Explicit integrable systems on two dimensional manifolds with a cubic first integral

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EXPLICIT INTEGRABLE SYSTEMS ON TWO DIMENSIONAL MANIFOLDS WITH A CUBIC FIRST INTEGRAL

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Abstract

A few years ago Selivanova gave an existence proof for some integrable models, in fact geodesic flows on two dimensional manifolds, with a cubic first integral. However the explicit form of these models hinged on the solution of a nonlinear third order ordinary differential equation which could not be obtained. We show that an appropriate choice of coordinates allows for integration and gives the explicit local form for the full family of integrable systems. The relevant metrics are described by a finite number of parameters and lead to a large class of models on the manifolds $\mathbb{S}^2$, $\mathbb{H}^2$ and $P^2(\mathbb{R})$ containing as special cases examples due to Goryachev, Chaplygin, Dullin, Matveev and Tsiganov.

1 Introduction

Let M be a n-dimensional smooth manifold with metric $g(X,Y) = g_{ij}\, X^i\, Y^j$ and let $T^*\, M$ be its cotangent bundle with coordinates (x, P), where P is a covector from $T^*_x\, M$. Let us recall that T^*M is a smooth symplectic $2n$-manifold with respect to the standard 2-form $\omega = dP_i \wedge dx^i$ which induces the Poisson bracket

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial P_i} - \frac{\partial f}{\partial P_i} \frac{\partial g}{\partial x^i} \right).$$

In $T^*\, M$ the geodesic flow is defined by the Hamiltonian

$$H = K + V, \qquad K = \frac{1}{2} \sum_{i,j=1}^{n} g^{ij}(x)\, P_i\, P_j, \qquad V = V(x), \tag{1}$$

where g^{ij} is the inverse metric of g_{ij}.

An "observable" $f:\ T^*M \to \mathbb{R}$, which can be written locally

$$f = \sum_{i_1+\cdots+i_n \leq m} f^{i_1,\cdots,i_n}(x)\, P_{i_1}\, \cdots P_{i_n}, \qquad \#(f) = m,$$

is a constant of motion iff $\{H, f\} = 0$. A hamiltonian system is said to be integrable in Liouville sense if there exist n constants of motion (including H) generically independent and in pairwise involution with respect to the Poisson bracket.

In what follows we will deal exclusively with integrable systems defined on two dimensional manifolds. An integrable system is just made out of 2 independent observables H and Q with $\{H, Q\} = 0$.

The general line of attack of this problem is based on the integer $m = \#(Q)$. For $m = 1$ M is a surface of revolution and for $m = 2$ M is a Liouville surface [3].

For higher values of m only particular examples have been obtained, some of which in explicit form. For $M = \mathbb{S}^2$ and $m = 3$ the oldest explicit examples (early twentieth century) were due to Goryachev and Chaplygin on the one hand and to Chaplygin on the other hand (see [2][p. 483] and [9] for the detailed references). On the same manifold with $m = 4$ there is the famous Kovalevskaya system [6] and some extension due to Goryachev (see [8] for the reference).

More recently there was a revival of this subject due to Selivanova [7], [8] and Kiyohara [5] who proved *existence* theorems of integrable systems for $m = 3, 4$ for the first author and for any $m \geq 3$ for the second author. As observed by Kiyohara himself for $m = 3$ the two classes of models are markedly different. Even more recently several new explicit examples for $m = 3$ were given by Dullin and Matveev [4] and Tsiganov [9].

In this work we will focus on Selivanova's integrable systems with a cubic first integral discussed in [7]. The existence theorems she proved are not explicit since there remains to solve a nonlinear ODE of third order. In Tsiganov's article too a non-linear ODE of fourth order four appears for which only special solutions could be obtained.

We will show that the solutions of these ODE are not required: the use of appropriate coordinates allows to get locally the explicit form of the full family of integrable system. Having the local form of the metric g on M one can determine the global structure of the manifold. In view of the many parameters exhibited by the metric, the global analysis gives rise to plenty of integrable models, some of which were discovered only recently.

The plan of the article is the following: in Section 2 we consider the class of models analyzed by Selivanova with the following leading terms for the cubic observable:

$$Q = p\,P_\phi^3 + 2q\,K\,P_\phi + \cdots, \qquad p \in \mathbb{R}, \quad q \geq 0,$$

and the general differential system resulting of $\{H, Q\} = 0$ is given.

In section 3 we first integrate the special case where $q = 0$: the differential system is reduced to a second order non-linear ODE. Its integration gives the local form of the integrable system and the global analysis detrmines the manifolds according to the parameters that appeared in the integration process.

In Section 4 we consider the general case $q > 0$. Here we have linearized, by an appropriate choice of the coordinates, the possibly non-linear ODE of third order. In Section 5, with the explicit local form of the metric, it is then straightforward (but lengthy because an enumeration of cases is required) to determine on which manifolds the metric is defined. We check that all the previously explicitly known integrable examples are indeed recovered.

2 Cubic first integral

Let us consider the hamiltonian (1) with

$$K = \frac{1}{2}\Big(P_\theta^2 + a(\theta)P_\phi^2\Big), \qquad V = f(\theta)\cos\phi + g(\theta), \qquad f(\theta) \not\equiv 0, \tag{2}$$

and the cubic observable

$$Q = Q_3 + Q_1, \tag{3}$$

with

$$\left\{\begin{array}{l} Q_3 = p\,P_\phi^3 + 2q\,K\,P_\phi, \qquad p \in \mathbb{R},\ q \geq 0, \\[2mm] Q_1 = \chi(\theta)\sin\phi\,P_\theta + \Big(\beta(\theta) + \gamma(\theta)\cos\phi\Big)P_\phi. \end{array}\right. \tag{4}$$

Lemma 1 *The constraint $\{H, Q\} = 0$ is equivalent to the following differential system:*

$$\begin{array}{lll} (a) & \chi\,\dot f = \gamma f, \qquad \chi\dot g = \beta\,f, \qquad \Big(\,\dot{} = D_\theta\Big), & \\[2mm] (b) & \dot\chi = -q\,f, \qquad \dot\beta = 2q\,\dot g, \qquad \dot\gamma + \chi\,a = 2q\,\dot f, \qquad a\gamma + \chi\,\dfrac{\dot a}{2} = 3(p + qa)f. & \end{array} \tag{5}$$

Proof: The relation $\{H, Q\} = 0$ splits into three constraints

$$\{K, Q_3\} = 0, \qquad \{K, Q_1\} + \{V, Q_3\} = 0, \qquad \{V, Q_1\} = 0. \tag{6}$$

The first is identically true, the second one is equivalent to the relations (5 b) while the last one is equivalent to (5 a) . $\square$

The special case $q = 0$ is rather difficult to obtain as the limit of the general case $q \neq 0$, so we will first work it out completely.

3 The special case $q = 0$

We can take $p = 1$ and obvious integrations give

$$\chi = \chi_0 > 0, \qquad \beta = \beta_0 \in \mathbb{R}, \qquad \gamma = \chi_0\,\frac{\dot{f}}{f}, \qquad \dot{g} = \frac{\beta_0}{\chi_0}\,f, \qquad a = -\frac{\dot{\gamma}}{\chi_0}, \tag{7}$$

and the last equation

$$\ddot{\gamma} + 2\,\frac{\dot{f}}{f}\,\dot{\gamma} + 6f = 0. \tag{8}$$

An appropriate choice of coordinates does simplify matters:

Lemma 2 *The differential equation for $u = \dot{f}$ as a function of the variable $x = f$ is given by*

$$u\left(\frac{uu'}{x}\right)' + cx = 0, \qquad c = \frac{6}{\chi_0} > 0. \qquad \Big(\,' = D_x\Big). \tag{9}$$

Proof: The relations in (7) become

$$g' = \frac{\beta_0}{\chi_0}\,\frac{x}{u}, \qquad \gamma = \chi_0\,\frac{u}{x}, \qquad a = -u\left(\frac{u}{x}\right)', \tag{10}$$

and (8) gives (9). $\square$

The solution of this ODE follows from

Lemma 3 *The general solution of (9) is given by*

$$u = -\frac{\zeta^2 + c_0}{2c}, \tag{11}$$

with

$$\zeta^3 + 3c_0\,\zeta - 2\rho = 0, \qquad 2(\rho - \rho_0) = 3c^2x^2, \tag{12}$$

and integration constants (ρ_0, c_0).

Proof: Let us define $\zeta' = -c\,x/u$. This allows a first integration of (9), giving $\dfrac{uu'}{x} = \zeta$. From this we deduce

$$cu' = -\zeta\zeta' \qquad \Longrightarrow \qquad 2c\,u = -\zeta^2 - c_0,$$

which in turn implies

$$\Big[\zeta^2 + c_0\Big]\zeta' = 2c^2\,x \qquad \Longrightarrow \qquad \zeta^3 + 3c_0\,\zeta - 2\rho = 0, \tag{13}$$

which concludes the proof. $\square$

It is now clear that the initial coordinates (θ, ϕ) chosen on S^2 will not lead, at least generically, to a simple form of the hamiltonian! To achieve a real simplification for the observables the symplectic coordinates change $(\theta, \phi, P_\theta, P_\phi) \;\to\; (\zeta, \phi, P_\zeta, P_\phi)$ gives:

Theorem 1 *Locally, the integrable system has for explicit form*

$$\left\{\begin{array}{l} H=\frac{1}{2}\left(F\,P_\zeta^2+\frac{G}{4F}\,P_\phi^2\right)+\chi_0\sqrt{F}\,\cos\phi-\beta_0\,\zeta,\\[2mm] Q=P_\phi^3-2\chi_0\Big(\sqrt{F}\,\sin\phi\,P_\zeta+(\sqrt{F}\,)'\,\cos\phi\,P_\phi\Big)+2\beta_0\,P_\phi,\end{array}\right.\qquad \Big(\,'=D_\zeta\Big),\tag{14}$$

with

$$F=-2\rho_0+3c_0\,\zeta+\zeta^3,\qquad\qquad G=9c_0^2+24\rho_0\,\zeta-18c_0\,\zeta^2-3\zeta^4.\tag{15}$$

Proofs: One may obtain these formulas by elementary computations and some scalings of χ_0, β_0 and H.

Alternatively, one can check that (15) implies the relations

$$G'=-12\,F,\qquad G=F'^2-2F\,F'',\tag{16}$$

which allows for a direct check of $\{H,Q\}=0$. As proved in [7] this system does not exhibit any linear or quadratic constant of motion and (H,Q) are algebraically independent. $\square$

We are now in position to analyze the global geometric aspects related to the metric

$$g=\frac{d\zeta^2}{F}+\frac{4F}{G}\,d\phi^2,\qquad\qquad \phi\in[0,2\pi).\tag{17}$$

One has first to impose the positivity of both F and G for this metric to be riemannian. This gives for ζ some interval I whose end-points are possible singularities of the metric. To ascertain that the metric is defined on some manifold one has to ensure that these singularities are apparent ones and not true curvature singularities.

Let us define, for the cubic F, its discriminant $\Delta=c_0^3+\rho_0^2$.

Theorem 2 *The metric (17):*

(i) For $\Delta<0$ is defined on $\mathbb{S}^2$ iff

$$F=(\zeta-\zeta_0)(\zeta-\zeta_1)(\zeta-\zeta_2),\qquad\qquad \zeta_0<\zeta<\zeta_1<\zeta_2.$$

The change of coordinates

$$\mathrm{sn}\,(u,k^2)=\sqrt{\frac{\zeta-\zeta_0}{\zeta_1-\zeta_0}},\qquad k^2=\frac{\zeta_1-\zeta_0}{\zeta_2-\zeta_0}\in\,(0,1),\tag{18}$$

gives for integrable system [1]

$$\left\{\begin{array}{l} H=\frac{1}{2}\left(P_u^2+\frac{D(u)}{s^2c^2d^2}P_\phi^2\right)+\chi_0\,k^2\,scd\,\cos\phi-\beta_0\,k^2s^2,\\[2mm] Q=4P_\phi^3-\chi_0\left(\sin\phi\,P_u+\frac{(scd)'}{scd}\,\cos\phi\,P_\phi\right)+2\beta_0\,P_\phi,\\[2mm] D(u)=(1-k^2s^4)^2-4k^2\,s^4c^2d^2,\qquad u\in(0,K).\end{array}\right.\tag{19}$$

[1]We use the shorthand notation: s, c, d respectively for $\mathrm{sn}\,(u,k^2)$, $\mathrm{cn}\,(u,k^2)$, $\mathrm{dn}\,(u,k^2)$.

(ii) For $\Delta = 0$ is defined on $\mathbb{H}^2$ iff

$$F = (\zeta - \zeta_1)^2(\zeta + 2\zeta_1), \qquad -2\zeta_1 < \zeta < \zeta_1, \quad (\zeta_1 > 0).$$

The change of coordinates

$$\zeta = \zeta_1(-2 + 3\,\tanh^2 u), \qquad u \in (0, +\infty), \tag{20}$$

gives for integrable system [2]

$$\left\{\begin{aligned} & H = M_1^2 + M_2^2 - \left(1 - \frac{3}{C^2}\right) M_3^2 + \chi_0\, T(1 - T^2)\, \cos\phi - \beta_0\, T^2, \\ & Q = 4M_3^3 - \chi_0\Big(M_1 - 3T\, \cos\phi\, M_3\Big) + 2\beta_0\, M_3. \end{aligned}\right. \tag{21}$$

(iii) For $\Delta > 0$ is not defined on a manifold.

Proof of (i): If $\Delta < 0$ the cubic F has three simple real roots $\zeta_0 < \zeta_1 < \zeta_2$. If we take $\zeta \in (\zeta_2, +\infty)$ then F is positive. The relation $G' = -12F$ shows that in this interval G is decreasing from $G(\zeta_2) = F'^2(\zeta_2) > 0$ to $-\infty$ and will vanish for some $\widehat{\zeta} > \zeta_2$. Hence to ensure positivity for F and G we must restrict ζ to the interval $(\zeta_2, \widehat{\zeta})$. Since at $\zeta = \widehat{\zeta}$ the function F does not vanish while G does, this point is a curvature singularity and the metric cannot be defined on a manifold.

The positivity of F is also ensured if we take $\zeta \in (\zeta_0, \zeta_1)$. In this interval G decreases monotonously from $G(\zeta_0)$ to $G(\zeta_1) = F'^2(\zeta_1) > 0$. Let us analyze the singularities at the end points. For ζ close to ζ_0 we have for approximate metric

$$g \approx \frac{4}{F'(\zeta_0)} \left[\frac{d\zeta^2}{4(\zeta - \zeta_0)} + \frac{F'^2(\zeta_0)}{G(\zeta_0)}\,(\zeta - \zeta_0)\, d\phi^2\right]. \tag{22}$$

The relation (16) gives $G(\zeta_0) = F'^2(\zeta_0)$, so the change of variable $\rho = \sqrt{\zeta - \zeta_0}$ allows to write

$$g \approx \frac{4}{F'(\zeta_0)}\Big(d\rho^2 + \rho^2\, d\phi^2\Big), \tag{23}$$

which shows that $\rho = 0$ is an *apparent* singularity, due to the choice of polar coordinates, which could removed by switching back to cartesian coordinates. So the point $\zeta = \zeta_0$ can be added to the manifold.

A similar argument works for $\zeta = \zeta_1$. In fact these end-points are geometrically the poles of the manifold and the index theorem for ∂_ϕ gives for Euler characteristic $\chi = 2$, showing that the manifold is indeed $\mathbb{S}^2$. Then, using the change of variable (18), it is a routine exercise in elliptic functions theory to operate the symplectic coordinates change $(\zeta, \phi, P_\zeta, P_\phi) \;\rightarrow\; (u, \phi, P_u, P_\phi)$ which, after several scalings of the observables and of their parameters, gives (19). Notice that one can also, by direct computation, check that $\{H, Q\} = 0$ from the formulas given in (19). $\square$

Proof of (ii): In this case we have

$$F = (\zeta + 2\zeta_1)(\zeta - \zeta_1)^2, \qquad G = -3(\zeta + 3\zeta_1)(\zeta - \zeta_1)^3, \qquad \zeta_1 = -\rho_0^{1/3}.$$

[2]We use the shorthand notation S, C, T respectively for $\sinh u$, $\cosh u$, $\tanh u$.

For $\zeta_1 < 0$ the positivity of F implies $\zeta \in (2|\zeta_1|, +\infty)$ and G decreases and vanishes for $\widehat{\zeta} = 3|\zeta_1|$ leading to a curvature singularity. The case $\zeta_1 = 0$ is also excluded since then $G \leq 0$ and the remaining case is $\zeta_1 > 0$. The positivity of F and G requires $\zeta \in (-2\zeta_1, \zeta_1)$. The singularity structure is most conveniently discussed thanks to the coordinates change (20) which brings the metric to the form

$$g = \frac{4}{3\zeta_1}\left\{du^2 + \frac{\sinh^2 u}{1 + 3\tanh^2 u}\, d\phi^2\right\}, \qquad u \in (0, +\infty), \tag{24}$$

from which we conclude that the manifold is $\mathbb{H}^2$. Then starting from (14), the symplectic change of coordinates $(\zeta, \phi, P_\zeta, P_\phi) \ \to \ (u, \phi, P_u, P_\phi)$, and some scalings, gives

$$\left\{\begin{array}{l} H = \dfrac{1}{2}\left(P_u^2 + \dfrac{(1 + 3\,T^2)}{S^2}\, P_\phi^2\right) + \chi_0\, T(1 - T^2)\cos\phi - \beta_0\, T^2, \\[3mm] Q = 4P_\phi^3 - \chi_0\left(\sin\phi\, P_u + \dfrac{1 - 3T^2}{T}\, \cos\phi\, P_\phi\right) + 2\beta_0\, P_\phi, \end{array}\right. \tag{25}$$

Defining the generators of the $so(2,1)$ Lie algebra in $T\mathbb{H}^2$ to be

$$M_1 = \sin\phi\, P_u + \frac{\cos\phi}{T}\, P_\phi, \qquad M_2 = \cos\phi\, P_u - \frac{\sin\phi}{T}\, P_\phi, \qquad M_3 = P_\phi, \tag{26}$$

transforms the observables (25) into (21). $\square$

Proof of (iii): For $\Delta > 0$ the cubic F has a single real zero ζ_0. The positivity of F requires that $\zeta \in (\zeta_0, +\infty)$. Since $G' = -12F$ the function G decreases from $G(\zeta_0)$ to $-\infty$. Since $G(\zeta_0) > 0$ there exists $\widehat{\zeta} > \zeta_0$ for which $G(\widehat{\zeta}) = 0$. So positivity restricts $\zeta \in (\zeta_0, \widehat{\zeta})$ and $\widehat{\zeta}$ is a curvature singularity showing that the metric cannot be defined on a manifold. $\square$

Remarks:

1. The integrable system (21) corresponds to the limit of (19) when $\zeta_2 \to \zeta_1$ or $k^2 \to 1$. Then the elliptic functions degenerate into hyperbolic functions. Let us emphasis that in this limit the observables behave smoothly while the manifold changes drastically . Let us also observe that H is globally defined on the manifold while Q is not.

2. In [7] Selivanova proved an existence theorem for an integrable system on S^2 with a cubic observable (case (i) of her Theorem 1.1). The observables are

$$\begin{array}{l} H = \dfrac{\psi'^2(y)}{2}\left(P_y^2 + P_\phi^2\right) + \dfrac{\psi'^2(y)}{2}(\psi(y) - \psi''(y))\, \cos\phi, \\[3mm] Q = P_\phi^3 - \dfrac{3}{2}\,\psi'(y)\, \sin\phi\, P_y + \dfrac{3}{2}\,\psi(y)\, \cos\phi\, P_\phi, \end{array} \qquad \Big(\ ' = D_y\Big), \tag{27}$$

where $\psi(y)$ is a solution of the ODE

$$\psi'\,\psi'' = \psi\,\psi'' - 2\psi''^2 + \psi'^2 + \psi^2, \qquad \psi(0) = 0,\ \psi'(0) = 1,\ \psi''(0) = \tau. \tag{28}$$

Comparing (27) and (14) for $\beta_0 = 0$ makes it obvious that we are dealing with the same integrable system, up to diffeomorphism. The *local* identification follows from

$$\psi(y) = -\frac{(\zeta^2 + c_0)}{2\sqrt{F}}, \qquad \frac{\sqrt{G}}{F}\, d\zeta = \pm\sqrt{3}\, dy, \tag{29}$$

and we have checked that the ODE (28) is a consequence of the relations (29) and (16). We see clearly that Selivanova's choice of the coordinate y led to a complicated ODE, very difficult to solve. In fact one should rather find coordinates such that the ODE becomes tractable, as we did.

4 Local structure of the integrable systems for $q > 0$

As already observed, if one insists in working with the variable θ, the differential system (5) can be reduced either to a third order [7] or to a fourth order [9] non-linear ODE. The key to a full integration of this system is again an appropriate choice of coordinates on the manifold.

Theorem 3 *Locally, the integrable system* (H, Q) *has for explicit form*

$$\left\{\begin{array}{l} H = \dfrac{1}{2\zeta}\Big(F\,P_\zeta^2 + \dfrac{G}{4\,F}\,P_\phi^2\Big) + \dfrac{\sqrt{F}}{2q\zeta}\,\cos\phi + \dfrac{\beta_0}{2q\zeta}, \\[2mm] Q = p\,P_\phi^3 + 2q\,H\,P_\phi - \sqrt{F}\,\sin\phi\,P_\zeta - (\sqrt{F}\,)'\,\cos\phi\,P_\phi, \end{array}\right. \qquad \Big(\ ' = D_\zeta\Big), \tag{30}$$

with the polynomials

$$F = c_0 + c_1\zeta + c_2\zeta^2 + \frac{p}{q}\,\zeta^3, \qquad G = F'^2 - 2F\,F''. \tag{31}$$

Proofs: Starting from (5) the functions β and g are easily determined to be

$$\beta = \frac{\beta_0}{\chi^2}, \qquad g = \frac{\beta_0}{2q\chi^2}. \tag{32}$$

The functions γ and a can be expressed in terms of f and its derivatives with respect to χ as

$$\gamma = -q\chi\,f', \qquad a = -q^2\left(f f'' + \frac{3}{\chi}\,f f'\right). \tag{33}$$

Then the last relation in (5) reduces to a second order *linear* ODE

$$\chi\,(f f')'' + 9\,(f f')' + \frac{15}{\chi}\,f f' = \frac{6p}{q^3}, \tag{34}$$

which is readily integrated to

$$f = \pm\sqrt{c_2 + f_1\,\chi^2 + \frac{c_1}{\chi^2} + \frac{c_0}{\chi^4}}, \qquad f_1 = \frac{p}{4q^3}. \tag{35}$$

The remaining functions become

$$\begin{array}{l} a = \dfrac{q^2}{f^2}\left(\dfrac{c_1^2 - 4c_0c_2}{\chi^6} - \dfrac{12c_0f_1}{\chi^4} - \dfrac{6c_1f_1}{\chi^2} - 4c_2f_1 - 3f_1^2\,\chi^2\right), \\[2mm] \gamma = \dfrac{q}{f}\left(-f_1\chi^2 + \dfrac{c_1}{\chi^2} + \dfrac{2c_0}{\chi^4}\right). \end{array} \tag{36}$$

The observables can be written, up to a scaling of the parameters, in terms of F and G defined by

$$\begin{aligned} &F = 4q^2\,\chi^4\,f^2 = c_0 + c_1\zeta + c_2\zeta^2 + g_1\zeta^3, \qquad g_1 = \frac{p}{q}, \qquad \zeta = \chi^2, \\ &G = 16q^2\,\chi^6\,f^2\,a = c_1^2 - 4c_0c_2 - 12c_0g_1\zeta - 6c_1g_1\zeta^2 - 4c_2g_1\zeta^3 - 3g_1^2\zeta^4. \end{aligned} \tag{37}$$

To simplify matters the symplectic change of coordinates $(\theta, \phi, P_\theta, P_\phi) \;\to\; (\zeta, \phi, P_\zeta, P_\phi)$. gives the required result, up to scalings.

Alternatively (37) implies the relations

$$G' = -12\frac{p}{q}\,F, \qquad G = F'^2 - 2F\,F'', \tag{38}$$

which allow a direct check of $\{H, Q\} = 0$. As proved in [7] this system does not exhibit any other conserved observable linear or quadratic in the momenta, and (H, Q) are algebraically independent. $\square$

Remarks:

1. The limit $q = 0$ is quite tricky: it is why we analyzed it separately in the previous section.

2. Let us observe that the kinetic parts of H in (14) and (30) are conformally related.

3. It is still possible to come back to the coordinate θ but the price to pay is the integration of the relation

$$\sqrt{\frac{\zeta}{F}}\,d\zeta = -d\theta, \tag{39}$$

which can be done using elementary functions for $c_0 = 0$.

5 The global structure

Let us now examine the global geometric aspects of the metric

$$g = \frac{\zeta}{F}\,d\zeta^2 + \frac{4\,\zeta\,F}{G}\,d\phi^2, \qquad \phi \in [0, 2\pi), \tag{40}$$

taking into account the following observations:

1. The positivity constraints are $\zeta F(\zeta) > 0$ and $G(\zeta) > 0$. They define the end-points of some interval I for ζ. In some cases, discussed in detail later on, one can obtain extensions beyond some of the end-points.

2. For the observables to be defined it is required that $F \geq 0 \quad \forall \zeta \in I$.

3. As already observed any point ζ_0 with $F(\zeta_0) \neq 0$ and $G(\zeta_0) = 0$ is a curvature singularity.

4. The point $\zeta = 0$ is a curvature singularity for $F(0) \neq 0$ and $G(0) \neq 0$.

In order to have a complete description of all the possible integrable models, we will present them in three sets:

1. The first set $p = 0$ with a simpler geometric structure.

2. The second set $p > 0$ somewhat similar to the $q = 0$ case.

3. The third set $p < 0$ which displays the richest structure.

5.1 First set of integrable models

Since $p = 0$ we have

$$F = c_0 + c_1\zeta + c_2\zeta^2 = c_2(\zeta - \zeta_1)(\zeta - \zeta_2), \qquad G = c_1^2 - 4c_0c_2, \qquad (c_0,\, c_1,\, c_2) \in \mathbb{R}^3. \quad (41)$$

Theorem 4 *In this set we have the following integrable models:*

(i) Iff $c_2 > 0$ *and* $0 < \zeta_2 < \zeta$ *the metric (40) is defined in* $\mathbb{H}^2$ *and*

$$\begin{cases} H = \dfrac{1}{2}\,\dfrac{M_1^2 + M_2^2 - M_3^2}{\rho + \cosh u} + \dfrac{\alpha\,\sinh u\,\cos\phi + \beta}{\rho + \cosh u}, & u \in (0, +\infty), \\[2ex] Q = H\,M_3 - \alpha\,M_1, \qquad \rho = \dfrac{\zeta_2 + \zeta_1}{\zeta_2 - \zeta_1} \in (-1, +\infty). \end{cases} \qquad (42)$$

(ii) Iff $c_2 < 0$ *and* $0 < \zeta_1 < \zeta < \zeta_2$ *the metric (40) is defined in* $\mathbb{S}^2$ *and*

$$\begin{cases} H = \dfrac{1}{2}\,\dfrac{L_1^2 + L_2^2 + L_3^2}{1 + \rho\cos\theta} + \dfrac{\alpha\,\rho\,\sin\theta\,\cos\phi + \beta}{1 + \rho\cos\theta}, & \theta \in (0, \pi), \\[2ex] Q = H\,L_3 + \alpha\,L_1, \qquad \rho = \dfrac{\zeta_2 - \zeta_1}{\zeta_2 + \zeta_1} \in (0, +1). \end{cases} \qquad (43)$$

(iii) Iff $c_2 = 0$ *the metric (40) is defined in* $\mathbb{R}^2$ *and*

$$\begin{cases} H = \dfrac{1}{2}\,\dfrac{P_x^2 + P_y^2}{1 + \rho^2(x^2 + y^2)} + \dfrac{2\alpha\,\rho^2\,x + \beta}{1 + \rho^2(x^2 + y^2)}, & (x, y) \in \mathbb{R}^2, \\[2ex] Q = H\,L_z - \alpha\,P_y, \qquad \rho > 0. \end{cases} \qquad (44)$$

In all cases α *and* β *are free parameters.*

Proof of (i): The positivity condition $G > 0$ shows that F has two real and distinct roots $\zeta_1 < \zeta_2$, so we will write

$$F = c_2(\zeta - \zeta_1)(\zeta - \zeta_2), \qquad G = c_2^2(\zeta_1 - \zeta_2)^2. \qquad (45)$$

Then imposing the positivity of ζF one has to deal with the iff part of the proof by an enumeration of all possible cases for the triplet $(0, \zeta_1, \zeta_2)$, including the possibility of one ζ_i

being zero. Taking into account the remarks given at the end of Section 4, one concludes that for $c_2 > 0$, we must take $\zeta > \zeta_2 > 0$. The change of coordinates

$$\zeta = \frac{\zeta_2 - \zeta_1}{2}\Big(\rho + \cosh u\Big), \qquad (\zeta_2, +\infty) \ \to \ (0, +\infty), \qquad \rho = \frac{\zeta_2 + \zeta_1}{\zeta_2 - \zeta_1}.$$

brings the metric (40) to the form

$$g = \frac{\zeta_2 - \zeta_1}{2c_2}\Big(\rho + \cosh u\Big)\Big(du^2 + \sinh^2 u\, d\phi^2\Big), \qquad u \in (0, +\infty), \tag{46}$$

which is conformal to the canonical metric on $\mathbb{H}^2$. Using the definitions (26) we obtain (42), up to scalings. $\Box$

Proof of (ii): For $c_2 < 0$ positivity requires either $0 < \zeta_1 < \zeta < \zeta_2$ or $\zeta_1 < \zeta < \zeta_2 < 0$. In both cases the change of coordinates

$$\zeta = \frac{\zeta_1 + \zeta_2}{2}\Big(1 + \rho\, \cos\theta\Big), \qquad (\zeta_1, \zeta_2) \to (\pi, 0), \qquad \rho = \frac{\zeta_2 - \zeta_1}{\zeta_2 + \zeta_1},$$

brings the metric (40) to one and the same form

$$g = \frac{\zeta_1 + \zeta_2}{2c_2}\Big(1 + \rho\, \cos\theta\Big)\Big(d\theta^2 + \sin^2\theta\, d\phi^2\Big), \qquad \theta \in (0, \pi), \tag{47}$$

which is conformal to the canonical metric on $\mathbb{S}^2$ for $\rho \in (0, +1)$. Using the $so(3)$ Lie algebra generators acting in $T^*\mathbb{S}^2$

$$L_1 = \sin\phi\, P_\theta + \frac{\cos\phi}{\tan\theta}\, P_\phi, \qquad L_2 = \cos\phi\, P_\theta - \frac{\sin\phi}{\tan\theta}\, P_\phi, \qquad L_3 = P_\phi, \tag{48}$$

one obtains (43), up to scalings. $\Box$

Proof of (iii): For $c_2 = 0$ we have $G = c_1^2 > 0$.

If $c_1 < 0$ we can write $F = |c_1|(\zeta_1 - \zeta)$ and positivity requires $\zeta \in (0, \zeta_1)$. If $\zeta_1 \neq 0$ then $\zeta = 0$ is a curvature singularity because $F(0)$ and $G(0)$ are not vanishing.

If $c_1 > 0$ we have $F = c_1(\zeta - \zeta_1)$. If $\zeta_1 < 0$ positivity requires either $\zeta > 0$, but $\zeta = 0$ is a curvature singularity, or $\zeta < \zeta_1$ and then F is negative. If $\zeta_1 = 0$ the metric becomes

$$g = \frac{1}{c_1}\Big(d\zeta^2 + 4\zeta^2\, d\phi^2\Big),$$

so to recover flat space we have to take $\widetilde{\phi} = 2\phi \in [0, 2\pi)$ and in H appears a term of the form $\cos(\widetilde{\phi}/2)$ which does not define a function in $\mathbb{R}^2$. Eventually, if $\zeta_1 > 0$ if we take $\zeta < 0$ the point $\zeta = 0$ is singular, so we are left with $\zeta > \zeta_1$. The change of coordinates

$$\zeta = \zeta_1(1 + \rho^2\, r^2), \quad \rho > 0, \qquad x = r\cos\phi, \quad y = r\sin\phi,$$

brings the metric (40) to the form

$$g = \frac{4\zeta_1^2\rho^2}{c_1}(1 + \rho^2\, r^2)(dx^2 + dy^2), \qquad (x,\, y) \in \mathbb{R}^2. \tag{49}$$

Using the $e(3)$ Lie algebra generators $(P_x, P_y, L_z = xP_y - yP_x)$ we obtain (44), up to scalings. $\square$

The remaining cases are given by $p \neq 0$. It is convenient to rescale F by $|p|/q$ and G by p^2/q^2 in order to have

$$F = \epsilon(\zeta^3 + f_0\zeta^2 + c_1\zeta + c_0), \quad \epsilon = \mathrm{sign}(p), \qquad G = F'^2 - 2FF'', \quad G' = -12\,\epsilon\,F, \tag{50}$$

and for the observables, up to scalings

$$\left\{\begin{array}{l} H = \dfrac{1}{2\zeta}\Big(F\,P_\zeta^2 + \dfrac{G}{4\,F}\,P_\phi^2\Big) + \alpha\,\dfrac{\sqrt{F}}{\zeta}\,\cos\phi + \dfrac{\beta}{\zeta}, \\[3mm] Q = \epsilon\,P_\phi^3 + 2\,H\,P_\phi - 2\alpha\Big(\sqrt{F}\,\sin\phi\,P_\zeta + (\sqrt{F}\,)'\,\cos\phi\,P_\phi\Big), \end{array}\right. \tag{51}$$

So the metric is still given by (40). We will denote by Δ_ϵ the discriminant of F according to the sign of ϵ.

5.2 Second set of integrable models

It is given by $p > 0$ or $\epsilon = +1$. We have:

Theorem 5 *The metric (40):*

(i) For $\Delta_+ < 0$ is defined on $\mathbb{S}^2$ iff

$$F = (\zeta - \zeta_0)(\zeta - \zeta_1)(\zeta - \zeta_2), \qquad 0 < \zeta_0 < \zeta < \zeta_1 < \zeta_2.$$

The integrable system, using the notations of Theorem 2 case (i), is

$$\left\{\begin{array}{l} H = \dfrac{1}{2\zeta_+(u)}\left(P_u^2 + \dfrac{D(u)}{s^2c^2d^2}P_\phi^2\right) + \alpha k^2\,\dfrac{scd}{\zeta_+(u)}\,\cos\phi + \dfrac{\beta}{\zeta_+(u)}, \\[3mm] Q = 4P_\phi^3 + 2H\,P_\phi - \alpha\left(\sin\phi\,P_u + \dfrac{(scd)'}{scd}\,\cos\phi\,P_\phi\right), \\[3mm] \zeta_+(u) = \rho + k^2\,\mathrm{sn}^2\,u, \qquad u \in (0, K), \quad \rho = \frac{\zeta_0}{\zeta_2 - \zeta_0} > 0. \end{array}\right. \tag{52}$$

(ii) For $\Delta_+ = 0$ is defined on $\mathbb{H}^2$ iff

$$F = (\zeta - \zeta_0)(\zeta - \zeta_1)^2, \qquad 0 < \zeta_0 < \zeta < \zeta_1.$$

The integrable system, using the notations of Theorem 2 case (ii), is

$$\left\{\begin{array}{l} H = \dfrac{1}{2\zeta_+(u)}\left\{M_1^2 + M_2^2 - \left(1 - \dfrac{3}{C^2}\right)M_3^2\right\} + \alpha\,\dfrac{T(1-T^2)}{\zeta_+(u)}\,\cos\phi + \dfrac{\beta}{\zeta_+(u)}, \\[3mm] Q = 4M_3^3 + 2H\,M_3 - \alpha\Big(M_1 - 3T\,\cos\phi\,M_3\Big), \\[3mm] \zeta_+(u) = \rho + \tanh^2 u, \quad u \in (0, +\infty), \qquad \rho = \frac{\zeta_0}{\zeta_1 - \zeta_0} > 0. \end{array}\right. \tag{53}$$

(iii) For $\Delta_+ > 0$ is not defined on a manifold.

Proof of (i): The iff part results from a case by case examination of all possible orderings of the 4-plet $(0, \zeta_0, \zeta_1, \zeta_2)$, including the possibility of one of the ζ_i being zero. We will not give the full details which can be easily checked by the reader, taking into account the remarks presented at the end of Section 4. The reader can check that with $F = (\zeta-\zeta_0)(\zeta-\zeta_1)(\zeta-\zeta_2)$ and $0 < \zeta_0 < \zeta < \zeta_1 < \zeta_2$, the polynomial F is positive and vanishes at the end-points $(\zeta_0,\, \zeta_1)$ while G is strictly positive. It follows that $\zeta = \zeta_0$ and $\zeta = \zeta_1$ are poles and the manifold is $\mathbb{S}^2$. Operating the same coordinate change as in Theorem 2, case (i), one obtains (52). $\square$

Proof of (ii): The polynomial G becomes $G = (\zeta_1 - \zeta)^3(3\zeta + \zeta_1 - 4\zeta_0)$. The change of variable

$$\zeta = (\zeta_1 - \zeta_0)(\rho + \mathrm{th}^2 u), \qquad (\zeta_0, \zeta_1) \to (0, +\infty), \qquad \rho = \frac{\zeta_0}{\zeta_1 - \zeta_0} > 0,$$

transforms the observables, up to scalings, into

$$\left\{\begin{array}{l} H = \dfrac{1}{2\zeta_+(u)}\left(P_u^2 + \dfrac{1+3T^2}{S^2}\,P_\phi^2\right) + \dfrac{\alpha}{\zeta_+(u)}\,T(1-T^2)\,\cos\phi + \dfrac{\beta}{\zeta_+(u)}, \\[3mm] Q = 4P_\phi^3 + 2H\,P_\phi - \alpha\,\sin\phi\,P_u - \alpha\,\dfrac{(1-3T^2)}{T}\,\cos\phi\,P_\phi, \\[3mm] \zeta_+(u) = \rho + \tanh^2 u. \end{array}\right. \tag{54}$$

Using the relations (26) one gets (53). $\square$

Proof of (iii): Examining all the possible cases gives no manifold for the metric. $\square$

5.3 Third set of integrable models

It is given by $p < 0$ or $\epsilon = -1$. It displays a richer structure and for clarity we will split up the description of the integrable systems into several theorems.

Theorem 6 *The metric (40) for $\Delta_- < 0$ is defined on $\mathbb{S}^2$ iff:*

(i) either $F = (\zeta - \zeta_0)(\zeta - \zeta_1)(\zeta_2 - \zeta), \quad \zeta_0 < \zeta_1 < \zeta < \zeta_2 \ (\zeta_1 > 0).$

The change of coordinates

$$\mathrm{sn}\,(u, k^2) = \sqrt{\frac{\zeta_2 - \zeta}{\zeta_2 - \zeta_1}}, \qquad k^2 = \frac{\zeta_2 - \zeta_1}{\zeta_2 - \zeta_0} \in\ (0,1), \tag{55}$$

gives for integrable system

$$\left\{\begin{array}{l} H = \dfrac{1}{2\zeta_-(u)}\left(P_u^2 + \dfrac{D(u)}{s^2c^2d^2}\,P_\phi^2\right) + \alpha\dfrac{k^2\,scd}{\zeta_-(u)}\,\cos\phi + \dfrac{\beta}{\zeta_-(u)}, \\[3mm] Q = -4\,P_\phi^3 + 2H\,P_\phi + \alpha\left(\sin\phi\,P_u + \dfrac{(scd)'}{scd}\,\cos\phi\,P_\phi\right), \\[3mm] \zeta_-(u) = k^2\Big(\rho - \mathrm{sn}^2\,u\Big), \qquad u \in (0, K), \qquad \rho = \frac{\zeta_2}{\zeta_2-\zeta_1} > 1. \end{array}\right. \tag{56}$$

(ii) or $F = (\zeta_0 - \zeta)(\zeta - \zeta_1)(\zeta - \zeta_2)$ *and* $G(0) = 0, \quad 0 < \zeta < \zeta_0 < \zeta_1 < \zeta_2.$
The integrable system is

$$\left\{\begin{array}{l} H = \dfrac{1}{2}\, f\,(L_1^2 + L_2^2) + \dfrac{1}{2}\left(\dfrac{h}{3f} - \cos^2\theta\, f\right)\dfrac{L_3^2}{\sin^2\theta} + \\[2ex] \qquad\qquad\qquad\qquad + \alpha\,\dfrac{\sin\theta\,\sqrt{f}}{(\cos^2\theta)^{1/3}}\,\cos\phi + \dfrac{\beta}{(\cos^2\theta)^{1/3}}, \\[2ex] Q = -\dfrac{4}{9}\,L_3^3 + 2H\,L_3 + 3\alpha\,(\cos\theta)^{1/3}\Big(\sqrt{f}\,L_1 + (\sqrt{f})'\,\cos\phi\,L_3\Big), \end{array}\right. \tag{57}$$

where $f(\theta) = \hat{f}(\cos\theta)$ *with*

$$\hat{f}(\mu) = \frac{\left(\mu^{2/3} - \frac{\zeta_1}{\zeta_0}\right)\left(\mu^{2/3} - \frac{\zeta_2}{\zeta_0}\right)}{\mu^{4/3} + \mu^{2/3} + 1}, \qquad \mu \in (-1, +1), \tag{58}$$

and $h(\theta) = \hat{h}(\cos\theta)$ *with*

$$\hat{h}(\mu) = -\mu^2 + \frac{4}{3}\Big(1 + \frac{\zeta_1 + \zeta_2}{\zeta_0}\Big)\mu^{4/3} - 2\Big(\frac{\zeta_1 + \zeta_2}{\zeta_0} + \frac{\zeta_1\zeta_2}{\zeta_0^2}\Big)\mu^{2/3} + 4\frac{\zeta_1\zeta_2}{\zeta_0^2}. \tag{59}$$

The parameter ζ_0 *is:*

$$\zeta_0 = \frac{\zeta_1\zeta_2}{(\sqrt{\zeta_1} + \sqrt{\zeta_2})^2} < \zeta_1. \tag{60}$$

Proof of (i): The change of variable indicated gives (56) by lengthy but straightforward computations. $\square$
Remark: The previous analysis does not describe appropriately the special case $\zeta_0 = 0$ for which elliptic functions are no longer required. In this case the coordinates change

$$\zeta = \frac{\zeta_1 + \zeta_2}{2} - \frac{\zeta_1 - \zeta_2}{2}\,\cos\theta, \qquad (\zeta_1, \zeta_2) \;\rightarrow\; (\pi, 0),$$

gives for the metric

$$g = d\theta^2 + \frac{\sin^2\theta}{1 + \sin^2\theta\, G(\cos\theta)}\, d\phi^2, \tag{61}$$

with

$$G(\mu) = \frac{3\mu^2 + 4\rho\mu + 1}{4(\rho + \mu)^2}, \qquad \rho = \frac{\zeta_2 + \zeta_1}{\zeta_2 - \zeta_1} > 1. \tag{62}$$

The integrable system is

$$\left\{\begin{array}{l} H = \dfrac{1}{2}\left(P_\theta^2 + \Big(\dfrac{1}{\sin^2\theta} + G(\cos\theta)\Big)P_\phi^2\right) + \alpha\,\dfrac{\sin\theta}{\sqrt{U}}\,\cos\phi + \dfrac{\beta}{U}, \\[2ex] Q = -P_\phi^3 + 2H\,P_\phi + 2\alpha\,\sqrt{U}\sin\phi\,P_\theta + 2\alpha\dfrac{(\sin\theta\sqrt{U})'}{\sin\theta}\,\cos\phi\,P_\phi., \\[2ex] U = \rho + \cos\theta, \end{array}\right. \tag{63}$$

on which we recognize the Dullin-Matveev system [4].

Proof of (ii): One has

$$G(0)=(\zeta_1-\zeta_2)^2\,\zeta_0^2-2\zeta_1\zeta_2(\zeta_1+\zeta_2)\,\zeta_0+\zeta_1^2\zeta_2^2.$$

Its vanishing determines uniquely ζ_0 in terms of (ζ_1,ζ_2) as given by (60). At this stage positivity requires $\zeta\in(0,\zeta_0)$. Let us make the change of variable $\zeta=\zeta_0\,\mu^{2/3}$. The metric becomes

$$g=\frac{4}{9}\left\{\frac{d\mu^2}{(1-\mu^2)\hat{f}(\mu)}+3(1-\mu^2)\frac{\hat{f}(\mu)}{\hat{h}(\mu)}\,d\phi^2\right\},\qquad \mu\in(0,1).$$

All the functions in the metric are *even* functions of μ: we can therefore take $\mu\in(-1,+1)$ extending the metric beyond $\mu=0$. One can check that the points $\mu=\pm1$ are poles and therefore we get again for manifold $\mathbb{S}^2$. The change of variable $\mu=\cos\theta$ with $\theta\in(0,\pi)$ gives then for result (57). □

Let us proceed to:

Theorem 7 *(a) The metric (40) for $\Delta_-=0$ is defined on $\mathbb{S}^2$ iff:*

(i) either $F=\zeta^2(\zeta_0-\zeta),\qquad 0<\zeta<\zeta_0,$ *and we have*

$$\left\{\begin{aligned}&H=\frac{1}{2}\Big(L_1^2+L_2^2+4L_3^2\Big)+\alpha\,\sin\theta\,\cos\phi+\frac{\beta}{\cos^2\theta},\qquad \theta\in(0,\pi),\\ &Q=-4\,L_3^3+2H\,L_3+\alpha\Big(\cos\theta\,L_1-2\sin\theta\,\cos\phi\,L_3\Big),\end{aligned}\right.\tag{64}$$

which is the Goryachev-Chaplygin top.

(ii) or $F=(\zeta-\zeta_1)^2(\zeta_0-\zeta)$ *and* $G(0)=0,\qquad 0<\zeta<\zeta_0.$
The integrable system is of the form (57) with the functions

$$\hat{f}(\mu)=\frac{\left(4-\mu^{2/3}\right)^2}{\mu^{4/3}+\mu^{2/3}+1},\qquad \hat{h}(\mu)=(4-\mu^{2/3})^3,\qquad \mu\in(-1,+1).\tag{65}$$

(b) The metric (40) for $\Delta_-=0$ is defined on $\mathbb{H}^2$ iff:

$$F=(\zeta-\zeta_1)^2(\zeta_0-\zeta),\qquad 0<\zeta_1<\zeta<\zeta_0.$$

The integrable system, in the notations of Theorem 2, case (ii), is

$$\left\{\begin{aligned}&H=\frac{1}{2\zeta_-(u)}\left\{M_1^2+M_2^2-\left(1-\frac{3}{C^2}\right)M_3^2\right\}+\alpha\,\frac{T(1-T^2)}{\zeta_-(u)}\,\cos\phi+\frac{\beta}{\zeta_-(u)},\\ &Q=-4M_3^3+2H\,M_3+\alpha\Big(M_1-3T\,\cos\phi\,M_3\Big),\\ &\zeta_-(u)=\rho-\tanh^2u,\quad u\in(0,+\infty),\qquad \rho=\tfrac{\zeta_0}{\zeta_0-\zeta_1}>1.\end{aligned}\right.\tag{66}$$

Proof of (a)(i): We have $F=\zeta^2(\zeta_0-\zeta)$ and $G=\zeta^3(4\zeta_0-3\zeta)$ and $\zeta\in(0,\zeta_0)$ from positivity. Taking for new variable θ such that $\zeta=\zeta_0\,\cos^2\theta$ we get for the metric

$$g=4\left(d\theta^2+\frac{\sin^2\theta}{1+3\sin^2\theta}\,d\phi^2\right),\qquad \theta\in(0,\pi/2).\tag{67}$$

As it stands the manifold is $P^2(\mathbb{R})$ (see [1][p. 268]). However we can also extend the metric taking $\theta \in (0,\pi)$: then the manifold extends to $\mathbb{S}^2$ since $\theta = 0$ and $\theta = \pi$ are poles and in this case we recover Goryachev-Chaplygin top. The observables can be transformed into (64). $\square$

Proof of (a)(ii): In this case we have $G(0) = \zeta_1^3(\zeta_1 - 4\zeta_0)$ which fixes $\zeta_0 = \zeta_1/4$. The argument then proceeds as in the proof of Theorem 6, case (ii). $\square$

Proof of (b): The proof is identical to the one for Theorem 5, case (ii), except for the change of coordinates, which is now

$$\zeta = \zeta_0 - (\zeta_0 - \zeta_1)\tanh^2 u : \qquad (\zeta_0, \zeta_1) \quad \to \quad (0, +\infty).$$

One gets (66) by similar arguments. $\square$

Theorem 8 *The metric (40) for $\Delta_- > 0$ is defined on $\mathbb{S}^2$ iff:*

$$F = (\zeta_0 - \zeta)(\zeta - \zeta_1)(\zeta - \overline{\zeta_1}) \quad \textit{and} \quad G(0) = 0, \quad 0 < \zeta < \zeta_0.$$

The integrable system is of the form (57) with the functions

$$\hat{f}(\mu) = \frac{\left(\mu^{2/3} - \frac{\zeta_1}{\zeta_0}\right)\left(\mu^{2/3} - \frac{\overline{\zeta_1}}{\zeta_0}\right)}{\mu^{4/3} + \mu^{2/3} + 1}, \qquad \mu \in (-1, +1), \tag{68}$$

and

$$\hat{h}(\mu) = -\mu^2 + \frac{4}{3}\Big(1 + \frac{\zeta_1 + \overline{\zeta_1}}{\zeta_0}\Big)\mu^{4/3} - 2\Big(\frac{\zeta_1 + \overline{\zeta_1}}{\zeta_0} + \frac{|\zeta_1|^2}{\zeta_0^2}\Big)\mu^{2/3} + 4\frac{|\zeta_1|^2}{\zeta_0^2}. \tag{69}$$

We have two possible values for ζ_0 which are

$$\zeta_0 = \frac{|\zeta_1|^2}{\zeta_1 + \overline{\zeta_1} \pm 2|\zeta_1|}. \tag{70}$$

Proof: We have

$$G(0) = (\zeta_1 - \overline{\zeta_1})^2\,\zeta_0^2 - 2(\zeta_1 + \overline{\zeta_1})|\zeta_1|^2\,\zeta_0 + |\zeta_1|^4.$$

Its vanishing gives for ζ_0 the roots (70). The subsequent analysis is identical to that already given in the proof of Theorem 6, case (ii). $\square$

It is interesting to examine the explicitly known integrable systems, with a metric defined in $\mathbb{S}^2$ and with a cubic observable already given in the literature:

1. The Goryachev-Chaplygin top given by Theorem 6, case (ii).
2. The Dullin-Matveev top [4] is given in the remark after Theorem 6.
3. If we restrict, in Theorem 8, the parameters according to

$$\zeta_0 = -(\zeta_1 + \overline{\zeta_1}) \quad \text{and} \quad \zeta_0^2 = |\zeta_1|^2, \qquad \Longrightarrow \qquad f = 1, \quad g = 4 - \mu^3,$$

we recover the Goryachev top

$$\begin{cases} H = \dfrac{1}{2}\left(L_1^2 + L_2^2 + \dfrac{4}{3}\,L_3^2\right) + \alpha\,\dfrac{\sin\theta}{(\cos^2\theta)^{1/3}}\,\cos\phi + \dfrac{\beta}{(\cos^2\theta)^{1/3}}, \\[2ex] Q = -\dfrac{4}{9}\,P_\phi^3 + 2H\,P_\phi + 3\alpha\,(\cos\theta)^{1/3}\,L_1. \end{cases} \tag{71}$$

The two new examples given by Tsiganov in [9] are not defined on a manifold.
Remarks:

1. All of the previous examples belong to the third set with $p < 0$.

2. Considering the genus of the algebraic curve $y^2 = \dfrac{F(\zeta)}{\zeta}$ let us observe that the Goryachev-Chaplygin and Dullin-Matveev systems have zero genus while the Goryachev system has genus one.

3. In general the potential V as well as the observable Q are not defined on the whole manifold.

6 Conclusion

We have exhaustively constructed all the integrable models, on two dimensional manifolds, characterized by the following form of the observables

$$\left\{\begin{array}{l} H = \frac{1}{2}\Big(P_\theta^2 + a(\theta)P_\phi^2\Big) + f(\theta)\,\cos\phi + g(\theta) \\ Q = p\,P_\phi^3 + q\,\Big(P_\theta^2 + a(\theta)P_\phi^2\Big)P_\phi + \chi(\theta)\sin\phi\,P_\theta + \Big(\beta(\theta) + \gamma(\theta)\cos\phi\Big)P_\phi \end{array}\right. \tag{72}$$

The main lesson from the failure of [7] to solve the problem has to do with the crucial role of the coordinates choice, which determines the structure of the ODE to be solved eventually. This is a familiar phenomenon to people dealing with Einstein equations: despite their diffeomorphism invariance, finding exact solutions relies on an adapted choice of coordinates which can simplify, or even linearize the differential system to be integrated.

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