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This paper considers the propagation of linear pulses driven by random polarization mode dispersion. Analytical expressions are derived for the probability density functions of the pulse width, timing displacement, and degree of polarization. The study is performed in Stokes space, and the frequency correlation between modes is shown to play an important role.

I. INTRODUCTION

The study of pulse propagation in a birefringent fiber with random birefringence has become of great interest for telecommunication applications. Recent experiments have shown that Polarization Mode Dispersion (PMD) is one of the main limitations on fiber transmission links [1]. PMD occurs because single-mode fibers are not really single-mode but can exhibit different group velocities. PMD is thus characterized by a Differential Group Delay (DGD) between the two modes associated with the fiber. PMD results in the splitting of a polarized input pulse into two orthogonal polarizations that arrive at the ouput at different times.

The effects of PMD are usually treated by means of the three-dimensional PMD vector. The PMD vector $\hat{\mathbf{r}}$ gives in Stokes space the relation between the output state and the frequency derivative of the output state: $\hat{\mathbf{s}}'(\omega) = \hat{\mathbf{r}}(\omega) \times \hat{\mathbf{s}}(\omega)$. The Principal States of Polarization (PSP) are defined as the states that satisfy $\hat{\mathbf{r}}(\omega) \times \hat{\mathbf{s}}(\omega) = \mathbf{0}$, so that no change in output polarization can be observed close to these states at first order in ω. However, for broadband pulses the first-order approach is not valid anymore, the changes of the PSP and the length of $\hat{\mathbf{r}}(\omega)$, i.e. the DGD, with respect to frequency cannot be neglected. As a consequence the frequency dependences of the PMD vectors and DGD are worth studying.

For a given frequency the properties of the PMD vector are well known [2–4]. In particular its elements are independent Gaussian random variables. Despite its obvious relevance the frequency dependences of the PMD vectors and DGD have been treated only in a few papers [5,6]. Furthermore each paper deals with the computation of one particular expectation: the expectation of the correlation between two PMD vectors [5], the expectation of the correlation between two square DGD [6]. We shall be able to extend these results by using an approach based on the theory on stochastic differential equations. This theory will give us the means to compute any expectation of any combination of PMD vectors or DGD at different frequencies. Furthermore, by applying and extending the theory of moments [7] we shall deduce from these formulas closed form expression for the means, variances and probability density functions of relevant quantities for the characterization of pulse propagation such as the time displacement, the time rms width, and the degree of polarization.

II. DESCRIPTION OF THE MODEL

The evolution of polarized fields in randomly birefringent fibers is governed by the coupled Schrödinger equations with random PMD between two modes (polarizations) [8]:

$$i\mathbf{A}_z + K_0\mathbf{A} + iK_1\mathbf{A}_t - \frac{\beta''}{2}\mathbf{A}_{tt} = 0 \qquad (1)$$

where $\mathbf{A}$ is the row vector $(A_x, A_y)^T$ that denotes the envelopes of the electric field in the two eigenmodes. The matrices K_0 and K_1 describe random fiber birefringence. The Group Velocity Dispersion (GVD) coefficient is the second derivative of the propagation constant with respect to frequency β'', which is positive (resp. negative) for normal (resp. anomalous) dispersion. One can eliminate the fast random birefringence variations that appear in Eq. (1) by means of a change of variables, that leads to the new vector equation:

$$i\mathbf{U}_z - \frac{\beta''}{2}\mathbf{U}_{tt} = iR\mathbf{U}_t \qquad (2)$$

where $\mathbf{U} \equiv M^{-1}\mathbf{A}$, $\mathbf{U} = (u, v)^T$ represents the slow evolution of the field envelopes in the reference frame of the local polarization eigenmodes, and the matrix M obeys the equation $iM_z + K_0M = 0$. R is a z-dependent matrix that involves high order PMD. It is associated with the coupling between the modes, as well as an accumulation of a mismatch between their phases.

The most commonly used model is the so-called retarded-plate model [9,10]. The birefringence strength

$\Delta\beta$ and its derivative $\Delta\beta'$ are constant; the birefringence angle is constant over elementary intervals with length Δz; at junctions between the fiber pieces with length Δz, a random axial rotation is added as well as an addition of random phase difference between the two field components, so that the Stokes vector obeys a random walk over the Poincaré sphere. For our numerical simulations we shall use this model. If Δz is small enough we can model this configuration by considering that the matrix R is [10]:

$$R(z) = m_1(z)\Sigma_1 + m_2(z)\Sigma_2 + m_3(z)\Sigma_3, \qquad (3)$$

where Σ_j are Pauli matrices:

$$\Sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the real-valued processes m_j are random white noises with autocorrelation functions

$$\langle m_j(z)m_j(z')\rangle = \sigma^2\delta(z-z') \text{ with } \sigma^2 = \frac{\Delta\beta'^2\Delta z}{12}. \quad (4)$$

In this model the so-called PMD parameter is given by $D_p = \sqrt{8/(3\pi)}\Delta\beta'\sqrt{\Delta z}$ [10]. Our parameter σ^2 is simply related to D_p through:

$$\sigma^2 = \frac{\pi}{32}D_p^2.$$

All models of random birefringence are eventually described by the white noise model (3) with some effective parameter σ (or D_p) as soon as the correlation length of the random fluctuations of birefringence parameters is much smaller than the other characteristic lengths of the problem [4,13,14].

For consistency, note that the usual GVD parameter is $D = -2\pi c\beta''/\lambda^2$ where λ is the carrier wavelength of the pulse (1.55 μm for standard optical fiber applications). Typical values of the PMD parameter D_p have been measured in the range 0.1–1 ps/$\sqrt{\text{km}}$ [11,12]. Dispersion shifted fibers (which are of particular interest for telecommunication) have been found to have particular high values [12]. The correlation length Δz of PMD varies around 0.1–1 km, and the GVD parameter D is between 1–20 ps/nm/km.

III. PULSE CHARACTERIZATION

Numerical simulations have been carried out for the determination of the width and degree of polarization of the pulse versus the propagation distance [9,15]. In these simulations the effect of GVD was usually neglected to examine the effect of PMD exclusively. Empirical formulas for the pulse width and degree polarization were proposed. Furthermore the numbers of samples necessary to get stable averaged values were found to be very large,

which tends to prove that the variances of the pulse width and degree of polarization are large as well. We shall confirm theoretically these observations. In this section we shall derive closed-form expressions for the average values, variances, and Probability Density Functions (PDF) of the time displacement, pulse width, and degree of polarization of a pulse driven by randomly varying birefringence. We add the time displacement because the fiber PMD not only induces optical pulse broadening but also differences in pulse arrival times between different fiber realizations.

If we neglect GVD then the driving equation for the field is:

$$\mathbf{U}_z = R(z)\mathbf{U}_t \qquad (5)$$

where the z-dependent matrix R is the random combination (3) of the Pauli matrices. We now define the parameters of the pulse that we shall study in the following sections.

Time displacement. We first introduce the time displacement T_c:

$$T_c = \frac{\int t(|u|^2 + |v|^2)dt}{\int (|u|^2 + |v|^2)dt} \qquad (6)$$

Pulse polarization. If the initial pulse is polarized along the u-axis, then the rotation of the pulse polarization can be characterized by the parameter P_r defined as the ratio of the energy on the u-axis over the total energy:

$$P_r = \frac{\int |u|^2 dt}{\int (|u|^2 + |v|^2)dt} \qquad (7)$$

The degree of polarization is defined by [17]:

$$P_d = \frac{\sqrt{s_1^2 + s_2^2 + s_3^3}}{s_0} \qquad (8)$$

in terms of the Stokes parameters:

$$s_0 = \int \left(|u|^2 + |v|^2\right)(t)dt,$$

$$s_1 = \int \left(|u|^2 - |v|^2\right)(t)dt,$$

$$s_2 = 2\int \text{Re}\,(u^*v)\,(t)dt,$$

$$s_3 = 2\int \text{Im}\,(u^*v)\,(t)dt.$$

We can also introduce the degree of polarization by the following way. We construct the new field components $(u_{\theta,\xi}, v_{\theta,\xi})$ as

$$u_{\theta,\xi}(t) = \cos(\theta)u(t) + \sin(\theta)e^{i\xi}v(t)$$
$$v_{\theta,\xi}(t) = -\sin(\theta)u(t) + \cos(\theta)e^{i\xi}v(t)$$

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where θ and ξ are determined so that the energy in the component $u_{\theta,\xi}$ is maximized. The angle θ is the so-called polarization angle while the ratio of the energy of the component $u_{\theta,\xi}$ over the total energy:

$$P_{d1} = \frac{\int |u_{\theta,\xi}|^2 dt}{\int (|u|^2 + |v|^2) dt} \qquad (9)$$

is related to the degree of polarization through the simple identity $P_d = 2P_{d1} - 1$. Note that $P_{d1} \geq P_r$. Furthermore the degree of polarization of a monochromatic pulse is always equal to 1, whatever the rotation of polarization. However, the spectral components of a short pulse tend to loose their polarization coherence so that the degree of polarization P_d of the corresponding pulse decays to zero.

Pulse width. It is not clear for practical applications whether the timing displacement should be eliminated from the computation of the pulse broadening. If the speed of the timing displacement is so slow that the clock recovery can follow the change, then the timing displacement has no influence on system performance and should be eliminated. Accordingly the pulse width should be defined by:

$$T_{w1}^2 = \frac{\int (t - T_c)^2 (|u|^2 + |v|^2) dt}{\int (|u|^2 + |v|^2) dt} \qquad (10)$$

If the PMD fluctuates more rapidly than the clock recovery can track, then it is necessary to include the timing displacement to properly simulate the transmission system:

$$T_{w2}^2 = \frac{\int t^2 (|u|^2 + |v|^2) dt}{\int (|u|^2 + |v|^2) dt} = T_{w1}^2 + T_c^2 \qquad (11)$$

We shall compute the mean values, variances, and PDF of these quantities for a general initial pulse u_0 polarized along the u-axis with rms width T_0. We shall also give explicit formulas in the case of a Gaussian pulse with rms width T_0:

$$u_0(t) = \exp\left(-\frac{t^2}{4T_0^2}\right) \qquad (12)$$

We shall also consider a sech pulse with rms width T_0 whose expression is:

$$u_0(t) = \text{sech}\left(\frac{\pi t}{2\sqrt{3}T_0}\right) \qquad (13)$$

IV. THE STOKES VECTOR

The Fourier components $\hat{\mathbf{U}} := (\hat{u}, \hat{v})^T$ of the field:

$$\hat{u}(\omega) = \int u(t) \exp(i\omega t) dt, \quad \hat{v}(\omega) = \int v(t) \exp(i\omega t) dt,$$

obey a system of ordinary differential equations:

$$\hat{\mathbf{U}}_z = i\omega R(z)\hat{\mathbf{U}}$$

There exist simple and exact analytical identities between the amount of broadening and Fourier components [7,16], as well as between the polarization degree, the time displacement and the Fourier components. These formulas are in fact nothing more than standard Parseval formula applied to well chosen quantities. A convenient representation of the polarization evolutions induced by the fluctuations m_j may be done in terms of the Stokes vector $\hat{\mathbf{s}}$ associated to the Fourier components of the field (u, v):

$$\hat{s}_1(\omega) = \left(|\hat{u}|^2 - |\hat{v}|^2\right)(\omega),$$
$$\hat{s}_2(\omega) = 2\text{Re}\left(\hat{u}^*\hat{v}\right)(\omega),$$
$$\hat{s}_3(\omega) = 2\text{Im}\left(\hat{u}^*\hat{v}\right)(\omega),$$

whose modulus $\sqrt{\hat{s}_1^2 + \hat{s}_2^2 + \hat{s}_3^2} = |\hat{u}|^2 + |\hat{v}|^2 = \hat{E}_0(\omega)$ is the spectral intensity:

$$\hat{E}_0(\omega) = \left|\int u_0(t)e^{i\omega t}dt\right|^2$$

which is a preserved quantity. In terms of the Stokes parameters the dynamics induced by PMD is simple:

$$\hat{\mathbf{s}}_z = 2\sigma\omega\dot{\mathbf{W}}(z) \times \hat{\mathbf{s}}$$

where $\sigma\dot{\mathbf{W}}(z)$ is the row vector $(m_3, m_1, m_2)^T(z)$. Thus the m_j appear as elementary infinitesimal generators of random rotations of the Stokes vector over the Poincaré sphere. Mathematically speaking the above equation should be understood as:

$$d\hat{s}_1(\omega) = 2\sigma\omega\left(\hat{s}_2 \circ dW^3 - \hat{s}_3 \circ dW^2\right)$$
$$d\hat{s}_2(\omega) = 2\sigma\omega\left(\hat{s}_3 \circ dW^1 - \hat{s}_1 \circ dW^3\right) \qquad (14)$$
$$d\hat{s}_3(\omega) = 2\sigma\omega\left(\hat{s}_1 \circ dW^2 - \hat{s}_2 \circ dW^1\right)$$

where $\circ$ stands for the Stratonovich stochastic integral and W^j are three independent Brownian motions. The correlation degree between the Stokes vectors at two nearby frequencies ω_1 and ω_2 is:

$$C(\omega_1, \omega_2) := \frac{\hat{\mathbf{s}}(\omega_1).\hat{\mathbf{s}}(\omega_2)}{\hat{E}_0(\omega_1)\hat{E}_0(\omega_2)} \qquad (15)$$

It is easy to check from Eq. (14) that C is a diffusion process with infinitesimal generator:

$$\mathcal{L} = 2\sigma^2\Delta\omega^2\frac{\partial}{\partial C}(1 - C^2)\frac{\partial}{\partial C}$$

where $\Delta\omega = \omega_1 - \omega_2$. This means that any expectation of a smooth function $f(C)$ of C satisfies the forward Kolmogorov equation:

$$\frac{\partial \langle f(C)\rangle}{\partial z} = \langle \mathcal{L}f(C)\rangle$$

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In particular the mean value of C decays exponentially as:

$$\langle C(\omega_1, \omega_2)\rangle = \exp\left(-4\Delta\omega^2\sigma^2 z\right)$$

This shows that PMD is a strongly frequency-dependent phenomenon, and that the Stokes vectors at two nearby frequencies become statistically independent as z increases. More precisely we can derive from the expression of the infinitesimal generator the Fokker-Planck equation for the PDF $p(z, C)$ satisfied by the correlation degree $C(\omega_1, \omega_2)$:

$$\frac{\partial p}{\partial z} = 2\sigma^2\Delta\omega^2\left((1 - C^2)\frac{\partial^2 p}{\partial C^2} - 2C\frac{\partial p}{\partial C}\right)$$

starting from the initial condition $p(0, C) = \delta(C - 1)$. This equation can be solved by means of an expansion over the Legendre polynomials [18]:

$$p(z, C) = \frac{1}{2}\sum_{n=0}^{\infty} P_n(C)\exp\left(-2n(n+1)\sigma^2\Delta\omega^2 z\right)\mathbf{1}_{C\in[-1,1]}$$

The PDF $p(z, .)$ is plotted in Figure 1 for different values of z. We can thus observe the transition from full correlation $p(0, C) = \delta(C - 1)$ at $z = 0$ to complete uncorrelation $p(\infty, C) = \frac{1}{2}\mathbf{1}_{C\in[-1,1]}$ for $\sigma^2\Delta\omega^2 z > 1$.
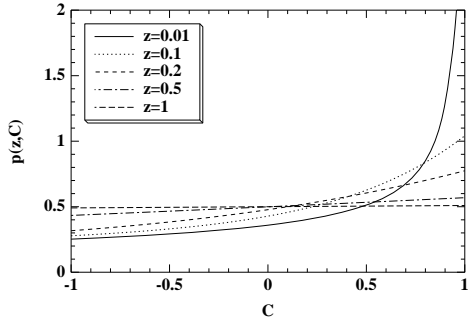

FIG. 1. *PDF of the correlation degree* $C(\omega_1, \omega_2)$ *for* $\sigma^2\Delta\omega^2 = 1$.

V. TIME DISPLACEMENT

In terms of the Fourier components the time displacement reads:

$$T_c = \frac{\int \hat{t}(\omega)\hat{E}_0(\omega)d\omega}{\int \hat{E}_0(\omega)d\omega} \tag{16}$$

where

$$\hat{t}(\omega) := \frac{\mathrm{Im}\left(\hat{u}^*(\omega)\hat{u}'(\omega)\right) + \mathrm{Im}\left(\hat{v}^*(\omega)\hat{v}'(\omega)\right)}{|\hat{u}|^2(\omega) + |\hat{v}|^2(\omega)},$$

and the prime stands for the derivative with respect to ω. The process $\bar{t}(\omega, z)$ obeys a simple equation in terms of the Stokes parameters:

$$d\hat{t} = \sigma\left(\hat{s}_1 \circ dW^1 + \hat{s}_2 \circ dW^2 + \hat{s}_3 \circ dW^3\right)$$

Computing the infinitesimal generator of $\hat{t}$:

$$\mathcal{L} = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial\hat{t}^2}$$

we get that $\hat{t}$ obeys the distribution of a Brownian motion with diffusion coefficient σ. Accordingly the PDF of $\hat{t}(\omega, z)$ is Gaussian:

$$p(z, \hat{t}) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{z}}\exp\left(-\frac{\hat{t}^2}{2\sigma^2 z}\right)\mathbf{1}_{\hat{t}\in\mathbb{R}}$$

For the following results the correlation function of $\hat{t}$ at two nearby frequencies is necessary. It is found that the process $(C, \hat{t}_1, \hat{t}_2)$, where $\hat{t}_1 = \hat{t}(\omega_1)$, $\hat{t}_2 = \hat{t}(\omega_2)$, and $C = C(\omega_1, \omega_2)$, is a diffusion process with infinitesimal generator:

$$\mathcal{L} = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial\hat{t}_1^2} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial\hat{t}_2^2} + \sigma^2 C\frac{\partial^2}{\partial\hat{t}_1\hat{t}_2}$$
$$+2\sigma^2\Delta\omega^2\frac{\partial}{\partial C}(1 - C^2)\frac{\partial}{\partial C}$$

We can thus deduce that:

$$\frac{d}{dz}\langle\hat{t}(\omega_1)\hat{t}(\omega_2)\rangle = \sigma^2\langle C(\omega_1, \omega_2)\rangle$$

which yields:

$$\langle\hat{t}(\omega_1)\hat{t}(\omega_2)\rangle = \frac{1 - \exp(-4(\omega_1 - \omega_2)^2\sigma^2 z)}{4(\omega_1 - \omega_2)^2}$$

This allows one to state that the time displacement has mean zero and variance:

$$\langle T_c^2\rangle = \frac{\int\int \hat{E}_0(\omega_1)\hat{E}_0(\omega_2)\langle\hat{t}(\omega_1)\hat{t}(\omega_2)\rangle\, d\omega_1\omega_2}{\left(\int \hat{E}_0(\omega)d\omega\right)^2} \tag{17}$$

If the input pulse has Gaussian shape with rms width T_0, then the variance of the time displacement can be written explicitly:

$$\langle T_c^2\rangle = \frac{T_0^2}{2}\left(\sqrt{1 + \frac{4\sigma^2 z}{T_0^2}} - 1\right) \tag{18}$$

This exact expression depends on the input shape, but the long-term growth rate of the variance is independent of the pulse shape. We can state in great generality that the mean square time displacement first grows linearly for $z \ll T_0^2/\sigma^2$, while the long-range growth is slower ($\sqrt{z}$):

$$\langle T_c^2\rangle \overset{\sigma^2 z\ll T_0^2}{\simeq} \sigma^2 z - 4(c_2 - c_4^2)\frac{\sigma^4 z^2}{T_0^2}$$
$$\overset{\sigma^2 z\gg T_0^2}{\simeq} \sqrt{\pi}c_1\sqrt{\sigma^2 z}$$

where

$$c_1 = \frac{\int \hat{E}_0(\omega)^2 d\omega}{\left(\int \hat{E}_0(\omega) d\omega\right)^2} = \frac{1}{2\pi} \frac{\int |u_0 \star u_0|^2(t) dt}{\left(\int |u_0|^2(t) dt\right)^2}$$

$$c_2 = \frac{\int \omega^2 \hat{E}_0(\omega) d\omega}{\int \hat{E}_0(\omega) d\omega} = \frac{\int |u_{0t}|^2 dt}{\int |u_0|^2 dt}$$

$$c_4 = \frac{\int \omega \hat{E}_0(\omega) d\omega}{\int \hat{E}_0(\omega) d\omega} = \frac{i \int u_{0t} u_0^* dt}{\int |u_0|^2 dt}$$

and the star $\star$ stands for the convolution. An interesting question concerns the statistical distribution of T_c. We can show on the one hand that the statistical distribution of T_c is Gaussian when $\sigma^2 z \ll T_0^2$ since T_c is then equal to σW_z^1 up to terms of order $\sigma^2 z / T_0^2$. On the other hand, when $\sigma^2 z \gg T_0^2$ the random variable T_c has also Gaussian statistics because Eq. (16) shows that it is the sum of a large number of uncorrelated components. The central limit theorem can then be invoked to show that T_c has Gaussian statistics with zero-mean and variance (17). Accordingly we may think that T_c has a Gaussian PDF in the general case. However a thorough study shows that the forth moment of T_c can be expanded as powers of $\sigma^2 z / T_0^2$, and the expansion of $\langle T_c^4 \rangle$ is different from the corresponding expansion of $3 \langle T_c^2 \rangle^2$. This establishes that T_c has not strictly Gaussian statistics, although numerical simulations show that the distribution is very close to be Gaussian. This resemblance is not surprising, since the expansion of the high order moments show that the moments of T_c obey the standard rules of Gaussian random variables up to order 3 with respect to $\sigma^2 z / T_0^2$:

$$\langle T_c^{2n} \rangle = \alpha_n \langle T_c^2 \rangle^n \left(1 + O(\frac{\sigma^6 z^3}{T_0^6})\right)$$

where $\alpha_n = (2n+1)(2n-1)...3$. In cases of practical applications for which $\sigma^2 z \leq T_0^2$, we may thus think at T_c as a Gaussian random variable up to a very good approximation.

A second noticeable problem is about the tail of the PDF of T_c. Although we have just noticed T_c has not pure Gaussian statistics, it is the linear combination of Gaussian random variables $\hat{t}$, so that the tail of the distribution of T_c will be imposed by the tail of some $\hat{t}(\omega)$. Accordingly:

$$\mathbb{P}\left(|T_c| \geq \alpha \langle T_c^2 \rangle^{1/2}\right) \stackrel{\alpha \gg 1}{\approx} \frac{1}{\sqrt{2\pi}\alpha} \exp\left(-\frac{\alpha^2}{2}\right)$$

Note that, if $\sigma^2 z \leq T_0^2$ or else if $\sigma^2 z \gg T_0^2$ the PDF of T_c is exactly:

$$p(t) = \frac{1}{\sqrt{2\pi} \langle T_c^2 \rangle^{1/2}} \exp\left(-\frac{t^2}{2 \langle T_c^2 \rangle}\right)$$

where $\langle T_c^2 \rangle$ is given by (17) in the general case and by (18) in case of a Gaussian pulse.

VI. PULSE WIDTH

In terms of the Fourier components the pulse width T_{w2} reads:

$$T_{w2}^2 = \frac{\int \hat{R}(\omega) \hat{E}_0(\omega) d\omega}{\int \hat{E}_0(\omega) d\omega}$$

where

$$\hat{R}(\omega) := \frac{|\hat{u}'|^2(\omega) + |\hat{v}'|^2(\omega)}{|\hat{u}|^2(\omega) + |\hat{v}|^2(\omega)}.$$

The process $\hat{R}(\omega, z)$ obeys a stochastic differential equation that reads:

$$d\hat{R} = \sigma \left(\hat{r}_1 \circ dW^1 + \hat{r}_2 \circ dW^2 + \hat{r}_3 \circ dW^3\right)$$

where the vector $\hat{\mathbf{r}}(\omega)$:

$$\hat{r}_1(\omega) = 2\text{Im}\left(\hat{u}'\hat{u}^* - \hat{v}'\hat{v}^*\right)$$
$$\hat{r}_2(\omega) = 2\text{Im}\left(\hat{u}'\hat{v}^* + \hat{v}'\hat{u}^*\right)$$
$$\hat{r}_3(\omega) = 2\text{Re}\left(\hat{u}'\hat{v}^* - \hat{v}'\hat{u}^*\right)$$

is solution of:

$$\hat{\mathbf{r}}_z = 2\sigma\omega\dot{\mathbf{W}}(z) \times \hat{\mathbf{r}} + 2\sigma\dot{\mathbf{W}}(z)$$

The vector $\hat{r}$ is the so-called PMD vector. Let us denote $\tau(\omega, z) = 4(\hat{R}(\omega, z) - \hat{R}(\omega, 0))$. We have $|\hat{\mathbf{r}}(\omega)|^2 = \tau(\omega, z)$ and τ is the so-called square Differential Group Delay (DGD). Furthermore τ is a diffusion process with infinitesimal generator:

$$\mathcal{L} = 8\sigma^2 \tau \frac{\partial^2}{\partial \tau^2} + 12\sigma^2 \frac{\partial}{\partial \tau}$$

which implies that $\tau(\omega, z)$ obeys a χ^2 distribution with three degrees of freedom. In other words the PDF of $\tau(\omega, z)$ is:

$$p(\tau) = \frac{\tau^{1/2}}{\sqrt{2\pi}(4\sigma^2 z)^{3/2}} \exp\left(-\frac{\tau}{8\sigma^2 z}\right) \mathbf{1}_{\tau \geq 0}$$

Note that the pulse width $T_{w2}(z)$ is thus always larger than $T_{w2}(0)$ whatever z and whatever the realization of the fiber since $\tau(z, \omega) \geq 0$. Besides the statistical distribution of τ is independent of ω. In particular its mean grows linearly with length:

$$\langle \tau \rangle = 12\sigma^2 z$$

The correlation function of τ at two nearby frequencies is required for the forthcoming statements. It is found that the process (τ_1, τ_2, C_p) where $\tau_1 = \tau(\omega_1)$, $\tau_2 = \tau(\omega_2)$, and $C_p = \hat{\mathbf{r}}(\omega_1).\hat{\mathbf{r}}(\omega_2)$ is a diffusion process with infinitesimal generator:

$$\mathcal{L} = 8\sigma^2\tau_1\frac{\partial^2}{\partial\tau_1^2} + 12\sigma^2\frac{\partial}{\partial\tau_1} + 8\sigma^2\tau_2\frac{\partial^2}{\partial\tau_2^2} + 12\sigma^2\frac{\partial}{\partial\tau_2}$$

$$+ 16\sigma^2 C_p\frac{\partial^2}{\partial\tau_1\partial\tau_2} + \sigma^2(12 - 4\Delta\omega^2 C_p)\frac{\partial}{\partial C_p}$$

$$+ 2\sigma^2\left(\Delta\omega^2(\tau_1\tau_2 - C_p^2) + \tau_1 + \tau_2 + 2C_p\right)\frac{\partial^2}{\partial C_p^2}$$

$$+ 8\sigma^2(C_p + \tau_1)\frac{\partial^2}{\partial C_p\partial\tau_1} + 8\sigma^2(C_p + \tau_2)\frac{\partial^2}{\partial C_p\partial\tau_2}$$

where $\Delta\omega = \omega_2 - \omega_1$. The complete expression of the generator represents a straightforward generalization of previously known results. For instance, it allows one to recover the expression of the expectation of the correlation degree between the PMD vectors at two nearby frequencies:

$$\langle C_p\rangle = \frac{3}{\Delta\omega^2}\left(1 - e^{-4\sigma^2\Delta\omega^2 z}\right)$$

which was first derived in [5]. We can also compute the covariance of the square DGD at two nearby frequencies ω_1 and ω_2:

$$\mathrm{Cov}(\tau(\omega_1), \tau(\omega_2)) = \frac{12}{\Delta\omega^4}\left(4\Delta\omega^2\sigma^2 z - 1 + e^{-4\Delta\omega^2\sigma^2 z}\right)$$
$$(19)$$

which is consistent with the formula derived in [6]. Further we can compute the expected value of any combination of C_p, τ_1, and τ_2. For instance, the second moment of the correlation degree of C_p is found to be:

$$\langle C_p^2\rangle = \frac{1}{24\Delta\omega^2}\left(4 - 9e^{-4\Delta\omega^2\sigma^2 z} + 5e^{-12\Delta\omega^2\sigma^2 z}\right.$$
$$\left. + 24\Delta\omega^2\sigma^2 z + 72\Delta\omega^4\sigma^4 z^2\right)$$

Once the statistical distribution of τ is known it is easy to compute the mean and variance of T_{w2}^2:

$$\langle T_{w2}^2\rangle = T_0^2 + 3\sigma^2 z \qquad (20)$$

where T_0 is the initial pulse width. Note that the linear growth of the square pulse width is a universal feature that does not depend on the shape of the initial pulse. This remarkable property is involved by the independence of the expected value of τ with respect to ω. However the variance of T_{w2}^2 depends on the shape of the initial pulse:

$$\mathrm{var}(T_{w2}^2) = \frac{\int\int \hat{E}_0(\omega_1)\hat{E}_0(\omega_2)\mathrm{Cov}(\tau(\omega_1), \tau(\omega_2))d\omega_1 d\omega_2}{16\left(\int \hat{E}_0(\omega)d\omega\right)^2}$$

In case of a Gaussian pulse:

$$\mathrm{var}(T_{w2}^2) = T_0\left(T_0^2 + 4\sigma^2 z\right)^{3/2} - 6T_0^2\sigma^2 z - T_0^4 \qquad (21)$$

More generally the following identity is valid whatever the input pulse:

$$\frac{d\,\mathrm{var}(T_{w2}^2)}{dz} = 12\sigma^2\langle T_c^2\rangle \qquad (22)$$

We can then deduce some properties of the variance:

$$\mathrm{var}(T_{w2}^2) \stackrel{\sigma^2 z \ll T_0^2}{\simeq} 6\sigma^4 z^2$$
$$\stackrel{\sigma^2 z \gg T_0^2}{\simeq} 8\sqrt{\pi}c_1(\sigma^2 z)^{3/2}$$

Let us now compute the PDF of T_{w2}^2:

$$\int_{t^2}^{\infty} p(u)du = \mathbb{P}(T_{w2}^2 \geq t^2)$$

It can be readily estimated in the asymptotic configurations $\sigma^2 z \ll T_0$ or $\gg T_0^2$. If $\sigma^2 z \ll T_0^2$ then the processes $\tau(\omega)$ are completely correlated for ω in the spectrum of the pulse so that the distribution of T_{w2}^2 is:

$$p(t^2) \stackrel{\sigma^2 z \ll T_0^2}{\simeq} \frac{\sqrt{t^2 - T_0^2}}{\sqrt{2\pi}(\sigma^2 z)^{3/2}}\exp\left(-\frac{t^2 - T_0^2}{2\sigma^2 z}\right).$$

If $\sigma^2 z \gg T_0^2$ then the processes τ are uncorrelated with respect to ω by Eq. (19), so that T_{w2}^2 is the sum of large number of uncorrelated components. The central limit theorem can then be invoked to establish that the statistics of T_{w2}^2 is Gaussian. Since the mean value and variance of T_{w2}^2 are known, this implies that the PDF of T_{w2}^2 is:

$$p(t^2) \stackrel{\sigma^2 z \gg T_0^2}{\simeq} \frac{1}{\sqrt{2\pi\,\mathrm{var}(T_{w2}^2)}}\exp\left(-\frac{(t^2 - \langle T_{w2}^2\rangle)^2}{2\,\mathrm{var}(T_{w2}^2)}\right).$$

Note that the decorrelation rate between $\tau(\omega_1)$ and $\tau(\omega_2)$ is very slow (as $1/\Delta\omega^2$), so that the second regime can be observed only when $\sigma^2 z$ is much larger than T_0^2. Besides the same argument shows that the first expression of the PDF should be valid even for $\sigma^2 z \sim T_0^2$, especially for the probability tail $p(t^2)$, $t^2 \gg T_0^2$, if we take care to take into account the exact expression of the variance of T_{w2}^2:

$$p(t^2) \stackrel{\sigma^2 z \lesssim T_0^2}{\simeq} \frac{\sqrt{t^2 - T_2^2}}{\sqrt{2\pi}\gamma_2^{3/2}}\exp\left(-\frac{t^2 - T_2^2}{2\gamma_2}\right)$$

where

$$\gamma_2 = \left(\mathrm{var}(T_{w2}^2)/6\right)^{1/2}$$
$$T_2^2 = \langle T_{w2}^2\rangle - \left(3\,\mathrm{var}(T_{w2}^2)/2\right)^{1/2}$$

If we consider T_{w1} the results read as follows. The mean value of T_{w1}^2 is simply the difference between the mean values of T_{w2}^2 and T_c^2. The variance of T_{w1}^2 is:

$$\mathrm{var}(T_{w1}^2) = \mathrm{var}(T_{w2}^2) + \mathrm{var}(T_c^2) - 2\mathrm{Cov}(T_{w2}^2, T_c^2)$$

Calculations show that

$$\mathrm{var}(T_{w1}^2) = \mathrm{var}(T_{w2}^2) - 2\langle T_c^2\rangle^2$$

In case of a Gaussian pulse we get:

$$\langle T_{w1}^2 \rangle = T_0^2 + 3\sigma^2 z - \frac{T_0^2}{2}\left(\sqrt{1 + \frac{4\sigma^2 z}{T_0^2}} - 1\right) \quad (23)$$

Note that this expression was also derived in Ref. [5]. The variance of T_{w1}^2 is:

$$\mathrm{var}(T_{w1}^2) = T_0^4 \sqrt{1 + \frac{4\sigma^2 z}{T_0^2}}\left(\sqrt{1 + \frac{4\sigma^2 z}{T_0^2}} - 1\right)^2 \quad (24)$$

For a general pulse shape, the mean value of T_{w1}^2 grows linearly:

$$\langle T_{w1}^2 \rangle \overset{\sigma^2 z \ll T_0^2}{\simeq} T_0^2 + 2\sigma^2 z \quad (25)$$

$$\overset{\sigma^2 z \gg T_0^2}{\simeq} 3\sigma^2 z \quad (26)$$

while the variance satisfies:

$$\mathrm{var}(T_{w1}^2) \overset{\sigma^2 z \ll T_0^2}{\simeq} 4\sigma^4 z^2$$

$$\overset{\sigma^2 z \gg T_0^2}{\simeq} 8\sqrt{\pi} c_1 (\sigma^2 z)^{3/2}$$

Let us now estimate the PDF of T_{w1}^2. If $\sigma^2 z \ll T_0^2$ then we know that $T_{w1}^2 = T_0^2 + \sigma^2 W_z^{2\,2} + \sigma^2 W_z^{3\,2} + O(\sigma^4 z^2 / T_0^2)$ so that it obeys a χ^2 distribution with two degrees of freedom:

$$p(t^2) \overset{\sigma^2 z \ll T_0^2}{=} \frac{1}{\sigma^2 z} \exp\left(-\frac{t^2 - T_0^2}{2\sigma^2 z}\right), \quad t^2 > T_0^2. \quad (27)$$

If $\sigma^2 z \gg T_0^2$ then T_{w2}^2 is the prevailing term in the expression of T_{w1}^2 so that their PDF are common. In the general configuration, T_{w2}^2 obeys a χ^2 distribution with three degrees of freedom, while T_c^2 is the square of a Gaussian random variable, so that we can guess that the PDF of T_{w1}^2 corresponds to a χ^2 distribution with a number d_1 of degrees of freedom which is between 2 and 3. Since we know the mean and variance of T_{w1}^2 we can fix the free parameters:

$$d_1 = \left(\langle T_{w1}^2 \rangle - T_0^2\right) / \left(\sigma^2 z\right)$$

so that:

$$p(t^2) = \frac{(t^2 - T_1^2)^{d_1/2 - 1}}{2^{d_1/2}\Gamma(d_1/2)\gamma_1^{d_1/2}} \exp\left(-\frac{t^2 - T_1^2}{2\gamma_1}\right) \quad (28)$$

where $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ and

$$\gamma_1 = \left(\mathrm{var}(T_{w1}^2)/(2d_1)\right)^{1/2}$$
$$T_1^2 = \langle T_{w1}^2 \rangle - \left(d_1 \mathrm{var}(T_{w1}^2)/2\right)^{1/2}$$

VII. ROTATION OF POLARIZATION

The rotation of the pulse polarization can be characterized by the ratio P_r of the energy on the input polarization state over the total energy. In terms of the first Stokes parameter the parameter P_r reads as:

$$P_r = \frac{1}{2} + \frac{1}{2}\frac{\int \hat{s}_1(\omega)d\omega}{\int \hat{E}_0(\omega)d\omega}$$

The computation of the mean and covariance of $\hat{s}_1$ is straightforward:

$$\langle \hat{s}_1(\omega)\rangle = \hat{E}_0(\omega)\exp(-4\omega^2\sigma^2 z)$$

$$\mathrm{Cov}(\hat{s}_1(\omega_1), \hat{s}_1(\omega_2)) = \hat{E}_0(\omega_1)\hat{E}_0(\omega_2)\left(\frac{1}{3}e^{-4(\omega_1-\omega_2)^2\sigma^2 z}\right.$$

$$\left. +\frac{2}{3}e^{-4(\omega_1^2+\omega_2^2+\omega_1\omega_2)\sigma^2 z} - e^{-4(\omega_1^2+\omega_2^2)\sigma^2 z}\right)$$

We can then state that the mean of P_r is:

$$\langle P_r \rangle = \frac{1}{2} + \frac{1}{2}\frac{\int \exp(-4\omega^2\sigma^2 z)\hat{E}_0(\omega)d\omega}{\int \hat{E}_0(\omega)d\omega}$$

which is equal for a Gaussian pulse to:

$$\langle P_r \rangle = \frac{1}{2} + \frac{1}{2}\left(1 + \frac{2\sigma^2 z}{T_0^2}\right)^{-1/2} \quad (29)$$

For a general pulse, we have:

$$\langle P_r \rangle \overset{\sigma^2 z \ll T_0^2}{\simeq} 1 - 2\sigma^2 z c_2$$

$$\overset{\sigma^2 z \gg T_0^2}{\simeq} \frac{1}{2} + \frac{c_3\sqrt{\pi}}{4\sqrt{\sigma^2 z}}$$

where $c_3 = \dfrac{\hat{E}_0(0)}{\int \hat{E}_0(\omega)d\omega} = \dfrac{1}{2\pi}\dfrac{\left|\int u_0 dt\right|^2}{\int |u_0|^2 dt}$.

The fluctuations around the mean value can be estimated by the computation of the variance of P_r:

$$\mathrm{var}(P_r) = \frac{1}{4}\frac{\int\int \mathrm{Cov}(\hat{s}_1(\omega_1), \hat{s}_1(\omega_2))d\omega_1 d\omega_2}{\left(\int \hat{E}_0(\omega)d\omega\right)^2}$$

For a Gaussian pulse:

$$\mathrm{var}(P_r) = \frac{1}{12}\left(1 + \frac{4\sigma^2 z}{T_0^2}\right)^{-1/2} - \frac{1}{4}\left(1 + \frac{2\sigma^2 z}{T_0^2}\right)^{-1}$$

$$+\frac{1}{6}\left(1 + \frac{3\sigma^2 z}{T_0^2}\right)^{-1/2}\left(1 + \frac{\sigma^2 z}{T_0^2}\right)^{-1/2} \quad (30)$$

For a general pulse, we have:

$$\mathrm{var}(P_r) \overset{\sigma^2 z \ll T_0^2}{\simeq} 16\sigma^4 z^2 c_2^2$$

$$\overset{\sigma^2 z \gg T_0^2}{\simeq} \frac{\sqrt{\pi} c_1}{6\sqrt{\sigma^2 z}}$$

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VIII. DEGREE OF POLARIZATION

The degree of polarization P_d can be expressed in terms of the frequency correlation degree of the Stokes vectors as:

$$P_d = \frac{\sqrt{\int\int C(\omega_1,\omega_2)\hat{E}_0(\omega_1)\hat{E}_0(\omega_2)d\omega_1 d\omega_2}}{\int \hat{E}_0(\omega)d\omega}$$

It is then easy to compute the mean of P_d^2. One then gets that the degree of polarization and the time displacement satisfy the equation:

$$\frac{d\left\langle T_c^2\right\rangle}{dz} = \sigma^2 \left\langle P_d^2\right\rangle$$

For a Gaussian pulse we have:

$$\left\langle P_d^2\right\rangle = \left(1 + \frac{4\sigma^2 z}{T_0^2}\right)^{-1/2}$$

For a general pulse the degree of polarization first decays linearly:

$$\left\langle P_d^2\right\rangle \overset{\sigma^2 z \ll T_0^2}{\simeq} 1 - 8\sigma^2 z(c_2 - c_4^2)$$

while it slowly decays to 0 for long propagation distances:

$$\left\langle P_d^2\right\rangle \overset{\sigma^2 z \gg T_0^2}{\simeq} \frac{\sqrt{\pi}c_1}{2\sigma\sqrt{z}}$$

More exactly, if $\sigma^2 z \ll T_0^2$, then the PDF of the random variable $1 - P_d$ is:

$$p(x) = \frac{1}{4\sigma^2(c_2 - c_4^2)z}\exp\left(-\frac{x}{4\sigma^2(c_2-c_4^2)z}\right).$$

If $\sigma^2 z \gg T_0^2$, then the PDF of P_d is:

$$p(x) = \frac{\sqrt{2}}{\sqrt{\pi}\alpha^{3/2}}x^2 \exp\left(-\frac{x^2}{2\alpha}\right)$$

where $\alpha(z) = (\sqrt{\pi}c_1)/(6\sigma\sqrt{z})$.

IX. INFLUENCE OF GVD

If we take into account both GVD and PMD, then the field $\mathbf{U} = (u,v)^T$ satisfies Eq. (2). In terms of the Fourier components $\hat{\mathbf{U}} = (\hat{u},\hat{v})^T$ this partial differential equation reads as a collection of ordinary differential equations:

$$\hat{\mathbf{U}}_z = i\omega R(z)\hat{\mathbf{U}} + i\frac{\beta''}{2}\omega^2\hat{\mathbf{U}}$$

Setting $\check{\mathbf{U}} = \hat{\mathbf{U}}\exp\left(-i\beta''\omega^2 z/2\right)$ the system of differential equations for $\check{\mathbf{U}}$ is the same as in the absence of GVD. The relevant quantities $\hat{E}_0(\omega)$, $\hat{R}(\omega)$, and $\hat{t}(\omega)$ can be expressed as:

$$\hat{E}_0(\omega) = \check{E}_0(\omega)$$
$$\hat{R}(\omega) = \check{R}(\omega) + \beta''^2\omega^2 z^2\check{E}_0(\omega) + 2\beta''\omega z\check{I}(\omega)$$
$$\hat{t}(\omega) = \check{t}(\omega) + \beta''\omega z\check{E}_0(\omega)$$

where $\check{E}_0$, $\check{R}$, and $\check{t}$ are defined in terms of $\check{u}$ and $\check{v}$ similarly as $\hat{E}_0$, $\hat{R}$, and $\hat{t}$. We know from the previous sections the statistical distributions of $\check{E}_0$, $\check{R}$, and $\check{t}$, so that it is easy to get the statistical distributions of $\hat{E}_0$, $\hat{R}$, and $\hat{t}$.

Let us first address the timing displacement. Only the mean value of the time displacement is modified by the presence of GVD:

$$\left\langle T_c\right\rangle = \beta''c_4 z$$
$$\mathrm{var}(T_c) = \mathrm{var}(T_c)|_{\beta''=0}$$

For the Gaussian pulse (12) we have $c_4 = 0$, which shows that GVD has no influence on the timing displacement. Let us now consider the pulse broadening. Both the mean value and variance of the pulse width are enhanced by the GVD:

$$\left\langle T_{w2}^2\right\rangle = T_0^2 + 3\sigma^2 z + c_2\beta''^2 z^2$$
$$\mathrm{var}(T_{w2}^2) = \mathrm{var}(T_{w2}^2)\big|_{\beta''=0}$$
$$+4\beta''^2 z^2\frac{\int\int \omega_1\omega_2 \left\langle \check{t}(\omega_1)\check{t}(\omega_2)\right\rangle \check{E}_0(\omega_1)\check{E}_0(\omega_2)d\omega_1 d\omega_2}{\left(\int \check{E}_0(\omega)d\omega\right)^2}$$

$$\left\langle T_{w1}^2\right\rangle = T_0^2 + 3\sigma^2 z + (c_2 + c_4^2)\beta''^2 z^2 + \mathrm{var}(T_c)|_{\beta''=0}$$
$$\mathrm{var}(T_{w1}^2) = \mathrm{var}(T_{w2}^2) - 2\mathrm{var}(T_c)^2$$

For the Gaussian pulse we have:

$$\left\langle T_{w2}^2\right\rangle = T_0^2 + 3\sigma^2 z + \frac{\beta''^2 z^2}{4T_0^2}$$
$$\mathrm{var}(T_{w2}^2) = T_0^4\left(\left(1 + \frac{4\sigma^2 z}{T_0^2}\right)^{3/2} - 6\frac{\sigma^2 z}{T_0^2} - 1\right)$$
$$+ \frac{\beta''^2 z^2}{4}\frac{\left(\sqrt{1 + \frac{4\sigma^2 z}{T_0^2}} - 1\right)^2}{\sqrt{1 + \frac{4\sigma^2 z}{T_0^2}}}$$

$$\left\langle T_{w1}^2\right\rangle = T_0^2 + 3\sigma^2 z + \frac{\beta''^2 z^2}{4T_0^2} - \frac{T_0^2}{2}\left(\sqrt{1 + \frac{4\sigma^2 z}{T_0^2}} - 1\right)$$
$$\mathrm{var}(T_{w1}^2) = T_0^4\sqrt{1 + \frac{4\sigma^2 z}{T_0^2}}\left(\sqrt{1 + \frac{4\sigma^2 z}{T_0^2}} - 1\right)^2$$
$$\times\left(1 + \frac{\beta''^2 z^2}{4T_0^4\left(1 + \frac{4\sigma^2 z}{T_0^2}\right)}\right)$$

The degree of polarization is not affected by the presence of GVD.

X. NUMERICAL SIMULATIONS

Here we solve numerically the propagation equation in absence of GVD. We consider a Gaussian pulse or a sech pulse as an initial condition with an initial rms width $T_0 = 4$ ps. We assume that the magnitude of linear birefringence is constant. In order to simulate the random fluctuations of birefringence we incorporate at fictitious junctions between adjacent fiber pieces with length Δz random axial rotation and addition of random phase difference between the field components. We take $\Delta\beta' = 2$ ps/km, and $\Delta z = 25$ m so that $\sigma^2 = 8.3 \ 10^{-3}$ ps^2/km. Note that this corresponds to a configuration where the usual parameter $D_p = 0.29$ ps/$\sqrt{\text{km}}$. The number of samples used to evaluate averaged values and variances are 10^4. Comparison of the numerical results and theoretical formulas show excellent agreement (Figures 2-3).
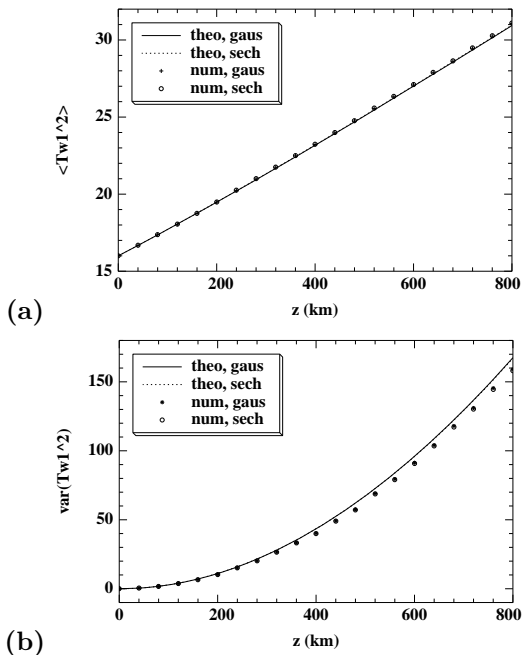

(a)

(b)

FIG. 2. *Square rms widths of pulses with gauss or sech shapes. The lines correspond to the theoretical values while the crosses and circles represent the results averaged over 10^4 numerical simulations. The mean values are plotted in picture a, the variances are plotted in picture b.*
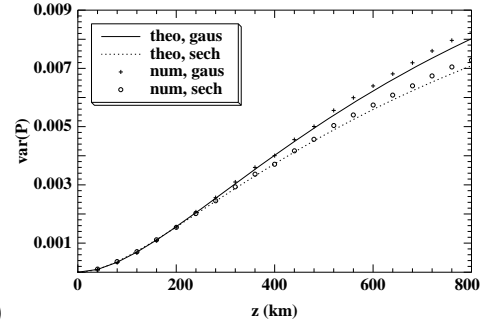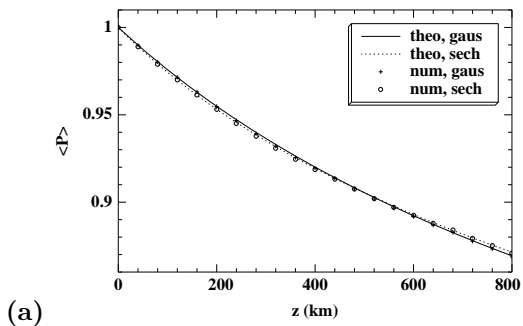

(a)

(b)

FIG. 3. *Rotations of the polarization of pulses with gauss or sech shapes. The lines correspond to the theoretical values while the crosses and circles represent the results averaged over 10^4 numerical simulations. The mean values are plotted in picture a, the variances are plotted in picture b.*

Another series of runs have been carried out where the local birefringence strength and angle vary with distance. We present in Figures 4-5 the results corresponding to two configurations where $\sigma^2 z/T_0^2 = 0.12$ ($\ll 1$) and $\sigma^2 z/T_0^2 = 2.2$ respectively. The agreements with the theoretical formulas are still excellent. In particular the numerical histograms of the pulse widths are found very close to the theoretical PDF.
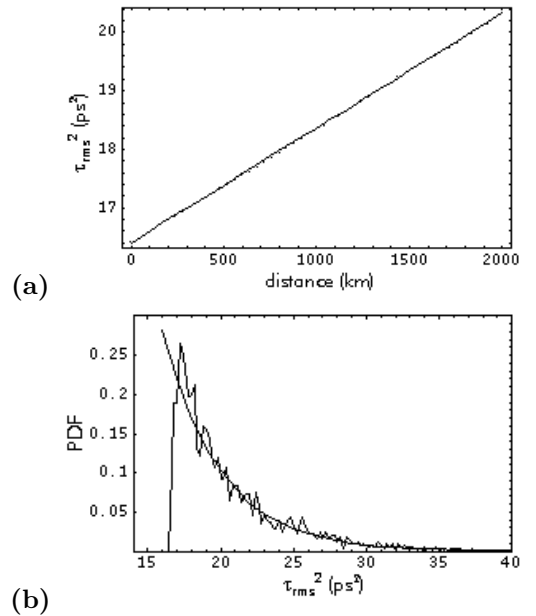

(a)

(b)

FIG. 4. *Mean square width (picture a) and PDF of the pulse width (picture b) of pulses with gauss shapes. The numerical values have been computed from 3000 runs. The PDF is plotted at $z = 2000$ km. Here $T_0^2 = 16.4$ ps^2 and $D_p = 0.1$ ps/$\sqrt{km}$ The PDF is plotted at $z = 2000$ km which corresponds to $\sigma^2 z/T_0^2 = 0.12$. The theoretical PDF is formula (27).*
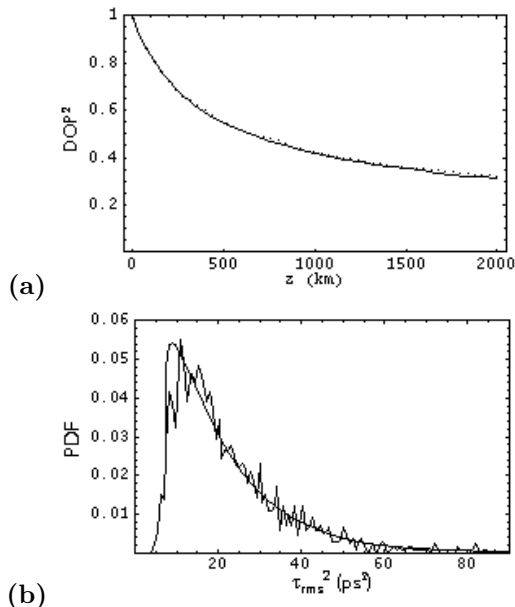
9

(a)

(b)

FIG. 5. *Degree of polarization (picture a) and PDF of the pulse width (picture b) of pulses with gauss shapes. The numerical values have been computed from 3000 runs. Here $T_0^2 = 3.6$ ps^2 and $D_p = 0.2$ $ps/\sqrt{km}$. The PDF is plotted at $z = 2000$ km which corresponds to $\sigma^2 z/T_0^2 = 2.2$. The theoretical PDF is formula (28).*

XI. CONCLUSION

In this paper we have studied behaviors of short pulses in fibers with randomly varying birefringence. We have derived closed form expressions for the statistical values of relevant quantities. The mean values, variances and PDF of the time displacement, time rms width and degree of polarization have been computed. This gives further insight in the PMD dynamic, as well as practical help for engineers in that not only the mean values averaged over an ensemble of fibers are given. As remarked in [6], the statistical properties of PMD in long fibers are uniquely determined by the mean DGD. It could be also interesting to take into account polarization-dependent losses. This could certainly be done, as the statistical distribution of the DGD was carried out in [19].

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