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Daniel Král'[*] Jean-Sébastien Sereni[†] Michael Stiebitz[‡]

Abstract

We prove that every n-vertex cubic bridgeless graph has at least $n/2$ perfect matchings and give a list of all 17 such graphs that have less than $n/2 + 2$ perfect matchings.

1 Introduction

Graphs considered in this paper can contain multiple edges but do not contain loops. A graph is *cubic* if every vertex has degree 3 and a subgraph is *spanning* if it contains all the vertices. A *perfect matching* is a spanning subgraph where every vertex has degree 1. A graph is *bridgeless* if it is connected and stays connected after removing any edge. Let us state the following classical theorem of Petersen [12].

Theorem 1 (Petersen, 1891). *Every cubic bridgeless graph G has a perfect matching.*

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In fact, a stronger statement is true, as was shown by Plesník [13].

Theorem 2 (Plesník, 1972)**.** *Let G be a bridgeless cubic graph. Every edge of G is contained in a perfect matching. Moreover, for any two edges e and f of G, there is a perfect matching avoiding both e and f.*

As a consequence, every cubic bridgeless graph has at least three perfect matchings. A natural question is what is the least number of perfect matchings that an n-vertex cubic bridgeless graph contains. Lovász and Plummer conjectured in the mid-1970s that this number grows exponentially with the number of vertices (see the book by Lovász and Plummer [9, Conjecture 8.1.8]).

Conjecture 1 (Lovász and Plummer, 1970s)**.** *Every cubic bridgeless graph with n vertices has at least $2^{\Omega(n)}$ perfect matchings.*

Edmonds, Lovász, and Pulleyblank [3, Theorem 5.1] proved the following theorem, which generalizes a result of Naddef [11] for matching covered bipartite graphs (see Section 2 for the relevant definitions).

Theorem 3 (Edmonds, Lovász, and Pulleyblank, 1982)**.** *The dimension of the perfect matching polytope of a connected and matching covered graph $G = (V, E)$ is*

$$|E| - |V| + 1 - b(G)\,,$$

where $b(G)$ is the number of bricks in the brick and brace decomposition of G.

This theorem implies that the dimension of the perfect matching polytope of a cubic bridgeless n-vertex graph is at least $n/4 + 1$. Since the vertices of the polytope correspond to distinct perfect matchings, we have the following lower bound on the number of perfect matchings of an n-vertex cubic bridgeless graph.

Theorem 4 (Edmonds, Lovász, and Pulleyblank, 1982)**.** *Every cubic bridgeless graph with n vertices has at least $n/4 + 2$ perfect matchings.*

If a cubic graph G has no non-trivial edge-cut of size 3, then Theorem 3 gives a better bound, since G must be a brick. A graph G is *cyclically k-edge-connected* if it has no edge-cut of size at most $k-1$ the removal of which yields at least two non-acyclic components. The following result is a simple consequence of Theorem 3.

Theorem 5 (Edmonds, Lovász, and Pulleyblank, 1982)**.** *Every cubic cyclically* 4*-edge-connected graph with* n *vertices has at least* $n/2+1$ *perfect matchings.*

Conjecture 1 has been verified for several special classes of graphs, one of them being bipartite graphs. The first non-trivial lower bound on the number of perfect matchings in cubic bridgeless bipartite graphs was obtained in 1969 by Sinkhorn [17] who proved a bound of $\frac{n}{2}$, thereby establishing a conjecture of Marshall. The same year, Minc [10] increased this lower bound by 2. Then, a bound of $\frac{3n}{2}-3$ was proved by Hartfiel and Crosby [7]. The first exponential bound, $6\cdot\left(\frac{4}{3}\right)^{n/2-3}$, was obtained in 1979 by Voorhoeve [18]. This was generalized to all regular bipartite graphs in 1998 by Schrijver [14], who thereby proved a conjecture of himself and Valiant [16]. His argument is involved, and we note that, as a particular case of a different and more general approach (using hyperbolic polynomials), Gurvits [6] managed to slightly improve the bound, as well as simplify the proof.

Recently, an important step towards a proof of Conjecture 1 has been made by Chudnovsky and Seymour [2] who proved the conjecture for planar graphs.

Theorem 6 (Chudnovsky and Seymour, 2008)**.** *Every cubic bridgeless planar graph with* n *vertices has at least* $2^{n/655978752}$ *perfect matchings.*

In this paper, we focus on proving a bound corresponding to that stated in Theorem 5 for all cubic bridgeless graphs, i.e., we remove the assumption that G is cyclically 4-edge-connected. In particular, we prove that every n-vertex cubic bridgeless graph G has at least $n/2$ perfect matchings and provide complete lists of such graphs with exactly $n/2$ and $n/2+1$ perfect matchings. Our main result is the following theorem.

Theorem 7. *Let* G *be a cubic bridgeless graph with* n *vertices. The graph* G *contains at least* $n/2+2$ *perfect matchings unless it is is one of the* 17 *exceptional graphs* $I_1,\dots,I_{10}$ *or* $H_0,\dots,H_6$ *which are depicted in Figures 2, 3, 4 and 6. The graph* H_0 *contains* $n/2$ *perfect matchings and the other exceptional graphs contain* $n/2+1$ *perfect matchings.*

Using our results, Esperet et al. [5] showed that every cubic bridgeless n-vertex graph has at least $3n/4-10$ perfect matchings and a superlinear bound has later been found [4].

2 Brick and brace decomposition

The brick and brace decomposition is one of the essential notions in the theory of perfect matchings. We explain the notion in general though we apply it only to cubic bridgeless graphs. We refer to the treatise of Schrijver [15, Chapter 37] for further exposition. Given a graph G and a subset X of vertices, $G - X$ is the subgraph obtained from G by removing the vertices of X. A graph G is *matching covered* if every edge of G is contained in a perfect matching. If V_1 and V_2 is a partition of a vertex set of G, then the edges with one end-vertex in V_1 and the other in V_2 form an *edge-cut*. An edge-cut is *non-trivial* if both V_1 and V_2 contain at least two vertices. An edge-cut E is *tight* if every perfect matching contains exactly one edge of E.

Let G be a matching covered graph with a non-trivial tight edge-cut E, which partitions the vertices of G into two sets V_1 and V_2. We decompose G into two simpler graphs G_1 and G_2 by *splitting along* E as follows: the graph G_i is obtained by contracting all the vertices of V_i to a single vertex, for $i \in \{1, 2\}$. Note that the structure of perfect matchings of G_1 and G_2 reflects the structure of perfect matchings of G: no matchings are lost by the splitting since every perfect matching uses exactly one edge of E. In particular, the graphs G_1 and G_2 are matching covered. If one or both of the new graphs contain a non-trivial tight edge-cut, we can again split along it. We continue until we obtain a multiset of graphs with no non-trivial tight edge-cuts. In doing so, we keep parallel edges that arise (which deviates from some other literature), since multiple edges play an important role regarding the number of perfect matchings. The following theorem of Lovász [8] states that splitting along non-trivial tight edge-cuts is independent of the order in that the edge-cuts were chosen.

Theorem 8 (Lovász, 1987)**.** *Let G be a matching covered graph. Up to multiplicities of edges, the multiset of graphs with no non-trivial tight edge-cuts obtained by splitting along non-trivial tight edge-cuts of G depends neither on the chosen edge-cuts nor on the order in which the splittings are performed.*

The graphs in the multiset obtained by splitting along non-trivial tight edge-cuts are of two kinds. Bipartite graphs with no non-trivial tight edge-cut are referred to as *braces*. They are characterized by the following property [3].

Theorem 9 (Edmonds, Lovász, and Pulleybank, 1982)**.** *A bipartite matching covered graph G has no non-trivial tight edge-cut if and only if for every pair*

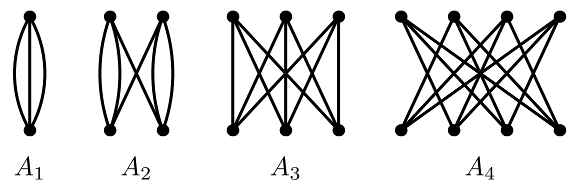


Figure 1: Cubic braces of order at most 4.

of subsets V and W from different color classes such that $|V| = |W| \leq 2$, the graph $G - (V \cup W)$ has a perfect matching.

If G is a cubic bridgeless graph that is a brace, shortly a *cubic brace*, we call the number of vertices in each color class of G the *order* of the brace. There is a unique cubic brace A_n of order n for each $n \in \{1, 2, 3, 4\}$. The braces $A_1, \ldots, A_4$ can be found in Figure 1.

Non-bipartite graphs that appear in the decomposition of a matching covered graph along its non-trivial tight edge-cuts are known as *bricks*. They are characterized as follows [3].

Theorem 10 (Edmonds, Lovász, and Pulleybank, 1982)**.** *A non-bipartite matching covered graph G has no non-trivial tight edge-cut if and only if it is 3-connected and for every two-element subset V of its vertices, the graph $G - V$ has a perfect matching.*

As in the case of braces, we refer to bricks that are cubic bridgeless graphs as to *cubic bricks*. Examples of cubic bricks can be found in Figure 2. Since the decomposition of a graph G along its non-trivial tight edge-cuts is formed by bricks and braces, it is called the *brick and brace decomposition* of G. Recall that this decomposition is unique by Theorem 8. The brick and brace decomposition is *non-trivial* if it contains at least two graphs, i.e., the brick and brace decomposition of G is non-trivial if and only if G is neither a brick nor a brace.

In the rest of this section, we deal with cubic bridgeless graphs only. Before our further considerations, let us state the following consequence of the structure of the perfect matching polytope of a cubic bridgeless graph G: *every tight edge-cut of G has size* 3. Indeed, if E is a tight edge-cut of G, then for every vector corresponding to a perfect matching of G, the coordinates of the edges of E sum-up to exactly 1. Thus, the same is true for every vector of the perfect matching polytope of G. The conclusion follows by observing that the vector with all coordinates equal to 1/3 belongs to the perfect matching polytope of G (which is a direct consequence of Edmonds' description of the perfect matching polytope of a graph). As a result, the graphs forming the brick and brace decomposition of a cubic bridgeless graph are also cubic and bridgeless. Furthermore, it follows from Theorem 10 that every cubic brick is a simple graph.

We now state and prove some basic lemmas on the brick and brace decompositions of cubic bridgeless graphs, and on cubic bricks and cubic braces. Though the reader can be familiar with some of these facts, we give their short proofs for completeness. Before our first lemma, we need two more definitions. A vertex v of a cubic graph G is *trimatched* if there exists a spanning subgraph H of G such that the degree of v in H is 3 and the degrees of the other vertices of H are 1. The graph G is *trimatched* if every vertex of G is trimatched. If G is a simple graph, then a vertex v with neighbors v_1, v_2 and v_3 is trimatched if and only if the graph $G - \{v, v_1, v_2, v_3\}$ has a perfect matching.

Lemma 11. *Every cubic brick G is trimatched.*

Proof. Let v be any vertex of G and v_1, v_2 and v_3 its neighbors. By Theorem 10, the graph $G - \{v_2, v_3\}$ has a perfect matching M. Since G is cubic, this perfect matching includes the edge vv_1. Since every cubic brick is simple, the perfect matching M together with the edges vv_2 and vv_3 is a spanning subgraph of G of the type sought. □

Using Lemma 11, we show that every non-trivial brick and brace decomposition contains a brace.

Lemma 12. *Every non-trivial brick and brace decomposition of a cubic bridgeless graph contains a brace.*

Proof. It is enough to prove that there is no graph with a brick and brace decomposition consisting of two bricks. Suppose on the contrary that G is such a graph. Let $E := \{v_1w_1, v_2w_2, v_3w_3\}$ be a non-trivial tight edge-cut of G: it partitions the vertex-set of G into two sets V_1 and V_2. Let G_1 and G_2 be the two bricks obtained by splitting along E. We may assume that G_1 contains the vertices v_i and we let u_1 be the vertex of G_1 obtained by contracting the vertices of V_2. Similarly, G_2 contains the vertices w_i and we let u_2 be the vertex of G_2 obtained by contracting the vertices of V_1.

By Lemma 11, both bricks G_1 and G_2 are trimatched. In particular, for $i \in \{1, 2\}$, the vertex u_i is trimatched in G_i. Let H_i be a spanning subgraph of G_i such that u_i has degree 3 in H_i and the other vertices have degree 1. The subgraphs H_1 and H_2 combine to a perfect matching of G including all three edges of E, which contradicts our assumption that E is tight. □

Let us now turn our attention to cubic braces. Again, we have to introduce a definition. An edge of a matching covered graph G is a *solo-edge* if it is contained in exactly one perfect matching. A matching covered graph is *matching double-covered* if it has no solo-edges.

Lemma 13. *Every cubic brace different from A_1 and A_2 (i.e. every cubic brace of order at least 3) is matching double-covered.*

Proof. Let G be a cubic brace. Since A_1 and A_2 are the only cubic braces of order at most 2, the order of the brace G is at least 3. Let uv be an edge of G and M a matching containing uv. Since the order of G is greater than 2, there exists an edge $u'v'$ not in M and not adjacent with uv. By Theorem 9, the graph $G - \{u, v, u', v'\}$ has a perfect matching M'. We can extend M' to G by adding the edges uv and $u'v'$. Thus, M and M' are two distinct perfect matchings of G containing the edge uv. Consequently, G has no solo-edge. □

We finish this section with a lemma on cubic graphs the decomposition of which contains a brace different from A_1 and A_2.

Lemma 14. *Every cubic bridgeless graph G with a brick and brace decomposition containing a brace different from A_1 and A_2 is matching double-covered.*

Proof. We proceed by induction on the number k of graphs in the brick and brace decomposition of G. If $k = 1$, then G is matching double-covered by Lemma 13. Assume that $k \geq 2$ and let us show that G is matching double-covered. To this end, let e be an edge of G. Consider any non-trivial tight edge-cut E of G. Let G_1 and G_2 be the graphs obtained from G by splitting along this edge-cut.

By Theorem 8, the brick and brace decomposition of G_1 or G_2 contains a brace different from A_1 and A_2. Assume that G_1 has this property. Thus, G_1 is matching double-covered by induction.

If e is in G_1, then G_1 contains two distinct perfect matchings containing e, and each of them can be extended to a perfect matching of G since G_2 is matching covered. Hence, e is not a solo-edge.

If e is in G_2, then a perfect matching of G_2 containing e can be extended to a perfect matching of G in at least two different ways, since G_1 is matching double-covered. Consequently, e is not a solo-edge either. □

3 Good cubic graphs

In this section, we present most of our tools for proving the lower bounds of Theorem 7 on the number of perfect matchings in a cubic bridgeless graph. Let us start with some terminology. An n-vertex cubic bridgeless graph G is *α-good* if G has $n/2+\alpha$ perfect matchings, and G is *$(\geq \alpha)$-good* if it has at least $n/2+\alpha$ perfect matchings. By Theorem 3, the dimension of the perfect matching polytope of an n-vertex cubic brick is $\frac{n}{2}$. A theorem of de Carvalho, Lucchesi, and Murty [1] charaterizes cubic bricks that have a simplex with $\frac{n}{2}+1$ vertices as the perfect matching polytope. Their result implies that every brick is (≥ 2)-good except the bricks $I_1, \ldots, I_{10}$ depicted in Figure 2.

Theorem 15 (de Carvalho, Lucchesi, and Murty, 2005)**.** *Every brick different from the* 10 *bricks $I_1, \ldots, I_{10}$ depicted in Figure 2 is* (≥ 2)*-good. All the bricks $I_1, \ldots, I_{10}$ are* 1*-good.*

Our lower bound argument is based on the analysis of the brick and brace decompositions of cubic bridgeless graphs. We have introduced the operation of *splitting along non-trivial tight edge-cuts* in Section 2. We now define the inverse operation. Let G_1 and G_2 be cubic bridgeless graphs, u a vertex of G_1 with neighbors u_1, u_2 and u_3, and v a vertex of G_2 with neighbors v_1, v_2 and v_3. Let G be the graph obtained from G_1 and G_2 by removing the vertices u and v and adding the edges u_1v_1, u_2v_2 and u_3v_3. We say that G is obtained by *gluing* the graphs G_1 and G_2, or more precisely from G_1 by *gluing G_2 through the vertex u*, or from G_2 by *gluing G_1 through the vertex v*. The gluing is a *solo-gluing* if for every $i \in \{1,2,3\}$, the edge uu_i is a solo-edge in G_1 or the edge vv_i is a solo-edge in G_2.

We now prove two lemmas giving lower bounds on the number of perfect matchings in graphs obtained by gluing smaller graphs. Before doing so, let us introduce one more definition. If G is a cubic bridgeless graph and v a vertex of G with neighbors v_1, v_2 and v_3, then the *pattern* of v is the triple (m_1, m_2, m_3) where m_i is the number of perfect matchings of G containing the edge vv_i for $i \in \{1,2,3\}$. We are now ready to prove the two lemmas.

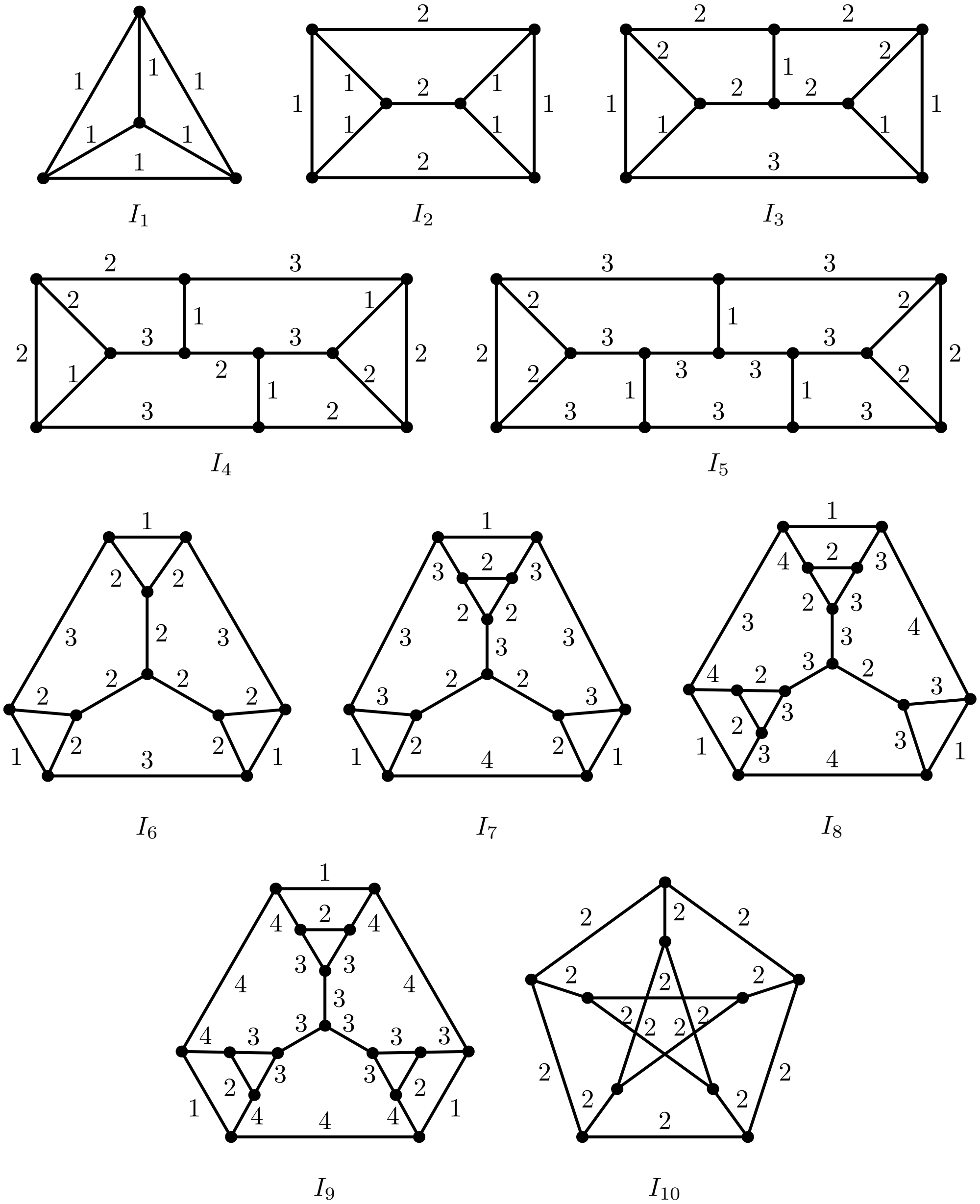


Figure 2: All 1-good bricks. The number near each edge indicates the number of perfect matchings containing this edge.

Lemma 16. *Let G be a cubic bridgeless graph obtained by gluing an α-good graph G_a and a β-good graph G_b. The graph G is $(\geq \alpha+\beta-1)$-good unless G is obtained by a solo-gluing, in which case G is $(\alpha+\beta-2)$-good.*

Proof. Let n_a be the number of vertices of G_a and n_b the number of vertices of G_b. Next, let v_a be the vertex of G_a such that G is obtained from G_a by gluing G_b through v_a. Similarly, v_b is the vertex of G_b such that G is obtained from G_b by gluing G_a through v_b. Finally, let $(m_{a,1}, m_{a,2}, m_{a,3})$ be the pattern of v_a in G_a and $(m_{b,1}, m_{b,2}, m_{b,3})$ the pattern of v_b in G_b.

Since G_a is α-good and G_b is β-good,

$$n_a/2+\alpha = m_{a,1}+m_{a,2}+m_{a,3} \tag{1}$$

and

$$n_b/2+\beta = m_{b,1}+m_{b,2}+m_{b,3}\,. \tag{2}$$

Observe that $xy \geq x+y-1$ for every positive integers x and y, with equality if and only $x=1$ or $y=1$. Hence, the definition of gluing and the fact that $m_{a,i} \geq 1$ and $m_{b,i} \geq 1$ yield that the number of perfect matchings of G is at least

$$m_{a,1}m_{b,1}+m_{a,2}m_{b,2}+m_{a,3}m_{b,3} \geq m_{a,1}+m_{a,2}+m_{a,3}+m_{b,1}+m_{b,2}+m_{b,3}-3 \tag{3}$$

with equality if and only if for every $i \in \{1,2,3\}$, at least one of the numbers $m_{a,i}$ and $m_{b,i}$ equals 1. Since G has n_a+n_b-2 vertices, (1), (2) and (3) imply that G is $(\geq \alpha+\beta-2)$-good. Moreover, G is $(\geq \alpha+\beta-1)$-good unless at least one of the numbers $m_{a,i}$ and $m_{b,i}$ equals 1 for every $i \in \{1,2,3\}$, i.e. unless G is obtained by a solo-gluing. □

In the final lemma of this section, we show that the bound from Lemma 16 can be improved if one of the glued graphs is matching double-covered.

Lemma 17. *Let G be a cubic bridgeless graph obtained by gluing an α-good graph G_a and a β-good graph G_b. If G_a is matching double-covered and G_b has at least five perfect matchings, then G is $(\alpha+\beta)$-good.*

Proof. Let us retain the notation from the proof of Lemma 16. Assume that $m_{b,1} \geq m_{b,2} \geq m_{b,3}$, and let p be the number of perfect matchings of G. It is still true that

$$p = m_{a,1}m_{b,1}+m_{a,2}m_{b,2}+m_{a,3}m_{b,3}\,. \tag{4}$$

First, assume that $m_{b,2} = m_{b,3} = 1$. Hence, $m_{b,1} \geq 3$ since G_b has at least five perfect matchings. Note that $m_{a,i} \geq 2$ for every $i \in \{1, 2, 3\}$ since G_a is matching double-covered. In particular, $m_{a,1}m_{b,1} \geq m_{a,1} + m_{b,1} + 1$ since $xy \geq x + y + 1$ for any $x \geq 2$ and $y \geq 3$. Thus, (4) translates to

$$\begin{aligned} p &\geq m_{a,1} + m_{b,1} + 1 + m_{a,2} + m_{a,3} \\ &= m_{a,1} + m_{a,2} + m_{a,3} + m_{b,1} + m_{b,2} + m_{b,3} - 1 \\ &= \frac{n_a}{2} + \frac{n_b}{2} + \alpha + \beta - 1\,, \end{aligned}$$

by the equations (1) and (2). Since the number of vertices of G is $n_a + n_b - 2$, we deduce that G is $(\alpha + \beta)$-good.

We next assume that both $m_{b,1}$ and $m_{b,2}$ are at least 2. Again recalling that $xy \geq x + y - 1$ for two positive integers x and y, with equality if and only if $x = 1$ or $y = 1$, we deduce from (4) that

$$p \geq m_{a,1} + m_{b,1} + m_{a,2} + m_{b,2} + m_{a,3} + m_{b,3} - 1 \geq \frac{n_a}{2} + \frac{n_b}{2} - 1.$$

Therefore, G is $(\alpha + \beta)$-good. □

4 Bipartite cubic graphs

In this section, we revisit a simple bound on the number of perfect matchings in bipartite graphs, which can be found in the book of Lovász and Plummer [9]. We need to slightly tune up the constants of the original bound so that they are good enough for our later considerations. Let us start by defining two auxiliary functions $f, g : \mathbb{N} \to \mathbb{N}$ recursively, as follows.

$$\begin{aligned} g(n) &= \begin{cases} 2 & \text{if } n = 1, \\ \left\lceil \frac{4}{3} g(n-1) \right\rceil & \text{otherwise,} \end{cases} \\ f(n) &= \left\lceil \frac{3}{2} g(n) \right\rceil \text{ for every } n \geq 1. \end{aligned}$$

The values of the functions $f(n)$ and $g(n)$ for small n can be found in Table 1.

We follow the lines of the proof of Theorem 8.1.7 from the book of Lovász and Plummer [9, Chapter 8] to prove the next lemma. In our further considerations, a bipartite graph is *near-cubic* if all its vertices have degree 3 except one vertex in each color class that has degree 2.

n	1	2	3	4	5	6	7	8	9	10
$g(n)$	2	3	4	6	8	11	15	20	27	36
$f(n)$	3	5	6	9	12	17	23	30	41	54

Table 1: The values $f(n)$ and $g(n)$ for $n \in \{1, \dots, 10\}$.

Lemma 18. *For each positive integer n, every cubic bipartite $2n$-vertex graph contains at least $f(n)$ perfect matchings and every near-cubic bipartite $2n$-vertex graph contains at least $g(n)$ perfect matchings.*

Proof. The proof proceeds by induction on n. The only cubic bipartite 2-vertex graph is the brace A_1, which has $3 = f(1)$ perfect matchings. The only near-cubic bipartite 2-vertex graph is obtained from A_1 by removing an edge: it has $2 = g(1)$ perfect matchings. Thus, the bounds claimed in the statement of the lemma hold if $n = 1$.

Assume that $n \geq 2$. Let us first consider a near-cubic bipartite $2n$-vertex graph G and let u and v be its vertices of degree 2. If u and v are adjacent, we show that G contains at least $f(n-1)$ perfect matchings. Indeed, let v' be the neighbor of u distinct from v and u' the neighbor of v distinct from u. Let G' be the graph obtained from G by removing the vertices u and v, and adding an edge between u' and v'. Since G' is a cubic bipartite graph, it contains at least $f(n-1)$ perfect matchings by the induction hypothesis. The perfect matchings of G' that contain the edge $u'v'$ can be converted to perfect matchings of G by replacing the edge $u'v'$ with the edges uv' and $u'v$, and those matchings of G' that avoid the edge $u'v'$ can be extended to perfect matchings of G by adding the edge uv. Since different perfect matchings of G' yield different perfect matchings of G, we deduce that G has at least $f(n-1)$ perfect matchings. The desired bound follows since $f(n-1) \geq \frac{4}{3}g(n-1)$.

We now consider the case where the vertices u and v are not adjacent. First, if u has only one neighbor v' (so there are two parallel edges between u and v'), then let G' be the subgraph of G obtained by removing u and v'. Since G' is a near-cubic bipartite graph, the induction hypothesis implies that G' has at least $g(n-1)$ perfect matchings. Each of these perfect matchings can be extended to a perfect matching of G in two different ways, by adding one of the two edges between u and v'. Since different perfect matchings of G' yield different perfect matchings of G, we deduce that G has at least $2 \cdot g(n-1)$ perfect matchings, which is more than $g(n)$. So assume now that u

has two distinct neighbors v_1 and v_2. For $i \in \{1,2\}$, let u_i and u_i' be the two neighbors of v_i different from u (these vertices need not be distinct). Finally, let G_1, G_2, G_3 and G_4 be the four graphs obtained from G by removing the vertex u, removing one of the four edges u_1v_1, $u_1'v_1$, u_2v_2 and $u_2'v_2$, and identifying the vertices v_1 and v_2. Each of the four graphs G_i is a near-cubic bipartite graph.

Every perfect matching of G_i corresponds to a perfect matching of G, e.g., any perfect matching of G_1 can be completed to a perfect matching of G by adding the edge uv_1 or uv_2. On the other hand, a perfect matching of G corresponds to perfect matchings in exactly three of the graphs $G_1, \ldots, G_4$ since it includes exactly one of the four edges u_1v_1, $u_1'v_1$, u_2v_2 and $u_2'v_2$. Hence, G has at least $4g(n-1)/3$ perfect matchings.

We have shown that G contains at least $4g(n-1)/3$ perfect matchings. Since the number of perfect matchings of G is an integer, G contains at least $g(n)$ perfect matchings, as asserted.

Assume now that H is a bipartite cubic graph. Let v be a vertex of H and v_1, v_2 and v_3 the three (non-necessarily all distinct) neighbors of v. For $i \in \{1,2,3\}$, let H_i be the near-cubic bipartite graph obtained by removing the edge vv_i. As shown before, H_i contains at least $g(n)$ perfect matchings. If M is a perfect matching of H, then M is also a perfect matching of exactly two of the graphs H_1, H_2 and H_3. Hence, H contains at least $3g(n)/2$ perfect matchings. Since the number of perfect matchings is an integer, H contains at least $f(n) = \lceil 3g(n)/2 \rceil$ perfect matchings. □

Lemma 19. *For each $n \geq 5$, every brace G of order n is $(\geq n+2)$-good.*

Proof. Since $g(5) = 8$, we infer that for all $n \geq 5$,

$$f(n) \geq \frac{3}{2} \cdot \left(\frac{4}{3}\right)^{n-5} \cdot 8 = \frac{4^{n-4}}{3^{n-6}} \geq 2n + 2\,.$$

By Lemma 18, G has at least $f(n) \geq 2n+2$ perfect matchings and thus G is $(\geq n+2)$-good. □

We finish this section by obtaining a simple constant lower bound on the number of perfect matchings in cubic bridgeless graphs, which turns out to be useful in our further considerations.

Lemma 20. *Every cubic bridgeless graph different from A_1, I_1 and I_2 has at least five perfect matchings.*

Proof. Let G be a cubic bridgeless graph. If G is a brace then, by Lemma 18, the graph G has at least five perfect matchings unless $G = A_1$. If G has a non-trivial brick and brace decomposition, then its decomposition contains a brace by Lemma 12, which cannot be A_1. Hence, the brace in the decomposition of G has at least five perfect matchings. Since the number of perfect matchings of a graph is at least the minimum of the number of perfect matchings of the graphs in its brick and brace decomposition (because every perfect matching of a graph in the decomposition can be extended to a perfect matching of the original graph), G has at least five perfect matchings.

It remains to consider the case where G is a brick. By Theorem 15, every n-vertex brick has at least $n/2 + 1$ perfect matchings. Hence, if G has less than five perfect matchings, then G has at most six vertices. But the only two bricks with at most six vertices are the bricks I_1 and I_2. $\square$

5 Single-brace cubic graphs

In this section, we analyze the number of perfect matchings in graphs the brick and brace decomposition of which contains exactly one brace. Such cubic bridgeless graphs are referred to as *single-brace* graphs. Before we proceed further, let us state a simple lemma on trimatched vertices in cubic graphs.

Lemma 21. *If G is a cubic bridgeless graph obtained from G' by gluing a graph G'' through a vertex v, then every vertex $w \neq v$ of G' that is trimatched in G' is also trimatched in G.*

Proof. Let H' be a spanning subgraph of G' such that the vertex w has degree 3 in H' and the other vertices of G' have degree 1. Let e be the edge of H' incident with v and let f be the edge corresponding to e in G''. Let M be a perfect matching of G'' that contains the edge f (recall that every cubic bridgeless graph is matching covered). The subgraph H' and the matching M combine to a spanning subgraph H of G where the degree of w is 3 and the degrees of other vertices are 1. Hence, the vertex w is trimatched in G. $\square$

Let us now apply Lemma 21 to establish the following auxiliary lemma restricting the set of vertices through which a brick can be glued to a brace.

Lemma 22. *Let G be a single-brace graph. If the brick and brace decomposition consists of a brace B of order n and bricks $B_1, \dots, B_k$, and the brace B is not A_2, then $k \le n$ and G can be obtained from B by gluing B_i through a vertex v_i of B for each $i \in \{1, \dots, k\}$ such that all the vertices v_i are in the same color class of B.*

Proof. The proof proceeds by induction on k, the conclusion holding trivially when $k = 1$. Assume that $k \ge 2$. Let us consider a non-trivial tight edge-cut E of G and let G_1 and G_2 be the two graphs obtained by splitting along the edge-cut E. By Theorem 8 and Lemma 12, one of the graphs G_1 and G_2 is a brick. By symmetry, we can assume that G_2 is the brick B_k. Let w be the vertex such that G is obtained from G_1 by gluing B_k through w.

By the induction hypothesis, for each $i \in \{1, \dots, k-1\}$, the graph G_1 is obtained from the brace B by gluing B_i through a vertex v_i, and the vertices $v_1, \dots, v_{k-1}$ are in the same color class of B. In order to finish the proof of the lemma, we have to exclude the following two cases.

- The vertex w is a vertex of one of the bricks $B_1, \dots, B_{k-1}$.
- The vertex w is not in the color class containing the vertices $v_1, \dots, v_{k-1}$.

To this end, we show that if w is of one of the above two types, then w is trimatched in G_1. Since G_2 is trimatched by Lemma 11, this would imply that the edge-cut E is not tight. If w is a vertex of one of the bricks, then it is trimatched by Lemma 21 (where we apply this lemma several times while gluing the bricks to construct G_1). Hence, we have to focus on the case where w is not in the color class containing the vertices $v_1, \dots, v_{k-1}$.

Since the brace A_1 does not appear in any non-trivial brick and brace decomposition and $B \ne A_2$, the brace B is simple (by Theorem 9). Let w' and w'' be two neighbors of w distinct from v_1, and let v' and v'' be two neighbors of v_1 distinct from w. By Theorem 9, the graph $B - \{v', v'', w', w''\}$ has a perfect matching. Adding the edges v_1v', v_1v'', ww' and ww'' to this perfect matching yields a spanning subgraph H_B of B, all the vertices of which have degree 1 except for the vertices v_1 and w, which both have degree 3. Along the brick and brace decomposition, using the fact that the bricks are trimatched by Lemma 11, the subgraph H_B can be extended to a spanning subgraph H of G_1 in which every vertex has degree 1 except the vertex w, which has degree 3. Hence, w is trimatched in G_1.

Since gluing a brick through a trimatched vertex does not create a new non-trivial tight edge-cut, the vertex w must belong to the same color class

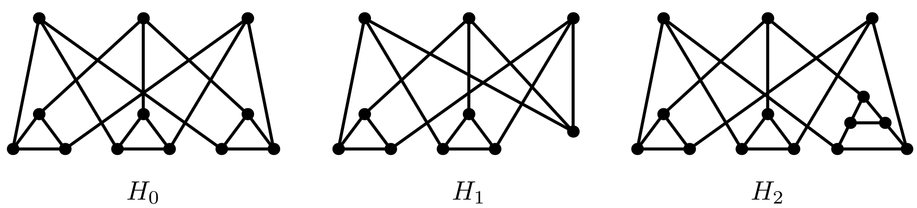


Figure 3: The exceptional graphs H_0, H_1 and H_2.

as $v_1, \ldots, v_{k-1}$. In particular, G can be obtained from the brace B by gluing the bricks $B_1, \ldots, B_k$ through vertices $v_1, \ldots, v_k$ contained in the same color class. Since each color class of B contains n vertices, the number k of bricks is at most n. The proof of the lemma is now finished. □

With Lemma 22, we are ready to consider single-brace graphs the decomposition of which contains the brace A_3.

Lemma 23. *If G is a single-brace graph that contains A_3 in its brick and brace decomposition, then G is (≥ 2)-good unless it is one of the graphs H_0, H_1 and H_2 depicted in Figure 3.*

Proof. By Lemma 22, the graph G is obtained from the brace A_3 by gluing at most three bricks through vertices of the same color class of A_3. Let i_1 be the number of bricks I_1 glued to A_3, i_2 the number of bricks I_2 glued to A_3, and i the number of other bricks glued to A_3. Thus, $i_1 + i_2 + i \leq 3$.

The graph A_3 is 3-good and matching double-covered (the latter being implied by Lemma 13). Since I_1 is 1-good, the graph G_1 obtained by gluing i_1 bricks I_1 to A_3 is $(\geq 3 - i_1)$-good by Lemma 16.

Let G_2 be the graph obtained from G_1 by gluing i_2 bricks I_2 according to the brick and brace decomposition of G. Note that I_2 is 1-good, and no vertex of I_2 is incident with three solo-edges. Moreover, the graph G_1 is matching double-covered by Lemma 14. Consequently, none of these i_2 gluings is a solo-gluing. Hence, the graph G_2 is $(\geq 3 - i_1)$-good by Lemma 16.

Finally, each of the remaining i bricks contains at least five perfect matchings by Lemma 20 and is (≥ 1)-good by Theorem 15. Since the graph G_2 is matching double-covered by Lemma 14, the final graph G is $(\geq 3 - i_1 + i)$-good by Lemma 17. Hence, if G is not (≥ 2)-good, then $i_1 \geq 2 + i$. Since $i_1 + i_2 + i \leq 3$, we deduce that $i = 0$ and $i_2 \in \{0, 1\}$. So, either $i_1 = 3$ and

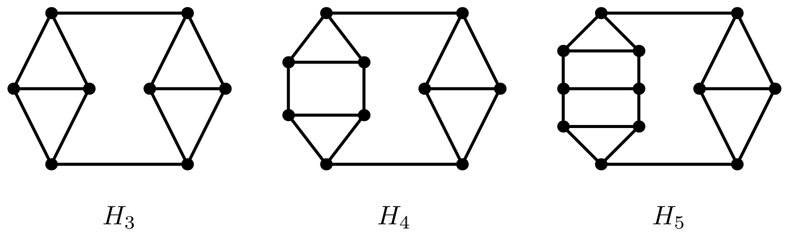


Figure 4: The exceptional graphs H_3, H_4 and H_5.

$i_2 = 0$, or $i_1 = 2$ and $i_2 = 0$, or $i_1 = 2$ and $i_2 = 1$. The graph G is then either H_0, H_1, or H_2, respectively. It is straightforward to verify that H_0 is 0-good and the graphs H_1 and H_2 are 1-good. □

Before we proceed to analyze single-brace graphs the brick and brace decomposition of which contains a brace of order at least 4, let us deal with the case where the decomposition contains the brace A_2.

Lemma 24. *If G is a single-brace graph that contains A_2 in its brick and brace decomposition, then G is (≥ 2)-good unless it is one of the graphs H_3, H_4 and H_5 depicted in Figure 4.*

Proof. Let $B = A_2$ be the brace and $B_1, \ldots, B_k$ the bricks forming the brick and brace decomposition of G. As in the proof of Lemma 22, it is possible to argue using Lemma 21 that G is obtained by gluing $B_1, \ldots, B_k$ through distinct vertices $v_1, \ldots, v_k$ of the brace B (this part of the proof only used the fact that every brick is trimatched). However, since B is not simple, it is not possible to argue that the vertices $v_1, \ldots, v_k$ are in the same color class of A_2 as in the proof of Lemma 22. In fact, they do not have to, as we shall see in what follows.

Although the vertices $v_1, \ldots, v_k$ do not have to be contained in the same color class of B, we assert it is still true that $k \leq 2$. Suppose on the contrary that $k \geq 3$. Then, two of the vertices v_i, say v_1 and v_2, are in the same color class of B. We show that the graph G' obtained from B by gluing the brick B_1 through the vertex v_1 and the brick B_2 through the vertex v_2 is trimatched. Since B_3 is a brick, and thus is trimatched by Lemma 11, this will eventually contradict the assumption that the edge-cut of G used to split off B_3 is tight.

Let u and u' be the vertices of the other color class of B than v_1 and v_2. By Lemma 21, all the vertices of G' except possibly u and u' are trimatched. Let us establish that the vertices u and u' are also trimatched in G'.

By symmetry, we can assume that u is joined by two parallel edges to v_1. Let u_1 and u_2 be the neighbors (in G) of u inside the brick B_1, u_3 the vertex of B_1 adjacent to u' and u_0 the remaining neighbor of u. Observe that u_0 is in the brick B_2. Since B_1 is trimatched, there exists a subgraph H' of G spanning B_1 that contains the edges uu_1, uu_2 and $u'u_3$ and every vertex of B_1 has degree 1 in H'. Adding to H' a perfect matching of B_2 containing the edge uu_0 yields a spanning subgraph H of G', in which the vertex u has degree 3 and the remaining vertices have degree 1. Since the case of the vertex u' is symmetric to that of u, we have proved that G' is trimatched. As argued before, the number of bricks in the brick and brace decomposition of G is at most 2, i.e. $k \leq 2$.

If $k = 0$, then $G = A_2$ which is 3-good. If $k = 1$, then G is (≥ 2)-good by Lemma 16 since every brick is (≥ 1)-good. If $k = 2$, then G is again (≥ 2)-good by Lemma 16 unless both B_1 and B_2 are 1-good bricks and both gluings are solo-gluings. Since the pattern of every vertex of A_2 is $(1, 2, 2)$, a gluing can be a solo-gluing only if the brick B_i contains a vertex of pattern $(1, 1, x)$ for some $x \in \mathbb{N}$. However, there are only three 1-good bricks containing a vertex of pattern $(1, 1, x)$; see Figure 2. In particular, both the bricks B_1 and B_2 must be one of the bricks I_1, I_2 and I_3.

Let us now argue that at least one of the bricks B_1 and B_2 is I_1. To this end, we prove that one of the two solo-gluings must be through a vertex of a brick with pattern $(1, 1, 1)$. This will yield the desired conclusion since, among I_1, I_2 and I_3, only I_1 contains a vertex with such a pattern. Let G' be the graph obtained from $B = A_2$ by solo-gluing I_2 or I_3. As argued before, the solo-gluing is through a vertex of the brick with pattern $(1, 1, x)$. By the structure of I_2 and I_3, it holds that $x \geq 2$. Let v be a vertex of G' that is not contained in the glued brick and e an edge incident with v. If e is contained in two different perfect matchings of A_2, then e is also contained in at least two different perfect matchings of G'. If e is contained in a single perfect matching of A_2, then this perfect matching can be extended in x different ways to the glued brick. Hence, every edge incident with v is in at least two different perfect matchings of G'. Since the choice of v was arbitrary among the vertices not contained in the brick, we deduce that only a brick containing a vertex with pattern $(1, 1, 1)$ can be solo-glued to G' (recall that gluing the second brick through a vertex contained in the first one would not

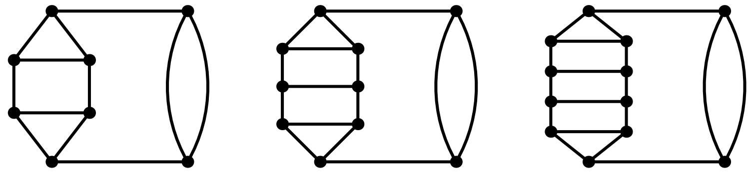


Figure 5: The graphs that can be obtained from the brace A_2 by solo-gluing the brick I_1 and one of the bricks I_1, I_2 and I_3 through vertices joined by parallel edges in A_2.

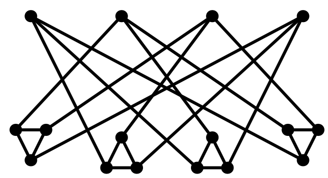
yield a non-trivial tight edge-cut). Hence, at least one of the bricks B_1 and B_2 is I_1.

By symmetry, we can assume henceforth that $B_1 = I_1$ and $B_2 \in \{I_1, I_2, I_3\}$. Let u and u' be vertices of one of the color classes of $B = A_2$. Let v be the vertex of the other color class joined by two parallel edges to u, and v' the vertex joined by two parallel edges to u'. By symmetry, the brick $B_1 = I_1$ is glued to $B = A_2$ through the vertex u. If the brick B_2 is glued through the vertex u' or the vertex v', we obtain one of the three 1-good graphs depicted in Figure 4. Note that although the brick B_2 can be glued in several non-symmetric ways, there is a unique way to solo-glue it. Finally, if the brick B_2 is glued through the vertex v, then the resulting graph is (≥ 2)-good. See Figure 5 for the three graphs that can be obtained in this way. □

It remains to analyze single-brace graphs the decomposition of which contains a brace of order at least 4.

Lemma 25. *If G is a single-brace graph that contains neither A_2 nor A_3 in its brick and brace decomposition, then G is (≥ 2)-good unless it is the graph H_6 depicted in Figure 6.*

Proof. Let B be the brace in the decomposition of G, n the order of B and $B_1, \ldots, B_k$ the bricks in the decomposition. By Lemma 22, $k \leq n$. Let i_1 be the number of bricks $B_1, \ldots, B_k$ isomorphic to the brick I_1. If the brace B is A_4, then B is 5-good. After gluing the i_1 bricks I_1, the resulting graph G' is $(\geq 5 - i_1)$-good by Lemma 16. Since G' is matching double-covered by Lemma 14, none of the gluings of the other $k - i_1$ bricks to G' is a solo-gluing. Hence, G is $(\geq 5 - i_1)$-good. We conclude that if G is not (≥ 2)-good, then $i_1 = 4$ and G is the exceptional graph H_6 depicted in Figure 6.



H_6

Figure 6: The exceptional graph H_6.

Assume now that B is not the brace A_4. Since B is also neither A_2 nor A_3 by the assumption of the lemma, B is $(n+2)$-good by Lemma 19. As in the previous paragraph, we argue that G is $(\geq n+2-i_1)$-good. Since $i_1 \leq k \leq n$ (the latter inequality is implied by Lemma 22), it follows that G is (≥ 2)-good. □

Lemmas 23–25 imply the following theorem. Note that every brace is (≥ 2)-good as shown in Section 4.

Theorem 26. *A single-brace graph G is (≥ 2)-good with the following exceptions:*

- *the graph H_0 which is 0-good, and*
- *the graphs $H_1, \ldots, H_6$ which are 1-good.*

The exceptional graphs are depicted in Figures 3, 4 and 6.

6 Multi-brace cubic graphs

In this section, we analyze cubic bridgeless graphs the brick and brace decomposition of which contains at least two braces. Before we do so, we establish two auxiliary lemmas. The first one asserts that almost every single-brace graph that is not (≥ 2)-good is trimatched.

Lemma 27. *The cubic graphs $H_0, \ldots, H_6$ are trimatched with the exception of H_1 which contains a single vertex that is not trimatched. The pattern of this vertex of H_1 is $(2, 2, 2)$.*

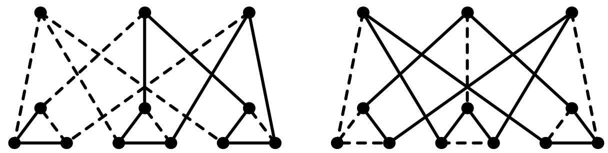


Figure 7: Spanning subgraphs of the graph H_0 showing that every vertex of H_0 is trimatched (symmetric cases are omitted). The edges contained in the subgraphs are dashed.

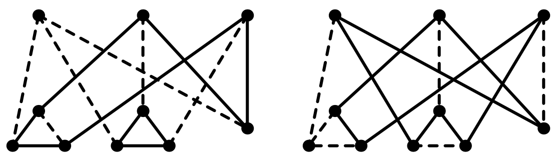


Figure 8: Spanning subgraphs of the graph H_1 showing that all but one vertices of H_1 are trimatched (symmetric cases are omitted). The edges contained in the subgraphs are dashed.

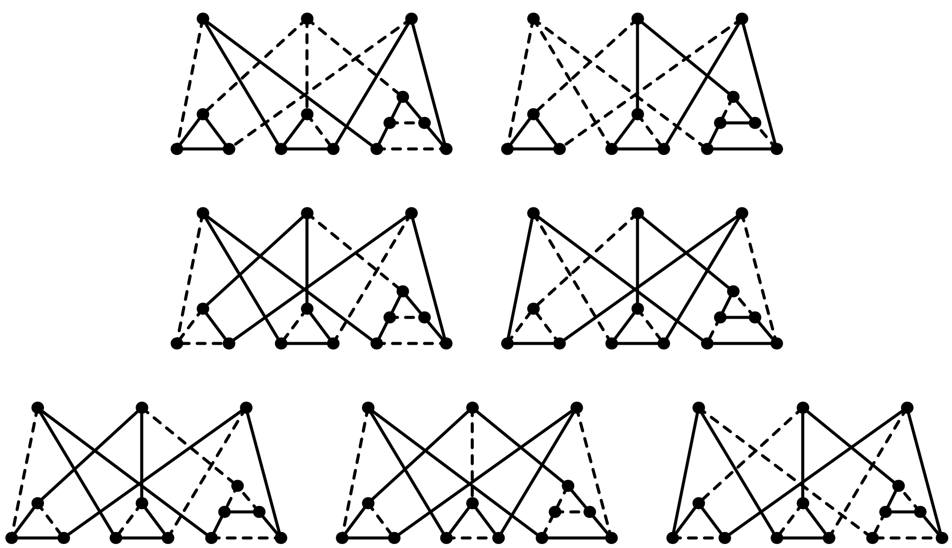


Figure 9: Spanning subgraphs of the graph H_2 showing that every vertex of H_2 is trimatched (symmetric cases are omitted). The edges contained in the subgraphs are dashed.

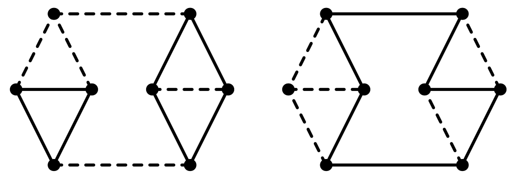


Figure 10: Spanning subgraphs of the graph H_3 showing that every vertex of H_3 is trimatched (symmetric cases are omitted). The edges contained in the subgraphs are dashed.

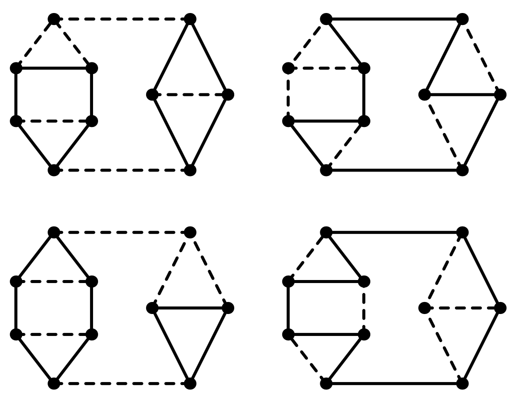


Figure 11: Spanning subgraphs of the graph H_4 showing that every vertex of H_4 is trimatched (symmetric cases are omitted). The edges contained in the subgraphs are dashed.

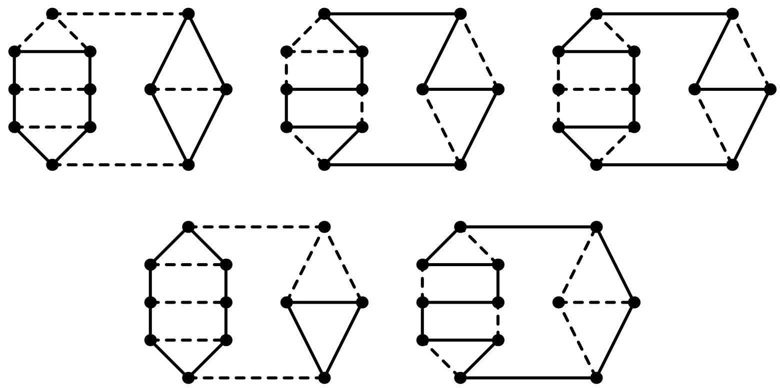


Figure 12: Spanning subgraphs of the graph H_5 showing that every vertex of H_5 is trimatched (symmetric cases are omitted). The edges contained in the subgraphs are dashed.

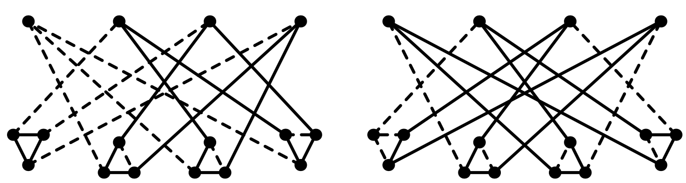


Figure 13: Spanning subgraphs of the graph H_6 showing that every vertex of H_6 is trimatched (symmetric cases are omitted). The edges contained in the subgraphs are dashed.

Proof. It is enough to exhibit spanning subgraphs of the graphs $H_0, \ldots, H_6$ satisfying the statement of the lemma. Such subgraphs can be found in Figures 7–13; the exceptional vertex of H_1 is the vertex of A_3 of the color class where the brick I_1 was glued through the other two vertices. □

In the next lemma, we restrict the structure of cubic bridgeless graphs that are not matching double-covered.

Lemma 28. *If G is a cubic bridgeless graph that is neither a brick nor the brace A_1, then every vertex of G is incident with at most one solo-edge.*

Proof. We proceed by induction on the number of vertices of G. If G has no non-trivial tight edge-cuts, then it must be a brace. If G is the brace A_2, then every vertex of G has pattern $(1, 2, 2)$ and the statement holds. Otherwise, G is matching double-covered by Lemma 14 and thus G has no solo-edges at all.

Assume that G has a non-trivial tight edge-cut $E = \{e_1, e_2, e_3\}$, and let G_1 and G_2 be the graphs obtained by splitting along E. Lemma 12 ensures that any non-trivial brick and brace decomposition contains at least one brace. Thus, we can assume that the brick and brace decomposition of G_1 contains a brace. By the induction hypothesis, every vertex of G_1 is incident with at most one solo-edge.

For $i \in \{1, 2\}$, let V_i be the set of vertices of G contained in G_i. Further, let v be the vertex of G_2 such that G is obtained from G_2 by gluing G_1 through v. In particular, $v \notin V_2$. Note that the edges e_1, e_2 and e_3 one-to-one correspond to the edges of G_2 incident with v. Since every vertex of G_1 is incident with at most one solo-edge, we can assume that, for each $i \in \{1, 2\}$, the graph G_2 admits a perfect matching that contains the edge e_i and can be extended to G_1 in at least two different ways.

Let w be any vertex of V_2 and let f_1, f_2 and f_3 be the three edges incident with w. We aim to show that at most one of these edges is a solo-edge. Since a cubic bridgeless graph is matching covered, there exists a perfect matching M_1 of G_2 containing the edge e_1. By symmetry, we can assume that M_1 also contains the edge f_1. Since any matching containing the edge e_1 can be extended to G_1 in at least two different ways, the edge f_1 is not a solo-edge.

On the other hand, Theorem 2 implies the existence of a perfect matching M_2 of G_2 avoiding both the edges e_3 and f_1. By symmetry, we may assume that M_2 contains the edge f_2. Since M_2 also contains the edge e_1 or e_2, it can be extended to G_1 in at least two different ways. Hence, the edge f_2 is not a solo-edge either. We conclude that every vertex of V_2 is incident with at least two edges that are not solo-edges.

Since the number of perfect matchings containing a given edge can only increase by gluing a graph through a vertex, every vertex of V_1 is incident with at most one solo-edge. This finishes the proof of the lemma. □

In the next lemma, we show that every cubic bridgeless graph G that is neither a brick nor a single-brace graph contains a non-trivial tight edge-cut with a useful property.

Lemma 29. *Let G be a cubic bridgeless graph that is neither a brick nor a single-brace graph. Then, G contains a non-trivial tight edge-cut E such that neither of the graphs obtained by splitting along E is a brick.*

Proof. We proceed by induction on the number K of graphs in the brick and brace decomposition of G. The result is true if $K = 2$. Indeed, since G is neither a brick nor a brace, Lemma 12 implies that the decomposition of G contains a brace. Since G is not a single-brace graph, we deduce that its decomposition is composed of two braces, which yields the sought conclusion. Assume now that $K > 2$ and the theorem holds for smaller values of K. Since G is neither a brick nor a brace, G contains a non-trivial tight edge-cut E. Let G_1 and G_2 be the two graphs obtained from G by splitting along E. By symmetry, we may assume that G_2 is a brick (otherwise E is the sought tight edge-cut). Hence, as G is not a single-brace graph, the brick and brace decomposition of G_1 contains at least two braces. Thus, by induction, G_1 contains a non-trivial tight edge-cut E' that splits G_1 into two graphs G'_1 and G'_2 such that neither of them is a brick, i.e., the brick and brace decomposition of both G'_1 and G'_2 contains a brace. Let v be the vertex of G_1 such that G is obtained from G_1 by gluing G_2 through v. By symmetry, we can assume that the vertex v is contained in G'_2.

We assert that E' is also a non-trivial tight edge-cut of G. Indeed, if G contains a perfect matching containing all three edges of E', then this matching uses exactly one edge of E because E is a tight edge-cut. Hence, the edge contained in E can be replaced with an edge of G_1 incident with v yielding a perfect matching of G_1 containing all three edges of E'.

Now split the graph G along the non-trivial tight edge-cut E'. One of the graphs obtained is the graph G'_1, which is not a brick. The other graph cannot be a brick either, since its brick and brace decomposition must contain a brace contained in the decomposition of G_2' (recall that Theorem 8 ensures that the brick and brace decomposition of G is unique). □

We are now ready to analyze cubic bridgeless graphs the brick and brace decomposition of which contains two or more braces. We start with the case of two braces, which will be the core of our inductive argument later.

Theorem 30. *If the brick and brace decomposition of a cubic bridgeless graph G contains two braces, then G is (≥ 2)-good.*

Proof. Since the brick and brace decomposition of G is non-trivial, G has a non-trivial tight edge-cut E. Let G_1 and G_2 be two graphs obtained from G by splitting along E. By Lemma 29, we can assume that neither G_1 nor G_2 is a brick. Hence, both G_1 and G_2 are single-brace graphs. By the definition of the brick and brace decomposition, neither G_1 nor G_2 can be the brace A_1. Note that both G_1 and G_2 have at least five perfect matchings by Lemma 20.

Assume first that G_1 is (≥ 2)-good. By Lemma 28, the gluing of G_1 and G_2 resulting in G is not a solo-gluing. Hence, if G_2 is (≥ 1)-good, then G is (≥ 2)-good by Lemma 16. If G_2 is not (≥ 1)-good, then G_2 must be the graph H_0 by Theorem 26. In particular, G_2 is matching double-covered. Consequently, G is (≥ 2)-good by Lemma 17 since G_1 has at least five perfect matchings. A symmetric argument applies if G_2 is (≥ 2)-good.

It remains to consider the case where neither G_1 nor G_2 is (≥ 2)-good. Theorem 26 yields that each of G_1 and G_2 is one of the graphs $H_0, \ldots, H_6$. For $i \in \{1, 2\}$, let v_i be the vertex of G_i such that G is obtained from G_i by gluing G_{3-i} through v_i. At least one of the vertices v_1 and v_2 is not trimatched, since the edge-cut E used to split G is tight. By Lemma 27 and symmetry, we can assume that G_1 is the graph H_1 and the pattern of v_1 in G_1 is $(2, 2, 2)$.

If G_2 is 1-good, then G is (≥ 2)-good by Lemma 17 since G_1 is 1-good and matching double-covered. The other case is that G_2 is not 1-good. Then Theorem 26 implies that G_2 is the graph H_0. Consequently, the pattern of v_2 is also $(2, 2, 2)$, and the graph G has at least $3 \cdot (2 \cdot 2) = 12$ perfect matchings. Since the number of vertices of G is $10 + 12 - 2 = 20$, the graph G is 2-good. □

Finally, we can prove the main theorem of this section.

Theorem 31. *If the brick and brace decomposition of a cubic bridgeless graph G contains at least two braces, then G is (≥ 2)-good.*

Proof. The proof proceeds by induction on the number of braces in the brick and brace decomposition of G. If the brick and brace decomposition of G contains exactly two braces, then G is (≥ 2)-good by Theorem 30. Assume now that the decomposition of G contains at least three braces. Let G_1 and G_2 be two graphs that can be obtained from G by splitting along a non-trivial tight edge-cut. By Lemma 29, we can assume that neither G_1 nor G_2 is a brick. By the definition of the brick and brace decomposition, neither G_1 nor G_2 is the brace A_1.

Since the brick and brace decomposition of G contains at least three braces, at least one of G_1 and G_2 is not a single-brace graph. By symmetry, we can assume that G_1 is not a single-brace graph, and thus G_1 is (≥ 2)-good by the induction hypothesis. The graph G_2 is (≥ 0)-good. This follows from Theorem 26 if G_2 is a single-brace graph, and from the induction hypothesis otherwise. By Lemma 28, the gluing of G_1 and G_2 resulting in G is not a solo-gluing. So, if G_2 is (≥ 1)-good, then G is (≥ 2)-good by Lemma 16. If G_2 is 0-good, then G_2 must be the graph H_0 by Theorem 26 and the induction hypothesis. In particular, G_2 is matching double-covered. Moreover, G_1 has at least five perfect matchings by Lemma 20. Hence, Lemma 17 implies that G is (≥ 2)-good. □

Theorems 15, 26 and 31 imply Theorem 7, the main result of this paper.

Acknowledgment

This research was done while the first two authors were visiting the third at Technische Universität Ilmenau. They thank their host for providing a perfect working environment.

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