Mathematics for 2d Interfaces

Claude Bardos, David Lannes

▶ **To cite this version:**

HAL Id: hal-00487357

https://hal.archives-ouvertes.fr/hal-00487357

Submitted on 28 May 2010

Mathematics for $2d$ Interfaces

Claude Bardos[*]and David Lannes[†]

Abstract

We present here a survey of recent results concerning the mathematical analysis of instabilities of the interface between two incompressible, non viscous, fluids of constant density and vorticity concentrated on the interface. This configuration includes the so-called Kelvin-Helmholtz (the two densities are equal), Rayleigh-Taylor (two different, nonzero, densities) and the water waves (one of the densities is zero) problems. After a brief review of results concerning strong and weak solutions of the Euler equation, we derive interface equations (such as the Birkhoff-Rott equation) that describe the motion of the interface. A linear analysis allows us to exhibit the main features of these equations (such as ellipticity properties); the consequences for the full, non linear, equations are then described. In particular, the solutions of the Kelvin-Helmholtz and Rayleigh-Taylor problems are necessarily analytic if they are above a certain threshold of regularity (a consequence is the illposedness of the initial value problem in a non analytic framework). We also say a few words on the phenomena that may occur below this regularity threshold. Finally, special attention is given to the water waves problem, which is much more stable than the Kelvin-Helmholtz and Rayleigh-Taylor configurations. Most of the results presented here are in $2d$ (the interface has dimension one), but we give a brief description of similarities and differences in the $3d$ case.

Contents

[*]Université Denis Diderot and Laboratory JLL Université Pierre et Marie Curie, Paris, France Pauli fellow; Wolfgang Pauli Institute Vienna (claude.bardos@gmail.com)

[†]DMA, Ecole Normale Suprieure et CNRS UMR 8553, 45, rue d'Ulm, 75005 Paris. (David.Lannes@ens.fr)

1   Introduction

This contribution is a survey of recent results concerning the mathematical analysis of instabilities in interfaces between two fluids of different densities and vorticity concentrated on the interface. In $2d$ this interface is a time dependent curve and in $3d$ a time dependent surface. This includes the two following extreme cases:

- The density of one of these fluids is infinitely small with respect to the density of the other and therefore can be taken equal to 0; this corresponds to the study of water waves or droplets in vacuum.

- The two fluids have the same density (in fact there is only one fluid) and the vorticity is concentrated on an interface.

Experiments and numerical simulation show that the range of possible behaviors for the interface is very rich. In some cases (especially in the water waves case) the interface is "stable" and its description has led to many studies in both the physical and mathematical literature. In other cases (this is the generic case when both densities are nonzero), the interface is "unstable". Our purpose is to show how mathematical analysis can contribute to the understanding of such instabilities. To do so, one starts from the most abstract model which is the initial value (or Cauchy) problem for the incompressible Euler equation. It models a configuration where viscosity and surface tension are neglected and therefore it is already too abstract to describe physical situations. As recalled below it admits solutions that would correspond to instantaneous creation or extinction of energy and as such would solve the energy crisis. However since in many cases the surface tension and viscosity are small, it turns out to be essential to study this limit case. We will consider the evolution of a fluid in a domain Ω which can be either the whole space $\Omega = \mathbb{R}^d$ a sub-domain $\Omega \subset \mathbb{R}^d$ or a periodic box $\Omega = (\mathbb{R}/\mathbb{Z})^d$. The natural quantities are

the density ρ the velocity $u(x,t)$ and the pressure $p(x,t)$. The equation of motion in Ω are then given by the Euler equations (or conservation of momentum),

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p = \rho \vec{\mathbf{g}}, \tag{1}$$

where the gravity $\vec{\mathbf{g}} \in \mathbb{R}^d$ is a constant vector, and the conservation of mass,

$$\partial_t \rho + \nabla \cdot (\rho u) = 0. \tag{2}$$

The fluid is also assumed to be incompressible; together with the conservation of mass, this yields the condition

$$\nabla \cdot u = 0, \tag{3}$$

which can be viewed as a constraint on the evolution of the fluid with the pressure as the Lagrange multiplier of this constraint. In the presence of boundary $(\Omega \neq \mathbb{R}^d)$ one has for all times t the following impermeability condition ,

$$\forall x \in \partial\Omega \qquad u(x,t) \cdot \vec{n} = 0 \text{ with } \quad \vec{n} \text{ the exterior normal on } \partial\Omega, \text{boundary of } \Omega. \tag{4}$$

If Ω is unbounded, it is also assumed that $u(t,\cdot)$ vanishes at infinity.

Besides the time variable t, two coordinate systems appear naturally. In the *Eulerian system*, $u(x,t)$ describes the velocity field at the point x and at the time t. In the *Lagrangian system*, we rather use the quantity $\widetilde{u}(x,t)$ that stands for the velocity of the fluid particle that was at the point x at the initial time $t = 0$. Equivalently, $\widetilde{u}(x,t) = u(X(x,t),t)$, where $t \mapsto X(x,t)$ solves the ordinary differential equation:

$$\dot{X}(x,t) = u(X(x,t),t), \qquad X(x,t=0) = x. \tag{5}$$

We consider solutions $u(x,t)$ which are continuous away from an interface $\Sigma(t)$ which defines a partition of $\Omega = \Omega_\pm(t) \cup \Sigma(t)$. We assume that on both sides of this interface the density of fluid $\rho = \rho_\pm$ is constant. It is important to keep in mind that the velocity of the fluid is well defined on $\Omega_\pm(t)$ and therefore that the equation (5), together with the incompressibility condition (3), defines two families of volume preserving diffeomorphisms from $\Omega_\pm(t = 0)$ onto $\Omega_\pm(t)$. However the velocity has no intrinsic definition on $\Sigma(t)$. Only the global evolution of this interface matters; as a consequence there is some freedom in the choice of the parametrization $r(t,\lambda)$ of the interface $\Sigma(t)$, and this choice will play an important rôle below.
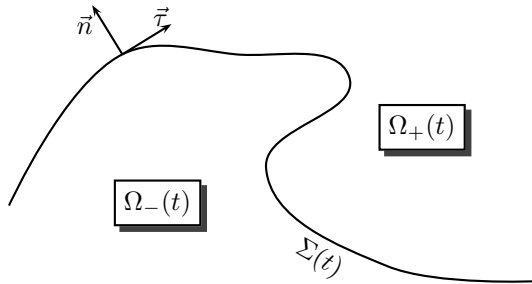
Observe now that as long as the solution is smooth, the vorticity $\omega = \nabla \wedge u$ is solution of the system of equations:

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u, \quad \nabla \cdot u = 0, \quad \nabla \wedge u = \omega, \quad \text{in } \Omega. \tag{6}$$

In $2d$ with a planar flow, the vorticity can be written $\omega = (0, 0, \partial_1 u_2 - \partial_2 u_1)^T$ and the term $\omega \cdot \nabla u$ vanishes. Still denoting by ω the *scalar vorticity* $\omega = \partial_1 u_2 - \partial_2 u_1$, the vorticity equation becomes

$$\partial_t \omega + u \cdot \nabla \omega = 0. \tag{7}$$

In any case, one has

$$\omega(x, 0) = 0 \text{ for } x \notin \Sigma(0) \Rightarrow \omega(x, t) = 0 \text{ for } x \notin \Sigma(t), \tag{8}$$

as long as the dynamics is well defined. Several contributions have been devoted to the case where $\omega(0)$ is not 0 away from $\Sigma(0)$ (cf. [36, 13, 49, 50, 64]). However since the evolution of the interface is the dominant effect we will consider only dynamics with 0 vorticity away from the interface. Therefore the only parameters are the Atwood number:

$$a = \frac{\rho_+ - \rho_-}{\rho_+ + \rho_-}, \qquad -1 \leq a \leq 1, \tag{9}$$

and the gravity $\vec{g}$. The Rayleigh-Taylor problem corresponds to $0 < |a| < 1 \Leftrightarrow \rho_+ \neq \rho_-$, $\rho_+ \rho_- \neq 0$, the Kelvin Helmholtz to $a = 0 \Leftrightarrow \rho_- = \rho_+$. Finally the situation where $a = \pm 1 \Leftrightarrow \rho_- \rho_+ = 0$ corresponds to a situation where one of the two densities vanishes and it describes the dynamics of water waves. The Rayleigh-Taylor and the Kelvin-Helmholtz problems are highly unstable while the stability of the water waves problem is ensured if an extra condition (*called in the literature Taylor Hypothesis or Rayleigh-Taylor sign condition*, see §7.3). The above names indicate that the study of these problems goes back to the end of the nineteenth century or the very beginning of the twentieth century.

The following facts should be kept in mind:

- The range of applications of the analysis is huge: From classical water waves (and application in oceanography) to plasma physic in laser confinement and astrophysics.

- The basic properties of the instabilities have already been established by the forerunners of the theory (which carries their names) by modal analysis of linearized problem near stationary solutions. A systematic use of the modal analysis can be found in the book of Chandrasekhar [12].

- The more recent contributions aim at a more systematic theory and allow for solutions that are not in a small neighborhood of stationary solutions.

- We are interested here in the appearance of singularities and focus therefore on the most singular component of the solution (this is the central idea of symbolic analysis). However, the most singular parts are not necessarily the most important for the applications (in the so called shallow water regime the terms neglected by the symbolic analysis are the most important ones, see §7.5).

For the sake of simplicity most of the presentation is devoted to $2d$ problems. Similarities and differences which appear with the $3d$ configurations are considered in Section 8.

2   Strong and weak solutions of the $2d$ Euler equation

When the density of both fluids is the same (say, $\rho + = \rho_- = 1$, and thus Atwood number $a = 0$), the problem is reduced to the "genuine" incompressible Euler equations (1), (3). In this case, the gravity does not play any significant role (it can be put into the pressure term), and we thus take $\vec{g} = 0$ in (1):

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0, \qquad \nabla \cdot u = 0. \tag{10}$$

By standard methods [40] one can show the following local existence theorem:

4

Theorem 1. *For any initial data $u_0 \in C^{1,\alpha}(\mathbb{R}^d)$ ($\alpha > 0$, $d = 2, 3$) with $\nabla \cdot u_0 = 0$ on $\mathbb{R}^d$, there exists a finite time $T^* > 0$ such that, with this initial data, one has a a unique classical solution to (10) on the time interval $[0, T^*]$.*

The above result is only local in time. T^* depends on the size of the initial data and to the best of our knowledge this is the only "general" result. Since we are concerned with solutions having their vorticity concentrated on a curve $\Sigma(t)$ (in $2d$, or a surface in $3d$), the above theorem does not help much and weak solutions have to be considered (for a precise meaning of what is meant by "weak solution", see for instance [40]). Using the conservation of the vorticity along Lagrangian trajectories implied by (7), one gets the following classical results:

Theorem 2. *1- (Yudovich [62]) With an initial vorticity $\omega_0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, the Euler equations (10) have a unique global weak solution which depends continuously on the data.*
2- If the initial vorticity is a bounded signed measure [18] or with a finite number of changes of sign [39], the Euler equations (10) have at least one weak solution $u(x,t) \in C_{\mathrm{weak}}(\mathbb{R}_t, L^2(\mathbb{R}^2))$.

However one should observe the big "gap" between the hypothesis of the point 1 and of the point 2 (this latter includes configurations allowing concentration of vorticity on an interface). Without the hypothesis $\omega \in L^\infty(\mathbb{R}^2)$ no uniqueness or stability result is available. Furthermore *wild solutions*, for instance with space time compact support have been constructed by Scheffer [48], Shnirelman [52] and finally by De Lellis and Székelyhidi [17] (see also the nice review paper by Villani [56]) who have obtained the following *instability, non uniqueness* theorem (note that the second point implies that the *wild solutions* are not that exotic, since they are obtained as limits of strong, smooth solutions of the Euler equations):

Theorem 3. *1- For any bounded domain $O \subset \mathbb{R}_x^d \times \mathbb{R}_t$ ($d = 2, 3$) there exists a weak solution $u \in L^\infty(\mathbb{R}_x^d \times \mathbb{R}_t)$ to (10) which satisfies the relation*

$$(x, t) \in O \Rightarrow |u(x, t)| = 1 \,, (x, t) \notin O \Rightarrow |u(x, t)| = 0.$$

2- Moreover, u is the limit in $L^2(\mathbb{R}_x^d \times \mathbb{R}_t)$ of a sequence of smooth, compactly supported, solutions u_k ($k \in \mathbb{N}$) to the Euler equations with smooth (and vanishing as $k \to \infty$) source terms f_k:

$$\partial_t u_k + \nabla \cdot (u_k \otimes u_k) + \nabla p_k = f_k, \qquad \nabla \cdot u_k = 0, \qquad f_k \to 0 \quad in \quad \mathcal{D}'(\mathbb{R}^d).$$

3 Weak solutions and interface equation

In this section we derive a closed system of equations for the interface $\Sigma(t)$ when the vorticity is a density concentrated on this curve, assumed that it satisfies the following smoothness assumption:

Definition 4. *1- A rectifiable curve Σ parametrized by its arc length, $\Sigma = \{r(s)\}$, is chord-arc if there exists two constants $0 < m < M < \infty$ such that one has:*

$$m|s_1 - s_2| \le |r(s_1) - r(s_2)| \le M|s_1 - s_2|.$$

A local version (near any point of Σ) of this definition is obtained with minor modifications.
2- We say that a curve $\Sigma \subset \Omega$ is regular if it is bounded away from $\partial\Omega$, rectifiable and chord-arc .

3.1 Biot-Savart formula and Hilbert Transform

From the incompressibility relation $\nabla \cdot u = 0$ one infers the existence of a *stream function* ψ such that

$$u(x,t) = \nabla^\perp \psi(x,t) = \left(-\partial_y \psi(x,t), \partial_x \psi(x,t)\right)^T. \tag{11}$$

Classical results of potential theory show that $u(x,t)$ converges to two values $u_\pm(x,t)$ on the two sides of the interface. The incompressibility condition implies the continuity across the interface of the normal[1] component of the velocity, but not of the tangential ones. Denoting by $u_\pm^\tau$ the trace of the tangential component of the velocity on both sides of the interface, and by $r(\cdot,t)$ a parametrization[2] of the interface, $\Sigma(t) = \{r(\lambda,t)\}$, standard distribution calculus shows that the scalar vorticity ω is a density measure supported on $\Sigma(t)$,

$$\forall \varphi \in C(\Omega), \qquad \langle \varphi, \omega \rangle = \int \varphi(r(\lambda,t)) \widetilde{\omega}(\lambda,t) d\lambda, \quad \text{with} \quad \widetilde{\omega}(\lambda,t) = -|\partial_\lambda r|(u_+^\tau - u_-^\tau)_{|(r(\lambda,t),t)};$$

the quantity $\widetilde{\omega}$ is called the *vorticity density* associated to the parametrization $r(\lambda,t)$. The vorticity density associated to the arc length parametrization is called the *vortex strength*, and will be denoted $\gamma(s,t)$ throughout this text,

$$\gamma(s(\lambda,t),t) = \frac{1}{|\partial_\lambda r(\lambda,t))|} \widetilde{\omega}(\lambda,t), \quad \text{with} \quad s(\lambda,t) = \int_0^\lambda |\partial_\lambda r(\lambda',t)| d\lambda'. \tag{12}$$

In terms of ψ, one has

$$\Delta \psi = \omega = \widetilde{\omega} \otimes \delta_{\Sigma(t)}. \tag{13}$$

From (11), (13) and the fact that u vanishes at infinity if Ω is unbounded, one can express $u(x,t)$ for all $x \notin \Sigma(t)$ in terms of the vorticity density $\widetilde{\omega}$ by a *Biot-Savart* type formula. Denoting by $G(x,y)$ the Green function of the Laplacian in Ω with Dirichlet boundary condition, one has

$$\begin{aligned}
\forall x \in \Omega(t) \backslash \Sigma(t), \qquad u(x,t) &= \int_\Omega \nabla_x^\perp G(x,y) \omega(t,y) dy \\
&= \int \nabla_x^\perp G(x,r(\lambda,t)) \widetilde{\omega}(\lambda,t) d\lambda.
\end{aligned} \tag{14}$$

For $x \in \Omega \backslash \Sigma(t)$ the function $u(t,x)$ is very smooth (indeed analytic). Though this is no more true for $x \in \Sigma(t)$, the formula (14) can be extended to the case $x \in \Sigma(t)$ provided the left hand side is replaced by

$$v = \frac{u_+ + u_-}{2},$$

and that on the right hand side the integral is defined as a principal value:

$$v(x,t) = \text{p.v.} \int \nabla^\perp G(x,r(\lambda,t)) \widetilde{\omega}(\lambda,t) d\lambda. \tag{15}$$

---

[1]It is of course assumed that the interface $\Sigma(t)$ is an orientable curve. We choose the orientation such that the normal vector $\vec{n}$ points from $\Omega_-(t)$ to $\Omega_+(t)$. The orientation of the tangent vector $\vec{\tau}$ is chosen such that $(\vec{\tau},\vec{n})$ is a direct orthonormal basis of $\mathbb{R}^2$

[2]Changing λ into $-\lambda$ if necessary, we always assume that $\partial_\lambda r$ is positively colinear to τ, that is, $\vec{\tau} = \frac{1}{|\partial_\lambda r|} \partial_\lambda r$.

In particular for $\Omega = \mathbb{R}^2$ one recovers the standard Biot-Savart formula:

$$v(x,t) = \frac{1}{2\pi} R_{\frac{\pi}{2}} \text{p.v.} \int \frac{x - r(\lambda, t)}{|x - r(\lambda, t)|^2} \widetilde{\omega}(\lambda, t) d\lambda, \tag{16}$$

with $R_{\frac{\pi}{2}}$ denoting the rotation of angle $\frac{\pi}{2}$. Furthermore, due to the ellipticity of the Laplacian and also for $\Omega \neq \mathbb{R}^2$, a local version of the above formula remains valid modulo a more regular term

$$v(x,t) = \frac{1}{2\pi} R_{\frac{\pi}{2}} \text{p.v.} \int_{\{r(\lambda, t) \in \mathcal{U}\}} \frac{x - r(\lambda, t)}{|x - r(\lambda, t)|^2} \widetilde{\omega}(\lambda, t) d\lambda + R_{\mathcal{U}}(\omega); \tag{17}$$

in (17), $x \mapsto R_{\mathcal{U}}(\omega)(x)$ is an analytic function and $\omega \mapsto R_{\mathcal{U}}(\omega)$ is a "regularizing operator" which depends on Ω and the open set $\mathcal{U}$.

Remark 1. Denoting by v^τ and v^n the tangential and normal components of v, one checks that:

$$v^\tau(x,t) = -\frac{1}{2\pi} \text{p.v.} \int_{\{r(\lambda, t) \in \mathcal{U}\}} \vec{n} \cdot \frac{x - r(\lambda, t)}{|x - r(\lambda, t)|^2} \widetilde{\omega}(\lambda, t) d\lambda + R_{\mathcal{U}}^\tau(\omega) \tag{18}$$

$$v^n(x,t) = \frac{1}{2\pi} \text{p.v.} \int_{\{r(\lambda, t) \in \mathcal{U}\}} \vec{\tau} \cdot \frac{x - r(\lambda, t)}{|x - r(\lambda, t)|^2} \widetilde{\omega}(\lambda, t) d\lambda + R_{\mathcal{U}}^n(\omega). \tag{19}$$

By a Taylor expansion of $r(\cdot, t)$ in the neighborhood of x, one checks that $(x - r(\lambda, t)) \cdot \vec{n} = O(|x - r(\lambda, t)|^2)$ while one only has $(x - r(\lambda, t)) \cdot \tau = O(|x - r(\lambda, t)|)$. It follows that v^τ is "one degree" more regular that v^n.

One of the main issues in the analysis of the Rayleigh-Taylor (and Kelvin-Helmholtz) problem is the ellipticity of the linearized operator. This property is local by its definition and its effects. Therefore it is convenient to write, with a Galilean transform, the equation (17) in the neighborhood of a point $r(\lambda, t) = (0, 0)$), with $\vec{\tau}, \vec{n}$ given by the coordinate axis, and where $\Sigma(t)$ is a graph $(x_1, \epsilon\sigma(x_1, t))$ which is a small perturbation of the x_1 axis in a neighborhood $|x_1| < \delta$ of the origin. In this setting (17) becomes

$$v_1(x,t) = -\frac{\epsilon}{2\pi} \text{p.v.} \int_{|x_1| < \delta} \frac{\sigma(x_1, t) - \sigma(x_1', t)}{(x_1 - x_1')^2 + \epsilon^2(\sigma(x_1, t) - \sigma(x_1', t))^2} \tilde{\omega}(x_1', t) dx_1', \tag{20}$$

$$v_2(x,t) = \frac{1}{2\pi} \text{p.v.} \int_{|x_1| < \delta} \frac{x_1 - x_1'}{(x_1 - x_1')^2 + \epsilon^2(\sigma(x_1, t) - \sigma(x_1', t))^2} \tilde{\omega}(x_1', t) dx_1', \tag{21}$$

where we omitted the more regular terms coming from $R_{\mathcal{U}}(\omega)$. We thus see that two operators will play an important role in the analysis of the linearized equation (see §4 below): the *Hilbert transform H* and $|D|$, defined respectively as

$$Hf(x_1) = \frac{1}{\pi} \text{p.v.} \int \frac{1}{x_1 - x_1'} f(x_1') dx_1' = F^{-1}(-i\text{sgn}(\xi)\hat{f}(\xi)), \tag{22}$$

$$|D|f(x_1) = \frac{1}{\pi} \text{p.v.} \int \frac{f(x_1) - f(x_1')}{(x_1 - x_1')^2} = \partial_{x_1}(Hf(x_1)) = F^{-1}(|\xi|\hat{f}(\xi)). \tag{23}$$

3.2 Jump relations and equations on the interface

With minimal regularity hypothesis on the interface and on the vorticity density one can deduce "jump relations". We have already seen that incompressibility relation $\nabla \cdot u = 0$ and the identity $\nabla \wedge u = \widetilde{\omega} \otimes \delta_{\Sigma(t)}$

lead to

$$[u^n] = 0 \quad \text{and} \quad [u^\tau] = \frac{-1}{|\partial_\lambda r|}\widetilde{\omega}, \tag{24}$$

where we use the standard notation $[u] = u_+ - u_-$.

From the conservation equations of mass (2) and momentum (1), written in the sense of distributions, we deduce further two equations for the evolution of the interface. If $r(\cdot, t)$ is a parametrization of the interface, $\Sigma(t) = \{r(\lambda, t)\}$, one gets (cf. [54], and [14] for a slightly different, but equivalent formulation):

$$(\partial_t r - v) \cdot \vec{n} = 0 \tag{25}$$

$$\partial_t(\frac{1}{2}\widetilde{\omega} - a|\partial_\lambda r|v^\tau) + \partial_\lambda\Big(\frac{1}{|\partial_\lambda r|}(\frac{1}{2}\widetilde{\omega} - a|\partial_\lambda r|v^\tau)(v - \partial_t r) \cdot \vec{\tau}\Big)$$

$$- a\partial_\lambda\Big(\frac{1}{8}\frac{\widetilde{\omega}^2}{|\partial_\lambda r|^2} - \frac{1}{2}|v|^2\Big) = -a|\partial_\lambda r|\vec{\mathbf{g}} \cdot \vec{\tau}. \tag{26}$$

Observe that the above system does not fully determines the evolution of the vector $r(\lambda, t)$. The equation (25) concerns only the normal component of its time derivative. This correspond to the fact already noticed that it is only the global evolution of the interface and not of its parametrization that matters. Different choices of the parametrization lead to the following formulas:

- When $a = 0$ (this corresponds to $\rho_- = \rho_+$ i.e. to the Kelvin-Helmholtz problem), it is convenient to choose a Lagrangian parametrization,

$$r_t(\lambda, t) = v(r(\lambda, t), t) = \frac{u_+ + u_-}{2}(r(\lambda, t)); \tag{27}$$

the equations (25) and (26) then reduce to

$$\partial_t\tilde{\omega} = 0 \quad \text{and thus} \quad \tilde{\omega}(\lambda, t) = \tilde{\omega}(\lambda, 0). \tag{28}$$

Using (15), we thus get

$$\partial_t r(\lambda, t) = \text{p.v.} \int \nabla_x^\perp G\big(r(\lambda, t), r(\lambda', t)\big)\tilde{\omega}(\lambda', 0)d\lambda' \tag{29}$$

or, when $\Omega = \mathbb{R}^2$,

$$\partial_t r(\lambda, t) = \frac{1}{2\pi}R_{\frac{\pi}{2}}\text{p.v.} \int \frac{r(\lambda, t) - r(\lambda', t)}{|r(\lambda, t) - r(\lambda', t)|^2}\tilde{\omega}(\lambda', 0)d\lambda'. \tag{30}$$

It is therefore natural to reparametrize the interface by the (time independent) arc length *of its initial position*, $\Sigma(t) = \{r(s_0^{-1}(s), t)\}$, where $s_0(\lambda) = \int_0^\lambda \partial_\lambda r(\lambda', 0)d\lambda'$. When $\Omega = \mathbb{R}^2$, one can further identify the fluid domain with the complex plane, i.e. $z(s, t) \sim r(s_0^{-1}(s), t)$ to obtain:

$$\partial_t\overline{z}(s, t) = \frac{1}{2\pi i}\text{p.v.} \int \frac{\gamma(s', 0)}{z(s, t) - z(s', t)}ds', \tag{31}$$

where $\gamma(s, 0)$ is the vortex strength (12) at $t = 0$.

If moreover $\gamma(s, 0)$ has *a distinguished sign* (which is the case when the initial vorticity is a Radon measure), then we can use the mapping

$$\alpha_0(s) = \int_0^s \gamma(s', 0)ds'$$

8

to perform another (time independent) reparametrization of the interface, namely, $\Sigma(t) = \{z(\alpha_0^{-1}(\alpha), t)\}$. One thus obtains the *Birkhoff-Rott equation* for the variable $\mathbf{z}(\alpha, t) = z(\alpha_0^{-1}(\alpha), t)$,

$$\partial_t \overline{\mathbf{z}}(\alpha, t) = \frac{1}{2\pi i} \text{p.v.} \int \frac{d\alpha'}{\mathbf{z}(\alpha, t) - \mathbf{z}(\alpha', t)}. \tag{32}$$

The Birkhoff-Rott equation corresponds to a parametrization by the circulation; indeed, one readily checks that $|\partial_\alpha \mathbf{z}(\alpha_0(s), t)| = \frac{1}{\gamma(s, 0)}$.

- When $a = -1$ (this corresponds to $\rho_+ = 0$, and thus to the water waves problem), several choices are possible. One can for instance, as in [54], choose a Lagrangian parametrization with the velocity given by the trace of the velocity field in $\Omega_-(r)$ at the surface,

$$r_t(\lambda, t) = u_-(r(\lambda, t), t) = u_-^n(r(\lambda, t), t)\vec{n} + u_-^\tau(r(\lambda, t), t)\vec{\tau}. \tag{33}$$

Another choice, adopted in [5], consists in parametrizing the surface by its (renormalized) arclength. It is also possible to use the particular case $a = -1$ of the following point when the surface is a graph.

- When the interface is a graph (we have seen that this hypothesis can always be used for "local analysis"), we choose the canonical parametrization by x_1, $\Sigma(t) = \{(x_1, \sigma(x_1, t)\}$ and the full system (including the Biot-Savart equation) becomes:

$$\partial_t \sigma + v_1 \partial_{x_1} \sigma = v_2 \tag{34}$$

$$\partial_t \left(\frac{1}{2} \widetilde{\omega} - a(v_1 + v_2 \partial_{x_1} \sigma) \right)$$
$$+ \partial_{x_1} \left\{ v_1 \left(\frac{1}{2} \widetilde{\omega} - a(v_1 + v_2 \partial_{x_1} \sigma) \right) - a \left(\frac{\widetilde{\omega}^2}{8(1 + \partial_{x_1} \sigma^2)} - \frac{|v|^2}{2} - g_1 x_1 - g_2 \sigma \right) \right\} = 0, \tag{35}$$

$$v_1(x_1, t) = -\frac{1}{2\pi} \text{p.v.} \int \frac{\sigma(x_1, t) - \sigma(x_1', t)}{(x_1 - x_1')^2 + (\sigma(x_1, t) - \sigma(x_1', t))^2} \widetilde{\omega}(x_1', t) dx_1', \tag{36}$$

$$v_2(x_1, t) = \frac{1}{2\pi} \text{p.v.} \int \frac{x_1 - x_1'}{(x_1 - x_1')^2 + (\sigma(x_1, t) - \sigma(x_1', t))^2} \widetilde{\omega}(x_1', t) dx_1'. \tag{37}$$

4 Linearization of the interface problem

To analyze the local properties of the above system (34)-(37), one introduces its linearized version near a stationary solution $\underline{\sigma}(x, t) = 0$, $\underline{u}_+(x, 0) = (-u_0, 0)$, and $\underline{u}_-(x, 0) = (u_0, 0)$ (and thus $\widetilde{\underline{\omega}} = 2u_0$). Looking for solutions of the form $\underline{\sigma} + \epsilon\sigma$, $\widetilde{\underline{\omega}} + \epsilon\widetilde{\omega}$, we are led to the following equations:

$$\partial_t \sigma - v_2 = 0, \tag{38}$$

$$\partial_t \left(\frac{1}{2} \widetilde{\omega} - a v_1 \right) + \partial_{x_1} \left(u_0 v_1 - \frac{1}{2} a u_0 \widetilde{\omega} + a g_2 \sigma \right) = 0, \tag{39}$$

$$\epsilon v_1(x_1, t) = -\epsilon \frac{1}{2\pi} \text{p.v.} \int \frac{\sigma(x_1, t) - \sigma(x_1', t)}{(x_1 - x_1')^2} (2u_0 + \epsilon\widetilde{\omega}(x_1', t)) dx_1' = -\epsilon u_0 |D| \sigma + O(\epsilon^2), \tag{40}$$

$$\epsilon v_2(x_1, t) = \frac{1}{2\pi} \text{p.v.} \int \frac{x_1 - x_1'}{(x_1 - x_1')^2} (2u_0 + \epsilon\omega(x_1', t)) dx_1' = \frac{1}{2} \epsilon H \widetilde{\omega} + O(\epsilon^2), \tag{41}$$

$$\tag{42}$$

9

where the operators $|D|$ and H are as defined in (22) and (23) (and where the identity $|D|1 = H1 = 0$ has been used to derive the second part of (40) and (41)).

At leading order in ϵ, we thus obtain the two following equations:

$$\partial_t \sigma - \frac{1}{2} H \widetilde{\omega} = 0 \,, \tag{43}$$

$$\partial_t \big(\frac{1}{2} \widetilde{\omega} + a u_0 |D| \sigma \big) + \partial_{x_1} \big(- u_0^2 |D| \sigma - \frac{1}{2} a u_0 \widetilde{\omega} + a g_2 \sigma \big) = 0 \,. \tag{44}$$

Taking the derivative with respect to t in (44) and using (43) one obtains finally

$$\partial_t^2 \widetilde{\omega} - 2 a u_0 \partial_{x_1 t}^2 \widetilde{\omega} + u_0^2 \partial_{x_1}^2 \widetilde{\omega} + a g_2 |D| \widetilde{\omega} = 0$$

or

$$(\partial_t - a u_0 \partial_{x_1})^2 \widetilde{\omega} + u_0^2 (1 - a^2) \partial_{x_1}^2 \widetilde{\omega} + a g_2 |D| \widetilde{\omega} = 0 \,. \tag{45}$$

The nature of Equation (45) depends on the parameters a, g_2 and u_0:

- If the Atwood number satisfies $a^2 < 1$ and if $u_0 \neq 0$, then the equation (45) is always *elliptic*, regardless of the sign of $a g_2$.

- If $a^2 = 1$ and $a g_2 > 0$, which corresponds to the water waves case (with downwards gravity), the equation (45) is then *nonstrictly hyperbolic* and the condition $a g_2 > 0$ is in fact the Levy condition ensuring the wellposedness of the Cauchy problem associated to it.

- If $u_0 = 0$ and $a g_2 > 0$, which corresponds to the Rayleigh-Taylor case ($0 < a < 1$) with the heavier fluid placed below the lighter one, then (45) has the same characteristics as in the case $a^2 = 1$, $a g_2 > 0$. The ill-posedness of the Rayleigh-Taylor problem, even for configuration close to the rest state, are therefore due to *nonlinear* effects.

- If $a^2 = 1$ (or $u_0 = 0$) and $a g_2 < 0$, which corresponds to the water waves case (or Rayleigh-Taylor) with the lighter fluid *below* the heavier one, the equation (45) is then *hypoelliptic*.

Remark 2. A Fourier type analysis of (45) with the introduction of *modes*

$$\omega(x, t) = e^{\tau t + i \xi x}$$

shows that all frequencies ξ such that

$$|\xi| \geq \frac{a g_2}{(1 - a^2) u_0^2}$$

lead to unstable modes (with $\Re \tau > 0$). The range of unstable frequencies increases with a going to zero which is the Kelvin-Helmholtz case. Without the nonlinear interactions, low frequencies would therefore remain stable for the Rayleigh-Taylor problem.

5 Consequences of the ellipticity of the Rayleigh-Taylor and Kelvin-Helmholtz problems

5.1 Analytic solutions

For a precise statement of the behavior of analytic solutions to (34)-(37), it is convenient to follow Duchon and Robert [19] and to introduce $\mathcal{B}_\rho$, the space of analytic functions which are Fourier transform

of bounded measures with exponential decay at infinity (keeping in mind the Payley Wiener theorem, this corresponds to functions that can be analytically extended in the strip $\{(x + iy), |y| < \rho\})$:

$$\mathcal{B}_\rho = \{u = \int d|\hat{u}|, \quad \|u\|_\rho = \int d|e^{\rho\xi}\hat{u}(\xi)| < \infty\}. \tag{46}$$

One then has the following theorem for the Kelvin-Helmholtz and Rayleigh-Taylor problems; it shows that one can construct *local* in time solutions to (34)-(37) if *both* the interface and the vorticity are prescribed (and analytic) at $t = 0$, and that *global* in time solutions exist if the initial vorticity is not arbitrarily chosen. Note that in the second point of the theorem, one considers a system on $(\partial_{x_1}\sigma, \widetilde{\omega})$ by taking the x_1 derivative of (34).

Theorem 5. *1- (see [55] and [54]) Let $\rho > 0$ and $(\sigma_0, \widetilde{\omega}_0) \in \mathcal{B}_\rho$, and assume further that σ_0 has zero mean value (i.e. $\widehat{\sigma_0}(\{0\}) = 0$). Then there exists a constant α such that the Cauchy problem (34)-(37) has a unique solution*

$$(\sigma(t,.), \tilde{\omega}(t,.)) \in \mathcal{B}_{\frac{\alpha\rho - |t|}{\alpha}}$$

defined for $|t| < \alpha\rho$ (and whose dependence on t is analytic).
2- (see [19]) There exists a constant $\epsilon > 0$ such that for all $\partial_{x_1}\sigma_0 \in \mathcal{B}_0$ satisfying

$$\int |\xi| d|\hat{\sigma}_0(\xi)| < \epsilon \quad and \quad \widehat{\partial_{x_1}\sigma_0}(\{0\}) = 0, \tag{47}$$

there exists $\alpha > 0$ such that the problem $\partial_{x_1}(34)$-(37) admits a unique solution $(\partial_{x_1}\sigma, \widetilde{\omega}) \in C(0, \infty; \mathcal{B}_\alpha)^2$ with the boundary conditions:

$$\partial_{x_1}\sigma(0, x_1) = \partial_{x_1}\sigma_0(x_1), \qquad \widehat{\widetilde{\omega}(0, \cdot)}(\{0\}) = 0.$$

Moreover, $\partial_{x_1}\sigma(\cdot, t)$ and $\widetilde{\omega}(\cdot, t)$ converge to 0 uniformly as t goes to ∞.

Remark 3.

1- The first point has been proven first for the Kelvin-Helmholtz problem ($a = 0$) in [55] and then extended to $a \neq 0$ in [54]. The basic new ingredient is contained in the relation between $\tilde{\omega}$ and $\tilde{\omega}/2 - a(v_1 + v_2\partial_{x_1}\sigma)$. The term $(v_1 + v_2\partial_{x_1}\sigma)$ is proportional to v^τ the tangent component of v and one has:

$$(v_1 + v_2\partial_{x_1}\sigma) = K(\tilde{\omega})$$

with K being a regularizing operator as observed in Remark 1. Therefore with Fredholm property and spectral analysis one can, for $|a| < 1$, invert the relation

$$\frac{1}{2}\tilde{\omega} - a(v_1 + v_2\partial_{x_1}\sigma) = (\frac{1}{2}I - aK)(\tilde{\omega}) \tag{48}$$

to obtain a well posed Cauchy problem for $(\sigma, \tilde{\omega})$ in the class of analytic functions (see for details [28], [54] and [8]).

2- The proof of the second point has only been done for $a = 0$ but the extension to $0 \leq |a| < 1$ does not seem too difficult.

Looking at the solution provided by the second point of the theorem on the interval $[0, T]$, with $T > 0$, and changing the time variable $t \mapsto T - t$, we have at our disposal solutions which are analytic for $0 \leq t < T$ and which at the time T exhibit any type of singularity compatible with the hypothesis (47). This is a much larger class of singularities than the one initially predicted by Meiron, Baker and Orsag for Kelvin-Helmholtz [44]. One can have examples for which the Hausdorff dimension of the graph $x_1 \mapsto \partial_{x_1}\sigma(x_1, T)$ may be of dimension greater than one. In particular in a periodic version of the above theorem one obtains starting, from analytic initial data at time $t = 0$, a solution that coincides at $t = T$ with the Weierstrass function,

$$\sigma(x_1, T) = \sum_{0 \leq n \leq \infty} 2^{-\frac{n}{2} - n} \cos(2^n x_1), \qquad \widehat{\partial_{x_1}\sigma}(\xi, T) = \sum_{0 \leq n \leq \infty} 2^{-\frac{n}{2}} (\delta(\xi + 2^n) + \delta(\xi - 2^n)).$$

The computations made in [44] and [43] led to the belief that the first appearance of singularity was the formation of a cusp in the curve $\partial_{x_1}\sigma(x_1, T)$ or $\omega(x_1, T)$. As already said, this turns out to be a special case of the above theorem, but that was also the motivation of the construction of Caflisch and Orellana [10] . With perturbation methods very close to the one used in [19], they construct solutions for Kelvin-Helmholtz problem which are analytic for $t > 0$ and which at the time $t = 0$ have an isolated singularity behaving like a cusp. In complex representation these solutions are constructed as analytic perturbation of the function:

$$z_0(\lambda, t) = (1 - i)\{(1 - e^{-\frac{t}{2} - i\lambda})^{1+\nu} - (1 - e^{-\frac{t}{2} + i\lambda})^{1+\nu}\}, \tag{49}$$

which turns out to be an exact solution of the linearized Birkhoff-Rott equation and which has the following properties: i) For any $t > 0$ $\lambda \mapsto z_0(t, \lambda)$ is analytic. ii) For $t = 0$ $\lambda \mapsto z(0, \lambda)$ does not belong to the space $C^{1+\nu}$ but belongs to any space $C^{1+\nu'}$ with $0 < \nu' < \nu$.

5.2 After the appearance of the first singularity

The mathematical properties of the interfaces may be used (with a contradiction argument) to obtain some results concerning the behavior of the solution (if it exists) after the appearance of the first singularities. As this turns out to be in many *free boundaries problems*, the method consists in showing that if the solution is locally (in space time) more regular than some threshold $\mathcal{S}$ then it is very regular (locally C^∞ or analytic). This implies in turn that after the appearance of the first singularity, the solution has to be less regular than the threshold $\mathcal{S}$. As a consequence, the determination of a threshold $\mathcal{S}$ that would imply C^∞ or analyticity is one of the challenges of the theory, and as such has been studied for instance by S. Wu [60] , Lebeau [33] and Lebeau-Kamotski [28]. For instance in [28] one has the following statement:

Theorem 6. *Assume that locally near a point (s_0, t_0) the curve $\Sigma(t) = \{r(s, t)\}$ parametrized by the arc length belongs to $C^{1+\nu}$ ($\nu > 0$) and is chord-arc; assume also that the density of vorticity is strictly positive and bounded at (s_0, t_0). Then in the neighborhood of (s_0, t_0), the curve and the density of vorticity are C^∞.*

Remark 4. If the vorticity is assumed to be positive and bounded for all s (and not only in a neighborhood of s_0), then the curve and density of vorticity are analytic. It is expected that the conclusion of Theorem 6 remains true if one replaces C^∞-regularity by analyticity.

The result of S. Wu [60] obtained independently and previously for Kelvin-Helmholtz is slightly sharper. Instead of $C^{1+\nu}$ one assumes that the function $r(t,s)$ belongs to H^1 and does not rolls up too fast. More precisely one has the following theorem (we give a non sharp version of the condition (50) to avoid the definition of technical functional spaces).

Theorem 7. *Assume that* $\mathbf{z} \in H^1([0,T], L^2_{loc}(\mathbb{R})) \cap L^2([0,T]; H^1_{loc}(\mathbb{R}))$ *is a solution of the Birkhoff-Rott equation (32), which is uniformly chord-arc on* $[0,T]$*:*

$$\exists m, M > 0, \quad \forall (t,\alpha,\beta) \in [0,T] \times \mathbb{R}^2, \qquad m|\alpha - \beta| \leq |\mathbf{z}(\alpha,t) - \mathbf{z}(\beta,t)| \leq M|\alpha - \beta|.$$

Then there exists a constant $c(m, M)$ *such that the relation:*

$$\sup_{[0,T]} \| \ln(\mathbf{z}_\alpha(\cdot, t)) \|_\infty \leq c(m, M) \tag{50}$$

implies the analyticity.

Remark 5. The above statements are in full agreement with previous theoretical and numerical results. The example of Caflish and Orellana corresponds to a singularity in $C^{1+\nu}$ and the Lebeau theorem shows that this function cannot be extended as a solution in $C^{1+\nu'}$ with $\nu' > 0$ after the singularity.
The Prandlt-Munk vortex, described in Section 6 below, is a non regular solution of the Birkhoff-Rott equation defined for all time. This is not in contradiction with the previous theorems because near the points $(\pm 1, \frac{t}{2})$ the vorticity density does not satisfy the Lebeau-Wu hypothesis. Numerical simulations done by Krasny [30] indicate the appearance after the singularity of spirals. On the other hand Theorem 7 implies that some spiral-type behavior would imply regularity and therefore is excluded. The catch lies in the fact that Krasny spirals do not seem to satisfy an hypothesis of the type (50)

Without entering into details it is interesting to give some comments on the proof of Theorem 7. As in any free boundary problem the challenge is to determine the threshold of regularity $\mathcal{S}$ as sharply as possible. The most delicate step is to prove that this threshold regularity implies a slightly better regularity; deducing analiticity (or C^∞ regularity) from this slightly better regularity is usually much easier.

In the present case, one shows that the assumptions of Theorem 7 lead to H^2-regularity. This can be reached from the Birkhoff-Rott equation by a localized (with a smooth real valued function $\eta(\alpha)$) variational estimate. With $V = \eta(\alpha)(1 + i) \ln \mathbf{z}_\alpha$ one obtains the formulas

$$\overline{V}_t - \frac{1}{2|\mathbf{z}_\alpha|^2} |D|V = F(\alpha, t),$$

for some function $F(\alpha, t)$ that can be explicitly computed, and thus

$$\int |\mathbf{z}_\alpha|^2 \left(\overline{V}_t - \frac{1}{2|\mathbf{z}_\alpha|^2} |D|V \right)^2 d\alpha = \int |\mathbf{z}_\alpha|^2 |F(\alpha, t)|^2 d\alpha. \tag{51}$$

On the other hand, one easily checks that

$$\int |\mathbf{z}_\alpha|^2 \left(\overline{V}_t - \frac{1}{2|\mathbf{z}_\alpha|^2} |D|V \right)^2 d\alpha = \int \left(|\mathbf{z}_\alpha|^2 |V_t|^2 + \frac{1}{4|\mathbf{z}_\alpha|^2} \big||D|V\big|^2 \right)$$
$$- \frac{1}{2} \Re \frac{d}{dt} \int V|D|V. \tag{52}$$

13

Since the assumptions of Theorem 7 imply that $m \leq |\mathbf{z}_\alpha| \leq M$, it is possible to deduce (after a bit of work!) from (51) and (52), a control of the quantity

$$\int q(t) \int (m^2 |V_t|^2 + \frac{1}{4M^2}||D|V|^2)d\alpha dt,$$

where $q(t)$ is a carefully chosen smooth, real valued function. Recalling that $V = \eta(\alpha)(1 + i)\ln \mathbf{z}_\alpha$, this suggests that it is possible to control the H^1_{loc}-norm (in space and time) of $\ln \mathbf{z}_\alpha$ by its L^∞-norm, which is finite by assumption.

6 Nonregular vortex sheets

6.1 The Prandtl-Munk vortex sheet

As described in Section 3, a solution of the Kelvin-Helmholtz problem (computed on the interface) should provide, with u given by the Biot-Savart relation (14), a weak solution of the incompressible Euler equation. The situation turns out not to be so simple for the following reasons:

(i) Relaxing regularity hypothesis on the interface, one can generate solutions of (30) which are not solutions of the Euler equation. And this pathology appears with initial data which satisfy the hypothesis of [39] and therefore have weak solutions (different from what is given by the Kelvin-Helmholtz equation (30)).

(ii) Even without the above pathology, one should observe that the Kelvin-Helmholtz problem generates solutions which do not satisfy (and there is a big "gap") the uniqueness stability criteria of Yudovich.

The point (i) is illustrated by the "Prandtl-Munk vortex sheet" which is notorious in fluid dynamics as the one that generates a circulation distribution that minimizes the induced drag on a plane wing [46]. With initial data:

$$\Sigma(0) = \{r(x_1, 0) = (x_1, 0)\}, \qquad \widetilde{\omega}(x_1, 0) = \frac{x_1}{\sqrt{1 - x_1^2}}\chi_{(-1,1)}(x_1),$$

where $\chi_{(-1,1)}$ is the characteristic function of the interval $(-1, 1)$. One observes (cf. [46]) that

$$r(x_1, t) = (x_1, -\frac{t}{2}), \qquad v = (0, -\frac{1}{2}), \qquad \widetilde{\omega}(x_1, t) = \widetilde{\omega}(x_1, 0)$$

is a solution of the equation (30). However the corresponding velocity $u(x, y, t)$ is *not* a solution of the Euler equation (10) and one has instead (see [38]):

$$\nabla \cdot v = 0, \qquad \partial_t u + \nabla \cdot (u \otimes u) + \nabla p = F,$$

with a nonzero forcing term F given by (with δ denoting the Dirac mass):

$$F(x_1, x_2) = \frac{\pi}{8}\Big((\delta_{(x_1+1, x_2+\frac{t}{2})} - \delta_{(x_1-1, x_2+\frac{t}{2})}), 0\Big).$$

The catch lies in the fact that the vorticity density is not regular enough in the neighborhood of the points $x_1 = \pm 1$ to justify the formal computations. On the other hand the initial data satisfy the hypothesis of [39] and there is therefore a weak solution of the Euler equations with these initial condition. This solution is singular enough near the same points to differ from the one given by (30) . We refer to [37] for a discussion of the equivalence at low regularity of the Birkhoff-Rott and Euler description of vortex sheets.

14

6.2 Regularization of vortex sheets

There exist various ways to regularize a vortex sheet, with different properties.

(i) Convolution by a sequence of functions converging to the Dirac distribution of analytic initial data with vorticity concentrated on a curve provides a regular solution of the Euler equation that does converge to the corresponding solution of the Kelvin-Helmholtz problem (in the existence domain of such solution), see [41].

(ii) The so-called α-regularization (one of the novel approaches for subgrid scale modeling of turbulence): In $\mathbb{R}^2$ the vorticity transport equation: is replaced by the α regularised equation:

$$\partial_t \omega_\alpha + u_\alpha \cdot \nabla \omega_\alpha = 0, u_\alpha = K_\alpha \omega_\alpha = (I - \alpha \Delta)^{-1} u \text{ with } u(x,t) = \frac{1}{2\pi} R_{\frac{\pi}{2}} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \omega_\alpha(t,y) dy. \text{ (53)}$$

For any $\alpha > 0$ the kernel K_α is regularizing. Therefore the above system is well posed (with global in time existence and uniqueness of the solution) for any measure valued initial data. Moreover any initial vorticity density supported by a curve $\Sigma(0)$ remains on the curve $\Sigma_\alpha(t)$ transported by the flow

$$\dot{X}(x,t) = u_\alpha(X(x,t),t), \; X(x,t=0) = x.$$

Eventually if the initial data (curve and vorticity density) are analytic, for $\alpha \to 0$ the curve $\Sigma_\alpha(t)$ and the corresponding vorticity density converge to the solution of the Kelvin Helmholtz problem as long as this solution, which exists locally in time, is defined; for details and further references see [7].

(iii) However and more annoying is the following fact: with an initial vorticity concentrated on a curve the Navier-Stokes has with viscosity $\nu > 0$ a well defined solution $u_\nu(x,t)$. Furthermore if the initial data satisfy the hypothesis of [39] this solution converges to a weak solution.

Weather this solution coincides with the one given by the Kelvin-Helmholtz problem (even with convenient hypothesis on the regularity of the curve and of the density of vorticity) is a completely open issue. With the "Prandtl-Munk vortex sheet" one can exhibit a counter-example (but with singular curve and vorticity) to this convergence: The corresponding initial data satisfy the hypothesis of [39]. Therefore, when the viscosity goes to 0, the corresponding solution of the Navier-Stokes converges to a weak solution of the Euler equation. On the other hand the "Prandtl-Munk vortex sheet" which is the solution of the Kelvin Helmholtz equation with the same data is not a weak solution of the Euler equation.

7 Water waves

The water waves problems refers to the case (like water in air) where one of the density (say ρ_+) is infinitely small with respect to the other and then taken equal to zero. As said in §3.2, various parametrizations are possible. Parametrizing the surface $\Sigma = \{r(t,\lambda)\}$ as in (33), one has

$$a = -1, \qquad r_t = u_-^n \vec{n} + u_-^\tau \vec{\tau}, \qquad r_\lambda = \vec{\tau}|r_\lambda|.$$

Therefore in the system (25)-(26), or equivalently

$$(\partial_t r - v) \cdot \vec{n} = 0,$$

$$\partial_t(\frac{\tilde{\omega}}{2} + (v \cdot r_\lambda)) + \partial_\lambda\Big\{\frac{1}{|r_\lambda|^2}(v - r_t) \cdot r_\lambda \left(\frac{\tilde{\omega}}{2} + (v \cdot r_\lambda)\right)\Big\} + \partial_\lambda\Big\{\frac{\tilde{\omega}^2}{8|r_\lambda|^2} - \frac{|v|^2}{2}\Big\} - \vec{g} \cdot r_\lambda = 0,$$

one may use the following relations

$$\frac{\tilde{\omega}}{2} + (v \cdot r_\lambda) = u_-^\tau |r_\lambda|,$$

$$\frac{1}{|r_\lambda|^2}(v - r_t) \cdot r_\lambda \left(\frac{\tilde{\omega}}{2} + (v \cdot r_\lambda)\right) = \left(\frac{1}{2} u_-^\tau u_+^\tau - \frac{|u_-^\tau|^2}{2}\right),$$

$$\left\{\frac{\tilde{\omega}^2}{8|r_\lambda|^2} - \frac{|v|^2}{2}\right\} = \left(-\frac{1}{2} u_-^\tau u_+^\tau - \frac{|u_-^n|^2}{2}\right),$$

to obtain the equation

$$\partial_t(u_-^\tau |\partial_\lambda r|) - \partial_\lambda \left(\frac{1}{2}(|u^n|^2 + |u^\tau|^2 + \vec{g} \cdot r\right) = 0. \tag{54}$$

A variant of (54) can also be derived if one chooses to parametrize the surface by its arclength (see for instance [28]); it is also worth commenting on the case where the surface is a graph, $\Sigma(t) = \{(x_1, \sigma(x_1, t))\}$. Starting from (34)-(35) and remarking that

$$\frac{1}{2}\tilde{\omega} + (v_1 + v_2 \partial_{x_1}\sigma) = \sqrt{1 + (\partial_{x_1}\sigma)^2}\, u_-^\tau,$$

one gets

$$\partial_t\left(\sqrt{1 + (\partial_{x_1}\sigma)^2}\, u_-^\tau\right) + \partial_{x_1}\left\{u_-^\tau(v - r_t) \cdot \vec{\tau}\right\} + \partial_{x_1}\left\{\frac{1}{8}(u_+^\tau - u_-^\tau)^2 - \frac{1}{2}|v|^2\right\} - \vec{g} \cdot r_{x_1} = 0.$$

Since $r_t = (0, \partial_t\sigma)^T = (0, \sqrt{1 + (\partial_{x_1}\sigma)^2}\, u_-^n)^T$, this is equivalent to

$$\partial_t\left(\sqrt{1 + (\partial_{x_1}\sigma)^2}\, u_-^\tau\right) + \partial_{x_1}\left\{\frac{1}{2}(u_-^\tau)^2 - u_-^\tau u_-^n - \frac{1}{2}(u_-^n)^2 + g\sigma\right\} = 0, \tag{55}$$

where we assumed that $\vec{g} = (0, -g)^T$.
Since $\nabla \cdot u = 0$ and $\nabla \wedge u = 0$ in $\Omega_-(t)$, and assuming that $\Omega_-(t)$ is simply connected, there exists a velocity potential $\Phi(x, t)$ which satisfies the relation:

$$\nabla\Phi(x, t) = u(x, t), \qquad \Delta\Phi(x, t) = 0 \quad \text{in} \quad \Omega_-(t).$$

Denoting by Ψ the trace of Φ at the surface, $\Psi(x_1, t) = \Phi(x_1, \sigma(x_1, t), t)$, one has

$$\partial_{x_1}\Psi = \partial_{x_1}\Phi_{|\Sigma(t)} + \partial_{x_1}\sigma \, \partial_{x_2}\Phi_{|\Sigma(t)} = \sqrt{1 + (\partial_{x_1}\sigma)^2}\, u_-^\tau.$$

Integrating with respect to x_1 in (55) yields therefore

$$\partial_t\Psi + \frac{1}{2}(u_-^\tau)^2 - u_-^\tau u_-^n - \frac{1}{2}(u_-^n)^2 + g\sigma = 0 \tag{56}$$

(note that adding if necessary a function depending on t only to Φ, the integration constant can be taken equal to 0).
Eventually it is convenient to introduce the Dirichlet Neumann operator $G(\sigma) : H^{\frac{1}{2}}(\mathbb{R}) \mapsto H^{-\frac{1}{2}}(\mathbb{R})$ associated to the resolution of the Dirichlet problem in $\Omega_-(t) \subset \Omega$ (with an additional homogeneous Neumann condition corresponding to (4) if there is a fixed boundary):

$$G(\sigma)\Psi = \sqrt{1 + (\partial_{x_1}\sigma)^2}\partial_{\vec{n}}\Phi, \tag{57}$$

16

where

$$\begin{cases} -\Delta\Phi = 0 \text{ in } \Omega(t), \\ \Phi = \Psi \text{ on } \partial\Omega_-(t)\backslash(\partial\Omega_-(t)\cap\partial\Omega), \qquad \partial_{\vec{n}}\Phi = 0 \text{ on } (\partial\Omega_-(t)\cap\partial\Omega). \end{cases}$$

Remarking that

$$\sqrt{1+(\partial_{x_1}\sigma)^2}\, u_-^\tau = \partial_{x_1}\Psi \quad \text{and} \quad \sqrt{1+(\partial_{x_1}\sigma)^2}\, u_-^n = G(\sigma)\Psi,$$

we deduce from (56) that the water waves equations can we written

$$\begin{cases} \partial_t\sigma = G(\sigma)\Psi, \\ \partial_t\Psi + g\sigma + \dfrac{1}{2}(\partial_{x_1}\Psi)^2 - \dfrac{1}{2(1+(\partial_{x_1}\sigma)^2)}(G(\sigma)\Psi + \partial_{x_1}\sigma\partial_{x_1}\Psi)^2 = 0, \end{cases} \tag{58}$$

where we assumed that $\vec{\mathbf{g}} = (0,-g)$ in the above expression.

Remark 6. The formulation (58) of the water waves equation corresponds to the Hamiltonian exhibited by Zakharov [63] and was written first under this form by Craig and Sulem [16]. As it is well known the equation (58) can be obtained directly from the Euler equation

$$\partial_t u + u\cdot\nabla u + \frac{1}{\rho_-}\nabla p = \vec{\mathbf{g}} \tag{59}$$

written on the domain $\Omega_-(t)$ occupied by the fluid. The above short derivation has been written to compare this problem with the Rayleigh-Taylor and Kelvin-Helmholtz equations and to emphasize similarities and differences.

7.1 Rayleigh-Taylor type behavior

In the water waves case, the linearized equation (45), with $u_0 = 0$, can be written

$$\partial_t^2\widetilde{\omega} - g_2|D|\widetilde{\omega} = 0. \tag{60}$$

It has already been noted in §4 that if $g_2 > 0$ (the fluid is above the surface), the operator appearing in (60) is hypoelliptic. The observations of Section 5 involving the ellipticity of the Kelvin Helmholtz problem could be adapted. *To the best of our knowledge this has not yet been worked out with full details except in the case of traveling waves which is the object of §7.4, and §8.2 for the extension to the 3d case.* Near a stationary solution with $g_2 > 0$ one may prove, with analytic initial data, local in time existence of an analytic solution. Relaxing one of the constraints at $t = 0$ one may adapt the construction of [19] or [10]. Hence by the change of variable $t \mapsto T - t$, one could construct solutions that exhibit a large class of singular behaviors. Finally since a Galilean transform changes the problem into an equation involving perturbations of a linear term of the form (60), the results of [33] and [60] could be adapted leading to a threshold of regularity $\mathcal{S}$ which for solutions more regular than this threshold would have C^∞ or "Gevrey" regularity. *In some sense the water waves problem with $ag_2 < 0$ shares the same behavior as the Rayleigh-Taylor problem and therefore carries the same name.*

7.2 The nonstrictly hyperbolic behavior

When $g_2 < 0$ (the fluid is below the surface), we have already commented in §4 that (60) is nonstrictly hyperbolic, and that the condition $g_2 < 0$ corresponds to the Levy condition on the subprincipal symbol.

17

The equation (60) roughly behaves as a wave equation with the positive self-adjoint operator $-\partial_x^2$ replaced by its square root.

An important observation is that the sign of g_2 is an *intrinsic object*. The equation (60) has been obtained by small perturbation analysis near a point where the x_1 axis is tangent to the interface and where the interface behaves like $(x_1, \sigma(x_1, t))$ therefore g_2 coincides (modulo small perturbations) with $\vec{g} \cdot \vec{n}$. With the full, nonlinear equations (59), the corresponding quantity is $-\partial_{\vec{n}} p$. Therefore the condition

$$-\partial_{\vec{n}} p > 0 \quad \text{on} \quad \Sigma(t) \tag{61}$$

should enforce the stability of the problem. This condition is called the *Rayleigh-Taylor stability criterion* (or *Taylor sign condition*). From a mathematical viewpoint, it is also the *Levy condition* ensuring the wellposedness of a nonstrictly hyperbolic equation. This is transparent on the formulation of the water waves problem given in Proposition 8 below. Following [24], we denote by $D_t = \partial_t + v \cdot \nabla$ the lagrangian derivative, and by $\mathcal{N}$ the operator $\mathcal{N} = \sqrt{1 + (\partial_{x_1} \sigma)^2} G(\sigma)$ (note that it is easy to define $\mathcal{N}$ even if $\Sigma(t)$ is not a graph); one then proves the following result for the interface curvature κ:

Proposition 8. *(cf Proposition 2.6 in [24]). For any smooth interface solution of the water wave equation one has:*

$$D_t^2 \kappa + (-\partial_{\vec{n}} p) \mathcal{N} \kappa = R, \tag{62}$$

where R is a remainder term consisting of lower order terms.

To prove this proposition one considers

$$\partial_t u_- + u_- \cdot \nabla u_- + \frac{1}{\rho_-} \nabla p_- = \vec{g} \tag{63}$$

on the interface $\partial \Omega_-(t)$ and apply the Laplace Beltrami operator $\Delta_{\Sigma(t)}$ to (63); using the important remark that at leading order, the Dirichlet-Neumann operator $G(\sigma)$ coincides with the square root of the Laplace Beltrami,

$$G(\sigma) = \sqrt{-\Delta_{\Sigma(t)}} + R_0, \qquad R_0 = \text{zero-th order operator,}$$

one can establish that modulo lower order terms one has:

$$\vec{n} \cdot \Delta_{\Sigma(t)}(\nabla p(t, x(t, \lambda))) = \partial_{\vec{n}} p \mathcal{N} \kappa,$$

and the result follows quite easily.

With the formula (62) one obtains local in time stability of the interface for initial data having a limited Sobolev regularity. Of course, this result can be obtained with other techniques (see for instance [36, 58, 31, 13, 49]).

Remark 7. In the whole space $\mathbb{R}$ the solution of the model dispersive equation (corresponding to (60) with $g_2 = -1$)

$$\partial_t^2 u + |D| u = 0, \qquad u(x, 0) = 0, \qquad u_t(x, 0) = \phi(x),$$

satisfies a dispersive estimate

$$\|u(\cdot, t)\|_{L^\infty} \leq C t^{-\frac{1}{2}} \|\phi(x)\|_{L^1}. \tag{64}$$

Therefore, one may expect to use dispersive effects, when the interface is a graph, to obtain *large time estimates* for the solution. This has been done in $2d$ (i.e. surface dimension $d-1=1$) by S. Wu [61]. The central idea in [61] is to transform the equations (inspired by Birkhoff normal forms) in order to increase the degree in the non linear terms and therefore to reduce their size. Though considerable difficulties make the implementation of this programme very delicate, it leads to an existence time of the order of

$$\exp\left(\frac{C}{\text{some norm of the initial data}}\right).$$

Since the dispersive effect in the formula (64) improves with the dimension, the situation for this problem is better in $3d$ (i.e. surface dimension $d-1=2$, see §8.2).

7.3   The Rayleigh-Taylor stability criterion

Our goal here is to comment on whether the Rayleigh-Taylor stability criterion (61) is satisfied for the water waves problem. Let us consider the case of water waves in finite depth. The boundary of $\Omega_-(t)$ is the union of two distinct pieces. A given boundary Γ_{fix} where the fluid is in contact with the boundary of the vessel, and the free boundary $\Sigma(t)$ between the region of the fluid of density ρ_1 (we take $\rho_-=1$ for the sake of simplicity), and the fluid of density $\rho_+=0$. On Γ_{fix} one assumes as usual the impermeability condition. One thus gets
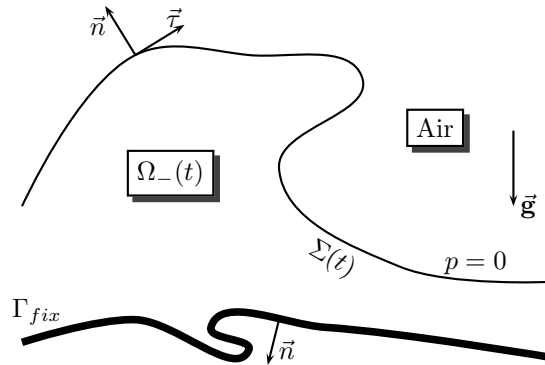
$$\partial_t u + \nabla(u \otimes u) = -\nabla p + \vec{\mathbf{g}}, \qquad \nabla \cdot u = 0, \qquad \nabla \wedge u = 0 \quad \text{in} \quad \Omega_-(t), \tag{65}$$

$$p = 0 \quad \text{on} \quad \Sigma(t), \qquad u \cdot \vec{n} = 0 \quad \text{on} \quad \Gamma_{fix}. \tag{66}$$

Note that neither $\Sigma(t)$ nor Γ_{fix} are assumed to be "globally" a graph. However one consider only configurations where

$$d(\Gamma_{fix}, \Sigma(t)) > h_{min} > 0, \tag{67}$$

and therefore where the influence of the fixed boundary on the free boundary involves only lower order terms (in terms of regularity).

Taking in particular into account the relation $\nabla \wedge u_- = 0$, one deduces from (65)-(66) that the pressure p satisfies

$$\begin{cases} -\Delta p = \sum_{ij} \partial_{x_i} u_j \partial_{x_j} u_i = |\nabla u|^2 \geq 0 \quad \text{in} \quad \Omega_-(t), \\ p = 0 \quad \text{on} \quad \Sigma(t), \qquad \partial_{\vec{n}} p = -\sum_{ij} u_i u_j \frac{\partial \vec{n}_i}{\partial x_j} + \vec{g} \cdot \vec{n} \quad \text{on} \quad \Gamma_{fix}. \end{cases}$$

Then the maximum principle implies that $p > 0$ in $\Omega_-(t)$ and therefore that

$$-\partial_{\vec{n}} p > 0 \quad \text{on} \quad \Sigma(t),$$

as soon as

$$\partial_{\vec{n}} p = -\sum_{ij} u_i u_j \frac{\partial \vec{n}_i}{\partial x_j} + \vec{g}.\vec{n} > 0 \quad \text{on} \quad \Gamma_{fix}. \tag{68}$$

For instance (68) turns out to be true for flat bottoms, since one then has

$$\sum_{ij} u_i u_j \frac{\partial \vec{n}_i}{\partial x_j} + \vec{g} \cdot \vec{n} = \vec{g} \cdot \vec{n} > 0;$$

the same argument works also in infinite depth (in fact, this is the configuration where it has been proved first [57, 58]), or for liquid drops[3].

Remark 8. Flat, infinite bottoms, or liquid drops are of course not the only configurations where the Rayleigh-Taylor stability criterion (61) is satisfied. Quite obviously, the same arguments show that (61) is also satisfied for "almost" flat bottoms. More generally, a condition on the curvature of the bottom can be derived [31, 3].

7.4   Ellipticity and regularity of traveling waves

Ellipticity results have been used above to derive instability results for the Kelvin-Helmholtz and Rayleigh-Taylor problems. Though such instabilities do not occur for surface waves (when the Taylor sign condition (61) is satisfied), an elliptic behavior can still be observed in some phenomena concerning water-waves. In particular, it is possible to derive *a priori* regularity results for traveling waves, both in $2d$ and in $3d$. Comments on the $3d$ case are postponed to §8.2.

Using the formulation (58) of the water-waves in terms of the surface elevation σ and the trace of the velocity potential at the free surface Ψ, a traveling wave is a solution to (58) of the form

$$(\sigma(t, x_1), \psi(t, x_1)) = (\underline{\sigma}(x_1 - ct), \underline{\psi}(x_1 - ct)),$$

where $c \in \mathbb{R}$ is the speed of the traveling wave. Consequently, $(\underline{\sigma}, \underline{\psi})$ must solve (we omit the underlines for the sake of clarity),

$$\begin{cases} G(\sigma)\psi + c\partial_{x_1}\sigma = 0, \\ g\sigma - c\partial_{x_1}\Psi + \frac{1}{2}|\partial_{x_1}\Psi|^2 - \frac{1}{2}\frac{(\partial_{x_1}\sigma\partial_{x_1}\Psi - c\partial_{x_1}\sigma)^2}{1 + |\partial_{x_1}\sigma|^2} = 0. \end{cases} \tag{69}$$

---

[3]However, this argument fails if the fluid is not irrotational. This is the reason why, in the study of liquid drops, it is *assumed* in [36, 49, 13] that the Rayleigh-Taylor condition holds. It also known that without this condition, the equations are ill-posed [20]

Existence of traveling waves in one surface dimension is known from the early works of Levi-Civita [34] and Nekrasov [45]. The fact that if the free boundary of these solutions is a C^1 curve, then it is also analytic is also a classical result that goes back to the works of Lewy [35] and Gerber [21, 22, 23]. These results use the fact that the system (69) is *elliptic*, a fact that can easily be observed on the linearized equations around the rest state $\sigma = 0$, $\psi = 0$. Recalling that $G[0] = |D|$, the linearized equations read indeed

$$\begin{cases} |D|\psi + c\partial_{x_1}\sigma = 0, \\ g\sigma - c\partial_{x_1}\psi = 0. \end{cases}$$

The symbol of this system is therefore

$$\begin{pmatrix} ic\xi_1 & |\xi_1| \\ g & -ic\xi_1 \end{pmatrix},$$

whose determinant is the elliptic symbol $c^2\xi_1^2 - g|\xi_1|$.

7.5 A word on shallow water asymptotics

Except for its influence, through (68), on the Rayleigh-Taylor criterion (61), the bottom does not play any role in the well-posedness theory of the water waves problem described above. By assuming (67), i.e. that the water depth never vanishes, one is ensured that the contribution of the bottom to the Dirichlet-Neumann operator $G(\sigma)\Psi$ is *analytic* (by the ellipticity of the potential equation $\Delta\Phi = 0$); therefore, *the contribution of the bottom is ignored in any symbolic description of $G(\sigma)\Psi$.* This is the reason why it is possible to obtain a local well-posedness theory for very rough bottoms, as shown recently in [1].

For the analysis of instabilities that interests us here, the influence of the bottom is thus irrelevant. This fact seems to contradict common sense observations, which show that the bottom often plays a important role in the dynamics of the surface (consider the shoaling of a wave as it approaches the shore for instance). The explanation of this apparent paradox is that the fact that the contribution of the bottom is *smooth* (and can therefore be ignored in the wellposedness theory) does not mean that it is *small*. For instance, consider the approximation of the Dirichlet-Neumann operator by the square root of the Laplace Beltrami of the surface,

$$G(\sigma) = \sqrt{-\Delta_{\Sigma(t)}} + R_0, \qquad R_0 = \text{zero-th order operator};$$

we have already seen that this symbolic approximation is very useful to prove the stability of the water waves problem. However, it is not uniform in the shallow water limit. More precisely, the operator norm $\|R_0\|_{L^2 \to L^2}$ grows to infinity in the shallow water limit (which corresponds to configurations where the ratio of the depth over the typical wavelength of the interface deformation is small). In order to obtain uniform estimates in the shallow water limit, one must take into account the contribution of the topography[4]. In some sense, the study of the interface is not decorrelated from the topography in the shallow water regime (the same phenomenon also occurs for interface waves, see for instance [11]). In order to focus on the instability phenomena at the interface, *we implicitly assumed throughout this paper that we are* not *in a shallow water type regime*. The study of this latter is however possible, albeit with different techniques, and a consequent literature has addressed this problem. A few examples are [15, 29, 51, 25, 3].

---

[4]In [32], a symbolic analysis "with tail" of the Dirichlet-Neumann operator is proposed, the tail being an infinitely smoothing perturbation of the principal symbol $\sqrt{-\Delta_{\Sigma(t)}}$ that takes into account the contribution of the bottom and removes the shallow water singularity

8 Differences between the $2d$ the $3d$ case

Most of this review has been devoted to interfaces in planar flows. Before concluding and without going to the details one should single out differences and similarities between the $2d$ and the $3d$ case.

In $3d$ the formula for the propagation of the vorticity is

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u;$$

as a consequence the $2d$ conservation relation does not exist any more. Therefore, there is no theorem for weak solutions nor global in time result for strong solutions. The only general available result concerns local existence in time of smooth solution, say, in $C^{1,\alpha}$. This regularity seems optimal as shown by the instability of the shear flow [40, 9]: more precisely, consider the vector field

$$U_{\mathrm{shf}}(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2))), \tag{70}$$

which is an obvious solution of the $3d$ incompressible Euler equation. With this type of flow one can construct solutions with the following property

$$U_{\mathrm{shf}}(x,0) \in C^{0,\sigma}, \quad \text{and for any} \quad t > 0, \quad U_{\mathrm{shf}}(x,t) \notin C^{0,\sigma}.$$

8.1 About the $3d$ Kelvin-Helmholtz and Rayleigh-Taylor problems

For smooth solutions, the generalization of the formulas (25)-(26) for the equations of the interface remain valid. Hence one can prove a local in time existence theorem for analytic initial data (cf. [54]).

However the ellipticity properties cease to be valid. As previously observed, these properties follow from the fact that the equation can be locally considered as a perturbation of the linearized problem. This linearized problem is elliptic in $2d$. In $3d$, one can still reduce the problem (see [55, 12]) to the situation where the interface is a graph $x_3 = \sigma(x_1, x_2, t)$, and where the solutions are perturbations of the stationary state $\underline{\sigma} = 0$, $\widetilde{\underline{\omega}}(x_1, x_2) = (\widetilde{\underline{\omega}}_1, \widetilde{\underline{\omega}}_2, 0)$. Then in Fourier variables the linearized problem in the Kelvin-Helmholtz case (the Rayleigh-Taylor case can be studied in the same way) is

$$\partial_t \begin{pmatrix} \widehat{\sigma} \\ \widehat{\omega}_1 \\ \widehat{\omega}_2 \\ \widehat{\omega}_3 \end{pmatrix} = \mathcal{A}(\xi) \begin{pmatrix} \widehat{\sigma} \\ \widehat{\omega}_1 \\ \widehat{\omega}_2 \\ \widehat{\omega}_3 \end{pmatrix},$$

where the operator $\mathcal{A}(\xi)$ is given by (denoting $\xi = |\xi|(\cos\theta, \sin\theta)$)

$$\mathcal{A}(\xi) = \begin{pmatrix} 0 & \frac{i}{2}\sin\theta & -\frac{i}{2}\cos\theta & 0 \\ -\frac{i}{2}|\xi|^2|\widetilde{\underline{\omega}}|^2\sin\theta & 0 & 0 & \frac{1}{2}(\xi \cdot \widetilde{\underline{\omega}})\sin\theta \\ \frac{i}{2}|\xi|^2|\widetilde{\underline{\omega}}|^2\cos\theta & 0 & 0 & -\frac{1}{2}(\xi \cdot \widetilde{\underline{\omega}})\cos\theta \\ 0 & -\frac{1}{2}(\xi \cdot \widetilde{\underline{\omega}})\sin\theta & \frac{1}{2}(\xi \cdot \widetilde{\underline{\omega}})\cos\theta & 0 \end{pmatrix}. \tag{71}$$

The eigenvalues of the matrix $\mathcal{A}(\xi)$ are

$$0 \text{ (with multiplicity 2)}, \quad \text{and} \quad \pm\frac{1}{2}|\xi \wedge \widetilde{\underline{\omega}}|.$$

Therefore, the operator $\partial_t - \mathcal{A}(\xi)$ is no more elliptic and perturbations transverse to the direction of streaming are thus unaffected [12].

22

8.2 About $3d$ water waves

8.2.1 A global existence result in $3d$

Local existence for $3d$ (surface dimension $d-1=2$) water waves under the Rayleigh-Taylor criterion (61) can be proved with tools that do not depend on the dimension [58, 31, 13, 49]. This is of course not the case if one wants to prove a global existence result based on the dispersive properties of the equation.
In order to describe the stability result for the water waves in $2d$ (surface dimension $d=1$), we used the equation on the curvature κ provided by Proposition 8. In [24], it is proven that such an equation still holds in $3d$ with the curvature replaced by the mean curvature, and a careful analysis of this equation allows the authors to obtain a global existence result for small datas. This is achieved by using several ingredients (in the same spirit as those in $2d$, see Remark 7, but with different techniques):

- In the whole space $\mathbb{R}^2$, the solution of the model dispersive equation

$$\partial_t^2 u + |D|u = 0, u(x,0) = 0, u_t(x,0) = \phi(x),$$

  satisfies a decay estimate

$$\|u(x.,t)\|_{L^\infty} \leq Ct^{-1}\|\phi\|_{L^1}. \tag{72}$$

- The use of a Birkhoff normal form to increase the degree of the non linearity. However, time resonances do not allow the construction of a standard normal form; the authors therefore introduce the notion of space resonance, which catches the fact that, for dispersive equations, wave packets do not have the same group velocity and therefore "do not see each other" in general.

8.2.2 Traveling waves in $3d$

Traveling waves also exist for $3d$ water waves (surface dimension $d-1=2$), but this fact is considerably more difficult to establish than in the $2d$ case and has only been solved very recently by Iooss and Plotnikov [27]. As in the $2d$ case, a simple look at the linearized equations is very instructive. The symbol of the linearized system is indeed given by

$$\begin{pmatrix} ic\xi_1 & |\xi| \\ g & -ic\xi_1 \end{pmatrix},$$

with $|\xi| = (|\xi_1|^2 + |\xi_2|^2)^{1/2}$. The determinant is then the symbol $c^2\xi_1^2 - g|\xi|$. This symbol is *not elliptic* since it vanishes on the (unbounded) region $\mathcal{C} = \{c^2\xi_2^2 = g|\xi|\}$. The equation defining $\mathcal{C}$ is a Schrödinger equation on the boundary that may propagate singularities.

Despite of this, Alazard and Métivier [2] managed to prove that smooth enough biperiodic (i.e. periodic in both space directions) traveling waves are automatically C^∞. By a careful paralinearization of the equations and two transformations inspired by the work of Iooss and Plotnikov, they show that the general situation is quite similar to what is observed in the linearization around the rest state. Then, they exhibit a diophantine condition that allow them to handle the degeneracy of the principal symbol.

Remark 9. As already mentioned, the sign of gravity also plays an important role. Were it negative, the symbol of the linearized equation would be weakly elliptic, even in the $3d$ case. This remains true for the full equations (69) if the Rayleigh-Taylor condition is not satisfied. The paralinearization strategy of Alazard-Métivier can then be used.

9   Conclusion

As a conclusion, we recall some of the basic threads that lead to the organization of this survey and add extra remarks and open problems.

All the analysis presented here is deduced from the incompressible Euler equation written in the sense of distributions. With this setting the stability of the Cauchy problem has been analyzed in the models problems of Rayleigh-Taylor (Atwood number $0 < |a| < 1$), Kelvin-Helmholtz ($a = 0$), and water waves ($|a| = 1$).

With an high frequency analysis it has been shown that for $|a| < 1$ the linearized problem is elliptic. Therefore the Cauchy problem is well posed only with analytic initial data. Moreover the solution whenever it exists is analytic even if the data (say for $t = 0$ or $T = t$) exhibit a singular behavior. To persist after the singularity the interface has to be very *wild* and this may be in agreement both with the numerical simulation of [30] and with physical observations.

The situation for the water waves may be different. When the Rayleigh-Taylor sign condition $-\partial_{\vec{n}} p > 0$ is not satisfied the linearized problem shares many similarities with the Rayleigh-Taylor equations. On the other hand with $-\partial_{\vec{n}} p > 0$ on the interface stability results can be obtained.

The above results given in $2d$ are compared in Section 8 with the situation in $3d$. The ellipticity (or hypoellipticity) of the underlying linearized problem is lost and therefore the situation may be very different for the Kelvin-Helmholtz and Rayleigh-Taylor problems. As a companion problem observe that with the shear flow

$$u(x_1, x_2, x_3, t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2)),$$

one can construct solutions defined for all time and with any "limited" degree of regularity on the interface. The same kind of difficulty is also encountered in some situation for the water waves problem (for the regularity of traveling waves for instance).

In the present description physical quantities like the surface tension and the viscosity have been ignored. This is based on the observation that such quantities are *small*[5] and that the understanding of the pathology of the phenomena when they are made equal to 0 should contribute to the understanding of what happens when they are small.

- The introduction of a surface tension creates a third order dispersive term and all the above equation become locally in time well posed [26, 4, 6]. This property does not persists when this

---

[5]this assertion is not very honest! Viscosity and surface tension are negligible for physical phenomena involving "reasonable" frequencies, but most of the instabilities described here occur at very high frequencies; the viscosity and surface tension terms should then play an important role. We refer to [32], where it is shown that a natural two-fluid generalization of the Rayleigh-Taylor stability criterion (61) is given by

$$[-\partial_z P] > \frac{1}{4} \frac{(\rho^+ \rho^-)^2}{T(\rho^+ + \rho^-)} \mathfrak{c}(\sigma)|\omega|^4_\infty,$$

where T is the surface tension coefficient, ω is the jump of the horizontal velocity at the interface, and $\mathfrak{c}(\sigma)$ is a geometrical constant.

However this "not very honest mathematical behavior" is not restricted to fluid mechanics and nevertheless contributes to the understanding of nature. An other basic example is the Maxwell equations; they involve the permittivity $\epsilon(x)$ and the magnetic permeability $\mu(x)$. These equations are the prototype of hyperbolic systems and the high frequency analysis is systematically used for their solutions. On the other hand ϵ and μ are derived from the Helmholtz equation under the hypothesis that the frequency is "moderate" . For higher frequencies k in the Helmholtz equation they would depend on k. Therefore in the space domain one obtains an equation with ϵ and μ acting as time convolution and no more as coefficients and the nature of the equations at high frequency should be changed!

surface tension goes to zero except for the water waves problem with the Rayleigh-Taylor sign condition [5, 49, 42].

- With the presence of a viscosity $\nu > 0$ the Euler equation is changed into the Navier-Stokes equation. When the density is constant ($\rho_+ = \rho_-$), this equation is in $2d$ reasonably understood even for initial data with a vorticity being a measure concentrated on a curve. When ν goes to zero (under reasonable hypothesis on the sign of the vorticity $\nabla \wedge u_\nu$) the corresponding solution converges to a weak solution of the Euler equation. Since there is no uniqueness theorem for weak solutions of the Euler equation, the connection between this limit and corresponding solutions of the Kelvin Helmholtz equation remains a fully open problem. Moreover with the Prandlt-Munk vortex described in Section 6 one can show the existence of situations where such convergence does not hold.

- Eventually one could consider a viscous perturbation of the water waves problem, i.e. a Navier-Stokes equation with a free boundary. This has already been studied (cf. for instance [53]). One may conjecture that in this case, with the Rayleigh-Taylor sign condition at the boundary, the convergence to the water waves solution should be true.

References

[1] T. ALAZARD, N. BURQ, C. ZUILY, *On the Water Waves Equations with Surface Tension*, submitted, http://arxiv.org/abs/0906.4406.

[2] T. ALAZARD, G. MÉTIVIER, *Paralinearization of the Dirichlet to Neumann operator, and regularity of three-dimensional water waves*, Communications in Partial Differential Equations **34** (2009), 1632-1704.

[3] B. ALVAREZ-SAMANIEGO, D. LANNES, *Large time existence for 3d water-waves and asymptotics* Invent. math. **171** (2008), 485-541.

[4] D.AMBROSE, *Well-posedness of vortex sheets with surface tension*, SIAM J. Math. Anal.**35** (2003), 211-244.

[5] D. M. AMBROSE, N. MASMOUDI, *The zero surface tension limit of two-dimensional water waves*, Comm. Pure Appl. Math. **58** (2005) 1287-1315.

[6] D. AMBROSE, N. MASMOUDI, *Well-posedness of 3D vortex sheets with surface tension*, Commun. Math. Sci. **5** (2007), 391-430.

[7] C. BARDOS, J. S. LINSHIZ, E. S. TITI, *Global regularity and convergence of a Birkhoff-Rott-alpha approximation of the dynamics of vortex sheets of the 2D Euler equations*, to appear in Comm. Pure Appl. Math. http://arxiv.org/abs/0902.3356.

[8] T. BEALE, T. HOU, J. LOWENGRUB, *Growth rates for the linearized motion of fluid interfaces away from equilibrium*, Comm. Pure Appl. Math. **46** (1993), 1269–1301.

[9] C. BARDOS, E. S. TITI, *Loss of smoothness and energy conserving rough weak solutions for the 3d Euler equations*, submitted, http://arxiv.org/abs/0906.2029.

[10] R. CAFLISCH, O. ORELLANA, *Singular solutions and ill-posedness for the evolution of vortex sheets* SIAM J. Math. Anal. **20(2)** (1989), 293–307.

[11] J. L. BONA, D. LANNES, J.-C. SAUT, *Asymptotic models for internal waves*, J. Math. Pures Appl. **89** (2008), 538566.

[12] S. CHANDRASEKHAR, *Hydrodynamic and hydromagnetic stability*, The International Series of Monographs on Physics Clarendon Press, Oxford 1961 xix+654 pp.

[13] D. COUTAND, S. SHKOLLER, *Well-posedness of the free-surface incompressible Euler equations with or without surface tension*, J. Amer. Math. Soc. **20** (2007), 829-930.

[14] A. CÓRDOBA, D. CÓRDOBA, F. GANCEDO, *Interface evolution: Water waves in 2-d*, Advances in Mathematics **223** (2010), 120-173.

[15] W. CRAIG, *An existence theory for water waves and the Boussinesq and the Korteweg-de-Vries scaling limits*, Comm. Partial Differential Equations **10** (1985), 787-1003.

[16] W. CRAIG, C. SULEM, *Numerical simulation of gravity waves*, J. Comput. Phys. **108** (1993) 73-83.

[17] C. DE LELLIS AND L. SZÉKELYHIDI JR., *On admissibility criteria for weak solutions of the Euler equations*, to appear in Arch. Ration. Mech. Anal., http://arxiv.org/abs/0712.3288.

[18] J.-M. DELORT, *Existence de nappes de tourbillon en dimension deux*, J. Amer. Math. Soc. **4** (1991), 553-586.

[19] J. DUCHON, R. ROBERT, *Global vortex sheet solutions of Euler equations in the plane*, J. Differential Equations **73** (1988), 215-224.

[20] D. G. EBIN, *The equations of motion of a perfect fluid with free boundary are not well posed* Comm. Partial Differential Equations **12** (1987), 1175-1201.

[21] R. GERBER, *Sur une condition de prolongement analytique des fonctions harmoniques*, C. R. Acad. Sci. Paris **233** (1951), 1560-1562.

[22] R. GERBER, *Sur les solutions exactes des équations du mouvement avec surface libre d'un liquide pesant*, J. Math. Pures Appl. **34** (1955), 185-299.

[23] R. GERBER, *Sur un lemme de représentation conforme*, C. R. Acad. Sci. Paris **241** (1955), 1440-1441.

[24] P. GERMAIN, N. MASMOUDI, J. SHATAH, *Global Solutions for the Gravity Water Waves Equation in Dimension 3*, submitted, http://arxiv.org/abs/0906.5343.

[25] T. IGUCHI, *A shallow water approximation for water waves*, J. Math. Kyoto Univ. **49** (2009), 13-55.

[26] T. IGUCHI, N. TANAKA, A. TANI, *On the two-phase free boundary problem for two-dimensional water waves*, Math. Ann. **309** (1997), 199-223.

[27] G. Iooss, P. Plotnikov, *Small divisor problem in the theory of three-dimensional water gravity waves*, Mem. Amer. Math. Soc. **200** (2009).

[28] V. Kamotski, G. Lebeau, *On 2D Rayleigh-Taylor instabilities*, Asymptot. Anal. **42** (2005), 1-27.

[29] T. Kano, T. Nishida, *A mathematical justification for Kortewegde Vries equation and Boussinesq equation of water surface waves*, Osaka J. Math. **23** (1986), 389413.

[30] R. Krasny, *Computation of vortex sheet roll-up in Trefftz plane*, J. Fluid Mech. **184** (1987), 123-155.

[31] D. Lannes, *Well-posedness of the water waves equations*, Journ. of the Amer. Math. Soc. **18(3)** (2005), 605-654.

[32] D. Lannes, *A stability criterion for two-fluid interfaces and applications*, submitted.

[33] G. Lebeau, *Régularité du problème de Kelvin–Helmholtz pour l'équation d'Euler 2d*, ESAIM: COCV **08**(2002), 801-825.

[34] T. Levi-Civita, *Détermination rigoureuse des ondes permanentes d'ampleur finie*, Math. Ann. **93** (1925), 264-314.

[35] H. Lewy, *A note on harmonic functions and a hydrodynamical application*, Proc. Amer. Math. Soc. **3** (1952), 111-113.

[36] H. Lindblad, *Well-posedness for the motion of an incompressible liquid with free surface boundary*, Ann. of Math. (2) **162** (2005), 109-194.

[37] M. C. Lopes Filho, H. J. Nussenzveig Lopes, S. Schochet, *A criterion for the equivalence of the Birkhoff-Rott and Euler descriptions of vortex sheet evolution*, Trans. Amer. Math. Soc. **359** (2007), 4125–4142

[38] M. C. Lopes Filho, H. J. Nussenzveig Lopes, M. O. Souza, *On the equation satisfied by a steady Prandtl-Munk vortex sheet*, Commun. Math. Sci. **1(1)** (2003), 68-73.

[39] M. C. Lopes Filho, H. J. Nussenzveig Lopes, Z. Xin, *Existence of vortex sheets with reflection symmetry in two space dimensions*, Arch. Ration. Mech. Anal. **158** (2001), 235-257.

[40] A.J. Majda, A.L. Bertozzi, *Vorticity and Incompressible Flows*, Cambridge University Press, Cambridge, UK, 2002.

[41] C. Marchioro, M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids* **96**, Applied Mathematical Sciences, Springer-Verlag, New York, 1994.

[42] M. Ming, Z. Zhang, *Well-posedness of the water-wave problem with surface tension*, J. Math. Pures Appl. **92** (2009), 429-455.

[43] D.W. Moore, *The spontaneous appearance of a singularity in the shape of an evolving vortex sheet*, Proc. Roy. Soc.London. Ser. A **365** (1979), 105-119.

[44] D. Meiron, G. Baker, S. Orszag, *Analytic structure of vortex sheet dynamics. I. Kelvin Helmholtz instability*, J. Fluid Mech. **114** (1982), 283-298.

[45] A. I. Nekrasov, *On waves of permanent type*, Izv. Ivanovo-Voznesensk. Politekhn. Inst., 1921.

[46] D.I. Pullin, J.D. Buntine, P.G. Saffman, *The spectrum of a stretched spiral vortex*, Phys. Fluids **6** (1994), 3010-3027.

[47] J.S. Smagorinsky, *General circulation model of the atmosphere*, Mon. weather Rev. **91** (1963), 99-164.

[48] V. Scheffer, *An inviscid flow with compact support in space-time*, J. of Geom. Anal. **3** (1993), 343-401.

[49] J. Shatah, C. Zeng, *Geometry and a priori estimates for free boundary problems of the Euler equation*, Comm. Pure Appl. Math. **61** (2008), 698-744.

[50] J. Shatah, C. Zeng, *A priori estimates for uid interface problems*, Comm. Pure Appl. Math. **61** (2008), 848876.

[51] G. Schneider, C.E. Wayne, *The long-wave limit for the water wave problem. I: The case of zero surface tension*, Commun. Pure Appl. Math. **53** (2000), 14751535.

[52] A. Shnirelman, *On the nonuniqueness of weak solutions of the Euler equations*, Comm. Pure Appl. Math. **50** (1997), 1261-1286.

[53] V. A. Solonnikov, *Unsteady motion of an isolated volume of a viscous incompressible fluid (Russian)*, Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), 1065-1087, 1118; translation in Math. USSR-Izv. **31** (1988), 381-405.

[54] C. Sulem, P.-L. Sulem, *Finite time analyticity for the two- and three-dimensional Rayleigh-Taylor instability*, Trans. Amer. Math. Soc. **287** (1981), 127-160.

[55] C. Sulem, P.-L. Sulem, C. Bardos, U. Frisch, *Finite time analyticity for the two- and three-dimensional Kelvin-Helmholtz instability*, Comm. Math. Phys. **80** (1981), 485-516.

[56] C. Villani, *Paradoxe de Scheffer-Shnirelman revu sous l'angle de l'intégration convexe (d'après C. De Lellis, L. Székelyhidi)*, Bourbaki Seminar, Exp. 1001 (November 2008).

[57] S. Wu, *Well-Posedeness in Sobolev spaces of the full water wave problem in $2 - D$*, Invent. Math. **130** (1997), 439-72.

[58] S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in 3-D*, J. Amer. Math. Soc. **12** (1999), 445-495.

[59] S. Wu, *Recent progress in mathematical analysis of vortex sheets*, Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), 233-242, Beijing, (2002).

[60] S. Wu, *Mathematical analysis of vortex sheets*, Comm. Pure Appl. Math. **59** (2006), 1065-1206.

[61] S. Wu, *Almost Global Wellposedness of the $2 - D$ full Water Wave Problem*, Invent. Math. **177** (2009), 45-135.

[62] V.I. Yudovich, *Non stationary flow of an ideal incompressible liquid*, Zh. Vych. Mat. **3** (1963), 1032-1066.

[63] V.E. ZAKHAROV, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, J. App. Mech. Techn. Phys. **9** (1968), 190-194.

[64] P. ZHANG, Z. ZHANG, *On the free boundary problem of three-dimensional incompressible Euler equations*, Comm. Pure Appl. Math. **61** (2008), 877-940.