Exact calculations for false discovery proportion with application to least favorable configurations

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EXACT CALCULATIONS FOR FALSE DISCOVERY PROPORTION WITH APPLICATION TO LEAST FAVORABLE CONFIGURATIONS

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In a context of multiple hypothesis testing, we provide several new exact calculations related to the false discovery proportion (FDP) of step-up and step-down procedures. For step-up procedures, we show that the number of erroneous rejections conditionally on the rejection number is simply a binomial variable, which leads to explicit computations of the c.d.f., the s-th moment and the mean of the FDP, the latter corresponding to the false discovery rate (FDR). For step-down procedures, we derive what is to our knowledge the first explicit formula for the FDR valid for any alternative c.d.f. of the p-values. We also derive explicit computations of the power for both step-up and step-down procedures. These formulas are "explicit" in the sense that they only involve the parameters of the model and the c.d.f. of the order statistics of i.i.d. uniform variables. The p-values are assumed either independent or coming from an equicorrelated multivariate normal model and an additional mixture model for the true/false hypotheses is used. This new approach is used to investigate new results which are of interest in their own right, related to least/most favorable configurations for the FDR and the variance of the FDP.

1. Introduction. When testing simultaneously m null hypotheses, the false discovery proportion (FDP) is defined as the proportion of errors among all the rejected hypotheses and the false discovery rate (FDR) is defined as the average of the FDP. Since its introduction by Benjamini and Hochberg (1995) [1], the FDR has become a widely used type I error criterion, because it is adaptive to the number of rejected hypotheses. However, the randomness of the denominator in the FDP expression makes the study of the FDP and of the FDR mathematically challenging.

There is a considerable number of papers that deal with the FDR control under different dependency conditions between the p-values (see for instance [1, 3–5, 27]). In the latter, the goal is, given a prespecified level α, to provide a procedure with a FDR smaller than α (for any value of the data law in a given distribution subspace, e.g. for some dependency assumptions). For instance, the famous linear step-up procedure (LSU), also called the Benjamini-Hochberg procedure [1] (based on the Simes's line [29]), has been proved to control the FDR under inde-

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pendence and under a positive dependence assumption (see [1, 3]). While controlling the FDR, one wants to maximize the power of the procedure, the power being generally defined as the averaged number of correct rejections divided by the number of false hypotheses.

In this paper, we deal with the "reversed" approach: given the procedure, we aim to compute the corresponding FDR, or more generally the s-th moment and the c.d.f. of the FDP, and the Power. For procedures using a constant thresholding and under a mixture model assuming independence between the p-values, Storey (2003) [30] addressed theses questions, while introducing the positive false discovery rate (pFDR) (see also Chi and Tan [7]). Considering step-up or step-down methods requires more efforts: when the p-values are all i.i.d. uniform, the exact distribution of the rejection number has been computed by Finner and Roters (2002) for step-up and step-down procedures, which leads to a computation of the FDR in the degenerate case where all the hypotheses are true [15]. When the p-values follow the particular "Dirac-uniform" configuration, that is, when the p-values associated to false hypotheses are equal to 0 and when the p-values associated to true hypotheses are i.i.d. uniform, the FDP distribution has been computed by Dickhaus (2008) for general step-up-down procedures (see Section 3.7 in [8]). For an arbitrary distribution of the p-values under the alternative, Ferreira and Zwinderman (2006) gave a first exact expression for the moment of the FDP of the LSU procedure under independence [12]. Together with other recent approaches (see, e.g. [4, 24, 25, 27]), this puts forward a connection between the FDR expression and the rejection number distribution in the step-up case, under independence of the p-values. Additionally, Sarkar (2002) found an exact formula for the FDR, which is valid for any step-up-down procedure [26]. However, it involves the c.d.f. of ordered components of dependent variables, by contrast with [8, 15] and present paper, using c.d.f. of ordered components of i.i.d. uniform variables, so obtaining substantially more explicit formulas.

Meanwhile, some of these approaches have also been investigated from the asymptotic point of view, when the number of hypotheses m tends to infinity; Chi (2007) computed the asymptotic rejection number distribution of the LSU procedure by introducing the criticality phenomenon [6], while Finner et al. (2007) computed the limiting FDR of the LSU procedure for positively correlated p-values (following an equicorrelated multivariate normal model) in the particular Dirac-uniform configuration [14]. In this paper, the point of view will be mainly nonasymptotic.

The new contributions of the present paper are as follows:

- For a step-up procedure using a threshold $(t_k)_k$, we proved that the distribution of the number of erroneous rejections conditionally on k rejections is a binomial variable of parameters k and $\mathrm{pFDR}(t_k) = \pi_0 t_k / G(t_k)$, where π_0 is the (averaged) proportion of true nulls and G is the c.d.f. of the p-values. This provides new explicit formulas for the c.d.f. of the FDP, the s-th moment of the FDP (providing a correction with respect to [12] for $s \geq 3$) and for the FDR, for any alternative distribution of the p-values. We also give an expression for the power, which yields a considerably less complex calculation than in [18], see Section 3.1.
- Considering a step-down procedure, a new explicit formula for the FDR and the power is

presented under any alternative distribution of the p-values. To our knowledge, this expression is the first one that clearly relates the FDR to (joint) rejection number distribution in the case of a step-down procedure and that is valid under any alternative, see Section 3.1.

- All the previous results, valid under independence between the p-values, can be easily extended to the case where the p-value family follows an equicorrelated multivariate normal model, by using a simple modification, see Section 3.2. However, this requires the use of a nonnegative correlation. The case of a possibly negative correlation is considered when only two hypotheses are tested, see Section 3.3.
- Our formulas corroborate the classical multiple testing results while they give rise to several new results. The two main corollaries hold under independence and are as follows. First, in Section 4.1.1, for the linear step-down procedure, we prove that a p-value configuration maximizing the FDR, i.e. a least favorable configuration (LFC) for the FDR, is the Dirac-uniform configuration. Additionally, considering a general step-down procedure, we define a new condition on the threshold ensuring that the Dirac-uniform configuration is still a LFC. As discussed in Section 4.1.1, this condition is different from the one of the step-up case. Second, we found an exact expression of the minimum and the maximum of the variance of the FDP of the LSU, these extrema being taken over some p-value configuration sets of interest. The latter allows to better understand the behavior of the FDP around the FDR. In particular, this puts forward that the convergence of the FDP towards the FDR is particularly slow in the sparse case, see Section 4.2.

All our formulas are valid nonasymptotically, that is, they hold for each $m \geq 2$. As a counterpart, they inevitably have a general form that can appear somewhat complex at first sight. For instance, denoting by Ψ_m the c.d.f. of the order statistics of m i.i.d. uniform variables on $(0,1)$, the FDR formula for step-up procedures requires the computation of Ψ_m at a given point of $\mathbb{R}^m$ (at most), while the FDR formula for step-down procedures requires the computation of Ψ_m at $2m$ different points of $\mathbb{R}^m$ (at most). However, let us underline what are to our opinion the two main interests of this exact approach:

- For some model parameter configurations and after possible simplifications, the formulas are usable for further theoretical studies (monotonicity with respect to a parameter, convergence when m tends to infinity,...), as in Theorem 4.1 and Theorem 4.3.
- For m not too large (say $m \leq 1000$), these formulas can be fully computed numerically, e.g. plotting exact graphs. Thus, they avoid using cumbersome and less accurate simulation experiments (extensively used in multiple testing literature), see for instance Section 4.1.2.

2. Preliminaries.

2.1. *Models for the p-value family.* On a given probability space, we consider a finite set of $m \geq 2$ null hypotheses, tested by a family of m p-values $\mathbf{p} = (p_i, i \in \{1, ..., m\})$. In this paper, for simplicity, we skip somewhat the formal definition of p-values by defining directly a p-value model, that is, by specifying the joint distribution of $\mathbf{p}$.

In what follows, we denote by $\mathcal{F}$ the set containing c.d.f.'s from $[0,1]$ into $[0,1]$ that are continuous and by $F_0(t) = t$ the c.d.f. of the uniform distribution over $[0,1]$ (we restricted our attention to the case where $F_0(t) = t$ only for simplicity, all our formulas will be valid for an arbitrary $F_0 \in \mathcal{F}$).

Definition 2.1 (Conditional p-value models).

- *The p-value family $\mathbf{p}$ follows the conditional independent model with parameters $H = (H_i)_{1\leq i\leq m} \in \{0,1\}^m$ and $F_1 \in \mathcal{F}$, that we denote by $\mathbf{p} \sim P^I_{(H,F_1)}$, if $\mathbf{p} = (p_i, i \in \{1,...,m\})$ is a family of mutually independent variables and for all i,*
$$p_i \sim \begin{cases} F_0 & \text{if } H_i = 0 \\ F_1 & \text{if } H_i = 1 \end{cases}.$$
- *The p-value family $\mathbf{p}$ follows the conditional equicorrelated multivariate normal model (say for short, conditional EMN model) with parameters $H = (H_i)_{1\leq i\leq m} \in \{0,1\}^m$, $\rho \in [-(m-1)^{-1}, 1]$, and $\mu > 0$, that we denote by $\mathbf{p} \sim P^N_{(H,\rho,\mu)}$, if for all i, $p_i \sim \overline{\Phi}(X_i + \mu H_i)$, where the vector $(X_i)_{1\leq i\leq m}$ is distributed as a $\mathbb{R}^m$-valued Gaussian vector with zero means and a covariance matrix having 1 on the diagonal and ρ elsewhere and where $\overline{\Phi}$ denotes the standard Gaussian distribution tail, that is, $\overline{\Phi}(z) = \mathbb{P}[Z \geq z]$ for $Z \sim \mathcal{N}(0,1)$. In that model, the marginal distributions of the p-values are thus given by:*
$$p_i \sim \begin{cases} F_0 & \text{if } H_i = 0 \\ F_1(t) = \overline{\Phi}\big(\overline{\Phi}^{-1}(t) - \mu\big) & \text{if } H_i = 1 \end{cases}.$$

The two above models are said "conditional" because the distribution of the p-values are defined conditionally on the value of the parameter $H = (H_i)_{1\leq i\leq m} \in \{0,1\}^m$. The latter determines which hypotheses are true or false: $H_i = 0$ (resp. 1) if the i-th null hypothesis is true (resp. false). We then denote by $\mathcal{H}_0(H) := \{i \in \{1,...,m\} \mid H_i = 0\}$ the set corresponding to the true null hypotheses and by $m_0(H) := |\mathcal{H}_0(H)|$ its cardinal. Analogously, we define $\mathcal{H}_1(H) := \{i \in \{1,...,m\} \mid H_i = 1\}$ and $m_1(H) := |\mathcal{H}_1(H)| = m - m_0(H)$.

To each one of the above models, we associate the "unconditional" version in which we endow the parameter H with the prior distribution $\mathcal{B}(1-\pi_0)^{\otimes m}$, making $(H_i)_{1\leq i\leq m} \in \{0,1\}^m$ being a sequence of i.i.d. Bernoulli with parameter $1-\pi_0$. On an intuitive point of view, this means that each hypothesis is true with probability π_0, independently from the other hypotheses. We thus define the following models for $\mathbf{p}$ (or more precisely for $(H, \mathbf{p})$):

Definition 2.2 (Unconditional p-value models).

- *The couple $(H, \mathbf{p})$ follows the unconditional independent model with parameters $\pi_0 \in [0,1]$ and $F_1 \in \mathcal{F}$, that we denote by $(H, \mathbf{p}) \sim \overline{P}^I_{(\pi_0,F_1)}$ if $H \sim \mathcal{B}(1-\pi_0)^{\otimes m}$ and the distribution of $\mathbf{p}$ conditionally to H is $P^I_{(H,F_1)}$, that is, conditionally on H, $\mathbf{p}$ follows the conditional independent model with parameters H and F_1. In that model, the p-values are i.i.d. with common c.d.f. $G(t) = \pi_0 F_0(t) + (1-\pi_0)F_1(t)$.*

- *The couple* $(H, \mathbf{p})$ *follows the unconditional equicorrelated multivariate normal model (say for short, unconditional EMN model) with parameters* $\pi_0 \in [0,1], \rho \in [-(m-1)^{-1}, 1]$, *and* $\mu > 0$, *that we denote by* $(H, \mathbf{p}) \sim \overline{P}^N_{(\pi_0,\rho,\mu)}$, *if* $H \sim \mathcal{B}(1-\pi_0)^{\otimes m}$ *and the distribution of* $\mathbf{p}$ *conditionally on* H *is* $P^N_{(H,\rho,\mu)}$, *that is, conditionally on* H, $\mathbf{p}$ *follows the conditional EMN model with parameters* H, ρ *and* μ.

An important point is that the quantities $m_0(H)$ and $m_1(H)$ are deterministic in the conditional models P^I, P^N, while they become random in the unconditional models $\overline{P}^I, \overline{P}^N$ with $m_0(H) \sim \mathcal{B}(m, \pi_0)$ and $m_1(H) \sim \mathcal{B}(m, 1-\pi_0)$.

The conditional independent model is one of the most standard p-value models and was for instance considered in the original paper of Benjamini and Hochberg (1995) [1]. Its unconditional version, also called the "random effects model", is very convenient and has been widely used since its introduction by Efron et al. (2001) [11], see for instance [17, 30].

The conditional EMN model is a simple instance of model introducing dependencies between the p-values. It corresponds to a one-sided testing on the mean of $X_i + \mu H_i$, simultaneously for all $1 \leq i \leq m$. It has become quite standard in recent FDR multiple testing literature; for instance, it was used in [14] with $\mu = \infty$ and it has been considered in [2, 5] for numerical experiments. Furthermore, Efron (2009) recently showed that the EMN model may also be viewed as an approximation for some non-equicorrelated models, which reinforces its interest for a practical use [10]. In this model, provided that $\rho \geq 0$, the p-values are positively regression dependent on each one on the subset $\mathcal{H}_0(H)$ (PRDS on $\mathcal{H}_0(H)$) which is one dependency condition that suffices for FDR control (see [3]). The unconditional version of this model is convenient because it provides exchangeable p-values (although not independent when $\rho \neq 0$).

Additionally, we will sometimes consider the "Dirac-uniform configuration" for the above models. In that configuration, all the p-values corresponding to false nulls ($H_i = 1$) are equal to zero, that is, F_1 is constantly equal to 1 for the independent models and $\mu = \infty$ for the EMN models. This configuration was introduced in [14] to increase the FDR as much as possible for the linear step-up procedures which thus appears as a "least favorable configuration" for the FDR (see also Section 4.1).

2.2. *Multiple testing procedures, FDP, FDR and power.* A *multiple testing procedure* R is defined as an algorithm which, from the data, aims to reject part of the null hypotheses. Below, we will consider, as is usually the case, multiple testing procedures which can be written as a function of the p-value family $\mathbf{p} = (p_i, i \in \{1, ..., m\})$. More formally, we define a multiple testing procedure as a measurable function R, which takes as input a realization of the p-value family $\mathbf{p} \in [0,1]^m$ and which returns a subset $R(\mathbf{p})$ of $\{1, ..., m\}$, corresponding to the rejected hypotheses (i.e. $i \in R(\mathbf{p})$ means that the i-th hypothesis is rejected by R for the observed p-values $\mathbf{p}$).

Particular multiple testing procedures are step-up and step-down procedures. First define a *threshold* as any nondecreasing sequence $\mathbf{t} = (t_k)_{1 \leq k \leq m} \in [0,1]^m$ (with $t_0 = 0$ by convention).

Next, for any threshold $\mathbf{t}$, the *step-up procedure* of threshold $\mathbf{t}$, denoted here by $\mathrm{SU}(\mathbf{t})$, rejects the i-th hypothesis if $p_i \leq t_{\hat{k}}$, with $\widehat{k} = \max\{k \in \{0,1,...,m\} \mid p_{(k)} \leq t_k\}$, where $p_{(1)} \leq p_{(2)} \leq ... \leq p_{(m)}$ denote the ordered p-values (with the convention $p_{(0)} = 0$). In particular, the procedure $\mathrm{SU}(\mathbf{t})$ using $t_k = \alpha k/m$ corresponds to the standard linear step-up procedure of Benjamini and Hochberg (1995) [1], denoted here by LSU. A less rejecting procedure uses a step-down algorithm; for any threshold $\mathbf{t}$, the *step-down procedure* of threshold $\mathbf{t}$, denoted here by $\mathrm{SD}(\mathbf{t})$, rejects the i-th hypothesis if $p_i \leq t_{\tilde{k}}$, with $\widetilde{k} = \max\{k \in \{0,1,...,m\} \mid \forall k' \leq k,\ p_{(k')} \leq t_{k'}\}$. Analogously to the step-up case, the procedure $\mathrm{SD}(\mathbf{t})$ using $t_k = \alpha k/m$ is called the linear step-down procedure and is denoted here by LSD.

Next, associated to any multiple testing procedure R and any configuration of true/false hypotheses $H \in \{0,1\}^m$, we introduce the false discovery proportion (FDP) of R as the proportion of true hypotheses in the set of the rejected hypotheses, that is,

$$\mathrm{FDP}(R,H) = \frac{|\mathcal{H}_0(H) \cap R|}{|R| \vee 1}, \tag{1}$$

where $|\cdot|$ denotes the cardinality function. Then, while for any multiple testing procedure R, the false discovery rate (FDR) is defined as the mean of the FDP (see [1]), the power is (generally) defined as the expected number of correctly rejected hypotheses divided by the number of false hypotheses. Of course, the FDR and the power depend on the distribution that generates the p-values, and we may use the models defined in Section 2.1. Formally, for any distribution P coming from a conditional model using parameter $H \in \{0,1\}^m$, we let

$$\mathrm{FDR}(R,P) = \mathbb{E}_{\mathbf{p}\sim P}[\mathrm{FDP}(R(\mathbf{p}),H)], \tag{2}$$

$$\mathrm{Pow}(R,P) = m_1(H)^{-1}\, \mathbb{E}_{\mathbf{p}\sim P}\left[|\mathcal{H}_1(H) \cap R(\mathbf{p})|\right]. \tag{3}$$

Similarly, for any p-value distribution $\overline{P}$ coming from an unconditional model, the FDR and the Power use an additional averaging over $H \sim \mathcal{B}(1-\pi_0)^{\otimes m}$ and are defined by:

$$\mathrm{FDR}(R,\overline{P}) = \mathbb{E}_{(H,\mathbf{p})\sim\overline{P}}[\mathrm{FDP}(R(\mathbf{p}),H)], \tag{4}$$

$$\mathrm{Pow}(R,\overline{P}) = (\pi_1 m)^{-1}\, \mathbb{E}_{(H,\mathbf{p})\sim\overline{P}}\left[|\mathcal{H}_1(H) \cap R(\mathbf{p})|\right]. \tag{5}$$

Remark that, for convenience, (5) is not exactly defined as the expectation of (3), because of the denominator. It corresponds precisely to the expected number of correctly rejected hypotheses divided by the *expected* number of false hypotheses.

In the paper, to simplify the notation, we sometimes drop the explicit dependency in $\mathbf{p}$, H or P, writing e.g. R instead of $R(\mathbf{p})$, $\mathcal{H}_0$ instead of $\mathcal{H}_0(H)$, $\mathrm{FDP}(R)$ instead of $\mathrm{FDP}(R,H)$ and $\mathrm{FDR}(R)$ instead of $\mathrm{FDR}(R,P)$.

2.3. *Some notation and useful results.* For any $k \geq 0$ and any threshold $\mathbf{t} = (t_1,...,t_k)$, we denote

$$\Psi_k(\mathbf{t}) = \Psi_k(t_1,...,t_k) = \mathbb{P}\left[U_{(1)} \leq t_1, ..., U_{(k)} \leq t_k\right]. \tag{6}$$

where $(U_i)_{1\le i\le k}$ is a sequence of k variables i.i.d. uniform on $(0,1)$ and with the convention $\Psi_0(\cdot)=1$. In practice, quantity (6) can be evaluated using Bolshev's recursion $\Psi_k(\mathbf{t}) = 1-\sum_{i=1}^{k}\binom{k}{i}(1-t_{k-i+1})^i\Psi_{k-i}(t_1,...,t_{k-i})$ or Steck's recursion $\Psi_k(\mathbf{t})=(t_k)^k-\sum_{j=0}^{k-2}\binom{k}{j}(t_k-t_{j+1})^{k-j}\Psi_j(t_1,...,t_j)$ (see [28], p. 366-369). Additionally, the following relation holds (see Lemma 2.1 in [15]): for all $k\in\mathbb{N}$ and $\nu_1,\nu_2\in\mathbb{R}$ such that $0\le\nu_1+\nu_2\le\nu_1+k\nu_2\le 1$,

$$\Psi_k(\nu_1+\nu_2,...,\nu_1+k\nu_2)=(\nu_1+\nu_2)(\nu_1+(k+1)\nu_2)^{k-1}. \tag{7}$$

From the Ψ_k's, we define the following useful quantities: for any threshold $\mathbf{t}=(t_k)_{1\le k\le m}$ and $k\ge 0$, $k\le m$, we let

$$\mathcal{D}_m(\mathbf{t},k)=\binom{m}{k}(t_k)^k\Psi_{m-k}\big(1-t_m,...,1-t_{k+1}\big), \tag{8}$$

$$\widetilde{\mathcal{D}}_m(\mathbf{t},k)=\binom{m}{k}(1-t_{k+1})^{m-k}\Psi_k\big(t_1,...,t_k\big). \tag{9}$$

Above, note that $(t_k)^k$ and $(1-t_{k+1})^{m-k}$ are correct when $k=0$ and $k=m$, even if $(t_j)_j$ is only defined for $1\le j\le m$. Note that Bolshev's recursion provides $\sum_{k=0}^m\mathcal{D}_m(\mathbf{t},k)=\sum_{k=0}^m\widetilde{\mathcal{D}}_m(\mathbf{t},k)=1$ for any threshold $\mathbf{t}$.

Finally, we will use the so-called Stirling numbers of the second kind, defined as coefficients $\left\{{s \atop \ell}\right\}$ for $s,\ell\ge 1$ by $\left\{{s \atop 0}\right\}=0$, $\left\{{s \atop \ell}\right\}=0$ for $\ell>s$, $\left\{{1 \atop 1}\right\}=1$ and the recurrence relation: for all $1\le\ell\le s{+}1$, $\left\{{s+1 \atop \ell}\right\}=\ell\left\{{s \atop \ell}\right\}+\left\{{s \atop \ell-1}\right\}$. For instance, $\left\{{3 \atop 1}\right\}=1$, $\left\{{3 \atop 2}\right\}=3$, $\left\{{3 \atop 3}\right\}=1$, $\left\{{4 \atop 1}\right\}=1$,$\left\{{4 \atop 2}\right\}=7$, $\left\{{4 \atop 3}\right\}=6$, $\left\{{4 \atop 4}\right\}=1$. From a combinatorial point of view, the coefficient $\left\{{s \atop \ell}\right\}$ counts the number of ways to partition a set of s elements into ℓ (nonempty) subsets. The latter is useful to compute the s-th moment of a binomial distribution: if $X\sim\mathcal{B}(n,q)$, we have $\forall s\ge 1$, $\mathbb{E}[X^s]=\sum_{\ell=1}^{s\wedge n}\frac{n!}{(n-\ell)!}\left\{{s \atop \ell}\right\}q^\ell$.

3. New formulas.

3.1. *Unconditional independent model, $m\ge 2$.*

3.1.1. *Step-up case.* Let us consider the unconditional independent model. Finner and Roters (2002) derived the exact distribution of the rejection number of any step-up procedure in the case of i.i.d. uniform p-values (i.e., when all the hypotheses are true) [15]. In the unconditional model, the latter can be generalized as follows: denoting $G(t)=\pi_0F_0(t)+(1-\pi_0)F_1(t)$ the common c.d.f. of the p-values, we have for $0\le k\le m$ that

$$\mathbb{P}\left[|\mathrm{SU}(\mathbf{t})|=k\right]=\mathcal{D}_m\big([G(t_j)]_{1\le j\le m},k\big). \tag{10}$$

(this is straightforward from [15] because G is continuous increasing).

Next, for the procedure $R(t)=\{i\mid p_i\le t\}$ using a *constant* threshold $t\in[0,1]$, Storey (2003) proved that the distribution of $|\mathcal{H}_0(H)\cap R(t)|$ conditionally on $|R(t)|=k$ is a binomial distribution $\mathcal{B}\big(k, \pi_0 F_0(t)/G(t)\big)$ (see proof of Theorem 1 in [30], see also Proposition 2.1 in [7]). Later, Chi (2007) proved that the distribution of $|\mathcal{H}_0(H) \cap \text{LSU}|$ conditionally on $|\text{LSU}| = k$ is asymptotically binomial (in a particular "supercritical" framework), see [6]. Here, we show that the latter holds non-asymptotically, for any step-up procedure, which, by using (10), gives exact formulas for the c.d.f. of the FDP, the s-th moment of the FDP, the FDR and the Power.

THEOREM 3.1. *When testing $m \geq 2$ hypotheses, consider a step-up procedure $SU(\mathbf{t})$ with threshold $\mathbf{t}$ and the notation of Section 2.3. Then for any parameter $\pi_0 \in [0,1]$ and $F_1 \in \mathcal{F}$, denoting $G(t) = \pi_0 F_0(t) + \pi_1 F_1(t)$, we have under the generating distribution $(H, \mathbf{p}) \sim \overline{P}^I_{(\pi_0, F_1)}$ of the unconditional independent model, for any $k \geq 1$,*

$$|\mathcal{H}_0(H) \cap SU(\mathbf{t})| \ \textit{conditionally on}\ |SU(\mathbf{t})| = k \quad \sim \ \mathcal{B}\left(k, \frac{\pi_0 F_0(t_k)}{G(t_k)}\right). \tag{11}$$

In particular, we derive the following formulas, for any $x \in (0,1)$, for any $s \geq 1$, denoting by $\left\{ {s \atop \ell} \right\}$ the Stirling number of second kind and by $\lfloor z \rfloor$ the largest integer smaller than or equal to z,

$$\mathbb{P}[FDP(SU(\mathbf{t}), H) \leq x] = \sum_{k=0}^{m} \sum_{j=0}^{\lfloor xk \rfloor} \binom{k}{j} \left(\frac{\pi_0 F_0(t_k)}{G(t_k)}\right)^j \left(\frac{\pi_1 F_1(t_k)}{G(t_k)}\right)^{k-j} \mathcal{D}_m\big([G(t_j)]_{1 \leq j \leq m}, k\big) \tag{12}$$

$$\mathbb{E}[FDP(SU(\mathbf{t}), H)^s] = \sum_{\ell=1}^{s \wedge m} \frac{m!}{(m-\ell)!} \left\{ {s \atop \ell} \right\} \pi_0^\ell \sum_{k=\ell}^{m} \frac{F_0(t_k)^\ell}{k^s} \mathcal{D}_{m-\ell}\big([G(t_{j+\ell})]_{1 \leq j \leq m-\ell}, k-\ell\big) \tag{13}$$

$$FDR(SU(\mathbf{t}), \overline{P}^I_{(\pi_0, F_1)}) = \pi_0 m \sum_{k=1}^{m} \frac{F_0(t_k)}{k} \mathcal{D}_{m-1}\big([G(t_{j+1})]_{1 \leq j \leq m-1}, k-1\big) \tag{14}$$

$$Pow(SU(\mathbf{t}), \overline{P}^I_{(\pi_0, F_1)}) = \sum_{k=1}^{m} F_1(t_k)\, \mathcal{D}_{m-1}\big([G(t_{j+1})]_{1 \leq j \leq m-1}, k-1\big). \tag{15}$$

We can apply Theorem 3.1 in the case where $t_k = \alpha k/m$, to deduce the following results for the LSU procedure of Benjamini and Hochberg (1995) [1]: first, (14) leads to $\text{FDR(LSU)} = \pi_0 \alpha$, recovering the well known result of Benjamini and Yekutieli [3] in the unconditional model. Second, (15) provides the exact expression

$$\text{Pow(LSU)} = \sum_{k=1}^{m} F_1(\alpha k/m) \binom{m-1}{k-1} (G(\alpha k/m))^{k-1} \Psi_{m-k}\big(1 - G(\alpha m/m), ..., 1 - G(\alpha(k+1)/m)\big).$$

Glueck et al. (2008) have obtained an exact expression for the power of the LSU under independence (in the conditional model) [18], but the corresponding formula was reported to have a complexity exponential in m, which is intractable for large m. Here, we obtained a much less complex formula, requiring (at most) the computation of the function Ψ_m at one point of $\mathbb{R}^m$. Third, formula (13) used with $t_k = \alpha k/m$ is similar to expression (2.1) in Theorem 2.1 of [12], in which a formula for the s-th moment of the FDP of the LSU was investigated (in the conditional model). Our formula (13) uses additional factors $\left\{ {s \atop \ell} \right\}$. As soon as $s \geq 3$, they are definitely needed and they seem forgotten in [12]; for instance, taking $\alpha = 1$, the corresponding linear step-up procedure rejects all the hypotheses and (13) reduces to the computation of the s-th moment of a binomial distribution, which uses at least one $\left\{ {s \atop \ell} \right\} > 1$, see Section 2.3.

Fourth, expression (12) used with $t_k = \alpha k/m$ yields what is to our knowledge the first exact expression for the c.d.f. of FDP(LSU), valid for any $m \geq 2$ and for any alternative c.d.f. F_1. For instance, taking a typical Gaussian setting where $F_1(t) = \overline{\Phi}\left(\overline{\Phi}^{-1}(t) - \mu\right)$, we are able to evaluate the probability $\mathbb{P}(\mathrm{FDP(LSU)} \leq c\,\alpha)$ for $c \geq 1$; for $\mu = 3$, $\alpha = 0.05$, $\pi_0 = 1 - 1/\sqrt{m}$, expression (12) provides $\mathbb{P}(\mathrm{FDP(LSU)} \leq \alpha) \simeq 0.724$, $\mathbb{P}(\mathrm{FDP(LSU)} \leq 2\alpha) \simeq 0.787$ for $m = 100$ and $\mathbb{P}(\mathrm{FDP(LSU)} \leq \alpha) \simeq 0.557$, $\mathbb{P}(\mathrm{FDP(LSU)} \leq 2\alpha) \simeq 0.826$ for $m = 1000$. This means that the LSU procedure, designed to control the FDR at level α, can have a FDP larger than 2α with a "non-negligible" probability, for some admittedly quite standard values of the model parameters. As m tends to infinity while the model parameters stay constant with m, the FDP converges to the FDR and the latter effect vanishes (see e.g. [21]). However, we show in Section 4.2 that the convergence can be slow in the sparse case.

Alternatively, some authors are interested in procedures R controlling the FDP, i.e. satisfying $\mathbb{P}[\mathrm{FDP}(R) \leq \alpha] \geq 1 - \gamma$, see e.g. [17, 20]. As a matter of fact, by using (11), we directly deduce that the latter FDP control is satisfied by the step-up procedure $\mathrm{SU}(\mathbf{t}^\star)$ using the *oracle* threshold defined by $t^\star_{m+1} = 1$ (by convention) and for any $1 \leq k \leq m$,

$$t^\star_k = \max\{t \in [0, t^\star_{k+1}] \mid \mathbb{P}[X \leq \alpha k] \geq 1 - \gamma \text{ for } X \sim \mathcal{B}(k, \pi_0 t/G(t))\},$$

with $t^\star_k = 0$ if the above set is empty. However, the latter threshold is unknown in practice, because it depends on the c.d.f. G and an interesting issue is to estimate it. Chi and Tan (2008) introduced the threshold $t^{CT}_k = \max\{t \in [0, t^{CT}_{k+1}] \mid \mathbb{P}[X \leq \alpha k] \geq 1 - \gamma \text{ for } X \sim \mathcal{B}(k, 1 \wedge (mt/k))\}$ (with the convention $t^{CT}_{m+1} = 1$). As a matter of fact, the latter can be seen as the empirical substitute of $t^\star_k$, because $G(t_{\hat{k}}) \simeq \mathbb{G}_m(t_{\hat{k}}) = \hat{k}/m$ for any step-up procedure rejecting $\hat{k}$ hypotheses ($\mathbb{G}_m$ denoting the e.c.d.f. of the p-values). Using the latter threshold, they established an asymptotic FDP control (as m tends to infinity). Here, a plausible explanation is that their procedure correctly mimics the oracle $\mathrm{SU}(\mathbf{t}^\star)$ (asymptotically).

3.1.2. *Step-down case.* In that section, we still consider the unconditional independent model, but we focus on the step-down case. By contrast with the step-up case, for a step-down procedure, the distribution of $|\mathcal{H}_0(H) \cap \mathrm{SD}(\mathbf{t})|$ conditionally on $|\mathrm{SD}(\mathbf{t})| = k$ is not binomial, in general. For instance, we prove in Section 6.3 that for $k \geq 1$, $k \leq m$,

$$\mathbb{P}[|\mathcal{H}_0(H) \cap \mathrm{SD}(\mathbf{t})| = k \mid |\mathrm{SD}(\mathbf{t}) = k] = \pi_0^k \frac{\Psi_k\big(F_0(t_1), ..., F_0(t_k)\big)}{\Psi_k\big(G(t_1), ..., G(t_k)\big)} =: a^k \tag{16}$$

$$\mathbb{P}[|\mathcal{H}_0(H) \cap \mathrm{SD}(\mathbf{t})| = 0 \mid |\mathrm{SD}(\mathbf{t}) = k] = \pi_1^k \frac{\Psi_k\big(F_1(t_1), ..., F_1(t_k)\big)}{\Psi_k\big(G(t_1), ..., G(t_k)\big)} =: (1-b)^k, \tag{17}$$

and it turns out that $a = b$ only for particular situations, such as $t_1 = ... = t_k$, $F_1(x) = x$ or $\pi_0 \in \{0, 1\}$. Also, in the Dirac-uniform configuration $F_1 = 1$ (an thus $G(t) = \pi_0 t + \pi_1$), we establish in Section 6.3 that, for $1 \leq j \leq k$,

$$\mathbb{P}[|\mathcal{H}_0(H) \cap \mathrm{SD}(\mathbf{t})| = j \mid |\mathrm{SD}(\mathbf{t}) = k] = \binom{k}{j} \pi_0^j \pi_1^{k-j} \frac{\Psi_j\big(t_{k-j+1}, ..., t_k\big)}{\Psi_k\big(\pi_0 t_1 + \pi_1, ..., \pi_0 t_k + \pi_1\big)}, \tag{18}$$

which is not binomial in general (see also Remark 3.3 below).

An exact expression for $\mathbb{P}[|\mathcal{H}_0(H) \cap \mathrm{SD}(\mathbf{t})| = j \mid |\mathrm{SD}(\mathbf{t})| = k]$ only involving functions of the type Ψ_i and valid for any j, $\mathbf{t}$ and F_1 seems considerably more difficult to derive. As a consequence, we now only focus on the calculation of the FDR and of the power. For this, let us recall the exact formula for the distribution of $|\mathrm{SD}(\mathbf{t})|$: for all $k \in \{0, ..., m\}$,

$$\mathbb{P}\left[|\mathrm{SD}(\mathbf{t})| = k\right] = \widetilde{\mathcal{D}}_m\big([G(t_j)]_{1 \leq j \leq m}, k\big). \tag{19}$$

(see formula (4) p.344 of [28] and [15] which can be directly generalized in the unconditional independent model because G is continuous increasing). The next result, based on the calculation of the distribution of $|\mathrm{SD}(\mathbf{t}')|$ conditionally on $|\mathrm{SD}(\mathbf{t})| = k$ (with $t'_j = t_{j+1}$), connects the FDR to distributions of the type (19) (see the proof in Section 6.2).

THEOREM 3.2. *For $m \geq 2$ hypotheses, consider the unconditional independent model $\overline{P}^I_{(\pi_0, F_1)}$, a step-down procedure $SD(\mathbf{t})$ with threshold $\mathbf{t}$ and the notation of Section 2.3. Then for any parameter $\pi_0 \in [0, 1]$ and $F_1 \in \mathcal{F}$ (denoting $G(t) = \pi_0 F_0(t) + \pi_1 F_1(t)$), we have*

$$\begin{aligned} FDR(SD(\mathbf{t}), \overline{P}^I_{(\pi_0, F_1)}) = & \pi_0 m \sum_{k=1}^m \sum_{k'=k}^m \frac{F_0(t_k)}{k'} \widetilde{\mathcal{D}}_{m-1}\big((G(t_j))_{1 \leq j \leq m-1}, k-1\big) \\ & \times \widetilde{\mathcal{D}}_{m-k}\left(\left(\frac{G(t_{k+j}) - G(t_k)}{1 - G(t_k)}\right)_{1 \leq j \leq m-k}, \, k' - k\right), \end{aligned} \tag{20}$$

$$Pow(SD(\mathbf{t}), \overline{P}^I_{(\pi_0, F_1)}) = \sum_{k=1}^m F_1(t_k) \widetilde{\mathcal{D}}_{m-1}\big([G(t_j)]_{1 \leq j \leq m-1}, k-1\big). \tag{21}$$

To the best of our knowledge, (20) is the first exact expression for a step-down procedure that relies the FDR to the (joint) distribution of rejection numbers, for any p-value alternative distribution. The main tool to get this result is Lemma A.2.

One straightforward consequence of (20) is that $\pi_0 m \sum_{k=1}^m \frac{t_k}{k} \widetilde{\mathcal{D}}_{m-1}\big([G(t_j)]_{1 \leq j \leq m-1}, k-1\big)$ is an upper-bound of FDR(SD($\mathbf{t}$)), which in particular proves that FDR(SD($\mathbf{t}$)) is always smaller than FDR(SU($\mathbf{t}$)), as soon as t_k/k is nondecreasing in k. While the latter result should probably be considered as well known, it is not trivial because when increasing the rejection number, both numerator and denominator are increasing within the FDP expression (for further developments on this issue, see Theorem 2 in [32]).

To illustrate (20), we may consider the case where $t_k = \alpha k/m$, that is, we may compute exactly the FDR of the LSD procedure. In the Dirac-uniform model $F_1 = 1$ and using (7), we deduce the following expression:

$$\begin{aligned}\mathrm{FDR}(\mathrm{LSD}, \overline{P}^I_{(\pi_0,F_1=1)}) =& \pi_0 \frac{\alpha^2}{m}\left(\pi_1 + \pi_0 \frac{\alpha}{m}\right) \sum_{k=1}^{m}\sum_{j=k}^{m} \frac{k}{j}\binom{m-1}{k-1}\binom{m-k}{j-k}\pi_0^{m-k} \\ &\times \left(\pi_1 + \pi_0\frac{\alpha k}{m}\right)^{k-2}\left(\frac{\alpha(j-k+1)}{m}\right)^{j-k-1}\left(1-\frac{\alpha(j+1)}{m}\right)^{m-j}. \end{aligned} \tag{22}$$

Finally, let us emphasize that expression (20) is also useful to investigate least favorable configurations for the FDR of step-down procedures (see Section 4.1.1).

REMARK 3.3. *If we only focus on the Dirac-uniform configuration, the FDP distribution of a given procedure R (rejecting any p-value equals to 0) only depends on the distribution of $|R \cap \mathcal{H}_0(H)|$ (conditionally on H), because $|R\cap\mathcal{H}_0(H)| = |R| - m_1(H)$. As done in Section 3.7 of [8], this leads to exact computations of the FDR for step-up, step-down and more general step-up-down procedures, in the particular Dirac-uniform configuration. In comparison, our approach is valid for an arbitrary alternative c.d.f. F_1, while it intrinsically uses the exchangeability of the p-values (which requires to use an unconditional model).*

3.2. *Unconditional EMN model with nonnegative correlation, $m \geq 2$.* In this subsection, our goal is to obtain results similar to Theorem 3.1 and Theorem 3.2, but this time in the unconditional EMN model of parameters π_0, ρ and μ, with a nonnegative correlation $\rho \in [0,1]$. In that case, we easily see that the joint distribution of the p-values can be realized as follows: for all i, $p_i = \overline{\Phi}(\sqrt{\rho}\ \overline{\Phi}^{-1}(U) + \sqrt{1-\rho}\ \overline{\Phi}^{-1}(U_i) + \mu H_i)$, where U, $(U_i)_i$ are all i.i.d. uniform on $(0,1)$ (and independent of $(H_i)_i$). This idea can be traced back to Stuart (1958) [31] and Owen and Steck (1962) [22]. As a consequence, conditionally on $U = u$, the p-values follow the unconditional independent model of parameters π_0, $F_0(\cdot \mid u,\rho)$ and $F_1(\cdot \mid u, \rho)$ where we let

$$F_0(t \mid u,\rho) = \overline{\Phi}\left(\frac{\overline{\Phi}^{-1}(t) - \sqrt{\rho}\ \overline{\Phi}^{-1}(u)}{\sqrt{1-\rho}}\right), \ F_1(t \mid u,\rho) = \overline{\Phi}\left(\frac{\overline{\Phi}^{-1}(t) - \sqrt{\rho}\ \overline{\Phi}^{-1}(u) - \mu}{\sqrt{1-\rho}}\right) \tag{23}$$

for $\rho \in [0,1)$ and $F_0(t \mid u, 1) = \mathbf{1}\{u \leq t\}$, $F_1(t \mid u,1) = \mathbf{1}\{u \leq \overline{\Phi}(\overline{\Phi}^{-1}(t) - \mu)\}$ for $\rho = 1$. As a result, to obtain formulas valid in the unconditional EMN model, we may directly use formulas holding in the unconditional independent model (with the above modified c.d.f.'s) and using an additional integration over $u \in (0,1)$. Hence, we deduce from Theorem 3.1 and Theorem 3.2 the following result (the formulas are not fully written for short):

COROLLARY 3.4. *For $m \geq 2$ hypotheses, consider the unconditional EMN model $\overline{P}^N_{(\pi_0,\rho,\mu)}$ with parameters $\pi_0 \in [0,1]$, $\mu > 0$ and $\rho \in [0,1]$ and let $G(t|u,\rho) = \pi_0 F_0(t|u,\rho) + \pi_1 F_1(t|u,\rho)$ using notation (23). Then, for any threshold $\mathbf{t}$, under the generating distribution $(H,\mathbf{p}) \sim \overline{P}^N_{(\pi_0,\rho,\mu)}$,* *the quantity* $\mathbb{P}[FDP(SU(\mathbf{t})) \leq x]$ *(resp.* $\mathbb{E}[FDP(SU(\mathbf{t}))^s]$*;* $FDR(SU(\mathbf{t}))$*;* $Pow(SU(\mathbf{t}))$*) is given by the RHS of* (12) *(resp.*(13)*;* (14)*;* (15)*), by replacing* $F_0(\cdot)$ *by* $F_0(\cdot \mid u,\rho)$*,* $F_1(\cdot)$ *by* $F_1(\cdot \mid u,\rho)$ *and* $G(\cdot)$ *by* $G(\cdot \mid u,\rho)$*, and by integrating over* u *with respect to the Lebesgue measure on* $(0,1)$*. Additionally, a similar result holds for step-down procedures using* (20) *and* (21)*.*

In particular, applying Corollary 3.4 for the LSU procedure, we obtain that $\mathrm{FDR}(\mathrm{LSU}, \overline{P}^N_{(\pi_0,\rho,\mu)})$ equals:
(24)

$$\pi_0 \sum_{k=1}^{m} \binom{m}{k} \int_0^1 F_0(\alpha k/m|u,\rho) G(\alpha k/m|u,\rho)^{k-1} \Psi_{m-k}\big((1 - G(\alpha(m-j+1)/m|u,\rho))_{1\leq j\leq m-k}\big) du.$$

An expression for $\lim_m \mathrm{FDR}(\mathrm{LSU}, \overline{P}^N_{(\pi_0,\rho,\mu)})$ was provided by Finner et al. (2007), by considering the asymptotic framework where m tends to infinity [14]. We compared the latter to the formula (24) by plotting the graph corresponding to their Figure 3 (not reported here). The results are qualitatively the same for $\pi_0 < 1$, but present major differences when $\pi_0 = 1$ and ρ is small. This is in accordance with the simulations reported in the concluding remarks of Section 5 in [14]. Hence, the asymptotic analysis may not reflect what happens for a realistically finite m, which can be seen as a limitation with respect to our non-asymptotic approach. As illustration, when $\pi_0 = 1$, Finner et al. (2007) proved that $\lim_{\rho\to 0} \lim_m \mathrm{FDR}(\mathrm{LSU}) = \overline{\Phi}(\sqrt{-2\log\alpha}) < \alpha$ whereas we have $\lim_{\rho\to 0} \mathrm{FDR}(\mathrm{LSU}) = \alpha$, as remarked in [14] using simulations and as proved formally in the next result.

COROLLARY 3.5. *For any* $m \geq 2$ *and for any threshold* $\mathbf{t}$*, the quantities* $FDR(SU(\mathbf{t}), \overline{P}^N_{(\pi_0,\rho,\mu)})$ *and* $FDR(SD(\mathbf{t}), \overline{P}^N_{(\pi_0,\rho,\mu)})$ *are continuous in any* $\pi_0 \in [0,1]$*, any* $\rho \in [0,1]$ *and any* $\mu > 0$*.*

Corollary 3.5 is a straightforward consequence of Corollary 3.4; to prove the continuity in $\rho = 1$, we may remark that for any u outside the set $\mathcal{S} = \{t_k, 1 \leq k \leq m\} \cup \{\overline{\Phi}(\overline{\Phi}^{-1}(t_k) - \mu), 1 \leq k \leq m\}$ of zero Lebesgue measure, the functions $F_0(t \mid u,\rho)$ and $F_1(t \mid u,\rho)$ are continuous in $\rho = 1$.

In particular, Corollary 3.5 shows that the limit of the FDR when ρ tends to 1 is given by the degenerated case $\rho = 1$. In the latter case, the FDR is particularly easy to compute because only one Gaussian variable is effective: for step-up procedures, $\mathrm{FDR}(\mathrm{SU}(\mathbf{t})) = \pi_0 t_m$; and for step-down procedures, $\mathrm{FDR}(\mathrm{SD}(\mathbf{t})) = \sum_{k=1}^m \binom{m}{k} \pi_0^k \pi_1^{m-k} \frac{k}{m} \min\{t_{m-k+1}, \overline{\Phi}(\overline{\Phi}^{-1}(t_1) - \mu)\}$ (the proof is left to the reader). For instance, under the special p-value configuration where $\rho = 1$ and $\pi_0 = 1$, the above FDR expressions yield $\mathrm{FDR}(\mathrm{SU}(\mathbf{t})) = t_m$ and $\mathrm{FDR}(\mathrm{SD}(\mathbf{t})) = t_1$. Thus, as $\rho \simeq 1$, the FDR value may considerably change as one considers a step-up or a step-down algorithm.

Going back to Corollary 3.4, let us mention that the latter can be used in order to evaluate the FDR control robustness under Gaussian equicorrelated positive dependence for any procedure (step-up or step-down) that controls the FDR under independence. For instance, the *adaptive* procedures of Blanchard and Roquain (2008) [5] (step-up using $t_k = \alpha \min\{1, (1-\alpha)k/(m-k+$

1)}) and Finner et al. (2009) [13] (step-up based upon $t_k = \alpha k/(m-(1-\alpha)k)$) have been proved to control the FDR at level α under independence (asymptotically for [13]). A simulation study was done in [5] in order to check if their respective FDR is still below α (or at least close to α) in the EMN model. Using our exact approach, we are able to reproduce their analysis without the errors due to the Monte-Carlo approximation. However, we underline that our approach uses *non-random* thresholds $\mathbf{t}$; this is not always the case for adaptive procedures (see e.g. [2, 5]).

3.3. *EMN model with a general correlation and $m = 2$.* When the correlation ρ is negative, the approach presented in the last section is not valid anymore and the problem seems considerably more difficult to tackle. We propose in this section to focus on the case where only two hypotheses are tested, which should hopefully give some hints concerning the behavior of the FDR under negative correlations for larger m. The next result follows from elementary integration and does not require the use of an unconditional model.

PROPOSITION 3.6. *For $m = 2$ hypotheses, consider the conditional EMN model $P^N_{(H,\rho,\mu)}$ with parameters $H = (H_1, H_2) \in \{0,1\}^2$ (generating $m_0 \in \{1,2\}$ true null hypotheses), $\rho \in [-1,1]$ and $\mu > 0$. Consider a threshold $\mathbf{t} = (t_1, t_2)$. Let $z_1 = \overline{\Phi}^{-1}(t_2)$ and $z_2 = \overline{\Phi}^{-1}(t_1)$. Then FDR($SU(\mathbf{t}), P^N_{(H,\rho,\mu)}$) is given by*

$$
\begin{array}{|l|l|}
\hline
\rho \in (-1,1), m_0 = 1 & \frac{1}{2}\int_0^{\overline{\Phi}(z_1-\mu)} \overline{\Phi}\left(\frac{z_1-\rho\overline{\Phi}^{-1}(w)}{\sqrt{1-\rho^2}}\right) dw + \int_{\overline{\Phi}(z_1-\mu)}^1 \overline{\Phi}\left(\frac{z_2-\rho\overline{\Phi}^{-1}(w)}{\sqrt{1-\rho^2}}\right) dw \\
\rho \in (-1,1), m_0 = 2 & t_1 + \int_{t_1}^{t_2} \overline{\Phi}\left(\frac{z_1-\rho\overline{\Phi}^{-1}(w)}{\sqrt{1-\rho^2}}\right) dw + \int_{t_2}^1 \overline{\Phi}\left(\frac{z_2-\rho\overline{\Phi}^{-1}(w)}{\sqrt{1-\rho^2}}\right) dw \\
\hline
\rho = -1, m_0 = 1 & \begin{cases} t_1 & \text{if } 0 < \mu \le 2z_1 \\ t_1 + \frac{1}{2}t_2 - \frac{1}{2}\overline{\Phi}(\mu - z_1) & \text{if } 2z_1 < \mu < z_1 + z_2 \\ \frac{1}{2}t_2 + \frac{1}{2}\overline{\Phi}(\mu - z_1) & \text{if } \mu \ge z_1 + z_2 \end{cases} \\
\rho = -1, m_0 = 2 & \begin{cases} 2t_1 & \text{if } 1/2 \ge t_2 \\ 2(t_1+t_2) - 1 & \text{if } 1/2 < t_2, t_1 + t_2 \le 1 \\ 1 & \text{if } 1/2 < t_2, t_1 + t_2 > 1 \end{cases} \\
\hline
\rho = 1, m_0 = 1 & \frac{1}{2}t_2 \\
\rho = 1, m_0 = 2 & t_2 \\
\hline
\end{array}, \tag{25}
$$

and $FDR(SD(\mathbf{t}), P^N_{(H,\rho,\mu)})$ *is given by*

(26)

$\rho \in (-1,1), m_0 = 1$	$\frac{1}{2}\int_0^{\overline{\Phi}(z_2-\mu)} \overline{\Phi}\left(\frac{z_1-\rho\overline{\Phi}^{-1}(w)}{\sqrt{1-\rho^2}}\right) dw + \frac{1}{2}\int_{\overline{\Phi}(z_2-\mu)}^{\overline{\Phi}(z_1-\mu)} \overline{\Phi}\left(\frac{z_2-\rho\overline{\Phi}^{-1}(w)}{\sqrt{1-\rho^2}}\right) dw + \int_{\overline{\Phi}(z_1-\mu)}^{1} \overline{\Phi}\left(\frac{z_2-\rho\overline{\Phi}^{-1}(w)}{\sqrt{1-\rho^2}}\right) dw$
$\rho \in (-1,1), m_0 = 2$	$t_1 + \int_{t_1}^{1} \overline{\Phi}\left(\frac{z_2-\rho\overline{\Phi}^{-1}(w)}{\sqrt{1-\rho^2}}\right) dw$
$\rho = -1, m_0 = 1$	$\begin{cases} t_1 & \text{if } 0 < \mu \leq z_1 + z_2 \\ \frac{1}{2}(t_1+t_2) - \frac{1}{2}\overline{\Phi}(\mu - z_2) + \frac{1}{2}\overline{\Phi}(\mu - z_1) & \text{if } z_1 + z_2 < \mu < 2z_2 \\ \frac{1}{2}t_2 + \frac{1}{2}\overline{\Phi}(\mu - z_1) & \text{if } \mu \geq 2z_2 \end{cases}$
$\rho = -1, m_0 = 2$	$\min(2t_1, 1)$
$\rho = 1, m_0 = 1$	$\frac{1}{2}\min(t_2, \overline{\Phi}(z_2 - \mu))$
$\rho = 1, m_0 = 2$	t_1

4. Application to least/most favorable configurations.

4.1. *Least favorable configurations for the FDR.* In order to study the FDR control, an interesting multiple testing issue is to determine which are the values of the model parameter F_1 (or μ) for which the FDR is maximum. The latter is called a least favorable configuration (LFC) for the FDR.

4.1.1. *Independent model.* Let us focus on the unconditional independent model. For a *step-up* procedure, expression (14) can be seen as $\pi_0 m \mathbb{E}_0[t_{\hat{k}}/\hat{k}]$ (where $\hat{k} = |\mathrm{SU}([G(t_{j+1})]_{1\leq j\leq m-1})|+1$ and $\mathbb{E}_0$ denotes the expectation with respect to $m-1$ i.i.d. uniform p-values). This shows that the behavior of the function $k \mapsto t_k/k$ is crucial to determine the LFC's of the FDR. Namely, if F_1 and F_1' are two c.d.f.'s such that for all $t \in [0,1]$, $F_1(t) \leq F_1'(t)$, then we have $\mathrm{FDR}(\mathrm{SU}(\mathbf{t}), \overline{P}^I_{(\pi_0,F_1)}) \leq \mathrm{FDR}(\mathrm{SU}(\mathbf{t}), \overline{P}^I_{(\pi_0,F_1')})$ when t_k/k is nondecreasing, while we have $\mathrm{FDR}(\mathrm{SU}(\mathbf{t}), \overline{P}^I_{(\pi_0,F_1)}) \geq \mathrm{FDR}(\mathrm{SU}(\mathbf{t}), \overline{P}^I_{(\pi_0,F_1')})$ when t_k/k is nonincreasing. The latter recovers a well known result of Benjamini and Yekutieli (2001) which was initially established in the conditional model, see Theorem 5.3 in [3]. As a consequence, the LFC for the FDR is either $F_1 = 1$ (Dirac-uniform) when t_k/k is nondecreasing, or $F_1 = 0$ ($F_1(x) = x$ if we only look at concave c.d.f.'s) when t_k/k is nonincreasing. In the border case of a linear threshold, the FDR does not depend on F_1 (e.g. it is equal to $\pi_0\alpha$ for the LSU), hence any configuration is a LFC.

An open problem is to determine the LFC's of a *step-down* procedure using a given threshold $\mathbf{t} = (t_j)_{1\leq j\leq m}$. Here, we introduce a new condition on the threshold $\mathbf{t}$ which provides that the Dirac-uniform configuration is a LFC for the FDR of the corresponding *step-down* procedure.

We define for any threshold $\mathbf{t} = (t_j)_{1\leq j\leq m}$ the following condition:

$$(A) \qquad k \in \{1,...,m\} \mapsto \sum_{i=0}^{m-k} \frac{t_k}{k+i} \widetilde{\mathcal{D}}_{m-k}\left(\left(\frac{t_{k+j}-t_k}{1-t_k}\right)_{1\leq j\leq m-k}, i\right) \text{ is nondecreasing.}$$

We now present the main result of this section, which uses Theorem 3.2 and is proved in Section 6.5.

THEOREM 4.1. *For $m \geq 2$ hypotheses, consider the unconditional independent model $\overline{P}^I_{(\pi_0,F_1)}$ and a step-down procedure SD($\mathbf{t}$) with a threshold $\mathbf{t} = (t_j)_{1\leq j\leq m}$ satisfying (A). Then for any $\pi_0 \in [0,1]$ and concave c.d.f. $F_1 \in \mathcal{F}$, we have*

$$FDR(SD(\mathbf{t}), \overline{P}^I_{(\pi_0,F_1)}) \leq FDR(SD(\mathbf{t}), \overline{P}^I_{(\pi_0,F_1=1)}), \tag{27}$$

meaning that the Dirac-uniform distribution is a least favorable configuration for the FDR of SD($\mathbf{t}$). Moreover, for $\alpha \in (0,1)$, the linear threshold $\mathbf{t} = (\alpha j/m)_{1\leq j\leq m}$ satisfies (A) and thus (27) holds for the linear step-down procedure LSD.

While condition (A) may be somehow difficult to state formally, it is very easy to check numerically because it only involves a finite set of real numbers. For instance, considering the threshold of Gavrilov et al. (2009) $t_k = \alpha k/(m+1-(1-\alpha)k)$ (for which the step-down procedure controls the FDR, see [16]), we may see that $(t_k)_k$ satisfies (A) for each $(\alpha, m) \in \{0.01, 0.05, 0.1, 0.2, 0.5, 0.9\} \times \{5, 10, 50, 100\}$, for instance. In fact, we were not able to find a value of (α, m) for which the corresponding threshold does not satisfy (A) (unfortunately, we have yet no formal argument proving (A) for any value of (α, m)). As a consequence, Theorem 4.1 states that the LFC for the procedure of Gavrilov et al. (2009) is still the Dirac-uniform configuration (over the class of concave c.d.f.'s), at least for the previously listed values of (α, m), which is a new interesting finding.

In comparison with the step-up case, for which the standard condition "$k \mapsto t_k/k$ nondecreasing" provides that the Dirac-uniform configuration is a LFC, the new sufficient condition (A) in the step-down case may be written as " $k \mapsto t_k/k \,.\, \Psi(\mathbf{t}, k)$ is nondecreasing", where $\Psi(\mathbf{t}, k) = \sum_{i=0}^{m-k} \frac{k}{k+i} \widetilde{\mathcal{D}}_{m-k}\left(\left(\frac{t_{k+j}-t_k}{1-t_k}\right)_{1\leq j\leq m-k}, i\right)$. It turns out that the additional function Ψ has a quite complex behavior, not necessarily connected to the one of t_k/k, so that there is no general relation between (A) and "$k \mapsto t_k/k$ nondecreasing"; for instance, on the one hand, (A) does not hold for the piecewise linear threshold defined by $t_k = \alpha pk/m$ for $1 \leq k \leq a$ and by $t_k = \alpha(pa-m)((a-m)m)^{-1}k + \alpha(1-(pa-m)(a-m)^{-1})$ for $a+1 \leq k \leq m$ (using e.g. $m = 50$, $p = 0.6$, $a = 4$, $\alpha = 0.5$), while t_k/k is nondecreasing. On the other hand, (A) holds for $t_k = 0.9\,(k/m)^{9/10}$ (using e.g. $m = 50$) while t_k/k is decreasing. In particular, for the latter threshold and considering only the set of concave c.d.f.'s, a LFC for FDR(SU($\mathbf{t}$)) is $F_1(x) = x$ while a LFC for FDR(SD($\mathbf{t}$)) is $F_1 = 1$ (and we checked numerically using Gaussian models that $F_1(x) = x$ is not a LFC for $\mathrm{FDR}(\mathrm{SD}(\mathbf{t}))$). This puts forward the complexity of the issue: whether we consider a step-up or a step-down procedure, the LFC's for the FDR may be different for some thresholds (e.g. $t_k = 0.9\,(k/m)^{9/10}$, $m = 50$) and they may coincide for some other thresholds (e.g. the one of [16] for suitable (α, m)).

4.1.2. *EMN model.* When the p-values follow the EMN model, Finner et al. (2007) conjectured that a LFC for the FDR of the LSU is still the Dirac-uniform distribution (see Section 1 in [14]). Here, we support this conjecture when $\rho \geq 0$ but we disprove it when $\rho < 0$.

In order to investigate this issue, we reported on Figure 1 the FDR of the LSU procedure against μ in the EMN model when $\rho > 0$ (left) and when $\rho < 0$ (right), by using Corollary 3.4 and Proposition 3.6. Under positive correlation, although each curve is not necessarily nondecreasing

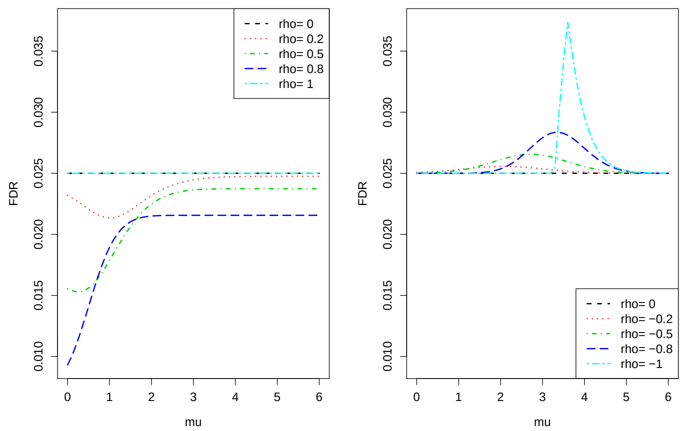

FIG 1. *FDR(LSU) against the mean* μ. *Left:* $\rho \geq 0$ *unconditional EMN model* $m = 100$ *and* $\pi_0 = 0.5$. *Right:* $\rho < 0$ *conditional EMN model with* $m = 2$ *and* $m_0 = 1$. $\alpha = 0.05$.

(e.g. for $\rho = 0.2$), the case $\mu = \infty$, close to the right most point of Figure 1, seems to be a LFC. A challenging problem is to state the latter formally. Under negative correlation and $m = 2$, however, $\mu = \infty$ is not a LFC anymore. As a matter of fact, in the case where $m = 2$, $m_0 = 1$ and $\mu = \infty$, the two p-values are independent (one p-value equals 0), so that the FDR equals $\alpha/2 = \alpha m_0/m$ which is not a maximum for the FDR, as we will show below.

Qualitatively, we observed the same behavior concerning the FDR of the LSD procedure.

Under negative correlation, the Dirac-uniform is not a LFC for the FDR and we can therefore legitimately ask what are the LFC's in that case. Here, we propose to solve this problem when $m = 2$ in the conditional EMN model. Let $z_1 = \overline{\Phi}^{-1}(\alpha)$ and $z_2 = \overline{\Phi}^{-1}(\alpha/2)$ and first consider the LSU procedure. Its FDR is plotted in Figure 2 (top). When $m_0 = 1$, we can check that

$(\rho, \mu) = (-1, z_1 + z_2)$ is a LFC because applying (25), the corresponding FDR is $3\alpha/4$, which equals the Benjamini-Yekutieli's (BY) upper-bound $(1+1/2+...+1/m)\alpha m_0/m$ [3] (valid under any dependency). Interestingly, in general, Guo and Rao (2008) states that the BY bound can be fulfilled using very specific dependency structures between the p-values (not necessarily including those coming from a EMN model) [19]. Here, we remark that the maximum value of the FDR still equals the BY bound for $m_0 = 1$, even for p-values coming from a EMN model. Next, for $m_0 = 2$, we may differentiate the corresponding expression in (25) in order to obtain that, assuming $\alpha \le 1/2$, the FDR (that does not depend on μ) attains its maximum in $\rho = (-z_1(z_1 - z_2) - \{(z_1^2 - z_1 z_2)^2 + 2\log(2)(z_1^2 - z_2^2) + 4\log(2)\}^{1/2})/(2\log(2)) \in (-1, 0)$.

Second, we consider the LSD procedure, whose FDR is plotted in Figure 2 (bottom). In the case where $m_0 = 2$, by differentiating the corresponding expression in (25) (that does not depend on μ), we are able to state that the FDR attains its maximum at $\rho = -1$ and that the value of the maximum is α. In particular, the FDR of the LSD procedure is always smaller than or equal to α when $m = 2$, in the conditional EMN model (for any m_0) and thus also in the unconditional EMN model, even for a negative correlation. An interesting open problem is to know whether this holds for larger m.

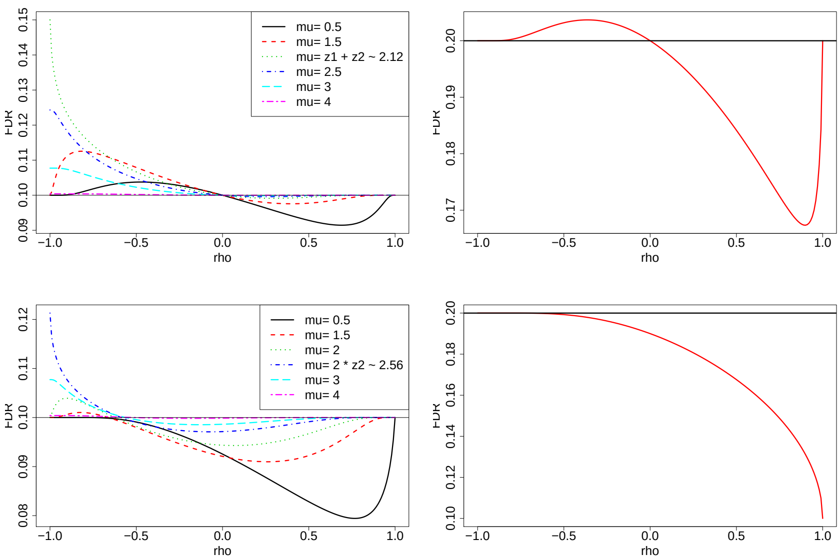

FIG 2. *FDR against the covariance $-1 \le \rho \le 1$ in the conditional EMN model. $m = 2$; $\alpha = 0.2$. Left: $m_0 = 1$, right: $m_0 = 2$. Top: LSU, bottom: LSD.*

REMARK 4.2. *Reiner-Benaim (2007) also studied the value of μ maximizing the FDR in the case $m = 2$ in the (conditional) EMN model with possibly negative correlation [23]. The latter work focused on the two-sided testing with $\rho \in \{-1, 1\}$, $m_0 = 1$ and $m = 2$.*

4.2. *Least/most favorable configurations for the variance of the FDP.* We focus here on the unconditional independent model and on the LSU procedure. Using (13) with $s = 2$, we easily derive the following expression for the variance of the FDP:

$$\mathbb{V}[\mathrm{FDP}(\mathrm{LSU})] = \alpha\pi_0 \sum_{k=1}^{m} \frac{1}{k}\mathcal{D}_{m-1}\big([G(\alpha(j+1)/m)]_{1\le j\le m-1}, k-1\big) - (\alpha\pi_0)^2/m. \tag{28}$$

As a consequence, by contrast with the FDR which is constantly equal to $\pi_0\alpha$ in that case, the variance of the FDP depends on the alternative p-value c.d.f. F_1. Moreover, since the sum in (28) equals $\mathbb{E}_0[(|\mathrm{SU}([G(\alpha(j+1)/m)]_{1\le j\le m-1})|+1)^{-1}]$ (where $\mathbb{E}_0$ denotes the expectation with respect to $m-1$ i.i.d. uniform p-values), the smaller F_1 (point-wise), the larger the variance. Therefore, over the set $F_1 \in \mathcal{F}$, the least favorable configuration for the variance (that is, the configuration where the variance is the largest) is given by $F_1 = 0$ while the most favorable configuration (that is, the configuration where the variance is the smallest) is the Dirac-uniform configuration $F_1 = 1$. Over the more "realistic" c.d.f. sets $\mathcal{F}' = \{F_1 \in \mathcal{F} \mid \forall x \in (0,1), F_1(x) \ge x\}$ and $\mathcal{F}_\varepsilon = \{F_1 \in \mathcal{F} \mid \forall x \in (0,1), F_1(x) \ge \varepsilon\}$, $0 < \varepsilon \le 1$ the least favorable configurations for the variance are given respectively by $F_1(x) = x$ and $F_1(x) = \varepsilon$. For these extreme configurations, the expression of the variance (28) can be simplified by using the next formula (proved in Section 6.6): for any threshold of the form $t_k = \beta + k\gamma$, $1 \le k \le m$, with $\beta, \gamma \ge 0$,

$$\mathbb{E}_0[(|\mathrm{SU}(\mathbf{t})|+1)^{-1}] = \frac{1}{\gamma-\beta}\left[\frac{(1+\gamma-\beta)^{m+1}-1}{m+1} - \gamma\big[(1+\gamma-\beta)^m - 1\big]\right] \tag{29}$$

for $\gamma \ne \beta$ and $\mathbb{E}_0[(|\mathrm{SU}(\mathbf{t})|+1)^{-1}] = 1 - m\gamma$ otherwise (where $\mathbb{E}_0$ denotes the expectation with respect to m i.i.d. uniform p-values) This leads to the following result.

THEOREM 4.3. *Consider the linear step-up procedure LSU in the unconditional independent model with parameters π_0 and F_1. Then for any $m \ge 2$, $\alpha \in (0,1)$, $\pi_0 \in [0,1]$ and $\varepsilon \in (0,1]$, under the generating distribution $(H, \mathbf{p}) \sim \overline{P}^I_{(\pi_0,F_1)}$, the following holds:*

$$\min_{F_1\in\mathcal{F}}\{\mathbb{V}[FDP(LSU)]\} = \min_{F_1\in\mathcal{F}'}\{\mathbb{V}[FDP(LSU)]\} = \min_{F_1\in\mathcal{F}_\varepsilon} \mathbb{V}[FDP(LSU)]\}$$

$$= \frac{\alpha\pi_0}{m}\frac{1-\pi_0^m}{1-\pi_0} - \frac{(\alpha\pi_0)^2}{m}\left(\frac{1-\pi_0^{m-1}}{1-\pi_0} + 1\right) \tag{30}$$

$$\max_{F_1\in\mathcal{F}}\{\mathbb{V}[FDP(LSU)]\} = \alpha\pi_0(1-\alpha\pi_0)$$

$$\max_{F_1\in\mathcal{F}'}\{\mathbb{V}[FDP(LSU)]\} = \alpha\pi_0(1-\alpha) + (1-\pi_0)\frac{\pi_0\alpha^2}{m}$$

$$\max_{F_1\in\mathcal{F}_\varepsilon}\{\mathbb{V}[FDP(LSU)]\} = \frac{\alpha\pi_0}{m}\frac{1-(1-(1-\pi_0)\varepsilon)^m}{(1-\pi_0)\varepsilon} - \frac{(\alpha\pi_0)^2}{m}\left(\frac{1-(1-(1-\pi_0)\varepsilon)^{m-1}}{(1-\pi_0)\varepsilon} + 1\right).$$

The proof is made in Section 6.6. Using Theorem 4.3, we are able to investigate the following asymptotic multiple testing issue: does the FDP converge to the FDR as m grows? Establishing the latter is crucial because when one establishes FDR $\leq \gamma$, one implicitly wants that for the observed realization ω, the control $\text{FDP}(\omega) \leq \gamma'$ still holds for $\gamma' \simeq \gamma$ (at least with high probability and when m is large). Here, the variance measure the L^2 distance between the FDP and the FDR, and since FDP $\in [0,1]$ the latter distance tends to zero if and only if the FDP converges to the FDR in probability. First, if $\pi_0 \in (0,1)$ does not depend on m, the convergence holds over the set of c.d.f.'s $\mathcal{F}_\varepsilon$, with a distance $(\mathbb{E}(\text{FDP} - \text{FDR})^2)^{1/2}$ converging to zero at rate $1/\sqrt{m}$. This corroborates previous asymptotic studies in the so-called "subcritical" case (see e.g. [6, 21]). By contrast, when considering the classes $\mathcal{F}$ and $\mathcal{F}'$ the convergence does not hold in the least favorable configurations $F_1(x) = 0$ and $F_1(x) = x$, respectively. The latter is quite intuitive because the denominator in the FDP does not converge to infinity anymore in that cases (see e.g. [15]), so these configurations can probably be considered as "marginal". Second, our non-asymptotic approach allows to make π_0 depends on m in the following way $1 - \pi_0 = 1 - \pi_{0,m} \sim m^{-\beta}$ with $0 < \beta \leq 1$, which corresponds to a classical "sparse" setting (see e.g. [9]). Expression (30) implies that, in this sparse case, the variance is always larger than a quantity of order $1/m^{1-\beta}$. In particular, when $1 - \pi_{0,m} \sim 1/m$, for any alternative c.d.f. F_1, the FDP does not converge to the FDR, and when $1 - \pi_{0,m} \sim m^{-\beta}$ with $0 < \beta < 1$, for all F_1, the convergence of the FDP towards the FDR is of order slower than $1/m^{(1-\beta)/2}$ (in L^2 norm). As illustration, for $m = 10000$, $1 - \pi_0 = 1/100$ and $\alpha = 0.05$, expression (30) gives $(\mathbb{E}(\text{FDP} - \text{FDR})^2)^{1/2} \geq 0.0217$, so the FDP has a distribution quite spread around the FDR $= \pi_0\alpha \simeq 0.05$. As a conclusion, considering a sparse signal slows down the convergence of the FDP to the FDR, so any FDR control should be interpreted with cautious, even in this very standard framework (independent p-values with the LSU procedure).

5. Extensions. Our approach is also useful to study the false non-discovery proportion (FNDP), that is, the proportion of false hypotheses among the non-rejected hypotheses, and in particular the false non-discovery rate (FNR), defined as the average of the FNDP. For this, we use the following duality property between step-up and step-down procedures: pointwise, the hypotheses rejected by $\text{SD}(\mathbf{t}, \mathbf{p})$ are exactly the hypotheses non-rejected by $\text{SU}(\overline{\mathbf{t}}, \overline{\mathbf{p}})$ with $\overline{p}_i = 1 - p_i$ and $\overline{t}_r = 1 - t_{m-r+1}$. Hence the distribution of the FNDP of a step-down procedure can be deduced from the distribution of the FDP of a step-up procedure. Precisely, for $0 \leq k \leq m-1$, the property (11) implies that the distribution of the erroneous non-rejection number $|\mathcal{H}_1(H) \cap (\text{SD}(\mathbf{t}))^c|$ conditionally on $|\text{SD}(\mathbf{t})| = k$ is binomial with parameters $m - k$ and $\pi_1(1 - F_1(t_{k+1}))/(1 - G(t_{k+1}))$. In particular, this leads to

$$\text{FNR}(\text{SD}(\mathbf{t}), \overline{P}^I_{\pi_0,F_1}) = m\pi_1 \sum_{k=0}^{m-1} \frac{1 - F_1(t_{k+1})}{m-k} \widetilde{\mathcal{D}}_{m-1}((G(t_j))_{1\leq j\leq m-1}, k).$$

Moreover, applying once more the duality property between step-up and step-down, we deduce from Section 3.1.2 that for a step-up procedure, the distribution of the erroneous non-rejection number conditional on the rejection number is not binomial, in general, while we can still obtain an explicit expression for the FNR.

Finally, since our formulas depend on the true parameters of the model, which are in general unknown in a statistical approach, one may formulate the concern of estimating these quantities in our formulas. We did not investigate in detail this issue as it would exceed the scope of this paper. Here, we simply notice that plugging convergent estimators of the parameters in our formulas will lead to convergent estimators for the corresponding quantity (e.g. $\mathbb{P}(\text{FDP} \leq x)$, FDR or Power), because our formulas are continuous in all the model parameters.

6. Proofs.

6.1. *Proof of Theorem 3.1.* Let us first prove (11) by computing the joint distribution of $|\mathcal{H}_0(H) \cap \text{SU}(\mathbf{t})|$ and $|\text{SU}(\mathbf{t})|$. In the independent unconditional model, we may use the exchangeability of $(H_i, p_i)_i$ to obtain for any $0 \leq j \leq \ell \leq m$,

$$\begin{aligned}
&\mathbb{P}[|\mathcal{H}_0(H) \cap \text{SU}(\mathbf{t})| = j, |\text{SU}(\mathbf{t})| = \ell] \\
&= \binom{\ell}{j}\binom{m}{\ell}\mathbb{P}[\mathcal{H}_0(H) \cap \text{SU}(\mathbf{t}) = \{1, ..., j\}, \text{SU}(\mathbf{t}) = \{1, ..., \ell\}] \\
&= \binom{\ell}{j}\binom{m}{\ell}\mathbb{P}[\text{SU}(\mathbf{t}) = \{1, ..., \ell\}, H_1 = ... = H_j = 0, H_{j+1} = ... = H_\ell = 1].
\end{aligned}$$

Next, by definition of a step-up procedure, we have $\text{SU}(\mathbf{t}) = \{i \mid p_i \leq t_{\hat{k}}\}$ with $\hat{k} = |\text{SU}(\mathbf{t})|$ (expression related to the "self-consistency" condition introduced in [4]). Using Lemma A.1, if $\hat{k}'_{(\ell)}$ denotes the number of rejections of the step-up procedure of threshold $(t_{j+\ell})_{1\leq j\leq m-\ell}$ over $m - \ell$ hypotheses and using the p-values $p_{\ell+1}$, ..., p_m, we have

$$\begin{aligned}
\text{SU}(\mathbf{t}) = \{1, ..., \ell\} &\Longleftrightarrow p_1 \leq t_\ell, ..., p_\ell \leq t_\ell,\ \hat{k} = \ell \\
&\Longleftrightarrow p_1 \leq t_\ell, ..., p_\ell \leq t_\ell,\ \hat{k}'_{(\ell)} = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{P}[|\mathcal{H}_0(H) \cap \text{SU}(\mathbf{t})| = j, |\text{SU}(\mathbf{t})| = \ell] \\
&= \binom{\ell}{j}\binom{m}{\ell}\mathbb{P}[p_1 \leq t_\ell, ..., p_\ell \leq t_\ell,\ \hat{k}'_{(\ell)} = 0, H_1 = ... = H_j = 0, H_{j+1} = ... = H_\ell = 1] \\
&= \binom{\ell}{j}\binom{m}{\ell}\mathbb{P}[p_1 \leq t_\ell, ..., p_j \leq t_\ell, H_1 = ... = H_j = 0] \\
&\quad \times \mathbb{P}[p_{j+1} \leq t_\ell, ..., p_\ell \leq t_\ell, H_{j+1} = ... = H_\ell = 1]\, \mathbb{P}[\hat{k}'_{(\ell)} = 0] \\
&= \binom{\ell}{j}\binom{m}{\ell}(\pi_0 F_0(t_\ell))^j (\pi_1 F_1(t_\ell))^{\ell-j}\Psi_{m-\ell}\big(1 - G(t_m), ..., 1 - G(t_{\ell+1})\big),
\end{aligned}$$

where we used the independence between the (H_i, p_i) in the second equality. This leads to (11) and then to (12). For (13), we use the Stirling numbers of second kind and the formula of the s-th moment of a binomial distribution of Section 2.3. Expression (14) is a direct consequence of (13) for $s = 1$. For the power computation, from (11), the distribution of $|\mathcal{H}_1(H) \cap \mathrm{SU}(\mathbf{t})|$ conditionally on $\hat{k} = |\mathrm{SU}(\mathbf{t})|$ is binomial with parameters $\hat{k}$ and $\pi_1 F_1(t_{\hat{k}})/G(t_{\hat{k}})$. Therefore, $\mathbb{E}[|\mathcal{H}_1(H) \cap \mathrm{SU}(\mathbf{t})|] = \mathbb{E}[\pi_1 \hat{k} F_1(t_{\hat{k}})/G(t_{\hat{k}})]$ and (15) follows.

6.2. *Proof of Theorem 3.2.* Let us prove the FDR expression (the proof for the power is similar). Define $\tilde{k} = |\mathrm{SD}(\mathbf{t})|$ and $\tilde{k}_{(1)}$, $\tilde{k}'_{(1)}$ as in Lemma A.2. We get by exchangeability of $(H_i, p_i)_i$ and independence of the p-values,

$$\begin{aligned}\mathrm{FDR}(R) &= \sum_{i=1}^m \mathbb{E}\left[\frac{\mathbf{1}\{p_i \leq t_{\tilde{k}}\}}{\tilde{k} \vee 1}\mathbf{1}\{H_i = 0\}\right] = m\mathbb{E}\left[\frac{\mathbf{1}\{p_1 \leq t_{\tilde{k}}\}}{\tilde{k} \vee 1}\mathbf{1}\{H_1 = 0\}\right] \\ &= m\mathbb{E}\left[\frac{\mathbf{1}\{p_1 \leq t_{\tilde{k}_{(1)}+1}\}}{\tilde{k}'_{(1)} + 1}\mathbf{1}\{H_1 = 0\}\right] = \pi_0 m \mathbb{E}\left[\frac{F_0(t_{\tilde{k}_{(1)}+1})}{\tilde{k}'_{(1)} + 1}\right].\end{aligned}$$

Therefore, expression (20) will be proved as soon as we state that for any k, k' with $0 \leq k \leq m$ and $k \leq k' \leq m$, that we have

$$\begin{aligned}&\mathbb{P}\left[|\mathrm{SD}(\mathbf{t})| = k, |\mathrm{SD}(\mathbf{t}')| = k'\right] \\ &= \widetilde{\mathcal{D}}_m\big((G(t_j))_{1\leq j\leq m}, k\big)\, \widetilde{\mathcal{D}}_{m-k}\left(\left(\frac{G(t_{k+1+j}) - G(t_{k+1})}{1 - G(t_{k+1})}\right)_{1\leq j\leq m-k}, k' - k\right),\end{aligned} \tag{31}$$

for any threshold $(t_j)_{1\leq j\leq m+1}$ and letting $\mathbf{t} = (t_j)_{1\leq j\leq m}$ and $\mathbf{t}' = (t_{j+1})_{1\leq j\leq m}$. To prove (31), remark that we may assumed that $G(x) = x$ up to replace $(t_j)_j$ by $(G(t_j))_j$ (because G is continuous increasing). Next, assume $k < k' < m$ and denote $L(r) = \sum_{i=1}^m \mathbf{1}\{p_i \leq t_r\}$. By definition of a step-down procedure, the probability $\mathbb{P}\left[|\mathrm{SD}(\mathbf{t})| = k, |\mathrm{SD}(\mathbf{t}')| = k'\right]$ is equal to

$$\begin{aligned}&\mathbb{P}\left[\forall j \leq k, L(j) \geq j, L(k+1) = k, \forall j, k+1 \leq j \leq k', L(j+1) \geq j, L(k'+2) = k'\right] \\ &= \binom{m}{k}\binom{m-k}{k'-k}\ \mathbb{P}\big[\forall j \leq k, L(j) \geq j, \forall j, k+1 \leq j \leq k', L(j+1) \geq j, \\ &\quad p_1, ..., p_k \leq t_{k+1} < p_{k+1}, ..., p_{k'} \leq t_{k'+2} < p_{k'+1}, ..., p_m\big] \\ &= \binom{m}{k}\binom{m-k}{k'-k}\ \mathbb{P}\Bigg[p_1, ..., p_k \leq t_{k+1} < p_{k+1}, ..., p_{k'} \leq t_{k'+2} < p_{k'+1}, ..., p_m\,, \\ &\quad \forall j \leq k, \sum_{i=1}^k \mathbf{1}\{p_i \leq t_j\} \geq j,\ \forall j, k+1 \leq j \leq k', \sum_{i=k+1}^{k'} \mathbf{1}\{p_i \leq t_{j+1}\} \geq j - k\Bigg] \\ &= \binom{m}{k}\binom{m-k}{k'-k}\mathbb{P}\left[p_{(1)} \leq t_1, ..., p_{(k)} \leq t_k\right](1 - t_{k'+2})^{m-k'} \\ &\quad \times \mathbb{P}\left[t_{k+1} < p_{(1)} \leq t_{k+2}, ..., p_{(k'-k)} \leq t_{k'+1}\right],\end{aligned}$$

where we used that the p-values are i.i.d. Simple computations give that

$$\mathbb{P}\left[t_{k+1} < p_{(1)} \leq t_{k+2}, ..., p_{(k'-k)} \leq t_{k'+1}\right] = \mathbb{P}\left[p_{(1)} \leq t_{k+2} - t_{k+1}, ..., p_{(k'-k)} \leq t_{k'+1} - t_{k+1}\right].$$

This leads to (31). The cases $k < k' = m$ and $k = k'$ are similar.

6.3. *Proof of* (16), (17) *and* (18). Let us state the following general expression: for $1 \leq j \leq k \leq m$,

$$\begin{aligned}(32) \qquad & \mathbb{P}[|\mathcal{H}_0(H) \cap \mathrm{SD}(\mathbf{t})| = j, |\mathrm{SD}(\mathbf{t})| = k] \\ &= \binom{k}{j} \pi_0^j \pi_1^{k-j} \frac{\mathbb{P}[U_{(1)} \leq t_1, ..., U_{(k)} \leq t_k \mid H_1 = ... = H_j = 0, H_{j+1} = ... = H_k = 1]}{\Psi_k\big(G(t_1), ..., G(t_k)\big)},\end{aligned}$$

where $(H_i, U_i)_{1 \leq i \leq k}$ is a sequence of i.i.d. variables following the unconditional independent model (for k hypotheses). Then, applying (32) in the case $j = k$, $j = 0$ and $F_1 = 1$ will lead to (16), (17) and (18), respectively. To state (32), we use that $(H_i, p_i)_i$ is a i.i.d. sequence:

$$\begin{aligned}&\mathbb{P}[|\mathcal{H}_0(H) \cap \mathrm{SD}(\mathbf{t})| = j, |\mathrm{SD}(\mathbf{t})| = k] \\ &= \binom{k}{j}\binom{m}{k} \mathbb{P}[\mathrm{SD}(\mathbf{t}) = \{1, ..., k\}, H_1 = ... = H_j = 0, H_{j+1} = ... = H_k = 1] \\ &= \binom{k}{j}\binom{m}{k} \mathbb{P}[\forall \ell \leq k, \sum_{i=1}^k \mathbf{1}\{p_i \leq t_\ell\} \geq \ell, \forall i \geq k+1, p_i > t_{k+1}, \\ &\quad H_1 = ... = H_j = 0, H_{j+1} = ... = H_k = 1] \\ &= \binom{k}{j} \pi_0^j \pi_1^{k-j} \mathbb{P}[\forall \ell \leq k, \sum_{i=1}^k \mathbf{1}\{p_i \leq t_\ell\} \geq \ell \mid H_1 = ... = H_j = 0, H_{j+1} = ... = H_k = 1] \\ &\quad \times \binom{m}{k} (1 - G(t_{k+1}))^{m-k},\end{aligned}$$

which leads to (32) by applying (19).

6.4. *Proof of Theorem 3.6.* We focus on the step-up case, when $\rho \in (-1, 1)$ and $m_0 = 1$ (the remaining cases are left to the reader). Without loss of generality, we may assume that the first coordinate correspond to the true null, that is, $H = (0, 1)$. In this context, the FDP takes one of the three values: $0, \frac{1}{2}, 1$, according to the location of the tests statistics $Y_i = X_i + \mu H_i$ with respect to the critical values z_1 and z_2. From the definition of a step-up procedure, we may define the two regions for $i \in \{1, 2\}$, $\mathcal{D}_i = \{(y_1, y_2) \in \mathbb{R}^2 \mid \mathrm{FDP}(y_1, y_2) = i/2\}$, where $\mathrm{FDP}(y_1, y_2)$ denotes the FDP of $\mathrm{SU}(\mathbf{t})$ taken in the p-values $\mathbf{p} = (\overline{\Phi}(y_1), \overline{\Phi}(y_2))$. The regions $\mathcal{D}_i$ are represented on Figure 3(**a**). Next, since (Y_1, Y_2) follows the EMN model, we may write for $i \in \{1, 2\}$, $\mathbb{P}(\mathrm{FDP}(Y_1, Y_2) = i/2) = (2\pi\sqrt{1-\rho^2})^{-1} \int\int_{\mathcal{D}_i} \exp\{-\frac{1}{2(1-\rho^2)}\big(y_1 -$

$\rho(y_2 - \mu)\big)^2 - \frac{1}{2}(y_2 - \mu)^2\}\, dy_1 dy_2 = (2\pi)^{-1} \int\int_{\widetilde{\mathcal{D}}_i} \exp\{-\frac{1}{2}(u^2 + v^2)\}\, dudv$ by using the substitution $u = (\sqrt{1-\rho^2})^{-1}\big(y_1 - \rho(y_2 - \mu)\big)$ and $v = y_2 - \mu$, and where the resulting integration domain $\widetilde{\mathcal{D}}_i$ is represented on Figure 3(**b**). Therefore, we obtain $\mathbb{P}(\mathrm{FDP} = 1/2) = (\sqrt{2\pi})^{-1} \int_{z_1-\mu}^{\infty} \exp\{-v^2/2\} \overline{\Phi} \left((z_1 - \rho v)/\sqrt{1-\rho^2}\right) dv$, and $\mathbb{P}(\mathrm{FDP} = 1) = (\sqrt{2\pi})^{-1} \int_{-\infty}^{z_1-\mu} \exp\{-v^2/2\} \overline{\Phi} \big((z_2$ -
and the final expression results by using the substitution $w = \overline{\Phi}(v)$.

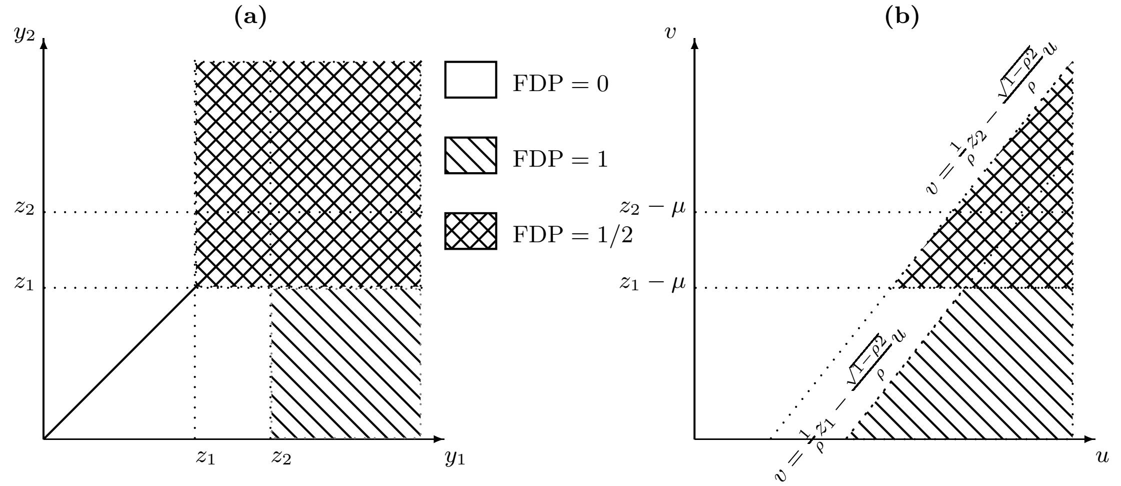

FIG 3. *Left:* $\mathcal{D}_i$. *Right:* $\widetilde{\mathcal{D}}_i$. *For* $i = 1$*(double cross-hatch) and* $i = 2$ *(simple cross-hatch). Graph for* $\rho < 0$.

6.5. *Proof of Theorem 4.1* . For any $t \in [0,1]$, let $G(t) = \pi_0 t + \pi_1 F_1(t)$ and $G_1(t) = \pi_0 t + \pi_1$. First, since F_1 is concave, we have for $t < t' \le 1$, that $(F_1(t') - F_1(t))/(t'-t) \ge (1 - F_1(t))/(1-t)$ and thus for $t \le t'$, we obtain the inequality

$$(G(t') - G(t))/(1 - G(t)) \ge (G_1(t') - G_1(t))/(1 - G_1(t)) \tag{33}$$

(by convention, the LHS (resp. RHS) of (33) is equal to 0 if $G(t) = 1$ (resp. $G_1(t) = 1$)). From expression (20), we obtain that $\mathrm{FDR}(\mathrm{LSD}, \overline{P}^I_{(\pi_0,F)})$ equals

$$\pi_0 m \sum_{k=1}^{m} \widetilde{\mathcal{D}}_{m-1}((G(t_j))_{1\le j\le m-1}, k-1) \sum_{i=0}^{m-k} \frac{t_k}{k+i} \widetilde{\mathcal{D}}_{m-k} \left(\left(\frac{G(t_{k+j}) - G(t_k)}{1 - G(t_k)} \right)_{1\le j\le m-k}, i \right)$$
$$\le \pi_0 m \sum_{k=1}^{m} \widetilde{\mathcal{D}}_{m-1}((G(t_j))_{1\le j\le m-1}, k-1) \sum_{i=0}^{m-k} \frac{t_k}{k+i} \widetilde{\mathcal{D}}_{m-k} \left(\left(\frac{G_1(t_{k+j}) - G_1(t_k)}{1 - G_1(t_k)} \right)_{1\le j\le m-k}, i \right)$$
$$= \pi_0 m \sum_{k=1}^{m} \widetilde{\mathcal{D}}_{m-1}((G(t_j))_{1\le j\le m-1}, k-1) \sum_{i=0}^{m-k} \frac{t_k}{k+i} \widetilde{\mathcal{D}}_{m-k} \left(\left(\frac{t_{k+j} - t_k}{1 - t_k} \right)_{1\le j\le m-k}, i \right),$$

where the inequality comes from (33) and because for a fixed k, the sum over i can be seen as the expectation of $t_k/(k+I)$ where I is the rejection number of a step-down procedure (point-wise nondecreasing in the threshold). Next, considering this time the sum over k as an expectation, since $G \leq G_1$ and since a step-down procedure is point-wise nondecreasing in the threshold, the proof is finished by using assumption (A).

Consider now the case where $t_j = \alpha j/m$ and let us prove that this threshold satisfies (A). For any $m \geq 2$ and $1 \leq k \leq m$, let us denote $S_{m,k} = \sum_{i=0}^{m-k} \frac{k}{k+i} \widetilde{\mathcal{D}}_{m-k}\left(\left(\frac{\alpha j/m}{1-\alpha k/m}\right)_{1\leq j\leq m-k}, i\right)$. Letting $a_{m,k} = \frac{k}{m}\left(1 - \alpha\,\frac{m-k}{m-\alpha k}\right)$ (increasing in k) and $b_{m,k} = \frac{m-\alpha}{m}\frac{m-k}{m-\alpha k}$ (decreasing in k), we may prove the following recursion (see the proof below):

$$S_{m,k} = a_{m,k} + b_{m,k} S_{m-1,k}. \tag{34}$$

Using (34) we propose to state that $(t_k)_k$ satisfies (A), that is,"$k \in \{1, ..., m\} \mapsto S_{m,k}$ is nondecreasing" by a recurrence on $m \geq 2$: the property is obviously true for $m = 2$. Assuming the property true for $m-1$, we obtain for any $2 \leq k \leq m-1$,

$$\begin{aligned} S_{m,k} - S_{m,k-1} &= a_{m,k} - a_{m,k-1} + b_{m,k}S_{m-1,k} - b_{m,k-1}S_{m-1,k-1} \\ &= a_{m,k} - a_{m,k-1} + (b_{m,k} - b_{m,k-1})S_{m-1,k} + b_{m,k-1}(S_{m-1,k} - S_{m-1,k-1}) \\ &\geq (a_{m,k} + b_{m,k}) - (a_{m,k-1} + b_{m,k-1}), \end{aligned}$$

because $S_{m-1,k} \leq 1$. Hence, since $a_{m,k} + b_{m,k} = 1 - \frac{\alpha}{m}\frac{m-k}{m-\alpha k}$, the quantity $a_{m,k}+b_{m,k}$ is increasing in k and $S_{m,k} - S_{m,k-1} \geq 0$. Also, we obviously have $S_{m,m} = 1 \geq S_{m,m-1}$, and the recurrence is completed.

We now finally state (34). Let for $1 \leq j \leq m-k$, $t_j = \frac{\alpha j/m}{1-\alpha k/m}$ and $t'_j = 1 - t_{m-k-j+1}$, so that

$$S_{m,k} = \mathbb{E}_0\left[\frac{k}{k + |\mathrm{SD}((t_j)_{1\leq j\leq m-k})|}\right] = \mathbb{E}_0\left[\frac{k}{m - |\mathrm{SU}((t'_j)_{1\leq j\leq m-k})|}\right],$$

where $\mathbb{E}_0$ denotes the expectation with respect to i.i.d. uniform p-values. Hence, denoting $t'_j = \beta + j\gamma$ with $\beta = \frac{m-m\alpha-\alpha}{m-\alpha k}$ and $\gamma = \frac{\alpha}{m-\alpha k}$, we obtain

$$\begin{aligned} \frac{S_{m,k}}{k} =& \frac{1}{m} + \frac{1}{m}\mathbb{E}_0\left[\frac{|\mathrm{SU}((t'_j)_{1\leq j\leq m-k})|}{m - |\mathrm{SU}((t'_j)_{i\leq j\leq m-k})|}\right] \\ =& \frac{1}{m} + \frac{1}{m}\sum_{j=1}^{m-k} \frac{j}{m-j}\binom{m-k}{j}(t'_j)^j \Psi_{m-k-j}\big(1 - t'_{m-k}, ..., 1 - t'_{j+1}\big) \\ =& \frac{1}{m} + \frac{m-k}{m}\sum_{j=1}^{m-k} \frac{t'_j}{m-j}\binom{m-k-1}{j-1}(t'_j)^{j-1} \Psi_{m-k-j}\big(1 - t'_{m-k}, ..., 1 - t'_{j+1}\big), \end{aligned}$$

so that $\frac{S_{m,k}}{k}$ equals

$$\frac{1}{m} - \gamma\frac{m-k}{m} + \frac{m-k}{m}(\beta + m\gamma)\sum_{j=0}^{m-1-k}\frac{(t'_{j+1})^j}{m-1-j}\binom{m-1-k}{j}\Psi_{m-1-k-j}\big(1-t'_{m-k},...,1-t'_{j+2}\big)$$

$$= \frac{1}{m} - \gamma\frac{m-k}{m} + \frac{m-k}{m}(\beta+m\gamma)\mathbb{E}_0\left[\frac{1}{m-1-|\mathrm{SU}((t'_{j+1})_{1\leq j\leq m-1-k})|}\right]$$

$$= \frac{1}{m} - \gamma\frac{m-k}{m} + \frac{m-k}{m}(\beta+m\gamma)\frac{S_{m-1,k}}{k},$$

and the recursion (34) is proved.

6.6. *Proofs for Section 4.2.* Let us first prove (29): denote for any $0 \leq i \leq m-1$, $u_i = \mathbb{E}_0[(|\mathrm{SU}((t_j)_{i+1\leq j\leq m})|+i+1)^{-1}]$ (where $\mathbb{E}_0$ denotes the expectation with respect to i.i.d. uniform p-values) and $u_m = 1/(m+1)$, $u_{m+1} = 0$, so that u_0 equals the quantity in (29). We may prove the following recursion relation: for any $0 \leq i \leq m$,

$$(i+1)u_i = (1-(m-i)\gamma) - (m-i)(\beta-\gamma)u_{i+1}. \tag{35}$$

Expression (35) is proved as follows: for $i < m$,

$$\begin{aligned}(i+1)u_i =& 1 - \mathbb{E}_0\left[\frac{|\mathrm{SU}((t_j)_{i+1\leq j\leq m})|}{|\mathrm{SU}((t_j)_{i+1\leq j\leq m})|+i+1}\right]\\
=& 1 - \sum_{k=1}^{m-i}\frac{k}{k+i+1}\binom{m-i}{k}(t_{k+i})^k\Psi_{m-i-k}\big(1-t_m,...,1-t_{k+i+1}\big)\\
=& 1-(m-i)\sum_{k=1}^{m-i}\frac{t_{k+i}}{k+i+1}\binom{m-i-1}{k-1}(t_{k+i})^{k-1}\Psi_{m-i-k}\big(1-t_m,...,1-t_{k+i+1}\big)\\
=& 1-(m-i)\sum_{k=0}^{m-(i+1)}\left(\gamma+\frac{\beta-\gamma}{k+(i+1)+1}\right)\binom{m-(i+1)}{k}(t_{k+(i+1)})^k\\
&\times\Psi_{m-(i+1)-k}\big(1-t_m,...,1-t_{k+(i+1)+1}\big).\end{aligned}$$

Next, we obtain that the solution of the recursion (35) is given by

$$u_i = \sum_{j=0}^{m-i}\frac{1-(m-(i+j))\gamma}{m-(i+j)}\ \frac{(m-i)\times\cdots\times(m-(i+j))}{(i+1)\times\cdots\times(i+j+1)}(\gamma-\beta)^j,$$

which leads to $u_0 = \sum_{j=0}^m \frac{1-(m-j)\gamma}{j+1}\binom{m}{j}(\gamma-\beta)^j$ and (29) results.

To prove Theorem 4.3, we combine (28) and (29), the latter using $m-1$ hypotheses and special values for β and γ: $\beta=\gamma=\pi_0\alpha/m$ for $F_1(x)=0$; $\beta=\gamma=\alpha/m$ for $F_1(x)=x$; $\beta=\pi_0\alpha/m+(1-\pi_0)\varepsilon$ and $\gamma=\pi_0\alpha/m$ for $F_1(x)=\varepsilon$.

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References.

[1] Benjamini, Y. and Hochberg, Y. (1995). Controlling the false discovery rate: a practical and powerful approach to multiple testing. *J. Roy. Statist. Soc. Ser. B*, 57(1):289–300.

[2] Benjamini, Y., Krieger, A. M., and Yekutieli, D. (2006). Adaptive linear step-up procedures that control the false discovery rate. *Biometrika*, 93(3):491–507.

[3] Benjamini, Y. and Yekutieli, D. (2001). The control of the false discovery rate in multiple testing under dependency. *Ann. Statist.*, 29(4):1165–1188.

[4] Blanchard, G. and Roquain, E. (2008). Two simple sufficient conditions for FDR control. *Electron. J. Stat.*, 2:963–992.

[5] Blanchard, G. and Roquain, E. (2009). Adaptive FDR control under independence and dependence. *J. Mach. Learn. Res.*, 10:2837–2871.

[6] Chi, Z. (2007). On the performance of FDR control: constraints and a partial solution. *Ann. Statist.*, 35(4):1409–1431.

[7] Chi, Z. and Tan, Z. (2008). Positive false discovery proportions: intrinsic bounds and adaptive control. *Statist. Sinica*, 18(3):837–860.

[8] Dickhaus, T. (2008). *False Discovery Rate and Asymptotics.* PhD thesis, Heinrich-Heine-Universität Düsseldorf.

[9] Donoho, D. and Jin, J. (2004). Higher criticism for detecting sparse heterogeneous mixtures. *Ann. Statist.*, 32(3):962–994.

[10] Efron, B. (2009). Correlated z -values and the accuracy of large-scale statistical estimates. Preprint.

[11] Efron, B., Tibshirani, R., Storey, J. D., and Tusher, V. (2001). Empirical Bayes analysis of a microarray experiment. *J. Amer. Statist. Assoc.*, 96(456):1151–1160.

[12] Ferreira, J. A. and Zwinderman, A. H. (2006). On the Benjamini-Hochberg method. *Ann. Statist.*, 34(4):1827–1849.

[13] Finner, H., Dickhaus, R., and Roters, M. (2009). On the false discovery rate and an asymptotically optimal rejection curve. *Ann. Statist.*, 37(2):596–618.

[14] Finner, H., Dickhaus, T., and Roters, M. (2007). Dependency and false discovery rate: asymptotics. *Ann. Statist.*, 35(4):1432–1455.

[15] Finner, H. and Roters, M. (2002). Multiple hypotheses testing and expected number of type I errors. *Ann. Statist.*, 30(1):220–238.

[16] Gavrilov, Y., Benjamini, Y., and Sarkar, S. K. (2009). An adaptive step-down procedure with proven fdr control under independence. *Ann. Statist.*, 37(2):619–629.

[17] Genovese, C. and Wasserman, L. (2004). A stochastic process approach to false discovery control. *Ann. Statist.*, 32(3):1035–1061.

[18] Glueck, D., Mandel, J., Karimpour-Fard, A., Hunter, L., and Muller, K. (2008). Exact calculations of average power for the benjamini-hochberg procedure. *International Journal of Biostatistics*, 4(1):1103–1103.

[19] Guo, W. and Rao, M. B. (2008). On control of the false discovery rate under no assumption of dependency. *Journal of Statistical Planning and Inference*, 138(10):3176–3188.

[20] Lehmann, E. L. and Romano, J. P. (2005). Generalizations of the familywise error rate. *Ann. Statist.*, 33:1138–1154.

[21] Neuvial, P. (2008). Asymptotic properties of false discovery rate controlling procedures under independence. *Electron. J. Stat.*, 2:1065–1110.

[22] Owen, D. B. and Steck, G. P. (1962). Moments of order statistics from the equicorrelated multivariate normal distribution. *Ann. Math. Statist.*, 33:1286–1291.

[23] Reiner-Benaim, A. (2007). FDR control by the BH procedure for two-sided correlated tests with implications to gene expression data analysis. *Biom. J.*, 49(1):107–126.
[24] Roquain, E. (2007). *Exceptional motifs in heterogeneous sequences. Contributions to theory and methodology of multiple testing*. PhD thesis, Université Paris XI.
[25] Roquain, E. and van de Wiel, M. (2009). Optimal weighting for false discovery rate control. *Electron. J. Stat.*, 3:678–711.
[26] Sarkar, S. K. (2002). Some results on false discovery rate in stepwise multiple testing procedures. *Ann. Statist.*, 30(1):239–257.
[27] Sarkar, S. K. (2008). On methods controlling the false discovery rate. *Sankhya, Ser. A*, 70:135–168.
[28] Shorack, G. R. and Wellner, J. A. (1986). *Empirical processes with applications to statistics*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York.
[29] Simes, R. J. (1986). An improved Bonferroni procedure for multiple tests of significance. *Biometrika*, 73(3):751–754.
[30] Storey, J. D. (2003). The positive false discovery rate: a Bayesian interpretation and the q-value. *Ann. Statist.*, 31(6):2013–2035.
[31] Stuart, A. (1958). Equally correlated variates and the multinormal integral. *J. Roy. Statist. Soc. Ser. B*, 20:373–378.
[32] Zeisel, A., Zuk, O., and Domany, E. (2009). Fdr control with adaptive procedures and fdr monotonicity.

Appendix

APPENDIX A: USEFUL LEMMAS

The following lemma is related to the proof of Theorem 2.1 in [12] and to Lemma 8.1 (i) in [25].

LEMMA A.1. *Consider a step-up procedure $SU(\mathbf{t})$ using a given threshold $\mathbf{t}$ testing m null hypotheses with p-values p_1, ..., p_m and rejecting $\hat{k} = |SU(\mathbf{t})|$ hypotheses. For a given $1 \le \ell \le m$, denote by $\hat{k}'_{(\ell)}$ the number of rejections of the step-up procedure of threshold $(t_{j+\ell})_{1\le j\le m-\ell}$ over $m-\ell$ hypotheses and using the p-values $p_{\ell+1}$, ..., p_m. Then we have point-wise*

$$\forall 1 \le i \le \ell,\, p_i \le t_{\hat{k}} \iff \forall 1 \le i \le \ell,\, p_i \le t_{\hat{k}'_{(\ell)}+\ell} \iff \hat{k} = \hat{k}'_{(\ell)} + \ell.$$

PROOF. First note that since $\hat{k}$ is nondecreasing in each coordinate of $(p_1, ..., p_m)$, we always have $\hat{k} \le \hat{k}'_{(\ell)} + \ell$. Second, since $p_{(k)} \le t_k$ is equivalent to $|\{1 \le j \le m \mid p_j \le t_k\}| \ge k$, the rejection number of SU($\mathbf{t}$) can be defined as $\widehat{k} = \max\{k \in \{0, 1, ..., m\} \mid |\{1 \le j \le m \mid p_j \le t_k\}| \ge k\}$. Hence, $\forall 1 \le i \le \ell$, $p_i \le t_{\hat{k}'_{(\ell)}+\ell}$ is equivalent to $|\{1 \le j \le m \mid p_j \le t_{\hat{k}'_{(\ell)}+\ell}\}| \ge \ell + |\{\ell+1 \le j \le m \mid p_j \le t_{\hat{k}'_{(\ell)}+\ell}\}|$ which is equivalent to $|\{1 \le j \le m \mid p_j \le t_{\hat{k}'_{(\ell)}+\ell}\}| \ge \ell + \hat{k}'_{(\ell)}$, by definition of $\hat{k}'_{(\ell)}$. As a consequence, since $\hat{k}$ is a maximum and since $\hat{k} \le \hat{k}'_{(\ell)} + \ell$, the latter is equivalent to $\hat{k}'_{(\ell)} + \ell = \hat{k}$. This establishes the second equivalence. The first equivalence easily comes from the second equivalence and using that $t_{\hat{k}} \le t_{\hat{k}'_{(\ell)}+\ell}$ because $(t_k)_k$ is a nondecreasing sequence. □

For step-down procedures, we use the next lemma.

LEMMA A.2. *Consider a step-down procedure* $SD(\mathbf{t})$ *using a given threshold* $\mathbf{t}$ *testing* m *null hypotheses with p-values* p_1, ..., p_m *and rejecting* $\tilde{k} = |SD(\mathbf{t})|$ *hypotheses. Denote by* $\tilde{k}_{(1)}$ *(resp.* $\tilde{k}'_{(1)}$*) the number of rejections of the step-up procedure of threshold* $(t_j)_{1\leq j\leq m-1}$ *(resp.* $(t_{j+1})_{1\leq j\leq m-1}$*) over* $m-1$ *hypotheses and using the p-values* p_2, ..., p_m*. Then we have pointwise*

$$p_1 \leq t_{\tilde{k}} \iff p_1 \leq t_{\tilde{k}_{(1)}+1} \iff \tilde{k} = \tilde{k}'_{(1)} + 1.$$

In the above lemma, we underline that the assertion $p_1 \leq t_{\tilde{k}'_{(1)}+1} \Longrightarrow p_1 \leq t_{\tilde{k}}$ is not true in general.

PROOF. Similarly to the step-up case, the rejection number of SD($\mathbf{t}$) can be defined as $\tilde{k} = \max\{k \in \{0,1,...,m\} \mid \forall k' \leq k,\ |\{1 \leq j \leq m \mid p_j \leq t_{k'}\}| \geq k'\}$. Also remark that we always have $\tilde{k}'_{(1)} + 1 \geq \tilde{k}$ and, by definition of $\tilde{k}$, for any j we have $p_j \leq t_{\tilde{k}} \Leftrightarrow p_j \leq t_{\tilde{k}+1}$. First prove that $p_1 \leq t_{\tilde{k}} \ \Leftrightarrow\ \tilde{k} = \tilde{k}'_{(1)} + 1$: using the definitions of $\tilde{k}$ and $\tilde{k}'_{(1)}$ we obtain $p_1 \leq t_{\tilde{k}} \Leftrightarrow |\{2 \leq j \leq m \mid p_j \leq t_{\tilde{k}+1}\}| < \tilde{k} \Leftrightarrow \tilde{k}'_{(1)} < \tilde{k} \Leftrightarrow \tilde{k}'_{(1)} + 1 = \tilde{k}$. Second, we prove $p_1 > t_{\tilde{k}} \Leftrightarrow p_1 > t_{\tilde{k}_{(1)}+1}$: since we obviously have $\tilde{k} \geq \tilde{k}_{(1)}$, we get $p_1 > t_{\tilde{k}} \Rightarrow p_1 > t_{\tilde{k}+1} \Rightarrow p_1 > t_{\tilde{k}_{(1)}+1}$. Conversely, if $p_1 > t_{\tilde{k}_{(1)}+1}$, we get $|\{1 \leq j \leq m \mid p_j \leq t_{\tilde{k}_{(1)}+1}\}| = |\{2 \leq j \leq m \mid p_j \leq t_{\tilde{k}_{(1)}+1}\}| < \tilde{k}_{(1)} + 1$ (by definition of $\tilde{k}_{(1)}$), hence $\tilde{k}_{(1)} + 1 > \tilde{k}$ (by definition of $\tilde{k}$), which implies $\tilde{k}_{(1)} = \tilde{k}$, thus $p_1 > t_{\tilde{k}+1}$ and finally $p_1 > t_{\tilde{k}}$. □

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