Canonical bases and affine Hecke algebras of type D

Peng Shan, Michela Varagnolo, Eric Vasserot

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P. Shan, M. Varagnolo, E. Vasserot

Abstract. We prove a conjecture of Miemietz and Kashiwara on canonical bases and branching rules of affine Hecke algebras of type D. The proof is similar to the proof of the type B case in [VV].

Introduction

Let $\mathbf{f}$ be the negative part of the quantized enveloping algebra of type $\mathrm{A}^{(1)}$. Lusztig's description of the canonical basis of $\mathbf{f}$ implies that this basis can be naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type A. This identification was mentioned in [G], and was used in [A]. More precisely, there is a linear isomorphism between $\mathbf{f}$ and the Grothendieck group of finite dimensional modules of the affine Hecke algebras of type A, and it is proved in [A] that the induction/restriction functors for affine Hecke algebras are given by the action of the Chevalley generators and their transposed operators with respect to some symmetric bilinear form on $\mathbf{f}$.

The branching rules for affine Hecke algebras of type B have been investigated quite recently, see [E], [EK1,2,3], [M] and [VV]. In particular, in [E], [EK1,2,3] an analogue of Ariki's construction is conjectured and studied for affine Hecke algebras of type B. Here $\mathbf{f}$ is replaced by a module ${}^\theta\mathbf{V}(\lambda)$ over an algebra ${}^\theta\mathbf{B}$. More precisely it is conjectured there that ${}^\theta\mathbf{V}(\lambda)$ admits a canonical basis which is naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type B. Further, in this identification the branching rules of the affine Hecke algebras of type B should be given by the ${}^\theta\mathbf{B}$-action on ${}^\theta\mathbf{V}(\lambda)$. This conjecture has been proved [VV]. It uses both the geometric picture introduced in [E] (to prove part of the conjecture) and a new kind of graded algebras similar to the KLR algebras from [KL], [R].

A similar description of the branching rules for affine Hecke algebras of type D has also been conjectured in [KM]. In this case $\mathbf{f}$ is replaced by another module ${}^\circ\mathbf{V}$ over the algebra ${}^\theta\mathbf{B}$ (the same algebra as in the type B case). The purpose of this paper is to prove the type D case. The method of the proof is the same as in [VV]. First we introduce a family of graded algebras ${}^\circ\mathbf{R}_m$ for m a non negative integer. They can be viewed as the Ext-algebras of some complex of constructible sheaves naturally attached to the Lie algebra of the group $SO(2m)$, see Remark 2.8. This complex enters in the Kazhdan-Lusztig classification of the simple modules of the affine Hecke algebra of the group $Spin(2m)$. Then we identify ${}^\circ\mathbf{V}$ with the sum of the Grothendieck groups of the graded algebras ${}^\circ\mathbf{R}_m$.

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The plan of the paper is the following. In Section 1 we introduce a graded algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$. It is associated with a quiver Γ with an involution θ and with a dimension vector ν. In Section 2 we consider a particular choice of pair (Γ,θ). The graded algebras ${}^\circ\mathbf{R}(\Gamma)_\nu$ associated with this pair (Γ,θ) are denoted by the symbol ${}^\circ\mathbf{R}_m$. Next we introduce the affine Hecke algebra of type D, more precisely the affine Hecke algebra associated with the group $SO(2m)$, and we prove that it is Morita equivalent to ${}^\circ\mathbf{R}_m$. In Section 3 we categorify the module ${}^\circ\mathbf{V}$ from [KM] using the graded algebras ${}^\circ\mathbf{R}_m$, see Theorem 3.28. The main result of the paper is Theorem 3.33.

0. Notation

0.1. Graded modules over graded algebras. Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. By a graded $\mathbf{k}$-algebra $\mathbf{R}=\bigoplus_d \mathbf{R}_d$ we'll always mean a $\mathbb{Z}$-graded associative $\mathbf{k}$-algebra. Let $\mathbf{R}$**-mod** be the category of finitely generated graded $\mathbf{R}$-modules, $\mathbf{R}$**-fmod** be the full subcategory of finite-dimensional graded modules and $\mathbf{R}$**-proj** be the full subcategory of projective objects. Unless specified otherwise all modules are left modules. We'll abbreviate

$$K(\mathbf{R})=[\mathbf{R}\text{-}\mathbf{proj}],\quad G(\mathbf{R})=[\mathbf{R}\text{-}\mathbf{fmod}].$$

Here $[\mathcal{C}]$ denotes the Grothendieck group of an exact category $\mathcal{C}$. Assume that the $\mathbf{k}$-vector spaces $\mathbf{R}_d$ are finite dimensional for each d. Then $K(\mathbf{R})$ is a free Abelian group with a basis formed by the isomorphism classes of the indecomposable objects in $\mathbf{R}$**-proj**, and $G(\mathbf{R})$ is a free Abelian group with a basis formed by the isomorphism classes of the simple objects in $\mathbf{R}$**-fmod**. Given an object M of $\mathbf{R}$**-proj** or $\mathbf{R}$**-fmod** let $[M]$ denote its class in $K(\mathbf{R})$, $G(\mathbf{R})$ respectively. When there is no risk of confusion we abbreviate $M=[M]$. We'll write $[M:N]$ for the composition multiplicity of the $\mathbf{R}$-module N in the $\mathbf{R}$-module M. Consider the ring $\mathcal{A}=\mathbb{Z}[v,v^{-1}]$. If the grading of $\mathbf{R}$ is bounded below then the $\mathcal{A}$-modules $K(\mathbf{R})$, $G(\mathbf{R})$ are free. Here $\mathcal{A}$ acts on $G(\mathbf{R})$, $K(\mathbf{R})$ as follows

$$vM=M[1],\quad v^{-1}M=M[-1].$$

For any M,N in $\mathbf{R}$**-mod** let

$$\mathrm{hom}_{\mathbf{R}}(M,N)=\bigoplus_d \mathrm{Hom}_{\mathbf{R}}(M,N[d])$$

be the $\mathbb{Z}$-graded $\mathbf{k}$-vector space of all $\mathbf{R}$-module homomorphisms. If $\mathbf{R}=\mathbf{k}$ we'll omit the subscript $\mathbf{R}$ in hom's and in tensor products. For any graded $\mathbf{k}$-vector space $M=\bigoplus_d M_d$ we'll write

$$\mathrm{gdim}(M)=\sum_d v^d \dim(M_d),$$

where dim is the dimension over $\mathbf{k}$.

0.2. Quivers with involutions. Recall that a quiver Γ is a tuple $(I, H, h \mapsto h', h \mapsto h'')$ where I is the set of vertices, H is the set of arrows and for each $h \in H$ the vertices $h', h'' \in I$ are the origin and the goal of h respectively. Note that the set I may be infinite. We'll assume that no arrow may join a vertex to itself. For each $i, j \in I$ we write

$$H_{i,j} = \{h \in H; h' = i, h'' = j\}.$$

We'll abbreviate $i \to j$ if $H_{i,j} \neq \emptyset$. Let $h_{i,j}$ be the number of elements in $H_{i,j}$ and set

$$i \cdot j = -h_{i,j} - h_{j,i}, \quad i \cdot i = 2, \quad i \neq j.$$

An involution θ on Γ is a pair of involutions on I and H, both denoted by θ, such that the following properties hold for each h in H

- $\theta(h)' = \theta(h'')$ and $\theta(h)'' = \theta(h')$,
- $\theta(h') = h''$ iff $\theta(h) = h$.

We'll always assume that θ has no fixed points in I, i.e., there is no $i \in I$ such that $\theta(i) = i$. To simplify we'll say that θ has no fixed point. Let

$$^{\theta}\mathbb{N}I = \{\nu = \sum_i \nu_i i \in \mathbb{N}I : \nu_{\theta(i)} = \nu_i, \ \forall i\}.$$

For any $\nu \in {}^{\theta}\mathbb{N}I$ set $|\nu| = \sum_i \nu_i$. It is an even integer. Write $|\nu| = 2m$ with $m \in \mathbb{N}$. We'll denote by ${}^{\theta}I^{\nu}$ the set of sequences

$$\mathbf{i} = (i_{1-m}, \dots, i_{m-1}, i_m)$$

of elements in I such that $\theta(i_l) = i_{1-l}$ and $\sum_k i_k = \nu$. For any such sequence $\mathbf{i}$ we'll abbreviate $\theta(\mathbf{i}) = (\theta(i_{1-m}), \dots, \theta(i_{m-1}), \theta(i_m))$. Finally, we set

$$^{\theta}I^m = \bigcup_{\nu} {}^{\theta}I^{\nu}, \quad \nu \in {}^{\theta}\mathbb{N}I, \quad |\nu| = 2m.$$

0.3. The wreath product. Given a positive integer m, let $\mathfrak{S}_m$ be the symmetric group, and $\mathbb{Z}_2 = \{-1, 1\}$. Consider the wreath product $W_m = \mathfrak{S}_m \wr \mathbb{Z}_2$. Write $s_1, \dots, s_{m-1}$ for the simple reflections in $\mathfrak{S}_m$. For each $l = 1, 2, \dots m$ let $\varepsilon_l \in (\mathbb{Z}_2)^m$ be -1 placed at the l-th position. There is a unique action of W_m on the set $\{1-m, \dots, m-1, m\}$ such that $\mathfrak{S}_m$ permutes $1, 2, \dots m$ and such that ε_l fixes k if $k \neq l, 1-l$ and switches l and $1-l$. The group W_m acts also on ${}^{\theta}I^{\nu}$. Indeed, view a sequence $\mathbf{i}$ as the map

$$\{1-m, \dots, m-1, m\} \to I, \quad l \mapsto i_l.$$

Then we set $w(\mathbf{i}) = \mathbf{i} \circ w^{-1}$ for $w \in W_m$. For each ν we fix once for all a sequence

$$\mathbf{i}_e = (i_{1-m}, \dots, i_m) \in {}^{\theta}I^{\nu}.$$

Let W_e be the centralizer of $\mathbf{i}_e$ in W_m. Then there is a bijection

$$W_e \backslash W_m \to {}^{\theta}I^{\nu}, \quad W_e w \mapsto w^{-1}(\mathbf{i}_e).$$

Now, assume that $m > 1$. We set $s_0 = \varepsilon_1 s_1 \varepsilon_1$. Let ${}^{\circ}W_m$ be the subgroup of W_m generated by $s_0, \dots, s_{m-1}$. We'll regard it as a Weyl group of type D_m such that $s_0, \dots, s_{m-1}$ are the simple reflections. Note that W_e is a subgroup of ${}^{\circ}W_m$. Indeed, if $W_e \not\subset {}^{\circ}W_m$ there should exist l such that ε_l belongs to W_e. This would imply that $i_l = \theta(i_l)$, contradicting the fact that θ has no fixed point. Therefore ${}^{\theta}I^{\nu}$ decomposes into two ${}^{\circ}W_m$-orbits. We'll denote them by ${}^{\theta}I^{\nu}_{+}$ and ${}^{\theta}I^{\nu}_{-}$. For $m = 1$ we set ${}^{\circ}W_1 = \{e\}$ and we choose again ${}^{\theta}I^{\nu}_{+}$ and ${}^{\theta}I^{\nu}_{-}$ in a obvious way.

1. The graded k-algebra $^\circ\mathbf{R}(\Gamma)_\nu$

Fix a quiver Γ with set of vertices I and set of arrows H. Fix an involution θ on Γ. Assume that Γ has no 1-loops and that θ has no fixed points. Fix a dimension vector $\nu \neq 0$ in ${}^\theta\mathbb{N}I$. Set $|\nu| = 2m$.

1.1. Definition of the graded k-algebra $^\circ\mathbf{R}(\Gamma)_\nu$. Assume that $m > 1$. We define a graded $\mathbf{k}$-algebra $^\circ\mathbf{R}(\Gamma)_\nu$ with 1 generated by $1_\mathbf{i}$, $\varkappa_l$, σ_k, with $\mathbf{i} \in {}^\theta I^\nu$, $l = 1, 2, \dots, m$, $k = 0, 1, \dots, m-1$ modulo the following defining relations

(a) $1_\mathbf{i}\, 1_{\mathbf{i}'} = \delta_{\mathbf{i},\mathbf{i}'} 1_\mathbf{i}, \quad \sigma_k 1_\mathbf{i} = 1_{s_k\mathbf{i}} \sigma_k, \quad \varkappa_l 1_\mathbf{i} = 1_\mathbf{i} \varkappa_l,$

(b) $\varkappa_l \varkappa_{l'} = \varkappa_{l'} \varkappa_l,$

(c) $\sigma_k^2 1_\mathbf{i} = Q_{i_k, i_{s_k(k)}}(\varkappa_{s_k(k)}, \varkappa_k) 1_\mathbf{i},$

(d) $\sigma_k \sigma_{k'} = \sigma_{k'} \sigma_k$ if $1 \leqslant k < k' - 1 < m - 1$ or $0 = k < k' \neq 2$,

(e) $(\sigma_{s_k(k)} \sigma_k \sigma_{s_k(k)} - \sigma_k \sigma_{s_k(k)} \sigma_k) 1_\mathbf{i} =$

$$= \begin{cases} \dfrac{Q_{i_k, i_{s_k(k)}}(\varkappa_{s_k(k)}, \varkappa_k) - Q_{i_k, i_{s_k(k)}}(\varkappa_{s_k(k)}, \varkappa_{s_k(k)+1})}{\varkappa_k - \varkappa_{s_k(k)+1}} 1_\mathbf{i} & \text{if } i_k = i_{s_k(k)+1}, \\ 0 & \text{else,} \end{cases}$$

(f) $(\sigma_k \varkappa_l - \varkappa_{s_k(l)} \sigma_k) 1_\mathbf{i} = \begin{cases} -1_\mathbf{i} & \text{if } l = k,\ i_k = i_{s_k(k)}, \\ 1_\mathbf{i} & \text{if } l = s_k(k),\ i_k = i_{s_k(k)}, \\ 0 & \text{else.} \end{cases}$

Here we have set $\varkappa_{1-l} = -\varkappa_l$ and

$$Q_{i,j}(u, v) = \begin{cases} (-1)^{h_{i,j}} (u - v)^{-i \cdot j} & \text{if } i \neq j, \\ 0 & \text{else.} \end{cases} \tag{1.1}$$

If $m = 0$ we set $^\circ\mathbf{R}(\Gamma)_0 = \mathbf{k} \oplus \mathbf{k}$. If $m = 1$ then we have $\nu = i + \theta(i)$ for some $i \in I$. Write $\mathbf{i} = i\theta(i)$, and

$$^\circ\mathbf{R}(\Gamma)_\nu = \mathbf{k}[\varkappa_1] 1_\mathbf{i} \oplus \mathbf{k}[\varkappa_1] 1_{\theta(\mathbf{i})}.$$

We'll abbreviate $\sigma_{\mathbf{i},k} = \sigma_k 1_\mathbf{i}$ and $\varkappa_{\mathbf{i},l} = \varkappa_l 1_\mathbf{i}$. The grading on $^\circ\mathbf{R}(\Gamma)_0$ is the trivial one. For $m \geqslant 1$ the grading on $^\circ\mathbf{R}(\Gamma)_\nu$ is given by the following rules :

$$\deg(1_\mathbf{i}) = 0,$$

$$\deg(\varkappa_{\mathbf{i},l}) = 2,$$

$$\deg(\sigma_{\mathbf{i},k}) = -i_k \cdot i_{s_k(k)}.$$

We define ω to be the unique involution of the graded $\mathbf{k}$-algebra $^\circ\mathbf{R}(\Gamma)_\nu$ which fixes $1_\mathbf{i}$, $\varkappa_l$, σ_k. We set ω to be identity on $^\circ\mathbf{R}(\Gamma)_0$.

1.2. Relation with the graded k-algebra ${}^\theta\mathbf{R}(\Gamma)_\nu$. A family of graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ was introduced in [VV, sec. 5.1], for λ an arbitrary dimension vector in $\mathbb{N}I$. Here we'll only consider the special case $\lambda = 0$, and we abbreviate ${}^\theta\mathbf{R}(\Gamma)_\nu = {}^\theta\mathbf{R}(\Gamma)_{0,\nu}$. Recall that if $\nu \neq 0$ then ${}^\theta\mathbf{R}(\Gamma)_\nu$ is the graded $\mathbf{k}$-algebra with 1 generated by $1_{\mathbf{i}}$, $\varkappa_l$, σ_k, π_1, with $\mathbf{i} \in {}^\theta I^\nu$, $l = 1, 2, \dots, m$, $k = 1, \dots, m-1$ such that $1_{\mathbf{i}}$, $\varkappa_l$ and σ_k satisfy the same relations as before and

$$\pi_1^2 = 1, \quad \pi_1 1_{\mathbf{i}} \pi_1 = 1_{\varepsilon_1 \mathbf{i}}, \quad \pi_1 \varkappa_l \pi_1 = \varkappa_{\varepsilon_1(l)},$$

$$(\pi_1\sigma_1)^2 = (\sigma_1\pi_1)^2, \quad \pi_1\sigma_k\pi_1 = \sigma_k \text{ if } k \neq 1.$$

If $\nu = 0$ then ${}^\theta\mathbf{R}(\Gamma)_0 = \mathbf{k}$. The grading is given by setting $\deg(1_{\mathbf{i}})$, $\deg(\varkappa_{\mathbf{i},l})$, $\deg(\sigma_{\mathbf{i},k})$ to be as before and $\deg(\pi_1 1_{\mathbf{i}}) = 0$. *In the rest of Section* 1 *we'll assume* $m > 0$. Then there is a canonical inclusion of graded $\mathbf{k}$-algebras

$$(1.2) \qquad {}^\circ\mathbf{R}(\Gamma)_\nu \subset {}^\theta\mathbf{R}(\Gamma)_\nu$$

such that $1_{\mathbf{i}}, \varkappa_l, \sigma_k \mapsto 1_{\mathbf{i}}, \varkappa_l, \sigma_k$ for $\mathbf{i} \in {}^\theta I^\nu$, $l = 1, \dots, m$, $k = 1, \dots, m-1$ and such that $\sigma_0 \mapsto \pi_1\sigma_1\pi_1$. From now on we'll write $\sigma_0 = \pi_1\sigma_1\pi_1$ whenever $m > 1$. The assignment $x \mapsto \pi_1 x \pi_1$ defines an involution of the graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}(\Gamma)_\nu$ which normalizes ${}^\circ\mathbf{R}(\Gamma)_\nu$. Thus it yields an involution

$$\gamma : {}^\circ\mathbf{R}(\Gamma)_\nu \to {}^\circ\mathbf{R}(\Gamma)_\nu.$$

Let $\langle\gamma\rangle$ be the group of two elements generated by γ. The smash product ${}^\circ\mathbf{R}(\Gamma)_\nu \rtimes \langle\gamma\rangle$ is a graded $\mathbf{k}$-algebra such that $\deg(\gamma) = 0$. There is an unique isomorphism of graded $\mathbf{k}$-algebras

$$(1.3) \qquad {}^\circ\mathbf{R}(\Gamma)_\nu \rtimes \langle\gamma\rangle \to {}^\theta\mathbf{R}(\Gamma)_\nu$$

which is identity on ${}^\circ\mathbf{R}(\Gamma)_\nu$ and which takes γ to π_1.

1.3. The polynomial representation and the PBW theorem. For any $\mathbf{i}$ in ${}^\theta I^\nu$ let ${}^\theta\mathbf{F}_{\mathbf{i}}$ be the subalgebra of ${}^\circ\mathbf{R}(\Gamma)_\nu$ generated by $1_{\mathbf{i}}$ and $\varkappa_{\mathbf{i},l}$ with $l = 1, 2, \dots, m$. It is a polynomial algebra. Let

$${}^\theta\mathbf{F}_\nu = \bigoplus_{\mathbf{i} \in {}^\theta I^\nu} {}^\theta\mathbf{F}_{\mathbf{i}}.$$

The group W_m acts on ${}^\theta\mathbf{F}_\nu$ via $w(\varkappa_{\mathbf{i},l}) = \varkappa_{w(\mathbf{i}),w(l)}$ for any $w \in W_m$. Consider the fixed points set

$${}^\circ\mathbf{S}_\nu = ({}^\theta\mathbf{F}_\nu)^{{}^\circ W_m}.$$

Regard ${}^\theta\mathbf{R}(\Gamma)_\nu$ and $\mathrm{End}({}^\theta\mathbf{F}_\nu)$ as ${}^\theta\mathbf{F}_\nu$-algebras via the left multiplication. In [VV, prop. 5.4] is given an injective graded ${}^\theta\mathbf{F}_\nu$-algebra morphism ${}^\theta\mathbf{R}(\Gamma)_\nu \to \mathrm{End}({}^\theta\mathbf{F}_\nu)$. It restricts via (1.2) to an injective graded ${}^\theta\mathbf{F}_\nu$-algebra morphism

$${}^\circ\mathbf{R}(\Gamma)_\nu \to \mathrm{End}({}^\theta\mathbf{F}_\nu).$$

Next, recall that ${}^\circ W_m$ is the Weyl group of type D_m with simple reflections $s_0, \dots, s_{m-1}$. For each w in ${}^\circ W_m$ we choose a reduced decomposition $\dot{w}$ of w. It has the following form

$$w = s_{k_1}s_{k_2}\cdots s_{k_r}, \quad 0 \leqslant k_1, k_2, \dots, k_r \leqslant m-1.$$

We define an element $\sigma_{\dot{w}}$ in ${}^\circ\mathbf{R}(\Gamma)_\nu$ by

$$(1.4) \qquad \sigma_{\dot{w}} = \sum_{\mathbf{i}} 1_{\mathbf{i}}\sigma_{\dot{w}}, \quad 1_{\mathbf{i}}\sigma_{\dot{w}} = \begin{cases} 1_{\mathbf{i}} & \text{if } r = 0 \\ 1_{\mathbf{i}}\sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_r} & \text{else,} \end{cases}$$

Observe that the element $\sigma_{\dot{w}}$ may depend on the choice of the reduced decomposition $\dot{w}$.

1.4. Proposition. *The* $\mathbf{k}$*-algebra* ${}^\circ\mathbf{R}(\Gamma)_\nu$ *is a free (left or right)* ${}^\theta\mathbf{F}_\nu$*-module with basis* $\{\sigma_{\dot w};\, w \in {}^\circ W_m\}$*. Its rank is* $2^{m-1}m!$*. The operator* $1_\mathbf{i}\sigma_{\dot w}$ *is homogeneous and its degree is independent of the choice of the reduced decomposition* $\dot w$*.*

Proof : The proof is the same as in [VV, prop. 5.5]. First, we filter the algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$ with $1_\mathbf{i}$, $\varkappa_{\mathbf{i},l}$ in degree 0 and $\sigma_{\mathbf{i},k}$ in degree 1. The *Nil Hecke algebra* of type D_m is the $\mathbf{k}$-algebra ${}^\circ\mathbf{NH}_m$ generated by $\bar\sigma_0, \bar\sigma_1, \dots, \bar\sigma_{m-1}$ with relations

$$\bar\sigma_k\bar\sigma_{k'} = \bar\sigma_{k'}\bar\sigma_k \text{ if } 1 \leqslant k < k'-1 < m-1 \text{ or } 0 = k < k' \neq 2,$$

$$\bar\sigma_{s_k(k)}\bar\sigma_k\bar\sigma_{s_k(k)} = \bar\sigma_k\bar\sigma_{s_k(k)}\bar\sigma_k, \quad \bar\sigma_k^2 = 0.$$

We can form the semidirect product ${}^\theta\mathbf{F}_\nu \rtimes {}^\circ\mathbf{NH}_m$, which is generated by $1_\mathbf{i}$, $\bar\varkappa_l$, $\bar\sigma_k$ with the relations above and

$$\bar\sigma_k\bar\varkappa_l = \bar\varkappa_{s_k(l)}\bar\sigma_k, \quad \bar\varkappa_l\bar\varkappa_{l'} = \bar\varkappa_{l'}\bar\varkappa_{l'}.$$

One proves as in [VV, prop. 5.5] that the map

$${}^\theta\mathbf{F}_\nu \rtimes {}^\circ\mathbf{NH}_m \to \mathrm{gr}({}^\circ\mathbf{R}(\Gamma)_\nu), \quad 1_\mathbf{i} \mapsto 1_\mathbf{i}, \quad \bar\varkappa_l \mapsto \varkappa_l, \quad \bar\sigma_k \mapsto \sigma_k.$$

is an isomorphism of $\mathbf{k}$-algebras.

□

Let ${}^\theta\mathbf{F}'_\nu = \bigoplus_\mathbf{i} {}^\theta\mathbf{F}'_\mathbf{i}$, where ${}^\theta\mathbf{F}'_\mathbf{i}$ is the localization of the ring ${}^\theta\mathbf{F}_\mathbf{i}$ with respect to the multiplicative system generated by

$$\{\varkappa_{\mathbf{i},l} \pm \varkappa_{\mathbf{i},l'};\, 1 \leqslant l \neq l' \leqslant m\} \cup \{\varkappa_{\mathbf{i},l};\, l = 1, 2, \dots, m\}.$$

1.5. Corollary. *The inclusion* ${}^\circ\mathbf{R}(\Gamma)_\nu \subset \mathrm{End}({}^\theta\mathbf{F}_\nu)$ *yields an isomorphism of* ${}^\theta\mathbf{F}'_\nu$*-algebras* ${}^\theta\mathbf{F}'_\nu \otimes_{{}^\theta\mathbf{F}_\nu} {}^\circ\mathbf{R}(\Gamma)_\nu \to {}^\theta\mathbf{F}'_\nu \rtimes {}^\circ W_m$*, such that for each* $\mathbf{i}$ *and each* $l = 1, 2, \dots, m$*,* $k = 0, 1, 2, \dots, m-1$ *we have*

$$\begin{aligned} &1_\mathbf{i} \mapsto 1_\mathbf{i}, \\ &\varkappa_{\mathbf{i},l} \mapsto \varkappa_l 1_\mathbf{i}, \\ &\sigma_{\mathbf{i},k} \mapsto \begin{cases} (\varkappa_k - \varkappa_{s_k(k)})^{-1}(s_k - 1)1_\mathbf{i} & \text{if } i_k = i_{s_k(k)}, \\ (\varkappa_k - \varkappa_{s_k(k)})^{h_{i_{s_k(k)},i_k}} s_k 1_\mathbf{i} & \text{if } i_k \neq i_{s_k(k)}. \end{cases} \end{aligned} \tag{1.5}$$

Proof: Follows from [VV, cor. 5.6] and Proposition 1.4.

□

Restricting the ${}^\theta\mathbf{F}_\nu$-action on ${}^\circ\mathbf{R}(\Gamma)_\nu$ to the $\mathbf{k}$-subalgebra ${}^\circ\mathbf{S}_\nu$ we get a structure of graded ${}^\circ\mathbf{S}_\nu$-algebra on ${}^\circ\mathbf{R}(\Gamma)_\nu$.

1.6. Proposition. *(a)* ${}^\circ\mathbf{S}_\nu$ *is isomorphic to the center of* ${}^\circ\mathbf{R}(\Gamma)_\nu$*.*

(b) ${}^\circ\mathbf{R}(\Gamma)_\nu$ *is a free graded module over* ${}^\circ\mathbf{S}_\nu$ *of rank* $(2^{m-1}m!)^2$*.*

Proof : Part (a) follows from Corollary 1.5. Part (b) follows from (a) and Proposition 1.4.

□

2. Affine Hecke algebras of type D

2.1. Affine Hecke algebras of type D. Fix p in $\mathbf{k}^\times$. For any integer $m \geqslant 0$ we define the extended affine Hecke algebra $\mathbf{H}_m$ of type D_m as follows. If $m > 1$ then $\mathbf{H}_m$ is the $\mathbf{k}$-algebra with 1 generated by

$$T_k, \quad X_l^{\pm 1}, \quad k = 0, 1, \dots, m-1, \quad l = 1, 2, \dots, m$$

satisfying the following defining relations :

(a) $X_l X_{l'} = X_{l'} X_l$,

(b) $T_k T_{s_k(k)} T_k = T_{s_k(k)} T_k T_{s_k(k)}$, $T_k T_{k'} = T_{k'} T_k$ if $1 \leqslant k < k'-1$ or $k = 0$, $k' \neq 2$,

(c) $(T_k - p)(T_k + p^{-1}) = 0$,

(d) $T_0 X_1^{-1} T_0 = X_2$, $T_k X_k T_k = X_{s_k(k)}$ if $k \neq 0$, $T_k X_l = X_l T_k$ if $k \neq 0, l, l-1$ or $k = 0$, $l \neq 1, 2$.

Finally, we set $\mathbf{H}_0 = \mathbf{k} \oplus \mathbf{k}$ and $\mathbf{H}_1 = \mathbf{k}[X_1^{\pm 1}]$.

2.2. Remarks. (a) The extended affine Hecke algebra $\mathbf{H}_m^{\mathrm{B}}$ of type B_m with parameters $p, q \in \mathbf{k}^\times$ such that $q = 1$ is generated by P, T_k, $X_l^{\pm 1}$, $k = 1, \dots, m-1$, $l = 1, \dots, m$ such that T_k, $X_l^{\pm 1}$ satisfy the relations as above and

$$P^2 = 1, \quad (PT_1)^2 = (T_1 P)^2, \quad PT_k = T_k P \text{ if } k \neq 1,$$

$$PX_1^{-1} P = X_1, \quad PX_l = X_l P \text{ if } l \neq 1.$$

See e.g., [VV, sec. 6.1]. There is an obvious $\mathbf{k}$-algebra embedding $\mathbf{H}_m \subset \mathbf{H}_m^{\mathrm{B}}$. Let γ denote also the involution $\mathbf{H}_m \to \mathbf{H}_m$, $a \mapsto PaP$. We have a canonical isomorphism of $\mathbf{k}$-algebras

$$\mathbf{H}_m \rtimes \langle \gamma \rangle \simeq \mathbf{H}_m^{\mathrm{B}}.$$

Compare Section 1.2.

(b) Given a connected reductive group G we call *affine Hecke algebra of* G the Hecke algebra of the extended affine Weyl group $W \ltimes P$, where W is the Weyl group of (G, T), P is the group of characters of T, and T is a maximal torus of G. Then $\mathbf{H}_m$ is the affine Hecke algebra of the group $SO(2m)$. Let $\mathbf{H}_m^e$ be the affine Hecke algebra of the group $Spin(2m)$. It is generated by $\mathbf{H}_m$ and an element X_0 such that

$$X_0^2 = X_1 X_2 \dots X_m, \quad T_k X_0 = X_0 T_k \text{ if } k \neq 0, \quad T_0 X_0 X_1^{-1} X_2^{-1} T_0 = X_0.$$

Thus $\mathbf{H}_m$ is the fixed point subset of the $\mathbf{k}$-algebra automorphism of $\mathbf{H}_m^e$ taking T_k, X_l to $T_k, (-1)^{\delta_{l,0}} X_l$ for all k, l. Therefore, if p is not a root of 1 the simple $\mathbf{H}_m$-modules can be recovered from the Kazhdan-Lusztig classification of the simple $\mathbf{H}_m^e$-modules via Clifford theory, see e.g., [Re].

2.3. Intertwiners and blocks of $\mathbf{H}_m$. We define

$$\mathbf{A} = \mathbf{k}[X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_m^{\pm 1}], \quad \mathbf{A}' = \mathbf{A}[\Sigma^{-1}], \quad \mathbf{H}'_m = \mathbf{A}' \otimes_{\mathbf{A}} \mathbf{H}_m,$$

where Σ is the multiplicative set generated by

$$1 - X_l X_{l'}^{\pm 1}, \quad 1 - p^2 X_l^{\pm 1} X_{l'}^{\pm 1}, \quad l \neq l'.$$

For $k = 0, \dots, m-1$ the intertwiner φ_k is the element of $\mathbf{H}'_m$ given by the following formulas

$$\varphi_k - 1 = \frac{X_k - X_{s_k(k)}}{pX_k - p^{-1}X_{s_k(k)}}\,(T_k - p). \tag{2.1}$$

The group ${}^\circ W_m$ acts on $\mathbf{A}'$ as follows

$$(s_k a)(X_1, \dots, X_m) = a(X_1, \dots, X_{k+1}, X_k, \dots, X_m) \text{ if } k \neq 0,$$

$$(s_0 a)(X_1, \dots, X_m) = a(X_2^{-1}, X_1^{-1}, \dots, X_m).$$

There is an isomorphism of $\mathbf{A}'$-algebras

$$\mathbf{A}' \rtimes {}^\circ W_m \to \mathbf{H}'_m, \quad s_k \mapsto \varphi_k.$$

The semi-direct product group $\mathbb{Z} \rtimes \mathbb{Z}_2 = \mathbb{Z} \rtimes \{-1, 1\}$ acts on $\mathbf{k}^\times$ by $(n, \varepsilon) : z \mapsto z^\varepsilon p^{2n}$. Given a $\mathbb{Z} \rtimes \mathbb{Z}_2$-invariant subset I of $\mathbf{k}^\times$ we denote by $\mathbf{H}_m$-$\mathbf{Mod}_I$ the category of all $\mathbf{H}_m$-modules such that the action of $X_1, X_2, \dots, X_m$ is locally finite with eigenvalues in I. We associate to the set I and to the element $p \in \mathbf{k}^\times$ a quiver Γ as follows. The set of vertices is I, and there is one arrow $p^2 i \to i$ whenever i lies in I. We equip Γ with an involution θ such that $\theta(i) = i^{-1}$ for each vertex i and such that θ takes the arrow $p^2 i \to i$ to the arrow $i^{-1} \to p^{-2} i^{-1}$. *We'll assume that the set I does not contain 1 nor -1 and that $p \neq 1, -1$.* Thus the involution θ has no fixed points and no arrow may join a vertex of Γ to itself.

2.4. Remark. We may assume that $I = \pm\{p^n;\, n \in \mathbb{Z}_{\mathrm{odd}}\}$. See the discussion in [KM]. Then Γ is of type A_∞ if p has infinite order and Γ is of type $\mathrm{A}_r^{(1)}$ if p^2 is a primitive r-th root of unity.

2.5. $\mathbf{H}_m$-modules versus ${}^\circ\mathbf{R}_m$-modules. Assume that $m \geqslant 1$. We define the graded $\mathbf{k}$-algebra

$${}^\theta\mathbf{R}_{I,m} = \bigoplus_\nu {}^\theta\mathbf{R}_{I,\nu}, \quad {}^\theta\mathbf{R}_{I,\nu} = {}^\theta\mathbf{R}(\Gamma)_\nu, \quad {}^\circ\mathbf{R}_{I,m} = \bigoplus_\nu {}^\circ\mathbf{R}_{I,\nu}, \quad {}^\circ\mathbf{R}_{I,\nu} = {}^\circ\mathbf{R}(\Gamma)_\nu,$$

$${}^\theta I^m = \bigsqcup_\nu {}^\theta I^\nu,$$

where ν runs over the set of all dimension vectors in ${}^\theta\mathbb{N}I$ such that $|\nu| = 2m$. When there is no risk of confusion we abbreviate

$${}^\theta\mathbf{R}_\nu = {}^\theta\mathbf{R}_{I,\nu}, \quad {}^\theta\mathbf{R}_m = {}^\theta\mathbf{R}_{I,m}, \quad {}^\circ\mathbf{R}_\nu = {}^\circ\mathbf{R}_{I,\nu}, \quad {}^\circ\mathbf{R}_m = {}^\circ\mathbf{R}_{I,m}.$$

Note that ${}^\theta\mathbf{R}_\nu$ and ${}^\theta\mathbf{R}_m$ are the same as in [VV, sec. 6.4], with $\lambda = 0$. Note also that the $\mathbf{k}$-algebra ${}^\circ\mathbf{R}_m$ may not have 1, because the set I may be infinite. We define ${}^\circ\mathbf{R}_m$-$\mathbf{Mod}_0$ as the category of all (non-graded) ${}^\circ\mathbf{R}_m$-modules such that the elements $\varkappa_1, \varkappa_2, \dots, \varkappa_m$ act locally nilpotently. Let ${}^\circ\mathbf{R}_m$-$\mathbf{fMod}_0$ and $\mathbf{H}_m$-$\mathbf{fMod}_I$ be the full subcategories of finite dimensional modules in ${}^\circ\mathbf{R}_m$-$\mathbf{Mod}_0$ and $\mathbf{H}_m$-$\mathbf{Mod}_I$ respectively. Fix a formal series $f(\varkappa)$ in $\mathbf{k}[[\varkappa]]$ such that $f(\varkappa) = 1 + \varkappa$ modulo $(\varkappa^2)$.

2.6. Theorem. *We have an equivalence of categories*

$$^\circ\mathbf{R}_m\text{-}\mathbf{Mod}_0 \to \mathbf{H}_m\text{-}\mathbf{Mod}_I, \quad M \mapsto M$$

which is given by

(*a*) X_l *acts on* $1_\mathbf{i}M$ *by* $i_l^{-1} f(\varkappa_l)$ *for each* $l = 1, 2, \dots, m$,

(*b*) *if* $m > 1$ *then* T_k *acts on* $1_\mathbf{i}M$ *as follows for each* $k = 0, 1, \dots, m-1$,

$$\frac{(pf(\varkappa_k) - p^{-1} f(\varkappa_{s_k(k)}))(\varkappa_k - \varkappa_{s_k(k)})}{f(\varkappa_k) - f(\varkappa_{s_k(k)})}\sigma_k + p \qquad \text{if } i_{s_k(k)} = i_k,$$

$$\frac{f(\varkappa_k) - f(\varkappa_{s_k(k)})}{(p^{-1}f(\varkappa_k) - pf(\varkappa_{s_k(k)}))(\varkappa_k - \varkappa_{s_k(k)})}\sigma_k + \frac{(p^{-2}-1)f(\varkappa_{s_k(k)})}{pf(\varkappa_k) - p^{-1}f(\varkappa_{s_k(k)})} \qquad \text{if } i_{s_k(k)} = p^2 i_k,$$

$$\frac{pi_k f(\varkappa_k) - p^{-1} i_{s_k(k)} f(\varkappa_{s_k(k)})}{i_k f(\varkappa_k) - i_{s_k(k)} f(\varkappa_{s_k(k)})}\sigma_k + \frac{(p^{-1}-p) i_k f(\varkappa_{s_k(k)})}{i_{s_k(k)} f(\varkappa_k) - i_k f(\varkappa_{s_k(k)})} \qquad \text{if } i_{s_k(k)} \neq i_k, p^2 i_k.$$

Proof : This follows from [VV, thm. 6.5] by Section 1.2 and Remark 2.2(a). One can also prove it by reproducing the arguments in loc. cit. by using (1.5) and (2.1). □

2.7. Corollary. *There is an equivalence of categories*

$$\Psi : {}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to \mathbf{H}_m\text{-}\mathbf{fMod}_I, \quad M \mapsto M.$$

2.8. Remarks. (a) Let $\mathfrak{g}$ be the Lie algebra of $G = SO(2m)$. Fix a maximal torus $T \subset G$. The group of characters of T is the lattice $\bigoplus_{l=1}^m \mathbb{Z}\,\varepsilon_l$, with Bourbaki's notation. Fix a dimension vector $\nu \in {}^\theta\mathbb{N}I$. Recall the sequence $\mathbf{i}_e = (i_{1-m}, \dots, i_{m-1}, i_m)$ from Section 0.3. Let $g \in T$ be the element such that $\varepsilon_l(g) = i_l^{-1}$ for each $l = 1, 2, \dots, m$. Recall also the notation ${}^\theta\boldsymbol{\mathcal{V}}_\nu$, $\mathbf{V}$, ${}^\theta E_\mathbf{V}$, and ${}^\theta G_\mathbf{V}$ from [VV]. Then $\mathbf{V}$ is an object of ${}^\theta\boldsymbol{\mathcal{V}}_\nu$, ${}^\theta G_\mathbf{V} = G_g$ is the centralizer of g in G, and

$${}^\theta E_\mathbf{V} = \mathfrak{g}_{g,p}, \quad \mathfrak{g}_{g,p} = \{x \in \mathfrak{g};, \mathrm{ad}_g(x) = p^2 x\}.$$

Let F_g be the set of all Borel Lie subalgebras of $\mathfrak{g}$ fixed by the adjoint action of g. It is a non connected manifold whose connected components are labelled by ${}^\theta I_+^\nu$. In Section 3.14 we'll introduce two central idempotents $1_{\nu,+}$, $1_{\nu,-}$ of ${}^\circ\mathbf{R}_\nu$. This yields a graded $\mathbf{k}$-algebra decomposition

$${}^\circ\mathbf{R}_\nu = {}^\circ\mathbf{R}_\nu 1_{\nu,+} \oplus {}^\circ\mathbf{R}_\nu 1_{\nu,-}.$$

By [VV, thm. 5.8] the graded $\mathbf{k}$-algebra ${}^\circ\mathbf{R}_\nu 1_{\nu,+}$ is isomorphic to

$$\mathrm{Ext}^*_{G_g}(\mathcal{L}_{g,p}, \mathcal{L}_{g,p}),$$

where $\mathcal{L}_{g,p}$ is the direct image of the constant perverse sheaf by the projection

$$\{(\mathfrak{b}, x) \in F_g \times \mathfrak{g}_{g,p};\, x \in \mathfrak{b}\} \to \mathfrak{g}_{g,p}, \quad (\mathfrak{b}, x) \mapsto x.$$

The complex $\mathcal{L}_{g,p}$ has been extensively studied by Lusztig, see e.g., [L1], [L2]. We hope to come back to this elsewhere.

(b) The results in Section 2.5 hold true if $\mathbf{k}$ is any field. Set $f(\varkappa) = 1 + \varkappa$ for instance.

2.9. Induction and restriction of $\mathbf{H}_m$-modules. For $i \in I$ we define functors

$$\begin{aligned} &E_i : \mathbf{H}_{m+1}\text{-}\mathbf{fMod}_I \to \mathbf{H}_m\text{-}\mathbf{fMod}_I, \\ &F_i : \mathbf{H}_m\text{-}\mathbf{fMod}_I \to \mathbf{H}_{m+1}\text{-}\mathbf{fMod}_I, \end{aligned} \tag{2.2}$$

where $E_iM \subset M$ is the generalized i^{-1}-eigenspace of the X_{m+1}-action, and where

$$F_iM = \mathrm{Ind}^{\mathbf{H}_{m+1}}_{\mathbf{H}_m \otimes \mathbf{k}[X^{\pm 1}_{m+1}]}(M \otimes \mathbf{k}_i).$$

Here $\mathbf{k}_i$ is the 1-dimensional representation of $\mathbf{k}[X^{\pm 1}_{m+1}]$ defined by $X_{m+1} \mapsto i^{-1}$.

3. Global bases of $^\circ\mathbf{V}$ and projective graded $^\circ\mathbf{R}$-modules

3.1. The Grothendieck groups of $^\circ\mathbf{R}_m$. The graded $\mathbf{k}$-algebra $^\circ\mathbf{R}_m$ is free of finite rank over its center by Proposition 1.6, a commutative graded $\mathbf{k}$-subalgebra. Therefore any simple object of $^\circ\mathbf{R}_m$**-mod** is finite-dimensional and there is a finite number of isomorphism classes of simple modules in $^\circ\mathbf{R}_m$**-mod**. The Abelian group $G(^\circ\mathbf{R}_m)$ is free with a basis formed by the classes of the simple objects of $^\circ\mathbf{R}_m$-**mod**. The Abelian group $K(^\circ\mathbf{R}_m)$ is free with a basis formed by the classes of the indecomposable projective objects. Both $G(^\circ\mathbf{R}_m)$ and $K(^\circ\mathbf{R}_m)$ are free $\mathcal{A}$-modules, where v shifts the grading by 1. We consider the following $\mathcal{A}$-modules

$$^\circ\mathbf{K}_I = \bigoplus_{m \geqslant 0} {}^\circ\mathbf{K}_{I,m}, \quad {}^\circ\mathbf{K}_{I,m} = K(^\circ\mathbf{R}_m),$$

$$^\circ\mathbf{G}_I = \bigoplus_{m \geqslant 0} {}^\circ\mathbf{G}_{I,m}, \quad {}^\circ\mathbf{G}_{I,m} = G(^\circ\mathbf{R}_m).$$

We'll also abbreviate

$$^\circ\mathbf{K}_{I,*} = \bigoplus_{m > 0} {}^\circ\mathbf{K}_{I,m}, \quad {}^\circ\mathbf{G}_{I,*} = \bigoplus_{m > 0} {}^\circ\mathbf{G}_{I,m}.$$

From now on, to unburden the notation we may abbreviate $^\circ\mathbf{R} = {}^\circ\mathbf{R}_m$, hoping it will not create any confusion. For any M, N in $^\circ\mathbf{R}$**-mod** we set

$$(M : N) = \mathrm{gdim}(M^\omega \otimes_{^\circ\mathbf{R}} N), \quad \langle M : N \rangle = \mathrm{gdim}\,\mathrm{hom}_{^\circ\mathbf{R}}(M, N),$$

where ω is the involution defined in Section 1.1. The Cartan pairing is the perfect $\mathcal{A}$-bilinear form

$$^\circ\mathbf{K}_I \times {}^\circ\mathbf{G}_I \to \mathcal{A}, \quad (P, M) \mapsto \langle P : M \rangle.$$

First, we concentrate on the $\mathcal{A}$-module $^\circ\mathbf{G}_I$. Consider the duality

$$^\circ\mathbf{R}\text{-}\mathbf{fmod} \to {}^\circ\mathbf{R}\text{-}\mathbf{fmod}, \quad M \mapsto M^\flat = \mathrm{hom}(M, \mathbf{k}),$$

with the action and the grading given by

$$(xf)(m) = f(\omega(x)m), \quad (M^\flat)_d = \mathrm{Hom}(M_{-d}, \mathbf{k}).$$

This duality functor yields an $\mathcal{A}$-antilinear map

$$^\circ\mathbf{G}_I \to {}^\circ\mathbf{G}_I, \quad M \mapsto M^\flat.$$

Let $^\circ B$ denote the set of isomorphism classes of simple objects of $^\circ\mathbf{R}$-$\mathbf{fMod}_0$. We can now define the upper global basis of $^\circ\mathbf{G}_I$ as follows. The proof is given in Section 3.21.

3.2. Proposition/Definition. *For each b in ${}^\circ B$ there is a unique selfdual irreducible graded ${}^\circ\mathbf{R}$-module ${}^\circ G^{\mathrm{up}}(b)$ which is isomorphic to b as a (non graded) ${}^\circ\mathbf{R}$-module. We set ${}^\circ G^{\mathrm{up}}(0) = 0$ and ${}^\circ\mathbf{G}^{\mathrm{up}} = \{{}^\circ G^{\mathrm{up}}(b);\, b \in {}^\circ B\}$. Hence ${}^\circ\mathbf{G}^{\mathrm{up}}$ is a $\mathcal{A}$-basis of ${}^\circ\mathbf{G}_I$.*

Now, we concentrate on the $\mathcal{A}$-module ${}^\circ\mathbf{K}_I$. We equip ${}^\circ\mathbf{K}_I$ with the symmetric $\mathcal{A}$-bilinear form

$$ {}^\circ\mathbf{K}_I \times {}^\circ\mathbf{K}_I \to \mathcal{A}, \quad (M, N) \mapsto (M : N). \tag{3.1} $$

Consider the duality

$$ {}^\circ\mathbf{R}\text{-}\mathbf{proj} \to {}^\circ\mathbf{R}\text{-}\mathbf{proj}, \quad P \mapsto P^\sharp = \mathrm{hom}_{{}^\circ\mathbf{R}}(P, {}^\circ\mathbf{R}), $$

with the action and the grading given by

$$ (xf)(p) = f(p)\omega(x), \quad (P^\sharp)_d = \mathrm{Hom}_{{}^\circ\mathbf{R}}(P[-d], {}^\circ\mathbf{R}). $$

This duality functor yields an $\mathcal{A}$-antilinear map

$$ {}^\circ\mathbf{K}_I \to {}^\circ\mathbf{K}_I, \quad P \mapsto P^\sharp. $$

Set $\mathcal{K} = \mathbb{Q}(v)$. Let $\mathcal{K} \to \mathcal{K}$, $f \mapsto \bar{f}$ be the unique involution such that $\bar{v} = v^{-1}$.

3.3. Definition. *For each b in ${}^\circ B$ let ${}^\circ G^{\mathrm{low}}(b)$ be the unique indecomposable graded module in ${}^\circ\mathbf{R}$-**proj** whose top is isomorphic to ${}^\circ G^{\mathrm{up}}(b)$. We set ${}^\circ G^{\mathrm{low}}(0) = 0$ and ${}^\circ\mathbf{G}^{\mathrm{low}} = \{{}^\circ G^{\mathrm{low}}(b);\, b \in {}^\circ B\}$, a $\mathcal{A}$-basis of ${}^\circ\mathbf{K}_I$.*

3.4. Proposition. *(a) We have $\langle {}^\circ G^{\mathrm{low}}(b) : {}^\circ G^{\mathrm{up}}(b')\rangle = \delta_{b,b'}$ for each b, b' in ${}^\circ B$.*

(b) We have $\langle P^\sharp : M\rangle = \overline{\langle P : M^\flat\rangle}$ for each P, M.

(c) We have ${}^\circ G^{\mathrm{low}}(b)^\sharp = {}^\circ G^{\mathrm{low}}(b)$ for each b in ${}^\circ B$.

The proof is the same as in [VV, prop. 8.4].

3.5. Example. Set $\nu = i + \theta(i)$ and $\mathbf{i} = i\theta(i)$. Consider the graded ${}^\circ\mathbf{R}_\nu$-modules

$$ {}^\circ\mathbf{R}_{\mathbf{i}} = {}^\circ\mathbf{R}1_{\mathbf{i}} = 1_{\mathbf{i}}{}^\circ\mathbf{R}, \quad {}^\circ\mathbf{L}_{\mathbf{i}} = \mathrm{top}({}^\circ\mathbf{R}_{\mathbf{i}}). $$

The global bases are given by

$$ {}^\circ\mathbf{G}^{\mathrm{low}}_\nu = \{{}^\circ\mathbf{R}_{\mathbf{i}},\, {}^\circ\mathbf{R}_{\theta(\mathbf{i})}\}, \quad {}^\circ\mathbf{G}^{\mathrm{up}}_\nu = \{{}^\circ\mathbf{L}_{\mathbf{i}},\, {}^\circ\mathbf{L}_{\theta(\mathbf{i})}\}. $$

For $m = 0$ we have ${}^\circ\mathbf{R}_0 = \mathbf{k} \oplus \mathbf{k}$. Set $\phi_+ = \mathbf{k} \oplus 0$ and $\phi_- = 0 \oplus \mathbf{k}$. We have

$$ {}^\circ\mathbf{G}^{\mathrm{low}}_0 = {}^\circ\mathbf{G}^{\mathrm{up}}_0 = \{\phi_+,\, \phi_-\}. $$

3.6. Definition of the operators e_i, f_i, e_i', f_i'. In this section we'll always assume $m > 0$ unless specified otherwise. First, let us introduce the following notation. Let $D_{m,1}$ be the set of minimal representative in ${}^\circ W_{m+1}$ of the cosets in ${}^\circ W_m \backslash {}^\circ W_{m+1}$. Write

$$D_{m,1;m,1} = D_{m,1} \cap (D_{m,1})^{-1}.$$

For each element w of $D_{m,1;m,1}$ we set

$$W(w) = {}^\circ W_m \cap w({}^\circ W_m)w^{-1}.$$

Let $\mathbf{R}_1$ be the $\mathbf{k}$-algebra generated by elements 1_i, $\varkappa_i$, $i \in I$, satisfying the defining relations $1_i\,1_{i'} = \delta_{i,i'}1_i$ and $\varkappa_i = 1_i\varkappa_i 1_i$. We equip $\mathbf{R}_1$ with the grading such that $\deg(1_i) = 0$ and $\deg(\varkappa_i) = 2$. Let

$$\mathbf{R}_i = 1_i\mathbf{R}_1 = \mathbf{R}_1 1_i, \quad \mathbf{L}_i = \mathrm{top}(\mathbf{R}_i) = \mathbf{R}_i/(\varkappa_i).$$

Then $\mathbf{R}_i$ is a graded projective $\mathbf{R}_1$-module and $\mathbf{L}_i$ is simple. We abbreviate

$${}^\theta\mathbf{R}_{m,1} = {}^\theta\mathbf{R}_m \otimes \mathbf{R}_1, \quad {}^\circ\mathbf{R}_{m,1} = {}^\circ\mathbf{R}_m \otimes \mathbf{R}_1.$$

There is an unique inclusion of graded $\mathbf{k}$-algebras

$$\begin{aligned} &{}^\theta\mathbf{R}_{m,1} \to {}^\theta\mathbf{R}_{m+1}, \\ &1_{\mathbf{i}} \otimes 1_i \mapsto 1_{\mathbf{i}'}, \\ &1_{\mathbf{i}} \otimes \varkappa_{i,l} \mapsto \varkappa_{\mathbf{i}',m+l}, \\ &\varkappa_{\mathbf{i},l} \otimes 1_i \mapsto \varkappa_{\mathbf{i}',l}, \\ &\pi_{\mathbf{i},1} \otimes 1_i \mapsto \pi_{\mathbf{i}',1}, \\ &\sigma_{\mathbf{i},k} \otimes 1_i \mapsto \sigma_{\mathbf{i}',k}, \end{aligned} \tag{3.2}$$

where, given $\mathbf{i} \in {}^\theta I^m$ and $i \in I$, we have set $\mathbf{i}' = \theta(i)\mathbf{i}i$, a sequence in ${}^\theta I^{m+1}$. This inclusion restricts to an inclusion ${}^\circ\mathbf{R}_{m,1} \subset {}^\circ\mathbf{R}_{m+1}$.

3.7. Lemma. *The graded ${}^\circ\mathbf{R}_{m,1}$-module ${}^\circ\mathbf{R}_{m+1}$ is free of rank $2(m+1)$.*

Proof : For each w in $D_{m,1}$ we have the element $\sigma_{\dot{w}}$ in ${}^\circ\mathbf{R}_{m+1}$ defined in (1.5). Using filtered/graded arguments it is easy to see that

$${}^\circ\mathbf{R}_{m+1} = \bigoplus_{w \in D_{m,1}} {}^\circ\mathbf{R}_{m,1}\sigma_{\dot{w}}.$$

□

We define a triple of adjoint functors $(\psi_!, \psi^*, \psi_*)$ where

$$\psi^* : {}^\circ\mathbf{R}_{m+1}\text{-}\mathbf{mod} \to {}^\circ\mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_1\text{-}\mathbf{mod}$$

is the restriction and $\psi_!$, ψ_* are given by

$$\psi_! : \begin{cases} {}^\circ\mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_1\text{-}\mathbf{mod} \to {}^\circ\mathbf{R}_{m+1}\text{-}\mathbf{mod}, \\ (M, M') \mapsto {}^\circ\mathbf{R}_{m+1} \otimes_{{}^\circ\mathbf{R}_{m,1}} (M \otimes M'), \end{cases}$$

$$\psi_* : \begin{cases} {}^\circ\mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_1\text{-}\mathbf{mod} \to {}^\circ\mathbf{R}_{m+1}\text{-}\mathbf{mod}, \\ (M, M') \mapsto \hom_{{}^\circ\mathbf{R}_{m,1}}({}^\circ\mathbf{R}_{m+1}, M \otimes M'). \end{cases}$$

First, note that the functors $\psi_!$, ψ^*, ψ_* commute with the shift of the grading. Next, the functor ψ^* is exact, and it takes finite dimensional graded modules to finite dimensional ones. The right graded ${}^\circ\mathbf{R}_{m,1}$-module ${}^\circ\mathbf{R}_{m+1}$ is free of finite rank. Thus $\psi_!$ is exact, and it takes finite dimensional graded modules to finite dimensional ones. The left graded ${}^\circ\mathbf{R}_{m,1}$-module ${}^\circ\mathbf{R}_{m+1}$ is also free of finite rank. Thus the functor ψ_* is exact, and it takes finite dimensional graded modules to finite dimensional ones. Further $\psi_!$ and ψ^* take projective graded modules to projective ones, because they are left adjoint to the exact functors ψ^*, ψ_* respectively. To summarize, the functors $\psi_!$, ψ^*, ψ_* are exact and take finite dimensional graded modules to finite dimensional ones, and the functors $\psi_!$, ψ^* take projective graded modules to projective ones.

For any graded ${}^\circ\mathbf{R}_m$-module Mwe write

$$f_i(M) = {}^\circ\mathbf{R}_{m+1} 1_{m,i} \otimes_{{}^\circ\mathbf{R}_m} M, \tag{3.3}$$
$$e_i(M) = {}^\circ\mathbf{R}_{m-1} \otimes_{{}^\circ\mathbf{R}_{m-1,1}} 1_{m-1,i} M.$$

Let us explain these formulas. The symbols $1_{m,i}$ and $1_{m-1,i}$ are given by

$$1_{m-1,i} M = \bigoplus_{\mathbf{i}} 1_{\theta(i)\mathbf{i}i} M, \quad \mathbf{i} \in {}^\theta I^{m-1}.$$

Note that $f_i(M)$ is a graded ${}^\circ\mathbf{R}_{m+1}$-module, while $e_i(M)$ is a graded ${}^\circ\mathbf{R}_{m-1}$-module. The tensor product in the definition of $e_i(M)$ is relative to the graded $\mathbf{k}$-algebra homomorphism

$${}^\circ\mathbf{R}_{m-1,1} \to {}^\circ\mathbf{R}_{m-1} \otimes \mathbf{R}_1 \to {}^\circ\mathbf{R}_{m-1} \otimes \mathbf{R}_i \to {}^\circ\mathbf{R}_{m-1} \otimes \mathbf{L}_i = {}^\circ\mathbf{R}_{m-1}.$$

In other words, let $e'_i(M)$ be the graded ${}^\circ\mathbf{R}_{m-1}$-module obtained by taking the direct summand $1_{m-1,i}M$ and restricting it to ${}^\circ\mathbf{R}_{m-1}$. Observe that if M is finitely generated then $e'_i(M)$ may not lie in ${}^\circ\mathbf{R}_{m-1}$-$\mathbf{mod}$. To remedy this, since $e'_i(M)$ affords a ${}^\circ\mathbf{R}_{m-1} \otimes \mathbf{R}_i$-action we consider the graded ${}^\circ\mathbf{R}_{m-1}$-module

$$e_i(M) = e'_i(M)/\varkappa_i e'_i(M).$$

3.8. Definition. *The functors e_i, f_i preserve the category ${}^\circ\mathbf{R}$-$\mathbf{proj}$, yielding $\mathcal{A}$-linear operators on ${}^\circ\mathbf{K}_I$ which act on ${}^\circ\mathbf{K}_{I,*}$ by the formula* (3.3) *and satisfy also*

$$f_i(\phi_+) = {}^\circ\mathbf{R}_{\theta(i)i}, \quad f_i(\phi_-) = {}^\circ\mathbf{R}_{i\theta(i)}, \quad e_i(\mathbf{R}_{\theta(j)j}) = \delta_{i,j}\phi_+ + \delta_{i,\theta(j)}\phi_-.$$

Let e_i, f_i denote also the $\mathcal{A}$-linear operators on ${}^\circ\mathbf{G}_I$ which are the transpose of f_i, e_i with respect to the Cartan pairing.

Note that the symbols $e_i(M)$, $f_i(M)$ have a different meaning if M is viewed as an element of ${}^\circ\mathbf{K}_I$ or if M is viewed as an element of ${}^\circ\mathbf{G}_I$. We hope this will not create any confusion. The proof of the following lemma is the same as in [VV, lem. 8.9].

3.9. Lemma. *(a) The operators e_i, f_i on ${}^\circ\mathbf{G}_I$ are given by*

$$e_i(M) = 1_{m-1,i}M \quad f_i(M) = \hom_{{}^\circ\mathbf{R}_{m,1}}({}^\circ\mathbf{R}_{m+1}, M \otimes \mathbf{L}_i), \quad M \in {}^\circ\mathbf{R}_m\text{-}\mathbf{fmod}.$$

*(b) For each $M \in {}^\circ\mathbf{R}_m$-**mod**, $M' \in {}^\circ\mathbf{R}_{m+1}$-**mod** we have*

$$(e_i'(M') : M) = (M' : f_i(M)).$$

*(c) We have $f_i(P)^\sharp = f_i(P^\sharp)$ for each $P \in {}^\circ\mathbf{R}$-**proj**.*

*(d) We have $e_i(M)^\flat = e_i(M^\flat)$ for each $M \in {}^\circ\mathbf{R}$-**fmod**.*

3.10. Induction of $\mathbf{H}_m$-modules versus induction of ${}^\circ\mathbf{R}_m$-modules. Recall the functors E_i, F_i on $\mathbf{H}$-$\mathbf{fMod}_I$ defined in (2.2). We have also the functors

$$\mathbf{for} : {}^\circ\mathbf{R}_m\text{-}\mathbf{fmod} \to {}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0, \quad \Psi : {}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to \mathbf{H}_m\text{-}\mathbf{fMod}_I,$$

where **for** is the forgetting of the grading. Finally we define functors

$$\begin{aligned} &E_i : {}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to {}^\circ\mathbf{R}_{m-1}\text{-}\mathbf{fMod}_0, \quad E_iM = 1_{m-1,i}M, \\ &F_i : {}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to {}^\circ\mathbf{R}_{m+1}\text{-}\mathbf{fMod}_0, \quad F_iM = \psi_!(M, \mathbf{L}_i). \end{aligned} \tag{3.4}$$

3.11. Proposition. *There are canonical isomorphisms of functors*

$$E_i \circ \Psi = \Psi \circ E_i, \quad F_i \circ \Psi = \Psi \circ F_i, \quad E_i \circ \mathbf{for} = \mathbf{for} \circ e_i, \quad F_i \circ \mathbf{for} = \mathbf{for} \circ f_{\theta(i)}.$$

Proof : The proof is the same as in [VV, prop. 8.17].

□

3.12. Proposition. *(a) The functor Ψ yields an isomorphism of Abelian groups*

$$\bigoplus_{m \geqslant 0} [{}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0] = \bigoplus_{m \geqslant 0} [\mathbf{H}_m\text{-}\mathbf{fMod}_I].$$

The functors E_i, F_i yield endomorphisms of both sides which are intertwined by Ψ.

(b) The functor **for** *factors to a group isomorphism*

$$^\circ\mathbf{G}_I/(v-1) = \bigoplus_{m \geqslant 0} [{}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0].$$

Proof : Claim (a) follows from Corollary 2.7 and Proposition 3.11. Claim (b) follows from Proposition 3.2.

□

3.13. Type D versus type B. We can compare the previous constructions with their analogues in type B. Let ${}^\theta\mathbf{K}$, ${}^\theta B$, ${}^\theta G^{\mathrm{low}}$, etc, denote the type B analogues of ${}^\circ\mathbf{K}$, ${}^\circ B$, ${}^\circ G^{\mathrm{low}}$, etc. See [VV] for details. We'll use the same notation for the functors ψ^*, $\psi_!$, ψ_*, e_i, f_i, etc, on the type B side and on the type D side. Fix $m > 0$ and $\nu \in {}^\theta\mathbb{N}I$ such that $|\nu| = 2m$. The inclusion of graded $\mathbf{k}$-algebras ${}^\circ\mathbf{R}_\nu \subset {}^\theta\mathbf{R}_\nu$ in (1.2) yields a restriction functor

$$\mathrm{res} : {}^\theta\mathbf{R}_\nu\text{-}\mathbf{mod} \to {}^\circ\mathbf{R}_\nu\text{-}\mathbf{mod}$$

and an induction functor

$$\mathrm{ind} : {}^\circ\mathbf{R}_\nu\text{-}\mathbf{mod} \to {}^\theta\mathbf{R}_\nu\text{-}\mathbf{mod}, \quad M \mapsto {}^\theta\mathbf{R}_\nu \otimes_{{}^\circ\mathbf{R}_\nu} M.$$

Both functors are exact, they map finite dimensional graded modules to finite dimensional ones, and they map projective graded modules to projective ones. Thus, they yield morphisms of $\mathcal{A}$-modules

$$\mathrm{res} : {}^\theta\mathbf{K}_{I,m} \to {}^\circ\mathbf{K}_{I,m}, \quad \mathrm{res} : {}^\theta\mathbf{G}_{I,m} \to {}^\circ\mathbf{G}_{I,m},$$

$$\mathrm{ind} : {}^\circ\mathbf{K}_{I,m} \to {}^\theta\mathbf{K}_{I,m}, \quad \mathrm{ind} : {}^\circ\mathbf{G}_{I,m} \to {}^\theta\mathbf{G}_{I,m}.$$

Moreover, for any $P \in {}^\theta\mathbf{K}_{I,m}$ and any $L \in {}^\theta\mathbf{G}_{I,m}$ we have

$$\begin{aligned} &\mathrm{res}(P^\sharp) = (\mathrm{res}P)^\sharp, \quad \mathrm{ind}(P^\sharp) = (\mathrm{ind}P)^\sharp \\ &\mathrm{res}(L^\flat) = (\mathrm{res}L)^\flat, \quad \mathrm{ind}(L^\flat) = (\mathrm{ind}L)^\flat. \end{aligned} \tag{3.5}$$

Note also that ind and res are left and right adjoint functors, because

$${}^\theta\mathbf{R}_\nu \otimes_{{}^\circ\mathbf{R}_\nu} M = \mathrm{hom}_{{}^\circ\mathbf{R}_\nu}({}^\theta\mathbf{R}_\nu, M)$$

as graded ${}^\theta\mathbf{R}_\nu$-modules.

3.14. Definition. *For any graded ${}^\circ\mathbf{R}_\nu$-module M we define the graded ${}^\circ\mathbf{R}_\nu$-module M^γ with the same underlying graded $\mathbf{k}$-vector space as M such that the action of ${}^\circ\mathbf{R}_\nu$ is twisted by γ, i.e., the graded $\mathbf{k}$-algebra ${}^\circ\mathbf{R}_\nu$ acts on M^γ by $a\,m = \gamma(a)m$ for $a \in {}^\circ\mathbf{R}_\nu$ and $m \in M$. Note that $(M^\gamma)^\gamma = M$, and that M^γ is simple (resp. projective, indecomposable) if M has the same property.*

For any graded ${}^\circ\mathbf{R}_m$-module M we have canonical isomorphisms of ${}^\circ\mathbf{R}$-modules

$$(f_i(M))^\gamma = f_i(M^\gamma), \quad (e_i(M))^\gamma = e_i(M^\gamma).$$

The first isomorphism is given by

$${}^\circ\mathbf{R}_{m+1}1_{m,i} \otimes_{{}^\circ\mathbf{R}_m} M \to {}^\circ\mathbf{R}_{m+1}1_{m,i} \otimes_{{}^\circ\mathbf{R}_m} M, \quad a \otimes m \mapsto \gamma(a) \otimes m.$$

The second one is the identity map on the vector space $1_{m,i}M$.

Recall that ${}^\theta I^\nu$ is the disjoint union of ${}^\theta I^\nu_+$ and ${}^\theta I^\nu_-$. We set

$$1_{\nu,+} = \sum_{\mathbf{i} \in {}^\theta I^\nu_+} 1_{\mathbf{i}}, \quad 1_{\nu,-} = \sum_{\mathbf{i} \in {}^\theta I^\nu_-} 1_{\mathbf{i}}.$$

3.15. Lemma. *Let M be a graded ${}^\circ\mathbf{R}_\nu$-module.*

(*a*) *Both $1_{\nu,+}$ and $1_{\nu,-}$ are central idempotents in ${}^\circ\mathbf{R}_\nu$. We have $1_{\nu,+} = \gamma(1_{\nu,-})$.*

(*b*) *There is a decomposition of graded ${}^\circ\mathbf{R}_\nu$-modules $M = 1_{\nu,+}M \oplus 1_{\nu,-}M$.*

(*c*) *We have a canonical isomorphism of ${}^\circ\mathbf{R}_\nu$-modules $\mathrm{res} \circ \mathrm{ind}(M) = M \oplus M^\gamma$.*

(*d*) *If there exists $a \in \{+,-\}$ such that $1_{\nu,-a}M = 0$, then there are canonical isomorphisms of graded ${}^\circ\mathbf{R}_\nu$-modules*

$$M = 1_{\nu,a}M, \quad 0 = 1_{\nu,a}M^\gamma, \quad M^\gamma = 1_{\nu,-a}M^\gamma.$$

Proof: Part (a) follows from Proposition 1.6 and the equality $\varepsilon_1({}^\theta I^\nu_+) = {}^\theta I^\nu_-$. Part (b) follows from (a), (c) is given by definition, and (d) follows from (a), (b).

□

Now, let m and ν be as before. Given $i \in I$, we set $\nu' = \nu + i + \theta(i)$. There is an obvious inclusion $W_m \subset W_{m+1}$. Thus the group W_m acts on ${}^\theta I^{\nu'}$, and the map

$${}^\theta I^\nu \to {}^\theta I^{\nu'}, \quad \mathbf{i} \mapsto \theta(i)\mathbf{i}i$$

is W_m-equivariant. Thus there is $a_i \in \{+,-\}$ such that the image of ${}^\theta I^\nu_+$ is contained in ${}^\theta I^{\nu'}_{a_i}$, and the image of ${}^\theta I^\nu_-$ is contained in ${}^\theta I^{\nu'}_{-a_i}$.

3.16. Lemma. *Let M be a graded ${}^\circ\mathbf{R}_\nu$-module such that $1_{\nu,-a}M = 0$, with $a = +,-$. Then we have*

$$1_{\nu',-a_i a} f_i(M) = 0, \quad 1_{\nu',a_i a} f_{\theta(i)}(M) = 0.$$

Proof: We have

$$\begin{aligned} 1_{\nu',-a_i a} f_i(M) &= 1_{\nu',-a_i a} {}^\circ\mathbf{R}_{\nu'} 1_{\nu,i} \otimes_{{}^\circ\mathbf{R}_\nu} M \\ &= {}^\circ\mathbf{R}_{\nu'} 1_{\nu',-a_i a} 1_{\nu,i} 1_{\nu,a} \otimes_{{}^\circ\mathbf{R}_\nu} M. \end{aligned}$$

Here we have identified $1_{\nu,a}$ with the image of $(1_{\nu,a}, 1_i)$ via the inclusion (3.2). The definition of this inclusion and that of a_i yield that

$$1_{\nu',a_i a} 1_{\nu,i} 1_{\nu,a} = 1_{\nu,a}, \quad 1_{\nu',-a_i a} 1_{\nu,i} 1_{\nu,a} = 0.$$

The first equality follows. Next, note that for any $\mathbf{i} \in {}^\theta I^\nu$, the sequences $\theta(i)\mathbf{i}i$ and $i\mathbf{i}\theta(i) = \varepsilon_{m+1}(\theta(i)\mathbf{i}i)$ always belong to different ${}^\circ W_{m+1}$-orbits. Thus, we have $a_{\theta(i)} = -a_i$. So the second equality follows from the first.

□

Consider the following diagram

$$\begin{array}{ccc} {}^\circ\mathbf{R}_\nu\text{-}\mathbf{mod} \times \mathbf{R}_i\text{-}\mathbf{mod} & \underset{\psi^*}{\overset{\psi_!}{\rightleftarrows}} & {}^\circ\mathbf{R}_{\nu'}\text{-}\mathbf{mod} \\ {\scriptstyle \mathrm{res}\times\mathrm{id}}\uparrow\downarrow{\scriptstyle \mathrm{ind}\times\mathrm{id}} & & {\scriptstyle \mathrm{res}}\uparrow\downarrow{\scriptstyle \mathrm{ind}} \\ {}^\theta\mathbf{R}_\nu\text{-}\mathbf{mod} \times \mathbf{R}_i\text{-}\mathbf{mod} & \underset{\psi^*}{\overset{\psi_!}{\rightleftarrows}} & {}^\theta\mathbf{R}_{\nu'}\text{-}\mathbf{mod}. \end{array}$$

3.17. Lemma. *There are canonical isomorphisms of functors*

$$\mathrm{ind}\circ\psi_! = \psi_!\circ(\mathrm{ind}\times\mathrm{id}),\quad \psi^*\circ\mathrm{ind} = (\mathrm{ind}\times\mathrm{id})\circ\psi^*,\quad \mathrm{ind}\circ\psi_* = \psi_*\circ(\mathrm{ind}\times\mathrm{id}),$$

$$\mathrm{res}\circ\psi_! = \psi_!\circ(\mathrm{res}\times\mathrm{id}),\quad \psi^*\circ\mathrm{res} = (\mathrm{res}\times\mathrm{id})\circ\psi^*,\quad \mathrm{res}\circ\psi_* = \psi_*\circ(\mathrm{res}\times\mathrm{id}).$$

Proof : The functor ind is left and right adjoint to res. Therefore it is enough to prove the first two isomorphisms in the first line. The isomorphism

$$\mathrm{ind}\circ\psi_! = \psi_!\circ(\mathrm{ind}\times\mathrm{id})$$

comes from the associativity of the induction. Let us prove that

$$\psi^*\circ\mathrm{ind} = (\mathrm{ind}\times\mathrm{id})\circ\psi^*.$$

For any M in ${}^\circ\mathbf{R}_{\nu'}$**-mod**, the obvious inclusion ${}^\theta\mathbf{R}_\nu\otimes\mathbf{R}_i\subset{}^\theta\mathbf{R}_{\nu'}$ yields a map

$$(\mathrm{ind}\times\mathrm{id})\,\psi^*(M) = ({}^\theta\mathbf{R}_\nu\otimes\mathbf{R}_i)\otimes_{{}^\circ\mathbf{R}_\nu\otimes\mathbf{R}_i}\psi^*(M)\to\psi^*({}^\theta\mathbf{R}_{\nu'}\otimes_{{}^\circ\mathbf{R}_\nu\otimes\mathbf{R}_i}M).$$

Combining it with the obvious map

$${}^\theta\mathbf{R}_{\nu'}\otimes_{{}^\circ\mathbf{R}_\nu\otimes\mathbf{R}_i}M\to{}^\theta\mathbf{R}_{\nu'}\otimes_{{}^\circ\mathbf{R}_{\nu'}}M$$

we get a morphism of ${}^\theta\mathbf{R}_\nu\otimes\mathbf{R}_i$-modules

$$(\mathrm{ind}\times\mathrm{id})\,\psi^*(M)\to\psi^*\,\mathrm{ind}(M).$$

We need to show that it is bijective. This is obvious because at the level of vector spaces, the map above is given by

$$M\oplus(\pi_{1,\nu}\otimes M)\to M\oplus(\pi_{1,\nu'}\otimes M),\quad m+\pi_{1,\nu}\otimes n\mapsto m+\pi_{1,\nu'}\otimes n.$$

Here $\pi_{1,\nu}$ and $\pi_{1,\nu'}$ denote the element π_1 in ${}^\theta\mathbf{R}_\nu$ and ${}^\theta\mathbf{R}_{\nu'}$ respectively.

□

3.18. Corollary. *(a) The operators e_i, f_i on ${}^\circ\mathbf{K}_{I,*}$ and on ${}^\theta\mathbf{K}_{I,*}$ are intertwined by the maps* ind, res*, i.e., we have*

$$e_i\circ\mathrm{ind} = \mathrm{ind}\circ e_i,\quad f_i\circ\mathrm{ind} = \mathrm{ind}\circ f_i,\quad e_i\circ\mathrm{res} = \mathrm{res}\circ e_i,\quad f_i\circ\mathrm{res} = \mathrm{res}\circ f_i.$$

(b) The same result holds for the operators e_i, f_i on ${}^\circ\mathbf{G}_{I,*}$ and on ${}^\theta\mathbf{G}_{I,*}$.

3.19. Now, we concentrate on non graded irreducible modules. First, let

$$\mathrm{Res}:{}^\theta\mathbf{R}_\nu\text{-}\mathbf{Mod}\to{}^\circ\mathbf{R}_\nu\text{-}\mathbf{Mod},\quad \mathrm{Ind}:{}^\circ\mathbf{R}_\nu\text{-}\mathbf{Mod}\to{}^\theta\mathbf{R}_\nu\text{-}\mathbf{Mod}$$

be the (non graded) restriction and induction functors. We have

$$\mathbf{for}\circ\mathrm{res} = \mathrm{Res}\circ\mathbf{for},\quad \mathbf{for}\circ\mathrm{ind} = \mathrm{Ind}\circ\mathbf{for}.$$

3.20. Lemma. *Let L, L' be irreducible* $^\circ\mathbf{R}_\nu$*-modules.*

(a) The $^\circ\mathbf{R}_\nu$*-modules L and L^γ are not isomorphic.*

(b) $\mathrm{Ind}(L)$ *is an irreducible* $^\theta\mathbf{R}_\nu$*-module, and every irreducible* $^\theta\mathbf{R}_\nu$*-module is obtained in this way.*

(c) $\mathrm{Ind}(L) \simeq \mathrm{Ind}(L')$ *iff* $L' \simeq L$ *or* L^γ.

Proof: For any $^\theta\mathbf{R}_\nu$-module $M \neq 0$, both $1_{\nu,+}M$ and $1_{\nu,-}M$ are nonzero. Indeed, we have $M = 1_{\nu,+}M \oplus 1_{\nu,-}M$, and we may suppose that $1_{\nu,+}M \neq 0$. The automorphism $M \to M$, $m \mapsto \pi_1 m$ takes $1_{\nu,+}M$ to $1_{\nu,-}M$ by Lemma 3.15(a). Hence $1_{\nu,-}M \neq 0$.

Now, to prove part (a), suppose that $\phi : L \to L^\gamma$ is an isomorphism of $^\circ\mathbf{R}_\nu$-modules. We can regard ϕ as a γ-antilinear map $L \to L$. Since L is irreducible, by Schur's lemma we may assume that $\phi^2 = \mathrm{id}_L$. Then L admits a $^\theta\mathbf{R}_\nu$-module structure such that the $^\circ\mathbf{R}_\nu$-action is as before and π_1 acts as ϕ. Thus, by the discussion above, neither $1_{\nu,+}L$ nor $1_{\nu,-}L$ is zero. This contradicts the fact that L is an irreducible $^\circ\mathbf{R}_\nu$-module.

Parts (b), (c) follow from (a) by Clifford theory, see e.g., [RR, appendix].

□

We can now prove Proposition 3.2.

3.21. Proof of Proposition 3.2. Let $b \in {}^\circ B$. We may suppose that $b = 1_{\nu,+}b$. By Lemma 3.20(b) the module $^\theta b = \mathrm{Ind}(b)$ lies in $^\theta B$. By [VV, prop. 8.2] there exists a unique selfdual irreducible graded $^\theta\mathbf{R}$-module $^\theta G^{\mathrm{up}}(^\theta b)$ which is isomorphic to $^\theta b$ as a non graded module. Set

$$^\circ G^{\mathrm{up}}(b) = 1_{\nu,+}\mathrm{res}(^\theta G^{\mathrm{up}}(^\theta b)).$$

By Lemma 3.15(d) we have $^\circ G^{\mathrm{up}}(b) = b$ as a non graded $^\circ\mathbf{R}$-module, and by (3.5) it is selfdual. This proves existence part of the proposition. To prove the uniqueness, suppose that M is another module with the same properties. Then $\mathrm{ind}(M)$ is a selfdual graded $^\theta\mathbf{R}$-module which is isomorphic to $^\theta b$ as a non graded $^\theta\mathbf{R}$-module. Thus we have $\mathrm{ind}(M) = {}^\theta G^{\mathrm{up}}(^\theta b)$ by loc. cit. By Lemma 3.15(d) we have also

$$M = 1_{\nu,+}\mathrm{res}(^\theta G^{\mathrm{up}}(^\theta b)).$$

So M is isomorphic to $^\circ G^{\mathrm{up}}(b)$.

□

3.22. The crystal operators on $^\circ\mathbf{G}_I$ and $^\circ B$. Fix a vertex i in I. For each irreducible graded $^\circ\mathbf{R}_m$-module M we define

$$\tilde e_i(M) = \mathrm{soc}\,(e_i(M)), \quad \tilde f_i(M) = \mathrm{top}\,\psi_!(M, \mathbf{L}_i), \quad \varepsilon_i(M) = \max\{n \geqslant 0;\, e_i^n(M) \neq 0\}.$$

3.23. Lemma. *Let M be an irreducible graded* $^\circ\mathbf{R}$*-module such that $e_i(M)$, $f_i(M)$ belong to* $^\circ\mathbf{G}_{I,*}$*. We have*

$$\mathrm{ind}(\tilde e_i(M)) = \tilde e_i(\mathrm{ind}(M)), \quad \mathrm{ind}(\tilde f_i(M)) = \tilde f_i(\mathrm{ind}(M)), \quad \varepsilon_i(M) = \varepsilon_i(\mathrm{ind}(M)).$$

In particular, $\tilde e_i(M)$ is irreducible or zero and $\tilde f_i(M)$ is irreducible.

Proof: By Corollary 3.18 we have $\mathrm{ind}(e_i(M)) = e_i(\mathrm{ind}(M))$. Thus, since ind is an exact functor we have $\mathrm{ind}(\tilde{e}_i(M)) \subset e_i(\mathrm{ind}(M))$. Since ind is an additive functor, by Lemma 3.20(b) we have indeed

$$\mathrm{ind}(\tilde{e}_i(M)) \subset \tilde{e}_i(\mathrm{ind}(M)).$$

Note that $\mathrm{ind}(M)$ is irreducible by Lemma 3.20(b), thus $\tilde{e}_i(\mathrm{ind}(M))$ is irreducible by [VV, prop. 8.21]. Since $\mathrm{ind}(\tilde{e}_i(M))$ is nonzero, the inclusion is an isomorphism. The fact that $\mathrm{ind}(\tilde{e}_i(M))$ is irreducible implies in particular that $\tilde{e}_i(M)$ is simple. The proof of the second isomorphism is similar. The third equality is obvious.

□

Similarly, for each irreducible ${}^\circ\mathbf{R}$-module b in ${}^\circ B$ we define

$$\tilde{E}_i(b) = \mathrm{soc}(E_i(b)), \quad \tilde{F}_i(b) = \mathrm{top}(F_i(b)), \quad \varepsilon_i(b) = \max\{n \geqslant 0;\, E_i^n(b) \neq 0\}.$$

Hence we have

$$\mathbf{for} \circ \tilde{e}_i = \tilde{E}_i \circ \mathbf{for}, \quad \mathbf{for} \circ \tilde{f}_i = \tilde{F}_i \circ \mathbf{for}, \quad \varepsilon_i = \varepsilon_i \circ \mathbf{for}.$$

3.24. Proposition. *For each b, b' in ${}^\circ B$ we have*

(a) $\tilde{F}_i(b) \in {}^\circ B$,

(b) $\tilde{E}_i(b) \in {}^\circ B \cup \{0\}$,

(c) $\tilde{F}_i(b) = b' \iff \tilde{E}_i(b') = b$,

(d) $\varepsilon_i(b) = \max\{n \geqslant 0; \tilde{E}_i^n(b) \neq 0\}$,

(e) $\varepsilon_i(\tilde{F}_i(b)) = \varepsilon_i(b) + 1$,

(f) if $\tilde{E}_i(b) = 0$ *for all* i *then* $b = \phi_\pm$.

Proof: Part (c) follows from adjunction. The other parts follow from [VV, prop. 3.14] and Lemma 3.23.

□

3.25. Remark. For any $b \in {}^\circ B$ and any $i \neq j$ we have $\tilde{F}_i(b) \neq \tilde{F}_j(b)$. This is obvious if $j \neq \theta(i)$. Because in this case $\tilde{F}_i(b)$ and $\tilde{F}_j(b)$ are ${}^\circ\mathbf{R}_\nu$-modules for different ν. Now, consider the case $j = \theta(i)$. We may suppose that $\tilde{F}_i(b) = 1_{\nu,+}\tilde{F}_i(b)$ for certain ν. Then by Lemma 3.16 we have $1_{\nu,+}\tilde{F}_{\theta(i)}(b) = 0$. In particular $\tilde{F}_i(b)$ is not isomorphic to $\tilde{F}_{\theta(i)}(b)$.

3.26. The algebra ${}^\theta\mathbf{B}$ and the ${}^\theta\mathbf{B}$-module ${}^\circ\mathbf{V}$. Following [EK1,2,3] we define a $\mathcal{K}$-algebra ${}^\theta\mathbf{B}$ as follows.

3.27. Definition. *Let ${}^\theta\mathbf{B}$ be the $\mathcal{K}$-algebra generated by e_i, f_i and invertible elements t_i, $i \in I$, satisfying the following defining relations*

(a) $t_i t_j = t_j t_i$ *and* $t_{\theta(i)} = t_i$ *for all* i, j,

(b) $t_i e_j t_i^{-1} = v^{i\cdot j + \theta(i)\cdot j} e_j$ *and* $t_i f_j t_i^{-1} = v^{-i\cdot j - \theta(i)\cdot j} f_j$ *for all* i, j,

(c) $e_i f_j = v^{-i\cdot j} f_j e_i + \delta_{i,j} + \delta_{\theta(i),j} t_i$ *for all* i, j,

$$(d) \quad \sum_{a+b=1-i\cdot j}(-1)^a e_i^{(a)}e_je_i^{(b)} = \sum_{a+b=1-i\cdot j}(-1)^a f_i^{(a)}f_jf_i^{(b)} = 0 \text{ if } i \neq j.$$

Here and below we use the following notation

$$\theta^{(a)} = \theta^a/\langle a\rangle!, \quad \langle a\rangle = \sum_{l=1}^{a} v^{a+1-2l}, \quad \langle a\rangle! = \prod_{l=1}^{m}\langle l\rangle.$$

We can now construct a representation of ${}^\theta\mathbf{B}$ as follows. By base change, the operators e_i, f_i in Definition 3.8 yield $\mathcal{K}$-linear operators on the $\mathcal{K}$-vector space

$${}^\circ\mathbf{V} = \mathcal{K}\otimes_{\mathcal{A}}{}^\circ\mathbf{K}_I.$$

We equip ${}^\circ\mathbf{V}$ with the $\mathcal{K}$-bilinear form given by

$$(M:N)_{KE} = (1-v^2)^m\,(M:N), \quad \forall M,N\in{}^\circ\mathbf{R}_m\text{-}\mathbf{proj}.$$

3.28. Theorem. *(a) The operators e_i, f_i define a representation of ${}^\theta\mathbf{B}$ on ${}^\circ\mathbf{V}$. The ${}^\theta\mathbf{B}$-module ${}^\circ\mathbf{V}$ is generated by linearly independent vectors ϕ_+ and ϕ_- such that for each $i\in I$ we have*

$$e_i\phi_\pm = 0, \quad t_i\phi_\pm = \phi_\mp, \quad \{x\in{}^\circ\mathbf{V};\, e_jx=0,\,\forall j\} = \mathbf{k}\,\phi_+\oplus\mathbf{k}\,\phi_-.$$

(b) The symmetric bilinear form on ${}^\circ\mathbf{V}$ is non-degenerate. We have $(\phi_a:\phi_{a'})_{KE} = \delta_{a,a'}$ for $a,a'=+,-$, and $(e_ix:y)=(x:f_iy)_{KE}$ for $i\in I$ and $x,y\in{}^\circ\mathbf{V}$.

Proof : For each i in I we define the $\mathcal{A}$-linear operator t_i on ${}^\circ\mathbf{K}_I$ by setting

$$t_i\phi_\pm = \phi_\mp \quad\text{and}\quad t_iP = v^{-\nu\cdot(i+\theta(i))}P^\gamma, \quad \forall P\in{}^\circ\mathbf{R}_\nu\text{-}\mathbf{proj}.$$

We must prove that the operators e_i, f_i, and t_i satisfy the relations of ${}^\theta\mathbf{B}$. The relations (a), (b) are obvious. The relation (d) is standard. It remains to check (c). For this we need a version of the Mackey's induction-restriction theorem. Note that for $m>1$ we have

$$D_{m,1;m,1} = \{e, s_m, \varepsilon_{m+1}\varepsilon_1\},$$

$$W(e) = {}^\circ W_m, \quad W(s_m) = {}^\circ W_{m-1}, \quad W(\varepsilon_{m+1}\varepsilon_1) = {}^\circ W_m.$$

Recall also that for $m=1$ we have set ${}^\circ W_1=\{e\}$.

3.29. Lemma. *Fix i, j in I. Let μ, ν in ${}^\theta\mathbb{N}I$ be such that $\nu+i+\theta(i)=\mu+j+\theta(j)$. Put $|\nu|=|\mu|=2m$. The graded $({}^\circ\mathbf{R}_{m,1},{}^\circ\mathbf{R}_{m,1})$-bimodule $1_{\nu,i}{}^\circ\mathbf{R}_{m+1}1_{\mu,j}$ has a filtration by graded bimodules whose associated graded is isomorphic to*

$$\delta_{i,j}\big({}^\circ\mathbf{R}_\nu\otimes\mathbf{R}_i\big)\oplus\delta_{\theta(i),j}\big(({}^\circ\mathbf{R}_\nu)^\gamma\otimes\mathbf{R}_{\theta(i)}\big)[d']\oplus A[d],$$

where A is equal to

$$({}^\circ\mathbf{R}_m 1_{\nu',i}\otimes\mathbf{R}_i)\otimes_{\mathbf{R}'}(1_{\nu',i}{}^\circ\mathbf{R}_m\otimes\mathbf{R}_i) \qquad \text{if } m>1,$$

$$({}^\circ\mathbf{R}_{\theta(\mathbf{j})}\otimes\mathbf{R}_i\otimes_{{}^\circ\mathbf{R}_1\otimes\mathbf{R}_1}{}^\circ\mathbf{R}_{\theta(\mathbf{i})}\otimes\mathbf{R}_j)\oplus({}^\circ\mathbf{R}_\mathbf{j}\otimes\mathbf{R}_i\otimes_{{}^\circ\mathbf{R}_1\otimes\mathbf{R}_1}{}^\circ\mathbf{R}_\mathbf{i}\otimes\mathbf{R}_j) \qquad \text{if } m=1.$$

Here we have set $\nu' = \nu - j - \theta(j)$, $\mathbf{R}' = {}^\circ\mathbf{R}_{m-1,1} \otimes \mathbf{R}_1$, $\mathbf{i} = i\theta(i)$, $\mathbf{j} = j\theta(j)$, $d = -i \cdot j$, and $d' = -\nu \cdot (i + \theta(i))/2$.

The proof is standard and is left to the reader. Now, recall that for $m > 1$ we have

$$f_j(P) = {}^\circ\mathbf{R}_{m+1} 1_{m,j} \otimes_{{}^\circ\mathbf{R}_{m,1}} (P \otimes \mathbf{R}_1), \quad e'_i(P) = 1_{m-1,i}P,$$

where $1_{m-1,i}P$ is regarded as a ${}^\circ\mathbf{R}_{m-1}$-module. Therefore we have

$$e'_i f_j(P) = 1_{m,i}{}^\circ\mathbf{R}_{m+1} 1_{m,j} \otimes_{{}^\circ\mathbf{R}_{m,1}} (P \otimes \mathbf{R}_1),$$
$$f_j e'_i(P) = {}^\circ\mathbf{R}_m 1_{m-1,j} \otimes_{{}^\circ\mathbf{R}_{m-1,1}} (1_{m-1,i}P \otimes \mathbf{R}_1).$$

Therefore, up to some filtration we have the following identities

- $e'_i f_i(P) = P \otimes \mathbf{R}_i + f_i e'_i(P)[-2]$,
- $e'_i f_{\theta(i)}(P) = P^\gamma \otimes \mathbf{R}_{\theta(i)}[-\nu \cdot (i + \theta(i))/2] + f_{\theta(i)} e'_i(P)[-i \cdot \theta(i)]$,
- $e'_i f_j(P) = f_j e'_i(P)[-i \cdot j]$ if $i \neq j, \theta(j)$.

These identities also hold for $m = 1$ and $P = {}^\circ\mathbf{R}_{\theta(i)i}$ for any $i \in I$. The first claim of part (a) follows because $\mathbf{R}_i = \mathbf{k} \oplus \mathbf{R}_i[2]$. The fact that ${}^\circ\mathbf{V}$ is generated by $\phi_\pm$ is a corollary of Proposition 3.31 below. Part (b) of the theorem follows from [KM, prop. 2.2(ii)] and Lemma 3.9(b).

□

3.30. Remarks. (a) The ${}^\theta\mathbf{B}$-module ${}^\circ\mathbf{V}$ is the same as the ${}^\theta\mathbf{B}$-module V_θ from [KM, prop. 2.2]. The involution $\sigma : {}^\circ\mathbf{V} \to {}^\circ\mathbf{V}$ in [KM, rem. 2.5(ii)] is given by $\sigma(P) = P^\gamma$. It yields an involution of ${}^\circ B$ in the obvious way. Note that Lemma 3.20(a) yields $\sigma(b) \neq b$ for any $b \in {}^\circ B$.

(b) Let ${}^\theta\mathbf{V}$ be the ${}^\theta\mathbf{B}$-module $\mathcal{K} \otimes_\mathcal{A} {}^\theta\mathbf{K}_I$ and let ϕ be the class of the trivial ${}^\theta\mathbf{R}_0$-module $\mathbf{k}$, see [VV, thm. 8.30]. We have an inclusion of ${}^\theta\mathbf{B}$-modules

$${}^\theta\mathbf{V} \to {}^\circ\mathbf{V}, \quad \phi \mapsto \phi_+ \oplus \phi_-, \quad P \mapsto \mathrm{res}(P).$$

3.31. Proposition. *For any $b \in {}^\circ B$ the following holds*

$$(a) \qquad \begin{cases} f_i({}^\circ G^{\mathrm{low}}(b)) = \langle \varepsilon_i(b) + 1 \rangle \, {}^\circ G^{\mathrm{low}}(\tilde{F}_i b) + \sum_{b'} f_{b,b'} \, {}^\circ G^{\mathrm{low}}(b'), \\ \\ b' \in {}^\circ B, \quad \varepsilon_i(b') > \varepsilon_i(b) + 1, \quad f_{b,b'} \in v^{2-\varepsilon_i(b')}\mathbb{Z}[v], \end{cases}$$

$$(b) \qquad \begin{cases} e_i({}^\circ G^{\mathrm{low}}(b)) = v^{1-\varepsilon_i(b)} \, {}^\circ G^{\mathrm{low}}(\tilde{E}_i b) + \sum_{b'} e_{b,b'} \, {}^\circ G^{\mathrm{low}}(b'), \\ \\ b' \in {}^\circ B, \quad \varepsilon_i(b') \geqslant \varepsilon_i(b), \quad e_{b,b'} \in v^{1-\varepsilon_i(b')}\mathbb{Z}[v]. \end{cases}$$

Proof: We prove part (a), the proof for (b) is similar. If ${}^\circ G^{\mathrm{low}}(b) = \phi_\pm$ this is obvious. So we assume that ${}^\circ G^{\mathrm{low}}(b)$ is a ${}^\circ\mathbf{R}_m$-module for $m \geqslant 1$. Fix $\nu \in {}^\theta\mathbb{N}I$

such that $f_i({}^\circ G^{\mathrm{low}}(b))$ is a ${}^\circ\mathbf{R}_\nu$-module. We'll abbreviate $1_{\nu,a} = 1_a$ for $a \in \{+,-\}$. Since ${}^\circ G^{\mathrm{low}}(b)$ is indecomposable, it fulfills the condition of Lemma 3.16. So there exists $a \in \{+,-\}$ such that $1_{-a} f_i({}^\circ G^{\mathrm{low}}(b)) = 0$. Thus, by Lemma 3.15$(c), (d)$ and Corollary 3.18 we have

$$f_i({}^\circ G^{\mathrm{low}}(b)) = 1_a \mathrm{res}\,\mathrm{ind} f_i({}^\circ G^{\mathrm{low}}(b)) = 1_a \mathrm{res}\, f_i\, \mathrm{ind}({}^\circ G^{\mathrm{low}}(b)).$$

Note that ${}^\theta b = \mathrm{Ind}(b)$ belongs to ${}^\theta B$ by Lemma 3.20(b). Hence (3.5) yields

$$\mathrm{ind}({}^\circ G^{\mathrm{low}}(b)) = {}^\theta G^{\mathrm{low}}({}^\theta b).$$

We deduce that

$$f_i({}^\circ G^{\mathrm{low}}(b)) = 1_a \mathrm{res} f_i({}^\theta G^{\mathrm{low}}({}^\theta b)).$$

Now, write

$$f_i({}^\theta G^{\mathrm{low}}({}^\theta b)) = \sum f_{{}^\theta b, {}^\theta b'}\, {}^\theta G^{\mathrm{low}}({}^\theta b'), \quad {}^\theta b' \in {}^\theta B.$$

Then we have

$$f_i({}^\circ G^{\mathrm{low}}(b)) = \sum f_{{}^\theta b, {}^\theta b'} 1_a \mathrm{res}({}^\theta G^{\mathrm{low}}({}^\theta b')).$$

For any ${}^\theta b' \in {}^\theta B$ the ${}^\circ\mathbf{R}$-module $1_a \mathrm{Res}({}^\theta b')$ belongs to ${}^\circ B$. Thus we have

$$1_a \mathrm{res}({}^\theta G^{\mathrm{low}}({}^\theta b')) = {}^\circ G^{\mathrm{low}}(1_a \mathrm{Res}({}^\theta b')).$$

If ${}^\theta b' \neq {}^\theta b''$ then $1_a \mathrm{Res}({}^\theta b') \neq 1_a \mathrm{Res}({}^\theta b'')$, because ${}^\theta b' = \mathrm{Ind}(1_a \mathrm{Res}({}^\theta b'))$. Thus

$$f_i({}^\circ G^{\mathrm{low}}(b)) = \sum f_{{}^\theta b, {}^\theta b'}\, {}^\circ G^{\mathrm{low}}(1_a \mathrm{Res}({}^\theta b')),$$

and this is the expansion of the lhs in the lower global basis. Finally, we have

$$\varepsilon_i(1_a \mathrm{Res}({}^\theta b')) = \varepsilon_i({}^\theta b')$$

by Lemma 3.23. So part (a) follows from [VV, prop. 10.11(b), 10.16].

□

3.32. The global bases of ${}^\circ\mathbf{V}$. Since the operators e_i, f_i on ${}^\circ\mathbf{V}$ satisfy the relations $e_i f_i = v^{-2} f_i e_i + 1$, we can define the modified root operators $\tilde{\mathbf{e}}_i$, $\tilde{\mathbf{f}}_i$ on the ${}^\theta\mathbf{B}$-module ${}^\circ\mathbf{V}$ as follows. For each u in ${}^\circ\mathbf{V}$ we write

$$u = \sum_{n \geqslant 0} f_i^{(n)} u_n \text{ with } e_i u_n = 0,$$
$$\tilde{\mathbf{e}}_i(u) = \sum_{n \geqslant 1} f_i^{(n-1)} u_n, \quad \tilde{\mathbf{f}}_i(u) = \sum_{n \geqslant 0} f_i^{(n+1)} u_n.$$

Let $\mathcal{R} \subset \mathcal{K}$ be the set of functions which are regular at $v = 0$. Let ${}^\circ\mathbf{L}$ be the $\mathcal{R}$-submodule of ${}^\circ\mathbf{V}$ spanned by the elements $\tilde{\mathbf{f}}_{i_1} \ldots \tilde{\mathbf{f}}_{i_l}(\phi_\pm)$ with $l \geqslant 0$, $i_1, \ldots, i_l \in I$. The following is the main result of the paper.

3.33. Theorem. *(a) We have*

$$^\circ\mathbf{L} = \bigoplus_{b\in {}^\circ B} \mathcal{R}\, {}^\circ\! G^{\mathrm{low}}(b), \quad \tilde{\mathbf{e}}_i({}^\circ\mathbf{L}) \subset {}^\circ\mathbf{L}, \quad \tilde{\mathbf{f}}_i({}^\circ\mathbf{L}) \subset {}^\circ\mathbf{L},$$

$$\tilde{\mathbf{e}}_i({}^\circ G^{\mathrm{low}}(b)) = {}^\circ G^{\mathrm{low}}(\tilde{E}_i(b)) \bmod v\,{}^\circ\mathbf{L}, \quad \tilde{\mathbf{f}}_i({}^\circ G^{\mathrm{low}}(b)) = {}^\circ G^{\mathrm{low}}(\tilde{F}_i(b)) \bmod v\,{}^\circ\mathbf{L}.$$

(b) The assignment $b \mapsto {}^\circ G^{\mathrm{low}}(b) \bmod v\,{}^\circ\mathbf{L}$ *yields a bijection from* ${}^\circ B$ *to the subset of* ${}^\circ\mathbf{L}/v{}^\circ\mathbf{L}$ *consisting of the* $\tilde{\mathbf{f}}_{i_1} \dots \tilde{\mathbf{f}}_{i_l}(\phi_\pm)$*'s. Further* ${}^\circ G^{\mathrm{low}}(b)$ *is the unique element* $x \in {}^\circ\mathbf{V}$ *such that* $x^\sharp = x$ *and* $x = {}^\circ G^{\mathrm{low}}(b) \bmod v\,{}^\circ\mathbf{L}$.

(c) For each b, b' *in* ${}^\circ B$ *let* $E_{i,b,b'}, F_{i,b,b'} \in \mathcal{A}$ *be the coefficients of* ${}^\circ G^{\mathrm{low}}(b')$ *in* $e_{\theta(i)}({}^\circ G^{\mathrm{low}}(b))$, $f_i({}^\circ G^{\mathrm{low}}(b))$ *respectively. Then we have*

$$E_{i,b,b'}|_{v=1} = [F_i \Psi \mathbf{for}({}^\circ G^{\mathrm{up}}(b')) : \Psi \mathbf{for}({}^\circ G^{\mathrm{up}}(b))],$$

$$F_{i,b,b'}|_{v=1} = [E_i \Psi \mathbf{for}({}^\circ G^{\mathrm{up}}(b')) : \Psi \mathbf{for}({}^\circ G^{\mathrm{up}}(b))].$$

Proof : Part (a) follows from [EK3, thm. 4.1, cor. 4.4], [E, Section 2.3], and Proposition 3.31. The first claim in (b) follows from (a). The second one is obvious. Part (c) follows from Proposition 3.11. More precisely, by duality we can regard $E_{i,b,b'}$, $F_{i,b,b'}$ as the coefficients of ${}^\circ G^{\mathrm{up}}(b)$ in $f_{\theta(i)}({}^\circ G^{\mathrm{up}}(b'))$ and $e_i({}^\circ G^{\mathrm{up}}(b'))$ respectively. Therefore, by Proposition 3.11 we can regard $E_{i,b,b'}|_{v=1}$, $F_{i,b,b'}|_{v=1}$ as the coefficients of $\Psi\mathbf{for}({}^\circ G^{\mathrm{up}}(b))$ in $F_i\Psi\mathbf{for}({}^\circ G^{\mathrm{up}}(b'))$ and $E_i\Psi\mathbf{for}({}^\circ G^{\mathrm{up}}(b'))$ respectively.

□

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