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Abstract

We prove that for every k and every $\varepsilon > 0$, there exists g such that every graph with tree-width at most k and odd-girth at least g has circular chromatic number at most $2 + \varepsilon$.

1 Introduction

It has been a challenging problem to prove the existence of graphs of arbitrarily high girth and chromatic number [2]. On the other hand, graphs with large girth that avoid a fixed minor are known to have low chromatic number (in particular, this applies to graphs embedded on a fixed surface). More precisely, as Thomassen observed [8], a graph that avoids a fixed minor and has large girth is 2-degenerate, and hence 3-colorable. Further, Galluccio, Goddyn and Hell [3] proved the following theorem, which essentially states that graphs with large girth that avoid a fixed minor are almost bipartite.

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Theorem 1 (Galluccio, Goddyn and Hell, 2001)**.** *For every graph H and every $\varepsilon > 0$, there exists an integer g such that the circular chromatic number of every H-minor free graph of girth at least g is at most $2 + \varepsilon$.*

A natural way to weaken the girth-condition is to require the graphs to have high odd-girth (the *odd-girth* is the length of a shortest odd cycle). However, Young [9] constructed 4-chromatic projective graphs with arbitrarily high odd-girth. Thus, the high odd-girth requirement is not sufficient to ensure 3-colorability, even for graphs embedded on a fixed surface. Klostermeyer and Zhang [4], though, proved that the circular chromatic number of every planar graph of sufficiently high odd-girth is arbitrarily close to 2. In particular, the same is true for K_4-minor free graphs, i.e. graphs with tree-width at most 2. We prove that the conclusion is still true for any class of graphs with bounded tree-width, which answers a question of Pan and Zhu [6, Question 6.2] also appearing as Question 8.12 in the survey by Zhu [10].

Theorem 2. *For every k and every $\varepsilon > 0$, there exists g such that every graph with tree-width at most k and odd-girth at least g has circular chromatic number at most $2 + \varepsilon$.*

Motivated by tree-width duality, Nešetřil and Zhu [5] proved the following theorem.

Theorem 3 (Nešetřil and Zhu, 1996)**.** *For every k and every $\varepsilon > 0$, there exists g such that every graph G with tree-width at most k and homomorphic to a graph H with girth at least g has circular chromatic number at most $2 + \varepsilon$.*

To see that Theorem 2 implies Theorem 3, observe that if G has an odd cycle of length g, then H has an odd cycle of length at most g.

2 Notation

A (p, q)*-coloring* of a graph is a coloring c of the vertices with colors from the set $\{0, \ldots, p-1\}$ such that the colors of any two adjacent vertices u and v satisfy $q \leq |c(u) - c(v)| \leq p - q$. The *circular chromatic number* $\chi_c(G)$ of a graph G is the infimum (and it can be shown to be the minimum) of the ratios p/q such that G has a (p, q)-coloring. For every finite graph G, it holds that $\chi(G) = \lceil \chi_c(G) \rceil$ and there is a (p, q)-coloring of G for every p and q with $p/q \geq \chi_c(G)$. In particular, the circular chromatic number of G is at most $2 + 1/k$ if and only if G is homomorphic to a cycle of length $2k + 1$.

The reader is referred to the surveys by Zhu [10, 11] for more information about circular colorings.

A *p-precoloring* is a coloring φ of a subset A of vertices of a graph G with colors from $\{0, \ldots, p-1\}$, and its *extension* is a coloring of the whole graph G that coincides with φ on A. The following lemma can be seen as a corollary of a theorem of Albertson and West [1, Theorem 1], and it is the only tool we use from this area.

Lemma 4. *For every p and q with $2 < p/q$, there exists d such that any p-precoloring of vertices with mutual distances at least d of a bipartite graph H extends to a (p,q)-coloring of H.*

A *k-tree* is a graph obtained from a complete graph of order $k+1$ by successively adding vertices joined to exactly k pairwise adjacent vertices. The *tree-width* of a graph G is the smallest k such that G is a subgraph of a k-tree. Graphs with tree-width at most k are also called *partial k-trees.*

A *rooted partial k-tree* is a partial k-tree G with $k+1$ distinguished vertices $v_1, \ldots, v_{k+1}$ such that there exists a k-tree G' that is a supergraph of G and the vertices $v_1, \ldots, v_{k+1}$ form a clique in G'. We also say that the partial k-tree is *rooted* at $v_1, \ldots, v_{k+1}$. If G is a partial k-tree rooted at $v_1, \ldots, v_{k+1}$ and G' is a partial k-tree rooted at $v'_1, \ldots, v'_{k+1}$, then the graph $G \oplus G'$ obtained by identifying v_i and v'_i is again a rooted partial k-tree (identify the cliques in the corresponding k-trees).

Fix p and q. If G is a rooted partial k-tree, then $\mathcal{F}(G)$ is the set of all p-precolorings of the $k+1$ distinguished vertices of G that can be extended to a (p,q)-coloring of G.

The next lemma is a standard application of results in the area of graphs of bounded tree-width [7].

Lemma 5. *Let k and N be positive integers such that $N \geq k+1$. If G is a partial k-tree with at least $3N$ vertices, then there exist partial rooted k-trees G_1 and G_2 such that G is isomorphic to $G_1 \oplus G_2$ and G_1 has at least $N+1$ and at most $2N$ vertices.*

If G is a partial k-tree rooted at $v_1, \ldots, v_{k+1}$, then its *type* is a $(k+1) \times (k+1)$ matrix M such that M_{ij} is the length of the shortest path between the vertices v_i and v_j. If there is no such path, M_{ij} is equal to ∞. Any matrix M that is a type of a partial rooted k-tree satisfies the triangle inequality (setting $\infty + x = \infty$ for any x). A symmetric matrix M whose entries are non-negative integers and ∞ (and zeroes only on the main diagonal) that satisfies the triangle inequality is a *type.* A type is *bipartite* if $M_{ij} + M_{jk} + M_{ik} \equiv 0 \mod 2$ for any three finite entries M_{ij}, M_{jk} and M_{ik}. Two bipartite types M

and M' are *compatible* if M_{ij} and M'_{ij} have the same parity whenever both of them are finite. We define a binary relation on bipartite types as follows: $M \preceq M'$ if and only if M and M' are compatible and $M_{ij} \le M'_{ij}$ for every i and j. Note that the relation $\preceq$ is a partial order.

We finish this section with the following lemma. Its straightforward proof is included to help us in familiarizing with the just introduced notation.

Lemma 6. *Let G^1 and G^2 be two bipartite rooted partial k-trees with types M^1 and M^2 such that there exists a bipartite type M^0 with $M^0 \preceq M^1$ and $M^0 \preceq M^2$. Then the types M^1 and M^2 are compatible, $G^1 \oplus G^2$ is a bipartite rooted partial k-tree and its type M satisfies $M^0 \preceq M$.*

Proof. The types M^1 and M^2 are compatible: if both M^1_{ij} and M^2_{ij} are finite, then M^0_{ij} is finite and has the same parity as M^1_{ij} and M^2_{ij}. Hence, the entries M^1_{ij} and M^2_{ij} have the same parity.

Let M be the type of $G^1 \oplus G^2$. Note that it does not hold in general that $M_{ij} = \min\{M^1_{ij}, M^2_{ij}\}$. We show that $M^0 \preceq M$ which will also imply that $G^1 \oplus G^2$ is bipartite since M^0 is a bipartite type. Consider a shortest path P between two distinguished vertices v_t and $v_{t'}$ and split P into paths $P_1, \dots, P_\ell$ delimited by distinguished vertices on P. Note that $\ell \le k$ since P is a path. Let $j_0 = t$ and let j_i be the index of the end-vertex of P_i for $i \in \{1, \dots, \ell\}$. In particular, $j_\ell = t'$. Each of the paths $P_1, \dots, P_\ell$ is fully contained in G^1 or in G^2 (possibly in both if it is a single edge). Since $M^0 \preceq M^1$ and $M^0 \preceq M^2$, the length of P_i is at least $M^0_{j_{i-1}j_i}$, and it has the same parity as $M^0_{j_{i-1}j_i}$. Since M^0 is a bipartite type (among others, it satisfies the triangle inequality), the length of P, which is $M_{tt'}$, has the same parity as $M^0_{j_0 j_\ell} = M^0_{tt'}$ and is at least $M^0_{tt'}$. This implies that $M^0 \preceq M$. □

3 The Main Lemma

In this section, we prove a lemma which forms the core of our argument. To this end, we first prove another lemma that asserts that for every k, p and q, the set of types of all bipartite rooted partial k-trees forbidding a fixed set of p-precolorings from extending (and maybe some other precolorings, too) has always a maximal element. We state the lemma slightly differently to facilitate its application.

Lemma 7. *For every k, p and q, there exists a finite number of (bipartite) types $M^1, \dots, M^m$ such that for any bipartite rooted partial k-tree G with type M, there exists a bipartite rooted partial k-tree G' with type M^i for some $i \in \{1, \dots, m\}$ such that $\mathcal{F}(G') \subseteq \mathcal{F}(G)$ and $M \preceq M^i$.*

Proof. Let $d \geq 2$ be the constant from Lemma 4 applied for p and q. Let $M^1, \ldots, M^m$ be all bipartite types with entries from the set $\{1, \ldots, D^{(k+1)^2}\} \cup \{\infty\}$ where $D = 4d$. Thus, m is finite and does not exceed $(D^{(k+1)^2} + 1)^{k(k+1)/2}$.

Let G be a bipartite rooted partial k-tree with type M. If M is one of the types $M^1, \ldots, M^m$, then there is nothing to prove (just choose i such that $M = M^i$). Otherwise, one of its entries is finite and exceeds $D^{(k+1)^2}$.

For $i \in \{1, \ldots, (k+1)^2\}$, let J^i be the set of all positive integers between D^{i-1} and $D^i - 1$ (inclusively). Let i_0 be the smallest integer such that no entry of M is contained in J^{i_0}. Since M has at most $k(k+1)/2$ different entries, such an index i_0 exists. Note that if $i_0 = 1$, then Lemma 4 implies that $\mathcal{F}(G)$ contains all possible p-precolorings, and the sought graph G' is the bipartite rooted partial k-tree composed of $k+1$ isolated vertices, with the all-∞ type.

Two vertices v_i and v_j at which G is rooted are *close* if M_{ij} is less than D^{i_0-1}. The relation $\approx$ of being close is an equivalence relation on $v_1, \ldots, v_{k+1}$. Indeed, it is reflexive and symmetric by the definition, and we now show that it is transitive. Suppose that M_{ij} and M_{jt} are both less than D^{i_0-1}. Then, the distance between v_i and v_t is at most $M_{ij} + M_{jt} \leq 2D^{i_0-1} - 2 \leq D^{i_0} - 1$ since $D \geq 2$. Consequently, by the choice of i_0, the distance between v_i and v_t is at most $D^{i_0-1} - 1$ and thus $v_i \approx v_t$.

Let $C_1, \ldots, C_\ell$ be the equivalence classes of the relation $\approx$. Note that $C_1, \ldots, C_\ell$ is a finer partition than that given by the equivalence relation of being connected.

Since G is bipartite, we can partition its vertices into two color classes, say red and blue. For every $i \in \{1, \ldots, \ell\}$, contract the closed neighborhood of a vertex v if v is a blue vertex and its distance from any vertex of C_i is at least D^{i_0-1} and keep doing so as long as such a vertex exists. Observe that the resulting graph is uniquely defined. After discarding the components that do not contain the vertices of C_i, we obtain a bipartite partial k-tree G_i rooted at the vertices of C_i: it is bipartite as we have always contracted closed neighborhoods of vertices of the same color (blue) to a single (red) vertex, and its tree-width is at most k since the tree-width is preserved by contractions. Moreover, the distance between any two vertices of C_i has not decreased since any path between them through any of the newly arising vertices has length at least $2D^{i_0-1} - 2 \geq D^{i_0-1}$.

Now, let G' be the bipartite rooted partial k-tree obtained by taking the disjoint union of $G_1, \ldots, G_\ell$. The type M' of G' can be obtained from the type of G: set M'_{ij} to be M_{ij} if the vertices v_i and v_j are close, and ∞ otherwise. Thus, M' is one of the types $M^1, \ldots, M^m$ and $M \preceq M'$. It remains to show that $\mathcal{F}(G') \subseteq \mathcal{F}(G)$.

Let $c \in \mathcal{F}(G')$ be a p-precoloring that extends to G', and recall that $D \geq 4$. For $i \in \{1, \ldots, \ell\}$, let A_i be the set of all red vertices at distance at most D^{i_0-1} and all blue vertices at distance at most $D^{i_0-1} - 1$ from C_i, and let R_i be the set of all red vertices at distance $D^{i_0-1} - 1$ or D^{i_0-1} from C_i. Set $B_i = A_i \setminus R_i$ (B_i is the "interior" of A_i and R_i its "boundary"). The extension of c to G_i naturally defines a coloring of all vertices of A_i: G_i is the subgraph of G induced by A_i with some red vertices of R_i identified (two vertices of R_i are identified if and only if they are in the same component of the graph $G - B_i$).

Let H be the following auxiliary graph obtained from G: remove the vertices of $B = B_1 \cup \cdots \cup B_\ell$ and, for $i \in \{1, \ldots, \ell\}$, identify every pair of vertices of R_i that are in the same component of $G - B$. Let R be the set of vertices of H corresponding to some vertices of $R_1 \cup \cdots \cup R_\ell$. Precolor the vertices of R with the colors given by the colorings of the graphs G_i (note that two vertices of R_i in the same component of $G - B_i$ are also in the same component of $G - B$, so this is well-defined). The graph H is bipartite as only red vertices have been identified. The distance between any two precolored vertices is at least d: consider two precolored vertices r and r' at distance at most $d - 1$. Let i and i' be such that $r \in R_i$ and $r' \in R_{i'}$. If $i = i'$, then r and r' are in the same component of $G - B$ and thus $r = r'$. If $i \neq i'$ then by the definition of R_i and $R_{i'}$, the vertex r is in G at distance at most D^{i_0-1} from some vertex v of C_i and r' is at distance at most D^{i_0-1} from some vertex v' of $C_{i'}$. So, the distance between v and v' is at most $2D^{i_0-1} + d - 1 \leq D^{i_0} - 1$. Since M has no entry from J^{i_0}, the vertices v and v' must be close and thus $i = i'$, a contradiction.

Since the distance between any two precolored vertices is at least d, the precoloring extends to H by Lemma 4 and in a natural way it defines a coloring of G. We conclude that every p-precoloring that extends to G' also extends to G and thus $\mathcal{F}(G') \subseteq \mathcal{F}(G)$. $\square$

We now prove our main lemma, which basically states that there is only a finite number of bipartite rooted partial k-trees that can appear in a minimal non-(p, q)-colorable graph with tree-width k and a given odd girth.

Lemma 8. *For every k, p and q, there exist a finite number m and bipartite rooted partial k-trees $G^1, \ldots, G^m$ with types $M^1, \ldots, M^m$ such that for any bipartite rooted partial k-tree G with type M there exists i such that $\mathcal{F}(G^i) \subseteq \mathcal{F}(G)$ and $M \preceq M^i$.*

Proof. Let $M^1, \ldots, M^m$ be the types from Lemma 7. We define the graph G^i as follows: for every p-precoloring c that does not extend to a bipartite partial rooted k-tree with type M^i, fix a partial rooted k-tree G^i_c with type

M^i such that c does not extend to G^i_c. Set $G^i = \bigoplus_c G^i_c$, where c runs over all such p-precolorings. If the above sum of partial k-trees is non-empty, then the type M of G^i is M^i. Indeed, $M \preceq M^i$ by the definition of G^i, and Lemma 6 implies that $M^i \preceq M$. If all the p-precolorings of the $k+1$ vertices in the root extend to each partial k-tree of type M^i, then let G^i be the graph consisting of $k+1$ isolated vertices. This happens in particular for the all-∞ type.

Let us verify the statement of the lemma. Let G be a bipartite rooted partial k-tree and let M be the type of G. If $\mathcal{F}(G)$ is composed of all p-precolorings, the sought graph G^i is the one composed of $k+1$ isolated vertices. Hence, we assume that $\mathcal{F}(G)$ does not contain all p-precolorings, i.e., there are p-precolorings that do not extend to G. By Lemma 7, there exists a bipartite rooted partial k-tree G' with type M' such that $M \preceq M' = M^i$ for some i and $\mathcal{F}(G') \subseteq \mathcal{F}(G)$. For every p-precoloring c that does not extend to G' (and there exists at least one such p-precoloring c), some graph G^i_c has been glued into G^i. Hence, $\mathcal{F}(G^i) \subseteq \mathcal{F}(G') \subseteq \mathcal{F}(G)$. Since the type of G^i is M^i, the conclusion of the lemma follows. □

4 Proof of Theorem 2

We are now ready to prove Theorem 2, which is recalled below.

Theorem 2. *For every k and every $\varepsilon > 0$, there exists g such that every graph with tree-width at most k and odd-girth at least g has circular chromatic number at most $2+\varepsilon$.*

Proof. Fix p and q such that $2 < p/q \leq 2+\varepsilon$. Let $G^1, \ldots, G^m$ be the bipartite partial k-trees from Lemma 8 applied for k, p and q. Set N to be the largest order of the graphs G^i and set g to be $3N$. We assert that each partial k-tree with odd-girth g has circular chromatic number at most p/q. Assume that this is not the case and let G be a counterexample with the fewest vertices.

The graph G has at least $3N$ vertices (otherwise, it has no odd cycles and thus is bipartite). By Lemma 5, G is isomorphic to $G_1 \oplus G_2$, where G_1 and G_2 are rooted partial k-trees and the number of vertices of G_1 is between $N+1$ and $2N$. By the choice of g, the graph G_1 has no odd cycle and thus it is a bipartite rooted partial k-tree. By Lemma 8, there exists i such that $\mathcal{F}(G^i) \subseteq \mathcal{F}(G_1)$ and $M_1 \preceq M^i$ where M_1 is the type of G_1 and M^i is the type of G^i. Let G' be the partial k-tree $G^i \oplus G_2$.

First, G' has fewer vertices than G since the number of vertices of G^i is at most N and the number of vertices of G_1 is at least $N+1$. Second, G'

has no (p,q)-coloring: if it had a (p,q)-coloring, then the corresponding p-precoloring of the $k+1$ vertices shared by G^i and G_2 would extend to G_1 since $\mathcal{F}(G^i) \subseteq \mathcal{F}(G_1)$ and thus G would have a (p,q)-coloring, too. Finally, G' has no odd cycle of length less than g: if it had such a cycle, replace any path between vertices v_j and $v_{j'}$ of the root of G^i with a path of at most the same length between them in G_1 (recall that $M_1 \preceq M^i$). If such paths for different pairs of v_j and $v_{j'}$ on the considered odd cycle intersect, take their symmetric difference. In this way, we obtain an Eulerian subgraph of $G = G_1 \oplus G_2$ with an odd number of edges such that the number of its edges is less than g. Consequently, this Eulerian subgraph has an odd cycle of length less than g, which violates the assumption on the odd-girth of G. We conclude that G' is a counterexample with fewer vertices than G, a contradiction. $\square$

We end by pointing out that the approach used yields an upper bound of $3(k+1) \cdot 2^{2^{p^{k+1}}((4d)^{(k+1)^2}+1)^{k^2}}$ for the smallest g such that all graphs with tree-width at most k and odd-girth at least g have circular chromatic number at most p/q, whenever $p/q > 2$. More precisely, the value of N cannot exceed $(k+1) \cdot 2^{2^{p^{k+1}}((4d)^{(k+1)^2}+1)^{k^2}}$. To see this, we consider all pairs $P = (C, M)$ where C is a set of p-precolorings of the root and M is a type such that there is a bipartite rooted partial k-tree of type M to which no coloring of C extends. Let n_P be the size of a smallest such partial k-tree. We obtain a sequence of at most $2^{p^{k+1}} \times \left((4d)^{(k+1)^2}+1\right)^{k^2}$ integers. The announced bound follows from the following fact: if the sequence is sorted in increasing order, then each term is at most twice the previous one.

Indeed, consider the tree-decomposition of the partial k-tree G_P chosen for the pair P. If the bag containing the root has a single child, then we delete a vertex of the root, and set a vertex in the single child to be part of the root. We obtain a partial k-tree to which some p-precolorings of C do not extend. Thus, $n_P \le 1 + n_{P'}$ for some pair P' and $n_{P'} < n_P$. If the bag containing the root has more than one child, then G_P can be obtained by identifying the roots of two smaller partial k-trees G and G'. By the minimality of G_P, the orders of G and G' are n_{P_1} and n_{P_2} for two pairs P_1 and P_2 such that $n_{P_i} < n_P$ for $i \in \{1,2\}$. This yields the stated fact, which in turn implies the given bound, since the smallest element of the sequence is $k+1$.

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