{-l}$. If $s_n^\omega 0r=s_n^\omega 0t0=s_l^\omega 0$, then we choose $U_{n,r}\in\Sigma^1_1(X^\omega)$ such that $\text{diam}_{d_{X^\omega}}(U_{n,r})\leq 2^{-l}$ and $(x_{s_n^\omega ir})_{i\in\omega}\in U_{n,r}\subseteq U_{n,t}$. $\square$

5 The Baire-measurable natural extension in infinite dimension works.

Theorem 1.7 is a consequence of Theorem 1.6, Lemma 2.1 and of the following result:

Theorem 5.1 $[\omega^\omega,\mathbb{A}_\omega]\preceq_{Bm}[\mathbb{G},\mathbb{A}_\omega\cap\mathbb{G}^\omega]$.

Notation. We define the following equivalence relation on the Baire space ω^ω, which is the analogous version of the usual equivalence relation $\mathbb{E}_0$ on the Cantor space 2^ω (see [H-K-L]):

$$\alpha\ \mathbb{E}_0^{\omega^\omega}\ \beta\ \ \Leftrightarrow\ \ \exists m\in\omega\ \ \forall n\geq m\ \ \ \alpha(n)=\beta(n).$$

Lemma 5.2 *There is a dense and $\mathbb{E}_0^{\omega^\omega}$-invariant G_δ subset G of ω^ω such that*

$$\forall\alpha\in G\ \ \forall l,m\in\omega\ \ \exists n\geq m\ \ \ s_n^\omega l\subseteq\alpha$$

(in particular, $G\subseteq\mathbb{G}$).

Proof. We set $G_0:=\{\alpha\in\omega^\omega\mid\forall l,m\in\omega\ \ \exists n\geq m\ \ \ s_n^\omega l\subseteq\alpha\}$. Note that G_0 is a dense G_δ subset of ω^ω. We also define, for $n,p\in\omega$, $f_n^p:\omega^\omega\to\{\alpha\in\omega^\omega\mid\alpha(n)=p\}$ by

$$f_n^p(\alpha)(m):=\begin{cases}\alpha(m)\ \text{ if }\ m\neq n,\\ \\ p\ \text{ if }\ m=n.\end{cases}$$

Note that f_n^p is onto, continuous, open, and has a clopen range. Then we set

$$D:=\{H\subseteq\omega^\omega\mid H\text{ is a dense }G_\delta\}$$

and we define $\Phi:D\to D$ by $\Phi(H):=H\cap\bigcap_{n,p\in\omega}(f_n^p)^{-1}(H)$. This allows us to define, for $q\in\omega$, $G_{q+1}:=\Phi(G_q)$, and we set $G:=\bigcap_{q\in\omega}G_q$. Note that $G\in D$. Moreover, if $\alpha\in G$ and $n,p\in\omega$, then $f_n^p(\alpha)\in G$. Indeed, let $q\in\omega$. Then $\alpha\in G_{q+1}\subseteq(f_n^p)^{-1}(G_q)$. Now if $\beta\ \mathbb{E}_0^{\omega^\omega}\ \alpha$, then there is $s\in\omega^{<\omega}$ such that $\beta=s(\alpha-\alpha||s|)$ (which means that $s\subseteq\beta$ and α, β agree from the coordinate $|s|$ on). We set, for $i\leq|s|$, $\beta_i:=(s|i)(\alpha-\alpha|i)$, so that $\beta_0=\alpha$ and $\beta_{|s|}=\beta$. Note that $\beta_{i+1}=f_i^{s(i)}(\beta_i)$ for each $i<|s|$, by induction on i. This proves that $\beta_i\in G$ for each $i\leq|s|$, by induction on i. In particular, $\beta\in G$ which is $\mathbb{E}_0^{\omega^\omega}$-invariant. This finishes the proof since $G\subseteq G_0$. $\square$

Notation. For each $l\in\omega$, we define an oriented graph $G^{\rightarrow}_{l+1}$ on ω^{l+1} as follows:

$$s\ G^{\rightarrow}_{l+1}\ s' \ \Leftrightarrow\ \exists n\in\omega\ \exists i\neq 0\ \exists t\in\omega^{<\omega}\ \ (s,s')=(s^\omega_n 0t, s^\omega_n it).$$

We denote by G_{l+1} the symmetrization of $G^{\rightarrow}_{l+1}$.

Lemma 5.3 *The graph* (ω^{l+1},G_{l+1}) *is connected and acyclic.*

Proof. We argue by induction on l. For $l=0$, we have

$$i\ G_1\ i' \ \Leftrightarrow\ (i=0 \text{ and } i'\neq 0) \text{ or } (i'=0 \text{ and } i\neq 0).$$

If $i<i'$, then $(i,0,i')$ is a G_1-walk from i to i' if $i\neq 0$, and (i,i') is a G_1-walk from i to i' if $i=0$. Thus (ω,G_1) is connected. Now if $(i_j)_{j\leq L}$ is a G_1-cycle, then either $i_0\neq 0$ and $i_1=i_{L-1}=0$, or $i_0=0$ and $i_2=0$. In both cases, this contradicts the fact that $(i_j)_{j\leq L}$ is a cycle. Thus (ω,G_1) is acyclic.

Assume that the result is true for l. Note that

$$si\ G_{l+2}\ s'i' \ \Leftrightarrow\ (s=s'=s^\omega_{l+1} \text{ and } i\ G_1\ i') \text{ or } (s\ G_{l+1}\ s' \text{ and } i=i').$$

We set, for $i\in\omega$, $E_i:=\{ti \mid t\in\omega^{l+1}\}$. Note that ω^{l+2} is the disjoint union of the E_i's, that the map $ti\mapsto t$ is an isomorphism from (E_i,G_{l+2}) onto (ω^{l+1},G_{l+1}), and that the map $s^\omega_{l+1}i\mapsto i$ is an isomorphism from $(\{s^\omega_{l+1}i \mid i\in\omega\},G_{l+2})$ onto (ω,G_1). In particular, (E_i,G_{l+2}) is connected and acyclic, and (ω^{l+2},G_{l+2}) is connected.

Now if $(t_j)_{j\leq L}$ is a G_{l+2}-cycle, then the sequence $[t_j(l+1)]_{j\leq L}$ is not constant. There are $j_0\leq L$ minimal with $t_{j_0}(l+1)\neq t_0(l+1)$, and $j_1>j_0$ minimal with $t_{j_1}(l+1)=t_0(l+1)$. Note that $t_{j_0-1}=t_{j_1}=s^\omega_{l+1}t_0(l+1)$. Thus $j_0=1$ and $j_1=L$. If $t_0(l+1)\neq 0$, then $t_1=t_{L-1}=s^\omega_{l+1}0$. If $t_0(l+1)=0$, then the sequence $[t_j(l+1)]_{0<j<L}$ is constant and $t_1=t_{L-1}=s^\omega_{l+1}t_1(l+1)$. In both cases, this contradicts the fact that $(t_j)_{j\leq L}$ is a cycle. Thus (ω^{l+2},G_{l+2}) is acyclic. $\square$

Notation. Lemma 5.3 and Theorem I.2.5 in [B] imply the existence, for each pair $\{s,s'\}$ of distinct vertices in ω^{l+1}, of a unique $s-s'$ path in (ω^{l+1},G_{l+1}). We will call it $p^{l+1}_{s,s'}$. If $s=s'$, then we set $p^{l+1}_{s,s'}:=<s>$. The proof of Lemma 5.3 shows that

$$p^{l+2}_{si,s'i'}=\begin{cases} <p^{l+1}_{s,s'}(0)i,...,p^{l+1}_{s,s'}(|p^{l+1}_{s,s'}|-1)i> \text{ if } i=i',\\ \\ <p^{l+1}_{s,s^\omega_{l+1}}(0)i,...,p^{l+1}_{s,s^\omega_{l+1}}(|p^{l+1}_{s,s^\omega_{l+1}}|-1)i,s^\omega_{l+1}0,p^{l+1}_{s^\omega_{l+1},s'}(0)i',...,p^{l+1}_{s^\omega_{l+1},s'}(|p^{l+1}_{s^\omega_{l+1},s'}|-1)i'> \\ \hfill \text{if } 0\neq i\neq i'\neq 0,\\ \\ <p^{l+1}_{s,s^\omega_{l+1}}(0)i,...,p^{l+1}_{s,s^\omega_{l+1}}(|p^{l+1}_{s,s^\omega_{l+1}}|-1)i,p^{l+1}_{s^\omega_{l+1},s'}(0)i',...,p^{l+1}_{s^\omega_{l+1},s'}(|p^{l+1}_{s^\omega_{l+1},s'}|-1)i'> \\ \hfill \text{otherwise.}\end{cases}$$

Lemma 5.4 *Let* $\beta\in\omega^\omega$. *Then* $[[\beta]_{\mathbb{E}_0^{\omega^\omega}},\mathbb{A}_\omega\cap([\beta]_{\mathbb{E}_0^{\omega^\omega}})^\omega]\preceq[G,\mathbb{A}_\omega\cap G^\omega]$.

Proof. We have seen that if $\alpha\ \mathbb{E}_0^{\omega^\omega}\ \beta$, then there is $s\in\omega^{<\omega}$ such that $\alpha=s(\beta-\beta||s|)$. We will construct $u(\alpha)\in G$ by induction on $|s|$.

• If $|s|=0$, then we simply choose $u(\beta)\in G$.

• If $|s|=1$, then we choose $n_0\in\omega$ such that $s^\omega_{n_0}\beta(0)\subseteq u(\beta)$, and we set

$$u[i(\beta-\beta|1)]:=s^\omega_{n_0}i[u(\beta)-u(\beta)|(n_0+1)]$$

if $i\neq\beta(0)$. Note that $u[i(\beta-\beta|1)]\ \mathbb{E}_0^{\omega^\omega}\ u(\beta)\in G$, so that $u[i(\beta-\beta|1)]\in G$. Moreover, we have $(u[i(\beta-\beta|1)])_{i\in\omega}\in\mathbb{A}_\omega$.

• Assume that $u(\alpha)\in G$ is constructed for $|s|\leq l+1$, which is the case for $l=0$. Let $\varphi:\omega^{l+1}\rightarrow\omega$ be a bijection with $\varphi(s^\omega_{l+1})=0$.

• We construct $(E_q)_{q\in\omega}\in[\mathcal{P}(\omega)]^\omega$ $\subseteq$-increasing such that $[\{\varphi^{-1}(p)\mid p\in E_q\},G_{l+1}]$ is connected for each $q\in\omega$ (see Lemma 5.3). We proceed by induction on q. We first set $E_0:=\{0\}$. Assume that E_q is constructed.

- If $E_q=\omega$, then we set $E_{q+1}:=\omega$.

- If $E_q\neq\omega$, then we use the paths $p^{l+1}_{s,s'}$ defined after Lemma 5.3. We choose $r\in\omega\setminus E_q$ minimal for which there is $p\in E_q$ such that $|p^{l+1}_{\varphi^{-1}(p),\varphi^{-1}(r)}|=2$. Such an r exists since if $m\in\omega\setminus E_q$, then there is $i<|p^{l+1}_{s^\omega_{l+1},\varphi^{-1}(m)}|$ minimal such that $\varphi[p^{l+1}_{s^\omega_{l+1},\varphi^{-1}(m)}(i)]\notin E_q$, and $|p^{l+1}_{p^{l+1}_{s^\omega_{l+1},\varphi^{-1}(m)}(i-1),p^{l+1}_{s^\omega_{l+1},\varphi^{-1}(m)}(i)}|=2$ since $i>0$. As $[\{\varphi^{-1}(p)\mid p\in E_q\},G_{l+1}]$ is connected, and acyclic by Lemma 5.3, there is a unique $p\in E_q$ such that $\varphi^{-1}(p)\ G_{l+1}\ \varphi^{-1}(r)$. There are $n\leq l$, $i_0\neq 0$ and $t\in\omega^{l-n}$ such that $\{\varphi^{-1}(p),\varphi^{-1}(r)\}=\{s^\omega_n 0t, s^\omega_n i_0t\}$. We set $E_{q+1}:=E_q\cup\{\varphi(s^\omega_n it)\mid i\in\omega\}$.

Claim 1 $\bigcup_{q\in\omega}\ E_q=\omega$.

Indeed, let $r\in\omega\setminus\{0\}$. By induction on $k\in\omega$ we see that $\varphi[p^{l+1}_{s^\omega_{l+1},\varphi^{-1}(r)}(1+k)]\in\bigcup_{q\in\omega}\ E_q$. Thus r is in $\bigcup_{q\in\omega}\ E_q$.

This allows us to define $q(s):=\text{min}\{q\in\omega\mid\varphi(s)\in E_q\}$, for $s\in\omega^{l+1}$.

Claim 2 *Let $n\leq l$, and $t\in\omega^{l-n}$. Then there is $i\in\omega$ such that $q(s^\omega_n it)<q(s^\omega_n jt)$ for each $j\neq i$. Moreover, $q(s^\omega_n jt)=q(s^\omega_n j't)$ if $j,j'\neq i$.*

Indeed, we argue by contradiction. Choose $i\neq j$ such that $q:=q^{n,t}_i=q^{n,t}_j$ is minimal among the $q^{n,t}_k$'s. By definition of E_0 we have $q\neq 0$. As $\varphi(s^\omega_n it)\in E_q\setminus E_{q-1}$, we have $E_{q-1}\neq\omega$. This implies the existence of $n'\leq l$ and $t'\in\omega^{l-n'}$ such that $E_q\setminus E_{q-1}\subseteq\{\varphi(s^\omega_{n'}it')\mid i\in\omega\}$. Thus $s^\omega_n it$ and $s^\omega_n jt$ differ at the coordinate n', which implies that $n=n'$ and $t=t'$. By construction of E_q there is $k\in\omega$ such that $s^\omega_n kt\in E_{q-1}$, which contradicts the minimality of q. This proves Claim 2.

• We have to construct $u(sk[\beta-\beta|(l+2)])\in G$ for $|s|=l+1$ and $k\neq\beta(l+1)$. We will construct $u(sk[\beta-\beta|(l+2)])$ by induction on $q(s)$.

- If $q(s)=0$, then $s=s^\omega_{l+1}$ and we choose $n_1\in\omega$ such that $s^\omega_{n_1}\beta(l+1)\subseteq u(s^\omega_{l+1}[\beta-\beta|(l+1)])$, and we set

$$u(s^\omega_{l+1}k[\beta-\beta|(l+2)]):=s^\omega_{n_1}k[u(s^\omega_{l+1}[\beta-\beta|(l+1)])-u(s^\omega_{l+1}[\beta-\beta|(l+1)])|(n_1+1)]$$

if $k\neq\beta(l+1)$. As before, $u(s^\omega_{l+1}k[\beta-\beta|(l+2)])\in G$. Moreover, $[u(s^\omega_{l+1}i[\beta-\beta|(l+2)])]_{i\in\omega}\in\mathbb{A}_\omega$.

- Assume that $u(sk[\beta-\beta|(l+2)])\in G$ is constructed for $q(s)\leq q$, which is the case for $q=0$. If $q(s)=q+1$, then $\varphi(s)\in E_{q+1}\setminus E_q$. This implies the existence of $n\leq l$, $t\in\omega^{l-n}$, $i_0\neq 0$ and of a unique $p\in E_q$ such that $\{\varphi^{-1}(p),s\}=\{s^\omega_n 0t, s^\omega_n i_0 t\}$.

Note that $q[\varphi^{-1}(p)]\leq q$, so that $\beta_p:=u(\varphi^{-1}(p)k[\beta-\beta|(l+2)])$ is defined and in G. We choose $n_{q+1}\in\omega$ such that $s^\omega_{n_{q+1}}[\varphi^{-1}(p)(n)]\subseteq\beta_p$, and we set

$$u(s^\omega_n itk[\beta-\beta|(l+2)]):=s^\omega_{n_{q+1}}i[\beta_p-\beta_p|(n_{q+1}+1)]$$

if $i\neq\varphi^{-1}(p)(n)$. This is licit by Claim 2, since only β_p is defined among the $u(s^\omega_n itk[\beta-\beta|(l+2)])$'s. As before, $u(s^\omega_n itk[\beta-\beta|(l+2)])\in G$. Moreover, $[u(s^\omega_n itk[\beta-\beta|(l+2)])]_{i\in\omega}\in\mathbb{A}_\omega$.

• Now $u:[\beta]_{\mathbb{E}_0^{\omega^\omega}}\rightarrow G$ is constructed. Assume that $(s^\omega_n i\gamma)_{i\in\omega}\in\mathbb{A}_\omega\cap([\beta]_{\mathbb{E}_0^{\omega^\omega}})^\omega$. We can write $\gamma=\tilde{t}\delta$, where $\tilde{t}\in\omega^{<\omega}$, $\delta=\beta-\beta|(n+1+|\tilde{t}|)$, and $\tilde{t}(|\tilde{t}|-1)\neq\beta(n+|\tilde{t}|)$ if $\tilde{t}\neq\emptyset$. We have to check that $[u(s^\omega_n i\gamma)]_{i\in\omega}\in\mathbb{A}_\omega$. We may assume that $\tilde{t}\neq\emptyset$. We set $k:=\tilde{t}(|\tilde{t}|-1)$ and also $t:=\tilde{t}|(|\tilde{t}|-1)$. Then $(s^\omega_n i\gamma)_{i\in\omega}=(s^\omega_n itk\delta)_{i\in\omega}$, and Claim 2 provides i. Now the construction of u shows that $[u(s^\omega_n i\gamma)]_{i\in\omega}\in\mathbb{A}_\omega$ (consider $l:=n+|t|$). □

Proof of Theorem 5.1. Using the axiom of choice, fix a selector $S:\omega^\omega\rightarrow\omega^\omega$ for $\mathbb{E}_0^{\omega^\omega}$, i.e., a map satisfying $\alpha\ \mathbb{E}_0^{\omega^\omega}\ \beta\ \Rightarrow\ S(\alpha)=S(\beta)\ \mathbb{E}_0^{\omega^\omega}\ \alpha$ for each $\alpha,\beta\in\omega^\omega$ (see 12.15 in [K]). We can write

$$\omega^\omega=G\cup\bigcup_{\beta\in S[\omega^\omega]\setminus G}[\beta]_{\mathbb{E}_0^{\omega^\omega}},$$

and this union is disjoint. By Lemma 5.4 there is $u_\beta:[\beta]_{\mathbb{E}_0^{\omega^\omega}}\rightarrow G$ such that

$$\mathbb{A}_\omega\cap([\beta]_{\mathbb{E}_0^{\omega^\omega}})^\omega\subseteq(u^\omega_\beta)^{-1}(\mathbb{A}_\omega\cap G^\omega),$$

for each $\beta\in\omega^\omega$. We define $u:\omega^\omega\rightarrow\mathbb{G}$ by

$$u(\alpha):=\begin{cases}\alpha\ \text{ if }\ \alpha\in G,\\ \\ u_\beta(\alpha)\ \text{ if }\ \alpha\in[\beta]_{\mathbb{E}_0^{\omega^\omega}}\ \text{ and }\ \beta\in S[\omega^\omega]\setminus G.\end{cases}$$

Now let U be an open subset of $\mathbb{G}$. Then $u^{-1}(U)=(G\cap U)\cup\bigcup_{\beta\in S[\omega^\omega]\setminus G}u_\beta^{-1}(U)$. The set $G\cap U$ is a G_δ subset of ω^ω, and $\bigcup_{\beta\in S[\omega^\omega]\setminus G}u_\beta^{-1}(U)\subseteq\omega^\omega\setminus G$ is meager. This proves that u is Baire-measurable.

Now let $(s^\omega_n i\gamma)_{i\in\omega}\in\mathbb{A}_\omega$. Note that $s^\omega_n i\gamma\ \mathbb{E}_0^{\omega^\omega}\ s^\omega_n j\gamma$ if $i,j\in\omega$. If $s^\omega_n 0\gamma\in G$, then

$$[u(s^\omega_n i\gamma)]_{i\in\omega}=(s^\omega_n i\gamma)_{i\in\omega}\in\mathbb{A}_\omega\cap G^\omega\subseteq\mathbb{A}_\omega\cap\mathbb{G}^\omega$$

since G is $\mathbb{E}_0^{\omega^\omega}$-invariant. If $s^\omega_n 0\gamma\notin G$, then there is $\beta\in S[\omega^\omega]\setminus G$ such that $s^\omega_n 0\gamma\in[\beta]_{\mathbb{E}_0^{\omega^\omega}}$. In this case we have $(s^\omega_n i\gamma)_{i\in\omega}\in\mathbb{A}_\omega\cap([\beta]_{\mathbb{E}_0^{\omega^\omega}})^\omega$. Thus $[u_\beta(s^\omega_n i\gamma)]_{i\in\omega}\in\mathbb{A}_\omega\cap G^\omega\subseteq\mathbb{A}_\omega\cap\mathbb{G}^\omega$ and $[u(s^\omega_n i\gamma)]_{i\in\omega}\in\mathbb{A}_\omega\cap\mathbb{G}^\omega$. This finishes the proof. $\square$

Question. Is it true that $[\omega^\omega,\mathbb{A}_\omega]\preceq_B[\mathbb{G},\mathbb{A}_\omega\cap\mathbb{G}^\omega]$? This would imply that we can replace “Baire measurable” with “Borel” in Theorem 1.7.

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