Convergence of some random functionals of discretized semimartingales

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Abstract

In this paper, we study the asymptotic behavior of sums of functions of the increments of a given semimartingale, taken along a regular grid whose mesh goes to 0. The function of the ith increment may depend on the current time, and also on the past of the semimartingale before this time. We study the convergence in probability of two types of such sums, and we also give associated central limit theorems. This extends known results when the summands are a function depending only on the increments, and this is motivated mainly by statistical applications.

Keywords: Contrast functions, Power variation, Limit theorems, Semimartingale.

1 Introduction

In many practical situations, one observes a random process X at discrete times and one wants to deduce from these observations, some properties on X. Take for example the specific case of a 1-dimensional diffusion-type process $X = X^\theta$ depending on a real-valued parameter θ, that is:

$$dX_s \;=\; \sigma(\theta, s)\, dW_s \;+\; a(\theta, s)\, ds, \tag{1.1}$$

where σ and a are (known) predictable functions on $\Omega \times \mathbb{R}_+$, and where W is a Brownian motion. We observe the values of X at times $i\Delta,\ i = 0, 1, 2, \cdots, n\Delta$, and the aim is to estimate θ. There are two cases: in the first one the observation window is arbitrarily large. In the second case (which is our concern here), the observation window is fixed, and so $\Delta = \Delta_n$ goes to 0 and $T = n\Delta_n$ is fixed.

Most known methods rely upon minimizing some contrast functions, like minus the log-likelihood, and those are typically expressed as "functionals" of the form:

$$\sum_{i=1}^{n} g_n \left(\sigma(\theta', (i-1)\Delta_n), X^\theta_{(i-1)\Delta_n}, X^\theta_{i\Delta_n} - X^\theta_{(i-1)\Delta_n}\right), \tag{1.2}$$

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with g_n possibly depending on n, see for example [5]. In other words, the asymptotic behavior (convergence, and if possible associated central limit theorems) of functionals like (1.2) is very important. This is why, for a function $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and a d-dimensional semimartingale X, we study the asymptotic behavior of the following two sequences of process

$$\left.\begin{array}{rcl} V^n(f,X)_t & = & \sum_{i=1}^{[t/\Delta_n]} f\left(\omega,(i-1)\Delta_n, X_{(i-1)\Delta_n}, X_{i\Delta_n}-X_{(i-1)\Delta_n}\right), \\ V'^n(f,X)_t & = & \Delta_n \sum_{i=1}^{[t/\Delta_n]} f\left(\omega,(i-1)\Delta_n, X_{(i-1)\Delta_n}, \frac{X_{i\Delta_n}-X_{(i-1)\Delta_n}}{\sqrt{\Delta_n}}\right), \end{array}\right\} \tag{1.3}$$

when $\Delta_n \to 0$. So, providing some basic tools for statistical problems is our main aim in this paper, although we do not study any specific statistical problem.

Another motivation for studying functionals like (1.3) is that they appear naturally in numerical approximations of stochastic differential equations like the Euler scheme or more sophisticated discretization schemes.

Let us now make two comments on the third argument of f in the processes in (1.3), namely $X_{(i-1)\Delta_n}$:

1. The functionals (1.3) are not changed if we replace f by $g(\omega,t,x) = f(\omega,t,X_{t-}(\omega),x)$, so apparently one could dispense with the dependency of f upon its third argument. However, we will need some Hölder continuity of $t \mapsto g(\omega,t,x)$ which is *not* satisfied by g defined as just above: so it is more convenient to single out the third argument.
2. One could replace $X_{(i-1)\Delta_n}$ by $Y_{(i-1)\Delta_n}$ for another semimartingale Y, say d'-dimensional. But those apparently more general functionals are like (1.3) with the $(d+d')$-dimensional pair $Z=(Y,X)$ instead of X.

When $f(\omega,s,z,x) \equiv f(x)$ (f is "deterministic"), (1.3) becomes:

$$\left.\begin{array}{rcl} V^n(f,X) & = & \sum_{i=1}^{[t/\Delta_n]} f\left(X_{i\Delta_n}-X_{(i-1)\Delta_n}\right), \\ V'^n(f,X) & = & \Delta_n \sum_{i=1}^{[t/\Delta_n]} f\left(\frac{X_{i\Delta_n}-X_{(i-1)\Delta_n}}{\sqrt{\Delta_n}}\right). \end{array}\right\} \tag{1.4}$$

When further $f(x) = |x|^r$, the processes $V^n(f,X)$ are known as the *realized power variations*, and of course $V'^n(f,X) = \Delta_n^{1-r/2}\, V^n(f,X)$.

The convergence of power variations is not new, see for example [10], an old paper by Lépingle. Recently they have been the object of a large number of papers, due to their applications in finance. Those applications are essentially the estimation of the volatility and tests for the presence or absence of jumps.

An early paper is Barndorff-Nielsen and Shephard [1], when X is a continuous Itô's semimartingale. Afterwards, many authors studied these type of processes: Mancini [11] studied the case where X is discontinuous with Lévy type jumps, in [7] Jacod studied the general case of a Lévy process, Corcuera, Nualart and Woerner in [3] studied the case of a fractional process, ..., the list is far from exhaustive. The results appear in their most general form for a continuous semimartingale in [2] and a discontinuous one in [6].

To give an idea of the expected results, let us mention that when X is a 1-dimensional Itô's semimartingale with diffusion coefficient σ and when f is continuous and "not too large near infinity" (depending on whether X is continuous or not) we have

$$V'^n(f,X)_t \stackrel{\mathbb{P}}{\longrightarrow} \int_0^t \rho_{\sigma_s}(f)\,ds,$$

(see for example [2]), where ρ_x is the law of the normal variable $\mathcal{N}(0,x^2)$ and $\rho_x(f)$ is the integral of f with respect to ρ_x.

In [1] Barndorff-Nielsen and Shephard give a central limit theorem for $V'^n(f,X)$, using a result of Jacod and Protter about a central limit theorem (or: CLT) for the Euler scheme for stochastic differential equations, see [9]. This CLT has been generalized in many papers, like [2] when X is continuous. If X is discontinuous, Jacod (in [6]) gives a CLT when the Blumenthal-Getoor index p of X is smaller than 1, and no CLT is known when $p > 1$.

Concerning $V^n(f,X)$, in the uni-dimensional case, Jacod extends some old results of Lépingle in [10]. In particular, if $f(x) \sim |x|^r$ near the origin and is continuous and X is an arbitrary semimartingale, then

$$V^n(f,X) \stackrel{\mathbb{P}}{\longrightarrow} D(f,X), \tag{1.5}$$

with

$$D(f,X)_t = \begin{cases} \sum_{s\leq t} f(\Delta X_s) \text{ if } r>2, \text{ or if } r\in(1,2) \text{ and } \langle X^c,X^c\rangle \equiv 0, \\ \sum_{s\leq t} f(\Delta X_s) + \langle X^c,X^c\rangle_t, \text{ if } r=2, \end{cases}$$

where ΔX_s is the jump of X at time s, and X^c denotes the continuous martingale part of X. Moreover, Jacod gives a central limit theorem for $V^n(f,X)$, first for Lévy processes in [7], second for semimartingales in [6].

The difficulty of the extended setting in the present paper is due to the fact that f is not any more deterministic and depends on all the variables (ω,s,z,x), as we have seen in the statistical problem. We want to know to which extent the earlier results remain valid in this setting, and especially the CLTs. Our concern is to exhibit reasonably general conditions on the test function f which ensure that the previously known results extend. Note also that for the CLT concerning $V'^n(f,X)$, and contrary to the existing literature, we do not always assume that $f(\omega,t,z,x)$ is even in x, although most applications concern the even case. The reader will also observe that in some cases there are additional terms due to the parameter z in $f(\omega,t,z,x)$.

The paper is organized as follows: in Sections 2 and 3 we state the Laws of large numbers and the CLT respectively, and in Sections 4 and 5 we give the proofs.

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2 Laws of large numbers

2.1 General notation

The basic process X is a d-dimensional semimartingale on a fixed filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We denote by $\Delta X_s = X_s - X_{s-}$ the jump of X at time s, and by I the set

$$I \;=\; \Big\{ r \geq 0: \;\sum_{s\leq t} ||\Delta X_s||^r \;<\; \infty \;\text{ a.s for all } t \Big\}.$$

Note that the set I always contains the interval $[2, \infty)$.

The optional and predictable σ-fields on $\Omega \times \mathbb{R}_+$ are denoted by $\mathcal{O}$ and $\mathcal{P}$, and if g is a function on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^l$ we call it optional (resp. predictable) if it is $\mathcal{O} \otimes \mathcal{R}^l$-measurable (resp. $\mathcal{P} \otimes \mathcal{R}^l$-measurable), where $\mathcal{R}^l$ is the Borel σ-field on $\mathbb{R}^l$.

The function f (unless otherwise stated) denotes a function from $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ into $\mathbb{R}^q$, for some $q \geq 1$. When $f(\omega, t, z, x)$ admits partial derivatives in z or x, we denote by $\nabla_z f$ or $\nabla_x f$ the corresponding gradients.

If M is a matrix, its transpose is M^t. The set of all $p \times q$ matrices is $\mathcal{M}(p, q)$, and $\mathcal{T}(p, q, r)$ is the set of all $p \times q \times r$-arrays.

For any $\sigma \in \mathcal{M}(d, m)$ we denote by ρ_σ the normal law $\mathcal{N}(0, \sigma\sigma^t)$, and by $\rho_\sigma(f(\omega, s, z, .))$ the integral of the function $x \mapsto f(\omega, s, z, x)$ with respect to ρ_σ.

We denote by $\mathcal{B}$ the set of all functions $\phi : \mathbb{R}^d \to \mathbb{R}_+$ bounded on compact.

A sequence (Z^n_t) of processes is said to converge u.c.p. (for: uniformly on compact sets and in probability) to Z_t, and written $Z^n \stackrel{u.c.p}{\to} Z$ or $Z^n_t \stackrel{u.c.p}{\to} Z_t$, if $\mathbb{P}\left(\sup_{s\leq t} ||Z^n_s - Z_s|| > \varepsilon\right)$ $\to 0$ for all $\varepsilon, t > 0$.

We write $Z^n \stackrel{\mathcal{L}-(s)}{\to} Z$ or $Z^n_t \stackrel{\mathcal{L}-(s)}{\to} Z_t$, if the process Z^n converge stably in law to Z, as processes (see [8] for details on the stable convergence).

We gather some important properties of f in the following definition.

Definition 2.1 *a) We say that f is of (random) polynomial growth if there exist a locally bounded process Γ (meaning: $\sup_{s\leq T_n} \Gamma_s \leq n$ for a sequence T_n of stopping times increasing a.s. to ∞), a function $\phi \in \mathcal{B}$, and a real $p \geq 0$ such that*

$$||f(\omega, s, z, x)|| \leq \Gamma_s(\omega)\phi(z)(1 + ||x||^p). \tag{2.1}$$

If we want to specify p, we say that f is at most of p-polynomial growth.

b) we say that f is locally equicontinuous in x (resp. (z, x)) if for all ω, all $T > 0$, and all compacts $\mathcal{K}, \mathcal{K}'$ in $\mathbb{R}^d$, the family of functions $(x \mapsto f(\omega, s, z, x))_{s\leq T, z\in\mathcal{K}'}$ (resp. $((z, x) \mapsto f(\omega, s, z, x))_{s\leq T}$) is equicontinuous on $\mathcal{K}$ (resp. $\mathcal{K} \times \mathcal{K}'$).

2.2 Assumptions

Let us start with the assumptions on X. For $V^n(f, X)$ we only need X to be an arbitrary semimartingale. For $V'^n(f)$ we need X to be an Itô semimartingale and a little more. Recall first that the property of X to be an Itô semimartingale is equivalent to the following: there are, possibly on an extension of the original probability space, an m-dimensional Brownian motion W (we may always take $m = d$) and a Poisson random measure $\underline{\mu}$ on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure $\underline{\nu}(ds, dy) = F(dy)\, ds$ with F is a σ-finite measure on $\mathbb{R}$, such that X can be written as

$$\begin{aligned} X_t &= X_0 + \int_0^t b_s\, ds + \int_0^t \sigma_{s-} dW_s + \int_0^t \int_{\mathbb{R}} h\,(\delta(s,y))\,(\underline{\mu} - \underline{\nu})(ds, dy) \\ &\quad + \int_0^t \int_{\mathbb{R}} h'\,(\delta(s,y))\,\underline{\mu}(ds, dy), \end{aligned} \tag{2.2}$$

for suitable "coefficients" b (predictable d-dimensional), σ (optional $d \times m$-dimensional), δ (predictable d-dimensional function on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$) and h is a truncation function from $\mathbb{R}^d$ into itself (continuous with compact support, equal to the identity on a neighborhood of 0), and $h'(x) := x - h(x)$.

Then we set:

Hypothesis (N_0)**:** The process X is an Itô's semimartingale, and its coefficients in (2.2) satisfy the following: b and $\int_{\mathbb{R}} (1 \wedge ||\delta(\omega, s, y)||^2)\, F(dy)$ are locally bounded, and σ is càdlàg. □

For the test function f we introduce the following, where A is an arbitrary subset of $\mathbb{R}^d$:

Hypothesis $(K[A])$**:** $f(\omega, t, z, x)$ is continuous in (z, x) on $\mathbb{R}^d \times A$ and if $(t_n, z_n, x_n) \to (t, z, x)$ with $x \in A$ and $t_n < t$, then $f(\omega, t_n, z_n, x_n)$ converges to a limit depending on (ω, t, z, x) only, and denoted by $f(\omega, t-, z, x)$. □

2.3 Results

The first two theorems concern the processes $V^n(f)$.

Theorem 2.2 *Let X be an arbitrary semimartingale, and let f satisfy $K(\mathbb{R}^d)$. Suppose there exist a neighborhood V of 0 on $\mathbb{R}^d$, a real $p > 2$, and for any $K > 0$, a locally bounded process Γ^K such that:*

$$\|z\| \leq K, \;\; x \in V \;\Rightarrow\; ||f(\omega, s, z, x)|| \;\leq\; \Gamma_s^K(\omega) \|x\|^p. \tag{2.3}$$

Then $V^n(f)$ converge a.s. for the Skorokhod topology to the process

$$D(f)_t \;=\; \sum_{s \leq t} f(s-, X_{s-}, \Delta X_s). \tag{2.4}$$

Remark 2.3 *This is one of the rare situations where one has almost sure convergence; see Section 3.1 of [4] for some other ones.*

Theorem 2.4 *Let X be an arbitrary semimartingale, and let f be optional, satisfy $(K(\mathbb{R}^d))$ and $f(\omega, s, z, 0) = 0$, and be C^2 in x on some neighborhood V of 0, and assume also*

- *For any $j,k \in \{1,\cdots,d\}$, the functions $\frac{\partial f}{\partial x_j}(\omega,s,x,z)$ and $\frac{\partial^2 f}{\partial x_j \partial x_k}(\omega,s,x,z)$ defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times V$ satisfy $(K[V])$.*
- *There exist $\phi \in \mathcal{B}$ and a locally bounded process Γ such that*

$$\sum_{j=1}^{d} \left(\left\| \frac{\partial f}{\partial x_j}(s,z,0) \right\| \;+\; \sum_{k=1}^{d} \left(\sup_{x \in V} \left\| \frac{\partial^2 f}{\partial x_j \partial x_k}(s,z,x) \right\| \right) \right) \;\leq\; \Gamma_s \phi(z).$$

Then $V^n(f)$ converge in probability, in the Skorokhod sense, to the process

$$\begin{aligned} D(f)_t \;&=\; \sum_{j=1}^{d} \int_0^t \frac{\partial f}{\partial x_j}(s-,X_{s-},0)\,dX_s + \frac{1}{2}\sum_{j,k=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x_j \partial x_k}(s-,X_{s-},0)\,d\langle X^{j,c},X^{k,c}\rangle_s \\ &\quad + \sum_{0<s\leq t} \left(f(s-,X_{s-},\Delta X_s) - \sum_{j=1}^{d} \Delta X_s^j \frac{\partial f}{\partial x_j}(s-,X_{s-},0) \right), \end{aligned} \tag{2.5}$$

where X^c is the continuous martingale part of X.

The two versions (2.4) and (2.5) of $D(f)$ agree when f satisfies the hypotheses of Theorem 2.2, so Theorem 2.4 extends Theorem 2.2 and gives the results in a more complete form. This result was not known even in the case when f only depends on x.

Remark 2.5 *Both theorems remain valid if the discretization grid is not regular, provided the successive discretization times are stopping times and the mesh goes to* 0 *(see Sections 3.5 and 4.5 of [4] for results of this type).*

Now we state the result about $V'^n(f)$.

Theorem 2.6 *Let f be optional, satisfy $(K(\mathbb{R}^d))$, be locally equicontinuous in x and with p-polynomial growth. Assume further that one of the following two conditions is satisfied:*

1. *X satisfies (N_0) and $p < 2$.*
2. *X satisfies (N_0) and is continuous.*

Then

$$V'^n(f) \;\overset{u.c.p.}{\longrightarrow}\; \int_0^t \rho_{\sigma_{s-}}\left(f(s-,X_{s-},.)\right) ds. \tag{2.6}$$

Remark 2.7 *Comparing with [2] or [6], we see that there is no additional term due to the third argument z in $f(\omega,s,z,x)$.*

In the discontinuous case (Hypothesis 1), the condition $p < 2$ simplifies the computations but is not optimal. The result remains true valid if there exist ϕ, $\phi' \in \mathcal{B}$ such that:

$$\phi'(x) \to 0, \text{ when } ||x|| \to \infty, \text{ and } ||f(\omega,s,z,x)|| \;\leq\; \Gamma_s(\omega)\phi(z)||x||^2\phi'(x).$$

3 Central limit theorems

In the framework of the CLT, one needs some additional assumptions both on X and on f, which depend on the problem at hand.

3.1 Assumptions on X

Hypothesis (N_1)**:** (N_0) is satisfied, and there exist a sequence (S_k) of stopping times increasing to ∞ and deterministic Borel functions (γ_k) such that:

$$||\delta(\omega,s,y)|| \;\leq\; \gamma_k(y) \text{ if } s \leq S_k(\omega) \quad \text{and} \quad \int_{\mathbb{R}} (1\wedge \gamma_k(y)^2)\,F(dy) \;<\infty. \qquad \square$$

The next assumption depends on a real $s \in [0,2]$:

Hypothesis $(N_2(s))$**:** (N_1) is satisfied, the mapping $s \mapsto \delta(\omega,s,y)$ is càglàd, and $\int_{\mathbb{R}}(1\wedge \gamma_k(y)^s)\,F(dy) \;<\infty$. Moreover, the process σ in (2.2) satisfies:

$$\sigma_t \;=\; \sigma_0 + \int_0^t \widetilde{b}_u du + \int_0^t \widetilde{\sigma}_u dW_u + M_t + \sum_{u\leq t} \Delta\sigma_u 1_{\{||\Delta\sigma_u||\geq 1\}}, \tag{3.7}$$

where

- $\widetilde{b}$ is predictable and locally bounded.
- $\widetilde{\sigma}$ is càdlàg, adapted with values in $\mathcal{T}(d,m,m)$.
- M is an $\mathcal{M}(d,m)$-valued local martingale, orthogonal to W and satisfying $||\Delta M_t|| \leq 1$ for all t. Its predictable quadratic covariation is $\langle M,M\rangle_t = \int_0^t a_u\,du,$, where a is locally bounded.
- The predictable compensator of $\sum_{u\leq t} 1_{\{||\Delta\sigma_u||\geq 1\}}$ is $\int_0^t \widetilde{a}_u\,du$, where $\widetilde{a}$ is locally bounded. $\square$

Clearly $(N_2(s)) \;\Rightarrow\; (N_2(s'))$, if $s<s'$.

Remark 3.1 *It is well known that the assumptions on σ in $(N_2(s))$ may be replaced by the following one (up to modifying the Poisson measure $\underline{\mu}$):*

$$\sigma_t \;=\; \sigma_0 + \int_0^t \widetilde{b}_u\,ds + \int_0^t \widetilde{\sigma}_u dW_u + \int_0^t \widetilde{v}_u\,dV_s + \int_{\mathbb{R}}\int_0^t k(\widetilde{\delta}(u,y))\star(\underline{\mu}-\underline{\nu})(du,dy)$$
$$+\int_{\mathbb{R}}\int_0^t k'(\widetilde{\delta}(u,y))\star\underline{\mu}(du,dy), \tag{3.8}$$

where $\widetilde{b}$ and $\widetilde{\sigma}$ are like in $(N_2(s))$ and

- *V is a l-dimensional Brownian motion independent of W.*
- *$\widetilde{v}$ takes its values in $\mathcal{T}(d,m,l)$, is progressively measurable and locally bounded.*

- $k(x)$ *is a truncation function on* $\mathbb{R}^d \times \mathbb{R}^m$ *and* $k'(x) := x - k(x)$.
- $\widetilde{\delta} : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathcal{M}(d,m)$ *is predictable and is such that:* $\int_{\mathbb{R}} (1 \wedge ||\widetilde{\delta}(u,y)||^2)\, F(dy)$ *is locally bounded.*

Of course, a, $\widetilde{a}$, $\widetilde{v}$ *and* $\widetilde{\delta}$ *are related, for example if* $k(x) = x1_{\{||x||<1\}}$, *one has* $\widetilde{v}_u^2 + \int_{\{||\widetilde{\delta}(u,y)||\leq 1\}} \widetilde{\delta}^2(u,y)\, F(dy) \;=\; a_u^2$ *and* $\widetilde{a}_u = \int_{\{||\widetilde{\delta}(u,y)||>1\}} F(dy)$.

3.2 Assumptions on the test function f

Hypothesis (M_1): f is optional and there exists a neighborhood V of 0 such that $f(\omega,s,z,x)$ is C^1 in (z,x), the functions $\nabla_x f$, $\nabla_z f$ are C^1 in x on V, and

$$f(\omega,s,z,0) = \nabla_x f(\omega,s,z,0) \equiv 0.$$

Moreover there are a locally bounded process Γ, a real $\alpha > \frac{1}{2}$, and some functions ϕ, ε and θ belonging to $\mathcal{B}$, with $\varepsilon(x) \to 0$ as $||x|| \to 0$ and $\theta(x) \leq \|x\|^2$ in the neighborhood of 0, such that:

$$\sum_{j,j'=1}^{d} \left(\left\| \frac{\partial^2 f}{\partial x_j \partial x_{j'}}(\omega,s,z,x) \right\| + \left\| \frac{\partial^2 f}{\partial x_j \partial z_{j'}}(\omega,s,z,x) \right\| \right) \leq \Gamma_s(\omega)\phi(z)||x||\varepsilon(x),$$

and for all $T > 0$ and $s,t \in [0,T]$,

$$\|f(\omega,t,z,x) - f(\omega,s,z,x)\| \leq \Gamma_T(\omega)\phi(z)\,|t-s|^{\alpha}\, \theta(x). \tag{3.9}$$

□

Hypothesis (M_2): $f(\omega,t,z,x)$ is optional, C^1 in (z,x), with $\nabla_x f$ and $\nabla_z f$ of (random) polynomial growth and locally equicontinuous in (z,x), and there are Γ, ϕ, α as in (M_1) and some $p > 0$ such that for all $T > 0$ and $s,t \in [0,T]$,

$$\|f(\omega,s,z,x) - f(\omega,t,z,x)\| \;\leq\; \Gamma_T(\omega)\phi(z)|t-s|^{\alpha}(1+\|x\|^p), \tag{3.10}$$

□

Hypothesis (M_2'): (M_2) is satisfied and moreover

$$\|f(\omega,s,z,x)\| + \|\nabla_x f(\omega,s,z,x)\| \;\leq\; \phi(z)\Gamma_s(\omega).$$ □

The previous hypotheses are fulfilled by most of the test functions used in statistics.

3.3 The results

In order to define the limiting processes, we need to expand the original space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, what we do as follows:

Consider an auxiliary space $(\Omega', \mathcal{F}', \mathbb{P}')$, which supports a q-dimensional Brownian motion $\overline{W}$ and some sequences $\{(U_p^k)_{1\leq k\leq m};\ (U_p^{'k})_{1\leq k\leq m};\ (\kappa_p)\}_{p\geq 1}$ of random variables, where the U_p^k and $U_p'^k$ are normal $\mathcal{N}(0,1)$ and the (κ_p) are uniform on $(0,1)$. We suppose all these variables and processes mutually independent.

Now set:

$$\widetilde{\Omega} = \Omega \times \Omega', \quad \widetilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}', \quad \widetilde{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'.$$

We then extend the variables and processes defined on Ω or Ω' on the space $\widetilde{\Omega}$, in the usual way.

Let (T_p) be an arbitrary sequence of stopping times exhausting the jumps of X (meaning: they are stopping times such that for all (ω, s) with $\Delta X_s(\omega) \neq 0$, there exists a unique p such that $T_p(\omega) = s$). We define on $\widetilde{\Omega}$ the filtration $(\widetilde{\mathcal{F}}_t)$ which is the smallest one satisfying the following conditions:

- $(\widetilde{\mathcal{F}}_t)$ is right continuous, and $\mathcal{F}_t \subset \widetilde{\mathcal{F}}_t$,
- $\overline{W}$ is adapted on $(\widetilde{\mathcal{F}}_t)$,
- the variables U_p^k, $U_p'^k$ and κ_p are $\widetilde{\mathcal{F}}_{T_p}$ measurable.

Now we are ready to give the results. We start with $V^n(f)$:

Theorem 3.2 *Suppose that X satisfies (N_1) and f satisfies (M_1), then*

$$\frac{1}{\sqrt{\Delta_n}} \left(V^n(f) - D(f)_{[t/\Delta_n]\Delta_n} \right) \xrightarrow{\mathcal{L}} F_t,$$

where the process F is

$$\begin{aligned} F_t \; = \; & \sum_{p:\ T_p \leq t} \sum_{j=1}^{d} \sum_{k=1}^{m} \Bigg(\left(\sqrt{\kappa_p}\, \sigma_{T_p-}^{j,k} U_p^k + \sqrt{1-\kappa_p}\, \sigma_{T_p}^{j,k} U_p'^k \right) \frac{\partial f}{\partial x_j} \left(T_p-, X_{T_p-}, \Delta X_{T_p} \right) \\ & - \sqrt{\kappa_p}\, \sigma_{T_p-}^{j,k} U_p^k \frac{\partial f}{\partial z_j} \left(T_p-, X_{T_p-}, \Delta X_{T_p} \right) \Bigg). \qquad (3.11) \end{aligned}$$

Remark 3.3 *The last term in (3.11) is due to the third argument of f, and does not appear in [6]. One could show that the theorem remains valid if, in the formula (3.9), $\theta(x) \leq \|x\|^p$ near the origin for some $p \in [0,2] \cap I$.*

It is useful to give some properties of the process F above. For this, under (M_1) and (N_1), one defines an $\mathcal{M}(q,q)$-valued process $C(f)$ as follows:

$$\begin{aligned} C(f)_t \; = \; & \frac{1}{2} \sum_{p:\ T_p \leq t} \sum_{j,j'=1}^{d} \sum_{k=1}^{m} \Bigg\{ \left(\sigma_{T_p-}^{j,k} \sigma_{T_p-}^{j',k} + \sigma_{T_p}^{j,k} \sigma_{T_p}^{j',k} \right) \\ & \times \left(\frac{\partial f}{\partial x_j} \right) \left(\frac{\partial f}{\partial x_{j'}} \right)^t \circ \left(T_p-, X_{T_p-}, \Delta X_{T_p} \right) \\ & - \sigma_{T_p-}^{j,k} \sigma_{T_p-}^{j',k} \left(\left(\frac{\partial f}{\partial x_j} \right) \left(\frac{\partial f}{\partial z_{j'}} \right)^t + \left(\frac{\partial f}{\partial z_j} \right) \left(\frac{\partial f}{\partial x_{j'}} \right)^t \right) \circ \left(T_p-, X_{T_p-}, \Delta X_{T_p} \right) \\ & + \sigma_{T_p-}^{j,k} \sigma_{T_p-}^{j',k} \left(\frac{\partial f}{\partial z_j} \right) \left(\frac{\partial f}{\partial z_{j'}} \right)^t \circ \left(T_p-, X_{T_p-}, \Delta X_{T_p} \right) \Bigg\}, \qquad (3.12) \end{aligned}$$

The following lemma is given without proof, since it is an immediate generalization of lemma 5.10 of [6].

Lemma 3.4 *If* (M_1) *and* (N_1) *are satisfied, then* $C(f)$ *is well defined and* F *is a semi-martingale on the extended space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$*. If further* $C(f)$ *is locally integrable, then* F *is a locally square-integrable martingale.*

Conditionally on $\mathcal{F}$*, the process* F *is a square integrable centered martingale with independent increments, its conditional variance is* $C(f)_t = \widetilde{\mathbb{E}}\{F_t^2|\mathcal{F}\}$*, its law is completely characterized by* X *and* $\sigma\sigma^t$ *and does not depend on the choice of the sequence* (T_p)*.*

Now we turn to $V'^n(f)$. Under (M_2) or $(M_3(r))$, one defines a process a taking its value in $\mathcal{M}(q,q)$ and satisfying for any $j,k \in \{1,\cdots,q\}$:

$$\sum_{l=1}^{q} a_t^{j,l}\, a_t^{l,k} \;=\; \rho_{\sigma_t}\left((f^j f^k)(t, X_t, .)\right) - \rho_{\sigma_t}\left(f^j(t, X_t, .)\right) \rho_{\sigma_t}\left(f^k(t, X_t, .)\right). \tag{3.13}$$

The process a, which may be chosen $(\mathcal{F}_t)$-adapted, is the square-root of the symmetric semi-definite positive element of $\mathcal{M}(m,m)$ whose components are given by the right side of (3.13).

Theorem 3.5 *Suppose* $f(\omega,s,z,x)$ *even in* x*, and assume that one of the following hypothesis is satisfied:*

- X *is continuous and satisfies* $(N_2(2))$ *and* f *satisfies* (M_2)*.*
- *one has* $(N_2(s))$ *for some* $s \leq 1$ *and* (M_2')*.*

Then

$$\frac{1}{\sqrt{\Delta_n}}\left(V'^n(f)_t \;-\; \int_0^t \rho_{\sigma_s} f(s, X_s, .)\, ds\right) \stackrel{\mathcal{L}-(s)}{\longrightarrow} L(f)_t,$$

where

$$L(f)_t \;=\; \int_0^t a_s\, d\overline{W}_s. \tag{3.14}$$

Remark 3.6 *Some times, one wants to apply the theorem for functions of the type* $f(\omega,s,z,x) = g(\omega,s,z)\|x\|^r$*, which are not any more* C^1 *in* x *on* $\mathbb{R}^d$ *when* $r \in (0,1]$*. Specifically, consider the following hypothesis:*

Hypothesis $(M_3(r))$**:** $f(\omega,s,z,x)$ *is optional and there is a closed subset* B *of* $\mathbb{R}^d$ *with Lebesgue measure* 0 *such that the application* $x \to f(\omega,t,z,x)$ *is* C^1 *on* B^c*. Moreover there are* $p \geq 0$ *and* α*,* ϕ *and* Γ *as in* (M_1) *such that for all* $T > 0$ *and* $s,t \in [0,T]$*,*

$$\left.\begin{array}{lll} \|f(\omega,s,z,x_1+x_2) - f(\omega,s,z,x_1)\| & \leq & \Gamma_T(\omega)\phi(z)\,(1+\|x_1\|^p)\,\|x_2\|^r. \\ \|f\,(\omega,s,z,x) - f\,(\omega,t,z,x)\,\| & \leq & \Gamma_T(\omega)\phi(z)|t-s|^\alpha\,(1+||x||^p)\,. \end{array}\right\} \tag{3.15}$$

Moreover,

- *if* $r = 1$ *then* $\nabla_x f$ *defined on* $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times B^c$ *is locally equicontinuous in* (z,x) *with at most polynomial growth.*

- *if $r \neq 1$, then for any element $C \in \mathcal{M}(d,d)$ and any $\mathcal{N}(0,C)$-random vector U, the distance from U to B has a density ψ_C on $\mathbb{R}_+$, satisfying $\sup_{x\in\mathbb{R}_+,\,\|C\|+\|C^{-1}\|<K} \psi_C(x) < \infty$ for all $K < \infty$. For any $x_1 \in B^c$,*

$$\|\nabla_x f(\omega,s,z,x_1)\| \;\leq\; \frac{\Gamma_s(\omega)\phi(z)(1+\|x_1\|^p)}{d(x_1,B)^{1-r}}, \tag{3.16}$$

and if $||x_2|| < \frac{d(x_1,B)}{2}$, then

$$\|\nabla_x f(\omega,s,z,x_1+x_2) \;-\; \nabla_x f(\omega,s,z,x_1)\| \;\leq\; \frac{\Gamma_s(\omega)\phi(z)(1+\|x_1\|^p)\|x_2\|}{d(x_1,B)^{2-r}}. \tag{3.17}$$

□

Then one can show that the results of theorem 3.5 remain valid if f satisfies $(M_3(r))$ for some $r \in (0,1]$ and X satisfies $N_2(2)$ with $\sigma\sigma^t$ everywhere invertible, if further one of the following condition is satisfied:

- *f satisfies $(M_3(r))$ and X is continuous,*
- *f satisfies $(M_3(r))$ and the real p in (3.15), (3.16) and (3.17) is always equal 0, while X satisfies $(N_2(s))$ and either $s \in [0,\frac{2}{3})$ and $r \in (0,1)$ or $s \in (\frac{2}{3},1)$ and $r \in (\frac{1-\sqrt{3s^2-8s+5}}{2-s},1)$.*

Our next objective is to generalize the CLT for $V'^n(f)$ in the case where f is not even. For this, we need some additional notation.

Let U be an $\mathcal{N}(0,Id_m)$ random vector, where Id_m is the identity matrix of order m (recall that m is the dimension of the Brownian motion W in $(N_2(s))$). We then denote by ρ', the law of U and by $\rho'(g_1(.))$ the integral of any function $g_1 : \mathbb{R}^m \to \mathbb{R}^q$ with respect to ρ' if it exists. If now $g_2 : \mathbb{R}^d \to \mathbb{R}^q$ and $x \in \mathcal{M}(d,m)$, we set: $\rho'(g_2(x.)) = \mathbb{E}\{g_2(xU)\}$.

For any $j \in \{1,\cdots,m\}$, we define the projection P_j on $\mathbb{R}^m$ by:

$$P_j(u) := u_j \;\text{ if }\; u = (u_1,\cdots,u_m).$$

Under (M_2) we define $w(1)$ and $w(2)$, two adapted processes taking their values respectively in the spaces $\mathcal{M}(q,m)$ and $\mathcal{M}(q,q)$, and such that for all $j,k \in \{1,\cdots,q\}$ and $j' \in \{1,\cdots,m\}$ we have

$$\left.\begin{array}{l} w(1)_s^{j,j'} \;=\; \rho'\left(f^j(s,X_s,\sigma_s.)P_{j'}(.)\right), \\ \sum_{l=1}^q w(2)_t^{j,l}\, w(2)_t^{l,k} \;=\; \rho'\left((f^j f^k)(s,X_s,\sigma_s.)\right) \\ \quad -\rho'\left(f^j(s,X_s,\sigma_s.)\right)\,\rho'\left(f^k(s,X_s,\sigma_s.)\right) - \sum_{l'=1}^m w(1)_t^{j,l'}\, w(1)_t^{l',k}. \end{array}\right\} \tag{3.18}$$

The process $w(2)$ is the square-root of the matrix whose components are given by the right side of the second equality in (3.18). Finally, under $(N_2(2))$ set

$$b' \;=\; b - \int_{\mathbb{R}} h(\delta(s,y))\, F(dy). \tag{3.19}$$

Theorem 3.7 *Assume either one of the following two assumptions:*

- *X satisfies $(N_2(2))$ and is continuous and f satisfies (M_2).*
- *We have $(N_2(s))$ for some $s \leq 1$ and f satisfies (M_2').*

If further $b' \equiv 0$ and $\widetilde{\sigma} \equiv 0$, we have

$$\frac{1}{\sqrt{\Delta_n}}\left(V'^n(f)_t - \int_0^t \rho' f(s, X_s, \sigma_s.)\, ds\right) \stackrel{\mathcal{L}-(s)}{\longrightarrow} L(f)_t,$$

where

$$L(f)_t \ := \ \int_0^t w(1)_s\, dW_s + \int_0^t w(2)_s\, d\overline{W}_s. \tag{3.20}$$

Remark 3.8 *Clearly, when f is even in x, the two versions of the process $L(f)$ in Theorems 3.5 and 3.7, agree. If X satisfies $(N_2(s))$ with $s \leq 1$, the hypotheses $b' = 0$ and $\widetilde{\sigma} = 0$ yield that X has the form:*

$$X_t = X_0 + \int_0^t \sigma_s\, dW_s + \sum_{s \leq t} \Delta X_s. \tag{3.21}$$

4 Proof of the laws of large numbers

4.1 Theorems 2.2 and 2.4

We start by stating two important lemmas, without proof. The first one is a (trivial) extension of what is done in Subsection 3.1 of [6], and the hypothesis $(K(\mathbb{R}))$ plays a crucial role there. The second one is a generalization of Itô's formula, and its proof can be found for example in [4] (see lemma 3.4.2).

Lemma 4.1 *Let X be an arbitrary semimartingale, and f be a function satisfying $(K[\mathbb{R}])$ and such that $f(s, z, x) = 0$ if $||x|| \leq \varepsilon$ for some $\varepsilon > 0$. Then*

$$V^n(f)_t \ - \sum_{s \leq [t/\Delta_n]\Delta_n} f(s-, X_{s-}, \Delta X_s).$$

converges in variation to 0 when $n \to \infty$, for each $\omega \in \Omega$.

Lemma 4.2 *Let X be a semimartingale and $f(\omega, u, z, x)$ be an optional function, C^2 in x. Then for any u, for almost all ω and for any $t \geq u$, one has:*

$$\begin{aligned} f(u, X_u, X_t) \ &= \ f(u, X_u, X_u) + \sum_{j=1}^d \int_{u+}^t \frac{\partial f}{\partial x_j}(u, X_u, X_{s-})\, dX_s \\ &+ \sum_{j,j'=1}^d \frac{1}{2}\int_{u+}^t \frac{\partial^2 f}{\partial x_j \partial x_{j'}}(u, X_u, X_{s-})\, d\langle X^{j,c}, X^{j',c}\rangle_s \\ &+ \sum_{u<s\leq t}\Big(f(u, X_u, X_s) - f(u, X_u, X_{s-}) - \sum_{j=1}^d \Delta X_s^j \frac{\partial f}{\partial x_j}(u, X_u, X_{s-})\Big). \end{aligned}$$

Now we are ready to prove the two theorems about $V^n(f)$.

Proof of Theorem 2.2: Since for any càdlàg process Y, the processes $Y_{[t\Delta_n]\Delta_n}$ converge pathwise to Y for the Skorokhod topology, it is sufficient to prove that the processes $V^n(f)_t - D(f)_{[t/\Delta_n]\Delta_n}$ converge u.c.p. to 0.

We suppose first that $\|X_t\| \leq C$ identically for some constant C. Let $t > 0$, and $S^n = \{0 = t_1^n < t_2^n < \cdots < t_{k^n}^n = t\}$ be a sequence of partitions of $[0,t]$ such that $\sup_i |t_i^n - t_{i-1}^n| \to 0$, when $n \to \infty$. According to Théorème 4 of [10], one has:

$$\sum_{i=1}^{k^n} |X^j_{t_i^n} - X^j_{t_{i-1}^n}|^p \longrightarrow \sum_{s\leq t} |\Delta X_s^j|^p \text{ a.s.},$$

for any $j \in \{1, \cdots, d\}$, and where X^j is the jth component of X.

Since the mappings $t \mapsto \sum_{s\leq t} |\Delta X_s^j|$ and $t \mapsto \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X^j|^p$ are increasing, we deduce that for almost all ω and for any real $t > 0$,

$$\limsup_n \sum_{i=1}^{[t/\Delta_n]} \|\Delta_i^n X\|^p \leq d^{p-1} \sum_{j=1}^d \sum_{s\leq t} |\Delta X_s^j|^p. \tag{4.1}$$

Let now $\psi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $1_{[-1,1]}(y) \leq \psi(y) \leq 1_{[-2,2]}(y)$. We then put for $y \in \mathbb{R}$ and $x \in \mathbb{R}^d$ and $\varepsilon > 0$:

$$\psi_\varepsilon(y) = \begin{cases} \psi(\frac{y}{\varepsilon}) & \text{if } \varepsilon < \infty \\ 1 & \text{if } \varepsilon = \infty, \end{cases} \qquad \Psi_\varepsilon(x) = \Pi_{j=1}^d \psi_\varepsilon(x_j). \tag{4.2}$$

Note that

$$\Psi_\varepsilon(x) = \begin{cases} 1 & \text{if } ||x|| \leq \varepsilon \\ 0 & \text{if } ||x|| > 2d\varepsilon, \end{cases}$$

and set, with the notation (2.4):

$$Z^n(f)_t = V^n(f)_t - D(f)_t, \tag{4.3}$$

Then

$$Z^n(f) = Z^n(f\Psi_\varepsilon) + Z^n(f(1-\Psi_\varepsilon)), \tag{4.4}$$

$$\limsup \sup_{t\leq T} \|Z^n(f)_t\| \leq \limsup \sup_{t\leq T} \|Z^n(f\Psi_\varepsilon)_t\| + \limsup \sup_{t\leq T} \|Z^n(f(1-\Psi_\varepsilon))_t\|, \tag{4.5}$$

for any $T > 0$. By Lemma 4.1, one has

$$\lim_{\varepsilon\to 0} \limsup_n \sup_{t\leq T} \|Z^n(f(1-\Psi_\varepsilon))_t\| = 0. \tag{4.6}$$

On the other hand, if $q \in (2,p)$ we have by (2.3) and $\|X\| \leq C$ and (4.1):

$$\begin{aligned} \sup_{s\leq t} \|Z^n(f\Psi_\varepsilon)_t\| &\leq (2d\varepsilon)^{p-q}\, \Gamma_t^{2C} \Big(\sum_{i=1}^{[t/\Delta_n]} \|\Delta_i^n X\|^q + \sum_{s\leq t} \|\Delta X_s\|^q \Big) \\ &\leq 2d^{p-1}(2d\varepsilon)^{p-q}\, \Gamma_t^{2C} \sum_{j=1}^d \sum_{s\leq t} |\Delta X_s^j|^q. \end{aligned}$$

Since $\sum_{s\leq t}|\Delta X^j_s|^q<\infty$, by letting $\varepsilon\to 0$ we conclude

$$\limsup_n \sup_{s\leq t} \ |Z^n(f\Psi_\varepsilon)_s| \ = 0,$$

which ends the proof in the case where X is bounded.

The general case is deduced by a classical method of "localization", for which we refer to Section 3 of [2] for details. □

Proof of Theorem 2.4: We use the previous notation, with $Z^n(f)$ is as in (4.3) and $D(f)$ as in (2.5). Recalling that (2.4) and (2.5) give the same process $D(f(1-\Psi_\varepsilon))$, we still have (4.6), and it is thus enough to prove that:

$$Z^n(f\Psi_\varepsilon) \ \longrightarrow^{u.c.p.} \ 0. \tag{4.7}$$

Set $f_\varepsilon := f\Psi_\varepsilon$. By the hypotheses on f, the function f_ε is C^2 in x if ε is small enough. We then apply lemma 4.2 to each $f_\varepsilon((i-1)\Delta_n, X_{(i-1)\Delta_n}, \Delta^n_i X)$, which gives $Z^n(f_\varepsilon)_t = \sum_{l=1}^3 Z^n(f_\varepsilon,l)_t$ where, with the notation $Y^n_s = X_s - X_{(i-1)\Delta_n}$ and $\phi^n(s):=(i-1)\Delta_n$ for $s\in((i-1)\Delta_n, i\Delta_n]$, we have

$$\begin{aligned}
Z^n(f_\varepsilon,1)_t &= \sum_{j=1}^d \int_0^{[t/\Delta_n]\Delta_n} \Big(\tfrac{\partial f_\varepsilon}{\partial x^j}(\phi^n(s), X_{\phi^n(s)}, Y^n_s) - \tfrac{\partial f_\varepsilon}{\partial x^j}(s-, X_{s-},0)\Big)\, dX^j_s,\\
Z^n(f_\varepsilon,2)_t &= \tfrac12 \sum_{j,k=1}^d \int_0^{[t/\Delta_n]\Delta_n} \Big(\tfrac{\partial^2 f_\varepsilon}{\partial x^j \partial x^k}(\phi^n(s), X_{\phi^n(s)}, Y^n_s)\\
&\quad -\tfrac{\partial^2 f_\varepsilon}{\partial x^j \partial x^k}(s-, Z_{s-},0)\Big)\, d\langle X^{c,j}, X^{c,k}\rangle_s,\\
Z^n(f,3)_t &= \sum_{s\leq t}\Big(f_\varepsilon(\phi^n(s), X_{\phi^n(s)}, Y^n_s) - f_\varepsilon(s-, X_{s-}, \Delta X_s)\\
&\quad - f_\varepsilon(\phi^n(s), X_{\phi^n(s)}, Y^n_{s-}) - \sum_{j=1}^d \Delta X^j_s \Big(\tfrac{\partial f_\varepsilon}{\partial x^j}(\phi^n(s), X_{\phi^n(s)}, Y^n_{s-})\\
&\quad - \tfrac{\partial f_\varepsilon}{\partial x^j}(s-, X_{s-},0)\Big)\Big).
\end{aligned}$$

Observe now that $\frac{\partial f_\varepsilon}{\partial x^j}(\phi^n(s), Z_{\phi^n(s)}, Y^n_{s-}) \to \frac{\partial f_\varepsilon}{\partial x^j}(s-, Z_{s-},0)$. Since $\frac{\partial f_\varepsilon}{\partial x^j}$ is dominated by a locally bounded processes, Lebesgue's theorem gives:

$$Z^n(f_\varepsilon,1) \ \to^{u.c.p.} \ 0.$$

The proof of $Z^n(f_\varepsilon,j)\to^{u.c.p.} 0$ for $j=2,3$ is similar, and we thus have (4.7).

4.2 Proof of Theorem 2.6

Let us start by strengthening the hypothesis (N_0) :

Hypothesis (LN_0)**:** (N_0) is satisfied, and the processes $b_s, \sigma_s, \int_{\mathbb{R}}(1\wedge||\delta(\omega,s,y)||^2)\,F(dy)$ and X_s are bounded by a constant. □

We also suppose that the process Γ which intervenes in (2.1) is uniformly bounded. Below, we denote all constants by K. Set

$$\beta_i^n = \sigma_{(i-1)\Delta_n} \frac{\Delta_i^n W}{\sqrt{\Delta_n}}. \tag{4.8}$$

Lemma 4.3 *Suppose (LN_0) satisfied and f optional, satisfying $(K(\mathbb{R}))$ and at most with polynomial growth. Then*

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} E\left\{f\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right) | \mathcal{F}_{(i-1)\Delta_n}\right\} \stackrel{u.c.p.}{\longrightarrow} \int_0^t H_{s-}\, ds, \tag{4.9}$$

when $n \to \infty$, where $H_s = \int_0^t \rho_{\sigma_s}\left(f(s, X_s, .)\right) ds$.

Proof: The left side of (4.9) is almost surely equal to $\Delta_n \sum_{i=1}^{[t/\Delta_n]} H_{(i-1)\Delta_n}$. This is a Riemann sum which therefore converges to $\int_0^t H_{s-}\, ds$ locally uniformly in t, because H is a càdlàg process. □

Lemma 4.4 *Let f be optional, locally equicontinuous in x and with at most p-polynomial growth. Assume further that X satisfies (LN_0) and either is continuous or $p < 2$. Then*

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} E\left(\left\| f\Big((i-1)\Delta_n, X_{(i-1)\Delta_n}, \frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big) - f\Big((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\Big)\right\|\right) \to 0.$$

Proof: We reproduce the proof of Lemma 4.4 (2) of [4] with some relevant changes. For any A, T, $\varepsilon > 0$, we define the variables

$$\begin{aligned} G_T(\varepsilon, A) &= \sup_{s\le T;\, ||x||\le A; ||z||\le K;\, ||y||\le\varepsilon} \|f(s,z,x+y) - f(s,z,x)\| \\ \zeta_i^n &= \left\| f\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \frac{\Delta_i^n X}{\sqrt{\Delta_n}}\right) - f((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n)\right\|. \end{aligned}$$

Then

$$\|\zeta_i^n\| \le G_t(\varepsilon, A) + \|\zeta_i^n\| \left(1_{\{\|\beta_i^n\|>A\}} + 1_{\{\|\Delta_i^n X/\sqrt{\Delta_n} - \beta_i^n\| > \varepsilon\}}\right). \tag{4.10}$$

Let q be a real such that $q > p$ if X is continuous and $q = 2$ if not. Then (2.1) with Γ a constant yields for all $B > 1$:

$$\|f(\omega, s, z, x)\| \le K\,\phi(z)\Big(B^{p-q}\|x\|^q + B^p\Big).$$

Also under (LN_0) one knows that:

$$\mathbb{E}\left\{\left\|\Delta_i^n X/\sqrt{\Delta_n} - \beta_i^n\right\|^q + \|\beta_i^n\|^q\right\} \le K.$$

Hence by (4.10):

$$\begin{aligned} \|\zeta_i^n\| \le\; & G_t(\varepsilon, A) + KB^p\left(1_{\{\|\beta_i^n\|>A\}} + 1_{\{\|\Delta X_i^n/\sqrt{\Delta_n} - \beta_i^n\|>\varepsilon\}}\right) \\ & + KB^{p-q}\left(\|\beta_i^n\|^q + \left\|\Delta_i^n X/\sqrt{\Delta_n} - \beta_i^n\right\|^q\right). \end{aligned}$$

It follows that

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\{\|\zeta_i^n\|\} \leq t\left(\mathbb{E}\{G_t(\varepsilon, A)\} + \frac{KB^p}{A} + KB^{p-q}\right)$$
$$+KB^p\varepsilon^{-2}\Delta_n \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\Big\{1 \wedge \Big\|\Delta_i^n X/\sqrt{\Delta_n} - \beta_i^n\Big\|^2\Big\}. \tag{4.11}$$

Next by lemma 4-1 of ([6])

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\Big\{1 \wedge \Big\|\Delta_i^n X/\sqrt{\Delta_n} - \beta_i^n\Big\|^2\Big\} \longrightarrow 0.$$

Then coming back to (4.11) and letting successively $n \to \infty,\ \varepsilon \to 0,\ A \to \infty$ and $B \to \infty$, we obtain the result. □

Proof of Theorem 2.6: We first prove the theorem under the stronger assumptions (LN_0) and Γ_t in (2.1) bounded. Set

$$U_t'^n := \Delta_n \sum_{i=1}^{[t/\Delta_n]} f\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \frac{\Delta_i^n X}{\sqrt{\Delta_n}}\right) - \int_0^t \rho_{\sigma_s}\left(f(s-, X_s, .)\right) ds.$$

Then $U_t'^n = \sum_{j=1}^3 U_t'^n(j)$, where

$$U_t'^n(1) = \Delta_n \sum_{i=1}^{[t/\Delta_n]} \left(f\Big((i-1)\Delta_n, X_{(i-1)\Delta_n}, \frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big) - f\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right)\right),$$

$$U_t'^n(2) = \Delta_n \sum_{i=1}^{[t/\Delta_n]} \Big(f\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right)$$
$$-\mathbb{E}\left\{f\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right) | \mathcal{F}_{(i-1)\Delta_n}\right\}\Big),$$

$$U_t'^n(3) = \Delta_n \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\left\{f\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right) | \mathcal{F}_{(i-1)\Delta_n}\right\} - \int_0^t \rho_{\sigma_s}\left(f\left(s-, X_s, .\right)\right) ds.$$

Observe first that $U_t'^n(2)$ is a martingale with respect to the filtration $(\mathcal{F}_{[t/\Delta_n]\Delta_n})_{t\geq 0}$, and its predictable quadratic variation is given by:

$$\langle U'^n(2)\rangle_t = \Delta_n^2 \sum_{i=1}^{[t/\Delta_n]} \Big(\mathbb{E}\Big\{f((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n)^2 | \mathcal{F}_{(i-1)\Delta_n}\Big\}$$
$$-\Big(\mathbb{E}\Big\{f((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n)\,|\mathcal{F}_{(i-1)\Delta_n}\Big\}\Big)^2\Big),$$

which satisfies $\langle U'^n(2)\rangle_t \leq Kt\Delta_n$. It follows by Doob's inequality, that $U_t'^n(2) \overset{u.c.p.}{\to} 0$. We have the same results for $U_t'^n(1)$ and $U_t'^n(3)$, respectively by lemma 4.4 and 4.3.

At this stage the theorem is proved under the stronger assumptions announced at the beginning of the proof, and as said in Theorem 2.2, the general case is obtained by a classical localization method.

5 Proof of the central limit theorems

5.1 Proof of theorem 3.2

We start again by strengthening our hypotheses:

Hypothesis (LN_1)**:** (N_1) is satisfied, and the processes b, σ and X are bounded. The functions $\gamma_k = \gamma$ do not depend on k and are bounded. □

Hypothèse (LM_1)**:** We have (M_1) and the process Γ is bounded. □

Under (LN_1), we have:

$$X_t = X_0 + \int_0^t b'_s\,ds + \int_0^t \sigma_s\,dW_s + \int_0^t \int_{\mathbb{R}} \delta(s,y)(\underline{\mu} - \underline{\nu})(ds,dy), \tag{5.12}$$

where

$$b'_s := b_s + \int_{\mathbb{R}} h'(\delta(s,y))\,F(dy). \tag{5.13}$$

For $\varepsilon > 0$, set:

$$E = \{\, y \in \mathbb{R},\ \gamma(y) > \varepsilon \,\} \quad \text{and} \quad N_t = 1_E \star \underline{\mu}_t, \tag{5.14}$$

and let $T'_1, \cdots, T'_p, \cdots$ be the successive jump times of N.

We state two important lemmas, the first of which is due to Jacod and Protter (Lemma 5.6 of [9]), and the second one is Lemma 5.9 of [6].

Lemma 5.1 *Suppose (LN_1) satisfied, and for each T'_p, denote by i^n_p the integer such that $(i^n_p - 1)\Delta_n < T'_p \le i^n_p\Delta_n$. Then the sequence of random variables*

$$\frac{1}{\sqrt{\Delta_n}}\Big(\sigma_{(i^n_p-1)\Delta_n}(W_{T'_p} - W_{(i^n_p-1)\Delta_n})\ ,\ \sigma_{T'_p}(W_{i^n_p\Delta_n} - W_{T'_p})\Big)_{p\ge1}$$

converges stably in law to

$$\Big(\sqrt{\kappa_p}\,\sigma_{T'_{p-}}U_p\ ,\ \sqrt{1-\kappa_p}\,\sigma_{T'_p}U'_p\Big)_{p\ge1},$$

where U_p is such that $U^t_p = (U^1_p, \cdots, U^m_p)$ and $U'^t_p = (U'^1_p, \cdots, U'^m_p)$.

Lemma 5.2 *Under the assumptions of lemma 5.1, on has:*

$$\frac{1}{\sqrt{\Delta_n}}\Big((X_{i^n_p\Delta_n} - X_{T'_p} - \sigma_{T'_p}(W_{i^n_p\Delta_n} - W_{T'_p})\Big) \xrightarrow{\mathbb{P}} 0,$$

$$\frac{1}{\sqrt{\Delta_n}}\Big(X_{T'_p-} - X_{(i^n_p-1)\Delta_n} - \sigma_{(i^n_p-1)\Delta_n}(W_{T'_p} - W_{(i^n_p-1)\Delta_n})\Big) \xrightarrow{\mathbb{P}} 0.$$

We are now ready to give the proof of the theorem.

The processes

$$W^n(f) = \frac{1}{\sqrt{\Delta_n}} \left(V^n(f)_t - \sum_{s \le [t/\Delta_n]\Delta_n} f(s-, X_{s-}, \Delta X_s) \right) \tag{5.15}$$

satisfy $W^n(f) = W^n(f,1) + W^n(f,2)$, where

$$\begin{aligned} W^n(f,1)_t &= \frac{1}{\sqrt{\Delta_n}} \left(V^n(f)_t - \sum_{s \le [t/\Delta_n]\Delta_n} f(\phi^n(s), X_{s-}, \Delta X_s) \right) \\ W^n(f,2)_t &= \frac{1}{\sqrt{\Delta_n}} \sum_{s \le [t/\Delta_n]\Delta_n} \left(f(\phi^n(s), X_{s-}, \Delta X_s) - f(s-, X_{s-}, \Delta X_s) \right), \end{aligned}$$

($\phi^n(s)$ is like in the previous section). (3.9) yields $W^n(f,2) \stackrel{u.c.p.}{\longrightarrow} 0$, and for all $\varepsilon > 0$ we have

$$W^n(f,1) = W^n(f(1-\Psi_\varepsilon),1) + W^n(f\Psi_\varepsilon,1), \tag{5.16}$$

where Ψ_ε is as in (4.2). Then the rest of the proof of Theorem 3.2 is divided in three steps.

Step 1: Here we study the convergence of the process $W^n(f(1-\Psi_\varepsilon),1)$. By subsection 3.1 of [6], for n large enough one has:

$$\begin{aligned} W^n(f(1-\Psi_\varepsilon)) &= \frac{1}{\sqrt{\Delta_n}} \sum_{p:\ T'_p \le [t/\Delta_n]\Delta_n} \Big(f(1-\Psi_\varepsilon)((i_p^n - 1)\Delta_n, X_{(i_p^n-1)\Delta_n}, \Delta_i^n X) \\ &\quad - f(1-\Psi_\varepsilon)((i_p^n-1)\Delta_n, X_{T'_p-}, \Delta X_{T'_p}) \Big), \\ &= \frac{1}{\sqrt{\Delta_n}} \sum_{p:\ T'_p \le [t/\Delta_n]\Delta_n} \left(\sum_{j=1}^d \left(\Delta_{i_p^n}^n X^j - \Delta X_{T'_p}^j \right) \right. \\ &\quad \times \frac{\partial f(1-\Psi_\varepsilon)}{\partial x_j}((i_p^n-1)\Delta_n, \overline{X}'^n_p, \overline{X}^n_p) \\ &\quad \left. + \sum_{j=1}^d \left(X^j_{(i_p^n-1)\Delta_n} - X^j_{T'_p-} \right) \frac{\partial f(1-\Psi_\varepsilon)}{\partial z_j}((i_p^n-1)\Delta_n, \overline{X}'^n_p, \overline{X}^n_p) \right), \end{aligned}$$

where $(\overline{X}'^n_p, \overline{X}^n_p)$ is between $(X_{(i_p^n-1)\Delta_n}, \Delta_i^n X)$ and $(X_{T'_p-}, \Delta X_{T'_p})$. Then by lemma 5.2 and 5.1, $W^n(f(1-\Psi_\varepsilon)$ converge stably in law to the process

$$\begin{aligned} F'(f(1-\Psi_\varepsilon))_t &:= \sum_{p:\ T'_p \le t} \sum_{j=1}^d \sum_{k=1}^m \Big(\left(\sqrt{\kappa_p}\, \sigma^{j,k}_{T'_p-} U_p^k + \sqrt{1-\kappa_p}\, \sigma^{j,k}_{T'_p} U'^k_p \right) \\ &\quad \times \frac{\partial f(1-\Psi_\varepsilon)}{\partial x_j}(T'_p-, X_{T'_p-}, \Delta X_{T'_p}) \\ &\quad - \sqrt{\kappa_p}\, \sigma^{j,k}_{T'_p-} U_p^k \frac{\partial f(1-\Psi_\varepsilon)}{\partial z_j}(T'_p-, X_{T'_p-}, \Delta X_{T'_p}) \Big), \end{aligned}$$

which has the same $\mathcal{F}$-conditional law than the process $F(f(1-\Psi_\varepsilon))$ associated with the function $f(1-\Psi_\varepsilon)$ by (3.11).

Step 2: Here we show that

$$F(f(1-\Psi_\varepsilon)) \stackrel{u.c.p.}{\longrightarrow} F(f) \quad \text{as } \varepsilon \to 0. \tag{5.17}$$

Recall the process $C(f)$ defined in (3.12), and set $f\Psi_\varepsilon = f_\varepsilon$. Under (LN_1) there exists a process A such that:

$$\forall T>0, \quad C(f_\varepsilon)_T \leq A_T, \quad \text{and} \quad \mathbb{E}(A_t) < \infty.$$

Since $C(f_\varepsilon)_T \to 0$ when $\varepsilon \to 0$, by Lebesgue's convergence theorem we have $\mathbb{E}(C(f_\varepsilon)_T) \to 0$. Furthermore by lemma 3.4, the process $F(f_\varepsilon)_t$ is a locally square integrable martingale and Doob's inequality yields that:

$$\widetilde{\mathbb{P}}\left(\sup_{t\leq T} \|F(f_\varepsilon)\| > \eta\right) \leq \frac{4}{\eta}\, \widetilde{\mathbb{E}}\left(F(f_\varepsilon)_t^2\right) = \frac{4}{\eta}\mathbb{E}\left(C(f_\varepsilon)_T\right)$$

hence $F(f\Psi_\varepsilon) \stackrel{u.c.p.}{\longrightarrow} 0$ when $\varepsilon \to 0$. Since $F(f) = F(f(1-\Psi_\varepsilon)) + F(f\Psi_\varepsilon)$, this implies (5.17).

Step 3: In this last step we show that

$$\lim_{\varepsilon\to 0} \limsup_n \mathbb{P}\left\{\sup_{t\leq T} \|W^n(f\Psi_\varepsilon, 1)\| > \eta\right\} = 0, \quad \forall \eta, T > 0. \tag{5.18}$$

Using Itô's formula of lemma 4.2, in a similar way than in the proof of theorem 2.4, we have $W^n(f_\varepsilon, 1) = \sum_{l=1}^5 W^n(f_\varepsilon, 1, l)$, where

$$\begin{aligned}
W^n(f_\varepsilon,1,1) &= \frac{1}{\sqrt{\Delta_n}} \sum_{j=1}^d \int_0^{[t/\Delta_n]\Delta_n} b'^j_s \frac{\partial f_\varepsilon}{\partial x_j}\left(\phi^n(s), X_{\phi^n(s)}, Y^n_{s-}\right) ds, \\
W^n(f_\varepsilon,1,2)_t &= \frac{1}{2\sqrt{\Delta_n}} \sum_{j,j'=1}^d \sum_{k=1}^m \int_0^{[t/\Delta_n]\Delta_n} \sigma^{j,k}_s \sigma^{j',k}_s \frac{\partial^2 f_\varepsilon}{\partial x_j \partial x_{j'}}\left(\phi^n(s), X_{\phi^n(s)}, Y^n_{s-}\right) ds, \\
W^n(f_\varepsilon,2,3)_t &= \frac{1}{\sqrt{\Delta_n}} \sum_{j=1}^d \sum_{k=1}^m \int_0^{[t/\Delta_n]\Delta_n} \sigma^{j,k}_s \frac{\partial f_\varepsilon}{\partial x_j}\left(\phi^n(s), X_{\phi^n(s)}, Y^n_{s-}\right) dW^k_s, \\
W^n(f_\varepsilon,1,4)_t &= \frac{1}{\sqrt{\Delta_n}} \sum_{j=1}^d \int_0^{[t/\Delta_n]\Delta_n} \int_0^{\mathbb{R}} \delta^j(s,y) \frac{\partial f_\varepsilon}{\partial x^j}\left(\phi^n(s), X_{\phi^n(s)}, Y^n_{s-}\right) (\underline{\mu} - \underline{\nu})(ds,dy), \\
W^n(f_\varepsilon,1,5)_t &= \frac{1}{\sqrt{\Delta_n}} \int_0^{[t/\Delta_n]\Delta_n} \int_0^{\mathbb{R}} \Big(f_\varepsilon\left(\phi^n(s), X_{\phi^n(s)}, Y^n_{s-} + \delta(s,y)\right) \\
&\quad - f_\varepsilon\left(\phi^n(s), X_{\phi^n(s)}, Y^n_{s-}\right) - \sum_{j=1}^d \delta^j(s,y) \frac{\partial f_\varepsilon}{\partial x_j}\left(\phi^n(s), X_{\phi^n(s)}, Y^n_{s-}\right) \\
&\quad - f_\varepsilon(\phi^n(s), X_{s-}, \delta(s,y))\Big)\, \underline{\mu}(ds,dy),
\end{aligned}$$

with Y_s^n and $\phi^n(s)$ as before.

Under (LN_1) and (LM_1), we have:

$$\left.\begin{array}{lcl} \mathbb{E}(||X_t - X_s||^p) & \leq & K|t-s|^{p/2}, \quad \forall p \in [0,2]. \\ \sum_{j=1}^d \left\| \frac{\partial f_\varepsilon}{\partial z_j}(s,z,x) \right\| & \leq & \alpha_\varepsilon(||x|| \wedge (2d\varepsilon))^2, \\ \sum_{j=1}^d \sum_{j'=1}^d \left\| \frac{\partial^2 f_\varepsilon}{\partial x_j \partial z_{j'}}(s,z,x) \right\| & \leq & \alpha_\varepsilon(||x|| \wedge (2d\varepsilon)), \\ \sum_{j=1}^d \sum_{j'=1}^d \left\| \frac{\partial^{k_1+k_2} f_\varepsilon}{\partial x_j^{k_1} \partial x_{j'}^{k_2}}(s,z,x) \right\| & \leq & \alpha_\varepsilon(||x|| \wedge (2d\varepsilon))^{3-(k_1+k_2)}, \end{array}\right\} \tag{5.19}$$

where $\alpha_\varepsilon \to 0$ when $\varepsilon \to 0$, $k_1 + k_2 \in \{0,1,2\}$, and $\frac{\partial^0(f_\varepsilon)}{\partial x_j^0} = f_\varepsilon$. We also have

$$\frac{\partial f_\varepsilon}{\partial x_j}\left(\phi^n(s), X_{\phi^n(s)}, Y_{s-}^n\right) = \sum_{j'=1}^d Y_{s-}^{n,j'} \frac{\partial^2 f_\varepsilon}{\partial x_j \partial x_{j'}}\left(\phi^n(s), X_{\phi^n(s)}, \overline{Y}_s^n\right),$$

where $\overline{Y}_s^n$ belongs to the segment joining Y_{s-}^n and 0, thus

$$\begin{aligned} &\int_0^{[t/\Delta_n]\Delta_n} \mathbb{E}\left\{\left\| \frac{\widehat{b}_s^j\, Y_{s-}^{n,j'}}{\sqrt{\Delta_n}} \frac{\partial^2 f_\varepsilon}{\partial x_j \partial x_{j'}}\left(X_{\phi^n(s)}, \overline{Y}_s^n\right) \right\|\right\} ds \\ \leq\ & K \int_0^t \left[\left(\mathbb{E}\left\{ \frac{\|Y_{s-}^{n,j'}\|^2}{\Delta_n} \right\} \right)^{1/2} \left(\mathbb{E}\left\{ \left\| \frac{\partial^2 f_\varepsilon}{\partial x_j \partial x_{j'}}\left(\phi^n(s), Z_{\phi^n(s)}, \overline{Y}_s^n\right) \right\|^2 \right\} \right)^{1/2} \right] ds \\ \leq\ & K \int_0^t \left(\mathbb{E}\left\{ \left\| \frac{\partial^2 f_\varepsilon}{\partial x_j \partial x_{j'}}\left(\phi^n(s), Z_{\phi^n(s)}, \overline{Y}_s^n\right) \right\|^2 \right\} \right)^{1/2} ds. \end{aligned} \tag{5.20}$$

Since $\frac{\partial^2 f_\varepsilon}{\partial x_j \partial x_{j'}}(\omega,s,z,0) = 0$, and $\frac{\partial^2 f_\varepsilon}{\partial x_j \partial x_{j'}}(\omega,s,z,x)$ satisfies $(K(V))$, one deduces by Lebesgue's theorem that (5.20) converge to 0, and thus

$$W^n(f_\varepsilon, 2, 1) \stackrel{u.c.p.}{\longrightarrow} 0.$$

Similarly we show that

$$W^n(f_\varepsilon, 2, 2) \stackrel{u.c.p.}{\longrightarrow} 0.$$

Next, the processes

$$\frac{1}{\sqrt{\Delta_n}} \int_0^{[t/\Delta_n]\Delta_n} \frac{\partial f_\varepsilon}{\partial x_j}\left(\phi^n(s), X_{\phi^n(s)}, Y_{s-}^n\right) \sigma_s^{j,k} dW_s^k$$

are martingale with respect to the filtration $(\mathcal{F}_{[t/\Delta_n]\Delta_n})$, hence by Doob's inequality and (5.19) one has:

$$\mathbb{P}\left\{ \sup_{t \leq T} \left\| \frac{1}{\sqrt{\Delta_n}} \int_0^{[t/\Delta_n]\Delta_n} \frac{\partial f_\varepsilon}{\partial x_j}\left(\phi^n(s), X_{\phi^n(s)}, Y_{s-}^n\right) \sigma_s^{j,k} dW_s^k \right\| > \eta \right\}$$

$$\leq \frac{1}{\eta^2\Delta_n}\int_0^T \mathbb{E}\left\{\left\|\frac{\partial f_\varepsilon}{\partial x}\left(\phi^n(s), X_{\phi^n(s)}, Y^n_{s-}\right)\sigma_s^{j,k}\right\|^2\right\} ds \leq \frac{KT\alpha_\varepsilon^2}{\eta^2},$$

and

$$\lim_{\varepsilon\to 0}\ \limsup_n\ \mathbb{P}\left\{\sup_{t\leq T}\|W^n(f_\varepsilon,1,3)_t\| > \eta\right\} = 0.$$

Similarly, we have:

$$\lim_{\varepsilon\to 0}\ \limsup_n\ \mathbb{P}\left\{\sup_{t\leq T}\|W^n(f_\varepsilon,1,4)_t\| > \eta\right\} = 0.$$

Now under (LM_1), separating the cases where $||x|| \leq ||x'||$ and $||x'|| \leq ||x||$, one shows that:

$$\left\|f_\varepsilon(\omega,s,z_1,x+x') - f_\varepsilon(s,z_1,x') - \sum_{j=1}^d x_j\frac{\partial f_\varepsilon}{\partial x_j}(\omega,s,z_1,x') - f_\varepsilon(\omega,s,z_2,x)\right\|$$

$$\leq K\alpha_\varepsilon||x||^2\left(\,||z_1-z_2|| + ||x'||\,\right).$$

Then $\mathbb{P}\left(\sup_{t\leq T}|W^n(f_\varepsilon,1,5)_t| > \eta\right)$ is smaller than

$$\begin{aligned}&\frac{1}{\eta}\,\mathbb{E}\Big\{\int_0^{[t/\Delta_n]\Delta_n}\int_{\mathbb{R}} \|\ f_\varepsilon(\phi^n(s), X_{\phi^n(s)}, Y^n_{s-}+\delta(s,y))\\
&- f_\varepsilon(\phi^n(s), X_{\phi^n(s)}, \delta(s,y)) - f_\varepsilon(\phi^n(s), X_{\phi^n(s)}, Y^n_{s-})\\
&- \delta(s,y)\frac{\partial f_\varepsilon}{\partial x}(\phi^n(s), X_{\phi^n(s)}, Y^n_{s-})\ \|\underline{\mu}(ds,dy)\ \Big\}\\
&\quad\leq K\alpha_\varepsilon\left(\int_0^t \mathbb{E}\left\{\frac{||Y^n_{s-}|| + ||Z_{\phi^n(s)} - Z_{s-}||}{\sqrt{\Delta_n}}\right\} ds\right) \leq K\alpha_\varepsilon t\end{aligned}$$

and thus

$$\lim_{\varepsilon\to 0}\ \limsup_n\ \mathbb{P}\left\{\sup_{t\leq T}\|W^n(f_\varepsilon,1,5)_t\| > \eta\right\} = 0.$$

This ends the proof under the reinforced assumptions (LN_1) and (LM_1). One finishes the proof by a classical localization procedure.

5.2 Proof of theorem 3.5 and 3.7.

As for the previous proofs, we first strengthen the hypotheses, and thanks to Remark 3.1, we adopt the form (3.8) for σ.

Hypothesis $(LN_2(s))$**:** We have $(N_2(s))$ and the processes b_s, $\widetilde{b}_s$, $\widetilde{\sigma}_s$, $\widetilde{v}_s$, $\Delta\sigma_s$, $\int_{\mathbb{R}}(1 \wedge ||\widetilde{\delta}(s,y)||^2)\,F(dy)$ are bounded. The functions $\gamma_k = \gamma$ do not depend on k and are also bounded. □

We denote (LM_2) (resp. (LM_2')) the hypothesis (M_2) (resp. (M_2')) with the additional condition that the process Γ is bounded.

Under $(LN_2(s))$ with $s \leq 1$, X can be write as:

$$X_t \;=\; X_0 + \int_0^t b'_s\, ds + \int_0^t \sigma_s\, dW_s + \int_{\mathbb{R}} \int_0^t \delta(s,y)\, \underline{\mu}(ds,dy), \tag{5.21}$$

where $b'_s = b_s - \int_{\mathbb{R}} h(\delta(s,y))\, F(dy)$, and under $(L_2(2))$ the process σ is writen:

$$\sigma_t \;=\; \sigma_0 + \int_0^t \widetilde{b}'_s\, ds + \int_0^t \widetilde{\sigma}_s\, dW_s + \int_0^t \widetilde{v}\, dV_s + \int_0^t \int_{\mathbb{R}} \widetilde{\delta}(s,y)\, (ds,dy), \tag{5.22}$$

with $\widetilde{b}'_s = \widetilde{b}_s + \int_{\mathbb{R}} k'\left(\widetilde{\delta}(s,y)\right) F(dy)$.

Let us now give some useful lemmas.

Lemma 5.3 *Suppose $(LN_2(2))$ satisfied and assume that f is optional, locally equicontinuous in x and at most with p-polynomial growth. If further, either X is continuous or $p < 1$, then:*

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} E\left(\left\| f\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \frac{\Delta_i^n X}{\sqrt{\Delta_n}}\right) - f((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n) \right\|^2 \right) \to 0. \tag{5.23}$$

Proof: The proof of this lemma is the same as for Lemma 4.4, the condition $p < 1$ come in because of the the square in (5.23). □

Set

$$U_t^n \;:=\; \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\Big\{ \Big(f\Big((i-1)\Delta_n, X_{(i-1)\Delta_n}, \frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big) - f\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right) \Big) | \mathcal{F}_{(i-1)\Delta_n} \Big\} \tag{5.24}$$

Lemma 5.4 *Suppose $(LN_2(2))$ and (M_2) satisfied and X continuous. Assume further that one of the following two conditions is satisfied:*

A. *The application $x \mapsto f(\omega, s, z, x)$ is even in x.*

B. *We have $b' = 0$ and $\widetilde{\sigma} = 0$.*

Then $U^n \stackrel{u.c.p.}{\longrightarrow} 0$.

Proof: A) Set

$$L_i^n := f\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \frac{\Delta_i^n X}{\sqrt{\Delta_n}}\right) - f\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right). \tag{5.25}$$

Then $L_i^n = L_i'^n + L_i''^n$, where

$$L_i'^n = \sum_{j=1}^d \left(\frac{\partial f}{\partial x_j} \left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \bar{\gamma}_i^n\right) - \frac{\partial f}{\partial x_j} \left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right)\right) \times \left(\frac{\Delta_i^n X^j}{\sqrt{\Delta_n}} - \beta_i^{n,j}\right),$$

$$L_i''^n = \sum_{j=1}^d \left(\frac{\Delta_i^n X^j}{\sqrt{\Delta_n}} - \beta_i^{n,j}\right) \frac{\partial f}{\partial x_j} \left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right),$$

for some random variable $\bar{\gamma}_i^n$ between $\frac{\Delta_i^n X}{\sqrt{\Delta_n}}$ and β_i^n. For any $\varepsilon,\ A > 0$, set

$$G_t^A(\omega, \varepsilon) = \sup_{s \le t;\, \|y\| \le \varepsilon;\, z \in \mathcal{K};\, \|x\| \le A} \left\{ \sum_{j=1}^d \left\| \frac{\partial f}{\partial x}(\omega, s, z, x+y) - \frac{\partial f}{\partial x}(\omega, s, z, x) \right\| \right\}.$$

Then:

$$\|L_i'^n\| \le K \left(G_t^A(\varepsilon) + \left(1 + \|\beta_i^n\|^p + \left\|\frac{\Delta_i^n X}{\sqrt{\Delta_n}} - \beta_i^n\right\|^p\right) \left(\frac{\|\beta_i^n\|}{A} + \frac{\left\|\frac{\Delta_i^n X}{\sqrt{\Delta_n}} - \beta_i^n\right\|}{\varepsilon}\right)\right) \times \left\|\frac{\Delta_i^n X}{\sqrt{\Delta_n}} - \beta_i^n\right\|.$$

Next, under the assumption $(N_2(s))$ (in particular the properties of σ), one shows that for all $q \ge 2$:

$$\mathbb{E}(\|\beta_i^n\|^q) \le K, \quad E\left\{\left\|\Delta_i^n X/\sqrt{\Delta_n} - \beta_i^n\right\|^q\right\} \le K\Delta_n. \tag{5.26}$$

Thus by a repeated use of Hölder inequality:

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E\{\|L_i'^n\|\} \le Kt \left[\left(E\left\{ \left(G_t^A(\varepsilon)\right)^2 \right\} \right)^{1/2} + \frac{\Delta_n^{1/4}}{\varepsilon} + \frac{1}{A} \right]. \tag{5.27}$$

Letting successively $n \to \infty$, then $\varepsilon \to 0$ and then $A \to \infty$, we obtain

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} E\{\|L_i'^n\|\} \longrightarrow 0. \tag{5.28}$$

Let us now turn to $L_i''^n$. Under $(LN_2(s))$ we have: $\frac{\Delta_i^n X}{\sqrt{\Delta_n}} - \beta_i^n = \widetilde{\xi}_i^n + \widehat{\xi}_i^n$, where

$$\widehat{\xi}_i^n = \frac{1}{\sqrt{\Delta_n}} \left(\int_{(i-1)\Delta_n}^{i\Delta_n} (b'_s - b'_{(i-1)\Delta_n})\, ds + \int_{(i-1)\Delta_n}^{i\Delta_n} \Big(\int_{(i-1)\Delta_n}^s \widetilde{b'}_u\, du + \int_{(i-1)\Delta_n}^s (\widetilde{\sigma}_u - \widetilde{\sigma}_{(i-1)\Delta_n})\, dW_u \Big) dW_s \right),$$

$$\widetilde{\xi}_i^n = \sqrt{\Delta_n} b'_{(i-1)\Delta_n} + \frac{1}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \Big(\widetilde{\sigma}_{(i-1)\Delta_n}(W_s - W_{(i-1)\Delta_n}) + \int_{(i-1)\Delta_n}^s \int_{\mathbb{R}} \widetilde{\delta}(u,y)(\underline{\mu} - \underline{\nu})(du, dy) + \int_{(i-1)\Delta_n}^s \widetilde{v}_u\, dV_u \Big) dW_s.$$

1) Here we show that for any $j \in \{1, \cdots, d\}$,

$$\mathbb{E}\left\{\widetilde{\xi}_i^{n,j}\frac{\partial f}{\partial x_j}\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right) \ |\mathcal{F}_{(i-1)\Delta_n}\right\} \ = \ 0. \qquad (5.29)$$

Since the function $x \to \frac{\partial f}{\partial x_j}(\omega, s, z, x)$ is odd, one clearly has:

$$\mathbb{E}\left\{b'^{j}_{(i-1)\Delta_n}\frac{\partial f}{\partial x_j}\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right) \ |\mathcal{F}_{(i-1)\Delta_n}\right\} \ = \ 0, \qquad (5.30)$$

and for any $k, \ k' \in \{1, \cdots, m\}$:

$$\mathbb{E}\left\{\widetilde{\sigma}^{j,k,k'}_{(i-1)\Delta_n}\Big(\int_{(i-1)\Delta_n}^{i\Delta_n}(W_s^{k'} - W_{(i-1)\Delta_n}^{k'})\, dW_s^k\Big)\times\right.$$

$$\left.\times\frac{\partial f}{\partial x_j}\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right) |\mathcal{F}_{(i-1)\Delta_n}\right\} \ = \ 0. \qquad (5.31)$$

Next consider the σ-field:

$$\mathcal{F}'_{(i-1)\Delta_n} \ = \ \mathcal{F}_{(i-1)\Delta_n}\bigvee\sigma(W_s - W_{(i-1)\Delta_n} : (i-1)\Delta_n \le s \le i\Delta_n).$$

Since W is independent of $\underline{\mu}$ and of V, for any $j, \ k$ as above one has:

$$\mathbb{E}\left\{\left(\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\int_{(i-1)\Delta_n}^{s}\int_{\mathbb{R}}\widetilde{\delta}^{j,k}(u,y)(\underline{\mu} - \underline{\nu})(du, dy)\right)\, dW_s^k\right)\times\right.$$

$$\left.\times\frac{\partial f}{\partial x_j}\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right) \ |\mathcal{F}_{(i-1)\Delta_n}\right\} \ = 0, \qquad (5.32)$$

and for any $j' \in \{1, \cdots, l\}$:

$$\mathbb{E}\left\{\Big(\int_{(i-1)\Delta_n}^{i\Delta_n}\Big(\int_{(i-1)\Delta_n}^{s}\widetilde{v}_u^{j,k,j'}\, dV_u^{j'}\Big)\, dW_s^k\Big)\times\right.$$

$$\left.\times\frac{\partial f}{\partial x_j}((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n) \ |\mathcal{F}_{(i-1)\Delta_n}\right\} \ = 0. \qquad (5.33)$$

From (5.30), (5.31), (5.32) and (5.33) we deduce (5.29).

2) In this step, we show that for all $j \in \{1, \cdots, d\}$,

$$\sqrt{\Delta_n}\sum_{i=1}^{[t/\Delta_n]}\mathbb{E}\left\{\left\|\widehat{\xi}_i^{n,j}\frac{\partial f}{\partial x_j}\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right)\right\| \ |\mathcal{F}_{(i-1)\Delta_n}\right\}\longrightarrow \ 0. \qquad (5.34)$$

By Hölder and Doob inequalities we have:

$$\mathbb{E}\{||\,\widehat{\xi}_i^n\,||^2\} \ \le \ K\left(\Delta_n^3 + \int_{(i-1)\Delta_n}^{i\Delta_n}\left(||b'_s - b'_{(i-1)\Delta_n}||^2 + ||\widetilde{\sigma}_s - \widetilde{\sigma}_{(i-1)\Delta_n}||^2\right)\, ds\right).$$

Since $\mathbb{E}\left\{\left\|\frac{\partial f}{\partial x}\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right)\right\|^2\right\} \le K$, it follows from a repeated use of Holdër inequality that

$$\sqrt{\Delta_n}\sum_{i=1}^{[t/\Delta_n]}\mathbb{E}\left\{\left\|\widehat{\xi}_i^{n,j}\frac{\partial f}{\partial x_j}\left((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\right)\right\| |\mathcal{F}_{(i-1)\Delta_n}\right\} \le Kt\Delta_n +$$

$$+Kt^{1/2}\left(\mathbb{E}\left\{\int_0^{[t/\Delta_n]\Delta_n}\left(||b'_s - b'_{[s/\Delta_n]\Delta_n}||^2 + ||\widetilde{\sigma}_s - \widetilde{\sigma}_{[s/\Delta_n]\Delta_n}||^2\right) ds\right\}\right)^{1/2}.$$

Since b' and $\widetilde{\sigma}$ have some continuity properties in s, we deduce by Lebesgue theorem that the last quantity tends to 0 when $n \to \infty$, hence (5.34).

B) The proof is the same than for (A), except for the fact that we have (5.30) and (5.31) because $b' = \widetilde{\sigma} = 0$. □

We give now another version of Lemma 5.4, in the case where X is discontinuous:

Lemma 5.5 *Suppose X satisfies $(LN_2(s))$ with $s \le 1$ and f satisfies (LM'_2). Assume further that either $f(\omega, s, z, x)$ is even in x or $b' = \widetilde{\sigma} = 0$. Then*

$$U^n \stackrel{u.c.p.}{\longrightarrow} 0, \quad \textit{when } n \to \infty.$$

Proof: Recall that under $(LN_2(s))$ with $s \le 1$, X is written as in (5.21). Set

$$X'_t := X_0 + \int_0^t b'_s\, ds + \int_0^t \sigma_s\, dW_s.$$

Let (ε_n) be a sequence such that: $\varepsilon_n \in]0,1]$ and $\varepsilon_n \to 0$ when $n \to \infty$, and set $E_n = \{x \in \mathbb{R}, \gamma(x) > \varepsilon_n\}$. Then

$$\begin{aligned}\frac{\Delta_i^n X}{\sqrt{\Delta_n}} &= \frac{\Delta_i^n X'}{\sqrt{\Delta_n}} + \frac{1}{\sqrt{\Delta_n}}\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{E_n^c}\delta(s,x)\,\underline{\mu}(ds,dx).\\ &\quad +\frac{1}{\sqrt{\Delta_n}}\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{E_n}\delta(s,x)\,\underline{\mu}(ds,dx).\end{aligned}$$

Set

$$\begin{aligned}\zeta_i^n(1) &:= \frac{1}{\sqrt{\Delta_n}}\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{E_n}\delta(s,y)\,\underline{\mu}(ds,dy),\\ \zeta_i^n(2) &:= \frac{1}{\sqrt{\Delta_n}}\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{E_n^c}\delta(s,y)\,\underline{\mu}(ds,dy).\end{aligned}$$

Then using the notation (5.25), one has $L_i^n = \sum_{j=1}^3 L_i^n(j)$, where

$$L_i^n(1) = f\Big((i-1)\Delta_n, X_{(i-1)\Delta_n}, \frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big) - f\Big((i-1)\Delta_n, X_{(i-1)\Delta_n}, \frac{\Delta_i^n X}{\sqrt{\Delta_n}} - \zeta_i^n(1)\Big),$$
$$L_i^n(2) = f\Big((i-1)\Delta_n, X_{(i-1)\Delta_n}, \frac{\Delta_i^n X}{\sqrt{\Delta_n}} - \zeta_i^n(1)\Big) - f\Big((i-1)\Delta_n, X_{(i-1)\Delta_n}, \frac{\Delta_i^n X'}{\sqrt{\Delta_n}}\Big),$$
$$L_i^n(3) \ = \ f\Big((i-1)\Delta_n, X_{(i-1)\Delta_n}, \frac{\Delta_i^n X'}{\sqrt{\Delta_n}}\Big) - f\Big((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n\Big).$$

The hypothesis (LM_2') gives the existence of a sequence of reals (K^m) such that

$$\|z\| \le m \ \Rightarrow \ \|f(\omega,s,z,x_1) - f(\omega,s,z,x_1+x_2)\| \ \le \ K^m(1 \wedge \|x_2\|).$$

Hence

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\Big\{ \|L_i^n(1)\| \,|\mathcal{F}_{(i-1)\Delta_n} \Big\} \le K\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\Big\{ (1 \wedge \|\zeta_i^n(1)\|) \,|\mathcal{F}_{(i-1)\Delta_n} \Big\}.$$

By the inequality (5.9) of lemma 5.3 of [6], we deduce:

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\left\{\|L_i^n(1)\|\right\} \ \le \ Kt\Delta_n^{1/2}\varepsilon_n^{-1}. \tag{5.35}$$

Next, set $\theta(y) = \int_{\{|\gamma(x)| \le y\}} |\gamma(x)|\, F(dx)$, which goes to 0 as $y \to 0$. One has

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\{\|L_i^n(2)\|\} \ \le \ K\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\{\|\zeta_i^n(2)\|\} \ \le \ Kt\theta(\varepsilon_n). \tag{5.36}$$

Finally, lemma 5.5 implies:

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \left\|\mathbb{E}\{L_i^n(3) \ |\mathcal{F}_{(i-1)\Delta_n}\}\right\| \ \longrightarrow^{u.c.p.} \ 0, \quad \text{when } n \to \infty. \tag{5.37}$$

By (5.35), (5.36) and (5.37) we have:

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \left\|\mathbb{E}\left\{ L_i^n \,|\mathcal{F}_{(i-1)\Delta_n}\right\}\right\| \ \le \ Kt\left(\Delta_n^{1/2}\varepsilon_n^{-1} + \theta(\varepsilon_n)\right)$$
$$+\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \left\|\mathbb{E}\left\{ L_i^n(3) \ |\mathcal{F}_{(i-1)\Delta_n}\right\}\right\|.$$

Choosing $\varepsilon_n = (1 \wedge \Delta_n^{1/4})$, we conclude:

$$\sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \left\|\mathbb{E}\{L_i^n \ |\mathcal{F}_{(i-1)\Delta_n}\}\right\| \ \longrightarrow^{u.c.p.} \ 0,$$

and this ends the proof. □

Set now

$$U_t'^n = \frac{1}{\sqrt{\Delta_n}}\Big(\Delta_n \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\{f((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta_i^n)|\mathcal{F}_{(i-1)\Delta_n}\} - \int_0^t \rho_{\sigma_s}(f(s, X_s, .))ds\Big). \tag{5.38}$$

Lemma 5.6 *If X satisfies $(LN_2(2))$ and f satisfies (LM_2), we have $U'^n \stackrel{u.c.p.}{\longrightarrow} 0$.*

Proof: We can assume without loss of generality that f is 1-dimensional. We also write the proof when the dimensions of X and σ are 1, since the multidimensional case is more cumbersome but similar to prove. We have $U_t'^n = U_t'^n(1) + U_t'^n(2) + U_t'^n(3)$, where

$$\begin{aligned} U_t'^n(1) &= \frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{[t/\Delta_n]} \int_{(i-1)\Delta_n}^{i\Delta_n} \Big(\rho_{\sigma_{(i-1)\Delta_n}}(f((i-1)\Delta_n, X_{(i-1)\Delta_n}, .)) \\ &\quad - \rho_{\sigma_s}(f((i-1)\Delta_n, X_s, .))\Big)\, ds, \\ U_t'^n(2) &= \frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{[t/\Delta_n]} \int_{(i-1)\Delta_n}^{i\Delta_n} \Big(\rho_{\sigma_s}(f((i-1)\Delta_n, X_s, .)) - \rho_{\sigma_s}(f(s, X_s, .))\Big)\, ds, \\ U_t'^n(3) &= \frac{1}{\sqrt{\Delta_n}} \int_{[t/\Delta_n]\Delta_n}^t \rho_{\sigma_s}(f(s, X_s, .))\, ds. \end{aligned}$$

Since f is at most with polynomial growth,

$$\frac{1}{\sqrt{\Delta_n}} \int_{[t/\Delta_n]\Delta_n}^t |\rho_{\sigma_s}(f(s, X_s, .))|\, ds \;\le\; K\sqrt{\Delta_n},$$

hence $U_t'^n(3) \stackrel{u.c.p.}{\longrightarrow} 0$. Otherwise Hypothesis (M_2), and in particular (3.10), implies

$$\frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{[t/\Delta_n]} \int_{(i-1)\Delta_n}^{i\Delta_n} |\rho_{\sigma_s}(f((i-1)\Delta_n, X_s, .)) - \rho_{\sigma_s}(f(s, X_s, .))|\, ds \le Kt\Delta_n^{\alpha-1/2},$$

hence $U_t'^n(2) \stackrel{u.c.p.}{\longrightarrow} 0$.

It remains to show that:

$$U_t'^n(1) \stackrel{u.c.p.}{\longrightarrow} 0. \tag{5.39}$$

The function $(z, x) \mapsto f(\omega, s, z, x)$ being C^1, so is the application $(w, z) \mapsto \rho_w(f(s, z, .))$. Set $F_{n,i}(\omega, w, z) := \rho_w(f(\omega, (i-1)\Delta_n, z, .))$ and $X_t'' = X_t - \int_0^t b_s ds$ and $\sigma_t'' = \sigma_t - \int_0^t \widetilde{b}_s' ds$. Then we have $U_t'^n(1) = -\sum_{j=1}^3 U_t'^n(1, j)$, where

$$U_t'^n(1,1) \;=\; \frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{[t/\Delta_n]} \int_{(i-1)\Delta_n}^{i\Delta_n} \Bigg(\int_{(i-1)\Delta_n}^s \Big(\widetilde{b}_u' \frac{\partial F_{n,i}}{\partial w}(\sigma_{(i-1)\Delta_n}, X_{(i-1)\Delta_n})$$

$$+ b'_u \frac{\partial F_{n,i}}{\partial z}(\sigma_{(i-1)\Delta_n}, X_{(i-1)\Delta_n})\Big)\, du\Big)\, ds$$

$$\begin{aligned} U'^n_t(1,2) &= \frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{[t/\Delta_n]} \int_{(i-1)\Delta_n}^{i\Delta_n} \Big((\sigma''_s - \sigma''_{(i-1)\Delta_n}) \frac{\partial F_{n,i}}{\partial w}(\sigma_{(i-1)\Delta_n}, X_{(i-1)\Delta_n}) \\ &\quad + (X''_s - X''_{(i-1)\Delta_n}) \frac{\partial F_{n,i}}{\partial z}(\sigma_{(i-1)\Delta_n}, X_{(i-1)\Delta_n}) \Big)\, ds \\ U'^n_t(1,3) &= \frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{[t/\Delta_n]} \int_{(i-1)\Delta_n}^{i\Delta_n} \Big(F_{n,i}(\sigma_s, X_s) - F_{n,i}(\sigma_{(i-1)\Delta_n}, X_{(i-1)\Delta_n}) \\ &\quad - (X_s - X_{(i-1)\Delta_n}) \frac{\partial F_{n,i}}{\partial z}(\sigma_{(i-1)\Delta_n}, X_{(i-1)\Delta_n}) \\ &\quad - (\sigma_s - \sigma_{(i-1)\Delta_n}) \frac{\partial F_{n,i}}{\partial w}(\sigma_{(i-1)\Delta_n}, X_{(i-1)\Delta_n}) \Big)\, ds. \end{aligned}$$

Since $\widetilde{b}'$, b' are bounded we have $\sup_{s \le t} |U'^n_t(1,1)| \le Kt\Delta_n^{1/2}$, hence $U'^n_t(1,1) \stackrel{u.c.p.}{\longrightarrow} 0$.

Next, the process $U'^n(1,2)$ is a martingale with respect to the filtration $(\mathcal{F}_{[t/\Delta_n]\Delta_n})$ and the expectation of its predictable bracket is smaller than $Kt\Delta_n$. Hence Doob's inequality yields $U^n_t(1,2) \stackrel{u.c.p.}{\longrightarrow} 0$.

Finally, if $\zeta^n_i(s)$ denotes the integrand in the definition of $U'^n_t(1,3)$, we have

$$\zeta^n_i(s) := (\sigma_s - \sigma_{(i-1)\Delta_n}) \left(\frac{\partial F_{n,i}}{\partial w}(\overline{\sigma}(i,n,s), \overline{X}(i,n,s)) - \frac{\partial F_{n,i}}{\partial w}(\sigma_{(i-1)\Delta_n}, X_{(i-1)\Delta_n}) \right)$$

$$+ \ (X_s - X_{(i-1)\Delta_n}) \left(\frac{\partial F_{n,i}}{\partial z}(\overline{\sigma}(i,n,s), \overline{X}(i,n,s)) - \frac{\partial F_{n,i}}{\partial z}(\sigma_{(i-1)\Delta_n}, X_{(i-1)\Delta_n}) \right), \quad (5.40)$$

with $(\overline{\sigma}(i,n,s), \overline{X}(i,n,s))$ in between $(\sigma_{(i-1)\Delta_n}, X_{(i-1)\Delta_n})$ and (σ_s, X_s). For A, $\varepsilon > 0$, set:

$$\begin{aligned} G_t(\varepsilon, A) = \sup \Big\{ &\left| \tfrac{\partial f}{\partial x}(s, z_1, x_1) - \tfrac{\partial f}{\partial x}(s, z_2, x_2) \right| + \left| \tfrac{\partial f}{\partial z}(s, z_1, x_1) - \tfrac{\partial f}{\partial z}(s, z_2, x_2) \right| : \\ &s \le t; \ \ |x_1|, \ |x_2| \le A; \ \ |x_1 - x_2| \le \varepsilon; \ |z_1|, \ |z_2| \le K; \ \ |z_1 - z_2| \le \varepsilon \Big\}, \end{aligned}$$

then by the properties of f, we have $G_t(\varepsilon, A) \to 0$ when $\varepsilon \to 0$. Therefore it follows from (5.40) that

$$\begin{aligned} |\zeta^n_i(s)| \ &\le \ K \Big((1+A) G_t(A\varepsilon, KA) + \frac{|\sigma_s - \sigma_{(i-1)\Delta_n}| + |X_s - X_{(i-1)\Delta_n}|}{\varepsilon} + \\ &\quad + (\mathbb{P}(|U| > A/K))^{1/2} \Big) \times \Big(|\sigma_s - \sigma_{(i-1)\Delta_n}| + |X_s - X_{(i-1)\Delta_n}| \Big), \end{aligned}$$

where U is a $\mathcal{N}(0,1)$ Gaussian variable.

Since under $(LN_2(2))$, $\mathbb{E}\left\{ |\sigma_t - \sigma_s|^2 + |Z_t - Z_s|^2 \right\} \le K|t-s|$, we deduce:

$$\begin{aligned} \frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{[t/\Delta_n]} \int_{(i-1)\Delta_n}^{i\Delta_n} \mathbb{E}\{|\zeta^n_i(s)|\}\, ds \ &\le \ Kt\Big((1+A)(\mathbb{E}\{G_t(A\varepsilon, KA)^2\})^{1/2} + \\ &\quad + \mathbb{P}(|U| > A/K)^{1/2} + \frac{\sqrt{\Delta_n}}{\varepsilon} \Big). \end{aligned}$$

Letting $n \to \infty$, then $\varepsilon \to 0$, and $A \to \infty$, we obtain $U'^n_t(1,3) \stackrel{u.c.p.}{\longrightarrow} 0$, hence (5.39). $\square$

The next lemmas are very important because they deal with the part of the processes having a non-trivial limit. We use the notation of Subsection 3. The first one is about the "even case" for f. Set

$$\overline{U}^n_t = \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \Big(f((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta^n_i) - \mathbb{E}\{ f((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta^n_i) \,|\mathcal{F}_{(i-1)\Delta_n}\} \Big). \tag{5.41}$$

Lemma 5.7 *Suppose* $(LN_2(2))$ *satisfied and* $f(\omega,s,z,x)$ *even in* x *with at most polynomial growth. Then* $\overline{U}^n_t \stackrel{\mathcal{L}-(s)}{\longrightarrow} L(f)_t$, *where* $L(f)_t = \int_0^t a_s\, d\overline{W}_s$ *is given by (3.14).*

Proof: Set

$$\xi^n_i = \sqrt{\Delta_n}\Big(f((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta^n_i) - \mathbb{E}\{ f((i-1)\Delta_n, X_{(i-1)\Delta_n}, \beta^n_i) |\mathcal{F}_{(i-1)\Delta_n}\}\Big),$$

then

$$E\{\xi^n_i \,|\mathcal{F}_{(i-1)\Delta_n}\} \;=\; 0. \tag{5.42}$$

For any $j,k \in \{1,\cdots,q\}$, we have:

$$\begin{aligned}\mathbb{E}\{\xi^{n,j}_i \xi^{n,k}_i |\mathcal{F}_{(i-1)\Delta_n}\} \;=\; & \Delta_n\Big(\rho_{\sigma_{(i-1)\Delta_n}}((f^j f^k)((i-1)\Delta_n, X_{(i-1)\Delta_n},.)) \\ & - \rho_{\sigma_{(i-1)\Delta_n}}(f^j((i-1)\Delta_n, X_{(i-1)\Delta_n},.)) \times \\ & \times \rho_{\sigma_{(i-1)\Delta_n}}(f^k((i-1)\Delta_n, X_{(i-1)\Delta_n},.))\Big).\end{aligned}$$

Then as in lemma 4.3, one shows that:

$$\left.\begin{array}{l} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\left\{ \left(\xi^{n,j}_i \xi^{n,k}_i\right) |\mathcal{F}_{(i-1)\Delta_n}\right\} \text{ converges u.c.p. to the process} \\ \int_0^t \left(\rho_{\sigma_s}((f^j f^k)(s,X_s,.)) - \rho_{\sigma_s}(f^j(s,X_s,.))\rho_{\sigma_s}(f^k(s,X_s,.))\right) ds \end{array}\right\} \tag{5.43}$$

Next for any $\varepsilon > 0$, we have:

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\{||\xi^n_i||^2 1_{\{||\xi^n_i||>\varepsilon\}}|\mathcal{F}_{(i-1)\Delta_n}\} \;\le\; \frac{1}{\varepsilon^2}\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\{||\xi^n_i||^4|\mathcal{F}_{(i-1)\Delta_n}\} \;\le\; \frac{Kt}{\varepsilon^2}\,\Delta_n. \tag{5.44}$$

Since f is even in x: $\forall\, j' \in \{1,\cdots,m\}$,

$$\mathbb{E}\left\{\xi^n_i \Delta^n_i W^{j'} \,|\mathcal{F}_{(i-1)\Delta_n}\right\} \;=\; 0. \tag{5.45}$$

If now N is a martingale orthogonal to W, by the proof of Proposition 4.1 (see (4.13)) of [2],

$$\mathbb{E}\{\xi^n_i \Delta^n_i N|\mathcal{F}_{(i-1)\Delta_n}\} = 0. \tag{5.46}$$

By (5.42), (5.43), (5.44), (5.45) and (5.46) we can apply theorem IX-7-28 of [6] which gives our lemma. $\square$

Remark 5.8 *In the previous lemma, the hypothesis on f is more than what we need, having $f(\omega,s,z,x)$ to be optional even in x and satisfying $(K(\mathbb{R}^d))$ and with at most polynomial growth would be enough.*

Now we deal with the case where $f(\omega,s,z,x)$ is not even in x.

Lemma 5.9 *Suppose that X and f satisfy respectively $(LN_2(2))$ and (LM_2), then $\overline{U}_t^n \stackrel{\mathcal{L}-(s)}{\longrightarrow} L(f)_t$, where $L(f)_t$ is given by (3.20).*

Proof: The proof goes as for lemma 5.7, except that (5.45) fails here, since $f(\omega,s,z,x)$ is not even in x. However we have

$$\mathbb{E}\Big\{\xi_i^{n,j}\Delta_i^n W^k|\mathcal{F}_{(i-1)\Delta_n}\Big\} = \sqrt{\Delta_n}\mathbb{E}\Big\{f^j((i-1)\Delta_n, X_{(i-1)\Delta_n},\beta_i^n)\Delta_i^n W^k)|\mathcal{F}_{(i-1)\Delta_n}\Big\},$$

and (as in the proof of lemma 4.3) one has:

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\left\{\xi_i^{n,j}\Delta_i^n W^k\,|\mathcal{F}_{(i-1)\Delta_n}\right\} \stackrel{u.c.p.}{\longrightarrow} \int_0^t w(1)_s^{j,k}\,ds. \tag{5.47}$$

Then taking account (5.47), and using once more theorem IX-7-28 of [6], we get this time Lemma 5.9. □

5.2.1 Proof of theorems 3.5 and 3.7:

We first prove the theorems under the strong hypotheses stated at the beginning of the Subsection 5.2. Set

$$W_t^n := \sqrt{\Delta_n}\left(V_t'^n - \int_0^t \rho_{\sigma_s}(f(s,X_s,.))\,ds\right).$$

Then, using the notation (5.24), (5.25), (5.38) and (5.41), we have:

$$W_t^n = \sqrt{\Delta_n}\sum_{i=1}^{[t/\Delta_n]}(L_i^n - \mathbb{E}\{L_i^n|\mathcal{F}_{(i-1)\Delta_n}\}) + \overline{U}_t^n + U_t^n + U_t'^n.$$

The process $\sqrt{\Delta_n}\sum_{i=1}^{[t/\Delta_n]}(L_i^n - \mathbb{E}\{L^n i|\mathcal{F}_{(i-1)\Delta_n}\})$ is a martingale with respect to the filtration $(\mathcal{F}_{[t/\Delta_n]\Delta_n})$, whose predictable bracket is smaller than $\Delta_n\mathbb{E}\{\|L_i^n\|^2|\mathcal{F}_{(i-1)\Delta_n}\}$. Hence Lemma 5.3 and Doob's inequality yield that

$$\sqrt{\Delta_n}\sum_{i=1}^{[t/\Delta_n]}(L_i^n - \mathbb{E}\{L_i^n|\mathcal{F}_{(i-1)\Delta_n}\}) \stackrel{u.c.p.}{\longrightarrow} 0.$$

Moreover $U^n \stackrel{u.c.p.}{\longrightarrow} 0$ by Lemmas 5.4 or 5.5, depending on the case. Next, Lemma 5.6 yields $U'^n \stackrel{u.c.p.}{\longrightarrow} 0$. Finally Lemma 5.7 for Theorem 3.5 and Lemma 5.9 for Theorem 3.7 give that $\overline{U}^n$ converges stably in law to the process $L(f)$ given respectively by (3.14) and (3.20).

At this stage, we have proved the theorems under the strong assumptions mentioned above. The general case is deduced by a "localization" procedure.

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