On the reconstruction of conductivity of bordered two-dimensional surface in $\mathbb{R}^3$ from electrical currents measurements on its boundary

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Abstract.
An electrical potential U on a bordered real surface X in $\mathbb{R}^3$ with isotropic conductivity function $\sigma > 0$ satisfies equation $d(\sigma d^c U)\big|_X = 0$, where $d^c = i(\bar\partial - \partial)$, $d = \bar\partial + \partial$ are real operators associated with complex (conforme) structure on X induced by Euclidien metric of $\mathbb{R}^3$. This paper gives exact reconstruction of conductivity function σ on X from Dirichlet-to-Neumann mapping $U\big|_{bX} \to \sigma d^c U\big|_{bX}$. This paper extends to the case of the Riemann surfaces the reconstruction schemes of R.Novikov [N2] and of A.Bukhgeim [B], given for the case $X \subset \mathbb{R}^2$. The paper extends and corrects the statements of [HM], where the inverse boundary value problem on the Riemann surfaces was firstly considered.

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0. Introduction

0.1. Reduction of inverse boundary value problem on a surface in $\mathbb{R}^3$ to the corresponding problem on affine algebraic Riemann surface in $\mathbb{C}^3$.

Let X be bordered oriented two-dimensional manifold in $\mathbb{R}^3$. Manifold X is equiped by complex (conformal) structure induced by Euclidean metric of $\mathbb{R}^3$. We say that X possesses an isotropic conductivity function $\sigma > 0$, if any electric potential u on bX generates electrical potential U on X, solving the Dirichlet problem:

$$U\big|_{bX} = u \ \text{ and } \ d\sigma d^c U\big|_X = 0, \tag{0.1}$$

where $d^c = i(\bar\partial - \partial)$, $d = \bar\partial + \partial$ and the Cauchy-Riemann operator $\bar\partial$ corresponds to complex (conformal) structure on X. Inverse conductivity problem consists in the reconstruction of $\sigma\big|_X$ from the mapping potential $U\big|_{bX} \to$ current $j = \sigma d^c U\big|_{bX}$ for solutions of (0.1). This mapping is called Dirichlet-to-Neumann mapping.

This problem is the special case of the following more general inverse boundary value problem, going back to I.M.Gelfand [Ge] and A.Calderon [C]: to find potential (2-forme) q on X in the equation

$$dd^c\psi = q\psi \tag{0.2}$$

from knowledge of Dirichlet-to-Neumann mapping $\psi\big|_{bX} \to d^c\psi\big|_{bX}$ for solutions of (0.2). Equation (0.2) is called in some context by stationary Schrödinger equation, in other context by monochromatic acoustic equation etc. Equation (0.1) can be reduced to the equation (0.2) with
$q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$ by the substitution $\psi = \sqrt{\sigma}U$.

Let restriction of Euclidean metric of $\mathbb{R}^3$ on X have (in local coordinates) the form

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2 = Adz^2 + 2Bdzd\bar{z} + \bar{A}d\bar{z}^2,$$

where $z = x + iy$, $B = \frac{E+G}{4}$, $A = \frac{E-G-2iF}{4}$. Put $\mu = \frac{\bar{A}}{B+\sqrt{B^2-|A|^2}}$. By classical results (going back to Gauss and Riemann) one can construct holomorphic embedding $\varphi: X \to \mathbb{C}^3$, using some solution of Beltrami equation: $\bar{\partial}\varphi = \mu\partial\varphi$ on X. Moreover, embedding φ can be chosen in such a way that $\varphi(X)$ belongs to smooth algebraic curve V in $\mathbb{C}^3$. Using existence of embedding φ we can identify further X with $\varphi(X)$.

0.2. Reconstruction schemes for the case $X \subset \mathbb{R}^2 \simeq \mathbb{C}$.

For the case $X = \Omega \subset \mathbb{R}^2$ the exact reconstruction scheme for formulated inverse problems was given in [N2], [N3] under some restriction (smallness assumption) for σ or q (see Corollary 2 of [N2]) . For the case of inverse conductivity problem, see (0.1), (0.2), when $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$, restriction on σ in this scheme was eliminated by A.Nachman [Na] by the reduction to the equivalent question for the first order system studied by R.Beals and R.Coifman [BC2]. Recently A.Bukhgeim [B] has found new original reconstruction scheme for inverse boundary value problem, see (0.2), without smallness assumption on q.

In a particular case, the scheme of [N2] for the inverse conductivity problem consists in the following. Let $\sigma(x) > 0$ for $x \in \bar{\Omega}$ and $\sigma \in C^{(2)}(\bar{\Omega})$. Put $\sigma(x) = 1$ for $x \in \mathbb{R}^2\backslash\bar{\Omega}$.

Let $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$.

From L.Faddeev [F1] result it follows: $\exists$ compact set $E \subset \mathbb{C}$ such that for each $\lambda \in \mathbb{C}\backslash E$ there exists a unique solution $\psi(z,\lambda)$ of the equation $dd^c\psi = q\psi = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\psi$, with asymptotics

$$\psi(z,\lambda)e^{-\lambda z} \stackrel{\text{def}}{=} \mu(z,\lambda) = 1 + o(1), \;\; z \to \infty.$$

Such solution can be found from the integral equation

$$\mu(z,\lambda) = 1 + \frac{i}{2}\int\limits_{\xi\in\Omega} g(z-\xi,\lambda)\frac{\mu(\xi,\lambda)dd^c\sqrt{\sigma}}{\sqrt{\sigma}}, \tag{0.3}$$

where the function

$$g(z,\lambda) = \frac{i}{(2\pi)^2}\int\limits_{w\in\mathbb{C}} \frac{e^{\lambda w-\bar{\lambda}\bar{w}}dw\wedge d\bar{w}}{(w+z)\bar{w}} = \frac{i}{2(2\pi)^2}\int\limits_{w\in\mathbb{C}} \frac{e^{i(w\bar{z}+\bar{w}z)}dw\wedge d\bar{w}}{w(\bar{w}-i\lambda)}$$

is called the Faddeev-Green function for the operator

$$\mu \mapsto \bar{\partial}(\partial + \lambda dz)\mu.$$

From [N2] it follows that $\forall\lambda\in\mathbb{C}\backslash E$ the function $\psi\big|_{b\Omega}$ can be found through Dirichlet-to-Neumann mapping by integral equation

$$\psi(z,\lambda)\big|_{b\Omega}=e^{\lambda z}+\int\limits_{\xi\in b\Omega}e^{\lambda(z-\xi)}g(z-\xi,\lambda)(\hat{\Phi}\psi(\xi,\lambda)-\hat{\Phi}_0\psi(\xi,\lambda)),\tag{0.4}$$

where $\hat{\Phi}\psi=\bar{\partial}\psi\big|_{b\Omega}$, $\hat{\Phi}_0\psi=\bar{\partial}\psi_0\big|_{b\Omega}$, $\psi_0\big|_{b\Omega}=\psi\big|_{b\Omega}$ and $\partial\bar{\partial}\psi_0\big|_{\Omega}=0$.

By results of [BC1], [GN] and [N2] it follows that $\psi(z,\lambda)$ satisfies $\bar{\partial}$-equation of Bers-Vekua type with respect to $\lambda\in\mathbb{C}\backslash E$:

$$\frac{\partial\psi}{\partial\bar{\lambda}}=b(\lambda)\bar{\psi},\quad\text{where}\tag{0.5}$$

$$\bar{\lambda}b(\lambda)=-\frac{1}{2\pi i}\int\limits_{z\in b\Omega}e^{\lambda z-\bar{\lambda}\bar{z}}\bar{\partial}_z\mu(z,\lambda)=\frac{1}{4\pi}\int\limits_{\Omega}e^{\lambda z-\bar{\lambda}\bar{z}}q\mu,\tag{0.6}$$

$$\psi(z,\lambda)e^{-\lambda z}=\mu(z,\lambda)\to 1,\ \lambda\to\infty,\ \forall\ z\in\mathbb{C}.\tag{0.7}$$

From [BC2] and [Na] it follows that for $q=\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$, $\sigma>0$, $\sigma\in C^{(2)}(\bar{\Omega})$ the exceptional set $E=\{\emptyset\}$ and function $\lambda\mapsto b(\lambda)$ belongs to $L^{2+\varepsilon}(\mathbb{C})\cap L^{2-\varepsilon}(\mathbb{C})$ for some $\varepsilon>0$. As a consequence function $\mu=e^{-\lambda z}\psi$ is a unique solution of the Fredholm integral equation

$$\mu(z,\lambda)+\frac{1}{2\pi}\int\limits_{\lambda'\in\mathbb{C}}b(\lambda')e^{\bar{\lambda}'\bar{z}-\lambda' z}\bar{\mu}(z,\lambda')\frac{d\lambda'\wedge d\bar{\lambda}'}{\lambda'-\lambda}=1.\tag{0.8}$$

Integral equations (0.4), (0.8) permit, starting from the Dirichlet-to-Neumann mapping, to find firstly the boundary values $\psi\big|_{b\Omega}$, secondly "$\bar{\partial}$-scattering data" $b(\lambda)$ and thirdly function $\psi\big|_{\Omega}$. From equality $dd^c\psi=\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\psi$ on X we find finally $\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$ on X.

The scheme of the Bukhgeim type [B] can be presented in the following way. Let $q=Qdd^c|z|^2$, where $Q\in C^{(1)}(\bar{\Omega})$, but potential Q is not necessary of the conductivity form $\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$. By variation of Faddeev statement and proof we obtain that $\forall\ a\in\mathbb{C}\ \exists$ compact set $E\subset\mathbb{C}$ such that $\forall\lambda\in\mathbb{C}\backslash E$ there exists a unique solution $\psi_a(z,\lambda)$ of the equation $dd^c\psi=q\psi$ with asymptotics

$$\psi_a(z,\lambda)e^{-\lambda(z-a)^2}=\mu_a(z,\lambda)=1+o(1),\ z\to\infty.$$

Such a solution can be found from integral equation (0.3), where kernel $g(z,\lambda)$ is replaced by kernel

$$g_a(z,\zeta,\lambda)=\frac{ie^{\lambda a^2-\bar{\lambda}\bar{a}^2}}{2\pi^2}\int\limits_{\mathbb{C}}\frac{e^{-\lambda(\zeta-\eta+a)^2+\bar{\lambda}(\bar{\zeta}-\bar{\eta}+\bar{a})^2}}{(\eta-z)(\bar{\zeta}-\bar{\eta})}d\eta\wedge d\bar{\eta}.$$

Kernel $g_a(z,\zeta,\lambda)$ can be called the Faddeev type Green function for the operator

$\mu \to \bar\partial(\partial + \lambda d(z-a)^2)\mu$. Equation $\bar\partial(\partial + \lambda d(z-a)^2)\mu = \frac{i}{2} q\mu$ and Green formula implies

$$\int\limits_{b\Omega} e^{\lambda(z-a)^2 - \bar\lambda(\bar z - \bar a)^2} \bar\partial\mu = \int\limits_{\Omega} e^{\lambda(z-a)^2 - \bar\lambda(\bar z - \bar a)^2} \frac{q\mu}{2i}. \tag{0.9}$$

Stationary phase method, applied to the integral in the right-hand side of (0.9), gives for $\tau \to \infty$, $\tau \in \mathbb{R}$, equality

$$\lim_{\substack{\tau\to\infty \\ \tau\in\mathbb{R}}} \frac{4\tau}{\pi i} \int\limits_{z\in b\Omega} e^{i\tau[(z-a)^2 + (\bar z - \bar a)^2]} \bar\partial_z \mu_a(z, i\tau) = Q(a). \tag{0.10}$$

Formula (0.10) means that values of potential Q in the arbitrary point a of Ω can be reconstructed from Dirichlet-to-Neumann mapping $\mu_a\big|_{b\Omega} \mapsto \bar\partial_z \mu_a\big|_{b\Omega}$ for family of functions $\mu_a(z, \lambda)$ depending on parameter $\lambda = i\tau$, $\tau > const$, where we assume that $\mu_a\big|_{b\Omega}$ is found using an analog of (0.4) for $\psi_a\big|_{b\Omega}$.

Bukhgeim's scheme works well at least $\forall\ Q \in C^{(1)}(\bar\Omega)$.

More constructive scheme of [N2] works quite well only in the absence of exceptional set E in the λ-plane for Faddeev type functions. In papers [BLMP], [Ts], [N3] it was constructed modified Faddeev-Green function permitting to solve inverse boundary problem (0.2), on the $\mathbb{R}^2 = \mathbb{C}$, at least, under some smallness assumptions on potential Q.

Let us note that the first uniqueness results in the two-dimensional inverse boundary value or scattering problems for (0.1) or (0.2) goes back to A.Calderon [C], V.Druskin [D], R.Kohn, M.Vogelius [KV], J.Sylvester, G.Uhlmann [SU] and R.Novikov [N1].

Note in this connection that the first seminal results on reconstruction of the two-dimensional Schrödinger operator H on the torus from the data "extracted" from the family of eigenfunctions (Bloch-Floquet) of single energy level $H\psi = E\psi$ were obtained in series of papers starting from B.Dubrovin, I.Krichever, S.P.Novikov [DKN], S.P.Novikov, A.Veselov [NV]. These results were obtained in connection with (2+1)- dimensional evolution equations.

This paper extends to the case of Riemann surfaces reconstruction procedures of [N2] and of [B]. The paper extends (and also corrects) the recent paper [HM2] where the inverse boundary value problem on Riemann surface was firstly considered.Earlier in [HM1] it was proved that if $X \subset \mathbb{R}^3$ possesses a constant conductivity then X with complex structure can be effectively reconstructed by at most three generic potential $\to$ current measurements on bX.

Very recently, motivated by [B] and [HM1], [HM2], C.Guillarmou and L.Tzou [GT] have obtained general identifiability result (without reconstruction procedure): if for all solutions of equations $dd^c u + q_j u = 0$, $q_j \in C^{(2)}(X)$, $j = 1, 2$, Cauchy datas $u\big|_{bX}$, $d^c u\big|_{bX}$, coincide, then $q_1 = q_2$ on X.

1. Preliminaries and main results

Let $\mathbb{C}P^3$ be complex projective space with homogeneous coordinates

$w = (w_0 : w_1 : w_2 : w_3)$. Let $\mathbb{C}P^2_\infty = \{w \in \mathbb{C}P^3 : \ w_0 = 0\}$. Then $\mathbb{C}P^3 \backslash \mathbb{C}P^2_\infty$ can be considered as the complex affine space with coordinates $z_k = w_k/w_0$, $k = 1, 2, 3$. By classical result of G. Halphen (see R.Hartshorne [H], ch.IV, § 6) any compact Riemann surface of genus g can be embedded in $\mathbb{C}P^3$ as projective algebraic curve $\tilde{V}$, which intersects $\mathbb{C}P^2_\infty$ transversally in $d > g$ points, where $d \geq 1$ if $g = 0$, $d \geq 3$ if $g = 1$ and $d \geq g + 3$ if $g \geq 2$. Without loss of generality one can suppose that

i) $V = \tilde{V} \backslash \mathbb{C}P^2_\infty$ is connected affine algebraic curve in $\mathbb{C}^3$ defined by polynomial equations $V = \{z \in \mathbb{C}^3 : \ p_1(z) = p_2(z) = p_3(z) = 0\}$ such that the rang of the matrix $[\frac{\partial p_1}{\partial z}(z), \frac{\partial p_2}{\partial z}(z), \frac{\partial p_3}{\partial z}(z)] \equiv 2 \ \forall \ z \in V$.

ii) $\tilde{V} \cap \mathbb{C}P^2_\infty = \{\beta_1, \ldots, \beta_d\}$, where

$$\beta_l = (0 : \beta_l^1 : \beta_l^2 : \beta_l^3), \ (\frac{\beta_l^2}{\beta_l^1}, \frac{\beta_l^3}{\beta_l^1}) \in \mathbb{C}^2, \ l = 1, 2, \ldots, d.$$

iii) For $r_0 > 0$ large enough

$$\det \begin{vmatrix} \frac{\partial p_\alpha}{\partial z_2} & \frac{\partial p_\alpha}{\partial z_3} \\ \frac{\partial p_\beta}{\partial z_2} & \frac{\partial p_\beta}{\partial z_3} \end{vmatrix} \neq 0 \ \text{ for } \ z \in V : \ |z_1| \geq r_0 \ \text{ and } \ \alpha \neq \beta.$$

iv) For $|z|$ large enough:

$$\frac{dz_2}{dz_1}|_{V_l} = \gamma_l + \frac{\gamma_l^0}{z_1^2} + O(\frac{1}{z_1^3}), \ \ \frac{dz_3}{dz_1}|_{V_l} = \tilde{\gamma}_l + \frac{\tilde{\gamma}_l^0}{z_1^2} + O(\frac{1}{z_1^3}),$$

where $\gamma_l, \ \tilde{\gamma}_l, \ \gamma_l^0, \ \tilde{\gamma}_l^0 \neq 0$, for $l = 1, \ldots, d$, $d \geq 2$.

Let $V_0 = \{z \in V : \ |z_1| \leq r_0\}$ and $V \backslash V_0 = \cup_{l=1}^d V_l$, where $\{V_l\}$ are connected components of $V \backslash V_0$. Let us equip V by Euclidean volume form $dd^c|z|^2$. Let $\tilde{W}^{1,\tilde{p}}(V) = \{F \in L^\infty(V) : \ \bar{\partial}F \in L^{\tilde{p}}_{0,1}(V)\}$, $\tilde{W}^{1,\tilde{p}}_{1,0}(V) = \{f \in L^\infty_{1,0}(V) : \ \bar{\partial}f \in L^{\tilde{p}}_{1,1}(V)\}$, $\tilde{p} > 2$. Let $H_{0,1}(V)$ denotes the space of antiholomorphic (0,1)-forms on V. Let $H^p_{0,1}(V) = H_{0,1}(V) \cap L^p_{0,1}(V)$, $1 < p < 2$.

Let $W^{1,p}(V) = \{F \in L^p(V) : \ \bar{\partial}F \in L^p_{0,1}(V)\}$.

From the Hodge-Riemann decomposition theorem (see [GH], [Ho]) $\forall \Phi_0 \in W^{1,p}_{0,1}(\tilde{V})$ we have
$\Phi_0 = \bar{\partial}(\bar{\partial}^* G \Phi_0) + \mathcal{H}\Phi_0$, where $\mathcal{H}\Phi_0 \in H_{0,1}(\tilde{V})$ and G is the Hodge-Green operator for the Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ on $\tilde{V}$ with the properties: $G(H_{0,1}(\tilde{V})) = 0$, $\bar{\partial}G = G\bar{\partial}$, $\bar{\partial}^*G = G\bar{\partial}^*$.

Straight generalization of Proposition 1 from [He] gives explicit operators:
$R_1 : \ L^p_{0,1}(V) \to L^{\tilde{p}}(V)$, $R_0 : \ L^p_{0,1}(V) \to \tilde{W}^{1,\tilde{p}}(V)$ and $\mathcal{H} : \ L^p_{0,1}(V) \to H^p_{0,1}(V)$, $1 < p < 2$, $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{2}$, such that $\forall \Phi \in L^p_{0,1}(V)$ we have decomposition of Hodge-Riemann type:

$$\Phi = \bar{\partial}R\Phi + \mathcal{H}\Phi, \quad \text{where} \quad R = R_1 + R_0,$$

$$R_1\Phi(z) = \frac{1}{2\pi i} \int\limits_{\xi \in V} \Phi(\xi) \wedge (dp_\alpha \wedge dp_\beta) \rfloor d\xi_1 \wedge d\xi_2 \wedge d\xi_3 \det[\frac{\partial p_\alpha(\xi)}{\partial \xi}, \frac{\partial p_\beta(\xi)}{\partial \xi}, \frac{\bar{\xi} - \bar{z}}{|\xi - z|^2}],$$

$$R_0\Phi(z) = (\bar{\partial}^* G(\bar{\partial}R_1\Phi - \Phi))(z) - (\bar{\partial}^* G(\bar{\partial}R_1\Phi - \Phi))(\beta_1),$$

$$(\bar{\partial}R_1\Phi - \Phi) \in W^{1,p}_{0,1}(\tilde{V}), \ G \ \text{ is the Hodge} - \text{Green operator for Laplacian } \ \bar{\partial}\bar{\partial}^*$$

$$\text{for } \ (0,1) - \text{forms } \ \text{on } \ \tilde{V},$$

(1,1)-form under sign of integral does not depend on the choice of indexes $\alpha, \beta = 1, 2, 3$, $\alpha \neq \beta$,

$$\mathcal{H}\Phi = \sum_{j=1}^{g} \Big(\int\limits_{V} \Phi \wedge \omega_j \Big) \bar{\omega}_j,$$

$\{\omega_j\}$ is orthonormal basis of holomorphic (1,0)-forms on $\tilde{V}$, i.e.

$$\int\limits_{V} \omega_j \wedge \bar{\omega}_k = \delta_{jk}, \quad j, k = 1, 2, \ldots, g.$$

Note that as a corollary of construction of R we have that $\lim\limits_{\substack{z \in V_1 \\ z \to \infty}} R\Phi(z) = R\Phi(\beta_1) = 0$.

Remark 1.1. If $V = \{z \in \mathbb{C}^2 : \ P(z) = 0\}$ be algebraic curve in $\mathbb{C}^2$ then formula for operator R_1 is reduced to the following:

$$R_1\Phi(z) = \frac{1}{2\pi i} \int\limits_{\xi \in V} \Phi(\xi) \frac{d\xi_1}{\frac{\partial P}{\partial \xi_2}} \det \Big[\frac{\partial P}{\partial \xi}(\xi), \frac{\bar{\xi} - \bar{z}}{|\xi - z|^2} \Big].$$

Remark 1.2. Based on [HP] one can construct an explicit formula not only for the main part R_1 of the R-operator, but for the whole operator $R = R_1 + R_0$.

Let $\varphi \in L^1_{1,1}(V) \cap L^\infty_{1,1}(V)$, $f \in \tilde{W}^{1,\tilde{p}}_{1,0}(V)$, $\lambda \in \mathbb{C}$, $\theta \in \mathbb{C}$.

Let

$$\hat{R}_\theta \varphi = R((dz_1 + \theta dz_2) \rfloor \varphi)(dz_1 + \theta dz_2),$$
$$R_{\lambda,\theta} f = e_{-\lambda,\theta} \overline{R(\overline{e_{\lambda,\theta} f})}, \quad \text{where} \quad e_{\lambda,\theta}(z) = e^{\lambda(z_1 + \theta z_2) - \bar{\lambda}(\bar{z}_1 + \bar{\theta}\bar{z}_2)}.$$

By straight generalization of Propositions 2, 3 from [He] the form $f = \hat{R}_\theta \varphi$ is a solution of $\bar{\partial} f = \varphi$ on V, function $u = R_{\lambda,\theta} f$ is a solution of

$$(\partial + \lambda(dz_1 + \theta dz_2))u = f - \mathcal{H}_{\lambda,\theta} f, \quad \text{where}$$
$$\mathcal{H}_{\lambda,\theta} f \stackrel{\text{def}}{=} e_{-\lambda,\theta} \overline{\mathcal{H}(\overline{e_{\lambda,\theta} f})}, \quad u \in W^{1,\tilde{p}}(V), \quad \tilde{p} > 2.$$

In addition, by straight generalization of Proposition 4 from [He] we have that

$$\bar{\partial}(\partial + \lambda(dz_1 + \theta dz_2))u = \varphi + \bar{\lambda}(d\bar{z}_1 + \bar{\theta} d\bar{z}_2) \wedge \mathcal{H}_{\lambda,\theta}(\hat{R}_\theta \varphi) \ \text{ on } \ V.$$

Definition 1.1. The kernel $g_{\lambda,\theta}(z, \xi)$, $z, \xi \in V$, $\lambda \in \mathbb{C}$, of integral operator $R_{\lambda,\theta} \circ \hat{R}_\theta$ is called in [He] the Faddeev type Green function for operator $\bar{\partial}(\partial + \lambda(dz_1 + \theta dz_2))$.

Definition 1.2. Let $g = genus\, \tilde{V}$. Let $\{\omega_j\}$, $j = 1, \ldots, g$, be orthonormal basis of holomorphic forms on $\tilde{V}$. Let $\{a_1, \ldots, a_g\}$ be different points (or effective divisor) on $V \backslash V_0$. Let

$$\Delta_\theta(\lambda) = \det \Big[\int\limits_{\xi \in V} \hat{R}_\theta(\delta(\xi, a_j)) \wedge \bar{\omega}_k(\xi) e_{\lambda,\theta}(\xi), \quad j, k = 1, \ldots, g, \Big]$$

where $\delta(\xi, a_j)$- Dirac (1,1)-form concentrated in $\{a_j\}$.

Let $E_\theta = \{\lambda \in \mathbb{C} : \ \Delta_\theta(\lambda) = 0\}$.

Definition 1.3. Parameter $\theta \in \mathbb{C}$ will be called generic if $\theta \notin \{\theta_1, \ldots, \theta_d\}$, where $\theta_l = -1/\gamma_l$. Divisor $\{a_1, \ldots, a_g\}$ on $V \backslash V_0$ will be called generic if

$$\det\Big[\frac{\omega_j}{dz_1}(a_k)\Big]_{j,k=1,\ldots,g} \neq 0.$$

Proposition 1.1. *Let parameter $\theta \in \mathbb{C}$ and divisor $\{a_1, \ldots, a_g\}$ on $V \backslash V_0$ be generic, where $V_0 = \{z \in V : \ |z_1| \leq r_0\}$, $g \geq 1$. Then for r_0 large enough we have inequalities:*

$$\overline{\lim}_{\lambda \to \infty} |\lambda^g \Delta_\theta(\lambda)| < \infty \quad \textit{and}$$
$$\forall \varepsilon > 0 \ \ \underline{\lim}_{\lambda \to \infty} |\lambda^g \Delta_\theta(\lambda)|_\varepsilon > 0, \quad \textit{where} \ \ |\lambda^g \Delta_\theta(\lambda)|_\varepsilon = \sup_{\{\lambda' : |\lambda' - \lambda| < \varepsilon\}} |(\lambda')^g \cdot \Delta_\theta(\lambda')|$$

Besides, the set E_θ is a closed nowhere dense subset of $\mathbb{C}$.

Let X be domain containing V_0 and relatively compact on V. Let $\sigma \in C^{(3)}(V)$, $\sigma > 0$, on V, $\sigma = 1$ on $V \backslash X$. Let Y be domain containing $\bar{X}$ and relatively compact on V. Let divisor $\{a_1, \ldots, a_g\}$ on $Y \backslash X$ and parameter $\theta \in \mathbb{C}$ be generic.

Definition 1.4. The functions $\psi_\theta(z, \lambda) = \sqrt{\sigma} F_\theta(z, \lambda) = \mu_\theta(z, \lambda) e^{\lambda(z_1 + \theta z_2)}$, $z \in V$, $\theta \in \mathbb{C} \backslash \{\theta_1, \ldots, \theta_d\}$, $\lambda \in \mathbb{C} \backslash E_\theta$, will be called the Faddeev type functions, associated with σ, θ and $\{a_1, \ldots, a_g\}$ if ψ_θ, F_θ, μ_θ satisfy correspondingly properties:

$$\begin{aligned}
&d\sigma d^c F_\theta = 2\sqrt{\sigma} e^{\lambda(z_1 + \theta z_2)} \sum_{j=1}^{g} C_{j,\theta}(\lambda) \delta(z, a_j), \\
&dd^c \psi_\theta = q\psi_\theta + 2e^{\lambda(z_1 + \theta z_2)} \sum_{j=1}^{g} C_{j,\theta}(\lambda) \delta(z, a_j), \\
&\bar{\partial}(\partial + \lambda(dz_1 + \theta dz_2))\mu_\theta = \frac{i}{2} q\mu_\theta + i \sum_{j=1}^{g} C_{j,\theta}(\lambda) \delta(z, a_j),
\end{aligned} \tag{1.1}$$

and the normalization condition

$$\lim_{\substack{z \in V_1 \\ z \to \infty}} \mu_\theta(z, \lambda) = 1, \tag{1.2}$$

where $\mu_\theta\big|_Y \in L^{\tilde{p}}(Y)$, $\mu_\theta\big|_{V \backslash Y} \in L^\infty(V \backslash Y)$, $\tilde{p} > 2$, $q = \frac{dd^c \sqrt{\sigma}}{\sqrt{\sigma}}$, $\{C_{j,\theta}\}$ are some functions of $\lambda \in \mathbb{C} \backslash E_\theta$.

Theorem 1.1. *Under the aforementioned notations and conditions, $\forall$ generic $\theta \in \mathbb{C}$, $\forall$ generic divisor $\{a_1, \ldots, a_g\} \subset V \backslash X$ and $\forall \ \lambda \in \mathbb{C} \backslash E_\theta : \ |\lambda| > const(V, \{a_j\}, \theta, \sigma)$ there exists unique Faddeev type function*

$$\psi_\theta(z, \lambda) = \sqrt{\sigma} F_\theta(z, \lambda) = e^{\lambda(z_1 + \theta z_2)} \mu_\theta(z, \lambda),$$

associated with conductivity function σ and divisor $\{a_1,\dots,a_g\}$. Moreover:

A) function $z\to\psi_\theta(z,\lambda)$ and parameters $\{C_{j,\theta}(\lambda)\}$ can be found from the following equations, depending on parameters $\theta\in\mathbb{C}$, $\lambda\in\mathbb{C}\backslash E_\theta$,

$$\begin{aligned}&\psi_\theta(z,\lambda)-\frac{i}{2}\int\limits_{\xi\in X} e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))}g_{\lambda,\theta}(z,\xi)\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\psi_\theta(z,\lambda)=\\ &e^{\lambda(z_1+\theta z_2)}+i\sum_{j=1}^{g}C_{j,\theta}(\lambda)g_{\lambda,\theta}(z,a_j)e^{\lambda(z_1+\theta z_2)},\end{aligned}\tag{1.3}$$

$$\begin{aligned}&2\sum_{j=1}^{g}C_{j,\theta}(\lambda)e_{\lambda,\theta}(a_j)\frac{\bar\omega_k}{d\bar z_1+\bar\theta d\bar z_2}(a_j)=\\ &-\int\limits_{z\in V} e^{-\bar\lambda(\bar z_1+\bar\theta\bar z_2)}\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\psi_\theta(z,\lambda)\frac{\bar\omega_k}{d\bar z_1+\bar\theta d\bar z_2}(z),\end{aligned}\tag{1.4}$$

where $k=1,2,\dots,g$ and $\{\omega_j\}$ is orthonormal basis of holomorphic forms on $\tilde V$;

B) functions $z\to\psi_\theta(z,\lambda)$ and parameters $\{C_{j,\theta}(\lambda)\}$ satisfy the following properties for $\lambda\in\mathbb{C}\backslash E_\theta:\ |\lambda|\ge const(V,\{a_j\},\theta,\sigma)$

$$\exists\lim_{\substack{z\to\infty,\ z\in V_l\\ l=1,2,\dots,d}}\frac{\bar z_1+\bar\theta\bar z_2}{\bar\lambda}e^{-\bar\lambda(\bar z_1+\bar\theta\bar z_2)}\Big(\frac{\partial\psi_\theta}{\partial\bar z_1}+\bar\theta\frac{\partial\psi_\theta}{\partial\bar z_2}\Big)=\lim_{\substack{z\to\infty\\ z\in V_l}}\psi_\theta e^{-\lambda(z_1+\theta z_2)}b_\theta(\lambda),\tag{1.5}$$

$$iC_{j,\theta}(\lambda)=(2\pi i)Res_{a_j}e^{-\lambda(z_1+\theta z_2)}\partial\psi_\theta\overset{\text{def}}{=}2\pi i\lim_{\varepsilon\to0}\int\limits_{|z-a_j|=\varepsilon}e^{-\lambda(z_1+\theta z_2)}\partial\psi_\theta,\tag{1.6}$$

$$\frac{\partial\psi_\theta(z,\lambda)}{\partial\bar\lambda}=b_\theta(\lambda)\overline{\psi_\theta(z,\lambda)},\tag{1.7}$$

$$\frac{\partial C_{j,\theta}(\lambda)}{\partial\bar\lambda}e^{\lambda(a_{j,1}+\theta a_{j,2})}=b_\theta(\lambda)\overline{C_{j,\theta}(\lambda)}e^{\bar\lambda(\bar a_{j,1}+\bar\theta\bar a_{j,2})}.\tag{1.8}$$

Besides,

$$\begin{aligned}&\bar\lambda b_\theta(\lambda)d=-\frac{1}{2\pi i}\int\limits_{z\in bX}e_{\lambda,\theta}(z)\bar\partial\mu(z)+i\sum_{j=1}^{g}C_{j,\theta}e_{\lambda,\theta}(a_j),\\ &|\lambda|\cdot|b_\theta(\lambda)|\le const(V,\{a_j\},\sigma)\frac{1}{(|\lambda|+1)^{1/3}}\frac{1}{|\Delta_\theta(\lambda)|(1+|\lambda|)^g},\\ &|C_{j,\theta}(\lambda)|\le const(V,\{a_j\},\sigma)\frac{1}{(|\lambda|+1)^{1/3}}\frac{1}{|\Delta_\theta(\lambda)|(1+|\lambda|)^g}.\end{aligned}\tag{1.9}$$

Remark 1.3. If $\|\ln\sqrt{\sigma}\|_{C^{(2)}(X)}\le const(V,\{a_j\},\theta)$ then the condition $\lambda\in\mathbb{C}\backslash E_\theta:\ \ |\lambda|\ge const(V,\{a_j\},\theta,\sigma)$ in Theorem 1.1 can be replaced by the condition $\lambda\in\mathbb{C}\backslash E_\theta$. Dependence of $const(V,\{a_j\},\theta,\sigma)$ of σ means its dependence only of $\|\ln\sqrt{\sigma}\|_{C^{(2)}(X)}$.

Definition 1.5. The functions $b_\theta(\lambda)$ and $\{C_{j,\theta}\}$ will be called "scattering" data for potential q.

Let $\hat{\Phi}(\psi\big|_{bX}) = \bar{\partial}\psi\big|_{bX}$ for all sufficiently regular solutions ψ of (0.2) in $\bar{X}$, where $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$. The operator Φ is equivalent to the Dirichlet-to-Neumann operator for (0.1). Let $\hat{\Phi}_0$ denote $\hat{\Phi}$ for $q \equiv 0$ on $\bar{X}$.

Theorem 1.2. *Under the conditions of Proposition 1.1 and Theorem 1.1, the following statements are valid:*

A) $\forall \lambda \in \mathbb{C}\backslash E_\theta :\ |\lambda| \geq const(V, \{a_j\}, \theta, \sigma)$ the restriction of $\psi_\theta(z,\lambda)$ on bX and data $\{C_{j,\theta}(\lambda)\}$ can be reconstructed from Dirichlet-to-Neumann data as unique solution of the Fredholm integral equation

$$\psi_\theta(z,\lambda)|_{bX} + \int\limits_{\xi\in bX} e^{\lambda[(z_1-\xi_1)+\theta(z_2-\xi_2)]} g_{\lambda,\theta}(z,\xi)(\hat{\Phi}-\hat{\Phi}_0)\psi_\theta(\xi,\lambda) = \tag{1.10}$$

$$e^{\lambda(z_1+\theta z_2)} + i\sum_{j=1}^{g} C_{j,\theta}(\lambda) g_{\lambda,\theta}(z,a_j) e^{\lambda(z_1+\theta z_2)}, \quad \textit{where}$$

$$\int\limits_{z\in bX} (z_1+\theta z_2)^{-k}(\partial + \lambda(dz_1+\theta dz_2))\mu_\theta(z,\lambda) = -\sum_{j=1}^{g}(a_{j,1}+\theta a_{j,2})^{-k} C_{j,\theta}(\lambda), \tag{1.11}$$

$k = 2,\ldots,g+1$, where (without restriction of generality) we suppose that values $\{a_{j,1}\}$ of the first coordinates of points $\{a_j\}$ are mutually different;

B) Function $\sigma(w)$, $w \in X$, can be reconstructed from Dirichlet-to- Neumann data

$$\psi_\theta\big|_{bX} \stackrel{\text{def}}{=} \mu_\theta\big|_{bX} e^{\lambda(z_1+\theta z_2)} \to \bar{\partial}\psi_\theta\big|_{bX}$$

by explicit formulas, where we assume that $\psi_\theta\big|_{bX}$ is found using (1.10), (1.11).

For the case $V = \{z \in \mathbb{C}^2 :\ P(z) = 0\}$, where P is a polynomial of degree N, this formula has the following form. Let $\{w_m\}$ be points of V, where $(dz_1+\theta dz_2)\big|_V(w_m) = 0$, $m = 1,\ldots,M$. Then for almost all θ values $\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}dd^c|z|^2}\big|_V(w_m)$ can be found from the following linear system

$$\tau(1+o(1))\frac{d^k}{d\tau^k}\Big(\int\limits_{z\in bX} e_{i\tau,\theta}(z)\bar{\partial}\mu_\theta(z,i\tau)\Big) =$$

$$\sum_{m=1}^{M}\frac{i\pi(1+|\theta|^2)}{2}\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}dd^c|z|^2}\bigg|_V(w_m)\times \tag{1.12}$$

$$\frac{|\frac{\partial P}{\partial z_1}(w)|^3\frac{d^k}{d\tau^k}\exp i\tau[(w_{m,1}+\theta w_{m,2})+(\bar{w}_{m,n}+\bar{\theta}\bar{w}_{m,2})]}{\big|\frac{\partial^2 P}{\partial z_1^2}\big(\frac{\partial P}{\partial z_2}\big)^2 - 2\frac{\partial^2 P}{\partial z_1\partial z_2}\big(\frac{\partial P}{\partial z_2}\big)\big(\frac{\partial P}{\partial z_1}\big)+\frac{\partial^2 P}{\partial z_2^2}\big(\frac{\partial P}{\partial z_1}\big)^2\big|(w_m)},$$

where $m,k = 1,\ldots,M$; $M = N(N-1)$, $\tau \in \mathbb{R}$, $\tau \to \infty$, $|\tau|^g|\Delta_\theta(i\tau)| \geq \varepsilon > 0$, ε- small enough. Determinant of system (1.12) is proportional to the determinant of Vandermonde.

C) If $g = 0$ and if $\theta = \theta(\lambda) = \lambda^{-2}$, then $\forall\ z \in X$ and $\forall \lambda \in \mathbb{C}$ function $\mu_\theta(z,\lambda) = \psi_\theta(z,\lambda)e^{-\lambda(z_1+\theta z_2}$ is unique solution of Fredholm integral equation

$$\mu_{\theta(\lambda)}(z,\lambda) + \frac{1}{2\pi i}\int_{\xi\in\mathbb{C}} b_{\theta(\xi)}(\xi)e^{\bar\xi(\bar z_1+\bar\theta(\xi)\bar z_2)-\xi(z_1+\theta(\xi)z_2)}\bar\mu_{\theta(\xi)}(z,\xi)\frac{d\xi\wedge d\bar\xi}{\xi-\lambda} = 1,$$

$$\text{where } \ |b_{\theta(\xi)}(\xi)| \le \frac{const(V)}{(1+|\xi|)^2},$$

and function $z \to \sigma(z)$, $z \in X$, can be found from equality

$$dd^c\psi_{\theta(\lambda)}(z,\lambda) = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}(z)\psi_{\theta(\lambda)}(z,\lambda), \quad z \in X.$$

Remark 1.4. Using the Faddeev type Green function constructed in [He], in [HM2] were obtained natural analogues of the main steps of the reconstruction scheme of [N2] on the Riemann surface V. In particular, under a smallness assumption on $\partial\log\sqrt{\sigma}$ the existence (and uniqueness) of the solution $\mu(z,\lambda)$ of the Faddeev type integral equation

$$\mu_\theta(z,\lambda) = 1 + \frac{i}{2}\int\limits_{\xi\in V} g_{\lambda,\theta}(z,\xi)\frac{\mu_\theta(\xi,\lambda)dd^c\sqrt{\sigma}}{\sqrt{\sigma}} + i\sum_{j=1}^{g} C_j g_{\lambda,\theta}(z,a_j), \ z \in V, \ \lambda \in \mathbb{C},$$

holds for any a priori fixed constants $C_1,\ldots,C_g$. However (and this fact was overlooked in [HM]) for $\lambda \in \mathbb{C}\backslash E$ there exists unique choise of constants $C_j(\lambda,\sigma)$ for which the integral equation above is equivalent to the differential equation

$$\bar\partial(\partial + \lambda(dz_1+\theta dz_2))\mu - \frac{i}{2}(\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\mu) + i\sum_{j=1}^{g} C_j\delta(z,a_j),$$

where $\delta(z,a_j)$ are Dirac measures concentrated in the points a_j.

2. Faddeev type functions on Riemann surfaces. Uniqueness

Let projective algebraic curve $\tilde V$ be embedded in $\mathbb{C}P^3$ and intersect $\mathbb{C}P^2_\infty = \{w \in \mathbb{C}P^3 : \ w_0 = 0\}$ transversally in $d > g$ points. Let $V = \tilde V\backslash\mathbb{C}P^2_\infty$, $V_0 = \{z \in V : \ |z_1| \le r_0\}$ and properties i)-iv) from § 1 be valid.

Proposition 2.1. *Let σ be positive function belonging to $C^{(2)}(V)$ such that $\sigma \equiv const = 1$ on $V\backslash X \subset V\backslash V_0 = \cup_{l=1}^d V_l$, where $\{V_l\}$ are connected components of $V\backslash\bar V_0$. Put $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$. Let $\{a_1,\ldots,a_g\}$ be generic divisor with support in $Y\backslash\bar X$, $\bar X \subset Y \subset \bar Y \subset V$. Let for generic $\theta \in \mathbb{C}$ and $\lambda \in \mathbb{C} : \ |\lambda| \ge const(V,\{a_j\},\theta,\sigma)$ function $z \mapsto \mu = \mu_\theta(z,\lambda)$ be such that:*

$$\begin{aligned}&\mu\big|_Y \in L^{\tilde p}(Y), \ \ \mu\big|_{V\backslash Y} \in L^\infty(V\backslash\bar Y),\\ &\bar\partial\mu\big|_Y \in L^p(Y), \ \ \bar\partial\mu\big|_{V\backslash\bar Y} \in L^{\tilde p}(V\backslash Y), \ \ 1 \le p < 2, \ \tilde p > 2,\end{aligned} \tag{2.1}$$

$$\bar\partial(\partial+\lambda(dz_1+\theta dz_2)\mu=\frac{i}{2}q\mu+i\sum_{j=1}^{g}C_j\delta(z,a_j)\ \ \textit{with some}\ \ C_j=C_{j,\theta}(\lambda)\ \ \textit{and} \tag{2.2}$$

$$\mu_\theta(z,\lambda)\to 0,\quad z\to\infty,\quad z\in V_1. \tag{2.3}$$

Then $\mu_\theta(z,\lambda)\equiv 0$, $z\in V$.

Remark 2.1. Proposition 2.1 is a corrected version of Proposition 2.1 of [HM2]. For the case $V=\mathbb{C}$ the equivalent result goes back to [BC2].

Lemma 2.1. *Let* $\psi=\sqrt{\sigma}F=e^{\lambda(z_1+\theta z_2)}\mu$, *where* μ *satisfies (2.1), (2.2) and*

$$F_1=\sqrt{\sigma}\partial F,\quad F_2=\sqrt{\sigma}\bar\partial F. \tag{2.4}$$

Then forms F_1, F_2 *satisfy the system of equations*

$$\begin{aligned}\bar\partial F_1+F_2\wedge\partial\ln\sqrt{\sigma}&=ie^{\lambda(z_1+\theta z_2)}\sum_{j=1}^{g}C_j\delta(z,a_j),\\ \partial F_2+F_1\wedge\bar\partial\ln\sqrt{\sigma}&=-ie^{\lambda(z_1+\theta z_2)}\sum_{j=1}^{g}C_j\delta(z,a_j).\end{aligned} \tag{2.5}$$

Proof of Lemma 2.1. From definition of F_1 and F_2 it follows that

$$\begin{aligned}&d\sigma d^cF=i[2\sigma\partial\bar\partial F-\bar\partial\sigma\wedge\partial F+\partial\sigma\wedge\bar\partial F]=\\ &2i\sqrt{\sigma}(\partial F_2+F_1\wedge\bar\partial\ln\sqrt{\sigma})=-2i\sqrt{\sigma}(\bar\partial F_1+F_2\wedge\partial\ln\sqrt{\sigma}).\end{aligned}$$

From (2.4) and (2.2) we deduce also that

$$d(\sigma d^cF)=\sqrt{\sigma}\big(dd^c\psi-\psi\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\big)=2\sqrt{\sigma}e^{\lambda(z_1+\theta z_2)}\sum_{j=1}^{g}C_j\delta(z,a_j).$$

These equalities imply (2.5).

Lemma 2.1 is proved.

Lemma 2.2. *Let* $\{b_m\}$ *be the points of* X, *where* $(dz_1+\theta dz_2)\big|_X(b_m)=0$.
Let $B^0=\cup_m\{b_m\}$ *and* $A^0=\cup_j\{a_j\}$.
Let $u_\pm=m_1\pm e_{-\lambda,\theta}(z)\bar m_2$, *where* $m_1=e^{-\lambda(z_1+\theta z_2)}f_1$, $m_2=e^{-\lambda(z_1+\theta z_2)}f_2$, $f_1=\sqrt{\sigma}\frac{\partial F}{\partial z_1}$, $f_2=\sqrt{\sigma}\frac{\partial F}{\partial\bar z_1}$. *Let also* $q_1=\frac{\partial\ln\sqrt{\sigma}}{\partial z_1}$ *and* $\delta_0(z,a_j)=\frac{\delta(z,a_j)}{dz_1\wedge d\bar z_1}$. *Then in conditions of Lemma 2.1*

$$\begin{aligned}&\sup_{z\in X}|\bar\partial u_\pm\big|_X(z)\cdot dist^2(z,B^0)|=O\big(\sup_{z\in X}|u_\pm dist(z,B^0)|\big)<\infty;\\ &u_\pm\big|_{V\backslash X}\in L^1(V\backslash X)\cap O(V\backslash(X\cup A^0))\end{aligned} \tag{2.6}$$

and system (2.5) is equivalent to the system

$$\begin{aligned}&\frac{\partial u_\pm}{\partial \bar z_1}d\bar z_1 = \mp(e_{-\lambda,\theta}(z)q_1\bar u_\pm)d\bar z_1+\\ &i\sum_{j=1}^{g}(C_j \pm \bar C_j e_{-\lambda,\theta}(z))\delta_0(z,a_j)d\bar z_1.\end{aligned} \tag{2.7}$$

Proof of Lemma 2.2. From (2.1) we deduce the property

$$\begin{aligned}&u_\pm\big|_Y \in L^p(Y), \quad 1\le p<2,\\ &u_\pm\big|_{V\backslash Y} \in L^{\tilde p}(V\backslash Y)\oplus L^\infty(V\backslash Y), \quad \tilde p>2.\end{aligned}$$

System (2.5) is equivalent to the system of equations

$$\begin{aligned}&\frac{\partial f_1}{\partial \bar z_1} = -f_2 q_1 + i e^{\lambda(z_1+\theta z_2)}\sum_{j=1}^{g} C_j\delta_0(z,a_j),\\ &\frac{\partial f_2}{\partial z_1} = -f_1 \bar q_1 + i e^{\lambda(z_1+\theta z_2)}\sum_{j=1}^{g} C_j\delta_0(z,a_j).\end{aligned}$$

This system and definition of m_1, m_2 imply

$$\begin{aligned}&\frac{\partial m_1}{\partial \bar z_1} = -q_1 m_2 + i\sum_{j=1}^{g} C_j\delta_0(z,a_j),\\ &\frac{\partial m_2}{\partial z_1} + \lambda m_2\big(1+\theta\frac{\partial z_2}{\partial z_1}\big) = -\bar q_1 m_1 + i\sum_{j=1}^{g} C_j\delta_0(z,a_j).\end{aligned}$$

From the last equalities and definition of $u_\pm$ we deduce

$$\frac{\partial u_\pm}{\partial \bar z_1} = \frac{\partial m_1}{\partial \bar z_1} \pm e_{-\lambda,\theta}(z)\big(\frac{\partial \bar m_2}{\partial \bar z_1} + \bar\lambda\big(1+\bar\theta\frac{\partial \bar z_2}{\partial \bar z_1}\big)\bar m_2\big) = -q_1 m_2 + i\sum_{j=1}^{g} C_j\delta_0(z,a_j)\pm$$

$$e_{-\lambda,\theta}(z)\big(\bar\lambda\big(1+\bar\theta\frac{\partial \bar z_2}{\partial \bar z_1}\big)\bar m_2 - \bar\lambda \bar m_2\big(1+\bar\theta\frac{\partial \bar z_2}{\partial \bar z_1}\big) - q_1\bar m_1 + i\sum_{j=1}^{g}\bar C_j\delta_0(z,a_j)\big) =$$

$$\mp(e_{-\lambda,\theta}(z)q_1\bar u_\pm) + i\sum_{j=1}^{g}(C_j \pm \bar C_j e_{-\lambda,\theta}(z))\delta_0(z,a_j).$$

Property (2.7) is proved.

For proving (2.6) we will use construction coming back to Bers and Vekua (see [Ro], [V]). Let $\beta_\pm$ be continuous on Y solutions of $\bar\partial$- equations

$$\bar\partial\beta_\pm = \pm e_{-\lambda,\theta}(z)q_1\frac{\bar u_\pm}{u_\pm}d\bar z_1,$$

where the right-hand side belongs to $L^{\infty}_{0,1}(Y)$.

Functions $v_{\pm} = u_{\pm}e^{-\beta_{\pm}}$ belongs to $\mathcal{O}(Y)$. Indeed, from (2.1), (2.2) it follows that $\mu \in W^{1,p}(Y) \cap W^{1,\tilde{p}}_{loc}(Y\backslash(A^0 \cup B^0))$. From this and from definition of $v_{\pm}$ we deduce that $\bar{\partial} v_{\pm} = q_1\bar{u}_{\pm}d\bar{z}_1 e^{-\beta_{\pm}} - q_1 u_{\pm}\frac{\bar{u}_{\pm}}{u_{\pm}}e^{-\beta_{\pm}}d\bar{z}_1 = 0$ on $Y\backslash(A^0 \cup B^0)$ and the following formula for $u_{\pm}$ is valid

$$u_{\pm}(z) = v_{\pm}(z)e^{\beta_{\pm}(z)}. \tag{2.8}$$

From this and (2.7), (2.8) we obtain (2.6).

Lemma 2.2 is proved.

Lemma 2.3. *Let $u_{\pm}$ be the functions from Lemma 2.2 and μ be the function from Lemma 2.1. Then*

$$u_{\pm} = \frac{\partial \mu}{\partial z_1} + \lambda\big(1 + \theta\frac{\partial z_2}{\partial z_1}\big)\mu - q_1\mu \pm e_{-\lambda,\theta}(z)\big(\frac{\partial \bar{\mu}}{\partial z_1} - q_1\bar{\mu}\big).$$

Proof of Lemma 2.3. We have

$$u_{\pm} = e^{-\lambda(z_1+\theta z_2)}f_1 \pm e^{-\lambda(z_1+\theta z_2)}\bar{f}_2 = e^{-\lambda(z_1+\theta z_2)}(f_1 \pm \bar{f}_2),$$

where

$$f_1 = \sqrt{\sigma}\frac{\partial F}{\partial z_1} = \sqrt{\sigma}\frac{\partial}{\partial z_1}\big(\frac{1}{\sqrt{\sigma}}e^{\lambda(z_1+\theta z_2)}\mu\big) =$$
$$e^{\lambda(z_1+\theta z_2)}\big(\frac{\partial \mu}{\partial z_1} + \lambda\big(1+\theta\frac{\partial z_2}{\partial z_1}\big)\mu - q_1\mu\big),$$
$$\bar{f}_2 = \sqrt{\sigma}\frac{\partial \bar{F}}{\partial z_1} = \sqrt{\sigma}\frac{\partial}{\partial z_1}\big(\frac{1}{\sqrt{\sigma}}e^{\bar{\lambda}(\bar{z}_1+\bar{\theta}\bar{z}_2)}\bar{\mu}\big) =$$
$$e^{\bar{\lambda}(\bar{z}_1+\bar{\theta}\bar{z}_2)}\big(\frac{\partial \bar{\mu}}{\partial z_1} - q_1\bar{\mu}\big).$$

This imply Lemma 2.3.

Lemma 2.4. *Let $\omega_1, \dots, \omega_g$ be orthonormal basis of holomorphic 1-forms on $\tilde{V}$. Let $\{a_1, \dots, a_g\}$ be generic divisor on $Y\backslash\bar{X}$, where $V_0 \subset \bar{X} \subset Y \subset V$. Put $\omega^0_{j,k} = \frac{\omega_k}{dz_1}(a_j)$. Let for some generic $\theta \in \mathbb{C}$ and $\lambda \in \mathbb{C}$ functions $u_{\pm}$ from Lemmas 2.2-2.3 satisfy (2.6), (2.7) with some $C_j = C_{j,\theta}(\lambda)$. Then*

$$\sup_j |C_{j,\theta}(\lambda)| \le const(V, \{a_j\}, \theta)\|\ln\sqrt{\sigma}\|^2_{W^{2,\infty}(X)}(1+|\lambda|)^{-1/3}\|u_{\pm}\|_{L^{\infty}(X,B^0)},$$

where $\|u_{\pm}\|_{L^{\infty}(X,B^0)} \stackrel{\text{def}}{=} \sup_{z\in X}|u_{\pm}(z)dist(z,B^0)|.$

Proof of Lemma 2.4. From condition iv) of section 1 we deduce $|\omega^0_{j,k}| < \infty$. From definition of generic divisor we obtain $\det[\omega^0_{j,k}] \neq 0$. From (2.7) and from definition of

Dirac measure $\forall\ k = 1, \ldots, g$ we deduce

$$\overline{\lim_{r\to\infty}}\Big(\int\limits_{\{z\in V:\ |z_1|=r\}} u_\pm \wedge \omega_k\Big) \pm \int\limits_X e_{-\lambda,\theta}(z)\frac{\partial \ln\sqrt{\sigma}}{\partial z_1}\bar u_\pm d\bar z_1 \wedge \omega_k =$$
$$i\int\limits_Y \sum_{j=1}^{g}(C_j \pm \bar C_j e_{-\lambda,\theta}(z))\delta_0(z,a_j)d\bar z_1 \wedge \omega_k = \tag{2.9}$$
$$i\sum_{j=1}^{g}(C_j \pm \bar C_j e_{-\lambda,\theta}(a_j))\omega^0_{j,k},\ \ j,k = 1,2,\ldots,g.$$

From estimates $\overline{\lim\limits_{r_n\to\infty}}\ \sup\limits_{\{z\in V:\ |z_1|=r_n\}} |u_\pm(z)| < \infty$, for some sequence $r_n \to \infty$, and $\frac{|\omega_k|}{dz_1} \le O\big(|\frac{1}{z_1^2}|\big)$, $z \in V\backslash Y$, $k = 1, \ldots, g$, we obtain

$$\overline{\lim_{r\to\infty}}\Big|\int\limits_{\{z\in V:\ |z_1|=r\}} u_\pm \wedge \omega_k\Big| = 0. \tag{2.10}$$

From (2.9), (2.10) and Kramers's formula we obtain

$$i(C_j \pm \bar C_j e_{-\lambda,\theta}(a_j)) =$$
$$\frac{\det[\omega^0_{1,k};\ldots;\omega^0_{j-1,k};\int\limits_X \pm e_{-\lambda,\theta}(z)\frac{\partial\ln\sqrt{\sigma}}{\partial z_1}\bar u_\pm d\bar z_1\wedge\omega_k;\omega^0_{j+1,k};\ldots;\omega^0_{g,k}]}{\det[\omega^0_{j,k}]}, \tag{2.11}$$

where $j, k = 1, \ldots, g$.

Let us prove estimate

$$\Big|\int\limits_X e_{-\lambda,\theta}(z)\frac{\partial \ln\sqrt{\sigma}}{\partial z_1}\bar u_\pm d\bar z_1 \wedge \omega_k\Big| \le$$
$$const(X,\theta)(1+|\lambda|)^{-1/3}\|\ln\sqrt{\sigma}\|^2_{W^{2,\infty}(X)}\cdot\|u_\pm\|_{L^\infty(X,B^0)}. \tag{2.12}$$

For $|\lambda| \le 1$ estimate follows directly, using that $\ln\sqrt{\sigma} \in W^{1,\infty}(X)$.

Let $B^\varepsilon = \cup_{m=1}^{M}\{z \in X:\ |z - b_m| \le \varepsilon\}$.

Let $\chi_{\varepsilon,\nu}$, $\nu = 1, 2$, be functions from $C^{(1)}(V)$ such that $\chi_{\varepsilon,1} + \chi_{\varepsilon,2} \equiv 1$ on V, $supp\,\chi_{\varepsilon,1} \subset B^{2\varepsilon}$, $supp\,\chi_{\varepsilon,2} \subset V\backslash B^\varepsilon$, $|d\chi_{\varepsilon,\nu}| = O(\frac{1}{\varepsilon})$, $\nu = 1, 2$.

Put $J^\varepsilon_\nu u_\pm = \int\limits_X \chi_{\varepsilon,\nu}(z)e_{-\lambda,\theta}(z)\frac{\partial\ln\sqrt{\sigma}}{\partial z_1}\bar u_\pm d\bar z_1 \wedge \omega_k$, $\nu = 1, 2$. We have directly:

$$|J^\varepsilon_1 u_\pm| \le const(X)\varepsilon\|\ln\sqrt{\sigma}\|_{W^{1,1}(X)}\cdot\|u_\pm\|_{L^\infty(X,B^0)}. \tag{2.13}$$

For $J_2^\varepsilon u_\pm$ we obtain by integration by parts:

$$J_2^\varepsilon u_\pm = -\frac{1}{\lambda}\int\limits_X \chi_{\varepsilon,2}\partial e_{-\lambda,\theta}(z)\frac{\partial \ln\sqrt{\sigma}}{\partial z_1}\bar{u}_\pm d\bar{z}_1\wedge\frac{\omega_k}{dz_1+\theta dz_2} = \frac{1}{\lambda}\int\limits_X e_{-\lambda,\theta}(z)\partial\Big(\chi_{\varepsilon,2}\frac{\partial\ln\sqrt{\sigma}}{\partial z_1}\bar{u}_\pm d\bar{z}_1\wedge\frac{\omega_k}{dz_1+\theta dz_2}\Big). \tag{2.14}$$

To estimate (2.14) we use (2.6) and the following properties: $|\partial\chi_{\varepsilon,2}| = O(\frac{1}{\varepsilon})$, $supp(\partial\chi_{\varepsilon,2})\subset B^{2\varepsilon}$,

$$\|\frac{\partial\ln\sqrt{\sigma}}{\partial z_1}d\bar{z}_1\wedge\partial\chi_{\varepsilon,2}u_\pm\frac{\omega_k}{dz_1+\theta dz_2}\|_{L^1_{0,1}(X)}\le \frac{const(X,\theta)}{\varepsilon}\|\ln\sqrt{\sigma}\|_{W^{1,\infty}(X)}\|u_\pm\|_{L^\infty(X,B^0)}$$
$$\|\frac{\partial^2\ln\sqrt{\sigma}}{\partial z_1^2}dz_1\wedge d\bar{z}_1\chi_{\varepsilon,2}u_\pm\frac{\omega_k}{dz_1+\theta dz_2}\|_{L^1_{1,1}(X)}\le |\ln\varepsilon|const(X,\theta)\|\ln\sqrt{\sigma}\|_{W^{2,\infty}(X)}\|u_\pm\|_{L^\infty(X,B^0)}$$
$$\|\frac{\partial\ln\sqrt{\sigma}}{\partial z_1}d\bar{z}_1\chi_{\varepsilon,2}u_\pm\wedge\partial\big(\frac{\omega_k}{dz_1+\theta dz_2}\big)\|_{L^1_{0,1}(X)}\le \frac{const(X,\theta)}{\varepsilon}\|\ln\sqrt{\sigma}\|_{W^{1,\infty}(X)}\|u_\pm\|_{L^\infty(X,B^0)}$$
$$\partial\bar{u}_\pm\big|_X = \mp(e_{\lambda,\theta}(z)\bar{q}_1\bar{u}_\pm)dz_1.$$

From (2.14), (2.6) and these properties we obtain

$$|J_2^\varepsilon u_\pm|\le|\ln\varepsilon|\frac{const(X,\theta)}{|\lambda|}\|\ln\sqrt{\sigma}\|_{W^{2,\infty}(X)}\cdot\|u_\pm\|_{L^\infty(X,B^0)}+ \frac{const(X,\theta)}{\varepsilon|\lambda|}\|\ln\sqrt{\sigma}\|_{W^{1,\infty}(X)}\cdot\|u_\pm\|_{L^\infty(X,B^0)}+ \frac{const(X,\theta,\delta)}{\varepsilon^{1+\delta}|\lambda|}\|\ln\sqrt{\sigma}\|_{W^{1,\infty}(X)}\cdot\|u_\pm\|_{L^\infty(X,B^0)}. \tag{2.15}$$

Putting in (2.13), (2.15) $\varepsilon = \frac{1}{\sqrt{\lambda}}$ and $\delta = 1/3$ we obtain (2.12) for $|\lambda|\ge 1$.

Inequalities (2.11), (2.12) imply estimate

$$|C_j\pm\bar{C}_j e_{-\lambda,\theta}(a_j)|\le const(X,\{a_j\},\theta)(1+|\lambda|)^{-1/3}\|\ln\sqrt{\sigma}\|^2_{W^{2,\infty}(X)}\cdot\|u_\pm\|_{L^\infty(X,B^0)}.$$

We obtained statement of Lemma 2.4.

Lemma 2.5. *Let functions $u_\pm$ satisfy (2.6), (2.7) and R - operator from section 1. Then*

$$\|R[e_{-\lambda,\theta}q_1\bar{u}_\pm d\bar{\xi}_1\|_{L^\infty(X,B^0)}\le const(X,\theta)(1+|\lambda|)^{-1/5}\|\ln\sqrt{\sigma}\|_{W^{2,\infty}(X)}\cdot\|u_\pm\|_{L^\infty(X,B^0)}.$$

Proof of Lemma 2.5.

Let $\chi_{\varepsilon,\nu}$, $\nu = 1,2$, be partition of unity from Lemma 2.4. Put $S^{\varepsilon}_{\nu}u_{\pm} = R[\chi_{\varepsilon,\nu}q_1\bar{u}_{\pm}d\bar{\xi}_1]$, $\nu = 1,2$. Using (2.6) and formula for operator R we deduce estimate

$$\|S^{\varepsilon}_1 u_{\pm}\|_{L^{\infty}(X,B^0)} = O(\varepsilon)\|\ln\sqrt{\sigma}\|_{W^{1,\infty}(X)}\|u_{\pm}\|_{L^{\infty}(X,B^0)}. \tag{2.16}$$

Let $R_{1,0}(\xi,z)$ be kernel of operator R. It means, in particular, that $\bar{\partial}_{\xi}R_{1,0}(\xi,z) = -\delta(\xi,z)$, where $\delta(\xi,z)$- Dirac (1,1)- measure, concentrated in the point $\xi = z$. We have

$$S^{\varepsilon}_2 u_{\pm} = \int\limits_X \chi_{\varepsilon,2}e_{-\lambda,\theta}q_1\bar{u}_{\pm}d\bar{\xi}_1R_{1,0}(\xi,z). \tag{2.17}$$

Integration by parts in (2.17) gives the following

$$\begin{gathered} S^{\varepsilon}_2 u_{\pm} = \frac{1}{\bar{\lambda}}\int\limits_X \bar{\partial}e_{-\lambda,\theta}(\xi)\frac{d\bar{\xi}_1}{d\bar{\xi}_1 + \bar{\theta}d\bar{\xi}_2}\chi_{\varepsilon,2}(\xi)q_1(\xi)\bar{u}_{\pm}(\xi)R_{1,0}(\xi,z) = \\ -\frac{1}{\bar{\lambda}}\int\limits_X e_{-\lambda,\theta}(\xi)\bar{\partial}\Big(\frac{d\bar{\xi}_1}{d\bar{\xi}_1 + \bar{\theta}d\bar{\xi}_2}\chi_{\varepsilon,2}(\xi)q_1(\xi)\bar{u}_{\pm}(\xi)\Big)R_{1,0}(\xi,z) + \\ \frac{1}{\bar{\lambda}}e_{-\lambda,\theta}(z)\frac{d\bar{\xi}_1}{d\bar{\xi}_1 + \bar{\theta}d\bar{\xi}_2}(z)\chi_{\varepsilon,2}(z)q_1(z)\bar{u}_{\pm}(z). \end{gathered} \tag{2.18}$$

To estimate (2.18) we use (2.6), properties of partition of unity $\{\chi_{\varepsilon,\nu}\}$ and inequalities

$$\begin{gathered} \Big|\frac{d\bar{\xi}_1}{d\bar{\xi}_1 + \bar{\theta}d\bar{\xi}_2}(\xi)\Big| = O\big(\frac{1}{dist(\xi,B^0)}\big), \quad \Big|\bar{\partial}\frac{d\bar{\xi}_1}{d\bar{\xi}_1 + \bar{\theta}d\bar{\xi}_2}(\xi)\Big| = O\big(\frac{1}{(dist(\xi,B^0))^2}\big), \\ |q_1(\xi)| = O\big(\frac{1}{dist(\xi,B^0)}\big), \quad |\bar{\partial}q_1(\xi)| = O\big(\frac{1}{(dist(\xi,B^0))^2}\big), \quad \xi \in X. \end{gathered} \tag{2.19}$$

From (2.19), (2.8) and from the formula for operator R we deduce estimate

$$\|S^{\varepsilon}_2 u_{\pm}\|_{L^{\infty}(X)} = O\big(\frac{1}{\varepsilon^4|\lambda|}\big)\|\ln\sqrt{\sigma}\|_{W^{2,\infty}(X)}\|u_{\pm}\|_{L^{\infty}(X,B^0)}. \tag{2.20}$$

Putting in (2.16), (2.20) $\varepsilon = \frac{1}{|\lambda|^{1/5}}$ we obtain statement of Lemma 2.5.

Proof of Proposition 2.1.

Let function μ satisfy conditions (2.1)-(2.3) and $u_{\pm}$ be functions defined in Lemma 2.2. Then by Lemma 2.3 we have

$$\begin{gathered} \lim_{\substack{z\to\infty \\ z\in V_1}} u_{\pm}(z,\lambda) = \lim_{\substack{z\to\infty \\ z\in V_1}}(m_1 \pm e_{-\lambda,\theta}(z)\bar{m}_2) = \\ \lim_{\substack{z\to\infty \\ z\in V_1}}[\lambda\big(1 + \theta\frac{dz_2}{dz_1}\big)\mu + \frac{\partial\mu}{\partial z_1} \pm e_{-\lambda,\theta}(z)\frac{\partial\bar{\mu}}{\partial z_1}] \to 0. \end{gathered} \tag{2.21}$$

Let

$$h_{\pm} = u_{\pm} \pm R[(e_{-\lambda,\theta}(z) q_1 \bar{u}_{\pm}) d\bar{z}_1 - i \sum_{j=1}^{g} (C_j \pm \bar{C}_j e_{-\lambda,\theta}(z)) \delta_0(z, a_j) d\bar{z}_1], \tag{2.22}$$

where R is the operator from section 1.

By Lemmas 2.2-2.5 and properties of operator R we have $h_{\pm} \in \mathcal{O}(V) \cap L^{\infty}(V)$ and $h_{\pm}(z,\lambda) \to 0$, $z \to \infty$, $z \in V_1$. By Liouville theorem, $h_{\pm}(z,\lambda) \equiv 0$ on V, $\lambda \in \mathbb{C}$. Then from (2.22) with $h_{\pm}(z,\lambda) \equiv 0$ and Lemmas 2.4, 2.5 it follows that $u_{\pm}(z,\lambda) \equiv 0$, $z \in V$, if $\lambda \in \mathbb{C} \backslash E_\theta :\ |\lambda| \geq const(V, \{a_j\}, \theta) \| \ln \sqrt{\sigma} \|^2_{W^{2,\infty}(X)}$. Property $u_{\pm}(z,\lambda) \equiv 0$, $z \in V$, implies by Lemma 2.3 equality $\frac{\partial \mu}{\partial \bar{z}_1} - \bar{q}_1 \mu = 0$, $z \in V$, where $\mu(z) \to \infty$ if $z \in V_1$, $z \to \infty$. The Liouville type theorem for generalized holomorphic functions ([Ro], theorem 7.1) implies $\mu \equiv 0$. Proposition 2.1 is proved.

3. Faddeev type functions on Riemann surface. Existence. Proof of Theorem 1.1A

Proposition 3.1. *Let conductivity σ and divisor $\{a_1, \ldots, a_g\}$ satisfy conditions of Proposition 2.1. Then $\forall$ generic $\theta \in \mathbb{C}$ and $\forall \lambda \in \mathbb{C} \backslash E_\theta :\ |\lambda| \geq const(V, \{a_j\}, \theta, \sigma)$ there exists unique Faddeev type function*

$$\begin{aligned} &\psi \stackrel{\text{def}}{=} \sqrt{\sigma} F \stackrel{\text{def}}{=} e^{\lambda(z_1 + \theta z_2)} \mu, \quad \text{where} \\ &\psi = \psi_\theta(z,\lambda),\ F = F_\theta(z,\lambda),\ \mu = \mu_\theta(z,\lambda), \end{aligned} \tag{3.1}$$

associated with σ and divisor $\{a_1, \ldots, a_g\}$, i.e.

$$\begin{aligned} &\bar{\partial}(\partial + \lambda(dz_1 + \theta dz_2))\mu = \frac{i}{2} q \mu + \sum_{j=1}^{g} C_j \delta(z, a_j), \quad \textit{for some} \ \ C_j = C_{j,\theta}(\lambda), \quad \textit{where} \\ &q = \frac{dd^c \sqrt{\sigma}}{\sqrt{\sigma}},\ \mu\big|_Y \in L^{\tilde{p}}(Y),\ \mu\big|_{V \backslash \bar{Y}} \in L^{\infty}(V \backslash \bar{Y}),\ \lim_{\substack{z \to \infty \\ z \in V_1}} \mu_\theta(z,\lambda) = 1. \end{aligned} \tag{3.1a}$$

In addition,

$$\begin{aligned} &\|\mu_\theta(z,\lambda) - \mu_\theta(\infty_l,\lambda)\|_{L^{\tilde{p}}(V)} \leq \frac{const(V, \{a_j\}, \theta, \sigma, \tilde{p}, \varepsilon)}{|\Delta_\theta(\lambda)| \cdot (1 + |\lambda|)^{g+1-\varepsilon}}, \\ &\textit{where} \quad \mu_\theta(\infty_l,\lambda) \stackrel{\text{def}}{=} \lim_{\substack{z \to \infty \\ z \in V_l}} \mu_\theta(z,\lambda), \quad l = 1, \ldots, d, \\ &\|\partial \mu\|_{L^p_{1,0}(Y)} + \|\partial \mu\|_{L^{\tilde{p}}_{1,0}(V \backslash Y)} \leq \frac{const(V, \{a_j\}, \theta, \sigma, p, \tilde{p}, \varepsilon)}{|\Delta_\theta(\lambda)| \cdot (1 + |\lambda|)^{g-\varepsilon}}, \quad p < 2,\ \tilde{p} > 2, \end{aligned} \tag{3.1b}$$

$$\begin{aligned} &\forall \ \textit{generic} \ \ \theta \in \mathbb{C} \ \ \textit{and} \ \ \lambda \in \mathbb{C} \backslash E_\theta :\ |\lambda| \geq const(V, \{a_j\}, \theta, \sigma), \\ &\frac{\partial \mu}{\partial \bar{\lambda}}\big|_Y \in W^{1,p}(Y), \ \ \frac{\partial \mu}{\partial \bar{\lambda}}\big|_{V_l \backslash Y} \in L^{\infty}(V_l \backslash Y) \cup W^{1,\tilde{p}}(V_l \backslash Y), \end{aligned} \tag{3.1c}$$

where $\{V_l\}$ are connected components of $V\backslash V_0$, $l = 1, \ldots, d$,

$$e_{\lambda,\theta}(z) = e^{\lambda(z_1+\theta z_2)-\bar\lambda(\bar z_1+\bar\theta\bar z_2)}.$$

Remark 3.1. Proposition 3.1 is a corrected version of Proposition 2.2 from [HM2]. For the case $V = \mathbb{C}$ the results of such a type goes back to [F1], [F2].

Lemma 3.1. *Under the conditions of Proposition 3.1, $\forall\lambda \in \mathbb{C}\backslash E_\theta$ function $z \to \mu_\theta(z,\lambda)$ belonging to $L^{\tilde p}(Y)$ on Y and to $L^\infty(V\backslash Y)$ on $V\backslash Y$ satisfies (3.1a) iff there exists $C_j = C_{j,\theta}(\lambda)$, $j = 1, \ldots, g$, such that*

$$\mu_\theta(z,\lambda) = 1 + \frac{i}{2}\int\limits_{\xi\in X} g_{\lambda,\theta}(z,\xi)q\mu_\theta(\xi,\lambda) + i\sum_{j=1}^{g} C_{j,\theta}(\lambda)g_{\lambda,\theta}(z,a_j) \tag{3.2}$$

and one of two equivalent conditions is valid

$$\begin{gathered}\mathcal{H}_{\lambda,\theta}\big(\hat R_\theta\big(\frac{i}{2}q\mu\big)\big) + i\sum_{j=1}^{g} C_{j,\theta}(\lambda)\mathcal{H}_{\lambda,\theta}(\hat R_\theta(\delta(z,a_j)) = 0 \ \text{ or} \\ (\partial + \lambda(dz_1+\theta dz_2))\mu_\theta(z,\lambda) \in H_{1,0}(V\backslash(X\cup_{j=1}^{g}\{a_j\}))\cap L^1_{1,0}(Y\backslash X),\end{gathered} \tag{3.3}$$

where $g_{\lambda,\theta}$ is Faddeev type Green function, $\hat R_\theta$, $\mathcal{H}_{\lambda,\theta}$ - operators defined in section 1.

Proof of Lemma 3.1. From Proposition 4 in [He] and from definition of Green function $g_{\lambda,\theta}(z,\xi)$ we deduce that integral equation (3.2) is equivalent to the following differential equation

$$\begin{gathered}\bar\partial(\partial + \lambda(dz_1+\theta dz_2))\mu = \frac{i}{2}q\mu + i\sum_{j=1}^{g} C_{j,\theta}\delta(z,a_j) + \\ \bar\lambda(d\bar z_1 + \bar\theta d\bar z_2))\times\big[\mathcal{H}_{\lambda,\theta}\big(\hat R_\theta\big(\frac{i}{2}q\mu\big)\big) + i\sum_{j=1}^{g} C_{j,\theta}\mathcal{H}_{\lambda,\theta}(\hat R_\theta(\delta(z,a_j))\big].\end{gathered} \tag{3.4}$$

Equation (3.4) is equivalent to (3.1a) if one of two equivalent conditions (3.3) is valid.
Lemma 3.1 is proved.

Lemma 3.2. *Let $\{a_1, \ldots, a_g\}$ be generic divisor in $Y\backslash\bar X$. Then for any generic $\theta \in \mathbb{C}$ and $\forall\lambda \in \mathbb{C}\backslash E_\theta$: $|\lambda| \ge const(V, \{a_j\}, \theta, \sigma)$, integral equation (3.2), (3.3) is uniquely solvable Fredholm integral equation in the space $\tilde W^{1,\tilde p}(V)$.*

Proof of Lemma 3.2. Let $\theta \in \mathbb{C}$ and $\lambda \in \mathbb{C}\backslash E_\theta$: $|\lambda| \ge const(V, \{a_j\}, \theta, \sigma)$. From (3.2), (3.3) we obtain integral equation for $\tilde\mu_\theta = \mu_\theta - 1$ and $\tilde C_{j,\theta}$:

$$\begin{gathered}\tilde\mu_\theta(z,\lambda) - \frac{i}{2}\int\limits_{\xi\in V} g_{\lambda,\theta}(z,\xi)q(\xi)\tilde\mu_\theta(\xi,\lambda) - i\sum_{j=1}^{g}\tilde C_{j,\theta}(\lambda)g_{\lambda,\theta}(z,a_j) = \\ \frac{i}{2}\int\limits_{\xi\in V} g_{\lambda,\theta}(z,\xi)q(\xi) + i\sum_{j=1}^{g} C^0_{j,\theta}(\lambda)g_{\lambda,\theta}(z,a_j).\end{gathered} \tag{3.5}$$

Parameters $\tilde{C}_j = \tilde{C}_{j,\theta}(\lambda)$, $j = 1, \ldots, g$, are defined by the equations:

$$
\begin{aligned}
&-i\sum_{j=1}^{g}\tilde{C}_j\int\limits_V \hat{R}_\theta(\delta(\xi,a_j))\bar{\omega}_k(\xi)e_{\lambda,\theta}(\xi) = \\
&\int\limits_{\xi\in V} e_{\lambda,\theta}(\xi)\hat{R}_\theta\big(\frac{i}{2}q\tilde{\mu}\big)\bar{\omega}_k(\xi), \quad k = 1,2,\ldots,g.
\end{aligned}
\tag{3.6}
$$

We remind that determinant of system (3.6) is exactly $\Delta_\theta(\lambda)$.

Parameters $C^0_{j,\theta}$ are defined by (3.6) with $C^0_{j,\theta}$ in place of $\tilde{C}_{j,\theta}$ and 1 in place of $\tilde{\mu}$. One can see also that $C^0_{j,\theta}(\lambda) = C_{j,\theta}(\lambda) - \tilde{C}_{j,\theta}(\lambda)$.

Let us prove that (3.5), (3.6) determine Fredholm integral equation in the space $\tilde{W}^{1,\tilde{p}}(V)$, $\tilde{p} > 2$.

Propositions 2, 3 of [He] imply that correspondance

$$\tilde{\mu} \mapsto R_{\lambda,\theta}\circ\big(\hat{R}_\theta\big(\frac{i}{2}q\tilde{\mu}\big) + i\sum_{j=1}^{g}\tilde{C}_{j,\theta}\hat{R}_\theta(\delta(z,a_j))\big)$$

define linear continuous mapping of $\tilde{W}^{1,\tilde{p}}(V)$ into itself. This mapping is compact because mapping $\tilde{\mu}\to q\tilde{\mu}$, $supp\, q\subset X$, from $\tilde{W}^{1,\tilde{p}}(V)$ into $L^{\tilde{p}}_{1,1}(X)$ is compact, operator $\hat{R}_\theta:\ L^{\tilde{p}}_{1,1}(X)\to\tilde{W}^{1,\tilde{p}}_{1,0}(V)$ and operator $R_{\lambda,\theta}:\ \tilde{W}^{1,\tilde{p}}_{1,0}(V)\to\tilde{W}^{1,\tilde{p}}(V)$ are bounded.

If for fixed $\lambda\notin E_\theta$ Fredholm equation (3.5), (3.6) is not solvable then corresponding homogeneous equation, when the right-hand side of (3.5) is replaced by zero, admits nontrivial solution $\tilde{\mu}^* = \mu^* - 1$.

By Lemma 3.1 function $\tilde{\mu}^*$ satisfies differential equation (2.2) with C_j replaced by $\tilde{C}_j$ and with property $\tilde{\mu}^*(z)\to 0$, $z\to\infty$, $z\in V_1$.

By Proposition 2.1, $\tilde{\mu}^*\equiv 0$ if $\lambda\in\mathbb{C}\backslash E_\theta:\ |\lambda|\geq const(V,\{a_j\},\theta,\sigma)$.

It means that equation (3.2), (3.3) is uniquely solvable Fredholm integral equation for any $\lambda\in\mathbb{C}\backslash E_\theta:\ |\lambda|\geq const(V,\{a_j\},\theta,\sigma)$.

Lemma 3.2 is proved.

Lemma 3.3. *Let $\{a_1,\ldots,a_g\}$ be generic divisor on $Y\backslash X$. Let $\lambda\in\mathbb{C}\backslash E_\theta$. Let μ be solution of integral equation (3.2), (3.3). Then relations (3.3) determining parameters $C_j = C_{j,\theta}(\lambda)$ are reduced to the following explicit formulas*

$$2i\sum_{j=1}^{g}C_{j,\theta}e_{\lambda,\theta}(a_j)\frac{\bar{\omega}_k}{d\bar{z}_1}(a_j) = \int\limits_{z\in X} e_{\lambda,\theta}(z)\big(i\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}} + 2\bar{\partial}\ln\sqrt{\sigma}\wedge\partial\ln\sqrt{\sigma}\big)\mu\frac{\bar{\omega}_k}{d\bar{z}_1}(z). \tag{3.7}$$

Proof of Lemma 3.3. By Lemma 3.1 equations (3.2), (3.3) are equivalent to the equation:

$$\bar{\partial}(\partial + \lambda(dz_1 + \theta dz_2))\mu = \frac{i}{2}q\mu + i\sum_{j=1}^{g}C_{j,\theta}\delta(z,a_j), \tag{3.8}$$

where $\mu = \mu_\theta(z,\lambda) \to 1$, $z \in V_1$, $z \to \infty$.

System (2.7) implies the following relation

$$\begin{aligned} &\overline{\lim_{R\to\infty}} \int\limits_{|z_1|=R} \bar{u}_\pm \wedge \bar{\omega}_k + i \int\limits_{z\in V\setminus X} \sum_{j=1}^{g} (\bar{C}_{j,\theta} \mp C_{j,\theta} e_{\lambda,\theta}(z)) \frac{\delta(z,a_j)}{d\bar{z}_1} \bar{\omega}_k = \\ &\mp \int\limits_{z\in X} e^{\lambda(z_1+\theta z_2)-\bar{\lambda}(\bar{z}_1+\bar{\theta}\bar{z}_2)} \bar{q}_1 u_\pm dz_1 \wedge \bar{\omega}_k, \end{aligned} \tag{3.9}$$

where $\bar{q}_1 = \frac{\partial \ln\sqrt{\sigma}}{\partial \bar{z}_1}$.

To obtain (3.9) we multiply the both sides of (2.7) by $\wedge\omega_k$, integrate on V and take conjugation.

From Lemma 2.3 and Lemma 3.2 it follows that

$$u_\pm(z) \to \lambda(1+\theta\gamma_l) \cdot \lim_{\substack{z\to\infty \\ z\in V_l}} \mu_\theta(z,\lambda), \; z \to \infty, \; z \in V_l,$$

$$\text{where} \;\; \gamma_l = \lim_{\substack{z\to\infty \\ z\in V_l}} \frac{\partial z_2}{\partial z_1}, \quad \lim_{\substack{z\to\infty \\ z\in V_1}} \mu_\theta(z,\lambda) = 1.$$

Existence of $\lim\limits_{\substack{z\to\infty \\ z\in V_l}} \mu_\theta(z,\lambda)$ follows from Lemma 4.1 below. This imply that

$$\overline{\lim_{R\to\infty}} \Big| \int\limits_{|z_1|=R} \bar{u}_\pm \wedge \bar{\omega}_k \Big| = \overline{\lim_{R\to\infty}} \Big| \int\limits_{|z_1|=R} \bar{\lambda}(1+\bar{\theta}\bar{\gamma}_l)\bar{\omega}_k \Big| = \lim_{R\to\infty} |\lambda| O\big(\frac{1}{R}\big) = 0. \tag{3.10}$$

From (3.9), (3.10) and definition of $u_\pm$ we obtain

$$\begin{aligned} &2i \sum_{j=1}^{g} \int\limits_{V\setminus X} C_j e_{\lambda,\theta}(z) \frac{\delta(z,a_j)}{d\bar{z}_1} \wedge \bar{\omega}_k = \int\limits_{z\in X} e_{\lambda,\theta}(z) \bar{q}_1 (u_+ + u_-) dz_1 \wedge \bar{\omega}_k = \\ &2 \int\limits_{z\in X} e^{-\bar{\lambda}(\bar{z}_1+\bar{\theta}\bar{z}_2)} \bar{q}_1 f_1 dz_1 \wedge \bar{\omega}_k, \quad \text{where} \;\; f_1 = \sqrt{\sigma}\frac{\partial F}{\partial z_1}. \end{aligned}$$

By Lemma 2.3 we have

$$\begin{aligned} &2 \int\limits_{z\in X} e^{-\bar{\lambda}(\bar{z}_1+\bar{\theta}\bar{z}_2)} \bar{q}_1 f_1 dz_1 \wedge \bar{\omega}_k = \\ &2 \int\limits_{z\in X} e_{\lambda,\theta}(z) \bar{q}_1 \big(\frac{\partial \mu}{\partial z_1} + \lambda\mu + \lambda\theta \frac{\partial z_2}{\partial z_1}\mu - q_1\mu\big) dz_1 \wedge \bar{\omega}_k. \end{aligned} \tag{3.11}$$

From definition of $\delta(z,a_j)$ we have

$$2i \sum_{j=1}^{g} \int\limits_{z\in V\setminus X} C_j e_{\lambda,\theta}(z) \frac{\delta(z,a_j)}{d\bar{z}_1} \wedge \bar{\omega}_k = -2i \sum_{j=1}^{g} C_j e_{\lambda,\theta}(a_j) \frac{\bar{\omega}_k}{d\bar{z}_1}(a_j). \tag{3.12}$$

By integration by part we have

$$
\begin{aligned}
&2\int\limits_{z\in X} e_{\lambda,\theta}(z)\bar q_1\big(\frac{\partial\mu}{\partial z_1}+\lambda\mu\big)dz_1\wedge\bar\omega_k = 2\int\limits_X e_{\lambda,\theta}(z)\frac{\partial\ln\sqrt{\sigma}}{\partial\bar z_1}\lambda\mu dz_1\wedge\bar\omega_k -\\
&-2\int\limits_X e_{\lambda,\theta}(z)\frac{\partial\ln\sqrt{\sigma}}{\partial\bar z_1}\big(\lambda\mu+\lambda\theta\frac{\partial z_2}{\partial z_1}\mu\big)dz_1\wedge\bar\omega_k - 2\int\limits_X e_{\lambda,\theta}(z)\frac{\partial^2\ln\sqrt{\sigma}}{\partial z_1\partial\bar z_1}\mu dz_1\wedge\bar\omega_k = \\
&-2\int\limits_X e_{\lambda,\theta}(z)\big(\frac{\partial^2\ln\sqrt{\sigma}}{\partial z_1\partial\bar z_1}+\frac{\partial\ln\sqrt{\sigma}}{\partial\bar z_1}\lambda\theta\frac{\partial z_2}{\partial z_1}\big)\mu dz_1\wedge\bar\omega_k.
\end{aligned}
\tag{3.13}
$$

Using (3.11), (3.12), (3.13) we obtain

$$
i\sum_{j=1}^{g} C_{j,\theta}e_{\lambda,\theta}(a_j)\frac{\bar\omega_k}{d\bar z_1}(a_j) = \int\limits_{z\in X} e_{\lambda,\theta}(z)\big(\frac{\partial^2\ln\sqrt{\sigma}}{\partial z_1\partial\bar z_1}+\big|\frac{\partial\ln\sqrt{\sigma}}{\partial\bar z_1}\big|^2\big)\mu dz_1\wedge\bar\omega_k.
$$

Lemma 3.3 is proved.

Proof of Proposition 3.1. a) By Lemmas 3.1-3.3 statement (3.1a) of Proposition is valid, i.e. there exists function $z\to\mu_\theta(z,\lambda)$, $z\in V$ with property (3.1a) $\forall\lambda\in\mathbb{C}\backslash E_\theta$: $|\lambda|\ge const(V,\{a_j\},\theta,\sigma)$.

b) Put $f_0=\hat R_\theta\big(\frac{i}{2}q\mu\big)$, $f_1=\hat R_\theta\big(i\sum\limits_{j=1}^{g}C_{j,\theta}\delta(z,a_j)\big)$ and $f=f_0+f_1$. By (3.2) we have $\mu-1=R_{\lambda,\theta}f=R_{\lambda,\theta}f_0+R_{\lambda,\theta}f_1$.

Put

$$
L^{p,\tilde p}_{0,q}(V)=\{u:\ u\big|_Y\in L^p_{0,q}(Y),\ \ u\big|_{V\backslash\bar Y}\in L^{\tilde p}_{0,q}(V\backslash Y)\},\quad 1\le p<2,\ \tilde p>2,\ q=0,1.
$$

By Proposition 3 ii$'$ from [He] we obtain

$$
\begin{aligned}
&\|\mu-\mu_\theta(\infty_l,\lambda)\|_{L^{\tilde p}(V_l\backslash Y)}\le\\
&const(V,\tilde p,\theta)\cdot\min(|\lambda|^{-1/2},|\lambda|^{-1})\big(\|f_0\|_{\tilde W^{1,\tilde p}_{1,0}(V)}+\sum_{j=1}^{g}|C_{j,\theta}|\big)\\
&\|\partial\mu\|_{L^{p,\tilde p}_{1,0}(V)}\le const(V,\tilde p,\theta)\big(\|f_0\|_{\tilde W^{1,\tilde p}_{1,0}(V)}+\sum_{j=1}^{g}|C_{j,\theta}|\big).
\end{aligned}
\tag{3.14}
$$

For proving estimates (3.1b) let us now estimate $\{C^0_{j,\theta}\}$.

In order to estimate $\{C^0_{j,\theta}\}$ we must use equations (3.6), where parameters $\{\tilde C_{j,\theta}\}$ are replaced by $\{C^0_{j,\theta}\}$ and function $\tilde\mu$ is replaced by 1. For modified equations (3.6)

1) we apply Kramer formula for solution of linear system and integration by parts in all integrals of this system, using $e_{\lambda,\theta}(z)(d\bar z_1+\bar\theta d\bar z_2)=\frac{1}{\lambda}\bar\partial e_{\lambda,\theta}(z)$. In addition, we

use: formula (1.2) for $\Delta_\theta(\lambda)$, formula $\bar\partial\hat R_\theta\big(\frac{i}{2}q\mu\big)=\frac{i}{2}q\mu$ and estimate of singular integral, containing $\bar\partial\big(\frac{\bar\omega_k}{d\bar z_1+\bar\theta d\bar z_2}\big)$. This gives inequality:

$$\sum_j |C^0_{j,\theta}(\lambda)|\le \frac{const(V,\{a_j\},\theta,\sigma)}{|\Delta_\theta(\lambda)|(1+|\lambda|)^g}.$$

ii) The equation (3.5) together with obtained inequality for $\sum|C^0_{j,\theta}(\lambda)|$, estimate of Faddeev type Green function $|g_{\lambda,\theta}(z,\xi)|=O\big(\frac{1}{|\lambda|^{1-\varepsilon}}\big)$ are used to obtain estimate (3.15) for $\sum|\tilde C_{j,\theta}(\lambda)|$ and $|\mu_\theta(\lambda)|$:

$$\begin{gathered}|\lambda|^{-\varepsilon}\|\mu\|_{\tilde W^{1,\tilde p}(V)}+\sum_j|\tilde C_{j,\theta}(\lambda)|\le\frac{const(V,\{a_j\},\theta,\sigma,\tilde p,\varepsilon)}{|\Delta_\theta(\lambda)|(1+|\lambda|)^g}\ \text{ and}\\ \|\mu-\mu(\infty_l,\cdot)\|_{L^{\tilde p}(V_l\backslash Y)}\le\frac{const(V,\{a_j\},\theta,\sigma,\tilde p,\varepsilon)}{|\Delta_\theta(\lambda)|(1+|\lambda|)^{g+1-\varepsilon}},\\ \text{where }\ \lambda\in\mathbb C\backslash E_\theta:\ |\lambda|\ge const(V,\{a_j\},\theta,\sigma,\varepsilon),\quad l=1,\dots,d,\ \mu_\theta(\infty_1,\lambda)=1.\end{gathered}\tag{3.15}$$

These estimates imply estimates (3.1b).

c) Differentiation of equation (3.2) with respect to $\bar\lambda$ gives equality

$$\begin{gathered}\frac{\partial\mu}{\partial\bar\lambda}-R_{\lambda,\theta}\circ\big(\hat R_\theta\big(\frac{i}{2}q\frac{\partial\mu}{\partial\bar\lambda}+i\sum_{j=1}^g\frac{\partial C_{j,\theta}(\lambda)}{\partial\bar\lambda}\delta(z,a_j)\big)=\\ (\bar z_1+\bar\theta\bar z_2)(\mu-1)-R_{\lambda,\theta}\big((\bar\xi_1+\bar\theta\bar\xi_2)\hat R_\theta\big(\frac{i}{2}q\mu+i\sum_{j=1}^g C_{j,\theta}\delta(z,a_j)\big).\end{gathered}\tag{3.16}$$

Equality (3.16) can be rewritten in the following form

$$\begin{gathered}\frac{\partial\mu}{\partial\bar\lambda}=\big(I-R_{\lambda,\theta}\circ\hat R_\theta\big(\frac{i}{2}q\cdot\big)\big)^{-1}\big[(\bar z_1+\bar\theta\bar z_2)(\mu-1)+R_{\lambda,\theta}\circ\hat R_\theta\big(i\sum_{j=1}^g\frac{\partial C_{j,\theta}}{\partial\bar\lambda}\delta(z,a_j)\big)-\\ R_{\lambda,\theta}\big((\bar\xi_1+\bar\theta\bar\xi_2)\hat R_\theta\big(\frac{i}{2}q\mu+i\sum_{j=1}^g C_{j,\theta}(\lambda)\delta(z,a_j)\big)\big].\end{gathered}\tag{3.17}$$

Using Propositions 2, 3 from [He], estimates from part (b) of this proof we obtain from (3.17)

$$\begin{gathered}e_{\lambda,\theta}(z)\frac{\partial\mu}{\partial\bar\lambda}\big|_Y\in W^{1,p}(Y),\\ e_{\lambda,\theta}(z)\frac{\partial\mu}{\partial\bar\lambda}\big|_{V_l}\in W^{1,\tilde p}(V_l\backslash Y)\cup L^\infty(V_l\backslash Y).\end{gathered}$$

Statement (3.1c) is proved.

Proposition 3.1 is proved.

4. Equation $\frac{\partial\mu(z,\lambda)}{\partial\bar\lambda} = b_\theta(\lambda)e_{-\lambda,\theta}(z)\bar\mu_\theta(z,\lambda)$. Proof of Theorem 1.1B

Proposition 4.1. *Let conductivity σ, divisor $\{a_1,\dots,a_g\}$ and θ satisfy the conditions of Proposition 2.1. Let function $\psi_\theta(z,\lambda) = e^{\lambda(z_1+\theta z_2)}\mu_\theta(z,\lambda)$ be the Faddeev type function, associated with σ, θ and divisor $\{a_1,\dots,a_g\}$. Then for $\lambda\in\mathbb{C}\backslash E_\theta$: $|\lambda|\geq const(V,\{a_j\},\theta,\sigma)$*

i) the following $\bar\partial$-equations take place

$$\frac{\partial\mu_\theta(z,\lambda)}{\partial\bar\lambda} = b_\theta(\lambda)e_{-\lambda,\theta}(z)\overline{\mu_\theta(z,\lambda)},\quad \text{if}\quad z\in V\backslash\{a_1,\dots,a_g\}, \tag{4.1}$$

$$\frac{\partial C_{j,\theta}(\lambda)}{\partial\bar\lambda} = b_\theta(\lambda)e_{-\lambda,\theta}(a_j)\overline{C_{j,\theta}(\lambda)},\quad j=1,\dots,g,\quad \text{where} \tag{4.2}$$

ii) function $b_\theta(\lambda)$ satisfies equations:

$$\begin{gathered} b_\theta(\lambda)\lim_{\substack{z\to\infty\\ z\in V_l}}\overline{\mu_\theta(z,\lambda)} = \lim_{\substack{z\to\infty\\ z\in V_l}}\frac{\bar z_1+\bar\theta\bar z_2}{\bar\lambda}e_{\lambda,\theta}(z)\frac{\partial\mu_\theta(z,\lambda)}{\partial(\bar z_1+\bar\theta\bar z_2)},\\ \bar\lambda b_\theta(\lambda)d = -\frac{1}{2\pi i}\int\limits_{z\in bX}e_{\lambda,\theta}(z)\bar\partial\mu_\theta(z,\lambda) + i\sum_{j=1}^{g}C_{j,\theta}(\lambda)e_{\lambda,\theta}(a_j),\ l=1,\dots,d \end{gathered} \tag{4.3}$$

and the inequality

$$|\lambda|(1+|\lambda|)^g|\Delta_\theta(\lambda)|\cdot|b_\theta(\lambda)| \leq const(V,\{a_j\},\theta,\sigma)\frac{1}{(|\lambda|+1)^{1/3}}. \tag{4.4}$$

Remark 4.1. For the case $V=\mathbb{C}$ this statement is obtained in [GN], [N2], [N3]. Proposition 4.1 is a corrected version of Proposition 3.2 of [HM2].

Lemma 4.1. *i) Let function $\mu=\mu_\theta(z,\lambda)$, $z\in V\backslash Y$, $\lambda\in\mathbb{C}\backslash E_\theta$: $|\lambda|\geq const(V,\{a_j\},\theta,\sigma)$ satisfy equation*

$$\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu = 0\ \text{ on }\ V\backslash Y \tag{4.5}$$

and the property

$$\begin{gathered} [\mu-\mu_\theta(\infty_l,\lambda)]\big|_{V_l\backslash Y}\in W^{1,\tilde p}(V_l\backslash\bar Y),\quad \text{where}\quad \tilde p>2,\\ \mu_\theta(\infty_l,\lambda)\stackrel{\text{def}}{=}\lim_{\substack{z\to\infty\\ z\in V_l}}\mu_\theta(z,\lambda),\ l=1,\dots,d. \end{gathered}$$

Then

$$\begin{gathered} A\stackrel{\text{def}}{=}\frac{\partial\mu}{\partial(z_1+\theta z_2)}+\lambda\mu\in\mathcal{O}(\tilde V\backslash\bar Y)\quad \text{and}\\ A\big|_{V_l\backslash Y} = \lambda\mu(\infty_l)+\sum_{k=1}^{\infty}A_{k,l}\frac{1}{(z_1+\theta z_2)^k}, \end{gathered}$$

$$\bar B \overset{\text{def}}{=} e_{\lambda,\theta}(z)\frac{\partial\mu}{\partial(\bar z_1+\bar\theta\bar z_2)} \in \overline{\mathcal{O}(\tilde V\backslash\bar Y)} \quad \textit{and}$$
$$\bar B\big|_{V_l\backslash Y} = \sum_{k=1}^{\infty} B_{k,l}\frac{1}{(\bar z_1+\bar\theta\bar z_2)^k},\ l=1,\ldots,d, \tag{4.6}$$

where $\mathcal{O}(\tilde V\backslash\bar Y)$ is the space of holomorphic functions on $(\tilde V\backslash\bar Y)$.

ii) Let

$$M\big|_{V_l} = \mu_\theta(\infty_l,\lambda) + \sum_{k=1}^{\infty}\frac{a_{k,l}(\lambda)}{(z_1+\theta z_2)^k} \quad \text{and} \quad \bar N\big|_{V_l} = \sum_{k=1}^{\infty}\frac{b_{k,l}(\lambda)}{(\bar z_1+\bar\theta\bar z_2)^k}$$

be formal series with coefficients determined by relations

$$\lambda a_{k,l} - (k-1)a_{k-1,l} = A_{k,l},\ \ \bar\lambda b_{k,l} - (k-1)b_{k-1,l} = B_{k,l},\ \ l=1,\ldots,d,\ \ k=1,2,\ldots.$$

Let

$$M_\nu\big|_{V_l} = \mu_\theta(\infty_l,\lambda) + \sum_{k=1}^{\nu}\frac{a_{k,l}}{(z_1+\theta z_2)^k},\quad \bar N_\nu\big|_{V_l} = \sum_{k=1}^{\nu}\frac{b_{k,l}}{(\bar z_1+\bar\theta\bar z_2)^k}. \tag{4.7}$$

Then function μ has the asymptotic decomposition

$$\mu\big|_{V_l} = M\big|_{V_l} + e_{-\lambda,\theta}(z)\bar N\big|_{V_l},\quad z_1\to\infty,$$
$$\text{i.e. } \mu\big|_{V_l} = M\big|_{V_l} + e_{-\lambda,\theta}(z)\bar N_\nu\big|_{V_l} + O\Big(\frac{1}{|z_1|^{\nu+1}}\Big).$$

Proof of Lemma 4.1.

i) From (4.5) it follows that

$$\partial\bar\partial(e^{\lambda(z_1+\theta z_2)}\mu(z,\lambda))\big|_{V\backslash\bar Y} = 0.$$

Thus $\bar\partial(e^{\lambda(z_1+\theta z_2)}\mu(z,\lambda)) = e^{\lambda(z_1+\theta z_2)}\bar\partial\mu$ is antiholomorphic form on $V\backslash\bar Y$ and $\partial\mu + \lambda\mu(dz_1+\theta dz_2)$ is holomorphic form on $V\backslash\bar Y$. From this, condition $\bar\partial\mu \in L^{\tilde p}_{0,1}(V\backslash\bar Y)$ and the Cauchy theorem it follows that

$$e^{\lambda(z_1+\theta z_2)}\bar\partial\mu\big|_{V_l\backslash\bar Y} = e^{\bar\lambda(\bar z_1+\bar\theta\bar z_2)}\bar B(d\bar z_1+\bar\theta d\bar z_2)\big|_{V_l\backslash\bar Y} =$$
$$e^{\bar\lambda(\bar z_1+\bar\theta\bar z_2)}\sum_{k=1}^{\infty}\frac{B_{k,l}}{(\bar z_1+\bar\theta\bar z_2)^k}(d\bar z_1+\bar\theta d\bar z_2)\big|_{V_l} \quad \text{and}$$
$$(\partial\mu+\lambda\mu(dz_1+\theta dz_2))\big|_{V_l\backslash\bar Y} = A(dz_1+\theta dz_2)\big|_{V_l\backslash\bar Y} =$$
$$\Big(\lambda\mu(\infty_l) + \sum_{k=1}^{\infty}\frac{A_{k,l}}{(z_1+\theta z_2)^k}\Big)(dz_1+\theta dz_2)\big|_{V_l\backslash\bar Y}.$$

It gives (4.6).

ii) From (4.6), (4.7) we obtain, first, that

$$\begin{aligned}&\bar\partial\mu\big|_{V_l} = e^{-\lambda(z_1+\theta z_2)}\bar\partial\big(e^{\bar\lambda(\bar z_1+\bar\theta\bar z_2)}\bar N_\nu\big)\big|_{V_l} + O\big(\frac{1}{|\bar z_1|^{\nu+1}}\big)\\ &\text{then}\ \ \mu\big|_{V_l} = M_\nu\big|_{V_l} + e_{-\lambda,\theta}(z)\bar N_\nu\big|_{V_l} + \tilde O\big(\frac{1}{|\bar z_1|^{\nu}}\big).\end{aligned} \tag{4.8}$$

Comparison of the last equality for different indexes ν and $\nu+1$ implies that $\tilde O\big(\frac{1}{|\bar z_1|^{\nu}}\big) = O\big(\frac{1}{|\bar z_1|^{\nu+1}}\big)$.

It gives statement of Lemma 4.1.

Lemma 4.2. *i) Functions M_ν and N_ν (congugated to $\bar N_\nu$) from decomposition (4.8) have the following properties:*

$$\begin{aligned}&\forall\ z\in\tilde V\backslash Y\ \exists\ \lim_{\nu\to\infty}\big(\frac{\partial M_\nu}{\partial(z_1+\theta z_2)} + \lambda M_\nu\big) \stackrel{\text{def}}{=} \frac{\partial M}{\partial(z_1+\theta z_2)} + \lambda M \ \ \textit{and}\\ &\exists\ \lim_{\nu\to\infty}\big(\frac{\partial N_\nu}{\partial(z_1+\theta z_2)} + \lambda N_\nu\big) \stackrel{\text{def}}{=} \frac{\partial N}{\partial(z_1+\theta z_2)} + \lambda N.\end{aligned}$$

ii) Functions $\frac{\partial M}{\partial(z_1+\theta z_2)} + \lambda M$ and $\frac{\partial N}{\partial(z_1+\theta z_2)} + \lambda N$ belongs to $\mathcal{O}(\tilde V\backslash Y)$ and

$$\begin{aligned}&\frac{\partial\mu}{\partial(\bar z_1+\bar\theta\bar z_2)} = e_{-\lambda,\theta}(z)\big(\frac{\partial\bar N}{\partial(\bar z_1+\bar\theta\bar z_2)} + \bar\lambda\bar N\big),\\ &\frac{\partial\mu}{\partial(z_1+\theta z_2)} + \lambda\mu = \frac{\partial M}{\partial(z_1+\theta z_2)} + \lambda M,\end{aligned} \tag{4.9}$$

$$\frac{\partial N}{\partial(z_1+\theta z_2)} + \lambda N \to 0, \quad z_1\to\infty. \tag{4.10}$$

Proof of Lemma 4.2.

Part i) and equalities (4.9), (4.10) from part ii) follow directly from (4.8).

Properties (4.8), (4.9), (4.10), property $\bar\partial\mu\in L^{p,\tilde p}_{0,1}$ (Proposition 3.1b) and extension property of bounded holomorphic functions through isolated singularities imply that

$$\frac{\partial M}{\partial(z_1+\theta z_2)} + \lambda M \ \ \text{and}\ \ \frac{\partial N}{\partial(z_1+\theta z_2)} + \lambda N$$

belongs to $\mathcal{O}(\tilde V\backslash Y)$.

Lemma 4.2 is proved.

Lemma 4.3. *Let $\psi_\theta(z,\lambda) = e^{\lambda(z_1+\theta z_2)}\mu_\theta(z,\lambda)$ be the Faddeev type function on V, associated with potential $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$ and divisor $\{a_1,\ldots,a_g\}$ on $Y\backslash\bar X$. Then*

$\forall \lambda \in \mathbb{C}\backslash E_\theta: \ |\lambda| \geq const(V, \{a_j\}, \theta, \sigma)$

$$\begin{aligned} & e_{\lambda,\theta}(z) \frac{\partial \mu}{\partial(\bar{z}_1 + \bar{\theta}\bar{z}_2)}\big|_{V_l\backslash \bar{Y}} = \sum_{k=1}^{\infty} B_{k,l}(\bar{z}_1 + \bar{\theta}\bar{z}_2)^{-k}, \quad \textit{where} \\ & B_{1,l} = -\frac{1}{2\pi i}\times \\ & \int\limits_{\{z\in V_l:\ |z_1|=r_1\}} e_{\lambda,\theta}(z) \frac{\partial \mu}{\partial(\bar{z}_1 + \bar{\theta}\bar{z}_2)}(d\bar{z}_1 + \bar{\theta} d\bar{z}_2) \ \forall\, r_1: \ Y \subset \{z \in V: \ |z_1| < r_1\}. \end{aligned} \tag{4.11}$$

Proof of Lemma 4.3.

Estimate of $\partial\mu$ from (3.1b) and the Cauchy theorem, applied to antiholomorphic function $e_{\lambda,\theta}(z)\frac{\partial \mu}{\partial(\bar{z}_1+\bar{\theta}\bar{z}_2)}\big|_{V_l\backslash\bar{Y}}$ implies (4.11).

Proof of Proposition 4.1.

Since ψ, μ are Faddeev type functions, we have the equations

$$\bar{\partial}(\partial + \lambda(dz_1 + \theta dz_2))\mu = \frac{i}{2} q\mu + i\sum_{j=1}^{\infty} C_{j,\theta}(\lambda)\delta(z, a_j),$$

$$dd^c\psi = q\psi + 2\sum_{j=1}^{g} e^{\lambda(z_1+\theta z_2)} C_{j,\theta}(\lambda)\delta(z, a_j).$$

Put $\psi_{\bar{\lambda}} = \frac{\partial \psi}{\partial \bar{\lambda}}$ and $\mu_{\bar{\lambda}} = \frac{\partial \mu}{\partial \bar{\lambda}}$.

We obtain

$$dd^c\psi_{\bar{\lambda}} = q\psi_{\bar{\lambda}} + 2\sum_{j=1}^{g} e^{\lambda(z_1+\theta z_2)} \frac{\partial C_{j,\theta}}{\partial \bar{\lambda}}(\lambda)\delta(z, a_j).$$

From Lemma 4.1 we deduce

$$\begin{aligned} & \frac{\partial \mu}{\partial(\bar{z}_1 + \bar{\theta}\bar{z}_2)}\big|_{V_l\backslash \bar{Y}} = e_{-\lambda,\theta}(z)\frac{B_{1,l}(\lambda)}{\bar{z}_1 + \bar{\theta}\bar{z}_2} + O\big(\frac{1}{|z_1|^2}\big), \text{ and} \\ & \big(\frac{\partial \mu}{\partial(z_1 + \theta z_2)} + \lambda\mu\big)\big|_{V_l\backslash \bar{Y}} = \lambda\mu(\infty_l) + \frac{A_{1,l}(\lambda)}{z_1 + \theta z_2} + O\big(\frac{1}{|z_1|^2}\big). \end{aligned} \tag{4.12}$$

From (4.6), (4.7), (4.8) we deduce

$$\mu\big|_{V_l\backslash \bar{Y}} = \mu_\theta(\infty_l, \lambda) + \frac{a_l(\lambda)}{z_1 + \theta z_2} + e_{-\lambda,\theta}(z)\frac{b_l(\lambda)}{\bar{z}_1 + \bar{\theta}\bar{z}_2} + O\big(\frac{1}{|z_1|^2}\big), \ z_1 \to \infty, \tag{4.13}$$

$$\text{where} \ \ \bar{\lambda} b_l(\lambda) \stackrel{\text{def}}{=} \bar{\lambda} b_{1,l}(\lambda) = B_{1,l}, \ \ \lambda a_l(\lambda) \stackrel{\text{def}}{=} \lambda a_{1,l}(\lambda) = A_{1,l}, \ \ l = 1, \ldots, d. \tag{4.14}$$

From (4.13) and (3.1c) we obtain for $l = 1, \dots, d$

$$\psi\big|_{V_l\backslash Y} = e^{\lambda(z_1+\theta z_2)}\mu =$$
$$e^{\lambda(z_1+\theta z_2)}\big(\mu_\theta(\infty_l,\lambda) + \frac{a_l(\lambda)}{z_1+\theta z_2} + e^{\bar\lambda(\bar z_1+\bar\theta\bar z_2)-\lambda(z_1+\theta z_2)}\frac{b_l(\lambda)}{\bar z_1+\bar\theta\bar z_2} + O(\frac{1}{|z_1|^2})\big),$$
$$\psi_{\bar\lambda}\big|_{V_l\backslash Y} = \frac{\partial\psi}{\partial\bar\lambda}\big|_{V_l\backslash Y} = e^{\bar\lambda(\bar z_1+\bar\theta\bar z_2)}\big[(\bar z_1+\bar\theta\bar z_2)\frac{b_l(\lambda)+e_{\lambda,\theta}(z)\frac{\partial\mu_\theta(\infty_l,\lambda)}{\partial\bar\lambda}}{\bar z_1+\bar\theta\bar z_2} + O(\frac{1}{|z_1|})\big] =$$
$$e^{\bar\lambda(\bar z_1+\bar\theta\bar z_2)}\big(b_l(\lambda) + e_{\lambda,\theta}(z)\frac{\partial\mu_\theta(\infty_l,\lambda)}{\partial\bar\lambda} + O(\frac{1}{|z_1|})\big).$$

For function $\mu_{\bar\lambda} = e^{-\lambda(z_1+\theta z_2)}\psi_{\bar\lambda}$ we obtain

$$\bar\partial(\partial + \lambda(dz_1+\theta dz_2))\mu_{\bar\lambda} = \frac{i}{2}q\mu_{\bar\lambda} + i\sum_{j=1}^{g}\frac{\partial C_{j,\theta}}{\partial\bar\lambda}\delta(z,a_j)$$
$$\text{and}\quad \mu_{\bar\lambda} = e_{-\lambda,\theta}(z)\big(b_l(\lambda) + e_{\lambda,\theta}(z)\frac{\partial\mu_\theta(\infty_l,\lambda)}{\partial\bar\lambda} + O(\frac{1}{|z_1|})\big),\quad z\in V_l.$$

For z_1 large enough function $e_{-\lambda,\theta}(z)\bar\mu_{\bar\lambda} \stackrel{\text{def}}{=} \varphi(z,\lambda)$ satisfies equation $\bar\partial(\partial + \lambda(dz_1+\theta dz_2))\varphi = 0$. From this, Lemma 4.1 and property $\overline{\lim\limits_{z\to\infty}}|\varphi(z,\lambda)|_V < \infty$ we deduce that $\varphi\big|_{V_l}(z,\lambda) \to const_l(\lambda) \stackrel{\text{def}}{=} \varphi(\infty_l,\lambda)$, if $z\in V_l$, $z\to\infty$, $l = 1,\dots,d$. So in the relations above we have $e_{\lambda,\theta}(z)\mu_{\bar\lambda}(\infty_l,\lambda) \equiv 0$, $l = 1,\dots,d$. Functions $e_{-\lambda,\theta}(z)\bar\mu_{\bar\lambda}$ and μ both satisfy equation $\bar\partial(\partial + \lambda(dz_1+\theta dz_2))\mu = \frac{i}{2}q\mu$ on $V\backslash\{a_1,\dots,a_g\}$. Besides $\overline{\mu\big|_{V_l}(z,\lambda)} \to \overline{\mu(\infty_l,\lambda)}$ and $e_{\lambda,\theta}(z)\mu_{\bar\lambda}(z,\lambda) \to b_l(\lambda)$, if $z\in V_l$, $z\to\infty$. Applying Proposition 2.1 we obtain

$$e_{\lambda,\theta}(z)\mu_{\bar\lambda} = b_l(\lambda)\bar\mu_\theta(z,\lambda)(\overline{\mu_\theta(\infty_l,\lambda)})^{-1},\quad l = 1,\dots,d.$$

This implies equalities (4.1), (4.2), where

$$b_\theta(\lambda) = \frac{b_l(\lambda)}{\overline{\mu_\theta(\infty_l,\lambda)}},\quad l = 1,\dots,d. \tag{4.15}$$

Asymptotic formula (4.3) follows from (4.11), (4.14) and (4.15). These formulas and Cauchy-Green formula imply also the following important expression for $b_\theta(\lambda)$:

$$\bar\lambda b_\theta(\lambda)d = -\frac{1}{2\pi i}\int\limits_{z\in bY} e_{\lambda,\theta}(z)\bar\partial\mu = -\frac{1}{2\pi i}\int\limits_{z\in bX} e_{\lambda,\theta}(z)\bar\partial\mu + i\sum_{j=1}^{g}C_{j,\theta}e_{\lambda,\theta}(a_j), \tag{4.16}$$

where

$$\int\limits_{z\in bX} e_{\lambda,\theta}(z)\bar\partial\mu = \int\limits_{X}\frac{1}{2i}e_{\lambda,\theta}(z)q\mu. \tag{4.17}$$

Equality (4.3) follows from (4.16). This equality together with estimate of $\{C_j\}$ from Lemma 2.4 and estimate through integration by parts of $\int\limits_X e_{\lambda,\theta} q\mu$ imply (4.4).

Proposition 4.1 is proved.

5. Reconstruction of function $\psi_\theta\big|_{bX}$ from Dirichlet-to-Neumann data on bX. Proof of Theorem 1.2A

Let X be domain containing V_0 and relativement compact in V with smooth (of classe $C^{(2)}$) boundry. Let $\sigma \in C^{(2)}(V)$, $\sigma > 0$ on V, $\sigma = 1$ on $V\backslash X$. Let $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$. Let $u \in C(bX)$ and $\tilde{u} \in W^{1,\tilde{p}}(X)$, $\tilde{p} > 2$, be solution of the Dirichlet problem $d\sigma d^c\tilde{u}\big|_X = 0$, $\tilde{u}\big|_{bX} = u$, where $d^c = i(\bar\partial - \partial)$, $d = \bar\partial + \partial$. Let $\tilde\psi = \sqrt{\sigma}\tilde{u}$ and $\psi = \sqrt{\sigma}u$. Then

$$dd^c\tilde\psi = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\tilde\psi = q\tilde\psi \quad \text{on} \quad X, \quad \tilde\psi\big|_{bX} = \psi. \tag{5.1}$$

Let ψ_0 be solution of Dirichlet problem

$$dd^c\psi_0\big|_X = 0, \quad \psi_0\big|_{bX} = \psi\big|_{bX}.$$

Let

$$\hat\Phi\psi = \bar\partial\tilde\psi\big|_{bX} \quad \text{and} \quad \hat\Phi_0\psi = \bar\partial\tilde\psi_0\big|_{bX}. \tag{5.2}$$

Operator $\psi\big|_{bX} \mapsto \bar\partial\tilde\psi\big|_{bX}$ is equivalent to the Dirichlet-to-Neumann operator $u\big|_{bX} \mapsto \sigma d^c\tilde{u}\big|_{bX}$.

Proposition 5.1. *Let $\psi = e^{\lambda(z_1+\theta z_2)}\mu$ be the Faddeev type function associated with potential $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$ (see Definition 1.4), generic divisor $\{a_1,\ldots,a_g\}$ with support in $V\backslash\bar{X}$ and generic $\theta \in \mathbb{C}$. Then $\forall\lambda \in \mathbb{C}\backslash E_\theta :\ |\lambda| \geq const(V,\{a_j\}\theta,\sigma)$ the restriction $\psi\big|_{bX}$ of ψ on bX can be found from Dirichlet-to-Neumann operator $\psi\big|_{bX} \to \sigma d^c\psi\big|_{bX}$ through the uniquely solvable Fredholm integral equation*

$$\mu_\theta(z,\lambda)\big|_{bX} + \int\limits_{\xi\in bX} g_{\lambda,\theta}(z,\zeta)m_{-\lambda}(\hat\Phi - \hat\Phi_0)m_\lambda\mu_\theta(\zeta,\lambda) = 1 + i\sum_{j=1}^{g} C_{j,\theta}(\lambda)g_{\lambda,\theta}(z,a_j), \tag{5.3}$$

$$i\sum_{j=1}^{g}(a_{j,1}+\theta a_{j,2})^{-k}C_{j,\theta}(\lambda) + \int\limits_{z\in bX}(z_1+\theta z_2)^{-k}(\partial + \lambda(dz_1+\theta dz_2))\mu = 0, \quad k = 2,\ldots,g+1,$$

where $g_{\lambda,\theta}(z,\xi)$- kernel of operator $R_{\lambda,\theta}\circ\hat{R}_\theta$,

$$m_{-\lambda}(\hat\Phi - \hat\Phi_0)m_\lambda\mu_\theta(\zeta,\lambda) = \int\limits_{w\in bX} e^{-\lambda(\zeta_1+\theta\zeta_2)}(\Phi(\zeta,w) - \Phi_0(\zeta,w))e^{\lambda(w_1+\theta w_2)}\mu_\theta(w,\lambda), \tag{5.4}$$

$\Phi(\zeta,w)$, $\Phi_0(\zeta,w)$ are kernels of operators $\hat\Phi$ and $\hat\Phi_0$, $m_{\pm\lambda}$ denote the multiplication operators by $e^{\pm\lambda(z_1+\theta z_2)}$, values $\{a_{j,1}\}$ of the first coordinate of points $\{a_j\}$ are supposed to be mutually different.

This proposition for the case $V=\mathbb{C}$ is equivalent to the second part of Theorem 1 from [N2].

Lemma 5.1. *Let $\psi = e^{\lambda(z_1+\theta z_2)}\mu$ be Faddeev type function of Proposition 5.1. Then $\forall\ z\in V\backslash X$ and $\forall\lambda\in\mathbb{C}\backslash E_\theta:\ |\lambda|\ge const(V,\{a_j\},\theta,\sigma)$ we have equalities*

$$\mu_\theta(z,\lambda) = 1 - \int\limits_{\xi\in bX} g_{\lambda,\theta}(z,\xi)\bar\partial\mu_\theta(\xi,\lambda) - \int\limits_{\xi\in bX}\mu_\theta(z,\xi)e^{\lambda(\xi_1+\theta\xi_2)}\partial\big(e^{-\lambda(\xi_1+\theta\xi_2)}g_{\lambda,\theta}(z,\xi)\big)+ i\sum_{j=1}^{g} C_{j,\theta}(\lambda)g_{j,\theta}(z,a_j) \tag{5.5}$$

and

$$-\int\limits_{\xi\in bX}(z_1+\theta z_2)^{-k}(\partial+\lambda(dz_1+\theta dz_2))\mu = \sum_{j=1}^{g}(a_{j,1}+\theta a_{j,2})^{-k} iC_{j,\theta}(\lambda),\ k=2,\ldots \tag{5.6}$$

Proof of Lemma 5.1.

The equation

$$\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu = \frac{i}{2}q\mu + i\sum_{j=1}^{g} C_{j,\theta}(\lambda)\delta(z,a_j), \tag{5.7}$$

where $supp\, q\subseteq X$ implies that (1,0)-form $f=(\partial+\lambda(dz_1+\theta dz_2))\mu$ is holomorphic on $(V\backslash(X\cup_{j=1}^{g}\{a_j\})$ and $Res_{a_j}(\partial+\lambda(dz_1+\theta dz_2))\mu = \frac{iC_j}{2\pi i}$. This and the property (4.12) imply that $\forall\lambda\in\mathbb{C}\backslash E_\theta$ and $\forall\ k\ge 2$ form $(z_1+\theta z_2)^{-k}f$ is holomorphic in the neighborhood of $(\tilde V\backslash V)$. By residue theorem applied to the form $(z_1+\theta z_2)^{-k}f$ on $\tilde V\backslash X$, we obtain

$$\int\limits_{z\in bX}(z_1+\theta z_2)^{-k}f(z,\lambda) = -2\pi i\sum_{j=1}^{g} Res_{a_j}(z_1+\theta z_2)^{-k}f(z,\lambda) = -(a_{j,1}+\theta a_{j,2})^{-k}(iC_{j,\theta}(\lambda)),$$

$k=2,3,\ldots$. Equality (5.6) is proved.

Let us prove now (5.5). Differential equation (5.7), where $\mu\big|_Y\in L^{\tilde p}(Y)$, $\mu\big|_{V\backslash\bar Y}\in L^\infty(V\backslash\bar Y)$, $\mu(z)\to 1$, $z\to\infty$, $z\in V_1$, is equivalent by Lemma 3.1 to the system of equations

$$\mu(z,\lambda) = 1 + R_{\lambda,\theta}\circ\hat R_\theta\big(\frac{i}{2}q\mu + i\sum_{j=1}^{g} C_j\delta(z,a_j)\big),\ z\in V,\ \text{ and} \tag{5.8}$$

$$\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu = 0,\quad z\in V\backslash(X\cup_{j=1}^{g}\{a_j\}). \tag{5.9}$$

These equations imply relations (5.6). Besides, we have equality

$$\int\limits_X g_{\lambda,\theta}(z,\xi)\frac{i}{2}q(\xi)\mu(\xi) = \int\limits_X g_{\lambda,\theta}(z,\xi)\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu.$$

Using Green-Riemann formula we obtain

$$\int_X e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))} g_{\lambda,\theta}(z,\xi)\partial\bar\partial\psi = \int_X \psi\partial\bar\partial\big(e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))} g_{\lambda,\theta}(z,\xi)\big)+$$
$$\int_{bX} e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))} g_{\lambda,\theta}(z,\xi)\bar\partial\psi + \int_{bX} \psi\partial\big(e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))} g_{\lambda,\theta}(z,\xi)\big).$$

For $z \in V\backslash X$ we have

$$\partial\bar\partial\big(e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))} g_{\lambda,\theta}(z,\xi)\big) = 0.$$

Then

$$-\int_{\xi\in X} g_{\lambda,\theta}(z,\xi)\big(\frac{i}{2}q\mu\big) = \int_{\xi\in bX} g_{\lambda,\theta}\bar\partial\mu + \int_{\xi\in bX} e^{\lambda(\xi_1+\theta\xi_2)}\mu\partial\big(e^{-\lambda(\xi_1+\theta\xi_2)} g_{\lambda,\theta}(z,\xi)\big). \quad (5.10)$$

From (5.8), (5.10) we deduce statement (5.5) of Lemma 5.1.

Proof of Proposition 5.1.

Let $\psi_0 : \ \bar\partial\partial\psi_0\big|_X = 0$ and $\psi_0\big|_{bX} = \psi$. By Green-Riemann formula $\forall\ z \in V\backslash X$ we have

$$\int_{\xi\in bX} \psi\partial\big(e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))} g_{\lambda,\theta}(z,\xi)\big) + \int_{\xi\in bX} e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))} g_{\lambda,\theta}(z,\xi)\bar\partial\psi_0 = 0. \quad (5.11)$$

Formulas (5.11) and (5.5), (5.6) imply

$$\psi(z,\lambda) = e^{\lambda(z_1+\theta z_2)} - \int_{bX} e^{\lambda((z_1-\xi_1)+\theta(z_2-\xi_2))} g_{\lambda,\theta}(z,\xi)(\bar\partial\psi(\xi) - \bar\partial\psi_0(\xi))+$$
$$i\sum_{j=1}^{g} e^{\lambda(z_1+\theta z_2)} C_j g_{\lambda,\theta}(z,a_j). \quad (5.12)$$

Formula (5.12), (5.6) are equivalent to (5.3). Integral equation (5.3) is the Fredholm equation in $C(bX)$, because operator $(\hat\Phi - \hat\Phi_0)$ is compact operator in $C(bX)$. Existence $\forall\lambda \in \mathbb{C}\backslash E_\theta$ of unique Faddeev type function $\psi = e^{\lambda(z_1+\theta z_2)}\mu$, associated with q and divisor $\{a_1,\ldots,a_g\}$ imply existence of solution of (5.3) with residue data
$iC_j = Res_{a_j}(\partial + \lambda(dz_1+\theta dz_2))\mu,\ \ j = 1,\ldots,g$. Let us prove uniqueness of solution (5.3) in $C(bX)$ with residue data $\{C_j\}$. Suppose $\mu \in C(bX)$ solves (5.3), (5.6). Consider this μ as Dirichlet data for equation $\bar\partial(\partial + \lambda(dz_1+\theta dz_2))\mu = \frac{i}{2}q\mu$ on X, solution of which well defines μ on $\bar X$.

Let us also define μ on $V\backslash\bar X$ by (5.5). Function $\mu(z,\lambda)$ defined in such a way on V belongs to $C(V\backslash \cup_{j=1}^{g} \{a_j\})$.

Let us show that μ satisfy (5.7). By Sohotsky-Plemelj jump formula $\forall\ z^* \in bX$ we have

$$\frac{i}{2}\mu(z^*) = \lim_{\substack{z\to z^*\\ z\in X}} \Big(\int\limits_{bX} g_{\lambda,\theta}\bar\partial\mu + \mu e^{\lambda(\xi_1+\theta\xi_2)}\partial\big(e^{-\lambda(\xi_1+\theta\xi_2)}g_{\lambda,\theta}\big)\Big) - \\ - \lim_{\substack{z\to z^*\\ z\in V\setminus X}} \Big(\int\limits_{bX} g_{\lambda,\theta}\bar\partial\mu + \mu e^{\lambda(\xi_1+\theta\xi_2)}\partial\big(e^{-\lambda(\xi_1+\theta\xi_2)}g_{\lambda,\theta}\big)\Big). \tag{5.13}$$

From (5.5) and (5.13) we deduce equality

$$\mu - \frac{i}{2}\mu = 1 - \int\limits_{\xi\in bX} g_{\lambda,\theta}\bar\partial\mu - \int\limits_{\xi\in bX} \mu e^{\lambda(\xi_1+\theta\xi_2)}\partial\big(e^{-\lambda(\xi_1+\theta\xi_2)}g_{\lambda,\theta}\big) + i\sum_{j=1}^{g} C_j g_{\lambda,\theta}(z,a_j),\ z\in X. \tag{5.14}$$

By Green-Riemann formula we have also

$$-\int\limits_{bX} g_{\lambda,\theta}\bar\partial\mu - \mu e^{\lambda(\xi_1+\theta\xi_2)}\partial\big(e^{-\lambda(\xi_1+\theta\xi_2)}g_{\lambda,\theta}\big) + i\sum_{j=1}^{g} C_{j,\theta}g_{\lambda,\theta}(z,a_j) =$$

$$-\int\limits_{X} \mu(\bar\partial(\partial + \lambda(dz_1+\theta dz_2)g_{\lambda,\theta}) + \int\limits_{X} g_{\lambda,\theta}\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu + i\sum_{j=1}^{g} C_{j,\theta}g_{\lambda,\theta}(z,a_j) =$$

$$\begin{cases} \frac{\mu}{2i} + \int\limits_X g_{\lambda,\theta}\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu + i\sum\limits_{j=1}^{g} C_{j,\theta}g_{\lambda,\theta}(z,a_j),\ z\in X, \\ \int\limits_X g_{\lambda,\theta}\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu + i\sum\limits_{j=1}^{g} C_{j,\theta}g_{\lambda,\theta}(z,a_j),\ z\in V\setminus(X\cup_{j=1}^{g}\{a_j\}). \end{cases} \tag{5.15}$$

Equalities (5.5), (5.6), (5.14) and (5.15) imply (3.3) and

$$\mu(z) = 1 + \int\limits_V g_{\lambda,\theta}\bar\partial(\partial+\lambda(dz_1+\theta dz_2))\mu + i\sum_{j=1}^{g} C_{j,\theta}g_{\lambda,\theta}(z,a_j) =$$

$$1 + R_{\lambda,\theta}\circ\hat R_\theta\big(\frac{i}{2}q\mu + i\sum_{j=1}^{g} C_{j,\theta}\delta(z,a_j)\big),\ z\in V.$$

By Lemma 3.1 function $\mu_\theta(z,\lambda)$ is the Faddeev type function associated with q and divisor $\{a_1,\dots,a_g\}$. The uniqueness of solution of (5.3) in $C(bX)$ with residue data $\{C_{j,\theta}\}$ follows now from uniqueness of the Faddeev type function.

6. Reconstruction of conductivity function from Dirichlet-to-Neumann data. Proof of Theorem 1.2B

We will obtain here exact formulas for reconstruction of conductivity function $\sigma\in C^{(3)}(V)$, $\sigma>0$, $\sigma\equiv 1$ on $V\setminus X$, from Dirichlet-to-Neumann data

$$\psi_\theta\big|_{bX} \to \bar\partial\psi_\theta\big|_{bX}$$

for Faddeev type functions $\psi_\theta(z,\lambda) = e^{\lambda(z_1+\theta z_2)}\mu_\theta(z,\lambda)$, $\theta \in \mathbb{C}\backslash\{\theta_1,\theta_d\}$, $\lambda \in \mathbb{C}\backslash E_\theta:\ |\lambda| \ge const(V,\{a_j\},\theta,\sigma)$, $\{a_1,\ldots,a_g\} \subset Y\backslash X$.

For simplicity of presentation we consider in detail the case of regular algebraic curves in $\mathbb{C}^2 \subset \mathbb{C}P^2$, only.

Let $\tilde V = \{\tilde z = (\tilde z_0 : \tilde z_1 : \tilde z_2) \in \mathbb{C}P^2:\ \tilde P(\tilde z) = 0\}$, where $\tilde P(\tilde z)$ homogeneous polynomial of degre N. Let $\mathbb{C}P^1_\infty = \{\tilde z : \mathbb{C}P^2 : \tilde z_0 = 0\}$. Put

$$\begin{aligned}
&\mathbb{C}^2 = \{\tilde z \in \mathbb{C}P^2 : \tilde z_0 \ne 0\},\quad z_1 = \frac{\tilde z_1}{\tilde z_0},\quad z_2 = \frac{\tilde z_2}{\tilde z_0},\quad P(z) = \tilde P(1,z_1,z_2),\\
&V = \{z \in \mathbb{C}^2:\ P(z) = 0\} = \tilde V \cap \mathbb{C}^2.
\end{aligned} \tag{6.1}$$

Without restriction of generality we suppose that $\tilde V$ be (regular) curve of degree $N \ge 2$ with property:

$$\begin{aligned}
&\tilde V \cap \mathbb{C}P^1_\infty = \{\beta_1,\ldots,\beta_d\},\quad \text{where}\ \ \beta_1,\ldots,\beta_d\ \ \text{be different points of}\ \ \mathbb{C}P^1_\infty,\\
&\beta_l = (0 : \beta_l^1 : \beta_l^2),\quad \frac{\beta_l^2}{\beta_l^1} \in \mathbb{C},\quad l = 1,\ldots,N,\\
&\frac{\partial P}{\partial z_2}(z) \ne 0,\quad \text{if}\quad z \in V:\ |z_1| \ge r_0 = const(V).
\end{aligned} \tag{6.2}$$

For $\theta \in \mathbb{C}$ let $\{w_m\}$ be points of V, where $(dz_1 + \theta dz_2)\big|_V(w_m) = 0$. Then for almost all $\theta \in \mathbb{C}$ the following relations are valid

$$\begin{aligned}
&\theta = \frac{\partial P}{\partial z_2}(w_m)\Big/\frac{\partial P}{\partial z_1}(w_m),\quad \frac{\partial P}{\partial z_1}(w_m) \ne 0,\\
&\Big[\frac{\partial^2 P}{\partial z_1^2}\Big(\frac{\partial P}{\partial z_2}\Big)^2 - 2\frac{\partial^2 P}{\partial z_1 \partial z_2}\Big(\frac{\partial P}{\partial z_2}\Big)\Big(\frac{\partial P}{\partial z_1}\Big) + \frac{\partial^2 P}{\partial z_2^2}\Big(\frac{\partial P}{\partial z_1}\Big)^2\Big](w_m) \ne 0.
\end{aligned}$$

Without restriction of generality it is sufficient to give proof under condition that $\theta = 0$, i.e. for points $w_m = (w_{m,1}, w_{m,2}) \in V$ such that

$$\frac{\partial P}{\partial z_1}(w_m) \ne 0,\quad \frac{\partial P}{\partial z_2}(w_m) = 0,\quad \frac{\partial^2 P}{\partial z_2^2}(w_m) \ne 0 \tag{6.3}$$

and also such that $\forall\ m$ the line $\{z \in \mathbb{C}^2:\ z_1 = w_{m,1}\}$ has tangency with X only in the single point w_m, $m = 1,\ldots,M$. By Hurwitz-Riemann formula $M = N(N-1)$. In the neighborhood of point $w_m \in V$ curve V can be represented in the form

$$\begin{aligned}
&V = \{(z_1,z_2) \in \mathbb{C}^2:\ z_1 = w_{m,1} +\\
&\Big(\frac{\partial P}{\partial z_1}(w_m)\Big)^{-1}\Big[-\frac{1}{2}\frac{\partial^2 P}{\partial z_2^2}(w_m)(z_2 - w_{m,2})^2 + O((z_2 - w_{m,2})^3)\Big].
\end{aligned} \tag{6.4}$$

The reconstruction formula for $\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}(w_m)$, $m = 1,\ldots,M$, will be obtained here by the stationary phase method, using formula (4.17).

Let μ be Faddeev type function (3.1) with properties (3.1a)-(3.1c) and with $\theta = 0$.

Below in this section we will write $\hat{R}_0$, $R_{\lambda,0}$, $e_{\lambda,0}$, μ_0, ψ_0, Δ_0, E_0, $C_{j,0}$ as $\hat{R}$, R_λ, e_λ, μ, ψ, Δ, E, C_j.

Let

$$f_0 = F_0 dz_1 = \frac{i}{2}\hat{R}(q\mu), \quad f_1 = F_1 dz_1 = i\sum_{j=1}^{g} C_j(\lambda)\hat{R}(\delta(\cdot, a_j)),$$

where $\mu = \mu(z,\lambda)$, $z \in V$, $\lambda \in \mathbb{C}\backslash E: \ |\lambda| \geq const(V, \{a_j\}, \sigma)$.

Lemma 6.1. *For $u_0 = R_\lambda f_0$ the following estimate holds:*

$$\|u_0(\cdot,\lambda) - \frac{F_0(\cdot,\lambda)}{\lambda}\|_{L^{9/4}(X)} \leq \frac{const(V,\tilde{p})}{|\lambda|^{7/5}}\|f_0(\cdot,\lambda)\|_{\tilde{W}^{2,\tilde{p}}_{1,0}(V)}.$$

Proof of Lemma 6.1. By Lemma 2.1 and Proposition 2 from [He] we have $f_0 \in \tilde{W}^{2,\tilde{p}}_{1,0}(V)$, $F_0 \in \tilde{W}^{1,p}(V)$. Using equality $\partial_z e_\lambda(z) = \lambda e_\lambda(z) dz_1$ and integration by parts formula $u_0 = R_\lambda f_0 = e_{-\lambda}(z)\overline{R(\overline{e_\lambda f_0})}$ can be transformed into the following

$$\begin{aligned} u_0(z) &= e_{-\lambda}(z)\overline{R_1(\overline{e_\lambda f_0})} + e_{-\lambda}(z)\overline{R_0(\overline{e_\lambda f_0})} = \\ &- \frac{e_{-\lambda}(z)}{2\pi i}\frac{1}{\lambda}\int\limits_V \frac{e_\lambda(\xi)\partial F_0 \wedge \overline{d\xi_1}\det\left[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \bar{\xi}_2}(\xi)\cdot|\xi - z|^2} - \\ &- \frac{e_{-\lambda}(z)}{2\pi i}\frac{1}{\lambda}\int\limits_V e_\lambda(\xi)F_0\partial\left(\frac{\det\left[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z\right]\wedge d\bar{\xi}_1}{\frac{\partial \bar{P}}{\partial \bar{\xi}_2}(\xi)\cdot|\xi - z|^2}\right) + e_{-\lambda}(z)\overline{R_0(\overline{e_\lambda f_0})}, \end{aligned} \tag{6.5}$$

where R_1, R_0 operators defined in section 1 (see remark 1.1). From (6.5), using Corollary 1.2 from [He], we deduce

$$\lambda u_0 - F_0 = -e_{-\lambda}(z)\overline{R_1(\overline{e_\lambda(\xi)\partial F_0})} - e_{-\lambda}(z)\overline{R_0(\overline{e_\lambda(\xi)\partial F_0})} \stackrel{\text{def}}{=} J_1(z) + J_0(z). \tag{6.6}$$

We will estimate further only term $J_1(z)$. Estimate for $J_0(z)$ is similar.

For $J_1(z)$ we have $J_1(z) = J_1^+(z) + J_1^-(z)$, where

$$J_1^{\pm}(z) = \frac{e_{-\lambda}(z)}{2\pi i}\int\limits_V \frac{e_\lambda(\xi)\chi_\rho^{\pm}(\xi)\partial F_0(\xi)\wedge\overline{d\xi_2}\det\left[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \bar{\xi}_1}(\xi)\cdot|\xi - z|^2},$$

$$\begin{aligned} &\chi_\rho^{\pm} \ \text{ be smooth functions such that } \ \chi_\rho^+ + \chi_\rho^- \equiv 1, \\ &\chi_\rho^+ = 1, \ \text{ if } \ \left|\frac{d\xi_1}{d\xi_2}\right| \leq \rho, \ \ supp\,\chi_\rho^+ \subset \{\xi : \ \left|\frac{d\xi_1}{d\xi_2}\right| \leq 2\rho\} \\ &\text{and } \ |d\chi_\rho^{\pm}| = O(\frac{1}{\rho}). \end{aligned} \tag{6.7}$$

Let $B_0 = \{z \in V: \ d\xi_1\big|_V(z) = 0\}$. Property $\bar\partial F_0 = dz_1 \rfloor \frac{i}{2} q\mu$ implies estimate $\bar\partial F_0 = O\big(\frac{1}{dist(z,B_0)}\big)dz_2$. From this, formula for $J_1^+(z)$ and Lemma 3.1 of [He] we obtain estimate for

$$J_1^+: \ \|J_1^+\|_{L^{9/4}(X)} = O(\rho^{2/3})\|f_0\|_{\tilde W^{2,\tilde p}_{1,0}(V)}. \qquad (6.8)$$

In order to estimate $J_1^-(z)$ we integrate by parts in the formula for J_1^-, using $\partial_z e_\lambda(z) = \lambda e_\lambda(z) dz_1$. Then inequalities

$$|\bar\partial F_0(z)| = O\big(\frac{1}{dist(z,B_0)}\big), \quad |\bar\partial\partial F_0(z)| = O\big(\frac{1}{dist(z,B_0)}\big)^2, \quad z \in X\backslash B_0$$

and inequality

$$\Big|\int\limits_{\rho\le|\xi_2|\le 1} \frac{d\xi_2\wedge d\bar\xi_2}{|\xi_2|^2(\bar\xi_2 - \bar z_2)}\Big| + \Big|\int\limits_{\rho\le|\xi_2|\le 1} \frac{d\xi_2\wedge d\bar\xi_2}{|\xi_2|(\bar\xi_2 - \bar z_2)^2}\Big| = O(\frac{1}{\rho})$$

imply estimate

$$\|J_1^-\|_{L^\infty(X)} = O(\frac{1}{|\lambda|\rho})\|f_0\|_{\tilde W^{2,\tilde p}_{1,0}(V)}. \qquad (6.9)$$

From (6.6), (6.8), (6.9) with $\rho = |\lambda|^{-3/5}$ we obtain statement of Lemma 6.1.

Lemma 6.2. *Let $q \in C^{(1)}_{1,1}(V)$, $supp\, q \subseteq X$, $f_0 = \frac{i}{2}\hat R(q\mu)$, $u_0 = R_\lambda f_0$. Then the following asymptotic estimate is valid*

$$\Big|\int\limits_X e_\lambda(z)q(z)u_0(z)\Big| = o(\frac{1}{|\lambda|}), \quad \textit{for}$$
$$\lambda \in \mathbb{C}: \ |\lambda| \ge const(V,\{a_j\},\sigma), \quad |\Delta(\lambda)(1+|\lambda|)^g| \ge \delta > 0, \quad \textit{for some sufficient small} \ \ \delta.$$

Proof of Lemma 6.2.

From Lemma 6.1, using estimate of μ from (3.1b), we obtain asymptotic relation in the space $L^{\tilde p}(V)$, $2 < \tilde p < 9/4$:

$$u_0(z,\lambda) = \frac{F_0(z,\lambda)}{\lambda} + O(\frac{1}{|\lambda|^{7/5}}) =$$
$$\frac{dz_1 \rfloor \frac{i}{2}\hat R(q)}{\lambda} + O(\frac{1}{|\lambda|^{7/5}}) \ \text{ if } \ |\Delta_\theta(\lambda)(1+|\lambda|)^g| \ge \delta > 0.$$

Putting this relation into $\int\limits_X e_\lambda(z)q(z)u_0(z)$, we obtain

$$\int\limits_X e_\lambda(z)q(z)u_0(z) = \frac{i}{2\lambda}\int\limits_X e_\lambda(z)q(z)(dz_1\rfloor \hat R(q)) + O(\frac{1}{|\lambda|^{7/5}}).$$

By Riemann-Lebesgue type theorem

$$\int\limits_X e_\lambda(z)q(z)(dz_1\rfloor\hat{R}(q)) = o(1) \ \text{ if } \ \lambda\to\infty, \ \ |\Delta(\lambda)|(1+|\lambda|)^g \geq \delta > 0.$$

This implies the statement of Lemma 6.2.

Lemma 6.3. *Let $q \in C^{(1)}_{1,1}(V)$, $supp\, q \subset X$. Let $w_1,\dots,w_M$ be the points, where $dz_1\big|_V(w_m) = 0$. Then the following consequence of stationary phase method is valid:*

$$\int\limits_V e^{i\tau(z_1+\bar{z}_1)}q(z) = \sum_m (1+o(1))\sum_{m=1}^{M} -\frac{\pi}{r}\frac{\left|\frac{\partial P}{\partial z_1}(w_m)\right|Q_2(w_m)}{\left|\frac{\partial^2 P}{\partial z_2^2}(w_m)\right|}e^{i\tau(w_{m,1}+\bar{w}_{m,1})}, \qquad (6.10)$$

where $Q_2(w_m) = \frac{q}{2idz_2\wedge d\bar{z}_2}(w_m)$.

Proof of Lemma 6.3 (see [Fe], Th.2.1).

Lemma 6.4. *Let $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}} \in C^{(1)}_{1,1}(X)$, $supp\, q \subset X$, $f_1 = i\sum\limits_{j=1}^{g} C_j(\lambda)\hat{R}(\delta(\cdot,a_j))$, $u_1 = R_\lambda f_1$. Then the following asymptotic estimate is valid*

$$\Big|\int\limits_X e_\lambda(z)q(z)u_1(z)\Big| = O(\frac{1}{|\lambda|^{3/2-\varepsilon}}), \ \ \textit{for} \ \ \lambda\in\mathbb{C}: \ |\lambda| \geq const(V,\{a_j\},\sigma,\varepsilon),$$

$$|\Delta(\lambda)(1+|\lambda|)^g| \geq \delta > 0, \ \ \delta \ \ \textit{for some sufficiently small} \ \ \delta.$$

Proof of Lemma 6.4. Using that $\{a_1,\dots,a_g\}$ be generic divisor, from estimate (3.7) (Lemma 3.3) we obtain inequality

$$\sup_{j,\lambda}|C_j(\lambda)| \leq const(V,\{a_j\})\sup_k\left|\int\limits_X e_\lambda(z)\big(i\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}} + 2\bar{\partial}\ln\sqrt{\sigma}\wedge\partial\ln\sqrt{\sigma}\big)\mu\frac{\bar{\omega}_k}{d\bar{z}_1}(z)\right|.$$

Let $\varepsilon > 0$ be small enough and $B_\varepsilon = \{z \in X : \left|\frac{dz_1}{dz_2}\right| < \varepsilon\}$. Then

$$\Big|\frac{\bar{\omega}_k(z)}{d\bar{z}_1}\Big|_X = O\big(\sum_{m=1}^{M}\frac{1}{|z_2-w_{m,k}|}\big), \ \ z\in X.$$

Let $\chi^\pm_\rho \in C^{(1)}(X)$ be functions with properties (6.7). Using that $\sigma \in C^{(3)}(X)$, $\mu \in \tilde{W}^{1,\tilde{p}}(X)$, $\partial_z e_\lambda(z) = \lambda e_\lambda(z)dz_1$ by integration by parts we obtain

$$\left|\int\limits_X \chi^-_\rho(z)e_\lambda(z)\big(i\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}} + 2\bar{\partial}\ln\sqrt{\sigma}\wedge\partial\ln\sqrt{\sigma}\big)\mu(z,\lambda)\frac{\bar{\omega}_k}{d\bar{z}_1}(z)\right| \leq$$

$$\frac{const(V,\sigma)}{\rho\lambda}.$$

We have also directly

$$\Big|\int\limits_X \chi_\rho^+(z)e_\lambda(z)\big(i\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}+2\bar\partial\ln\sqrt{\sigma}\wedge\partial\ln\sqrt{\sigma}\big)\mu(z,\lambda)\frac{\bar\omega_k}{d\bar z_1}(z)\Big|\le$$
$$const(V,\sigma)\rho.$$

These estimates with $\rho=\frac{1}{\sqrt{|\lambda|}}$ and estimates for Faddeev type Green function $|R_\lambda\circ\hat R(\delta(\cdot,a_j)|=O(\frac{1}{|\lambda|^{1-\varepsilon}})$ from Theorem 4 of [He] imply statement of Lemma 6.4.

Proposition 6.1. *Under conditions (6.1)-(6.4), for* $\lambda=i\tau:\ \tau\in\mathbb{R},\ |\tau^g\Delta(i\tau)|\ge\delta>0$, *$\delta$- small enough, the following formula is valid*

$$\int\limits_{z\in bX} e_{i\tau}(z)\bar\partial_z\mu(z,i\tau)=\int\limits_{z\in X} e_{i\tau}(z)\frac{q\mu}{2i}=$$
$$\frac{1+o(1)}{\tau}\sum_{m=1}^M\frac{\pi i}{2}\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}dd^c|z|^2}\big|_V(w_m)e^{i\tau(w_{m,1}+\bar w_{m,1})}\big|\frac{\partial^2P}{\partial z_2^2}(w_m)\big|^{-1}\frac{\partial P}{\partial z_1}(w_m). \tag{6.11}$$

Proof of Proposition 6.1 and Theorem 1.2B. From Lemma 3.1 we have equality

$$\mu=1+R_\lambda\circ\hat R\big(\frac{i}{2}q\mu\big)+R_\lambda\circ\hat R\big(i\sum_{j=1}^g C_j\delta(z,a_j)\big)=1+u_0+u_1. \tag{6.12}$$

Let $\delta>0$ be small enough. Estimates of Lemmas 6.2, 6.4 and (6.12) give asymptotic equality

$$\mu=1+o\big(\frac{1}{\lambda}\big) \tag{6.13}$$

under conditions $\lambda\in\mathbb{C}:\ |\lambda|\ge const(V,\{a_j\},\sigma),\ |\Delta(\lambda)(1+|\lambda|)^g|\ge\delta>0$.

By Proposition 1.1, $\forall\varepsilon>0$ we have inequality

$$\underline{\lim}_{\lambda\to\infty}|\lambda^g\Delta(\lambda)|_\varepsilon=\delta(\varepsilon)>0,\ \ \text{where}\ \ |\lambda^g\Delta(\lambda)|_\varepsilon=\sup_{\{\lambda':\ |\lambda'-\lambda|\le\varepsilon\}}|\lambda'\Delta(\lambda')|.$$

So for any $\varepsilon>0$ and any positive $\delta<\delta(\varepsilon)$ there exists r such that the set $\{\lambda\in\mathbb{C}:\ |\Delta(\lambda)(1+|\lambda|)^g|\ge\delta>0\}$ intersects any disque $\{\lambda':\ |\lambda-\lambda'|<\varepsilon\}$, with $|\lambda|\ge r$. This property, Lemma 6.3 and property (6.13) imply Proposition 6.1.

Theorem 1.2B follows from Proposition 1. Indeed, stationary phase method permits differentiation of (6.11) with respect to τ, keeping (in our case) terms of order $\frac{1}{\tau}$. Differentiation of the right-hand side of (6.11) gives for $\theta=0$ the right-hand side of (1.12).

Theorem 1.2B is proved.

Remark 6.1. To obtain version of Proposition 6.1 with arbitrary generic θ from Proposition 6.1 with $\theta=0$ it is sufficient to change coordinate system: $\tilde z_1=z_1+\theta z_2$, $\tilde z_2=z_2$.

Remark 6.2. Proposition 6.1 can be reformulated also as formula for reconstruction of conductivity function from scattering data $b_\theta(i\tau)$ and $C_{j,\theta}(i\tau)$. Indeed,by formula (4.16), we have

$$\int\limits_{bX} e_{i\tau,\theta}(z)\bar\partial\mu(z,i\tau) = -2\pi\big[\tau b_\theta(i\tau)d + \sum_{j=1}^{g} C_{j,\tau}(i\tau)e_{i\tau,\theta}(a_j),$$

where d is defined in section 1.

7. Proof of Proposition 1.1

For simplicity of presentation we give proof only for the case when V is algebraic curve in $\mathbb{C}^2$. Proposition 1.1 will be obtained here as a corollary of the following statement.

Proposition 7.1. *Let $\theta \in \mathbb{C}\backslash\{\theta_1,\dots,\theta_N\}$, $\delta = \delta(\theta) = \inf\limits_{l}|\theta-\theta_l|$, $V_0 = \{z \in V : |z_1| \le r_0(\delta)\}$, $r_0(\delta) = \frac{const(V)}{\sqrt{\delta}}$. Let $\{b_m\}$ be the points of V, where $(dz_1+\theta dz_2)\big|_V(b_m) = 0$, $m = 1,\dots,M$, and $\{a_1,\dots,a_g\}$ be the points of generic divisor in $V\backslash\bar V_0$. Then $\forall\, j,k = 1,\dots,g$ and for $\lambda = i\tau$, where $\tau\in\mathbb{R}$, large enough, such that $|\Delta_\theta(i\tau)| \ge \delta > 0$, the following asymptotic equality is valid*

$$\int\limits_V \hat R_\theta(\delta(\xi,a_j))\wedge\bar\omega_k(\xi)e_{\lambda,\theta}(\xi) = -\frac{1}{\bar\lambda}e_{\lambda,\theta}(a_j)\frac{\bar\omega_k}{d\bar\xi_1+\bar\theta d\bar\xi_2}(a_j)-$$

$$-\frac{\pi}{|\lambda|}\sum_{m=1}^{M}\exp\big[\lambda(b_{m,1}+\theta b_{m,2})-\bar\lambda(\bar b_{m,1}+\bar\theta\bar b_{m,2})\big]K_{j,k}(b_m,a_j)+O\big(\frac{1}{|\lambda|^2}\big),$$

where

$$K_{j,k}(b_m,a_j) = \frac{|\frac{\partial P}{\partial z_1}(b_m)|^3\hat R_\theta(\delta(b_m,a_j))\wedge\bar\omega_k(b_m)(1+|\theta|^2)}{|\frac{\partial^2 P}{\partial z_1^2}\big(\frac{\partial P}{\partial z_2}\big)^2 - 2\frac{\partial^2 P}{\partial z_1\partial z_2}\frac{\partial P}{\partial z_2}\frac{\partial P}{\partial z_1} + \frac{\partial^2 P}{\partial z_2^2}\big(\frac{\partial P}{\partial z_1}\big)^2|dd^c|z|^2\big|_V(b_m)}. \qquad (7.1)$$

Lemma 7.1. *Let $V\backslash V_0 = \cup_{l=1}^{g}V_l$ be a curve with properties i)-iv) of section 1. Then $\forall\theta\ne\theta_1,\dots,\theta_d$ any point w, where $(dz_1+\theta dz_2)\big|_V(w) = 0$, belongs to $V_0 = \{z\in V: |z_1| \le r_0(\delta)\}$, where $r_0(\delta) = const(V)/\sqrt{\delta}$, $\delta = \min\limits_l|\theta-\theta_l|$.*

Proof of Lemma 7.1. For any point $w\in V\backslash V_0$, where $(dz_1+\theta dz_2)\big|_V(w) = 0$, definition $\theta_l = -\frac{1}{\gamma_l}$,
$l = 1,\dots,d$, and property iii) of Section 1 imply for some $l = l(w)$ equality

$$0 = (dz_1+\theta dz_2)\big|_V(w) = \big[1+\theta\big(\gamma_l+\frac{\gamma_l^0}{w_1^2}+O\big(\frac{1}{w_1^3}\big)\big)\big]dz_1 =$$
$$\big[1+\theta\gamma_l+O\big(\frac{\theta}{w_1^2}\big)\big]dz_1 = \gamma_l\big[(\theta-\theta_l)+O\big(\frac{\theta}{w_1^2}\big)\big]dz_1.$$

This gives equality $\theta\big(1+O\big(\frac{1}{w_1^2}\big)\big) = \theta_l$. This equality together with inequality $|\theta-\theta_l|\ge\delta$ implies inequality $|w_1| \le \frac{const(V)}{\sqrt{\delta}} = r_0(\delta)$.

Lemma 7.1 is proved.

Let further

$$A_{\varepsilon,j} = \{z \in V: \ |z - a_j| \le \varepsilon\}, \ A_\varepsilon = \cup_{j=1}^{g} A_{\varepsilon,j},$$
$$B_{\varepsilon,m} = \{z \in V: \ |z - b_m| \le \varepsilon\}, \ B_\varepsilon = \cup_{m=1}^{M} B_{\varepsilon,m}.$$

Lemma 7.2 *Let $r_0(\delta)$, $\delta = \delta(\theta)$ be as in Lemma 7.1. Let χ^{A_ε}, χ^{B_ε} be smooth functions with properties*

$$\chi^{A_\varepsilon}\big|_{A_\varepsilon} = 1, \ \ \chi^{A_\varepsilon}\big|_{V\backslash A_{2\varepsilon}} = 0, \ \ |d\chi^{A_\varepsilon}| = O(\frac{1}{\varepsilon}),$$
$$\chi^{B_\varepsilon}\big|_{B_\varepsilon} = 1, \ \ \chi^{B_\varepsilon}\big|_{V\backslash B_{2\varepsilon}} = 0, \ \ |d\chi^{B_\varepsilon}| = O(\frac{1}{\varepsilon}).$$

Then for any $\varepsilon > 0$ small enough we have $B_{2\varepsilon} \cap A_{2\varepsilon} = \{\emptyset\}$ and $\forall \ j,k = 1,\ldots,g$

$$\Delta^{j,k}_{\theta,\varepsilon}(\lambda) \stackrel{\text{def}}{=} \int\limits_{\xi \in V} (1 - \chi^{A_\varepsilon} - \chi^{B_\varepsilon})\hat{R}(\delta(\xi, a_j)) \wedge \bar{\omega}_k(\xi) e_{\lambda,\theta}(\xi) = O\big(\frac{1}{\lambda^2}\big).$$

Proof of Lemma 7.2. By Lemma 7.1, any point b_m, where $(dz_1 + \theta dz_2)\big|_V(b_m) = 0$ belongs to $\{z \in V: \ |z_1| \le r_0\}$. Under the conditions of Lemma 7.2, any a_j from $\{a_1,\ldots,a_g\}$ belongs to $\{z \in V: \ |z_1| > r_0(\delta)\}$, $\delta = \delta(\theta)$.

Then $B_{2\varepsilon} \cap A_{2\varepsilon} = \{\emptyset\}$ if ε is small enough. From definition of $\Delta^{j,k}_{\theta,\varepsilon}$ and equality $\bar\partial \hat{R}_\theta(\delta(\varepsilon, a_j))\big|_{V\backslash\{a_j\}} = 0$ we obtain

$$\begin{aligned}
\Delta^{j,k}_{\theta,\varepsilon}(\lambda) = \frac{1}{\bar\lambda}\int\limits_V (1 - \chi^{A_\varepsilon} - \chi^{B_\varepsilon})\hat{R}_\theta(\delta(\xi, a_j)) \wedge \frac{\bar\omega_k}{d\bar\xi_1 + \bar\theta d\bar\xi_2}\bar\partial e_{\lambda,\theta}(\xi) = \\
-\frac{1}{\bar\lambda}\int\limits_V (1 - \chi^{A_\varepsilon} - \chi^{B_\varepsilon})\hat{R}_\theta(\delta(\xi, a_j)) \wedge \bar\partial\big(\frac{\bar\omega_k}{d\bar\xi_1 + \bar\theta d\bar\xi_2}\big) e_{\lambda,\theta}(\xi) - \\
\frac{1}{\bar\lambda}\int\limits_V \bar\partial(\chi^{A_\varepsilon} + \chi^{B_\varepsilon})\hat{R}_\theta(\delta(\xi, a_j)) \wedge \frac{\bar\omega_k(\xi)}{d\bar\xi_1 + \bar\theta d\bar\xi_2} e_{\lambda,\theta}(\xi) + \\
\frac{1}{\bar\lambda}\lim_{r\to\infty}\int\limits_{\{\xi \in V: \ |\xi_1| = r\}} \hat{R}_\theta(\delta(\xi, a_j)) \wedge \frac{\bar\omega_k}{d\bar\xi_1 + \bar\theta d\bar\xi_2} e_{\lambda,\theta}(\xi).
\end{aligned} \tag{7.2}$$

From asymptotic estimates $|\hat{R}_\theta(\delta(\xi, a_j))| = O(|d\xi_1|)$ and $|\bar\omega_k| = O\big(\frac{d\bar\xi_1}{\bar\xi_1^2}\big)$, $\xi_1 \to \infty$, and property $\inf\limits_l |\theta - \theta_l| > 0$ we obtain vanishing of the last term of the right-hand side of (7.2).

Property $(d\xi_1 + \theta d\xi_2)\big|_{V\backslash B_\varepsilon} \ne 0$ permits to integrate other terms of the right-hand side of (7.2) by parts once more and to obtain statement of Lemma 7.2.

Lemma 7.3 *For any $k, j \in \{1, \ldots, g\}$, $\theta \notin \{\theta_1, \ldots, \theta_d\}$ and any $\varepsilon > 0$ we have the asymptotic equality*

$$\int\limits_V \chi^{A_{\varepsilon,j}} \hat{R}_\theta(\delta(\xi, a_j)) \wedge \bar{\omega}_k(\xi) e_{\lambda,\theta}(\xi) = -\frac{1}{\bar{\lambda}} e_{\lambda,\theta}(a_j) \frac{\bar{\omega}_k}{d\bar{\xi}_1 + \bar{\theta} d\bar{\xi}_2}(a_j) + \left(\frac{1}{\lambda^2}\right).$$

Proof of Lemma 7.3. Integration by parts of the left-hand side, equality $\bar{\partial}\hat{R}(\delta(\xi, a_j)) = \delta(\xi, a_j)$ and inequality $(d\xi_1 + \theta d\xi_2)\big|_{A_{\varepsilon,j}} \neq 0$ imply statement of Lemma 7.3.

Lemma 7.4 *Under the conditions of Lemmas 7.1, 7.2, $\forall \delta > 0$, $\theta : \inf\limits_l |\theta - \theta_l| > \delta$, $\forall\ j, k = 1, \ldots, g$,*

$$\int\limits_V \chi^{B_\varepsilon} \hat{R}_\theta(\delta(\xi, a_j)) \wedge \bar{\omega}_k(\xi) e_{i\tau,\theta}(\xi) =$$

$$-\frac{\pi}{|\lambda|} \sum_{m=1}^{M} \exp[\lambda(b_{m,1} + \theta b_{m,2}) - \bar{\lambda}(\bar{b}_{m,1} + \bar{\theta}\bar{b}_{m,2})] K_{j,k}(b_m, a_j) + O\left(\frac{1}{|\lambda|^2}\right),$$

where $\theta = \theta(b_m)$, $m = 1, \ldots, M$, and $K_{j,k}(b_m, a_j)$ are defined by (7.1).

Proof of Lemma 7.4. This statement is consequence of the classical result of the stationary phase method [Fe], applied to the left-hand side, taking into account the following equality for $e_{\lambda,\theta}(z)$ in the neighborhood of the stationary points $b_m \in V$, $m = 1, \ldots, M$,

$$e_{\lambda,\theta}(z) = \exp[\lambda(b_{m,1} + \theta b_{m,2}) - \bar{\lambda}(\bar{b}_{m,1} + \bar{\theta}\bar{b}_{m,2})] \times \exp[\lambda A(z_2 - b_{m,2})^2 - \bar{\lambda}\bar{A}(\bar{z}_2 - \bar{b}_{m,2})^2],$$

where

$$A = -\frac{\left(\frac{\partial^2 P}{\partial z_1^2}\theta^2 - 2\frac{\partial^2 P}{\partial z_1 \partial z_2}\theta + \frac{\partial^2 P}{\partial z_2^2}\right)(b_m)(z_2 - b_{m,2})^2(1 + O(z_2 - b_{m,2}))}{2\left(\frac{\partial P}{\partial z_1}\right)(b_m)}.$$

We use here $z_2, \bar{z}_2$ as coordinates of integration.
Lemma 7.4 is proved.

Proof of Proposition 7.1. Proposition 7.1 follows from Lemmas 7.2-7.4.
In the proof of Proposition 1.1 we will apply also the following statement about exponential polynomials discovered by L.Ehrenpreis [E] and reinforced by C.Berenstein and M.Dostal [BD].

Proposition 7.2. ([E], [BD]) *Let $Q(\xi)$ be an exponential polynomial*

$$Q(\xi) = \sum_{k=1}^{N} q_k(\xi) e^{<\alpha_k, \xi>},$$

where $\{q_k\}$ are polynomials of $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, $\alpha_k = \{\alpha_{k,1}, \ldots, \alpha_{k,n}\} \in \mathbb{C}^n$, $k = 1, \ldots, N$.

Let $h(\xi) = \max_k Re < \alpha_k, \xi >$. *Then* $\forall \varepsilon > 0$ $\exists$ *constant* $C = C(\varepsilon, Q) > 0$ *such that*

$$|Q(\xi)|_\varepsilon \stackrel{\text{def}}{=} \sup_{\{\xi' \in \mathbb{C}:\ |\xi' - \xi| < \varepsilon\}} |Q(\xi')| \geq \frac{1}{C} e^{h(\xi)}.$$

The final part of the proof of Proposition 1.1 consists of the following. Proposition 7.1 and definition of $\Delta_\theta(\lambda)$ imply asymptotic equality

$$|\lambda|^g \Delta_\theta(\lambda) = \det\big(-\frac{\lambda}{\bar\lambda} e_{\lambda,\theta} \frac{\bar\omega_k}{d\bar\xi_1 + \bar\theta d\bar\xi_2}(a_j) - \pi \sum_{m=1}^{M} \exp[\lambda(b_{m,1} + \theta b_{m,2}) - \bar\lambda(\bar b_{m,1} + \bar\theta \bar b_{m,2})] K_{j,k}(b_m, a_j)\big) + O\big(\frac{1}{|\lambda|}\big), \tag{7.3}$$

where $j, k = 1, \ldots, g$.

The determinant of the right-hand side of (7.3) is an exponential polynomial $Q(\lambda, \bar\lambda)$ of the form

$$Q(\lambda, \bar\lambda) = \sum_{k=1}^{N} q_k(\lambda, \bar\lambda) e^{\lambda\alpha_k - \bar\lambda\bar\alpha_k}, \tag{7.4}$$

where $\lambda \in \mathbb{C}$, $\alpha_k \in \mathbb{C}$, $k = 1, \ldots, N$. Coefficient $q_k(\lambda, \bar\lambda)$ of exponential polynomial $Q(\lambda, \bar\lambda)$ and complex frequences $\{\alpha_k\}$ depend on V, $\{a_j\}$, θ, $\{b_m\}$. Applying Proposition 7.2 to the exponential polynomial (7.4) we obtain uniformly for $\lambda \in \mathbb{C}$ estimate

$$|Q(\lambda, \bar\lambda)|_\varepsilon \geq \frac{1}{C(\varepsilon, Q)} e^{\max_k Re\,(\lambda\alpha_k - \bar\lambda\bar\alpha_k)} = \frac{1}{C(\varepsilon, Q)}. \tag{7.5}$$

The both inequalities of Proposition 1.1 follow from (7.3)-(7.5).

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