# Central limit theorems for multiple Skorohod integrals

Ivan Nourdin, David Nualart

# Central limit theorems for multiple Skorohod integrals

Ivan Nourdin
Laboratoire de Probabilités et Modèles Aléatoires
Université Pierre et Marie Curie
Boîte courrier 188
4 Place Jussieu, 75252 Paris Cedex 5, France
ivan.nourdin@upmc.fr

David Nualart*
Department of Mathematics
University of Kansas
Lawrence, Kansas 66045, USA
nualart@math.ku.edu

September 2, 2009

**Abstract**

In this paper, we prove a central limit theorem for a sequence of multiple Skorohod integrals using the techniques of Malliavin calculus. The convergence is stable, and the limit is a conditionally Gaussian random variable. Some applications to sequences of multiple stochastic integrals, and renormalized weighted Hermite variations of the fractional Brownian motion are discussed.

**Key words**: central limit theorem, fractional Brownian motion, Malliavin calculus.

**2000 Mathematics Subject Classification**: 60F05, 60H05, 60G15, 60H07.

*The work of D. Nualart is supported by the NSF Grant DMS-0604207

# 1 Introduction

Consider a sequence of random variables $\{F_n, n \geq 1\}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Suppose that the $\sigma$-field $\mathcal{F}$ is generated by an isonormal Gaussian process $X = \{X(h), h \in \mathfrak{H}\}$ on a real separable infinite-dimensional Hilbert space $\mathfrak{H}$. This just means that $X$ is a centered Gaussian family of random variables indexed by the elements of $\mathfrak{H}$, and such that, for every $h, g \in \mathfrak{H}$,

$$E[X(h)X(g)] = \langle h, g \rangle_{\mathfrak{H}}. \tag{1.1}$$

Suppose that the sequence $\{F_n, n \geq 1\}$ is normalized, that is, $E(F_n) = 0$ and $\lim_{n\to\infty} E(F_n^2) = 1$. A natural problem is to find suitable conditions ensuring that $F_n$ converges in law towards a given distribution. When the random variables $F_n$ belong to the $q$th Wiener chaos of $X$ (for a fixed $q \geq 2$), then it turns out that the following conditions are equivalent:

(i) $F_n$ converges in law to $N(0,1)$;

(ii) $\lim_{n\to\infty} E[F_n^4] = 3$;

(iii) $\lim_{n\to\infty} \|DF_n\|_{\mathfrak{H}}^2 = q$ in $L^2(\Omega)$.

Here, $D$ stands for the derivative operator in the sense of Malliavin calculus (see Section 2 below for more details). More precisely, the following bound is in order, where $N$ denotes a standard Gaussian random variable:

$$\begin{aligned}\sup_{z\in\mathbb{R}} |P(F_n \leq z) - P(N \leq z)| &\leqslant \sqrt{E\left[\left(1 - \frac{1}{q}\|DF_n\|_{\mathfrak{H}}^2\right)^2\right]} && (1.2)\\ &\leqslant \sqrt{\frac{q-1}{3q}}\sqrt{\left|E(F_n^4) - 3\right|}. && (1.3)\end{aligned}$$

The equivalence between conditions (i) and (ii) was proved in Nualart and Peccati [22] by means of the Dambis, Dubins and Schwarz theorem. It implies that the convergence in distribution of a sequence of multiple stochastic integrals towards a Gaussian random variable is completely determined by the asymptotic behavior of their second and fourth moments, which represents a drastic simplification of the classical "method of moments and diagrams" (see, for instance, the survey by Peccati and Taqqu [26], as well as the references therein). The equivalence with condition (iii) was proved later by Nualart and Ortiz-Latorre [21] using tools of Malliavin calculus. Finally, the Berry-Esseen's type bound (1.2) is taken from Nourdin and Peccati [16], while (1.3) was shown in Nourdin, Peccati and Reinert [17].

Peccati and Tudor [27] also obtained a multidimensional version of the equivalence between (i) and (ii). In particular, they proved that, given a sequence $\{F_n, n \geq 1\}$ of $d$-dimensional random vectors such that $F_n^i$ belongs to the $q_i$th Wiener chaos for $i = 1, \dots, d$, where $1 \leqslant q_1 \leqslant \dots \leqslant q_d$, then if the covariance matrix of $F_n$ converges to the $d \times d$ identity matrix $I_d$, the convergence in distribution to each component towards the law $N(0,1)$ implies the convergence in distribution of the whole sequence $F_n$ towards the standard centered Gaussian law $N(0, I_d)$.

Recent examples of application of these results are, among others, the study of $p$-variations of fractional stochastic integrals (Corcuera *et al.* [4]), quadratic functionals of bivariate Gaussian processes (Deheuvels *et al.* [5]), self-intersection local times of fractional Brownian motion (Hu and Nualart [7]), approximation schemes for scalar fractional differential equations (Neuenkirch and Nourdin [12]), high-frequency CLTs for random fields on homogeneous spaces (Marinucci and Peccati [10, 11] and Peccati [23]), needlets analysis on the sphere (Baldi *et al.* [1]), estimation of self-similarity orders (Tudor and Viens [31]), weighted power variations of iterated Brownian motion (Nourdin and Peccati [15]) or bipower variations of Gaussian processes with stationary increments (Barndorff-Nielsen *et al.* [2]).

Since the works by Nualart and Peccati [22] and Peccati and Tudor [27], great efforts have been made to find similar statements in the case where the limit is not necessarily Gaussian. In the references [24] and [25], Peccati and Taqqu propose sufficient conditions ensuring that a given sequence of multiple Wiener-Itô integrals converges stably towards mixtures of Gaussian random variables. In another direction, Nourdin and Peccati [14] proved an extension of the above equivalence (i) – (iii) for a sequence of random variables $\{F_n, n \geq 1\}$ in a fixed $q$th Wiener chaos, $q \geq 2$, where the limit law is $2\,G_{\nu/2} - \nu$, $G_{\nu/2}$ being the Gamma distribution with parameter $\nu/2$.

The purpose of the present paper is to study the convergence in distribution of a sequence of random variables of the form $F_n = \delta^q(u_n)$, where $u_n$ are random variables with values in $\mathfrak{H}^{\otimes q}$ (the $q$th tensor product of $\mathfrak{H}$) and $\delta^q$ denotes the multiple Skorohod integral (that is, $\delta^2(u) = \delta(\delta(u))$, $\delta^3(u) = \delta(\delta(\delta(u)))$, and so on), towards a mixture of Gaussian random variables. Our main abstract result, Theorem 3.1, roughly says that under some technical conditions, if $\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}}$ converges in $L^1(\Omega)$ to a nonnegative random variable $S^2$, then the sequence $F_n$ converges stably to a random variable $F$ with conditional characteristic function $E\left(e^{i\lambda F} \middle| X\right) = E\left(e^{-\frac{\lambda^2}{2}S^2}\right)$. Notice that if $u_n$ is deterministic, then $F_n$ belongs to the $q$th Wiener chaos, and we have a sequence of the type considered above. In particular, if $S^2$ is also deterministic, we recover the fact that condition (iii) above implies the convergence in distribution to the law $N(0,1)$.

We develop some particular applications of Theorem 3.1 in the following directions. First, we consider a sequence of random variables in a fixed Wiener chaos and we derive new criteria for the convergence to a mixture of Gaussian laws. Second, we show the convergence in law of the sequence $\delta^q(u_n)$, where $q \geq 2$ and $u_n$ is a $q$-parameter process of the form

$$u_n = n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \mathbf{1}_{(k/n,(k+1)/n]^q},$$

towards the random variable $\sigma_{H,q} \int_0^1 f(B_s) dW_s$, where $B$ is a fractional Brownian motion with Hurst parameter $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$, $W$ is a standard Brownian motion independent of $B$, and $\sigma_{H,q}$ denotes some positive constant. This convergence allows us to establish a new asymptotic result for the behavior of the weighted $q$th Hermite variation of the fractional Brownian motion with Hurst parameter $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$, which complements and provides a new perspective to the results proved by Nourdin [13], Nourdin, Nualart and Tudor [18], and Nourdin and Réveillac [19]. The reader is referred to Section 5 for a detailed description of these results.

The paper is organized as follows. In Section 2, we present some preliminary results about Malliavin calculus. Section 3 contains the statement and the proof of the main abstract result. In Section 4, we apply it to sequences of multiple stochastic integrals, while Section 5 focuses on the applications to the weighted Hermite variations of the fractional Brownian motion.

# 2 Preliminaries

Let $\mathfrak{H}$ be a real separable infinite-dimensional Hilbert space. For any integer $q \geq 1$, let $\mathfrak{H}^{\otimes q}$ be the $q$th tensor product of $\mathfrak{H}$. Also, we denote by $\mathfrak{H}^{\odot q}$ the $q$th symmetric tensor product.

Suppose that $X = \{X(h),\, h \in \mathfrak{H}\}$ is an isonormal Gaussian process on $\mathfrak{H}$, defined on some probability space $(\Omega, \mathcal{F}, P)$. Recall that this means that the covariance of $X$ is given in terms of the scalar product of $\mathfrak{H}$ by (1.1). Assume from now on that $\mathcal{F}$ is generated by $X$.

For every integer $q \geq 1$, let $\mathcal{H}_q$ be the $q$th Wiener chaos of $X$, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where $H_q$ is the $q$th Hermite polynomial defined by

$$H_q(x) = \frac{(-1)^q}{q!} e^{x^2/2} \frac{d^q}{dx^q}\big(e^{-x^2/2}\big).$$

We denote by $\mathcal{H}_0$ the space of constant random variables. For any $q \geq 1$, the mapping $I_q(h^{\otimes q}) = q! H_q(X(h))$ provides a linear isometry between $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!}\, \|\cdot\|_{\mathfrak{H}^{\otimes q}}$) and $\mathcal{H}_q$ (equipped with the $L^2(\Omega)$ norm). For $q = 0$, by convention $\mathcal{H}_0 = \mathbb{R}$, and $I_0$ is the identity map.

It is well-known (Wiener chaos expansion) that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_q$. That is, any square integrable random variable $F \in L^2(\Omega)$ admits the following chaotic expansion:

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{2.1}$$

where $f_0 = E[F]$, and the $f_q \in \mathfrak{H}^{\odot q}$, $q \geq 1$, are uniquely determined by $F$. For every $q \geq 0$, we denote by $J_q$ the orthogonal projection operator on the $q$th Wiener chaos. In particular, if $F \in L^2(\Omega)$ is as in (2.1), then $J_q F = I_q(f_q)$ for every $q \geq 0$.

Let $\{e_k,\, k \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$, $g \in \mathfrak{H}^{\odot q}$ and $r \in \{0, \ldots, p \wedge q\}$, the $r$th contraction of $f$ and $g$ is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1,\ldots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}. \tag{2.2}$$

Notice that $f \otimes_r g$ is not necessarily symmetric. We denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of $f$ and $g$ while, for $p = q$, $f \otimes_q g = \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}$.

In the particular case $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a measurable space and $\mu$ is a $\sigma$-finite and non-atomic measure, one has that $\mathfrak{H}^{\odot q} = L^2_s(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q})$ is the space of symmetric and square integrable functions on $A^q$. Moreover, for every $f \in \mathfrak{H}^{\odot q}$, $I_q(f)$ coincides with the multiple Wiener-Itô integral of order $q$ of $f$ with respect to $X$ (introduced by Itô in

[8]) and (2.2) can be written as

$$(f \otimes_r g)(t_1, \ldots, t_{p+q-2r}) = \int_{A^r} f(t_1, \ldots, t_{p-r}, s_1, \ldots, s_r) \\ \times g(t_{p-r+1}, \ldots, t_{p+q-2r}, s_1, \ldots, s_r) d\mu(s_1) \ldots d\mu(s_r).$$

Let us now introduce some basic elements of the Malliavin calculus with respect to the isonormal Gaussian process $X$. We refer the reader to Nualart [20] for a more detailed presentation of these notions. Let $\mathcal{S}$ be the set of all smooth and cylindrical random variables of the form

$$F = g\left(X(\phi_1), \ldots, X(\phi_n)\right), \tag{2.3}$$

where $n \geq 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is a infinitely differentiable function with compact support, and $\phi_i \in \mathfrak{H}$. The Malliavin derivative of $F$ with respect to $X$ is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DF \;=\; \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\left(X(\phi_1), \ldots, X(\phi_n)\right) \phi_i.$$

By iteration, one can define the $q$th derivative $D^q F$ for every $q \geq 2$, which is an element of $L^2(\Omega, \mathfrak{H}^{\odot q})$.

For $q \geq 1$ and $p \geq 1$, $\mathbb{D}^{q,p}$ denotes the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{\mathbb{D}^{q,p}}$, defined by the relation

$$\|F\|_{\mathbb{D}^{q,p}}^p \;=\; E\left[|F|^p\right] + \sum_{i=1}^{q} E\left(\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p\right).$$

The Malliavin derivative $D$ verifies the following chain rule. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and if $F = (F_1, \ldots, F_n)$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F) DF_i.$$

We denote by $\delta$ the adjoint of the operator $D$, also called the divergence operator. The operator $\delta$ is also called the Skorohod integral because in the case of the Brownian motion it coincides with the anticipating stochastic integral introduced by Skorohod in [30]. A random element $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of $\delta$, noted Dom$\delta$, if and only if it verifies

$$\left|E\left(\langle DF, u\rangle_{\mathfrak{H}}\right)\right| \leq c_u \sqrt{E(F^2)}$$

for any $F \in \mathbb{D}^{1,2}$, where $c_u$ is a constant depending only on $u$. If $u \in \mathrm{Dom}\delta$, then the random variable $\delta(u)$ is defined by the duality relationship (called 'integration by parts formula'):

$$E(F\delta(u)) = E\big(\langle DF, u\rangle_{\mathfrak{H}}\big), \tag{2.4}$$

which holds for every $F \in \mathbb{D}^{1,2}$. The formula (2.4) extends to the multiple Skorohod integral $\delta^q$, and we have

$$E\left(F\delta^q(u)\right) = E\left(\langle D^q F, u\rangle_{\mathfrak{H}^{\otimes q}}\right) \tag{2.5}$$

for any element $u$ in the domain of $\delta^q$ and any random variable $F \in \mathbb{D}^{q,2}$. Moreover, $\delta^q(h) = I_q(h)$ for any $h \in \mathfrak{H}^{\odot q}$.

The following property will be extensively used in the paper.

**Lemma 2.1** *Let $q \geq 1$ be an integer. Suppose that $F \in \mathbb{D}^{q,2}$, and let $u$ be a symmetric element in* $\mathrm{Dom}\delta^q$*. Assume that, for any $0 \leq r + j \leq q$, $\left\langle D^r F, \delta^j(u)\right\rangle_{\mathfrak{H}^{\otimes r}} \in L^2(\Omega, \mathfrak{H}^{\otimes q-r-j})$. Then, for any $r = 0, \ldots, q-1$, $\langle D^r F, u\rangle_{\mathfrak{H}^{\otimes r}}$ belongs to the domain of $\delta^{q-r}$ and we have*

$$F\delta^q(u) = \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r}\left(\langle D^r F, u\rangle_{\mathfrak{H}^{\otimes r}}\right). \tag{2.6}$$

*(We use the convention that $\delta^0(v) = v$, $v \in \mathbb{R}$, and $D^0 F = F$, $F \in L^2(\Omega)$.)*

**Proof.** We prove this lemma by induction on $q$. For $q = 1$ it reads $F\delta(u) = \delta(Fu) + \langle DF, u\rangle_{\mathfrak{H}}$, and this formula is well-known, see e.g. [20, Proposition 1.3.3]. Suppose the result is true for $q$. Then, if $u$ belongs to the domain of $\delta^{q+1}$, by the induction hypothesis applied to $\delta(u)$,

$$F\delta^{q+1}(u) = F\delta^q(\delta(u)) = \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r}\left(\langle D^r F, \delta(u)\rangle_{\mathfrak{H}^{\otimes r}}\right). \tag{2.7}$$

On the other hand

$$\langle D^r F, \delta(u)\rangle_{\mathfrak{H}^{\otimes r}} = \delta\left(\langle D^r F, u\rangle_{\mathfrak{H}^{\otimes r}}\right) + \left\langle D^{r+1} F, u\right\rangle_{\mathfrak{H}^{\otimes r}}. \tag{2.8}$$

Finally, substituting (2.8) into (2.7) yields the desired result. ■

For any Hilbert space $V$, we denote by $\mathbb{D}^{k,p}(V)$ the corresponding Sobolev space of $V$-valued random variables (see [20, page 31]). The operator $\delta^q$

is continuous from $\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})$ to $\mathbb{D}^{k-q,p}$, for any $p > 1$ and any integers $k \geq q \geq 1$, that is, we have

$$\|\delta^q(u)\|_{\mathbb{D}^{k-q,p}} \leq c_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})} \tag{2.9}$$

for all $u \in \mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})$, and some constant $c_{k,p} > 0$. These estimates are consequences of Meyer inequalities (see [20, Proposition 1.5.7]). In particular, these estimates imply that $\mathbb{D}^{q,2}(\mathfrak{H}^{\otimes q}) \subset \mathrm{Dom}\delta^q$ for any integer $q \geq 1$.

We will also use the following commutation relationship between the Malliavin derivative and the Skorohod integral (see [20, Proposition 1.3.2])

$$D\delta(u) = u + \delta(Du), \tag{2.10}$$

for any $u \in \mathbb{D}^{2,2}(\mathfrak{H})$. By induction we can show the following formula for any symmetric element $u$ in $\mathbb{D}^{j+k,2}(\mathfrak{H}^{\otimes j})$

$$D^k\delta^j(u) = \sum_{i=0}^{j\wedge k} \binom{k}{i}\binom{j}{i} i!\delta^{j-i}(D^{k-i}u). \tag{2.11}$$

We will make use of the following formula for the variance of a multiple Skorohod integral. Let $u, v \in \mathbb{D}^{2q,2}(\mathfrak{H}^{\otimes q}) \subset \mathrm{Dom}\delta^q$ be two symmetric functions. Then

$$\begin{aligned} E(\delta^q(u)\delta^q(v)) &= E(\langle u, D^q(\delta^q(v))\rangle_{\mathfrak{H}^{\otimes q}}) \\ &= \sum_{i=0}^{q} \binom{q}{i}^2 i! E\left(\left\langle u, \delta^{q-i}(D^{q-i}v)\right\rangle_{\mathfrak{H}^{\otimes q}}\right) \\ &= \sum_{i=0}^{q} \binom{q}{i}^2 i! E\left(\left\langle D^{q-i}u, D^{q-i}v\right\rangle_{\mathfrak{H}^{\otimes(2q-i)}}\right). \end{aligned} \tag{2.12}$$

The operator $L$ is defined on the Wiener chaos expansion as

$$L = \sum_{q=0}^{\infty} -qJ_q,$$

and is called the infinitesimal generator of the Ornstein-Uhlenbeck semigroup. The domain of this operator in $L^2(\Omega)$ is the set

$$\mathrm{Dom}L = \{F \in L^2(\Omega) : \sum_{q=1}^{\infty} q^2 \|J_qF\|^2_{L^2(\Omega)} < \infty\} = \mathbb{D}^{2,2}.$$

There is an important relation between the operators $D$, $\delta$ and $L$ (see [20, Proposition 1.4.3]). A random variable $F$ belongs to the domain of $L$ if and only if $F \in \mathrm{Dom}\,(\delta D)$ (i.e. $F \in \mathbb{D}^{1,2}$ and $DF \in \mathrm{Dom}\delta$), and in this case

$$\delta DF = -LF. \tag{2.13}$$

Note also that a random variable $F$ as in (2.1) is in $\mathbb{D}^{1,2}$ if and only if

$$\sum_{q=1}^{\infty} qq! \|f_q\|^2_{\mathfrak{H}^{\otimes q}} < \infty,$$

and, in this case, $E\left(\|DF\|^2_{\mathfrak{H}}\right) = \sum_{q\geq 1} qq!\|f_q\|^2_{\mathfrak{H}^{\otimes q}}$. If $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ (with $\mu$ non-atomic), then the derivative of a random variable $F$ as in (2.1) can be identified with the element of $L^2(A \times \Omega)$ given by

$$D_a F = \sum_{q=1}^{\infty} qI_{q-1}\left(f_q(\cdot, a)\right), \quad a \in A. \tag{2.14}$$

Finally, we need the definition of stable convergence (see, for instance, the original paper [29], or the book [9] for an exhaustive discussion of stable convergence).

**Definition 2.2** *Let $F_n$ be a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$, and suppose that $F$ is a random variable defined on an enlarged probability space $(\Omega, \mathcal{G}, P)$, with $\mathcal{F} \subseteq \mathcal{G}$. We say that $F_n$ converges* $\mathcal{G}$-stably *to $F$ (or only* stably *when the context is clear) if, for any continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$ and any bounded $\mathcal{F}$-measurable random variable $Z$, we have $E\left[f(F_n)Z\right] \to E\left[f(F)Z\right]$ as $n$ tends to infinity.*

# 3 Convergence in law of multiple Skorohod integrals

As in the previous section, $X = \{X(h), h \in \mathfrak{H}\}$ is an isonormal Gaussian process associated with a real separable infinite-dimensional Hilbert space $\mathfrak{H}$. The next theorem is the main abstract result of the present paper.

**Theorem 3.1** *Fix an integer $q \geq 1$, and suppose that $F_n$ is a sequence of random variables of the form $F_n = \delta^q(u_n)$, for some symmetric functions $u_n$ in $\mathbb{D}^{2q,2q}(\mathfrak{H}^{\otimes q})$. Suppose moreover that the sequence $F_n$ is bounded in $L^1(\Omega)$, and that:*

*(i)* $\left\langle u_n, (DF_n)^{\otimes k_1} \otimes \ldots \otimes (D^{q-1}F_n)^{\otimes k_{q-1}} \otimes h \right\rangle_{\mathfrak{H}^{\otimes q}}$ *converges in* $L^1(\Omega)$ *to zero, for all integers* $r, k_1, \ldots, k_{q-1} \geq 0$ *such that*

$$k_1 + 2k_2 + \ldots + (q-1)k_{q-1} + r = q,$$

*and all* $h \in \mathfrak{H}^{\otimes r}$*;*

*(ii)* $\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}}$ *converges in* $L^1(\Omega)$ *to a nonnegative random variable* $S^2$*.*

*Then,* $F_n$ *converges stably to a random variable with conditional Gaussian law* $N(0, S^2)$ *given* $X$*.*

**Remark 3.2** When $q = 1$, condition (i) of the theorem is that $\langle u_n, h \rangle_{\mathfrak{H}}$ converges to zero in $L^1(\Omega)$, for each $h \in \mathfrak{H}$. When $q = 2$, condition (i) means that $\langle u_n, h \otimes g \rangle_{\mathfrak{H}^{\otimes 2}}$, $\langle u_n, DF_n \otimes h \rangle_{\mathfrak{H}^{\otimes 2}}$ and $\langle u_n, DF_n \otimes DF_n \rangle_{\mathfrak{H}^{\otimes 2}}$ converge to zero in $L^1(\Omega)$, for each $h, g \in \mathfrak{H}$. And so on.

**Proof of Theorem 3.1.** Taking into account Definition 2.2, it suffices to show that for any $h_1, \ldots, h_m \in \mathfrak{H}$, the sequence

$$\xi_n = (F_n, X(h_1), \ldots, X(h_m))$$

converges in distribution to a vector $(F_\infty, X(h_1), \ldots, X(h_m))$, where $F_\infty$ satisfies, for any $\lambda \in \mathbb{R}$,

$$E(e^{i\lambda F_\infty} | X(h_1), \ldots, X(h_m)) = e^{-\frac{\lambda^2}{2} S^2}. \tag{3.1}$$

Since the sequence $F_n$ is bounded in $L^1(\Omega)$, the sequence $\xi_n$ is tight. Assume that $(F_\infty, X(h_1), \ldots, X(h_m))$ denotes the limit in law of a certain subsequence of $\xi_n$, denoted again by $\xi_n$.

Let $Y = \phi(X(h_1), \ldots, X(h_m))$, with $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^m)$ ($\phi$ is infinitely differentiable, bounded, with bounded partial derivatives of all orders), and consider $\phi_n(\lambda) = E\left(e^{i\lambda F_n} Y\right)$ for $\lambda \in \mathbb{R}$. The convergence in law of $\xi_n$, together with the fact that $F_n$ is bounded in $L^1(\Omega)$, imply that

$$\lim_{n\to\infty} \phi_n'(\lambda) = \lim_{n\to\infty} iE\left(F_n e^{i\lambda F_n} Y\right) = iE(F_\infty e^{i\lambda F_\infty} Y). \tag{3.2}$$

On the other hand, by (2.5) and the Leibnitz rule for $D^q$, we obtain

$$\begin{aligned}
\phi_n'(\lambda) &= iE(F_n e^{i\lambda F_n} Y) = iE\left(\delta^q(u_n) e^{i\lambda F_n} Y\right) \\
&= iE\left(\left\langle u_n, D^q\left(e^{i\lambda F_n} Y\right)\right\rangle_{\mathfrak{H}^{\otimes q}}\right) \\
&= i\sum_{a=0}^{q} \binom{q}{a} E\left(\left\langle u_n, D^a\left(e^{i\lambda F_n}\right) \widetilde{\otimes} D^{q-a} Y\right\rangle_{\mathfrak{H}^{\otimes q}}\right) \\
&= i\sum_{a=0}^{q} \binom{q}{a} \sum \frac{a!}{k_1! \dots k_a!} (i\lambda)^{k_1+\dots+k_a} \\
&\quad \times E\left(e^{i\lambda F_n} \left\langle u_n, (DF_n)^{\otimes k_1} \widetilde{\otimes} \dots \widetilde{\otimes} (D^a F_n)^{\otimes k_a} \widetilde{\otimes} D^{q-a} Y\right\rangle_{\mathfrak{H}^{\otimes q}}\right) \\
&= i\sum_{a=0}^{q} \binom{q}{a} \sum \frac{a!}{k_1! \dots k_a!} (i\lambda)^{k_1+\dots+k_a} \\
&\quad \times E\left(e^{i\lambda F_n} \left\langle u_n, (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^a F_n)^{\otimes k_a} \otimes D^{q-a} Y\right\rangle_{\mathfrak{H}^{\otimes q}}\right),
\end{aligned}$$

where the second sum in the two last equalities runs over all sequences of integers $(k_1, \dots, k_a)$ such that $k_1 + 2k_2 + \dots + ak_a = a$, due to the Faá di Bruno's formula. By condition (i), this yields that

$$\phi_n'(\lambda) = -\lambda E\left(e^{i\lambda F_n} \langle u_n, D^q F_n\rangle_{\mathfrak{H}^{\otimes q}} Y\right) + R_n,$$

with $R_n$ converging to zero as $n \to \infty$. Using condition (ii) and (3.2), we obtain that

$$iE(F_\infty e^{i\lambda F_\infty} Y) = -\lambda E\left(e^{i\lambda F_\infty} S^2 Y\right).$$

Since $S^2$ is defined through condition (ii), it is in particular measurable with respect to $X$. Thus, the following linear differential equation verified by the conditional characteristic function of $F_\infty$ holds:

$$\frac{\partial}{\partial\lambda} E(e^{i\lambda F_\infty} | X(h_1), \dots, X(h_m)) = -\lambda\, S^2\, E(e^{i\lambda F_\infty} | X(h_1), \dots, X(h_m)).$$

By solving it, we obtain (3.1), which yields the desired conclusion. ■

The next corollary provides stronger but easier conditions for the stable convergence.

**Corollary 3.3** *For a fixed $q \geq 1$, suppose that $F_n$ is a sequence of random variables of the form $F_n = \delta^q(u_n)$, for some symmetric functions $u_n$ in $\mathbb{D}^{2q,2q}(\mathfrak{H}^{\otimes q})$. Suppose moreover that the sequence $F_n$ is bounded in $\mathbb{D}^{q,p}$ for all $p \geq 2$, and that:*

*(i')* $\langle u_n, h\rangle_{\mathfrak{H}^{\otimes q}}$ *converges to zero in* $L^1(\Omega)$ *for all* $h \in \mathfrak{H}^{\otimes q}$*; and* $u_n \otimes_l D^l F_n$ *converges to zero in* $L^2(\Omega; \mathfrak{H}^{\otimes(q-l)})$ *for all* $l = 1, \ldots, q-1$*;*

*(ii)* $\langle u_n, D^q F_n\rangle_{\mathfrak{H}^{\otimes q}}$ *converges in* $L^1(\Omega)$ *to a nonnegative random variable* $S^2$*.*

*Then,* $F_n$ *converges stably to a random variable with conditional Gaussian law* $N(0, S^2)$ *given* $X$*.*

**Proof.** It suffices to show that condition (i') implies condition (i) in Theorem 3.1. When $k_a \neq 0$ for $1 \leq a \leq q-1$, we have, for all $h \in \mathfrak{H}^{\otimes r}$ (with $r = q - k_1 - 2k_2 - \ldots - ak_a$),

$$\begin{aligned}
& \left| \left\langle u_n, (DF_n)^{\otimes k_1} \otimes \ldots \otimes (D^a F_n)^{\otimes k_a} \otimes h \right\rangle_{\mathfrak{H}^{\otimes q}} \right| \\
= & \left| \left\langle u_n \otimes_a D^a F_n, \right. \right. \\
& \left. \left. \quad (DF_n)^{\otimes k_1} \otimes \ldots \otimes (D^{a-1} F_n)^{\otimes k_{a-1}} \otimes (D^a F_n)^{\otimes (k_a - 1)} \otimes h \right\rangle_{\mathfrak{H}^{\otimes(q-a)}} \right| \\
\leq & \; \|u_n \otimes_a D^a F_n\|_{\mathfrak{H}^{\otimes(q-a)}} \\
& \times \left\| (DF_n)^{\otimes k_1} \otimes \ldots \otimes (D^{a-1} F_n)^{\otimes k_{a-1}} \otimes (D^a F_n)^{\otimes (k_a - 1)} \otimes h \right\|_{\mathfrak{H}^{\otimes(q-a)}}.
\end{aligned}$$

The second factor is bounded in $L^2(\Omega)$, and the first factor converges to zero in $L^2(\Omega)$, for all $a = 1, \ldots, q-1$. In the case $a = 0$ we have that $\langle u_n, h\rangle_{\mathfrak{H}^{\otimes q}}$ converges to zero in $L^1(\Omega)$, for all $h \in \mathfrak{H}^{\otimes q}$, by condition (i'). This completes the proof. ■

# 4 Multiple stochastic integrals

Suppose that $\mathfrak{H}$ is a Hilbert space $L^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a measurable space and $\mu$ is a $\sigma$-finite and non-atomic measure.

Fix an integer $m \geq 2$, and consider a sequence of multiple stochastic integrals $\{F_n = I_m(g_n), n \geq 1\}$ with $g_n \in \mathfrak{H}^{\odot m}$. We would like to apply Theorem 3.1 with $q = 1$ to the sequence $F_n$. To do this, we represent each $F_n$ as

$$F_n = \delta(u_n), \quad \text{with } u_n = I_{m-1}(\widehat{g}_n),$$

for $\widehat{g}_n \in \mathfrak{H}^{\otimes m}$ some function which is symmetric in the first $m-1$ variables.

Notice that, from (2.14), we have $DF_n = mI_{m-1}(g_n)$. Hence, since $F_n = -\frac{1}{m} LF_n = \frac{1}{m}\delta(DF_n)$ by (2.13), $g_n$ is always a possible choice for $\widehat{g}_n$. (In this case, $\widehat{g}_n$ is symmetric in all the variables.) However, as observed, for instance, in Example 4.2 below, the choice $\widehat{g}_n = g_n$ does not allow to conclude in general.

**Proposition 4.1** *For a fixed integer $m \geq 2$, let $F_n$ be a sequence of random variables of the form $F_n = I_m(g_n)$, with $g_n \in \mathfrak{H}^{\odot m}$. Suppose moreover that $F_n$ is bounded in $L^2(\Omega)$ and that $F_n = \delta(u_n)$, where $u_n = I_{m-1}(\widehat{g}_n)$, for $\widehat{g}_n \in \mathfrak{H}^{\otimes m}$ some function which is symmetric in the first $m-1$ variables. Finally, assume that:*

*(a) $\langle \widehat{g}_n \otimes_{m-1} \widehat{g}_n, h^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}}$ converges to zero for all $h \in \mathfrak{H}$;*

*(b) $\langle u_n, DF_n \rangle_{\mathfrak{H}}$ converges in $L^1(\Omega)$ to a non negative random variable $S^2$.*

*Then, $F_n$ converges stably to a random variable with conditional Gaussian law $N(0, S^2)$ given $X$.*

**Proof.** It suffices to apply Theorem 3.1 to $u_n = I_{m-1}(\widehat{g}_n)$ and $q = 1$. Indeed, we have

$$\begin{aligned} E\left(\langle u_n, h\rangle_{\mathfrak{H}}^2\right) &= E\left(\langle I_{m-1}(\widehat{g}_n), h\rangle_{\mathfrak{H}}^2\right) = E\left(I_{m-1}(\widehat{g}_n \otimes_1 h)^2\right) \\ &= (m-1)!\,\|\widehat{g}_n \otimes_1 h\|^2_{\mathfrak{H}^{\otimes(m-1)}} \\ &= (m-1)!\,\left\langle \widehat{g}_n \otimes_{m-1} \widehat{g}_n, h^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}} \to 0, \end{aligned}$$

which implies condition (i) in Theorem 3.1, see also Remark 3.2. Condition (ii) in Theorem 3.1 follows from (b). ∎

**Example 4.2** (see also [28, Proposition 2.1] or [24, Proposition 18] for two different proofs using other techniques). Suppose that $\{W_t,\ t \in [0,1]\}$ is a standard Brownian motion. (This corresponds to $A = [0,1]$ and $\mu$ the Lebesgue measure.) Assume that $m = 2$ and take $g_n(s,t) = \frac{1}{2}\sqrt{n}(s \vee t)^n$. Then

$$F_n = I_2(g_n) = \sqrt{n}\int_0^1 t^n W_t dW_t,$$

and

$$D_s F_n = \sqrt{n} s^n W_s + \sqrt{n}\int_s^1 t^n W_t dW_t.$$

We can take $u_n(t) = \sqrt{n} t^n W_t$, that is, $\widehat{g}_n(s,t) = \sqrt{n} t^n \mathbf{1}_{[0,t]}(s)$. In this case,

$$(\widehat{g}_n \otimes_1 \widehat{g}_n)(s,t) = n s^n t^n (s \wedge t),$$

which converges to zero weakly in $L^2(\Omega)$, and

$$\langle u_n, DF_n\rangle_{\mathfrak{H}} = \int_0^1 nt^{2n}W_t^2 dt + n\int_0^1 t^n W_t \left(\int_0^t s^n W_s dW_s\right) dt,$$

which converges in $L^2(\Omega)$ to $\frac{1}{2}W_1^2$. Therefore, conditions (a) and (b) of Proposition 4.1 are satisfied with $S^2 = \frac{1}{2}W_1^2$, and $F_n$ converges in distribution to $\frac{1}{\sqrt{2}}W_1 \times N$, with $N \sim N(0,1)$. One easily see on this particular example that the choice $\widehat{g}_n = g_n$ does not allows us to conclude in general (except when $S^2$ is deterministic); indeed, one can check here that $\langle u_n, DF_n\rangle_{\mathfrak{H}} = \frac{1}{m}\|DF_n\|^2_{\mathfrak{H}}$ does not converge in $L^1(\Omega)$.

If we take $\widehat{g}_n = g_n$ and $S^2 = 1$, then condition (b) coincides with condition (iii) in the introduction. In this case, Nualart and Peccati criterion combined with Lemma 6 in [21] tells us that, if the sequence of variances converges to one, then condition (a) is automatically satisfied.

On the other hand, we can also apply Theorem 3.1 with $u_n = g_n$. In this way, applying Corollary 3.3, we obtain that the following conditions imply that $F_n$ converges to a normal random variable $N(0,1)$ independent of $X$:

($\alpha$) $g_n$ converges weakly to zero;

($\beta$) $\|g_n \otimes_l g_n\|_{\mathfrak{H}^{\otimes 2(q-l)}}$ converges to zero for all $l = 1, \dots, q-1$;

($\gamma$) $q!\|g_n\|^2_{\mathfrak{H}^{\otimes q}}$ converges to 1.

Indeed, notice first that if $g_n$ is bounded in $\mathfrak{H}^{\odot q}$, then $F_n$ is bounded in all the Sobolev spaces $\mathbb{D}^{q,p}$, $p \geq 2$. Then, condition (ii) in Corollary 3.3 follows from ($\gamma$) and the equality $D^q\left(I_q(g_n)\right) = q!g_n$. On the other hand, condition (i') in Corollary 3.3 follows from (ii) and

$$\begin{aligned}
E\left[\left\|g_n \otimes_l D^l F_n\right\|^2_{\mathfrak{H}^{\otimes(q-l)}}\right] &= \frac{q!^2}{(q-l)!^2} E\left[\|g_n \otimes_l I_{q-l}(g_n)\|^2_{\mathfrak{H}^{\otimes(q-l)}}\right] \\
&= \frac{q!^2}{(q-l)!^2} E\left[\|I_{q-l}(g_n \otimes_l g_n)\|^2_{\mathfrak{H}^{\otimes(q-l)}}\right] \\
&= \frac{q!^2}{(q-l)!} \left\|g_n \widetilde{\otimes}_l g_n\right\|^2_{\mathfrak{H}^{\otimes 2(q-l)}} \\
&\leq \frac{q!^2}{(q-l)!} \|g_n \otimes_l g_n\|^2_{\mathfrak{H}^{\otimes 2(q-l)}}.
\end{aligned}$$

In this way we recover the fact that condition (iii) in the introduction implies the normal convergence.

# 5 Weighted Hermite variations of the fractional Brownian motion

## 5.1 Description of the results

The fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B = \{B_t,\, t \geq 0\}$ with the covariance function

$$E(B_s B_t) = R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{5.1}$$

From (5.1), it follows that $E|B_t - B_s|^2 = (t-s)^{2H}$ for all $0 \leqslant s < t$ and that, for each $a > 0$, the process $\{a^{-H} B_{at},\, t \geq 0\}$ is also a fBm with Hurst parameter $H$ (self-similarity property). As a consequence, the sequence $\{B_j - B_{j-1},\, j = 1, 2, \ldots\}$ is stationary, Gaussian and ergodic, with correlation given by

$$\rho_H(n) = \frac{1}{2}\left[|n+1|^{2H} - 2|n|^{2H} + |n-1|^{2H}\right], \tag{5.2}$$

which behaves as $H(2H-1)|n|^{2H-2}$ as $n$ tends to infnity.

Set $\Delta B_{k/n} = B_{(k+1)/n} - B_{k/n}$, where $k = 0, 1, \ldots, n$, and $n \geq 1$. The ergodic theorem combined with the self-similarity property implies that the sequence $n^{2H-1}\sum_{k=0}^{n-1}\left(\Delta B_{k/n}\right)^2$ converges, almost surely and in $L^1(\Omega)$, to $E(B_1^2) = 1$. Moreover, it is well-known (see, e.g., [3]) that, provided $H \in (0, \frac{3}{4})$, a central limit theorem holds: the sequence

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\left(n^{2H}\left(\Delta B_{k/n}\right)^2 - 1\right) = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_2\left(n^H \Delta B_{k/n}\right) \tag{5.3}$$

converges in law to $N(0, \sigma_H^2)$ as $n \to \infty$, for some constant $\sigma_H > 0$. (Notice also that, by normalizing with $\sqrt{n \log n}$ instead of $\sqrt{n}$, the central limit theorem continues to hold in the critical case $H = \frac{3}{4}$.) When $H > \frac{3}{4}$, the situation is very different. Indeed, we have in contrast that

$$n^{1-2H}\sum_{k=0}^{n-1}\left(n^{2H}\left(\Delta B_{k/n}\right)^2 - 1\right) = n^{1-2H}\sum_{k=0}^{n-1} H_2\left(n^H \Delta B_{k/n}\right)$$

converges in $L^2(\Omega)$. More generally, consider an integer $q \geq 2$. If $H < 1 - \frac{1}{2q}$, then the sequence

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} H_q\left(n^H \Delta B_{k/n}\right) \tag{5.4}$$

converges in law to $N(0,\sigma_{q,H}^2)$ (for some constant $\sigma_{q,H}>0$), whereas, if $H>1-\frac{1}{2q}$, then the sequence

$$n^{q-qH-1}\sum_{k=0}^{n-1} H_q\left(n^H \Delta B_{k/n}\right)$$

converges in $L^2(\Omega)$.

Some unexpected results happen when we introduce a weight of the form $f(B_{k/n})$ in (5.4). In fact, a new critical value ($H=\frac{1}{2q}$) plays an important role. More precisely, consider the following sequence of random variables:

$$G_n=\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(B_{k/n})H_q(n^H\Delta B_{k/n}). \tag{5.5}$$

Here, the integer $q\geq 2$ is fixed and the function $f:\mathbb{R}\to\mathbb{R}$ is supposed to satisfy some suitable regularity and growth conditions. In [13, 18], the following convergences as $n\to\infty$ are shown:

- If $H<\frac{1}{2q}$, then

$$n^{qH-\frac{1}{2}}\,G_n \overset{L^2(\Omega)}{\longrightarrow} \frac{(-1)^q}{2^q q!}\int_0^1 f^{(q)}(B_s)ds. \tag{5.6}$$

- If $\frac{1}{2q}<H<1-\frac{1}{2q}$, then

$$G_n \overset{\text{stably}}{\longrightarrow} \sigma_{H,q}\int_0^1 f(B_s)dW_s, \tag{5.7}$$

  where $W$ is a Brownian motion independent of $B$, and

$$\sigma_{H,q}^2=q!\sum_{r\in\mathbb{Z}}\rho_H(r)^q<\infty. \tag{5.8}$$

- If $H=1-\frac{1}{2q}$, then

$$\frac{G_n}{\sqrt{\log n}} \overset{\text{stably}}{\longrightarrow} \sqrt{\frac{2}{q!}}\left(1-\frac{1}{2q}\right)^{q/2}\left(1-\frac{1}{q}\right)^{q/2}\int_0^1 f(B_s)dW_s,$$

  where $W$ is a Brownian motion independent of $B$.

- If $H > 1 - \frac{1}{2q}$, then

$$n^{q(1-H)-\frac{1}{2}} G_n \overset{L^2(\Omega)}{\longrightarrow} \int_0^1 f(B_s) dZ_s^{(q)},$$

where $Z^{(q)}$ denotes the Hermite process of order $q$ canonically constructed from $B$ (see [18] for the details).

In addition, when $q = 2$ and $H = \frac{1}{4}$, it was shown in [19] that $G_n$ converges stably to a linear combination of the limits in (5.7) and (5.6). (The proof of this last result follows an approach similar to the proof of our Theorem 3.1, and allows to derive a change of variable formula for the fBm of Hurst index $\frac{1}{4}$, with a correction term that is an ordinary Itô integral with respect to a Brownian motion that is independent of $B$.) But the convergence of $G_n$ in the critical case $H = \frac{1}{2q}$, $q \geq 3$, was open till now.

In the present paper, we are going to show that Theorem 3.1 provides a proof of the following new result, valid for any integer $q \geq 2$ and any index $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$:

$$G_n - n^{-\frac{1}{2}-qH} \frac{(-1)^q}{2^q q!} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n}) \overset{\text{stably}}{\longrightarrow} \sigma_{H,q} \int_0^1 f(B_s) dW_s. \tag{5.9}$$

(See Theorem 5.3 below for a precise statement.) Notice that (5.9) provides a new proof of (5.7) in the case $H \in \left(\frac{1}{2q}, \frac{1}{2}\right)$ (without considering two different levels of discretization $n \leqslant m$, as in [18]). More importantly, in the critical case $H = \frac{1}{2q}$, convergence (5.9) yields:

$$G_n \overset{\text{stably}}{\longrightarrow} \frac{(-1)^q}{2^q q!} \int_0^1 f^{(q)}(B_s) ds + \sigma_{1/(2q),q} \int_0^1 f(B_s) dW_s.$$

Hence, the understanding of the asymptotic behavior of the weighted Hermite variations of the fBm is now complete (indeed, the case $H = \frac{1}{2q}$, $q \geq 3$, was the only remaining case, as mentioned in the discussion above).

The main idea of the proof of (5.9) is a decomposition of the random variable $G_n$ using equation (2.6). The term with $r = 0$ is a multiple Skorohod integral of order $q$ and, by Theorem 5.2 below, it converges in law for any $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$. The term with $r = q$ behaves as $-n^{-\frac{1}{2}-qH} \frac{(-1)^q}{2^q q!} \sum_{k=0}^{n} f^{(q)}(B_{k/n})$. The remaining terms ($1 \leq r \leq q-1$) converge to zero in $L^2(\Omega)$.

## 5.2 Some preliminaries on the fractional Brownian motion

Before proving (5.9), we need some preliminaries on the Malliavin calculus associated with the fBm and some technical results (see [20, Chapter 5]).

In the following we assume $H \in \left(0, \frac{1}{2}\right)$. We denote by $\mathcal{E}$ the set of step functions on $[0, 1]$. Let $\mathfrak{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathfrak{H}} = R_H(t, s) = \frac{1}{2}\left(s^H + t^H - |t-s|^H\right).$$

The mapping $\mathbf{1}_{[0,t]} \to B_t$ can be extended to a linear isometry between the Hilbert space $\mathfrak{H}$ and the Gaussian space spanned by $B$. We denote this isometry by $\phi \to B(\phi)$. In this way $\{B(\phi), \phi \in \mathfrak{H}\}$ is an isonormal Gaussian space. (In fact, we know that the space $\mathfrak{H}$ coincides with $I_{0+}^{H-\frac{1}{2}}(L^2[0, 1])$, where

$$I_{0+}^{H-\frac{1}{2}} f(x) = \frac{1}{\Gamma(H - \frac{1}{2})} \int_0^x (x-y)^{H-\frac{3}{2}} f(y) dy$$

is the left-sided Liouville fractional integral of order $H - \frac{1}{2}$, see [6].)

From now on, we will make use of the notation

$$\begin{aligned} \varepsilon_t &= \mathbf{1}_{[0,t]}, \\ \partial_{k/n} &= \varepsilon_{(k+1)/n} - \varepsilon_{k/n} = \mathbf{1}_{(k/n,(k+1)/n]}, \end{aligned}$$

for $t \in [0, 1]$, $n \geq 1$, and $k = 0, \ldots, n-1$. Notice that $H_q(n^H \Delta B_{k/n}) = n^{qH} I_q(\partial_{k/n}^{\otimes q})$.

We need the following technical lemma.

**Lemma 5.1** *Recall that $H < \frac{1}{2}$. Let $n \geq 1$ and $k = 0, \ldots, n-1$. We have*

*(a)* $\left|E(B_r(B_t - B_s))\right| \leqslant (t-s)^{2H}$ *for any* $r \in [0, 1]$ *and* $0 \leqslant s < t \leqslant 1$.

*(b)* $\left|\left\langle \varepsilon_t,\ \partial_{k/n} \right\rangle_{\mathfrak{H}}\right| \leqslant n^{-2H}$ *for any* $t \in [0, 1]$.

*(c)* $\sup_{t \in [0,1]} \sum_{k=0}^{n-1} \left|\left\langle \varepsilon_t,\ \partial_{k/n} \right\rangle_{\mathfrak{H}}\right| = O(1)$ *as $n$ tends to infinity.*

*(d) For any integer $q \geq 2$,*

$$\sum_{k=0}^{n-1} \left|\left\langle \varepsilon_{k/n},\ \partial_{k/n} \right\rangle_{\mathfrak{H}}^q - \frac{(-1)^q}{2^q n^{2qH}}\right| = O(n^{-2H(q-1)}) \quad \text{as } n \text{ tends to infinity.} \tag{5.10}$$

*(e) Recall the definition (5.2) of $\rho_H$. We have*

$$\left\langle \partial_{j/n},\ \partial_{k/n} \right\rangle_{\mathfrak{H}} = n^{-2H} \rho_H(k-j).$$

*Consequently, for any integer $q \geq 1$, we can write*

$$\sum_{k,j=0}^{n-1} \left| \left\langle \partial_{j/n},\ \partial_{k/n} \right\rangle_{\mathfrak{H}} \right|^q = O(n^{1-2qH}) \quad \text{as } n \text{ tends to infinity.} \tag{5.11}$$

**Proof.** We have

$$\begin{aligned} E(B_r(B_t - B_s)) &= \frac{1}{2}\left(r^{2H} + t^{2H} - |t-r|^{2H}\right) - \frac{1}{2}\left(r^{2H} + s^{2H} - |s-r|^{2H}\right) \\ &= \frac{1}{2}(t^{2H} - s^{2H}) + \frac{1}{2}\left(|s-r|^{2H} - |t-r|^{2H}\right). \end{aligned}$$

Using the inequality $|b^{2H} - a^{2H}| \leqslant |b-a|^{2H}$ for any $a, b \in [0,1]$, we deduce (a). Property (b) is an immediate consequence of (a). To show property (c) we use

$$\left\langle \varepsilon_t,\ \partial_{k/n} \right\rangle_{\mathfrak{H}} = \frac{1}{2n^{2H}} \left[(k+1)^{2H} - k^{2H} - |k+1-nt|^{2H} + |k-nt|^{2H}\right].$$

Property (d) follows from

$$\left\langle \varepsilon_{k/n},\ \partial_{k/n} \right\rangle_{\mathfrak{H}} = \frac{1}{2n^{2H}} \left[(k+1)^{2H} - k^{2H} - 1\right],$$

and

$$\begin{aligned} \left| \left\langle \varepsilon_{k/n},\ \partial_{k/n} \right\rangle_{\mathfrak{H}}^q - \frac{(-1)^q}{2^q n^{2qH}} \right| &= \frac{1}{2^q n^{2qH}} \left| \left[(k+1)^{2H} - k^{2H} - 1\right]^q - (-1)^q \right| \\ &= \frac{1}{2^q n^{2qH}} \sum_{i=1}^{q} \binom{q}{i} \left[(k+1)^{2H} - k^{2H}\right]^i \\ &\leq \frac{1}{2^q n^{2qH}} \left[(k+1)^{2H} - k^{2H}\right] \sum_{i=1}^{q} \binom{q}{i}. \end{aligned}$$

Finally, property (e) follows from

$$\sum_{k,j=0}^{n-1} \left| \left\langle \partial_{j/n},\ \partial_{k/n} \right\rangle_{\mathfrak{H}} \right|^q \leq n^{-2qH} \sum_{k,j=0}^{n-1} |\rho_H(j-k)|^q \leq n^{1-2qH} \sum_{r \in \mathbb{Z}} |\rho_H(r)|^q.$$

■

### 5.3 An auxiliary convergence result

From now on, we fix $q \geq 2$ and we make use of the following hypothesis on $f : \mathbb{R} \to \mathbb{R}$:

**(H)** $f$ belongs to $\mathcal{C}^{2q}$ and, for any $p \geq 2$ and $i = 0, \ldots, 2q$,

$$E(\sup_{t\in[0,1]} |f^{(i)}(B_t)|^p) < \infty. \tag{5.12}$$

Notice that a sufficient condition for (5.12) to hold is that $f$ satisfies an exponential growth condition of the form $\left|f^{(2q)}(x)\right| \leqslant ke^{c|x|^p}$ for some constants $c, k > 0$ and $0 < p < 2$.

The aim of this section is to prove the following auxiliary convergence result.

**Theorem 5.2** *Suppose $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$, and let $f$ be a function satisfying Hypothesis **(H)**. Consider the sequence of $q$-parameter step processes defined by*

$$u_n = n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n})\partial_{k/n}^{\otimes q}. \tag{5.13}$$

*Then $u_n \in \mathrm{Dom}\delta^q$, and $\delta^q(u_n)$ converges stably to $\sigma_{H,q} \int_0^1 f(B_s)dW_s$, where $W$ is a Brownian motion independent of $B$, and $\sigma_{H,q} > 0$ is defined in (5.8).*

**Proof.** The fact that $u_n$ belongs to $\mathrm{Dom}\delta^q$ is a consequence of the inclusion $\mathbb{D}^{q,2}(\mathfrak{H}^{\otimes q}) \subset \mathrm{Dom}\delta^q$ and hypothesis **(H)**. We are now going to show that the sequence $F_n = \delta^q(u_n)$ satisfies the conditions of Theorem 3.1. We make use of the notation

$$\alpha_{k,j} = \left\langle \varepsilon_{k/n}, \partial_{j/n} \right\rangle_{\mathfrak{H}}, \ \beta_{k,j} = \left\langle \partial_{k/n}, \partial_{j/n} \right\rangle_{\mathfrak{H}}, \tag{5.14}$$

for $k, j = 0, \ldots, n-1$ and $n \geq 1$. Also $C$ will denote a generic constant.

*Step 1.* Let us show first that $F_n$ is bounded in $L^2(\Omega)$. Taking into account the continuity of the Skorohod integral from the space $\mathbb{D}^{q,2}(\mathfrak{H}^{\otimes q})$ into $L^2(\Omega)$ (see (2.9)), it suffices to show that $u_n$ is bounded in $\mathbb{D}^{q,2}(\mathfrak{H}^{\otimes q})$. Actually we are going to show that $u_n$ is bounded in $\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes k})$ for any integer $k \leq 2q$ and any real number $p \geq 2$. Using the estimate (5.11) we obtain

$$\|u_n\|_{\mathfrak{H}^{\otimes q}}^2 = n^{2qH-1} \sum_{k,j=0}^{n-1} f(B_{k/n})f(B_{j/n})\beta_{k,j}^q \leq C \sup_{0\leq t\leq 1} |f(B_t)|^2 .$$

Moreover for any integer $k \geq 1$,

$$D^k u_n = n^{qH-\frac{1}{2}} \sum_{j=0}^{n-1} f^{(k)}(B_{j/n}) \varepsilon_{j/n}^{\otimes k} \otimes \partial_{j/n}^{\otimes q},$$

and we obtain in the same way

$$\begin{aligned} \left\| D^k u_n \right\|^2_{\mathfrak{H}^{\otimes(q+k)}} &= n^{2qH-1} \sum_{l,j=0}^{n-1} f^{(k)}(B_{l/n}) f^{(k)}(B_{j/n}) \left\langle \varepsilon_{l/n}, \varepsilon_{j/n} \right\rangle^k \beta_{l,j}^q \\ &\leq C \sup_{0 \leq t \leq 1} \left| f^{(k)}(B_t) \right|^2. \end{aligned}$$

Then the result follows from hypothesis **(H)**.

*Step 2.* Let us show condition (i) of Theorem 3.1. Fix some integers $r, k_1, \ldots, k_{q-1} \geq 0$ such that $k_1 + 2k_2 + \ldots + (q-1)k_{q-1} + r = q$. Let $h \in \mathfrak{H}^{\otimes r}$. We claim that $\left\langle u_n, (DF_n)^{\otimes k_1} \otimes \ldots \otimes (D^{q-1}F_n)^{\otimes k_{q-1}} \otimes h \right\rangle_{\mathfrak{H}^{\otimes q}}$ converges to zero in $L^1(\Omega)$. Suppose first that $r \geq 1$. Without loss of generality, we can assume that $h$ has the form $g \otimes \varepsilon_t$, with $g \in \mathfrak{H}^{\otimes(r-1)}$. Set $\Phi_n = (DF_n)^{\otimes k_1} \otimes \ldots \otimes (D^{q-1}F_n)^{\otimes k_{q-1}} \otimes g$. Then we can write

$$\langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathfrak{H}^{\otimes q}} = n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \left\langle \partial_{k/n}^{\otimes(q-1)}, \Phi_n \right\rangle_{\mathfrak{H}^{\otimes(q-1)}} \left\langle \partial_{k/n}, \varepsilon_t \right\rangle_{\mathfrak{H}}.$$

As a consequence,

$$\begin{aligned} E\left( \left| \langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathfrak{H}^{\otimes q}} \right| \right) &\leq n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} E\left( \left| f(B_{k/n}) \left\langle \partial_{k/n}^{\otimes(q-1)}, \Phi_n \right\rangle_{\mathfrak{H}^{\otimes(q-1)}} \right| \right) \\ &\quad \times \left| \left\langle \partial_{k/n}, \varepsilon_t \right\rangle_{\mathfrak{H}} \right|. \end{aligned}$$

Condition (c) of Lemma 5.1 implies

$$\sum_{k=0}^{n-1} \left| \left\langle \partial_{k/n}, \varepsilon_t \right\rangle_{\mathfrak{H}} \right| \leq C.$$

Hence,

$$E\left( \left| \langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathfrak{H}^{\otimes q}} \right| \right) \leq C n^{H-\frac{1}{2}} \left( E\left( \|\Phi_n\|^2_{\mathfrak{H}^{\otimes(q-1)}} \right) \right)^{\frac{1}{2}}.$$

On the other hand

$$\|\Phi_n\|^2_{\mathfrak{H}^{\otimes(q-1)}} = \|g\|^2_{\mathfrak{H}^{\otimes(r-1)}} \prod_{m=1}^{q-1} \|D^m F_n\|^{2k_m}_{\mathfrak{H}^{\otimes m}},$$

and applying the generalized Hölder's inequality

$$\begin{aligned} E\left(\|\Phi_n\|^2_{\mathfrak{H}^{\otimes(q-1)}}\right) &\leq C \prod_{m=1}^{q-1} \left(E\left(\|D^m F_n\|^{2k_m(q-1)}_{\mathfrak{H}^{\otimes m}}\right)\right)^{\frac{1}{q-1}} \\ &= C \prod_{m=1}^{q-1} \|D^m F_n\|^{2k_m}_{L^{2k_m(q-1)}(\Omega;\mathfrak{H}^{\otimes m})}. \end{aligned}$$

By Meyer's inequalities (2.9), for any $1 \leq m \leq q-1$ and any $p \geq 2$, we obtain, using Step 1, that

$$\begin{aligned} \|D^m F_n\|_{L^p(\Omega;\mathfrak{H}^{\otimes m})} &= \|D^m \delta^q(u_n)\|_{L^p(\Omega;\mathfrak{H}^{\otimes m})} \\ &\leq \|\delta^q(u_n)\|_{\mathbb{D}^{m,p}} \leq C\,\|u_n\|_{\mathbb{D}^{m+q,p}(\mathfrak{H}^{\otimes q})} \leq C. \end{aligned}$$

Therefore,

$$E\left(\left|\langle u_n, \Phi_n \otimes \varepsilon_t\rangle_{\mathfrak{H}^{\otimes q}}\right|\right) \leq C n^{H-\frac{1}{2}},$$

which converges to zero as $n$ tends to infinity because $H < \frac{1}{2}$.

Suppose now that $r = 0$. In this case, we have $\Phi_n = (DF_n)^{\otimes k_1} \otimes \cdots \otimes (D^{q-1}F_n)^{\otimes k_{q-1}}$. Then

$$\left\langle \partial^{\otimes q}_{j/n}, \Phi_n \right\rangle_{\mathfrak{H}^{\otimes q}} = \left\langle \partial_{j/n}, DF_n \right\rangle^{k_1}_{\mathfrak{H}} \cdots \left\langle \partial^{\otimes(q-1)}_{j/n}, D^{q-1}F_n \right\rangle^{k_{q-1}}_{\mathfrak{H}^{\otimes(q-1)}}. \tag{5.15}$$

From (5.15) and (5.13) we obtain

$$\langle u_n, \Phi_n\rangle_{\mathfrak{H}^{\otimes q}} = n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \prod_{m=1}^{q-1} \left\langle \partial^{\otimes m}_{j/n}, D^m F_n \right\rangle^{k_m}_{\mathfrak{H}^{\otimes m}}. \tag{5.16}$$

Notice that for any $m = 1, \ldots, q-1$, the term $\left\langle \partial^{\otimes m}_{j/n}, D^m F_n \right\rangle_{\mathfrak{H}^{\otimes m}}$ can be estimated by $n^{-mH}\,\|D^m F_n\|_{\mathfrak{H}^{\otimes m}}$. Then, taking into account that

$$\sup_n E\left(\|D^m F_n\|^p_{\mathfrak{H}^{\otimes m}}\right) < \infty$$

for any $p \geq 2$, and that $\sum_{m=1}^{q-1} m k_m = q$, we obtain for $E\left(\left|\langle u_n, \Phi_n\rangle_{\mathfrak{H}^{\otimes q}}\right|\right)$ an estimate of the form $C\sqrt{n}$, which is unfortunately not satisfactory. For this reason, a finer analysis of the terms $\left\langle \partial^{\otimes m}_{j/n}, D^m F_n \right\rangle_{\mathfrak{H}^{\otimes m}}$ is required.

First we are going to apply formula (2.11) to compute the derivative $D^m F_n$, $m = 1, \ldots, q-1$:

$$\begin{aligned} D^m F_n &= \sum_{i=0}^{m} \binom{m}{i}\binom{q}{i} i! \delta^{q-i}(D^{m-i} u_n) \\ &= n^{qH-\frac{1}{2}} \sum_{i=0}^{m} \binom{m}{i}\binom{q}{i} i! \sum_{l=0}^{n-1} \left( \varepsilon_{l/n}^{\otimes(m-i)} \otimes \partial_{l/n}^{\otimes i} \right) \\ &\quad \times \delta^{q-i}\left( f^{(m-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right). \end{aligned} \tag{5.17}$$

Set $\Psi_n^{m,j} = \left\langle \partial_{j/n}^{\otimes m}, D^m F_n \right\rangle_{\mathfrak{H}^{\otimes m}}$, and recall the definition of $\alpha_{k,j}$ and $\beta_{k,j}$ from (5.14). From (5.17) we obtain

$$\begin{aligned} \Psi_n^{m,j} &= n^{qH-\frac{1}{2}} \sum_{i=0}^{m} \binom{m}{i}\binom{q}{i} i! \sum_{l=0}^{n-1} \alpha_{l,j}^{m-i} \beta_{l,j}^{i}\, \delta^{q-i}\left( f^{(m-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right) \\ &= \sum_{i=0}^{m} \Phi_n^{i,m,j}, \end{aligned} \tag{5.18}$$

with

$$\Phi_n^{i,m,j} = n^{qH-\frac{1}{2}} \binom{m}{i}\binom{q}{i} i! \sum_{l=0}^{n-1} \alpha_{l,j}^{m-i} \beta_{l,j}^{i} \delta^{q-i}\left( f^{(m-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right).$$

By Meyer inequalities (2.9) we obtain, using also assumption **(H)**, that, for any $p \geq 2$,

$$\begin{aligned} \left\| \delta^{q-i}\left( f^{(m-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right) \right\|_{L^p} &\leq C \left\| f^{(m-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right\|_{\mathbb{D}^{q-i,p}(\mathfrak{H}^{\otimes q-i})} \\ &\leq C n^{-(q-i)H}. \end{aligned} \tag{5.19}$$

Using Lemma 5.1 (b) and (e) we have $\left| \alpha_{l,j}^{m-i} \right| \leq C n^{-(m-i)2H}$ and $\sum_{l=0}^{n-1} \left| \beta_{l,j}^{i} \right| \leq C n^{-2iH}$. Therefore, for any $i \geq 1$, we have

$$\left\| \Phi_n^{i,m,j} \right\|_{L^p} \leq C n^{iH-\frac{1}{2}} \sum_{l=0}^{n-1} \left| \alpha_{l,j}^{m-i} \beta_{l,j}^{i} \right| \leq C n^{-\frac{1}{2}-2mH+iH}. \tag{5.20}$$

On the other hand, if $i = 0$, Lemma 5.1 (c) and (5.19) yield

$$\left\| \Phi_n^{0,m,j} \right\|_{L^p} \leq C n^{-\frac{1}{2}-2mH+2H}. \tag{5.21}$$

Notice that the estimate for the $L^p(\Omega)$-norm of $\Phi_n^{0,m,j}$ in the case $i=0$ is worst than for $i \geq 1$. We will see later that, for $p=2$, we can get a better estimate for $\Phi_n^{0,m,j}$.

Because $\sum_{m=1}^{q-1} k_m \geq 2$, the number of factors in $\prod_{m=1}^{q-1} \left\langle \partial_{j/n}, D^m F_n \right\rangle_{\mathfrak{H}^{\otimes m}}^{k_m}$ is at least two. As a consequence, we can write

$$\langle u_n, \Phi_n \rangle_{\mathfrak{H}^{\otimes q}} = n^{qH-\frac{1}{2}} \sum_{j=0}^{n-1} f(B_{j/n}) \Psi_n^{\mu,j} \Psi_n^{\nu,j} \Theta_n^j,$$

for some $\mu$, $\nu$ (not necessarily distinct), where

$$\Theta_n^j = \left(\Psi_n^{\mu,j}\right)^{k_\mu - 1} \left(\Psi_n^{\nu,j}\right)^{k_\nu - 1} \prod_{\substack{m=1 \\ m \neq \mu,\nu}}^{q-1} \left(\Psi_n^{m,j}\right)^{k_m}. \tag{5.22}$$

Consider the decomposition

$$\langle u_n, \Phi_n \rangle_{\mathfrak{H}^{\otimes q}} = A_n + B_n,$$

where

$$\begin{aligned} A_n &= n^{qH-\frac{1}{2}} \sum_{j=0}^{n-1} f(B_{j/n}) \left( \sum_{i=0}^{\mu} \sum_{k=0}^{\nu} \mathbf{1}_{i+k\geq 1} \Phi_n^{i,\mu,j} \Phi_n^{k,\nu,j} \right) \Theta_n^j, \\ B_n &= n^{qH-\frac{1}{2}} \sum_{j=0}^{n-1} f(B_{j/n}) \Phi_n^{0,\mu,j} \Phi_n^{0,\nu,j} \Theta_n^j. \end{aligned}$$

From (5.22) and the estimate $\|\Psi_n^{m,j}\|_{L^p} \leq C n^{-mH}$, for all $p \geq 2$ and $1 \leq m \leq q$, we obtain

$$\left\| \Theta_n^j \right\|_{L^p} \leq C n^{-H(q-\mu-\nu)}. \tag{5.23}$$

Then, from (5.20), (5.21) and (5.23) we obtain

$$\begin{aligned} E\left(|A_n|\right) &\leq C n^{qH+\frac{1}{2}} n^{-H(q-\mu-\nu)} \Big( \sum_{i=1}^{\mu} \sum_{k=1}^{\nu} n^{-1-2(\mu+\nu)H+(i+k)H} \\ &\quad + \sum_{i=1}^{\mu} n^{-1-2(\mu+\nu)H+iH+2H} + \sum_{k=1}^{\nu} n^{-1-2(\mu+\nu)H+kH+2H} \Big) \\ &= C n^{-\frac{1}{2}} + n^{-\frac{1}{2}+2H-\mu H} + n^{-\frac{1}{2}+2H-\nu H}, \end{aligned}$$

which converges to zero as $n$ tends to infinity, because $\mu, \nu \geq 1$ and $H < \frac{1}{2}$.

For the term $B_n$ using again the estimates (5.21) and (5.23) we get

$$
\begin{aligned}
E(|B_n|) &\le Cn^{qH+\frac{1}{2}-H(q-\mu-\nu)-1-2H(\mu+\nu)+4H} = Cn^{-\frac{1}{2}-H(\mu+\nu)+4H} \\
&\le Cn^{-\frac{1}{2}+2H},
\end{aligned}
$$

which converges to zero as $n$ tends to infinity if $H < \frac{1}{4}$. To handle the case $H \in \left[\frac{1}{4}, \frac{1}{2}\right)$ we need more precise estimates for the $L^2(\Omega)$-norm of $\Phi_n^{0,\nu,j}$. We have, using formula (2.12)

$$
\begin{aligned}
E\left[\left(\Phi_n^{0,\nu,j}\right)^2\right] &= \binom{q}{i}^2 \binom{m}{i}^2 i!^2 E\left(\left| n^{qH-\frac{1}{2}} \sum_{l=0}^{n-1} \alpha_{l,j}^{\nu} \delta^q \left(f^{(\nu)}(B_{l/n}) \partial_{l/n}^{\otimes q}\right)\right|^2\right) \\
= \; & n^{2qH-1} \binom{q}{i}^2 \binom{m}{i}^2 i!^2 \sum_{l,l'=0}^{n-1} \alpha_{l,j}^{\nu} \alpha_{l',j}^{\nu} \\
& \times E\left(\delta^q \left(f^{(\nu)}(B_{l/n}) \partial_{l/n}^{\otimes q}\right) \delta^q \left(f^{(\nu)}(B_{l'/n}) \partial_{l'/n}^{\otimes q}\right)\right) \\
= \; & n^{2qH-1} \binom{q}{i}^2 \binom{m}{i}^2 i!^2 \sum_{l,l'=0}^{n-1} \alpha_{l,j}^{\nu} \alpha_{l',j}^{\nu} \sum_{i=0}^{q} \binom{q}{i}^2 i! \alpha_{l,l'}^{q-i} \alpha_{l',l}^{q-i} \beta_{l,l'}^{2i} \\
& \times E\left(f^{(\nu+q-i)}(B_{l/n}) \, f^{(\nu+q-i)}(B_{l'/n})\right) \\
= \; & \sum_{i=0}^{q} R_n^i.
\end{aligned}
$$

If $i \ge 1$, then $\sum_{l,l'=0}^{n-1} \beta_{l,l'}^{2i} \le Cn^{1-4iH}$, and we obtain an estimate of the form $\left\|R_n^i\right\|_{L^2} \le Cn^{\gamma}$, where

$$
\gamma = \frac{1}{2}\left(2qH - 1 - 4\nu H - 4(q-i)H + 1 - 4iH\right) = -qH - 2\nu H.
$$

For $i = 0$, then $\sup_n \sum_{l,l'=0}^{n-1} \left|\alpha_{l,l'} \alpha_{l',l}\right| < \infty$, and we get

$$
\gamma = \frac{1}{2}\left(2qH - 1 - 2H(2\nu + 2q - 2)\right) = -qH - 2\nu H - \frac{1}{2} + 2H.
$$

We have obtained the estimate

$$
\left\|\Phi_n^{0,\nu,j}\right\|_{L^2} \le Cn^{-qH-2\nu H+2H-\frac{1}{2}}. \tag{5.24}
$$

Fix $\frac{1}{4qH} < \alpha < 1$. This choice is possible because $\frac{1}{4qH} < 1$. We have, by Hölder's inequality,

$$
E(|B_n|) \le Cn^{qH-\frac{1}{2}} \sum_{j=0}^{n-1} \left\|\Phi_n^{0,\mu,j}\right\|_{L^2}^{\alpha} \left\|\Phi_n^{0,\nu,j}\right\|_{L^2}^{\alpha} \left\|\left|\Phi_n^{0,\mu,j}\Phi_n^{0,\nu,j}\right|^{1-\alpha} \Theta_n^j\right\|_{L^{\frac{1}{1-\alpha}}}.
$$

Using (5.24), (5.21) and (5.23) we obtain

$$E\left(|B_n|\right) \le Cn^{\gamma}, \tag{5.25}$$

where

$$\begin{aligned}
\gamma &= qH + \frac{1}{2} + \left[-2qH - 2(\mu+\nu)H + 4H - 1\right]\alpha \\
&\quad -H(q-\mu-\nu) + (1-\alpha)(-1-2H(\mu+\nu)+4H) \\
&= -\frac{1}{2} + 4H - H(\mu+\nu) - 2\alpha qH \\
&\le -\frac{1}{2} + 2H - 2\alpha qH \le \frac{1}{2} - 2\alpha qH < 0,
\end{aligned}$$

because $H < \frac{1}{2}$. Therefore $E\left(|B_n|\right)$ converges to zero as $n$ tends to infinity.

*Step 3.* Let us show condition (ii). We have

$$\langle u_n, D^q F_n\rangle_{\mathfrak{H}^{\otimes q}} = n^{qH-\frac{1}{2}} \sum_{j=0}^{n-1} f(B_{j/n}) \left\langle \partial_{j/n}^{\otimes q}, D^q F_n \right\rangle_{\mathfrak{H}^{\otimes q}}.$$

From (5.18) we get

$$\left\langle \partial_{j/n}^{\otimes q}, D^q F_n \right\rangle_{\mathfrak{H}^{\otimes q}} = n^{qH-\frac{1}{2}} \sum_{i=0}^{q} \binom{q}{i}^2 i! \sum_{l=0}^{n-1} \alpha_{l,j}^{q-i} \beta_{l,j}^{i} \delta^{q-i}\left(f^{(q-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)}\right).$$

Therefore, we can make the decomposition

$$\langle u_n, D^q F_n\rangle_{\mathfrak{H}^{\otimes q}} = A_n + B_n + C_n,$$

where

$$\begin{aligned}
A_n &= n^{2qH-1} q! \sum_{l,j=0}^{n-1} \beta_{l,j}^{q} f(B_{l/n}) f(B_{j/n}), \\
B_n &= n^{2qH-1} \sum_{i=1}^{q-1} \binom{q}{i}^2 i! \sum_{l,j=0}^{n-1} \alpha_{l,j}^{q-i} \beta_{l,j}^{i} f(B_{j/n}) \delta^{q-i}\left(f^{(q-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)}\right), \\
C_n &= n^{2qH-1} \sum_{l,j=0}^{n-1} \alpha_{l,j}^{q} f(B_{j/n}) \delta^{q}\left(f^{(q)}(B_{l/n}) \partial_{l/n}^{\otimes(q)}\right).
\end{aligned}$$

The term $A_n$ converges to a nonnegative square integrable random variable. Indeed,

$$\begin{aligned} A_n &= \frac{q!}{2^q n} \sum_{k,j=0}^{n-1} f(B_{k/n}) f(B_{j/n}) \left( |k-j+1|^{2H} + |k-j-1|^{2H} - 2|k-j|^{2H} \right)^q \\ &= \frac{q!}{2^q n} \sum_{p=-\infty}^{\infty} \sum_{j=0 \vee -p}^{(n-1)\wedge(n-1-p)} f(B_{j/n}) f(B_{(j+p)/n}) \left( |p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H} \right)^q , \end{aligned}$$

which converges in $L^1(\Omega)$ to

$$q! \left( \sum_{k\in\mathbb{Z}} \rho_H(k)^q \right) \int_0^1 f(B_s)^2 ds \; .$$

Then, it suffices to show that the terms $B_n$ and $C_n$ converge to zero in $L^2(\Omega)$. For the term $B_n$ we can write, using the fact that $\sum_{l,j=0}^{n-1} \left| \alpha_{l,j}^{q-i} \beta_{l,j}^{i} \right| \leq Cn^{-2qH+1}$

$$\begin{aligned} E\left(|B_n|\right) &\leq Cn^{2qH-1} \sum_{i=1}^{q-1} \sum_{l,j=0}^{n-1} \left| \alpha_{l,j}^{q-i} \beta_{l,j}^{i} \right| \left\| \delta^{q-i} \left( f^{(q-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right) \right\|_{L^2} \\ &\leq C \sum_{i=1}^{q-1} n^{-H(q-i)}, \end{aligned}$$

which converges to zero as $n$ tends to infinity. Finally, for the term $C_n$ we can write

$$E\left(|C_n|\right) \leq Cn^{qH+\frac{1}{2}} \sup_j \left\| \Phi_n^{0,q,j} \right\|_{L^2} \leq Cn^{\frac{1}{2}-2qH+\left(2H-\frac{1}{2}\right)\vee 0},$$

and $\frac{1}{2} - 2qH + \left(2H - \frac{1}{2}\right) \vee 0 < 0$, because if $2H - \frac{1}{2} \leq 0$ this is true due to $\frac{1}{2} - 2qH < 0$, and if $2H - \frac{1}{2} \geq 0$, then we get $2H(1-q) < 0$. This completes the proof of Theorem 5.2. ■

## 5.4 Proof of the stable convergence (5.9)

As a consequence of Theorem 5.2, we can derive the following result, which is nothing but (5.9):

**Theorem 5.3** *Suppose that $f$ is a function satisfying Hypothesis **(H)**. Let $G_n$ be the sequence of random variables defined in (5.5). Then, provided $H \in (\frac{1}{4q}, \frac{1}{2})$, we have*

$$G_n - n^{-\frac{1}{2}-qH} \frac{(-1)^q}{2^q q!} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n}) \overset{\text{stably}}{\longrightarrow} \sigma_{H,q} \int_0^1 f(B_s) dW_s,$$

*where $W$ is a Brownian motion independent of $B$ and $\sigma_{H,q} > 0$ is defined by (5.8).*

**Proof.** We recall first that $H_q\left(n^H\left(\Delta B_{k/n}\right)\right) = \frac{1}{q!} n^{qH} \delta^q(\partial_{k/n}^{\otimes q})$. Then, using (2.6) yields

$$f(B_{k/n})\delta^q(\partial_{k/n}^{\otimes q}) = \sum_{r=0}^{q} \binom{q}{r} \alpha_{k,k}^r \delta^{q-r}(f^{(r)}(B_{k/n})\partial_{k/n}^{\otimes(q-r)}),$$

where $\alpha_{k,k}$ is defined in (5.14). As a consequence,

$$\begin{aligned} G_n &= \frac{1}{q!} n^{qH-\frac{1}{2}} \sum_{r=0}^{q} \sum_{k=0}^{n-1} \binom{q}{r} \alpha_{k,k}^r \delta^{q-r}(f^{(r)}(B_{k/n})\partial_{k/n}^{\otimes(q-r)}) \\ &= \frac{1}{q!} \delta^q(u_n) + \sum_{r=1}^{q-1} \delta^{q-r}(v_n^{(r)}) + R_n, \end{aligned}$$

where $u_n$ is defined in (5.13),

$$v_n^{(r)} = \frac{1}{q!} \binom{q}{r} n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \alpha_{k,k}^r \; f^{(r)}(B_{k/n}) \partial_{k/n}^{\otimes(q-r)},$$

and

$$R_n = \frac{1}{q!} n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \alpha_{k,k}^q f^{(q)}(B_{k/n}).$$

The proof will be done in two steps.

*Step 1* We first show that if $H \in \left(0, \frac{1}{2}\right)$, and $r = 1, \ldots, q-1$, $\delta^{q-r}(v_n^{(r)})$ converges to zero in $L^2(\Omega)$ as $n$ tends to infinity. It suffices to show that $v_n^{(r)}$

converges to zero in the norm of the space $\mathbb{D}^{q-r,2}(\mathfrak{H}^{\otimes(q-r)})$. For $0 \le m \le q-r$, we can write, using the notation $\beta_{k,l}$ defined by (5.14),

$$
\begin{aligned}
E\left(\left\|D^m v_n^{(r)}\right\|^2_{\mathfrak{H}^{\otimes(q-r+m)}}\right) &= \left(\frac{1}{q!}\binom{q}{r}\right)^2 n^{2qH-1} \\
&\quad \times \sum_{k,l=0}^{n-1} E\left(f^{(r+m)}(B_{k/n}) f^{(r+m)}(B_{l/n})\right) \\
&\quad \times \alpha^r_{k,k}\alpha^r_{l,l}\alpha^m_{k,l}\beta^{q-r}_{k,l} \\
&\le C n^{2qH-1} n^{-2H(2r-2+m+q-r)} \\
&= C n^{2H-1-2Hm},
\end{aligned}
$$

which converges to zero as $n$ tends to infinity.

*Step 2* To complete the proof it suffices to check that

$$
R_n - n^{-\frac{1}{2}-qH} \frac{(-1)^q}{2^q q!} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n})
$$

converges to zero in $L^2(\Omega)$ as $n$ tends to infinity. This follows from (5.10) and the estimates

$$
\begin{aligned}
&\left\| \frac{1}{q!} n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \alpha^q_{k,k} f^{(q)}(B_{k/n}) - \frac{(-1)^q}{2^q q!} n^{-\frac{1}{2}-qH} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n}) \right\|_{L^2} \\
\le\ & C n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \left| \alpha^q_{k,k} - \frac{1}{2^q n^{2qH}} \right| \le C\ n^{-qH+2H-\frac{1}{2}}.
\end{aligned}
$$

Notice that $-qH + 2H - \frac{1}{2} < 0$. The proof is now complete. ■

**Acknowledgments**: We are very grateful to Giovanni Peccati for numerous helpful comments, especially about Section 4. We also wish to thank an anonymous Referee for his/her careful reading.

# References

[1] P. Baldi, G. Kerkyacharian, D. Marinucci and D. Picard (2009). Subsampling Needlet Coefficients on the Sphere. *Bernoulli* **15**, no. 2, 438-463.

[2] O. Barndorff-Nielsen, J. Corcuera, M. Podolskij and J. Woerner (2009). Bipower variations for Gaussian processes with stationary increments. *J. Appl. Probab.* **46**, no. 1, 132-150.

[3] P. Breuer and P. Major (1983). Central limit theorems for nonlinear functionals of Gaussian fields. *J. Multivariate Anal.* **13**, no. 3, 425-441.

[4] J.M. Corcuera, D. Nualart and J.H.C. Woerner (2006). Power variation of some integral fractional processes. *Bernoulli* **12**, 713-735.

[5] P. Deheuvels, G. Peccati and M. Yor (2006). On quadratic functionals of the Brownian sheet and related processes. *Stoch. Proc. Appl.* **116**, 493-538.

[6] L. Decreusefond and A. S. Üstünel (1998). Stochastic analysis of the fractional Brownian motion. *Potential Analysis* **10**, 177-214.

[7] Y. Hu and D. Nualart (2005). Renormalized self-intersection local time for fractional Brownian motion. *Ann. Probab.* **33**, no. 3, 948-983.

[8] K. Itô (1951). Multiple Wiener integral. *J. Math. Soc. Japan* **3**, 157-169.

[9] J. Jacod and A.N. Shiryayev (1987). *Limit Theorems for Stochastic Processes.* Springer-Verlag, Berlin, Heidelberg, New York.

[10] D. Marinucci and G. Peccati (2007). High-frequency asymptotics for subordinated stationary fields on an Abelian compact group. *Stoch. Proc. Appl.* **118**, no. 4, 585-613.

[11] D. Marinucci and G. Peccati (2007). Group representations and high-resolution central limit theorems for subordinated spherical random fields. *Bernoulli*, in revision.

[12] A. Neuenkirch and I. Nourdin (2007). Exact rate of convergence of some approximation schemes associated to SDEs driven by fractional Brownian motion. *J. Theoret. Probab.* **20**, no. 4, 871-899.

[13] I. Nourdin (2008). Asymptotic behavior of weighted quadratic and cubic variations of fractional Brownian motion. *Ann. Probab.* **36**, no. 6, 2159-2175.

[14] I. Nourdin and G. Peccati (2009). Non-central convergence of multiple integrals. *Ann. Probab.* **37**, no. 4, 1412-1426.

[15] I. Nourdin and G. Peccati (2007). Weighted power variations of iterated Brownian motion. *Electron. J. Probab.* **13**, 1229-1256.

[16] I. Nourdin and G. Peccati (2009). Stein's method on Wiener chaos. *Probab. Theory Relat. Fields* **145**, no. 1, 75-118.

[17] I. Nourdin, G. Peccati and G. Reinert (2009). Invariance principles for homogeneous sums: universality of Wiener chaos. Preprint.

[18] I. Nourdin, D. Nualart and C.A. Tudor (2007). Central and non-central limit theorems for weighted power variations of fractional Brownian motion. *Ann. Inst. H. Poincaré Probab. Statist.*, in revision.

[19] I. Nourdin and A. Réveillac (2008). Asymptotic behavior of weighted quadratic variations of fractoinal Brownian motion: the critical case $H = 1/4$. *Ann. Probab.*, to appear.

[20] D. Nualart (2006). The Malliavin calculus and related topics. Springer-Verlag, Berlin, 2nd edition.

[21] D. Nualart and S. Ortiz-Latorre (2007). Central limit theorems for multiple stochastic integrals and Malliavin calculus. *Stoch. Proc. Appl.* **118**, no. 4, 614-628.

[22] D. Nualart and G. Peccati (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33**, no. 1, 177-193.

[23] G. Peccati (2007). Gaussian approximations of multiple integrals. *Electron. Comm. Probab.* **12**, 350-364.

[24] G. Peccati and M.S. Taqqu (2008). Stable convergence of multiple Wiener-Itô integrals. *J. Theoret. Probab.* **21**, no. 3, 527-570.

[25] G. Peccati and M.S. Taqqu (2007). Stable convergence of generalized $L^2$ stochastic integrals and the principle of conditioning. *Electron. J. Probab.* **12**, no. 15, 447-480.

[26] G. Peccati and M.S. Taqqu (2008). Moments, cumulants and diagram formulae for non-linear functionals of random measures. Preprint.

[27] G. Peccati and C.A. Tudor (2005). Gaussian limits for vector-valued multiple stochastic integrals. In: Séminaire de Probabilités XXXVIII , 247-262. Lecture Notes in Math. **1857**, Springer-Verlag, Berlin.

[28] G. Peccati and M. Yor (2004). Four limit theorems for quadratic functionals of Brownian motion and Brownian bridge. In: *Asymptotic Methods in Stochastics*. AMS, Fields Institute Communication Series, 75-87.

[29] A. Rényi (1963). On stable sequences of events. *Sankyha* **25 A**, 293-302.

[30] A. V. Skorohod (1975). On a generalization of a stochastic integral. *Theory Probab. Appl.* **20**, 219-233.

[31] C.A. Tudor and F. Viens (2007). Variations and estimators for the selfsimilarity order through Malliavin calculus. *Ann. Probab.*, to appear.