An Effective Extension of the Wagner Hierarchy to Blind Counter Automata

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Abstract. The extension of the Wagner hierarchy to blind counter automata accepting infinite words with a Muller acceptance condition is effective. We determine precisely this hierarchy.

1   Introduction

Regular ω-languages are accepted by (deterministic) Muller automata. Finite machines having a stronger expressive power when reading infinite words have also been investigated [Sta97a]. Recently Engelfriet and Hoogeboom studied **X**-automata, i.e. automata equipped with a storage type **X**, including the cases of pushdown automata, Turing machines, Petri nets [EH93]. A way to investigate the expressive power of such machines is to study the topological complexity of the ω-languages they accept. For deterministic machines, it is shown in [EH93] that every **X**-automaton accepts boolean combinations of $\mathbf{\Pi}_2^0$-sets. Hence in order to distinguish the different storage types it turned out that the study of the Wadge hierarchy is suitable. The Wadge hierarchy is a great refinement of the Borel hierarchy, recently studied by Duparc [Dup99a]. The Wadge hierarchy of ω-regular languages has been determined in an efective way by Wagner [Wag79]. Several extensions of this hierarchy have been recently determined as the extension to deterministic pushdown automata, to k-blind counter automata, [DFR01] [Dup99b] [Fin00b]. We present here the extension to (one) blind counter automata, which is the first known **effective** extension. We study Muller blind counter automata (MBCA), and define chains and superchains as Wagner did for Muller automata. The essential difference between the two hierarchies relies on the existence of superchains of transfinite length $\alpha < \omega^2$ for MBCA. The hierarchy is effective and leads to effective winning strategies in Wadge games between MBCA. The hierarchy of Muller automata equipped with several blind counters is presented in a non effective way in [Fin00b][DFR01].

2 Regular and Blind Counter ω-languages

We assume the reader to be familiar with the theory of formal languages and of ω-regular languages, see for example [HU69] ,[Tho90]. We first recall some definitions and results concerning ω-regular languages and omega pushdown automata and introduce blind counter automata as a special case of pushdown automata [Tho90] [Sta97a].

When Σ is a finite alphabet, a finite string (word) over Σ is any sequence $x = x_1 \ldots x_k$, where $x_i \in \Sigma$ for $i = 1, \ldots, k$,and k is an integer ≥ 1. The length of x is k, denoted by $|x|$. If $|x| = 0$, x is the empty word denoted by λ.

we write $x(i) = x_i$ and $x[i] = x(1) \ldots x(i)$ for $i \leq k$ and $x[0] = \lambda$. $\Sigma^\star$ is the set of finite words over Σ. The first infinite ordinal is ω. An ω-word over Σ is an ω -sequence $a_1 \ldots a_n \ldots$, where $a_i \in \Sigma, \forall i \geq 1$. When σ is an ω-word over Σ, we write $\sigma = \sigma(1)\sigma(2) \ldots \sigma(n) \ldots$ and $\sigma[n] = \sigma(1)\sigma(2) \ldots \sigma(n)$ the finite word of length n, prefix of σ. The set of ω-words over the alphabet Σ is denoted by Σ^ω. An ω-language over an alphabet Σ is a subset of Σ^ω.

The usual concatenation product of two finite words u and v is denoted $u.v$ (and sometimes just uv). This product is extended to the product $u.v$ of a finite word u and an ω-word v.

For $V \subseteq \Sigma^\star$, $V^\omega = \{\sigma = u_1 \ldots u_n \ldots \in \Sigma^\omega / u_i \in V, \forall i \geq 1\}$ is the ω-power of V.

R. Mc Naughton established that the expressive power of deterministic Muller automata (DMA) is equal to the expressive power of non deterministic Muller automata (MA) [Tho90]. An ω-language is regular iff it is accepted by a Muller automaton. The class REG_ω of ω-regular languages is the ω-Kleene closure of the class REG of (finitary) regular languages where the ω-Kleene closure of a family L of finitary languages is :

$$\omega - KC(L) = \{\cup_{i=1}^n U_i.V_i^\omega / U_i, V_i \in L, \forall i \in [1, n]\}$$

We now define the (blind) one counter machines which we assume here to be realtime and deterministic, and the corresponding classes of blind counter ω-languages.

Definition 1. *A (realtime deterministic) pushdown machine (PDM) is a 6-tuple $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$, where K is a finite set of states, Σ is a finite input alphabet, Γ is the finite pushdown alphabet, $q_0 \in K$ is the initial state, $Z_0 \in \Gamma$ is the start symbol, and δ is a mapping from $K \times \Sigma \times \Gamma$ into $K \times \Gamma^\star$.*
If $\gamma \in \Gamma^+$ describes the pushdown store content, the leftmost symbol will be assumed to be on " top" of the store. A configuration of a PDM is a pair (q, γ) where $q \in K$ and $\gamma \in \Gamma^\star$.
For $a \in \Sigma$, $\gamma, \beta \in \Gamma^\star$ and $Z \in \Gamma$, if (p, β) is in $\delta(q, a, Z)$, then we write $a : (q, Z\gamma) \mapsto_M (p, \beta\gamma)$.
$\mapsto_M^\star$ is the transitive and reflexive closure of $\mapsto_M$. (The subscript M will be

omitted whenever the meaning remains clear).

Let $\sigma = a_1 a_2 \ldots a_n \ldots$ be an ω-word over Σ. An infinite sequence of configurations $r = (q_i, \gamma_i)_{i \geq 1}$ is called a run of M on σ, starting in configuration (p, γ), iff:

1. *$(q_1, \gamma_1) = (p, \gamma)$*
2. *for each $i \geq 1$, $a_i : (q_i, \gamma_i) \mapsto_M (q_{i+1}, \gamma_{i+1})$*

For every such run, $In(r)$ is the set of all states entered infinitely often during run r.

A run r of M on σ, starting in configuration (q_0, Z_0), will be simply called " a run of M on σ ".

A one counter machine is a PDM such that $\Gamma = \{Z_0, I\}$ where Z_0 is the bottom symbol and always remains at the bottom of the store. So the pushdown store is used like a counter whose value is the integer n if the content of the pushdown store is $I^n Z_0$.

A one blind counter machine is a one counter machine such that every transition which is enabled at zero level is also enabled at non zero level, i.e. if $\delta(q, a, Z_0) = (p, I^n Z_0)$, for some $p, q \in K$, $a \in \Sigma$ and $n \geq 0$, then $\delta(q, a, I) = (p, I^{n+1})$. But the converse may not be true, i.e. some transition may be enabled at non zero level but not at zero level.

Definition 2. *A Muller (realtime deterministic) blind counter automaton (**MBCA**) is a 7-tuple $\mathcal{A} = (K, \Sigma, \Gamma, \delta, q_0, Z_0, \mathcal{F})$ where $\mathcal{A}' = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$ is a (realtime deterministic) one blind counter machine and $\mathcal{F} \subseteq 2^K$ is the collection of designated state sets.*

The ω-language accepted by M is $L(\mathcal{A}) = \{\sigma \in \Sigma^\omega \ / \ \text{there exists a run } r \text{ of } \mathcal{A} \text{ on } \sigma \text{ such that } In(r) \in \mathcal{F}\}$.

*The class of ω-languages accepted by **MBCA** will be denoted **BC**.*

Remark 3. *Machines we call here one blind counter machines are sometimes called one partially blind counter machines as in [Gre78].*

Remark 4. *If M is a deterministic pushdown machine , then for every $\sigma \in \Sigma^\omega$, there exists at most one run r of M on σ determined by the starting configuration. Each ω-language accepted by a Muller deterministic pushdown automaton (DMPDA) can be accepted by a DMPDA such that for every $\sigma \in \Sigma^\omega$, there exists such a run of M on σ.*

*But this is not true for **MBCA** because some words x may be rejected by an MBCA $\mathcal{A}$ because the machine $\mathcal{A}$ blocks at zero level when reading x. This is connected with the fact that the class **BC** is not closed under complementation as it is shown by the following example.*

Example 5. *It is easy to see that the ω-language $L = \{a^n b^p c^\omega \ / \ p \leq n\}$ is accepted by a deterministic MBCA, but its complement is not accepted by any deterministic MBCA because $L' = \{a^n b^p c^\omega \ / \ p > n\}$ is not accepted by any deterministic MBCA.*

3   Topology

We assume the reader to be familiar with basic notions of topology which may be found in [Kur66][LT94] [Sta97a] [PP98].

Topology is an important tool for the study of ω-languages, and leads to characterization of several classes of ω-languages.
For a finite alphabet X, we consider X^ω as a topological space with the Cantor topology (see [LT94] [Sta97a] [PP98]). The open sets of X^ω are the sets in the form $W.X^\omega$, where $W \subseteq X^\star$. A set $L \subseteq X^\omega$ is a closed set iff its complement $X^\omega - L$ is an open set. The class of open sets of X^ω will be denoted by $\mathbf{G}$ or by $\mathbf{\Sigma^0_1}$. The class of closed sets will be denoted by $\mathbf{F}$ or by $\mathbf{\Pi^0_1}$. Closed sets are characterized by the following:

Proposition 6. *A set $L \subseteq X^\omega$ is a closed set of X^ω iff for every $\sigma \in X^\omega$,*
$[\forall n \geq 1, \exists u \in X^\omega$ such that $\sigma(1)\ldots\sigma(n).u \in L]$ implies that $\sigma \in L$.

Define now the next classes of the Hierarchy of Borel sets of finite rank:

Definition 7. *The classes $\mathbf{\Sigma^0_n}$ and $\mathbf{\Pi^0_n}$ of the Borel Hierarchy on the topological space X^ω are defined as follows:*
$\mathbf{\Sigma^0_1}$ is the class of open sets of X^ω.
$\mathbf{\Pi^0_1}$ is the class of closed sets of X^ω.
$\mathbf{\Pi^0_2}$ or $\mathbf{G_\delta}$ is the class of countable intersections of open sets of X^ω.
$\mathbf{\Sigma^0_2}$ or $\mathbf{F_\sigma}$ is the class of countable unions of closed sets of X^ω.
And for any integer $n \geq 1$:
$\mathbf{\Sigma^0_{n+1}}$ is the class of countable unions of $\mathbf{\Pi^0_n}$-subsets of X^ω.
$\mathbf{\Pi^0_{n+1}}$ is the class of countable intersections of $\mathbf{\Sigma^0_n}$-subsets of X^ω.

There is a nice characterization of $\mathbf{\Pi^0_2}$-subsets of X^ω. First define the notion of W^δ:

Definition 8. *For $W \subseteq X^\star$, let:*
$W^\delta = \{\sigma \in X^\omega / \exists^\omega i$ such that $\sigma[i] \in W\}$.
($\sigma \in W^\delta$ iff σ has infinitely many prefixes in W).

Then we can state the following Proposition:

Proposition 9. *A subset L of X^ω is a $\mathbf{\Pi^0_2}$-subset of X^ω iff there exists a set $W \subseteq X^\star$ such that $L = W^\delta$.*

Mc Naughton's Theorem implies that every ω-regular language is a boolean combination of G_δ-sets, hence a $\mathbf{\Delta^0_3} = (\mathbf{\Pi^0_3} \cap \mathbf{\Sigma^0_3})$-set. This result holds in fact for every ω-language accepted by a deterministic $\mathbf{X}$-automaton in the sense of [EH93], i.e. an automaton equipped with a storage type $\mathbf{X}$, including the case of the Turing machine. A way to distinguish the expressive power of finite machines reading ω-words is the Wadge hierarchy which we now introduce.

Definition 10. *For $E \subseteq X^\omega$ and $F \subseteq Y^\omega$, E is said to be Wadge reducible to F ($E \leq_W F$) iff there exists a continuous function $f : X^\omega \to Y^\omega$, such that $E = f^{-1}(F)$.*
E and F are Wadge equivalent iff $E \leq_W F$ and $F \leq_W E$. This will be denoted by $E \equiv_W F$. And we shall say that $E <_W F$ iff $E \leq_W F$ but not $F \leq_W E$.
A set $E \subseteq X^\omega$ is said to be self dual iff $E \equiv_W (X^\omega - E)$, and otherwise it is said to be non self dual.

The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation. The equivalence classes of $\equiv_W$ are called wadge degrees.
WH is the class of Borel subsets of finite rank of a set X^ω, where X is a finite set, equipped with $\leq_W$ and with $\equiv_W$.
For $E \subseteq X^\omega$ and $F \subseteq Y^\omega$, if $E \leq_W F$ and $E = f^{-1}(F)$ where f is a continuous function from X^ω into Y^ω, then f is called a continuous reduction of E to F. Intuitively it means that E is less complicated than F because to check whether $x \in E$ it suffices to check whether $f(x) \in F$ where f is a continuous function. Hence the Wadge degree of an ω-language is a measure of its topological complexity.

Remark 11. *In the above definition, we consider that a subset $E \subseteq X^\omega$ is given together with the alphabet X. This is necessary as it is shown by the following example.*
Let $E = \{0,1\}^\omega$ considered as an ω-language over the alphabet $X = \{0,1\}$ and let $F = \{0,1\}^\omega$ be the same ω-language considered as an ω-language over the alphabet $Y = \{0,1,2\}$. Then E is an open and closed subset of $\{0,1\}^\omega$ but F is a closed and non open subset of $\{0,1,2\}^\omega$. It is easy to check that $E <_W F$ hence E and F are not Wadge equivalent.

Then we can define the Wadge class of a set F:

Definition 12. *Let F be a subset of X^ω. The wadge class of F is $[F]$ defined by: $[F] = \{E / E \subseteq Y^\omega$ for a finite alphabet Y and $E \leq_W F\}$.*

Recall that each Borel class $\mathbf{\Sigma_n^0}$ and $\mathbf{\Pi_n^0}$ is a Wadge class.

There is a close relationship between Wadge reducibility and games which we now introduce. Define first the Wadge game $W(A, B)$ for $A \subseteq X_A^\omega$ and $B \subseteq X_B^\omega$:

Definition 13. *The Wadge game $W(A, B)$ is a game with perfect information between two players, player 1 who is in charge of A and player 2 who is in charge of B.*
Player 1 first writes a letter $a_1 \in X_A$, then player 2 writes a letter $b_1 \in X_B$, then player 1 writes a letter $a_2 \in X_A$, and so on . . .
The two players alternatively write letters a_n of X_A for player 1 and b_n of X_B for player 2.
After ω steps, the player 1 has written an ω-word $a \in X_A^\omega$ and the player 2 has written an ω-word $b \in X_B^\omega$.
The player 2 is allowed to skip, even infinitely often, provided he really write an

ω-word in ω steps.
The player 2 wins the play iff $[a \in A \leftrightarrow b \in B]$, i.e. iff
$[(a \in A$ and $b \in B)$ or $(a \notin A$ and $b \notin B$ and b is infinite$)]$.

Recall that a strategy for player 1 is a function $\sigma : (X_B \cup \{s\})^\star \to X_A$. And a strategy for player 2 is a function $f : X_A^+ \to X_B \cup \{s\}$.
σ is a winning stategy (w.s.) for player 1 iff he always wins a play when he uses the strategy σ, i.e. when the nth letter he writes is given by $a_n = \sigma(b_1 \ldots b_{n-1})$, where b_i is the letter written by player 2 at step i and $b_i = s$ if player 2 skips at step i.
A winning strategy for player 2 is defined in a similar manner.

Martin's Theorem states that every Gale-Stewart Game $G(X)$ (see [Tho90] [PP98] for more details), with X a borel set, is determined and this implies the following :

Theorem 14 (Wadge). *Let $A \subseteq X_A^\omega$ and $B \subseteq X_B^\omega$ be two Borel sets, where X_A and X_B are finite alphabets. Then the Wadge game $W(A, B)$ is determined: one of the two players has a winning strategy. And $A \leq_W B$ iff the player 2 has a winning strategy in the game $W(A, B)$.*

Recall that a set X is well ordered by a binary relation $<$ iff $<$ is a linear order on X and there is not any strictly decreasing (for $<$) infinite sequence of elements in X.

Theorem 15 (Wadge). *Up to the complement and $\equiv_W$, the class of Borel subsets of finite rank of X^ω, for X a finite alphabet, is a well ordered hierarchy. There is an ordinal $|WH|$, called the length of the hierarchy, and a map d_W^0 from WH onto $|WH|$, such that for all $A, B \in WH$:*
$d_W^0 A < d_W^0 B \leftrightarrow A <_W B$ and
$d_W^0 A = d_W^0 B \leftrightarrow [A \equiv_W B$ or $A \equiv_W B^-]$.

Remark 16. *We do not give here the ordinal $|WH|$. Details may be found in [Dup99a].*

4   Wagner Hierarchy and its Extension to Blind Counter Automata

Consider now ω-regular languages. Landweber studied first the topological properties of ω-regular languages. He characterized the ω-regular languages in each of the Borel classes $\mathbf{F}, \mathbf{G}, \mathbf{F}_\sigma, \mathbf{G}_\delta$, and showed that one can decide, for an effectively given ω-regular language L, whether L is in $\mathbf{F}, \mathbf{G}, \mathbf{F}_\sigma$, or $\mathbf{G}_\delta$.
It turned out that an ω-regular language is in the class $\mathbf{G}_\delta$ iff it is accepted by a deterministic Büchi automaton. These results were refined by K. Wagner who studied the Wadge Hierarchy of ω-regular languages. In fact there is an effective version of the Wadge Hierarchy restricted to ω-regular languages:

Theorem 17 (Corollary of Büchi-Landweber's Theorem [BL69]). *For A and B some ω-regular sets, one can effectively decide which player has a w.s. in the game $W(A, B)$ and the winner has a w.s. given by a transducer.*

The hierarchy obtained on ω-regular languages is now called the Wagner hierarchy and has length ω^ω. Wagner [Wag79] gave an automata structure characterization, based on notion of chain and superchain, for an automaton to be in a given class and showed that the Wadge degree of an ω-regular language is computable. Wilke and Yoo proved in [WY95] that this can be done in polynomial time. Wagner's hierarchy has been recently studied by Carton and Perrin in connection with the theory of ω-semigroups [CP97] [CP98] [PP98] and by Selivanov in [Sel98].

We present in this paper an extension of the Wagner hierarchy to the class of blind counter ω-languages, using analogous notions of chains and superchains. We shall first define positive and negative loops, next chains and superchains. A crucial fact which allows this definition is the following lemma:

Lemma 18. *Let $\mathcal{A} = (K, \Sigma, \Gamma, \delta, q_0, Z_0, \mathcal{F})$ be a MBCA and $x \in \Sigma^\omega$ such that there exists an infinite run $r = (q_i, I^{n_i} Z_0)_{i \geq 1}$ of $\mathcal{A}$ over x such that $Inf(r) = F \subseteq K$. Then there exist infinitely many integers i such that for all $j \geq i$, $n_j \geq n_i$. Among these integers there exist infinitely many integers i_k, $k \geq 1$, and a state $q \in K$ such that for all $k \geq 1$, $q_{i_k} = q$. Then there exist two integers s, s' such that between steps i_s and $i_{s'}$ of the run r, $\mathcal{A}$ enters in every state of F and in not any other state of K, because $Inf(r) = F$.*

Proof. With the hypotheses of the lemma, assume that $r = (q_i, I^{n_i} Z_0)_{i \geq 1}$ is an infinite run of M over x. If there exist only finitely many integers i such that for all $j \geq i$, $n_j \geq n_i$, then there exists a largest one l. But then if j_0 is an integer $> l$ there exists an integer $j_1 > j_0$ such that $n_{j_1} < n_{j_0}$. By induction one could construct a sequence of integers $(j_k)_{k \geq 0}$ such that for all k, $n_{j_{k+1}} < n_{j_k}$. This would lead to a contradiction because every integer n_i is positive.

Then there exist infinitely many integers i such that $\forall j \geq i$, $n_j \geq n_i$. The set of states is finite, hence there exists a state $q \in K$ and infinitely many such integers i_k, $k \geq 1$, such that for all $k \geq 1$, $q_{i_k} = q$ and $n_{i_k} > 0$ or for all $k \geq 1$, $q_{i_k} = q$ and $n_{i_k} = 0$. Now if $Inf(r) = F$, the states not in F occur only finitely many times during run r thus there exist two integers $s < s'$ such that the set of states $\mathcal{A}$ enters between steps i_s and $i_{s'}$ of the run r is exactly F.

Remark 19. *The proof of Lemma 18 relies on a simple property of local minima of functions mapping natural numbers to themselves. A similar argument is due to Linna [Lin77].*

Then we shall write

(a) $(q, I) \overset{F}{\underset{}{\mapsto}}^\star (q, I^+)$ if $n_{i_s} > 0$ and $n_{i_{s'}} > n_{i_s}$

(b) $(q, I) \overset{F}{\underset{}{\mapsto}}^\star (q, I^=)$ if $n_{i_s} > 0$ and $n_{i_{s'}} = n_{i_s}$

(c) $(q, Z_0) \overset{F}{\mapsto}^{\star} (q, Z_0)$ if $n_{i_s} = 0$ and $n_{i_{s'}} = 0$

The set F is said to be an essential set (of states) and we shall say that in the case (a) there exists a loop $L(q, I, F, +)$, in the case (b) there exists a loop $L(q, I, F, =)$, in the case (c) there exists a loop $L(q, Z_0, F, =)$. Such a loop is positive if $F \in \mathcal{F}$ and it is negative if $F \notin \mathcal{F}$. We then denote the loop $L(q, I, F, =)$ by $L^+(q, I, F, =)$ or $L^-(q, I, F, =)$ and similarly in the other cases.

Lemma 20. *The set of essential sets and the set of positive and negative loops of a MBCA is effectively computable.*

This follows from the decidability of the emptiness problem for context free languages accepted by pushdown automata.

We assume now some familiarity with the Wagner hierarchy as presented in [Wag79] [Sta97a]. The next step is to define, following Wagner's study, the (alternating) chains. Let E^+ (respectively E^-) be the set of essential sets in $\mathcal{F}$ (respectively not in $\mathcal{F}$). An alternating chain of length n is in the form

$$F_1 \subset F_2 \subset F_3 \subset \dots F_n$$

where $F_i \in E^+$ iff $F_{i+1} \in E^-$ for $1 \leq i < n$. It is a positive chain if $F_1 \in E^+$ and a negative chain if $F_1 \in E^-$.

As in the case of Muller automata [Sta97a], one can see that if F is a maximal essential set then all (alternating) chains of maximal length contained in F have the same sign (positive or negative) because in every chain of maximal length contained in F one can replace the last essential set by F itself. Let then $l(F)$ be the maximal length of chains contained in F and $s(F)$ be the sign of these chains.

We now define the first invariant of the MBCA $\mathcal{A}$ as $m(\mathcal{A})$ being the maximal length of chains of essential sets. Lemma 18 is crucial because it makes every essential set F_i of a chain $F_1 \subset F_2 \subset F_3 \subset \dots F_n$ to be indefinitely reachable from (q, I) (respectively (q, Z_0)) if there exists a loop $L(q, I, F_n, + \ or \ =)$, (respectively $L(q, Z_0, F_n, =)$).

The great difference between the case of Muller automata and the case of MBCA comes with the notion of superchain. Briefly speaking in a MA $\mathcal{A}$ a superchain of length n is a sequence $S_1, \dots, S_n$ of chains of length $m(\mathcal{A})$ such that for every integer i, $1 \leq i < n$, S_{i+1} is reachable from S_i and S_{i+1} is positive iff S_i is negative. In the case of MA, S_i cannot be reachable from S_{i+1} otherwise there would exist a chain of length $> m(\mathcal{A})$.

But in the case of MBCA, in such a superchain, S_i may be reachable from S_{i+1} but **with a reachability which is limited by the counter**. This leads to the notion of superchains of length ω, where ω is the first infinite ordinal, and next of length α where α is an ordinal $< \omega^2$.

An example of a MBCA $\mathcal{A}$ with m($\mathcal{A}$)$= m$ and a superchain of length ω is obtained from two MA $\mathcal{B}$ and $\mathcal{B}'$ such that the graph of $\mathcal{B}$ is just constituted by a positive chain of length m with a maximal essential set $F_m = \{q_1, \ldots q_m\}$ and the graph of $\mathcal{B}'$ is just constituted by a negative chain of length m with a maximal essential set $F'_m = \{q'_1, \ldots q'_m\}$. The behaviour of the MBCA $\mathcal{A}$ is as follows: at the beginning of an infinite run, the counter may be increased up to a counter value N; then there exist transitions from state q_1 to q'_1 and conversely from state q'_1 to q_1 but these transitions make the counter value decrease. Moreover $\mathcal{A}$ has also the transitions of the two MA $\mathcal{B}$ and $\mathcal{B}'$ but these transitions do not change the counter value. Then one can see thet after a first transition from state q_1 to q'_1 or from q'_1 to q_1 the number of such transitions is bounded by the counter value N, but this initial value may be chosen $> n_0$ where n_0 is any given integer.

Let then $\mathcal{A}$ be a MBCA such that m($\mathcal{A}$)$= m$ and such that $\mathcal{A}$ has positive and negative chains of length m. A superchain of length ω is formed by two maximal loops $L^+(q, I, F_m, + \ or \ =)$ and $L^-(q', I, F'_m, + \ or \ =)$ of such chains, i.e. F_m is the last element of a positive chain of length m and F'_m is the last element of a negative chain of length m; moreover, for all $p_0 > 1$, configurations $(q, I^p Z_0)$ are reachable for integers $p > p_0$, and there exist transitions implying that

$$(q, I^p Z_0) \mapsto^\star (q', I^{p'} Z_0) \mapsto^\star (q, I^{p''} Z_0)$$

for some integers p, p', p''. the MBCA $\mathcal{A}$ having not any chain of length $> m$, it holds that $p'' < p$, because otherwise there would exist an essential set $F \supseteq F_m \cup F'_m$ and then there would exist a chain of length $> m$. And the loop $L^+(q, I, F_m, + \ or \ =)$ is in fact $L^+(q, I, F_m, =)$ and similarly $L^-(q', I, F'_m, + \ or \ =)$ is $L^-(q', I, F'_m, =)$
One can informally say that F_m is reachable from F'_m and conversely but after such transitions the counter value has decreased hence there is a limitation to this reachability.

Lemma 21. *The set of superchains of length ω of a MBCA is effectively computable.*

Now one can define superchains of length $\omega.p$ for an integer $p \geq 1$. Informally speaking a superchain of length $\omega.p$ is a sequence $\Omega_1, \ldots, \Omega_p$ of superchains of length ω such that any state q of an essential set of Ω_{i+1} is reachable with unbounded values of the counter from any state of an essential set of Ω_i. It is now easy to define superchains of length $\omega.p + s \geq 1$, (with p, s some integers ≥ 0), which are a sequence of a superchain of length s followed by a superchain of length $\omega.p$.
In the case $s > 0$, the superchain is said to be positive if it begins with a positive chain and it is said to be negative if it begins with a negative chain.
In the case $s = 0$, we consider now that a superchain: $\Omega_1, \ldots, \Omega_p$, of length $\omega.p$, is given with a loop L. Then it is said to be positive (respectively, negative) if Ω_1

is formed by two maximal loops $L^+(q, I, F_m, =)$ and $L^-(q', I, F'_m, =)$ of chains of length $m(\mathcal{A}) = m$ and configurations $(q, I^p Z_0)$ are reachable for unbounded values of $p \geq 1$ from the positive loop L (respectively, from the negative loop L).

We define now the second invariant of the MBCA $\mathcal{A}$ as $n(\mathcal{A})$ being the maximal length of superchains ($n(\mathcal{A}) < \omega^2$). The MBCA is said to be prime if all superchains of length $n(\mathcal{A})$ have the same sign, i.e. all are positive or all are negative. Denote $s(\mathcal{A}) = 0$ if $\mathcal{A}$ is not prime, $s(\mathcal{A}) = 1$ if all longest superchains are positive, and $s(\mathcal{A}) = -1$ if all longest superchains are negative.

Lemma 22. *Let $\mathcal{A}$ be a MBCA. Then $n(\mathcal{A})$ and $s(\mathcal{A})$ are computable. Moreover the set of superchains of length $n(\mathcal{A})$ is computable.*

We can now follow Wagner's study and define for α an ordinal $< \omega^2$ and m an integer ≥ 1:

$C_m^\alpha = \{L(\mathcal{A}) \ / \ s(\mathcal{A}) = 1 \text{ and } m(\mathcal{A}) = m \text{ and } n(\mathcal{A}) = \alpha\}$
$D_m^\alpha = \{L(\mathcal{A}) \ / \ s(\mathcal{A}) = -1 \text{ and } m(\mathcal{A}) = m \text{ and } n(\mathcal{A}) = \alpha\}$
$E_m^\alpha = \{L(\mathcal{A}) \ / \ s(\mathcal{A}) = 0 \text{ and } m(\mathcal{A}) = m \text{ and } n(\mathcal{A}) = \alpha\}$

Using the Wadge game, one can now show that each class C_m^α or D_m^α defines a Wadge degree, i.e. all ω-languages in the same class C_m^α or D_m^α are Wadge equivalent. In other words C_m^α and D_m^α are the restrictions to the class **BC** of some Wadge degrees.
Moreover when $\alpha = n$ is an integer, this degree corresponds to the degree obtained in the Wagner hierarchy for the classes C_m^n or D_m^n.

The classes C_m^α, D_m^α, and E_m^α, for m an integer ≥ 1 and α a non null ordinal $< \omega^2$, form the coarse structure of the Wadge hierarchy of **BC**. It is a strict extension of the coarse structure of the Wagner hierarchy studied in [Wag79] and it satisfies the following Theorem.

Theorem 23. *Let $\mathcal{A}$ and $\mathcal{B}$ be two MBCA accepting the ω-languages $L(\mathcal{A})$ and $L(\mathcal{B})$. Then it holds that:*

1. *If $m(\mathcal{A}) < m(\mathcal{B})$, then $L(\mathcal{A}) <_W L(\mathcal{B})$.*
2. *If $m(\mathcal{A}) = m(\mathcal{B})$, and $n(\mathcal{A}) < n(\mathcal{B})$, then $L(\mathcal{A}) <_W L(\mathcal{B})$.*
3. *If $m(\mathcal{A}) = m(\mathcal{B})$, $n(\mathcal{A}) = n(\mathcal{B})$, $s(\mathcal{A}) = 1$ or $s(\mathcal{A}) = -1$, and $s(\mathcal{B}) = 0$, then $L(\mathcal{A}) <_W L(\mathcal{B})$.*
4. *If $m(\mathcal{A}) = m(\mathcal{B})$, $n(\mathcal{A}) = n(\mathcal{B})$, $s(\mathcal{A}) = 1$ and $s(\mathcal{B}) = -1$, then $L(\mathcal{A})$ and $L(\mathcal{B})$ are non self dual and $L(\mathcal{A}) \equiv_W L(\mathcal{B})^-$.*

From this Theorem one can easily infer that the integer $m(\mathcal{A})$, the ordinal $n(\mathcal{A})$, and $s(\mathcal{A}) \in \{-1, 0, 1\}$, are invariants of the ω-language $L(\mathcal{A})$ and not only of the MBCA $\mathcal{A}$:

Corollary 24. *Let $\mathcal{A}$ and $\mathcal{B}$ be two MBCA accepting the same ω-language $L(\mathcal{A}) = L(\mathcal{B})$. Then $m(\mathcal{A}) = m(\mathcal{B})$, $n(\mathcal{A}) = n(\mathcal{B})$, and $s(\mathcal{A}) = s(\mathcal{B})$.*

One can give a canonical member in each of the classes C_m^α, D_m^α, and E_m^α, for m an integer ≥ 1 and α a non null ordinal $< \omega^2$. And one can easily deduce that the length of the coarse structure of the Wadge hierarchy of blind counter ω-languages is the ordinal ω^3, while the length of the coarse structure of the Wagner hierarchy was the ordinal ω^2.

The coarse structure of the class **BC** is effective but it is not exactly the Wadge hierarchy of **BC**, because each class E_m^α is the union of countably many (restrictions of) Wadge degrees. We can next define a sort of derivation as Wagner did for Muller automata.

Two MBCA $\mathcal{A}$ and $\mathcal{B}$ in the same class E_m^α have essentially the same "most difficult parts" because they have positive and negative superchains of length $n(\mathcal{A}) = n(\mathcal{B})$. Hence, in the case of Muller automata (then α is an integer), Wagner's idea was to cut off the superchains of length $n(\mathcal{A}) = n(\mathcal{B})$ of $\mathcal{A}$ and $\mathcal{B}$; this way one get some new automata $\partial\mathcal{A}$ and $\partial\mathcal{B}$ which are called the derivations of $\mathcal{A}$ and $\mathcal{B}$ and the comparison of $\mathcal{A}$ and $\mathcal{B}$ with regard to $\leq_W$ is reduced to the comparison of their derivations $\partial\mathcal{A}$ and $\partial\mathcal{B}$.

In the case of MBCA one do as in the case of MA but with some modification. We first define the derivation $\partial\mathcal{A}$ of a MBCA in E_m^α: $\mathcal{A} = (K, \Sigma, \Gamma, \delta, q_0, Z_0, \mathcal{F})$ as follows.
Let ∂K be the set of states in K from which some positive **and** some negative superchains of length $n(\mathcal{A})$ are reachable. In fact for each such $q \in \partial K$, it may exist an integer n_q such that positive **and** negative superchains of length $n(\mathcal{A})$ are reachable only from configurations $(q, I^n Z_0)$ with $n \geq n_q$. And these integers n_q are effectively computable. Let us define now

$$\partial\mathcal{A} = (\partial K, \Sigma, \Gamma = \{I, Z_0\}, \partial\delta, q_0, Z_0, \partial\mathcal{F})$$

where $\partial\delta$ is defined by:

for each $q \in \partial K$, $a \in \Sigma$, $Z \in \Gamma$:
$\partial\delta(q, a, Z) = \delta(q, a, Z)$ if $\delta(q, a, Z) = (p, \gamma)$ for some $\gamma \in \Gamma^\star$ and $p \in \partial K$.
Otherwise $\partial\delta(q, a, Z)$ is undefined.

And $\partial\mathcal{F} = \{F \ / \ F \subseteq \partial K \text{ and } F \in \mathcal{F}\}$

We consider now the MBCA $\partial\mathcal{A}$ given with the integers n_q, for $q \in \partial K$. Then we study the loops of $\partial\mathcal{A}$ as above but **we keep only loops in the form** $L(q, Z_0 \text{ or } I, F, +or-)$ **such that state** q **is reachable with a counter value** $n \geq n_q$. We can next define chains and superchains for $\partial'\mathcal{A} = (\partial\mathcal{A}, (n_q)_{q \in \partial K})$. We define $m(\partial'\mathcal{A})$, $n(\partial'\mathcal{A})$, and $s(\partial'\mathcal{A})$, and it holds that $m(\partial'\mathcal{A}) < m(\mathcal{A})$. We then

attribute a class $C_{m(\partial' \mathcal{A})}^{n(\partial' \mathcal{A})}$, $D_{m(\partial' \mathcal{A})}^{n(\partial' \mathcal{A})}$, or $E_{m(\partial' \mathcal{A})}^{n(\partial' \mathcal{A})}$, to $\partial' \mathcal{A}$ as we did for $\mathcal{A}$. It may happen that there does not exist any loop for $\partial' \mathcal{A} = (\partial \mathcal{A}, (n_q)_{q \in \partial K})$; in that case we associate the class E to $\partial' \mathcal{A}$. Now we can iterate this process and associate to the MBCA $\mathcal{A}$ a name $N(\mathcal{A})$ which is inductively defined by:

1. If $\mathcal{A}$ is prime and $s(\mathcal{A}) = 1$, then $N(\mathcal{A}) = C_{m(\mathcal{A})}^{n(\mathcal{A})}$.
2. If $\mathcal{A}$ is prime and $s(\mathcal{A}) = -1$, then $N(\mathcal{A}) = D_{m(\mathcal{A})}^{n(\mathcal{A})}$.
3. If $\mathcal{A}$ is not prime then $N(\mathcal{A}) = E_{m(\mathcal{A})}^{n(\mathcal{A})} N(\partial' \mathcal{A})$.

This name depends only on the ω-language $L(\mathcal{A})$ accepted by the MBCA $\mathcal{A}$ and is effectively computable. We can write it in a similar fashion as in Wagner's study: we associate with each blind counter ω-language $L(\mathcal{A})$ in **BC** a name in the form:

$$N(\mathcal{A}) = E_{m_1}^{\alpha_1} \dots E_{m_k}^{\alpha_k} H_{m_{k+1}}^{\alpha_{k+1}}$$

where $m_1 > m_2 > \dots > m_k > m_{k+1}$ are integers; each α_i is an ordinal $< \omega^2$; and $H \in \{C, D\}$, or in the form:

$$N(\mathcal{A}) = E_{m_1}^{\alpha_1} \dots E_{m_k}^{\alpha_k} E$$

which we shall simply denote by

$$N(\mathcal{A}) = E_{m_1}^{\alpha_1} \dots E_{m_k}^{\alpha_k}$$

where $m_1 > m_2 > \dots > m_k$ are integers and each α_i is an ordinal $< \omega^2$.

One can show that each such name is really the name of an ω-language in **BC**. And the Wadge relation $\leq_W$ is now computable because of the following result.

Theorem 25. *Let $\mathcal{A}$ and $\mathcal{B}$ be two MBCA accepting the ω-languages $L(\mathcal{A})$ and $L(\mathcal{B})$. Assume that the names associated with the MBCA $\mathcal{A}$ and $\mathcal{B}$ are:*

$$N(\mathcal{A}) = E_{m_1}^{\alpha_1} \dots E_{m_k}^{\alpha_k} H_{m_{k+1}}^{\alpha_{k+1}}$$

$$N(\mathcal{B}) = E_{m_1'}^{\alpha_1'} \dots E_{m_l'}^{\alpha_l'} H'^{\alpha_{l+1}'}_{m_{l+1}'}$$

where ($H = E$ or $H = C$ or $H = D$), and ($H' = E$ or $H' = C$ or $H' = D$). Then $L(\mathcal{A}) \leq_W L(\mathcal{B})$ if there exists an integer $j \leq min(k+1, l+1)$ such that $m_i = m_i'$ and $n_i = n_i'$ for $1 \leq i \leq j$ and one of the two following properties holds.

1. $j = k + 1 \leq l + 1$ *and* $H' = E$ *or* $H = H'$.
2. $j < min(k+1, l+1)$ *and*
   $m_{j+1} < m_{j+1}'$ *or* ($m_{j+1} = m_{j+1}'$ *and* $\alpha_{j+1} < \alpha_{j+1}'$).

Then the structure of the Wadge hierarchy of ω-languages in $\mathbf{BC}$ is completely determined. One can show that a blind counter ω-language $L(\mathcal{A})$, where $\mathcal{A}$ is a MBCA, is in the class $\boldsymbol{\Delta_2^0}$ iff $\mathrm{m}(\mathcal{A}) < 2$, i.e. iff the name of $\mathcal{A}$ is in the form C_1^α, D_1^α, or E_1^α, for $\alpha < \omega^2$. Thus the Wadge hierarchy restricted to the class $\mathbf{BC} \cap \boldsymbol{\Delta_2^0}$ has length ω^2, while the Wadge hierarchy restricted to $REG_\omega \cap \boldsymbol{\Delta_2^0}$ has length ω. The Wadge hierarchy of $\mathbf{BC} \cap \boldsymbol{\Delta_2^0}$ is then a great extension of the Wagner hierarchy restricted to the class $\boldsymbol{\Delta_2^0}$. This phenomenon is still true for larger Wadge degrees and non $\boldsymbol{\Delta_2^0}$-sets. Considering the length of the whole hierarchy of $\mathbf{BC}$ we get the following:

Corollary 26. *(a) The length of the Wadge hierarchy of blind counter ω-languages in $\boldsymbol{\Delta_2^0}$ is ω^2.*
(b) The length of the Wadge hierarchy of blind counter ω-languages is the ordinal ω^ω (hence it is equal to the length of the Wagner hierarchy).

Once the structures of two MBCA $\mathcal{A}$ and $\mathcal{B}$ are determined as well as their names $N(\mathcal{A})$ and $N(\mathcal{B})$ are effectively computed, one can construct winning strategies in Wadge games $W(L(\mathcal{A}), L(\mathcal{B}))$ and $W(L(\mathcal{B}), L(\mathcal{A}))$. These strategies may be defined by blind counter transducers, and this extends Wagner's result to blind counter automata.

5 Concluding Remarks

This extended abstract is still a very summarized presentation of our results, which will need exposition of many other details we could not include in this paper [Fin00a].

We have considered above deterministic real time blind counter automata, which form a subclass of the class of deterministic pushdown automata and of the class of deterministic k-blind counter automata. The Wadge hierarchies of ω-languages in each of these classes have been determined in a non effective way, by other methods, in [Dup99b] [Fin99b] [Fin00b], and these results had been announced in the survey [DFR01]. The Wadge degrees in these hierarchies may be described with similar names

$$N(\mathcal{A}) = E_{m_1}^{\alpha_1} \dots E_{m_k}^{\alpha_k} H_{m_{k+1}}^{\alpha_{k+1}}$$

where $m_1 > m_2 > \dots > m_k > m_{k+1}$ are integers ≥ 1 and $H \in \{C, D, E\}$, and

1. each α_i is an ordinal $< \omega^{k+1}$, in the case of k-**blind counter automata.**
2. each α_i is an ordinal $< \omega^\omega$, in the case of **deterministic pushdown automata.**

We will further extend the results of the present paper in both directions to get decidability results and effective winning strategies in Wadge games. The above case of (one) blind counter automata already introduces some of the fundamental ideas which we will apply in further cases.

Another problem is to study the complexity of the problem: " determine the Wadge degree of a blind counter ω-language ", extending this way the results of Wilke and Yoo to blind counter ω-languages.

Further study would be the investigation of links between the problems of simulation and bisimulation [Jan00] [JKM00] [JMS99] [Kuc00] and the problem of finding winning strategies in Wadge games.

A Wadge game between two blind counter ω-languages, whose complements are also blind counter ω-languages, can easily be reduced to a Gale-stewart game, (see [Tho95] [PP98]), with a winning set accepted by a deterministic 2-blind-counter automaton. This suggests that Walukiewicz's result, the proof of the existence of effective winning strategies in a Gale-stewart game with a winning set accepted by a deterministic pushdown automaton, [Wal96], could be extended to the case of a winning set accepted by a deterministic multi blind counter automata, giving additional results as asked by Thomas in [Tho95].

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