# The structural lambda-calculus

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**Abstract.** Inspired by a recent graphical formalism for $\lambda$-calculus based on Linear Logic technology, we introduce an untyped structural $\lambda$-calculus, called $\lambda\mathtt{j}$, which combines action at a distance with exponential rules decomposing the substitution by means of weakening, contraction and dereliction. Firstly, we prove fundamental properties such as confluence and preservation of $\beta$-strong normalisation. Secondly, we use $\lambda\mathtt{j}$ to describe known notions of developments and superdevelopments, and introduce a more general one called $\mathtt{XL}$-development. Then we show how to reformulate Regnier's $\sigma$-equivalence in $\lambda\mathtt{j}$ so that it becomes a strong bisimulation. Finally, we prove that explicit composition or de-composition of substitutions can be added to $\lambda\mathtt{j}$ while still preserving $\beta$-strong normalisation.

## 1 Introduction

Computer science has been greatly influenced by Linear Logic [8], especially because it provides a mechanism to explicitly control the use of resources by limiting the liberal use of the *structural rules* of weakening and contraction. Erasure and duplication are restricted to formulas marked with an *exponential* modality ?, and can only act on non-linear proofs marked with a bang modality !. Intuitionistic and Classical Logic can thus be encoded by a fragment containing such modalities, notably Multiplicative Exponential Linear Logic (MELL).

MELL proofs can be represented by sequent trees, but MELL Proof-Nets [8] provide a better-suited geometrical representation of proofs that eliminates irrelevant syntactical details. They have been extensively used to develop different encodings of intuitionistic logic/$\lambda$-calculus, giving rise to the geometry of interaction [9].

Normalisation of proofs (*i.e. cut elimination*) in MELL Proof-Nets is performed using three groups of rules, *multiplicative*, *exponential* and *commutative*. Non-linear proofs are distinguished by surrounding *boxes* which are handled by exponential rules: erasure, duplication and linear use correspond respectively to a cut elimination step involving a box and either a *weakening*, a *contraction* or a *dereliction*. The commutative rule allows to *compose* non-linear resources.

Different cut elimination systems [6, 15, 13], called *explicit substitution* (ES) calculi, were explained in terms of, or inspired by, the notion of reduction of MELL Proof-Nets. All of them use the idea that the content of a substitution/cut is a non-linear resource, *i.e.* a box that can be composed with another one by means of commutative rules. They also have in common an operational semantics defined in terms of a *propagation system* in which a substitution traverses a term to reach the variable occurrences.

A graph formalism for $\lambda$-terms inspired by Intuitionistic MELL has recently been proposed [1]. It avoids boxes by representing them through additional edges called *jumps*, and has no commutative reduction rule. This paper studies the term formalism, called $\lambda\mathtt{j}$-calculus, resulting from the reading back of the graphs $\lambda\mathtt{j}$-dags (and their reductions) by means of their sequentialisation theorem [1].

No rule of $\lambda\mathtt{j}$ propagates cuts, as the constructors in a term interact *at a distance*, *i.e.* they work modulo positions of cuts. Action at a distance is not a complete novelty [21, 4, 22], but none of the previous approaches faithfully reflect resource control as suggested by Linear Logic. We propose to recognise such behaviour as a new paradigm, more primitive than ES, particularly because propagations can be added on top of action at a distance (as we shall show). Despite the absence of commutative rules in $\lambda\mathtt{j}$, cuts can be composed, but in a different (more natural) way.

Similarly to formalisms [16] inspired by Proof-Nets, cut elimination is defined in terms of the number of free occurrences of variables in a term, here called *multiplicities*. More precisely, the weakening-box rule (resp. dereliction-box and contraction-box) applies to terms that are of the form $t[x/u]$ when $|t|_x = 0$ (resp. $|t|_x = 1$ and $|t|_x > 1$). The computation is, however, performed without propagating $[x/u]$, which we call a *jump* to stress that such action at a distance is really different from propagation in ES calculi. The rules of $\lambda\mathtt{j}$ therefore combine action at a distance, due to the tight correspondance with a graphical formalism, with exponential rules, due to the strong affinity with Linear Logic. Because of the weakening and contraction rules we call our language the *structural* $\lambda$-calculus.

Some calculi using either distance or multiplicities already exist, but without combining the two: only together those concepts unleash their full expressive power. Indeed, [4, 22] use distance rules to refine $\beta$-reduction, but add ES to the syntax without distinguishing between dereliction and contraction. This causes the formalism to be less expressive than $\lambda\mathtt{j}$ as discussed in Sections 4 and 6. Milner defines a $\lambda$-calculus with ES inspired by another graphical formalism, Bigraphs [21], where cuts also act at a distance. Again, he neither distinguishes between dereliction and contraction, nor does his $\beta$-rule exploit distance. The same goes for [29, 23].

This paper studies the $\lambda\mathtt{j}$-calculus focusing on four different aspects:

- **Basic properties**: Section 2 presents the calculus while Section 3 shows full composition, simulation of one-step $\beta$-reduction, confluence, and preservation of $\beta$-strong normalisation (PSN). Particularly, we prove PSN using a modular technique [14], which results in an extremely short formal argument thanks to the absence of propagations.
- **Developments**: The $\lambda\mathtt{j}$-calculus is a powerful, elegant and concise tool for studying $\beta$-reduction. As an example, in Section 4 we analyse the redex creation mechanism of $\lambda$-calculus, using normal forms of certain subsystems of $\lambda\mathtt{j}$ to characterise the result of full developments [12, 31] and full superdevelopments [17]. By adding *more distance* to the previous subsystems, we characterise the result of a new, more powerful notion of development, which we call $\mathtt{XL}$-development.

– **Operational equivalence**: Section 5 studies an operational equivalence $\equiv_{\mathtt{o}}$ which equates $\lambda\mathtt{j}$-terms behaving the same way but differing only in the positioning of their jumps. The relation $\equiv_{\mathtt{o}}$ includes a reformulation of Regnier's $\sigma$-equivalence [26], but also contains commutation for independent jumps. We show that $\equiv_{\mathtt{o}}$ is a strong bisimulation on $\lambda\mathtt{j}$-terms. Interestingly, this result holds only because of distance.
– **(De)composition of jumps**: In Section 6 we consider two further extensions of the system devised in Section 5, including, respectively, explicit composition and decomposition of jumps. We prove both new reduction relations to be confluent modulo $\equiv_{\mathtt{o}}$ and to enjoy PSN. The two systems, reintroducing some propagation rules, bridge the gap with traditional ES calculi and implementations. The PSN proofs in this section are the more technically demanding proofs of this paper, and a non-trivial contribution to the theory of termination proofs of ES calculi.

## 2 The Calculus

The set $\mathcal{T}$ of **terms** is defined by the following grammar:

$$t ::= x \text{ (variable)} \mid \lambda x.t \text{ (abstraction)} \mid t\ t \text{ (application)} \mid t[x/t] \text{ (closure)}$$

The object $[x/t]$, which is not a term itself, is called a **jump**. A term without jumps is a **$\lambda$-term**. We use the notation $\overline{v}_n^1$ for a list of terms $v_1 \dots v_n$, $t\ \overline{v}_n^1$ for $(\dots(t\ v_1)\dots v_n)$ and $t[x_i/u_i]_n^1$ for $t[x_1/u_1]\dots[x_n/u_n]$ $(n \geq 0)$.

**Free** and **bound** variables of $t$, respectively written $\mathtt{fv}(t)$ and $\mathtt{bv}(t)$, are defined as usual. The constructors $\lambda x.u$ and $u[x/v]$ bind the free occurrences of $x$ in $u$. The congruence generated by renaming of bound variables is called **$\alpha$-conversion**. Thus for example $(\lambda y.x)[x/y] =_\alpha (\lambda y'.x')[x'/y]$.

The **multiplicity of the variable $x$ in the term $t$** is defined as the number of free occurrences of $x$ in $t$, written $|t|_x$. We use $|t|_\Gamma$ for $\Sigma_{x\in\Gamma}|t|_x$. When $|t|_x = n \geq 2$, we write $t_{[y]_x}$ for the **non-deterministic replacement** of $i$ $(1 \leq i \leq n-1)$ occurrences of $x$ in $t$ by a *fresh* variable $y$. Thus, $(x\ z)[z/x]_{[y]_x}$ may denote either $(y\ z)[z/x]$ or $(x\ z)[z/y]$ but not $(y\ z)[z/y]$.

A **(meta-level) substitution** is a finite function from variables to terms. We use $\mathtt{var}(\sigma)$ to denote the variables of the domain and the codomain of the substitution $\sigma$. We denote by $\mathtt{id}$ the empty substitution. **Substitution** is defined, as usual, modulo $\alpha$-conversion so that the capture of variables is avoided. The **application** of a **substitution** $\sigma$ to a term $t$ is defined by induction on the structure of $t$ as follows:

$$\begin{array}{lll}
x\sigma & := \sigma(x) & \text{if } x \in \mathtt{dom}(\sigma) \\
y\sigma & := y & \text{if } x \notin \mathtt{dom}(\sigma) \\
(\lambda y.u)\sigma & := \lambda y.u\sigma & \text{if } y \notin \mathtt{var}(\sigma) \\
u_1[y/u_2]\sigma & := u_1\sigma[y/u_2\sigma] & \text{if } y \notin \mathtt{var}(\sigma) \\
(u_1\ u_2)\sigma & := (u_1\sigma)(u_2\sigma) &
\end{array}$$

**Lemma 1.** *Let $t$ be a term such that $|t|_x = n$. Then $t\{x/v\}$ satisfies the following properties.*

– *If* $n = 0$*, then* $t\{x/v\} = t$*.*
– *If* $n > 1$*, then* $t\{x/v\} = t_{[y]_x}\{x/v\}\{y/v\}$*.*
– *If* $n = 1$*, then*
  - $t = x$ *implies* $x\{x/v\} = v$*,*
  - $t = \lambda y.u$ & $x \neq y$ & $y \notin \mathtt{fv}(v)$ *implies* $t\{x/v\} = \lambda y.u\{x/v\}$*,*
  - $t = u_1\ u_2$ & $x \in \mathtt{fv}(u_1)$ *implies* $t\{x/v\} = u_1\{x/v\}u_2$*,*
  - $t = u_1\ u_2$ & $x \in \mathtt{fv}(u_2)$ *implies* $t\{x/v\} = u_1u_2\{x/v\}$*,*
  - $t = u_1[y/u_2]$ & $x \in \mathtt{fv}(u_1)$ *implies* $t\{x/v\} = u_1\{x/v\}[y/u_2]$*,*
  - $t = u_1[y/u_2]$ & $x \in \mathtt{fv}(u_2)$ *implies* $t\{x/v\} = u_1[y/u_2\{x/v\}]$*.*

We use juxtaposition of substitutions to denote composition so that $\tau\sigma$ is the substitution given by $x(\tau\sigma) := (x\tau)\sigma$. Composition enjoys the following well-known property.

**Lemma 2 (Composition).** *Let* $t, u, v$ *be terms such that* $x \notin \mathtt{fv}(v)$*. Then* $t\{x/u\}\{y/v\} = t\{y/v\}\{x/u\{y/v\}\}$*.*

*Proof.* By induction on $t$.

Besides $\alpha$-conversion, we consider the following rewriting rules:

| | | | |
|---|---|---|---|
| (dB) | $(\lambda x.t)\mathtt{L}\ u$ | $\to t[x/u]\mathtt{L}$ | |
| (w) | $t[x/u]$ | $\to t$ | if $\lvert t\rvert_x = 0$ |
| (d) | $t[x/u]$ | $\to t\{x/u\}$ | if $\lvert t\rvert_x = 1$ |
| (c) | $t[x/u]$ | $\to t_{[y]_x}[x/u][y/u]$ | if $\lvert t\rvert_x \geq 2$ |

where we use the (meta)notation L for a list $[x_i/u_i]_n^1$ with $n \geq 0$.

Note that dB reformulates the classical B-rule of ES calculi as a *distance* rule which skips the jumps affecting the abstraction of the redex. This same rule notably appears in weak ES calculi [19] to avoid the the $\beta$-redexes that are hidden by blocked substitutions. Here, the dB-rule is the natural term counterpart of a graphical and *local* rule in proof-nets and $\lambda j$-dags. Section 4 puts the expressiveness of this concept in evidence. The rules w, d and c are to be understood as the weakening, dereliction and contraction rules in $\lambda j$-dags.

It is worth noting that $\lambda\mathtt{j}$ allows to *compose* jumps, as for example reduction from $t = y[x/zy][y/v]$ *computes* the (simultaneous) jumps in $y[x/zv][y/v]$. Usually, the so-called *composition* of the two jumps of $t$ rather yields $y[y/v][x/zy[y/v]]$. We will study this more structural notion in Section 6.

The **rewriting relation** $\to_{\lambda\mathtt{j}}$ (resp. $\to_{\mathtt{j}}$) is generated by all (resp. all expect dB) the previous rewriting rules modulo $\alpha$-conversion. The j-rewriting rules are based on global side conditions, which may seem difficult to implement. However, if implementation is done via graphical formalisms (such as proof-nets, bigraphs, $\lambda j$-dags), these conditions become local and completely harmless.

Now consider any reduction relation $\mathcal{R}$. A term $t$ is said to be in **$\mathcal{R}$-normal form**, written $\mathcal{R}$-nf, if there is no $u$ so that $t \to_{\mathcal{R}} u$. We use $\mathcal{R}(t)$ to denote the *unique* $\mathcal{R}$-nf of $t$, when it exists. A term $t$ is **$\mathcal{R}$-strongly normalising** or **$\mathcal{R}$-terminating**, written $t \in \mathcal{SN}_{\mathcal{R}}$, if there is no infinite $\mathcal{R}$-reduction sequence starting at $t$, in which case $\eta_{\mathcal{R}}(t)$ denotes the **maximal length of a $\mathcal{R}$-reduction sequence starting at** $t$. The relation $\mathcal{R}$ is called **complete** if it is strongly normalising and confluent. For a sequence of $\mathcal{R}$-strongly normalising terms $\overline{t}_n^1$ we write $\eta_{\mathcal{R}}(\overline{t}_n^1)$ for $\Sigma_{i=1}^n\ \eta_{\mathcal{R}}(t_i)$. An inductive definition of $\mathcal{SN}_{\mathcal{R}}$ is usually given by:

$$t \in \mathcal{SN}_{\mathcal{R}} \text{ iff } \forall s\ (t \to_{\mathcal{R}} s \text{ implies } s \in \mathcal{SN}_{\mathcal{R}})$$

Define **$\mathcal{R}$-reduction of meta-level substitutions** by $\gamma \to_{\mathcal{R}}^* \gamma'$ iff $\mathtt{dom}(\gamma) = \mathtt{dom}(\gamma')$ and $\forall\ x \in \mathtt{dom}(\gamma) : \gamma(x) \to_{\mathcal{R}}^* \gamma'(x)$.

The following properties hold for all the reduction relations in this paper.

**Lemma 3 (Stability of Reduction by Substitution).**

- *If $t \to_{\mathcal{R}} t'$, then $t\sigma \to_{\mathcal{R}} t'\sigma$. So that $t\sigma \in \mathcal{SN}_{\mathcal{R}}$ implies $t'\sigma \in \mathcal{SN}_{\mathcal{R}}$ and $\eta_{\mathcal{R}}(t'\sigma) < \eta_{\mathcal{R}}(t\sigma)$.*
- *If $\gamma \to_{\mathcal{R}} \gamma'$, then $t\gamma \to_{\mathcal{R}}^* t\gamma'$. So that $t\gamma \in \mathcal{SN}_{\mathcal{R}}$ implies $t\gamma' \in \mathcal{SN}_{\mathcal{R}}$ and $\eta_{\mathcal{R}}(t\gamma') \leq \eta_{\mathcal{R}}(t\gamma)$.*

## 3 Main Properties

In this section we prove some sanity properties of the calculus: full composition, simulation of one-step $\beta$-reduction, confluence and PSN. Since the first three can easily be shown using standard rewriting technology, we concentrate on proving PSN, which usually is tricky, but turns out to be surprisingly simple in our case.

**Lemma 4 (Full Composition (FC)).** *Let $t, u \in \mathcal{T}$. Then $t[x/u] \to_{\mathtt{j}}^+ t\{x/u\}$. Moreover, $|t|_x \geq 1$ implies $t[x/u] \to_{\mathtt{d,c}}^+ t\{x/u\}$.*

*Proof.* By induction on $|t|_x$.

- If $|t|_x = 0$, then $t[x/u] \to_{\mathtt{w}} t = t\{x/u\}$.
- If $|t|_x = 1$, then $t[x/u] \to_{\mathtt{d}} t\{x/u\}$.
- If $|t|_x \geq 2$, then

$$\begin{array}{l} t[x/u] \to_{\mathtt{c}} t_{[y]_x}[y/u][x/u] \to_{\mathtt{j}}^+ \ (i.h.) \\ \qquad t_{[y]_x}\{y/u\}[x/u] \to_{\mathtt{j}}^+ \ (i.h.)\ t_{[y]_x}\{y/u\}\{x/u\} = t\{x/u\} \end{array}$$

**Corollary 1 (Simulation).** *Let $t \in \lambda$-term. If $t \to_\beta t'$, then $t \to_{\lambda\mathtt{j}}^+ t'$.*

The following notion, which counts the maximal number of free occurrences of a variable $x$ that may appear during a $\mathtt{j}$-reduction sequence from a term $t$, will be useful for various proofs. The **potential multiplicity** of the variable $x$ in the term $t$, written $\mathtt{M}_x(t)$, is defined for $\alpha$-equivalence classes as follows: if $x \notin \mathtt{fv}(t)$, then $\mathtt{M}_x(t) := 0$; otherwise:

$$\begin{array}{ll} \mathtt{M}_x(x) & := 1 \\ \mathtt{M}_x(\lambda y.u) & := \mathtt{M}_x(u) \\ \mathtt{M}_x(u\ v) & := \mathtt{M}_x(u) + \mathtt{M}_x(v) \\ \mathtt{M}_x(u[y/v]) & := \mathtt{M}_x(u) + \mathtt{max}(1, \mathtt{M}_y(u)) \cdot \mathtt{M}_x(v) \end{array}$$

Potential multiplicities enjoy the following properties.

**Lemma 5.** *Let* $t \in \mathcal{T}$.

1. *If* $u \in \mathcal{T}$ *and* $y \notin \mathtt{fv}(u)$, *then* $\mathtt{M}_y(t) = \mathtt{M}_y(t\{x/u\})$.
2. *If* $|t|_x \geq 2$, *then* $\mathtt{M}_z(t) = \mathtt{M}_z(t_{[y]_x})$ *and* $\mathtt{M}_x(t) = \mathtt{M}_x(t_{[y]_x}) + \mathtt{M}_y(t_{[y]_x})$.
3. *If* $t \to_{\mathtt{j}} t'$, *then* $\mathtt{M}_y(t) \geq \mathtt{M}_y(t')$.

*Proof.* By induction on $t$.

We consider multisets of integers. We use $[\,]$ to denote the empty multiset, $\sqcup$ to denote multiset union and $n \cdot [a_1, \ldots, a_n]$ to denote $[n \cdot a_1, \ldots, n \cdot a_n]$.

The **j-measure** of $t \in \mathcal{T}$, written $\mathtt{jm}(t)$, is given by:

$$\begin{array}{ll}
\mathtt{jm}(x) & := [\,] \\
\mathtt{jm}(\lambda x.t) & := \mathtt{jm}(t) \\
\mathtt{jm}(tu) & := \mathtt{jm}(t) \sqcup \mathtt{jm}(u) \\
\mathtt{jm}(t[x/u]) & := [\mathtt{M}_x(t)] \sqcup \mathtt{jm}(t) \sqcup \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{jm}(u)
\end{array}$$

**Lemma 6.** *Let* $t \in \mathcal{T}$. *Then,*

1. $\mathtt{jm}(t) = \mathtt{jm}(t_{[y]_x})$.
2. *If* $u \in \mathcal{T}$, *then* $\mathtt{jm}(t) \sqcup \mathtt{jm}(u) \geq \mathtt{jm}(t\{x/u\})$.

*Proof.* By induction on $t$. The first property is straightforward so that we only show the second one.

- $t = x$. Then $\mathtt{jm}(x) \sqcup \mathtt{jm}(u) = [\,] \sqcup \mathtt{jm}(u) = \mathtt{jm}(x\{x/u\})$.
- $t = y \neq x$. Then $\mathtt{jm}(y) \sqcup \mathtt{jm}(u) = [\,] \sqcup \mathtt{jm}(u) \geq [\,] = \mathtt{jm}(y\{x/u\})$.
- $t = t_1[y/t_2]$. W.l.g we assume $y \notin \mathtt{fv}(u)$. Then,

$$\begin{array}{ll}
\mathtt{jm}(t_1[y/t_2]) \sqcup \mathtt{jm}(u) & = \\
[\mathtt{M}_y(t_1)] \sqcup \mathtt{jm}(t_1) \sqcup \mathtt{max}(1, \mathtt{M}_y(t_1)) \cdot \mathtt{jm}(t_2) \sqcup \mathtt{jm}(u) & \geq_{i.h.\ \&\ L.5:1} \\
[\mathtt{M}_y(t_1\{x/u\})] \sqcup \mathtt{jm}(t_1\{x/u\}) \sqcup \mathtt{max}(1, \mathtt{M}_y(t_1\{x/u\})) \cdot \mathtt{jm}(t_2\{x/u\}) & = \\
\mathtt{jm}(t_1\{x/u\}[y/t_2\{x/u\}]) &
\end{array}$$

- All the other cases are straightforward.

**Lemma 7.** *Let* $t \in \mathcal{T}$.

1. $t_0 \equiv_\alpha t_1$ *implies* $\mathtt{jm}(t_0) = \mathtt{jm}(t_1)$.
2. $t_0 \to_{\mathtt{j}} t_1$ *implies* $\mathtt{jm}(t_0) > \mathtt{jm}(t_1)$.

*Proof.* By induction on the relations. The first point is straightforward, so that we only show the second one.

- $t_0 = t[x/u] \to_{\mathtt{w}} t = t_1$, with $|t|_x = 0$. Then $\mathtt{jm}(t_0) = \mathtt{jm}(t) \sqcup 1 \cdot \mathtt{jm}(u) \sqcup [0] > \mathtt{jm}(t) = \mathtt{jm}(t_1)$.
- $t_0 = t[x/u] \to_{\mathtt{d}} t\{x/u\} = t_1$, with $|t|_x = 1$.
  Then $\mathtt{jm}(t_0) = \mathtt{jm}(t) \sqcup 1 \cdot \mathtt{jm}(u) \sqcup [1] > \mathtt{jm}(t) \sqcup \mathtt{jm}(u) \geq_{L.\ 6:2} \mathtt{jm}(t\{x/u\}) = \mathtt{jm}(t_1)$.

– $t_0 = t[x/u] \to_{\mathtt{c}} t_{[y]_x}[x/u][y/u] = t_1$, with $|t|_x \geq 2$ and $y$ fresh. Then,

$$\begin{array}{ll}
\mathtt{jm}(t_0) & = \\
\mathtt{jm}(t) \sqcup \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{jm}(u) \sqcup [\mathtt{M}_x(t)] & = \\
\mathtt{jm}(t) \sqcup \mathtt{M}_x(t) \cdot \mathtt{jm}(u) \sqcup [\mathtt{M}_x(t)] & = \\
\mathtt{jm}(t) \sqcup (\mathtt{M}_x(t_{[y]_x}) + \mathtt{M}_y(t_{[y]_x})) \cdot \mathtt{jm}(u) \sqcup [\mathtt{M}_x(t)] & =_{L.6:1} \\
\mathtt{jm}(t_{[y]_x}) \sqcup (\mathtt{M}_x(t_{[y]_x}) + \mathtt{M}_y(t_{[y]_x})) \cdot \mathtt{jm}(u) \sqcup [\mathtt{M}_x(t)] & > \\
\mathtt{jm}(t_{[y]_x}) \sqcup \mathtt{M}_x(t_{[y]_x}) \cdot \mathtt{jm}(u) \sqcup [\mathtt{M}_x(t_{[y]_x})] \sqcup \mathtt{M}_y(t_{[y]_x}) \cdot \mathtt{jm}(u) \sqcup [\mathtt{M}_y(t_{[y]_x})] & = \\
\mathtt{jm}(t_{[y]_x}) \sqcup \mathtt{M}_x(t_{[y]_x}) \cdot \mathtt{jm}(u) \sqcup [\mathtt{M}_x(t_{[y]_x})] \sqcup \mathtt{M}_y(t_{[y]_x}[x/u]) \cdot \mathtt{jm}(u) \sqcup [\mathtt{M}_y(t_{[y]_x}[x/u])] & = \\
\mathtt{jm}(t_{[y]_x}[x/u]) \sqcup \mathtt{M}_y(t_{[y]_x}[x/u]) \cdot \mathtt{jm}(u) \sqcup [\mathtt{M}_y(t_{[y]_x}[x/u])] & = \\
\mathtt{jm}(t_1) &
\end{array}$$

– $t_0 = t[x/u] \to t'[x/u] = t_1$, where $t \to t'$. Then

$$\begin{array}{ll}
\mathtt{jm}(t_0) & = \\
\mathtt{jm}(t) \sqcup \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{jm}(u) \sqcup [\mathtt{M}_x(t)] & >_{i.h.} \\
\mathtt{jm}(t') \sqcup \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{jm}(u) \sqcup [\mathtt{M}_x(t)] & \geq_{L.\ 5:3} \\
\mathtt{jm}(t') \sqcup \mathtt{max}(1, \mathtt{M}_x(t')) \cdot \mathtt{jm}(u) \sqcup [\mathtt{M}_x(t')] & = \\
\mathtt{jm}(t_1) &
\end{array}$$

– $t_0 = t[x/u] \to t[x/u'] = t_1$, where $u \to u'$. Then

$$\begin{array}{ll}
\mathtt{jm}(t_0) & = \\
\mathtt{jm}(t) \sqcup \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{jm}(u) \sqcup [\mathtt{M}_x(t)] & >_{i.h.} \\
\mathtt{jm}(t) \sqcup \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{jm}(u') \sqcup [\mathtt{M}_x(t)] & = \\
\mathtt{jm}(t_1) &
\end{array}$$

– All the other cases are straightforward

**Lemma 8.** *The* $\mathtt{j}$*-reduction relation is complete.*

*Proof.* Using the Modular Abstract Theorem 5.

Confluence of calculi with ES can be easily proved by using Tait and Martin Löf's technique (see for example the case of $\lambda\mathtt{es}$ [13]). This technique is based on the definition of a simultaneous reduction relation $\Rrightarrow_{\lambda\mathtt{j}}$ which enjoys the diamond property.

The **simultaneous reduction relation** $\Rrightarrow_{\lambda\mathtt{j}}$ is defined on terms in $\mathtt{j}$-normal form as follows:

– $x \Rrightarrow_{\lambda\mathtt{j}} x$
– If $t \Rrightarrow_{\lambda\mathtt{j}} t'$, then $\lambda x.t \Rrightarrow_{\lambda\mathtt{j}} \lambda x.t'$
– If $t \Rrightarrow_{\lambda\mathtt{j}} t'$ & $u \Rrightarrow_{\lambda\mathtt{j}} u'$, then $t\ u \Rrightarrow_{\lambda\mathtt{j}} t'\ u'$
– If $t \Rrightarrow_{\lambda\mathtt{j}} t'$ and $u \Rrightarrow_{\lambda\mathtt{j}} u'$, then $(\lambda x.t)\ u \Rrightarrow_{\lambda\mathtt{j}} \mathtt{j}(t'[x/u'])$

**Lemma 9.** *If* $t \Rrightarrow_{\lambda\mathtt{j}} t'$*, then* $t \to^*_{\lambda\mathtt{j}} t'$*.*

*Proof.* By induction on $t \Rrightarrow_{\lambda\mathtt{j}} t'$.

**Lemma 10.** *If* $t \to_{\lambda\mathtt{j}} t'$*, then* $\mathtt{j}(t) \Rrightarrow_{\lambda\mathtt{j}} \mathtt{j}(t')$*.*

*Proof.* By induction on $t \to_{\lambda \mathtt{j}} t'$.

**Lemma 11.** *The relation $\Rrightarrow_{\lambda \mathtt{j}}$ enjoys the diamond property.*

*Proof.* By induction on $\Rrightarrow_{\lambda \mathtt{j}}$ and case analysis.

**Theorem 1 (Confluence).** *For all $t, u_1, u_2 \in \mathcal{T}$, if $t \to^*_{\lambda \mathtt{j}} u_i$ $(i = 1, 2)$, then $\exists v$ s.t. $u_i \to^*_{\lambda \mathtt{j}} v$ $(i = 1, 2)$.*

*Proof.* Let $t \to^*_{\lambda \mathtt{j}} t_i$ for $i = 1, 2$. Lemma 10 gives $\mathtt{j}(t) \Rrightarrow^*_{\lambda \mathtt{j}} \mathtt{j}(t_i)$ for $i = 1, 2$. Lemma 11 implies $\Rrightarrow_{\lambda \mathtt{j}}$ is confluent so that $\exists s$ such that $\mathtt{j}(t_i) \Rrightarrow^*_{\lambda \mathtt{j}} s$ for $i = 1, 2$. We can then close the diagram with $t_i \to^*_{\mathtt{j}} \mathtt{j}(t_i) \to^*_{\lambda \mathtt{j}} s$ by Lemma 9.

Remark that confluence does not use termination of the $\mathtt{j}$-(sub)calculus. To give a formal termination proof for it we introduce the following notions.

We now discuss PSN. A reduction system $\mathcal{R}$ is said to enjoy the **PSN property** w.r.t. another system $\mathcal{S}$ iff every term which is $\mathcal{S}$-strongly normalising is also $\mathcal{R}$-strongly normalising. Here PSN will mean PSN w.r.t. $\beta$-reduction.

The proof of PSN can be stated in terms of the **IE** property which relates termination of **I**mplicit substitution to termination of **E**xplicit substitution. A reduction system $\mathcal{R}$ enjoys the **IE property** iff for $n \geq 0$ and for all $t, u, \overline{v}^1_n \in \lambda$-terms: $u \in \mathcal{SN}_{\mathcal{R}}$ and $t\{x/u\}\overline{v}^1_n \in \mathcal{SN}_{\mathcal{R}}$ imply $t[x/u]\overline{v}^1_n \in \mathcal{SN}_{\mathcal{R}}$.

**Theorem 2 (IE implies PSN).** *A reduction relation $\mathcal{R}$ enjoys PSN if $\mathcal{R}$ verifies the* **IE***-property and the following:*

**(F0)** *If $\overline{t}^1_n \in \lambda$-terms in $\mathcal{SN}_{\mathcal{R}}$, then $x\overline{t}^1_n \in \mathcal{SN}_{\mathcal{R}}$.*
**(F1)** *If $u \in \lambda$-term in $\mathcal{SN}_{\mathcal{R}}$, then $\lambda x.u \in \mathcal{SN}_{\mathcal{R}}$.*
**(F2)** *The only $\mathcal{R}$-reducts of a $\lambda$-term $(\lambda x.u)v\overline{t}^1_n$ are $u[x/v]\overline{t}^1_n$ and those coming from internal reduction on $u, v, \overline{t}^1_n$.*

Intuitively, the first two requirements (F0) and (F1) mean that head-normal forms are stable under $\mathcal{R}$. The last requirement (F2) means that the head-redex can only be refined by $\mathcal{R}$, but nothing else.

*Proof.* We show $t \in \mathcal{SN}_{\mathcal{R}}$ by induction on the definition of $t \in \mathcal{SN}_{\beta}$ (as in [30]):

- If $t = x\overline{t}^1_n$ with $t_i \in \mathcal{SN}_{\beta}$, then (i.h.) $t_i \in \mathcal{SN}_{\mathcal{R}}$ and thus (**F0**) $x\overline{t}^1_n \in \mathcal{SN}_{\mathcal{R}}$.
- If $t = \lambda x.u$ with $u \in \mathcal{SN}_{\beta}$, then (i.h.) $u \in \mathcal{SN}_{\mathcal{R}}$ and thus (**F1**) $\lambda x.u \in \mathcal{SN}_{\mathcal{R}}$.
- If $t = (\lambda x.u)v\overline{t}^1_n$, with $u\{x/v\}\overline{t}^1_n \in \mathcal{SN}_{\beta}$ and $v \in \mathcal{SN}_{\beta}$, then (i.h.) both terms are in $\mathcal{SN}_{\mathcal{R}}$, **IE** gives $U = u[x/v]\overline{t}^1_n \in \mathcal{SN}_{\mathcal{R}}$, so in particular $u, v, \overline{t}^1_n \in \mathcal{SN}_{\mathcal{R}}$. We show $t \in \mathcal{SN}_{\mathcal{R}}$ by induction on $\eta_{\mathcal{R}}(u) + \eta_{\mathcal{R}}(v) + \Sigma_i\ \eta_{\mathcal{R}}(t_i)$. For that, we show that every $\mathcal{R}$-reduct of $t$ is in $\mathcal{SN}_{\mathcal{R}}$.
  Now, if $t \to_{\mathcal{R}} t'$ is an internal reduction, apply the i.h. Otherwise, **F2** gives $t \to_{\mathcal{R}} u[x/v]t_1 \dots t_n = U$ which is in $\mathcal{SN}_{\mathcal{R}}$.

**Theorem 3 (IE for $\lambda \mathtt{j}$).** $\lambda \mathtt{j}$ *enjoys the* **IE** *property.*

*Proof.* We show the following more general statement. For all terms $t, \overline{u}_m^1$ ($m \geq 1$), $\overline{v}_n^1$ ($n \geq 0$), if $\overline{u}_m^1 \in \mathcal{SN}_{\lambda\mathtt{j}}$ & $t\{x_i/u_i\}_m^1 \overline{v}_n^1 \in \mathcal{SN}_{\lambda\mathtt{j}}$, then $t[x_i/u_i]_m^1 \overline{v}_n^1 \in \mathcal{SN}_{\lambda\mathtt{j}}$, where $x_i \neq x_j$ for $i,j = 1 \ldots m$ and $x_i \notin \mathtt{fv}(u_j)$ for $i,j = 1 \ldots m$. The **IE** property then holds by taking $m = 1$.

Suppose $\overline{u}_m^1 \in \mathcal{SN}_{\lambda\mathtt{j}}$ & $t\{x_i/u_i\}_m^1 \overline{v}_n^1 \in \mathcal{SN}_{\lambda\mathtt{j}}$. We show $T = t[x_i/u_i]_m^1 \overline{v}_n^1 \in \mathcal{SN}_{\lambda\mathtt{j}}$ by induction on $\langle \eta_{\lambda\mathtt{j}}(t\{x_i/u_i\}_m^1 \overline{v}_n^1), \mathtt{o}_{\overline{x}_m^1}(t), \eta_{\lambda\mathtt{j}}(\overline{u}_m^1)\rangle$ where $\mathtt{o}_{x_i}(t) = 3^{|t|_{x_i}}$ and $\mathtt{o}_{\overline{x}_m^1}(t) = \Sigma_{i \in m} \mathtt{o}_{x_i}(t)$.

To show $T \in \mathcal{SN}_{\lambda\mathtt{j}}$ it is sufficient to show that every $\lambda\mathtt{j}$-reduct of $T$ is in $\mathcal{SN}_{\lambda\mathtt{j}}$. Since $m \geq 1$, then we can write $[x_i/u_i]_m^1 = [x_i/u_i]_{j-1}^1 [x_j/u_j][x_i/u_i]_m^{j+1}$.

- $T \to_{\lambda\mathtt{j}} t[x_i/u_i]_{j-1}^1 [x_j/u'_j][x_i/u_i]_m^{j+1} \overline{v}_n^1 = T'$ with $u_j \to_{\lambda\mathtt{j}} u'_j$. Then we have that $\eta_{\lambda\mathtt{j}}(t\{x_i/u_i\}_{j-1}^1 \{x_j/u'_j\}\{x_i/u_i\}_m^{j+1} \overline{v}_n^1) \leq \eta_{\lambda\mathtt{j}}(t\{x_i/u_i\}_m^1 \overline{v}_n^1)$ and $\mathtt{o}_{\overline{x}_m^1}(t) = \mathtt{o}_{\overline{x}_{j-1}^1 x_j \overline{x}_m^{j+1}}(t)$, and $\eta_{\lambda\mathtt{j}}(\overline{u}_{j-1}^1 u'_1 \overline{u}_m^{j+1}) < \eta_{\lambda\mathtt{j}}(u_1, \overline{u}_m^2)$. Since $\overline{u}_{j-1}^1 u'_1 \overline{u}_m^{j+1} \in \mathcal{SN}_{\lambda\mathtt{j}}$ and $t\{x_i/u_i\}_{j-1}^1 \{x_j/u'_j\}\{x_i/u_i\}_m^{j+1} \overline{v}_n^1 \in \mathcal{SN}_{\lambda\mathtt{j}}$ by Lemma 3, then we conclude by the i.h..
- $T \to_{\lambda\mathtt{j}} t'[x_i/u_i]_m^1 \overline{v}_n^1 = T'$ with $t \to_{\lambda\mathtt{j}} t'$. Then we have that $\eta_{\lambda\mathtt{j}}(t'\{x_i/u_i\}_m^1 \overline{v}_n^1) < \eta_{\lambda\mathtt{j}}(t\{x_i/u_i\}_m^1 \overline{v}_n^1)$. We conclude by the i.h. since $t'\{x_i/u_i\}_m^1 \overline{v}_n^1 \in \mathcal{SN}_{\lambda\mathtt{j}}$ by Lemma 3.
- $T \to_{\lambda\mathtt{j}} t[x_i/u_i]_m^1 v_1 \ldots v'_i \ldots v_n = T'$ with $v_i \to_{\lambda\mathtt{j}} v'_i$. We have that $\eta_{\lambda\mathtt{j}}(t\{x_i/u_i\}_m^1 v_1 \ldots v'_i \ldots v_n) < \eta_{\lambda\mathtt{j}}(t\{x_i/u_i\}_m^1 \overline{v}_n^1)$. We conclude by the i.h. since $t\{x_i/u_i\}_m^1 v_1 \ldots v'_i \ldots v_n \in \mathcal{SN}_{\lambda\mathtt{j}}$.
- $T \to_{\mathtt{w}} t[x_i/u_i]_{j-1}^1 [x_i/u_i]_m^{j+1} \overline{v}_n^1$, with $|t|_{x_j} = 0$.We have that the measure $\eta_{\lambda\mathtt{j}}(t\{x_i/u_i\}_{j-1}^1 \{x_i/u_i\}_m^{j+1} \overline{v}_n^1)$ is equal to $\eta_{\lambda\mathtt{j}}(t\{x_i/u_i\}_m^1 \overline{v}_n^1)$, but $\mathtt{o}_{\_}(t)$ decreases since $\mathtt{o}_{\overline{x}_{j-1}^1 \overline{x}_m^{j+1}}(t) < \mathtt{o}_{\overline{x}_m^1}(t)$. We can conclude by the i.h. since by hypothesis $t\{x_i/u_i\}_{j-1}^1 \{x_i/u_i\}_m^{j+1} \overline{v}_n^1 = t\{x_i/u_i\}_m^1 \overline{v}_n^1 \in \mathcal{SN}_{\lambda\mathtt{j}}$.
- $T \to_{\mathtt{d}} t[x_i/u_i]_{j-1}^1 \{x_j/u_j\}[x_i/u_i]_m^{j+1} \overline{v}_n^1$ with $|t|_{x_j} = 1$. Then we have that $\eta_{\lambda\mathtt{j}}(t\{x_i/u_i\}_{j-1}^1 \{x_j/u_j\}\{x_i/u_i\}_m^{j+1} \overline{v}_n^1) = \eta_{\lambda\mathtt{j}}(t\{x_i/u_i\}_m^1 \overline{v}_n^1)$. Also, the jumps are independent, so that $\overline{x}_{j-1}^1 \overline{x}_m^{j+1} \cap \mathtt{fv}(u_j) = \emptyset$ implies $\mathtt{o}_{\overline{x}_{j-1}^1 \overline{x}_m^{j+1}}(t\{x_j/u_j\}) < \mathtt{o}_{\overline{x}_m^1}(t)$.
  We conclude since $t\{x_i/u_i\}_{j-1}^1 \{x_j/u_j\}\{x_i/u_i\}_m^{j+1} \overline{v}_n^1 = t\{x_i/u_i\}_m^1 \overline{v}_n^1 \in \mathcal{SN}_{\lambda\mathtt{j}}$ by hypothesis.
- $T \to_{\mathtt{c}} t_{[y]_{x_j}} [x_i/u_i]_{j-1}^1 [x_j/u_j][y/u_j][x_i/u_i]_m^{j+1} \overline{v}_n^1$ with $|t|_{x_j} \geq 2$ and $y$ fresh. Then,
  $\eta_{\lambda\mathtt{j}}(t_{[y]_{x_j}} \{x_i/u_i\}_{j-1}^1 \{x_j/u_j\}\{y/u_j\}\{x_i/u_i\}_m^{j+1} \overline{v}_n^1) = \eta_{\lambda\mathtt{j}}(t\{x_i/u_i\}_m^1 \overline{v}_n^1))$ and $\mathtt{o}_{\overline{x}_{j-1}^1 x_j y \overline{x}_m^{j+1}}(t_{[y]_{x_j}}) < \mathtt{o}_{\overline{x}_m^1}(t)$. In order to apply the i.h. to $t_{[y]_{x_j}}$ we need.
  - $\overline{u}_{j-1}^1, u_j, u_j, \overline{u}_m^{j+1} \in \mathcal{SN}_{\lambda\mathtt{j}}$. This holds by hypothesis.
  - $t_{[y]_{x_1}} \{x_i/u_i\}_{j-1}^1 \{x_j/u_j\}\{y/u_j\}\{x_i/u_i\}_m^{j+1} \overline{v}_n^1 \in \mathcal{SN}_{\lambda\mathtt{j}}$. This holds since the term is equal to $t\{x_i/u_i\}_m^1 \overline{v}_n^1$ which is $\mathcal{SN}_{\lambda\mathtt{j}}$ by hypothesis.
- $T = (\lambda x.t')[x_i/u_i]_m^1 v_1 \overline{v}_n^2 \to_{\mathtt{dB}} t'[x/v_1][x_i/u_i]_m^1 \overline{v}_n^2 = T'$. By hypothesis $U = (\lambda x.t')\{x_i/u_i\}_m^1 v_1 \overline{v}_n^2 \in \mathcal{SN}_{\lambda\mathtt{j}}$. Using full composition we obtain

$$\begin{array}{l} U \to_{\mathtt{dB}} t'\{x_i/u_i\}_m^1 [x/v_1] \overline{v}_n^2 \to_{\lambda\mathtt{j}}^+ \\ \qquad t'\{x_i/u_i\}_m^1 \{x/v_1\} \overline{v}_n^2 = t'\{x/v_1\}\{x_i/u_i\}_m^1 \overline{v}_n^2 = U' \end{array}$$

  Thus $\eta_{\lambda\mathtt{j}}(U') < \eta_{\lambda\mathtt{j}}(U)$. To conclude $T' \in \mathcal{SN}_{\lambda\mathtt{j}}$ by the i.h. we then need

- $v_1, \overline{u}_m^1 \in \mathcal{SN}_{\lambda\mathtt{j}}$. But $\overline{u}_m^1 \in \mathcal{SN}_{\lambda\mathtt{j}}$ holds by hypothesis and $t\{x_i/u_i\}_m^1 \overline{v}_n^1 \in \mathcal{SN}_{\lambda\mathtt{j}}$ implies $v_1 \in \mathcal{SN}_{\lambda\mathtt{j}}$.
- $U' = t'\{x/v_1\}\{x_i/u_i\}_m^1 \overline{v}_n^2 \in \mathcal{SN}_{\lambda\mathtt{j}}$ which holds since $\eta_{\lambda\mathtt{j}}(U') < \eta_{\lambda\mathtt{j}}(U)$.

In contrast to known PSN proofs for calculi with ES and composition of substitutions [3, 13, 15], we get a very concise and simple proof of the **IE** property, and thus of PSN, due to the fact that $\lambda\mathtt{j}$ has no propagation rule. Indeed, since $\lambda\mathtt{j}$-reduction enjoys the **IE**-property and **F0**, **F1** and **F2** in Theorem 2 are straightforward for the $\lambda\mathtt{j}$-calculus, we get:

**Corollary 2 (PSN for $\lambda\mathtt{j}$).** *Let $t \in \lambda$-term. If $t \in \mathcal{SN}_\beta$, then $t \in \mathcal{SN}_{\lambda\mathtt{j}}$.*

## 4 Developments and All That

In $\lambda$-calculus creation of redexes can be classified in three types [18]:

**(Type 1)** $((\lambda x.\lambda y.t)\ u)\ v \to_\beta (\lambda y.t\{x/u\})\ v$.
**(Type 2)** $(\lambda x.x)\ (\lambda y.t)\ u \to_\beta (\lambda y.t)\ u$.
**(Type 3)** $(\lambda x.C[x\ v])\ (\lambda y.u) \to_\beta C\{x/\lambda y.u\}[(\lambda y.u)\ v\{x/\lambda y.u\}]$

When $\lambda$-terms are considered as trees, the first and second type create a redex *upward*, while the third creates it *downward*, which is the dangerous kind of creation since it may lead to divergence.

According to the previous classification, different ways to compute a term can be defined. A reduction sequence starting at $t$ is a **development** [12] (resp. a **full development**) if only (resp. all the) residuals of redexes (resp. all the redexes) of $t$ are contracted. A more liberal notion, called **L-development** here, and known as superdevelopment [17], allows to also reduce created redexes of type 1 and 2. A major result states that all developments (resp. L-developments) of a $\lambda$-term are finite, and that the results of all full developments (resp. full L-developments) coincide.

Note that reductions of type 1 and 2 are acceptable because the created redex is *hidden* in the initial term, so that non-termination only happens when creating redexes of type 3. However, *linear creations* of type 3 - *i.e.* creations which do not involve duplications - are also safe, and infinite reductions only happen if redexes created after duplication are reduced - we call such cases *non-linear creations* of type 3. As an example, consider $\Omega = (\lambda x.x\ x)\ (\lambda x.x\ x)$ whose infinite reduction involves only non-linear creations of the third type. These observations suggest that banning the third type of creation is excessive: it is sufficient to avoid non-linear ones. This extended form of L-development needs a language capable of distinguishing between the different linear/erasing/duplicating nature of redexes. This section extends the notion of L-development to that of **XL-development**, which also reduces linearly created redexes of type 3, and provides a finiteness result.

The following table summarises the behaviour of each computational notion studied in this section on the $\lambda$-term $u_0 = (I\ I)\ ((\lambda z.z\ y)\ I)$, where $I = \lambda x.x$.

$$\begin{array}{ll} \text{full development of } u_0 & = I\ (I\ y) \\ \text{full L-development of } u_0 & = I\ y \\ \text{full XL-development of } u_0 & = y \end{array} \qquad (1)$$

The specification of all the reduction subsystems used in this section exploits the idea of *multiplicity*. Thus, the $\lambda\mathtt{j}$-calculus provides a uniform and expressive framework to reason about creation of redexes in $\lambda$-calculus.

A **development** (resp. full development) of a term $t$ is a reduction sequence in which only (resp. all the) residuals of redex occurrences (resp. all the redex occurrences) that already exist in $t$ are contracted. There are many proofs of finiteness of developments, like [27, 31, 12, 30]. The **result of a full development** of a $\lambda$-term is unique and can simply be defined by induction on the structure of terms as follows:

$$\begin{array}{lll} x^\circ & := x & \\ (\lambda x.t)^\circ & := \lambda x.t^\circ & \\ ((\lambda x.t)\ u)^\circ & := t^\circ\{x/u^\circ\} & \\ (t\ u)^\circ & := t^\circ\ u^\circ & \text{if } t \neq \lambda \end{array}$$

Remark that $t^\circ \neq \lambda$ implies $t \neq \lambda$. This notion can be extended to jumps in two different ways. The first way inductive definition can be given by

$$\begin{array}{lll} x^\bullet & := x & \\ (\lambda y.t)^\bullet & := \lambda y.t^\bullet & \\ ((\lambda x.t)\ u)^\bullet & := t^\bullet[x/u^\bullet] & \\ (t\ u)^\bullet & := t^\bullet\ u^\bullet & \text{if } t \neq \lambda \\ (t[x/u])^\bullet & := t^\bullet\{x/u^\bullet\} & \end{array}$$

For a $\lambda$-term $t$ we have $t^\circ \neq t^\bullet$; in particular $t^\bullet \to_{\mathtt{j}}^{+} t^\circ$.

The second way consists in taking the opposite view, with the purpose to simulate developments of $\lambda$-calculus. This can be obtained with a function ${}^{\bullet\bullet}$ which is exactly as ${}^\bullet$ except for:

$$((\lambda x.t)\ u)^{\bullet\bullet} := t^{\bullet\bullet}\{x/u^{\bullet\bullet}\}$$

Developments are thus defined by induction on terms, but it is well-known that the function ${}^{\bullet\bullet}$ can also be defined in a more operational way. Let $\mathtt{B}$ be the rewriting rule $(\lambda x.t)u \to_{\mathtt{B}} t[x/u]$, which is the restriction of our $\mathtt{dB}$-rule to a *proximity* action. This relation is trivially *complete* so that we use $\mathtt{B}(t)$ for the (unique) $\mathtt{B}$-nf of the term $t$.

**Corollary 3.** *Let $t \in \lambda$-term. Then $t^\circ = \mathtt{j}(\mathtt{B}(t))$.*

*Proof.* By induction on $t$.

- Case $t = x$. Then $x^\circ = x = x^{\bullet\bullet} = \mathtt{j}(\mathtt{B}(x))$.
- Case $t = \lambda x.u$. Then $(\lambda x.u)^\circ = \lambda x.u^\circ =_{i.h.} \lambda x.u^{\bullet\bullet} = \lambda x.\mathtt{j}(\mathtt{B}(u))$.
- Case $T = u\ v$, where $t \neq \lambda$. We then have $(u\ v)^\circ = u^\circ\ v^\circ =_{i.h.} u^{\bullet\bullet}\ v^{\bullet\bullet} = (u\ v)^{\bullet\bullet}$. Similarly as $u\ v$ is not a redex $\mathtt{j}(\mathtt{B}(u\ v)) = \mathtt{j}(\mathtt{B}(u))\ \mathtt{j}(\mathtt{B}(v))$ and we conclude using the i.h.
- Case $t = (\lambda x.u)v$. We have $t^\circ = u^\circ\{x/v^\circ\} =_{i.h.} u^{\bullet\bullet}\{x/v^{\bullet\bullet}\} = ((\lambda x.u)v)^{\bullet\bullet} = t^{\bullet\bullet}$. Similarly $u^\circ\{x/v^\circ\} =_{i.h.} \mathtt{j}(\mathtt{B}(u))\{x/\mathtt{j}(\mathtt{B}(v))\} = \mathtt{j}(\mathtt{B}(u)[x/\mathtt{B}(v)]) = \mathtt{j}(\mathtt{B}(u\,v))$.

Developments can be extended to **L-developments** which also reduce created redexes of type 1 and 2 and are always finite. The **result of a full L-development** of a $\lambda$-term is unique and admits the following inductive definition [17]:

$$\begin{array}{lll} x^{\circ\circ} & := x & \\ (\lambda x.t)^{\circ\circ} & := \lambda x.t^{\circ\circ} & \\ (t\ u)^{\circ\circ} & := t^{\circ\circ}\ u^{\circ\circ} & \text{if } t^{\circ\circ} \neq \lambda \\ (t\ u)^{\circ\circ} & := t_1\{x/u^{\circ\circ}\} & \text{if } t^{\circ\circ} = \lambda x.t_1 \end{array}$$

Remark that $t^{\circ\circ} \neq \lambda$ implies $t \neq \lambda$.

Let us recover $t^{\circ\circ}$ by means of our language $\lambda\mathtt{j}$. The key to operationally describe the first type of creation is the *distance* dB-rule, whose (unique) nf will be noted $\mathtt{dB}(t)$. Replacing our definition of development $\mathtt{j}(\mathtt{B}(t))$ with $\mathtt{j}(\mathtt{dB}(t))$ gives: $\mathtt{dB}(((\lambda x.\lambda y.t)u)v) =$

$$\mathtt{dB}((\lambda y.t)[x/u]\ v) = \mathtt{dB}(t[y/v][x/u]) = \mathtt{dB}(t)[y/\mathtt{dB}(v)][x/\mathtt{dB}(u)]$$

Then, computing jumps, we get: $\mathtt{j}(\mathtt{dB}(((\lambda x.\lambda y.t)u)v))$:

$$\mathtt{j}(\mathtt{dB}(t)[y/\mathtt{dB}(v)][x/\mathtt{dB}(u)]) = \mathtt{j}(\mathtt{dB}(t))\{x/\mathtt{j}(\mathtt{dB}(u))\}\{y/\mathtt{j}(\mathtt{dB}(v))\}$$

$$\mathtt{j}(M) = \mathtt{j}(\mathtt{dB}(t))\{x/\mathtt{j}(\mathtt{dB}(u))\}\{y/\mathtt{j}(\mathtt{dB}(v))\}$$

And we are done. Now, to specify L-developments within our language $\lambda\mathtt{j}$ we also need to capture the second type of creation. We would therefore need to use $\mathtt{dB} \cup \mathtt{d} \cup \mathtt{w}$ instead of dB, but our (distance) d-rule turns out to be too powerful since created redexes of type 3 would also be captured as shown by the term $(\lambda x.x\ t)(\lambda y.u)$, where $x \notin \mathtt{fv}(t)$. Thus, the reduction d is restricted to act only on variables, written md (for *minimal dereliction*), so that $\rightarrow_{\mathtt{md}}$ is the context closure of the rule $x[x/u] \rightarrow u$. We then let A be the relation $\mathtt{dB} \cup \mathtt{md} \cup \mathtt{w}$.

**Lemma 12.** *The reduction relation* $\rightarrow_{\mathtt{A}}$ *is complete.*

*Proof.* Termination of A is straightforward. Confluence follows from local confluence (straightforward by case-analysis) and Newman's Lemma.

Interestingly, $\rightarrow_{\mathtt{A}}$ cannot be weakened to $\rightarrow_{\mathtt{dB}\cup\mathtt{md}}$ as illustrated by the term $s = ((\lambda x.((\lambda y.x)\ t))\ \lambda z.z)\ u$. Now, to prove that $\mathtt{j}(\mathtt{A}(\cdot))$ is an L-development some technical lemmas are needed.

**Lemma 13.** *A term in* A*-nf has either a* V-Form $x$*, an* A-Form $(u\ v)[x_i/s_i]^1_n$*, or an* L-Form $(\lambda x.v)[x_i/s_i]^1_n$*, where* $u, v, \overline{s}^1_n$ $(n \geq 0)$ *are* A*-nfs,* $u$ *is not an* L-Form*, and* $|M[x_i/s_i]^1_j|_{x_{j+1}} \geq 1$ *for* $j = 1, \ldots, n-1$ *and* ($M = u\ v$ *or* $M = \lambda x.v$).

*Proof.* By induction on $t$.

- If $t$ is a variable or an abstraction $\lambda x.u$, then we are done, since $u$ is necessarily an A-nf.
- If $t$ is an application $t_1\ t_2$, then $t_1$ and $t_2$ are necessarily A-nfs. The subterm $t_1$ cannot have the shape $(\lambda y.t_1')[y_i/v_i]^1_k$ otherwise $t$ would be dB-reducible.

- If $t$ is a closure, it has the general form $u[x_i/v_i]^1_n$ $(n \geq 1)$ where $u, \overline{v}^1_n$ are $\mathtt{A}$-nfs. We reason by induction on $n$.
  If $n = 1$, then $|u|_{x_1} \geq 1$ because $t$ is in $\mathtt{w}$-nf. Also, $u \neq x_1$ because $t$ is in $\mathtt{md}$-nf. Thus, $u$ is an application or an abstraction. As before, if $u$ is an application $t_1\ t_2$ the subterm $t_1$ cannot have the shape $(\lambda y.t'_1)[y_i/v_i]^1_k$ otherwise $t$ would be $\mathtt{dB}$-reducible.
  If $n > 1$, then $U = u[x_i/v_i]^1_{n-1}$ already verifies the statement by the i.h. We still need to show that $|U|_{x_n} \geq 1$, which is straightforward since $|U|_{x_n} = 0$ would imply that the term is not in $\mathtt{w}$-nf.

**Lemma 14.** *If* $\mathtt{j}(\mathtt{A}(T)) = \lambda x.t$ *then* $\mathtt{A}(T)$ *is an* $\mathtt{L\text{-}Form}$ *.*

*Proof.* By Lemma 13 $\mathtt{A}(T)$ is a $\mathtt{V\text{-}Form}$, an $\mathtt{A\text{-}Form}$, or an $\mathtt{L\text{-}Form}$. In the two first cases $\mathtt{j}(\mathtt{A}(T))$ cannot be a $\lambda$-abstraction, so that we trivially conclude.

**Lemma 15.** *Let* $t = (\lambda x.v)[x_i/v_i]^1_n$ $(n \geq 0)$ *and* $u$ *be* $\mathtt{A}$*-nfs. Then the sequence* $tu \to^+_{\mathtt{A}} \mathtt{A}(tu)$ *can be decomposed into* $tu \to_{\mathtt{dB}} v[x/u][x_i/v_i]^1_n \to^*_{\mathtt{md}\cup\mathtt{w}} \mathtt{A}(tu)$*.*

*Proof.* By Lemma 12 every term $tu$ has a unique $\mathtt{A}$-nf s.t. $tu \to^*_{\mathtt{A}} \mathtt{A}(tu)$. Then, if $tu \to^*_{\mathtt{A}} t'$ for some $t'$ in $\mathtt{A}$-nf, then $t'$ is necessarily $\mathtt{A}(tu)$. Thus, since $tu \to_{\mathtt{dB}} v[x/u][x_i/v_i]^1_n = s$, it is sufficient to show that $s$ can be $(\mathtt{md} \cup \mathtt{w})$-reduced to a $\mathtt{A}$-nf. We proceed by cases.

- If $x \notin \mathtt{fv}(v)$, then $s \to_{\mathtt{w}} v[x_i/v_i]^1_n$. We show that $v[x_i/v_i]^1_n \to^*_{\mathtt{md}\cup\mathtt{w}} v'$, for some $v'$ in $\mathtt{A}$-nf. We proceed by induction on $n$.
  If $n = 0$, then $v \to^*_{\mathtt{md}} v$, which is a $\mathtt{A}$-nf.
  If $n > 0$, then by the i.h. $v[x_i/v_i]^1_n \to^*_{\mathtt{md}\cup\mathtt{w}} v'[x_n/v_n]$, with $v'$ in $\mathtt{A}$-nf.
  If $x_n \notin \mathtt{fv}(v')$, then $v'[x_n/v_n] \to_{\mathtt{w}} v'$ and we are done.
  If $x_n = v'$, then $v'[x_n/v_n] \to_{\mathtt{md}} v_n$ and we are also done.
  If $x_n \in \mathtt{fv}(v')$ and $x_n \neq v'$, then $v'[x_n/v_n]$ is in $\mathtt{A}$-nf and we are done.
- If $x \in \mathtt{fv}(v)$ and $x = v$, then $x_1 \notin \mathtt{fv}(\lambda x.v)$ so that $t$ is $\mathtt{w}$-reducible which leads to a contradiction with the hypothesis.
- If $x \in \mathtt{fv}(v)$ and $x \neq v$, then $s$ is in $\mathtt{A}$-nf.

**Corollary 4.** *Let* $t$ *be a* $\lambda$*-term. Then* $t^{\circ\circ} = \mathtt{j}(\mathtt{A}(t))$*.*

*Proof.* By induction on $t$.

- Case $t = x$. Then $x^{\circ\circ} = x = \mathtt{j}(\mathtt{A}(x))$.
- Case $t = \lambda x.u$. Then $(\lambda x.u)^{\circ\circ} = \lambda x.u^{\circ\circ} =_{i.h.} \lambda x.\mathtt{j}(\mathtt{A}(u)) = \mathtt{j}(\mathtt{A}(t))$.
- Case $t = u\ v$, where $u^{\circ\circ} \neq \lambda$. By the i.h. $u^{\circ\circ} = \mathtt{j}(\mathtt{A}(u))$, hence $\mathtt{j}(\mathtt{A}(u)) \neq \lambda$ and $\mathtt{A}(u) \neq \lambda$. We then have $(u\ v)^{\circ\circ} = u^{\circ\circ}\ v^{\circ\circ} =_{i.h.} \mathtt{j}(\mathtt{A}(u))\ \mathtt{j}(\mathtt{A}(v)) = \mathtt{j}(\mathtt{A}(u)\ \mathtt{A}(v)) = \mathtt{j}(\mathtt{A}(u\ v))$.
- Case $t = u\ v$, where $u^{\circ\circ} = \lambda x.u_1 =_{i.h.} \mathtt{j}(\mathtt{A}(u))$ and $v^{\circ\circ} = \mathtt{j}(\mathtt{A}(v))$. By Lemma 14 $\mathtt{A}(u)$ is an $\mathtt{L\text{-}Form}$ $(\lambda x.u_2)[x_i/s_i]^1_n$ so that in particular $u_1 = \mathtt{j}(u_2[x_i/s_i]^1_n)$. Hence $\mathtt{A}(u)\ \mathtt{A}(v) \to_{\mathtt{dB}} u_2[x/\mathtt{A}(v)][x_i/s_i]^1_n = s$. By Lemma 15 $\mathtt{A}(u)\ \mathtt{A}(v) \to_{\mathtt{dB}} s \to^*_{\mathtt{md}\cup\mathtt{w}} \mathtt{A}(\mathtt{A}(u)\ \mathtt{A}(v)) = \mathtt{A}(u\ v)$ so that we get:

$$\begin{array}{ll}
\mathtt{j}(\mathtt{A}(u\ v)) & = \\
\mathtt{j}((\mathtt{md}\cup\mathtt{w})(s)) & = \\
\mathtt{j}(s) & = \\
\mathtt{j}(u_2[x/\mathtt{A}(v)][x_i/s_i]^1_n) & = \\
\mathtt{j}(u_2)\{x/\mathtt{j}(\mathtt{A}(v))\}\{x_i/\mathtt{j}(s)_i\}^1_n & =_{L.\ 2} \\
\mathtt{j}(u_2)\{x_i/\mathtt{j}(s)_i\}^1_n\{x/\mathtt{j}(\mathtt{A}(v))\} & = \\
\mathtt{j}(u_2[x_i/s_i]^1_n)\{x/\mathtt{j}(\mathtt{A}(v))\} & =_{i.h.} \\
u_1\{x/v^{\circ\circ}\} & = (u\ v)^{\circ\circ}
\end{array}$$

It is now natural to relax the previous relation $\mathtt{A}$ from $\mathtt{dB}\cup\mathtt{md}\cup\mathtt{w}$ to $\mathtt{dB}\cup\mathtt{d}\cup\mathtt{w}$, in other words, to also allow unrestricted $\mathtt{d}$-steps. Thus $\mathtt{L}$-developments are extended to **$\mathtt{XL}$-developments**, which also allow linear creations of type 3. Completeness of this extended notion is stated as follows:

**Lemma 16.** *The reduction relation $\rightarrow_{\mathtt{dB}\cup\mathtt{d}\cup\mathtt{w}}$ is complete.*

*Proof.* Since $\mathtt{dB}\cup\mathtt{d}\cup\mathtt{w}\subseteq\mathtt{j}$, which is terminating (Lemma 8), then $\mathtt{dB}\cup\mathtt{d}\cup\mathtt{w}$ is terminating as well. To show confluence it is sufficient to show local confluence, which is straightforward by case-analysis, then apply Newman's Lemma.

The result of a **full $\mathtt{XL}$-development** of a $\lambda$-term $t$, noted $t^{\circ\circ\circ}$, is defined by $\mathtt{j}((\mathtt{dB}\cup\mathtt{d}\cup\mathtt{w})(t))$ where $(\mathtt{dB}\cup\mathtt{d}\cup\mathtt{w})(t)$ denotes the (unique) $(\mathtt{dB}\cup\mathtt{d}\cup\mathtt{w})$-nf of $t$. This notion extends $\mathtt{L}$-developments in a deterministic way, *i.e.* provides a complete reduction relation for $\lambda$-terms, more liberal than $\mathtt{L}$-developments.

It is well known that every **affine** $\lambda$-term $t$ (*i.e.* a term where no variable has more than one occurrence in $t$) is $\beta$-strongly normalising (the number of constructors strictly diminishes with each step). Moreover, $\beta$-reduction of affine terms can be performed in $\lambda\mathtt{j}$ using only $\mathtt{dB}\cup\mathtt{d}\cup\mathtt{w}$, *i.e.* $\beta$-nf$(t) = (\mathtt{dB}\cup\mathtt{d}\cup\mathtt{w})(t)$. Thus:

**Corollary 5.** *Let $t$ be an affine $\lambda$-term. Then $t^{\circ\circ\circ} = \beta$-nf$(t)$.*

We hope that our extended notion of $\mathtt{XL}$-development can be applied to obtain more expressive solutions for higher-order matching problems, which arise for example in higher-order logic programming, logical frameworks, program transformations, etc. Indeed, the approach of higher-order matching in *untyped* frameworks [7, 5], which currently uses $\mathtt{L}$-developments, may be improved using $\mathtt{XL}$-developments, as suggested by example (1) at the beginning of this section.

## 5 Bisimilar Terms

The simplicity of the $\lambda\mathtt{j}$-calculus naturally suggests the study of some operational equivalence which should equate terms that differ only concerning the positioning of their jumps but *behave identically*. For instance, if $y\notin\mathtt{fv}(u)$, then $\lambda y.t[x/u]$ and $(\lambda y.t)[x/u]$ behave equivalently: there is a bijection between their redexes and their reducts, *i.e.* they are *bisimilar*. This idea is reminiscent of Regnier's equivalence on $\lambda$-terms [25], here written $\sigma^R$:

$$\begin{array}{lll}
(\lambda x.\lambda y.t)\ u \equiv_{\sigma^R_1} \lambda y.((\lambda x.t)\ u) & & \text{if } y\notin\mathtt{fv}(u) \\
(\lambda x.t\ v)\ u \equiv_{\sigma^R_2} (\lambda x.t)\ u\ v & & \text{if } x\notin\mathtt{fv}(v)
\end{array}$$

Reduction of the dB-redexes in the previous equations yields the following $\sigma$-equivalence notion, now on $\lambda\mathtt{j}$-terms:

$$(\lambda y.t)[x/u] \equiv_{\sigma_1} \lambda y.t[x/u] \text{ if } y \notin \mathtt{fv}(u)$$
$$(t\ v)[x/u] \equiv_{\sigma_2} t[x/u]\ v \quad \text{if } x \notin \mathtt{fv}(v)$$

This is not very surprising since $\sigma^R$-equivalence was introduced by noting that the two terms of each equation represent the *same* MELL proof-net modulo multiplicative redexes, which correspond exactly to the dB-redexes of the $\lambda\mathtt{j}$-calculus. Regnier proved that $\sigma^R$-equivalent terms have the same maximal $\beta$-reduction length. However, this does not imply that $\sigma^R$-equivalence is a strong bisimilarity on $\lambda$-terms. Indeed, take $\lambda$-terms $t_0 = ((\lambda x.\lambda y.y)\ z)\ w \equiv_{\sigma_1^R} (\lambda y.((\lambda x.y)\ z))\ w = t_1$. Both share the same $\beta$-normal form $w$ and $\eta_\beta(t_0) = \eta_\beta(t_1)$. Nevertheless, $t_0$ has one redex, while $t_1$ has two redexes, and the redex of $t_1$ involving $w$ has no corresponding redex in $t_0$. They also differ in terms of creation of redexes: the result of the full development of $t_0$ has a *created* redex, while the result of the full development of $t_1$ is the normal form of the term. Our reformulation of $\sigma^R$, however, equates two $\lambda\mathtt{j}$-terms $t'_0$ and $t'_1$ which are strongly bisimilar:

$$t_0 \rightarrow_{\mathtt{dB}} t'_0 = (\lambda y.y)[x/z]\ w \quad \equiv_{\sigma_1} \quad (\lambda y.y[z/x])\ w = t'_1\ {}_{\mathtt{dB}}\!\leftarrow t_1 \tag{2}$$

Actually, bisimulation holds also for permutation of *independent* jumps [13]:

$$t[x/u][y/v] \equiv_{\mathtt{CS}} t[y/v][x/u] \qquad \text{if } y \notin \mathtt{fv}(u)\ \&\ x \notin \mathtt{fv}(v)$$

While CS should naturally remain an equivalence, $\sigma$ has often been restricted to being considered a *reduction* relation [26], for no good reason. Here, we add CS and $\sigma$ to $\lambda\mathtt{j}$ without any trouble, in particular without loosing the PSN property (Corollary 9). The **operational equivalence** relation generated by $\mathtt{o} = \{\alpha, \mathtt{CS}, \sigma_1, \sigma_2\}$ realises a strong bisimulation, proved by induction on $\equiv_{\mathtt{CS},\sigma_1,\sigma_2}$ and using the following preliminary lemma:

**Lemma 17.** *For all $t, t' \in \mathcal{T}$ and substitution $\gamma$*

1. *If $t \equiv_{\mathtt{o}} t'$, then $t\gamma \equiv_{\mathtt{o}} t'\gamma$.*
2. *If $\gamma \equiv_{\mathtt{o}} \gamma'$, then $t\gamma \equiv_{\mathtt{o}} t\gamma'$.*

**Proposition 1 (Strong Bisimulation).** *For all $t, u, u' \in \mathcal{T}$ s.t. $t \equiv_{\mathtt{o}} u \rightarrow_{\lambda\mathtt{j}} u'$ $\exists t'$ s.t. $t \rightarrow_{\lambda\mathtt{j}} t' \equiv_{\mathtt{o}} u'$.*

*Proof.* Here we consider $\equiv_{\mathtt{CS},\sigma_1,\sigma_2}$ as an atomic step of equivalence and rephrase the statement as: for all $t_0, t_1, s_1 \in \mathcal{T}$ and $n > 0$ s.t. $t_0 \equiv^n_{\mathtt{CS},\sigma_1,\sigma_2} t_1 \rightarrow_{\lambda\mathtt{j}} s_1$ there exists $s_0$ s.t. $t_0 \rightarrow_{\lambda\mathtt{j}} s_0 \equiv^*_{\mathtt{CS},\sigma_1,\sigma_2} s_1$. The proof is by induction on $n$. If $n = 1$ then we reason by induction on $t_0 \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t_1$, considered as an atomic step of equivalence:

- Let $t_0 = t[x/u][y/v] \equiv_{\mathtt{CS}} t[y/v][x/u] = t_1$ with $y \notin \mathtt{fv}(u)$ and $x \notin \mathtt{fv}(v)$ . If $t_1 \rightarrow_{\lambda\mathtt{j}} s_1$ because
  - $t \rightarrow_{\lambda\mathtt{j}} t'$ then $t_0 = t[x/u][y/v] \rightarrow_{\lambda\mathtt{j}} t'[x/u][y/v] = s_0 \equiv_{\mathtt{CS}} s_1 = t'[y/v][x/u]$.
  - $u \rightarrow_{\lambda\mathtt{j}} u'$ or $v \rightarrow_{\lambda\mathtt{j}} v'$ then it is similar to the previous case.

- $t[y/v][x/u] \to_{\mathtt{d}} t\{y/v\}[x/u] = s_1$ then $t_0 \to_{\mathtt{d}} t[x/u]\{y/v\} = s_1$.
- $t[y/v][x/u] \to_{\mathtt{c}} t_{[z]_y}[y/v][z/v][x/u] = s_1$ then $t_0 \to_{\mathtt{c}} t_{[z]_y}[x/u][y/v][z/v] \equiv^2_{\mathtt{CS}} s_1$.
- $t[y/v][x/u] \to_{\mathtt{w}} t[x/u] = s_1$ then $t_0 \to_{\mathtt{d}} t[x/u] = s_1$.
- The three cases where it is $[x/u]$ which is reduced are similar to the last three cases.

– Let $t_0 = (\lambda y.t)[x/u] \equiv_{\sigma_1} \lambda y.t[x/u] = t_1$ with $y \notin \mathtt{fv}(u)$. If $t_1 \to_{\lambda\mathtt{j}} s_1$ because
  - $t \to_{\lambda\mathtt{j}} t'$ then $t_0 = (\lambda y.t)[x/u] \to_{\lambda\mathtt{j}} (\lambda y.t')[x/u] = s_0 \equiv_{\sigma_1} s_1 = \lambda y.t'[x/u]$.
  - $u \to_{\lambda\mathtt{j}} u'$ then it is similar to the previous case.
  - $\lambda y.t[x/u] \to_{\mathtt{d}} \lambda y.t\{x/u\} = s_1$ then $t_0 \to_{\mathtt{d}} (\lambda y.t)\{x/u\} = s_1$.
  - $\lambda y.t[x/u] \to_{\mathtt{c}} \lambda y.t_{[z]_x}[z/u][x/u] = s_1$ then $t_0 \to_{\mathtt{c}} (\lambda y.t_{[z]_x})[z/u][x/u] \equiv^2_{\sigma_1} s_1$.
  - $\lambda y.t[x/u] \to_{\mathtt{w}} \lambda y.t = s_1$ then $t_0 \to_{\mathtt{d}} \lambda y.t = s_1$.

– Let $t_0 = (t\ v)[x/u] \equiv_{\sigma_2} t[x/u]\ v = t_1$ with $x \notin \mathtt{fv}(v)$. If $t_1 \to_{\lambda\mathtt{j}} s_1$ because
  - $t \to_{\lambda\mathtt{j}} t'$ then $t_0 = (t\ v)[x/u] \to_{\lambda\mathtt{j}} (t'\ v)[x/u] = s_0 \equiv_{\sigma_2} s_1 = t'[x/u]\ v$.
  - $u \to_{\lambda\mathtt{j}} u'$ or $v \to_{\lambda\mathtt{j}} v'$ then it is similar to the previous case.
  - $t[x/u]\ v \to_{\mathtt{d}} t\{x/u\}\ v = s_1$ then $t_0 \to_{\mathtt{d}} (t\ v)\{x/u\} = s_1$.
  - $t[x/u]\ v \to_{\mathtt{c}} t_{[z]_x}[z/u][x/u]\ v = s_1$ then $t_0 \to_{\mathtt{d}} (t\ v)_{[z]_x}[z/u][x/u] = (t_{[z]_x}\ v)[z/u][x/u] \equiv^2_{\sigma_2} s_1$.
  - $t[x/u]\ v \to_{\mathtt{w}} t\ v = s_1$ then $t_0 \to_{\mathtt{d}} t\ v = s_1$.
  - $t_0 = ((\lambda y.t')\mathtt{L}\ v)[x/u]$ and $t_1 = (\lambda y.t')\mathtt{L}[x/u]\ v \to_{\mathtt{dB}} t'[y/v]\mathtt{L}[x/u] = s_1$. Then, $t_0 = ((\lambda y.t')\mathtt{L}\ v)[x/u] \to_{\mathtt{dB}} t'[y/v]\mathtt{L}[x/u] = s_1$

– The inductive cases:
  - If $t_0 = \lambda x.t \equiv_{\mathtt{CS},\sigma_1,\sigma_2} \lambda x.t' = t_1 \to_{\lambda\mathtt{j}} \lambda x.t'' = s_1$ then $t \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t' \to_{\lambda\mathtt{j}} t''$ and by the i.h. there exists $t'''$ s.t. $t \to_{\lambda\mathtt{j}} t''' \equiv^*_{\mathtt{CS},\sigma_1,\sigma_2} t''$. Then, $s_0 = \lambda x.t'''$ is s.t. $t_0 \to_{\lambda\mathtt{j}} s_0 \equiv^*_{\mathtt{CS},\sigma_1,\sigma_2} s_1$.
  - If $t_0 = t\ v \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t'\ v = t_1 \to_{\lambda\mathtt{j}} t'\ v' = s_1$ then $t\ v \to_{\lambda\mathtt{j}} t\ v' \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t'\ v'$. So $s_0 = t\ v'$.
  - The case $t_0 = t\ v \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t\ v' = t_1 \to_{\lambda\mathtt{j}} t'\ v' = s_1$ is analogous to the previous one.
  - If $t_0 = t\ v \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t'\ v = t_1 \to_{\lambda\mathtt{j}} t''\ v = s_1$ then $t \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t' \to_{\lambda\mathtt{j}} t''$ and by the i.h. there exists $t'''$ s.t. $t \to_{\lambda\mathtt{j}} t''' \equiv^*_{\mathtt{CS},\sigma_1,\sigma_2} t''$. Then, $s_0 = t'''\ v$ is s.t. $t_0 \to_{\lambda\mathtt{j}} s_0 \equiv^*_{\mathtt{CS},\sigma_1,\sigma_2} s_1$.
  - The case $t_0 = t\ v \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t\ v' = t_1 \to_{\lambda\mathtt{j}} t\ v'' = s_1$ is analogous to the previous one.
  - If $t_0 = (\lambda y.t)\mathtt{L}\ v \equiv_{\mathtt{CS},\sigma_1,\sigma_2} (\lambda y.t')\mathtt{L}\ v = t_1 \to_{\mathtt{dB}} t'[y/v]\mathtt{L}$ then $(\lambda y.t)\mathtt{L}\ v \to_{\mathtt{dB}} t[y/v]\mathtt{L} \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t'[y/v]\mathtt{L}$.
  - The cases $t_0 = (\lambda y.t)\mathtt{L}\ v \equiv_{\mathtt{CS},\sigma_1,\sigma_2} (\lambda y.t)\mathtt{L}'\ v = t_1 \to_{\mathtt{dB}} t[y/v]\mathtt{L}'$ and $t_0 = (\lambda y.t)\mathtt{L}\ v \equiv_{\mathtt{CS},\sigma_1,\sigma_2} (\lambda y.t)\mathtt{L}\ v' = t_1 \to_{\mathtt{dB}} t[y/v']\mathtt{L}$ are analogous to the previous one.
  - $t_0 = (\lambda y.t)\mathtt{L}\ v \equiv_{\sigma_1} (\lambda y.t[x/u])\mathtt{L}'\ v = t_1 \to_{\mathtt{dB}} t[x/u][y/v]\mathtt{L}'$, where $\mathtt{L} = [x/u]\mathtt{L}'$; As $y \notin \mathtt{fv}(u)$ and $x \notin \mathtt{fv}(v)$ does not contain any variable bounded by $\mathtt{L}$, then $t_0 = (\lambda y.t)\mathtt{L}\ v \to_{\mathtt{dB}} t[y/v]\mathtt{L} = t[y/v][x/u]\mathtt{L}' \equiv_{\mathtt{CS}} t[x/u][y/v]\mathtt{L}'$.
  - The case $t_0 = (\lambda y.t[x/u])\mathtt{L}'v \equiv_{\sigma_1} (\lambda y.t)\mathtt{L}\ v = t_1 \to_{\mathtt{dB}} t[y/v]\mathtt{L}'$ , where $\mathtt{L} = [x/u]\mathtt{L}'$ is analogous to the previous one.
  - $t_0 = (\lambda y.t)\mathtt{L}\ v \equiv_{\sigma_2} ((\lambda y.t)\mathtt{L}'\ v)[x/u] = t_1 \to_{\mathtt{dB}} t[y/v]\mathtt{L}'[x/u] = t[y/v]\mathtt{L}$, where $\mathtt{L} = \mathtt{L}'[x/u]$; then, $t_0 = (\lambda y.t)\mathtt{L}\ v \to_{\mathtt{dB}} t[y/v]\mathtt{L} = s_1$.

- The case $t_0 = ((\lambda y.t)\mathtt{L}\ v)[x/u] \equiv_{\sigma_2} ((\lambda y.t)\mathtt{L}[x/u]\ v) = t_1$ has been already treated before.
- If $t_0 = t\ [x/u] \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t'\ [x/u] = t_1 \to_{\lambda\mathtt{j}} t'\ [x/u'] = s_1$ then $t\ [x/u] \to_{\lambda\mathtt{j}} t\ [x/u'] \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t'\ [x/u']$.
- The case $t_0 = t\ [x/u] \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t\ [x/u'] = t_1 \to_{\lambda\mathtt{j}} t'\ [x/u'] = s_1$ is analogous to the previous one.
- If $t_0 = t\ [x/u] \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t'\ [x/u] = t_1 \to_{\lambda\mathtt{j}} t''\ [x/u] = s_1$ then $t \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t' \to_{\lambda\mathtt{j}} t''$ and by the i.h. there exists $t'''$ s.t. $t \to_{\lambda\mathtt{j}} t''' \equiv^*_{\mathtt{CS},\sigma_1,\sigma_2} t''$. Then, $s_0 = t'''[x/u]$ is s.t. $t_0 \to_{\lambda\mathtt{j}} s_0 \equiv^*_{\mathtt{CS},\sigma_1,\sigma_2} s_1$.
- The case $t_0 = t\ [x/u] \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t\ [x/u'] = t_1 \to_{\lambda\mathtt{j}} t\ [x/u''] = s_1$ is analogous to the previous one.
- If $t_0 = t\ [x/u] \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t\ [x/u'] = t_1 \to_{\mathtt{w}} t$ then $t_0 \to_{\mathtt{w}} t = s_1$.
- If $t_0 = t\ [x/u] \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t'\ [x/u] = t_1 \to_{\mathtt{w}} t$ then $t_0 \to_{\mathtt{w}} t \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t' = s_1$.
- If $t_0 = t\ [x/u] \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t\ [x/u'] = t_1 \to_{\mathtt{c}} t_{[y]_x}[x/u'][y/u']$ then $t_0 \to_{\mathtt{c}} t = t_{[y]_x}[x/u][y/u] \equiv^2_{\mathtt{CS},\sigma_1,\sigma_2} t_{[y]_x}[x/u'][y/u']$.
- If $t_0 = t\ [x/u] \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t'\ [x/u] = t_1 \to_{\mathtt{c}} (t')_{[y]_x}[x/u][y/u]$ observe that the equivalences does not change the number of occurrences and the renaming does not change the position of substitutions so that we can write $(t')_{[y]_x} = (t_{[y]_x})'$ and conclude with $t_0 \to_{\mathtt{c}} t = t_{[y]_x}[x/u][y/u] \equiv_{\mathtt{CS},\sigma_1,\sigma_2} (t_{[y]_x})'[x/u][y/u]$.
- If $t_0 = t\ [x/u] \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t\ [x/u'] = t_1 \to_{\mathtt{d}} t\{x/u'\}$. Then, $t\ [x/u] \to_{\mathtt{d}} t\{x/u\} \equiv^*_{\mathtt{CS},\sigma_1,\sigma_2} t\{x/u'\}$ where the last equivalence is obtained by lemma 17:2.
- If $t_0 = t\ [x/u] \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t'\ [x/u] = t_1 \to_{\mathtt{d}} t'\{x/u\}$. Then, $t\ [x/u] \to_{\mathtt{d}} t\{x/u\} \equiv^*_{\mathtt{CS},\sigma_1,\sigma_2} t'\{x/u\}$ where the last equivalence is obtained by lemma 17:1.

If $n > 1$ then we have $t_0 \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t'_0 \equiv^{n-1}_{\mathtt{CS},\sigma_1,\sigma_2} t_1 \to_{\lambda\mathtt{j}} s_1$. By the i.h. there exists $s'_0$ s.t. $t'_0 \to_{\lambda\mathtt{j}} s'_0 \equiv^*_{\mathtt{CS},\sigma_1,\sigma_2} s_1$. Then, applying the i.h. once more to $t_0 \equiv_{\mathtt{CS},\sigma_1,\sigma_2} t'_0 \to_{\lambda\mathtt{j}} s'_0$ (where the equivalence step is atomic), we get $s_0$ s.t. $t_0 \to_{\lambda\mathtt{j}} s_0 \equiv^*_{\mathtt{CS},\sigma_1,\sigma_2} s'_0 \equiv^*_{\mathtt{CS},\sigma_1,\sigma_2} s_1$.

Such bisimulation implies that two o-equivalent terms share the same maximal reduction length. Moreover, the strong bisimulation would not hold without distance rules. Indeed, the two $\sigma_1$-equivalent terms $t'_0$ and $t'_1$ in (2) do not have the same B-redexes but the same dB-redexes.

## 6 (De)composing Substitutions

Explicit substitution (ES) calculi may or may not include rewriting rules to *explicitly* compose substitutions. One often adds them to recover confluence on terms with metavariables. However, naïve rules may break the PSN property, so that *safe* composition rules are needed to recover both PSN and confluence on terms with metavariables [13]. The $\lambda\mathtt{j}$-calculus is peculiar as it allows to compose substitutions, but only *implicitly*. Indeed, a term $t[x/u][y/v]$ s.t. $y \in \mathtt{fv}(u)$ & $y \in \mathtt{fv}(t)$ reduces in various steps to $t[x/u\{y/v\}][y/v]$, but not to the explicit composition $t[x/u[y/v]][y/v]$. One of the aims of this section is adding *explicit composition* to $\lambda\mathtt{j}$ keeping PSN and confluence.

The second aim of this section concerns *explicit decomposition*. Indeed, some calculi [24, 20, 28, 11, 10] explicitly *decompose* substitutions, *i.e.* reduce $t[x/u[y/v]]$ to $t[x/u][y/v]$. We show that even in such a case PSN and confluence still hold.

Composition (boxing) and decomposition (unboxing) are dual systems:

| The **Boxing** system $\mathtt{b}$ | The **Unboxing** system $\mathtt{u}$ |
|---|---|
| if $x \notin \mathtt{fv}(t)$ & $x \in \mathtt{fv}(v)$ : | if $x \notin \mathtt{fv}(t)$ & $x \in \mathtt{fv}(v)$ : |
| $(t\ v)[x/u] \ \to_{\mathtt{ab}} t\ v[x/u]$ | $t\ v[x/u] \ \to_{\mathtt{au}} (t\ v)[x/u]$ |
| $t[y/v][x/u] \to_{\mathtt{sb}} t[y/v[x/u]]$ | $t[y/v[x/u]] \to_{\mathtt{su}} t[y/v][x/u]$ |

The boxing system reflects the commutative box-box rule of Linear Logic, the unboxing system is obtained by reversing its rules. Moreover, we consider the system modulo the $\mathtt{o}$-equivalence. Choosing a particular orientation for $\sigma_1$ and $\sigma_2$ leads to a full set of propagating rules, that is, something closer to traditional ES calculi. We prefer, however, to work modulo an equivalence to obtain a more general result. Remark that the constraint $x \notin \mathtt{fv}(t)$ for the unboxing rules does not limit their applicability, as it can always be satisfied through $\alpha$-equivalence.

*Digression.* It is natural to wonder if one could also work modulo (de)composition, *i.e.* adding two more general axioms:

$$\begin{array}{lll}(t\ v)[x/u] & \equiv_{\sigma_3} t\ v[x/u] & \text{if } x \notin \mathtt{fv}(t) \\ t[y/v][x/u] & \equiv_{\sigma_4} t[y/v[x/u]] & \text{if } x \notin \mathtt{fv}(t)\end{array}$$

The answer is no, as these last two congruences break the PSN property, if naïvely added. For example: let $u = (z\ z)[z/y]$, then

$$\begin{array}{ll}t = u[x/u] = (z\ z)[z/y][x/u] & \equiv_{\sigma_4} (z\ z)[z/y[x/u]] \quad \to_{\mathtt{c}} \\ (z_1\ z_2)[z_1/y[x/u]][z_2/y[x/u]] & \to^{+}_{\mathtt{d}} y[x/u]\ (y[x/u]) \ \equiv_{\sigma_2,\sigma_3,\alpha} \\ (y\ y)[x_1/u][x/u] & \equiv_{\sigma_4} (y\ y)[x_1/u[x/u]]\end{array}$$

*i.e.* $t$ reduces to a term containing $t$. Now, take $(\lambda x.((\lambda z.zz)y))\ ((\lambda z.zz)y) \in \mathcal{SN}_\beta$ which reduces to $t$, so that it is no longer strongly normalising in the $\lambda\mathtt{j}$-calculus extended by the five previous equations $\{\mathtt{CS}, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$.

Such a counter-example can be avoided imposing the constraint "$x \in \mathtt{fv}(v)$" to $\sigma_3$ and $\sigma_4$ (note that such constraint is also found in the definition of the boxing system). Nevertheless, $\lambda\mathtt{j}$-reduction modulo the *constrained* equivalences $\{\mathtt{CS}, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is an incredibly subtle and complex relation. For instance, $\mathtt{w}$-steps cannot be postponed, nor can the use of equivalences. Two natural canonical representations of the equivalence classes are obtained by pushing jumps towards the variables, or as far away from them as possible. None of them is stable by reduction, so working with equivalence classes is impossible. The PSN property for this calculus, if it holds, is very challenging.

One of the difficulties is that the equivalence $\{\mathtt{CS}, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is not a bisimulation: observe that the reducts $(xx_1)[x/y[y/z]][x_1/y[y/z]]$ of $t_2 = (xx)[x/y[y/z]]$ and $(xx_1)[x/y][x_1/y][y/z]$ of $t_3 = (xx)[x/y][y/z]$ are no longer equivalent. Nevertheless, $t_2$ and $t_3$ share the same normal form, and thus are still operationally equivalent, but in a weaker sense.

From here on we use the letter $\mathtt{p}$ to denote a **parameter** which represents any of the propagation systems $\{\mathtt{b},\mathtt{u}\}$. For every $\mathtt{p} \in \{\mathtt{b},\mathtt{u}\}$ we consider its associated **structural reduction system** $\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}$, written $\lambda\mathtt{j}_\mathtt{b}/\mathtt{o}$ and $\lambda\mathtt{j}_\mathtt{u}/\mathtt{o}$ respectively, defined by the reduction relation $\mathtt{dB} \cup \mathtt{j} \cup \mathtt{p}$ modulo the equivalence relation $\mathtt{o}$, a relation which is denoted by $(\mathtt{dB} \cup \mathtt{j} \cup \mathtt{p})/\mathtt{o}$. Both structural systems have good properties.

**Theorem 4 (Confluence Modulo).** *For all* $t_1, t_2 \in \mathcal{T}$*, if* $t_1 \equiv_\mathtt{o} t_2$ *and* $t_i \rightarrow^*_{\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}} u_i$ $(i = 1,2)$*, then* $\exists v_i$ $(i = 1,2)$ *s.t.* $u_i \rightarrow^*_{\lambda\mathtt{j}} v_i$ $(i = 1,2)$ *and* $v_1 \equiv_\mathtt{o} v_2$*.*

*Proof.* Straightforward, by interpreting $t$ into $\mathtt{j}(t)$ and using Theorem 1.

To prove PSN for $\lambda\mathtt{j}_\mathtt{b}/\mathtt{o}$ and $\lambda\mathtt{j}_\mathtt{u}/\mathtt{o}$ it is sufficient, according to Theorem 2, to show the **IE** property. However, a simple inductive argument like the one used for $\lambda\mathtt{j}$-reduction relation does no longer work. Therefore we shall show the **IE** property by adapting the technique in [13]. This has proven a challenging venture, so that this section presents the perhaps most important technical achievement in this paper. We split the proof into the following steps:

1. Define a labelling to mark some $\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}$-strongly normalising terms used within jumps. Thus for example $t[\![x/u]\!]$ means that $u \in \mathcal{T}$ and $u \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}}$.
2. Enrich the original $\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}$-reduction system with a relation used only to propagate terminating labelled jumps. Let $\mathcal{J}_\mathtt{p}/\mathtt{O}$ be the resulting calculus.
3. Show that $u \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}}$ and $t\{x/u\}\overline{v}^1_n \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}}$ imply $t[\![x/u]\!]\overline{v}^1_n \in \mathcal{SN}_{\mathcal{J}_\mathtt{p}/\mathtt{O}}$.
4. Show that $t[\![x/u]\!]\overline{v}^1_n \in \mathcal{SN}_{\mathcal{J}_\mathtt{p}/\mathtt{O}}$ implies $t[x/u]\overline{v}^1_n \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}}$.

In Sections 6.1 and 6.2 points 1 and 2 are developed, while Section 6.3 deals with points 3 and 4.

### 6.1 The Labelled Systems

Each labelled system is defined by a set of labelled terms together with a set of reduction rules and axioms.

**Definition 1 (Labelled Terms).** *Let* $\mathtt{p} \in \{\mathtt{b},\mathtt{u}\}$*. The set* $\mathbb{T}_\mathtt{p}$ *of* **labelled p-terms** *is generated using the following grammar:*

$$t ::= x \mid tt \mid \lambda x.t \mid t[x/t] \mid t[\![x/v]\!] \ (v \in \mathcal{T} \cap \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}})$$

Now consider the following reduction subsystems:

The **Labelled Equations** $\underline{\mathtt{CS}}$:

$$\begin{array}{llll}
t\llbracket x/u\rrbracket[y/v] & \equiv_{\underline{\mathtt{CS}}_1} & t[y/v]\llbracket x/u\rrbracket & \text{if } y \notin \mathtt{fv}(u) \ \&\ x \notin \mathtt{fv}(v) \\
t\llbracket x/u\rrbracket\llbracket y/v\rrbracket & \equiv_{\underline{\mathtt{CS}}_2} & t\llbracket y/v\rrbracket\llbracket x/u\rrbracket & \text{if } y \notin \mathtt{fv}(u) \ \&\ x \notin \mathtt{fv}(v)
\end{array}$$

The **Labelled Equations** $\underline{\sigma}$:

$$\begin{array}{llll}
(\lambda y.t)\llbracket x/u\rrbracket & \equiv_{\underline{\sigma}_1} & \lambda y.t\llbracket x/u\rrbracket & \text{if } y \notin \mathtt{fv}(u) \\
(tv)\llbracket x/u\rrbracket & \equiv_{\underline{\sigma}_2} & t\llbracket x/u\rrbracket v & \text{if } x \notin \mathtt{fv}(v)
\end{array}$$

The **Labelled Jumping** system $\underline{\mathtt{j}}$:

$$\begin{array}{llll}
t\llbracket x/u\rrbracket & \rightarrow_{\underline{\mathtt{w}}} & t & \text{if } |t|_x = 0 \\
t\llbracket x/u\rrbracket & \rightarrow_{\underline{\mathtt{d}}} & t\{x/u\} & \text{if } |t|_x = 1 \\
t\llbracket x/u\rrbracket & \rightarrow_{\underline{\mathtt{c}}} & t_{[y]_x}\llbracket x/u\rrbracket\llbracket y/u\rrbracket & \text{if } |t|_x \geq 2
\end{array}$$

The **Labelled Boxing** system $\underline{\mathtt{b}}$:

$$\begin{array}{llll}
(tv)\llbracket x/u\rrbracket & \rightarrow_{\underline{\mathtt{ab}}} & tv\llbracket x/u\rrbracket & \text{if } x \notin \mathtt{fv}(t) \ \&\ x \in \mathtt{fv}(v) \\
t[y/v]\llbracket x/u\rrbracket & \rightarrow_{\underline{\mathtt{sb}}} & t[y/v\llbracket x/u\rrbracket] & \text{if } x \notin \mathtt{fv}(t) \ \&\ x \in \mathtt{fv}(v)
\end{array}$$

The **Labelled Unboxing** system $\underline{\mathtt{u}}$:

$$\begin{array}{llll}
tv\llbracket x/u\rrbracket & \rightarrow_{\underline{\mathtt{au}}} & (tv)\llbracket x/u\rrbracket & \text{if } x \in \mathtt{fv}(v) \\
t[y/v\llbracket x/u\rrbracket] & \rightarrow_{\underline{\mathtt{su}}_1} & t[y/v]\llbracket x/u\rrbracket & \text{if } x \in \mathtt{fv}(v) \\
t\llbracket y/v[x/u]\rrbracket & \rightarrow_{\underline{\mathtt{su}}_2} & t\llbracket y/v\rrbracket\llbracket x/u\rrbracket & \text{if } x \in \mathtt{fv}(v)
\end{array}$$

The **Generalised** $\mathtt{dB}$ rule:

$$(\lambda x.t)\mathbb{L}u \quad \rightarrow_{\mathtt{gdB}} \quad t[x/u]\mathbb{L}$$

where $\mathbb{L}$ is a list of jumps, some of which, potentially all, may be labelled. Note that $\mathtt{dB}$-reduction on the set $\mathcal{T}$ just is a particular case of $\mathtt{gdB}$-reduction on $\mathbb{T}_{\mathtt{p}}$. The **equivalence relation** $\underline{\alpha}$ (resp. $\underline{\mathtt{o}}$) is generated by axiom $\alpha$ (resp. $\{\underline{\alpha}, \underline{\mathtt{CS}}, \underline{\sigma}\}$) on labelled terms. The **equivalence relation** $\mathtt{O}$ is generated by $\mathtt{o} \cup \underline{\mathtt{o}}$. The **reduction relation** $\mathcal{J}_{\mathtt{p}}$ (resp. $\mathcal{J}_{\mathtt{p}}/\mathtt{O}$) is generated by ($\mathtt{gdB} \cup \mathtt{j} \cup \underline{\mathtt{j}} \cup \mathtt{p} \cup \underline{\mathtt{p}}$) (resp. $\mathtt{gdB} \cup \mathtt{j} \cup \underline{\mathtt{j}} \cup \mathtt{p} \cup \underline{\mathtt{p}}$ modulo $\mathtt{O}$). The relation $\mathcal{J}_{\mathtt{p}}$ can be understood as the union of two disjoint reduction relations, respectively called forgettable and persistent. Forgettable reductions do not create persistent redexes, and they are strongly normalising (Lemmas 21 and 22). These two facts imply that termination of $\mathcal{J}_{\mathtt{p}}$ does not depend on its forgettable subsystem.

The **forgettable** reduction relation $\rightarrow_{\mathtt{Fp}}$:

**Action on labelled jumps:** If $t \rightarrow_{\underline{\mathtt{j}},\underline{\mathtt{p}}} t'$, then $t \rightarrow_{\mathtt{Fp}} t'$.

**Action Inside labelled jump:** If $v \rightarrow_{\lambda \mathtt{j}_{\mathtt{p}}/\mathtt{o}} v'$, then $u\llbracket x/v\rrbracket \rightarrow_{\mathtt{Fp}} u\llbracket x/v'\rrbracket$.

**Closure by non-labelling contexts:** If $t \rightarrow_{\mathtt{Fp}} t'$, then $tu \rightarrow_{\mathtt{Fp}} t'u$, $ut \rightarrow_{\mathtt{Fp}} ut'$, $\lambda x.t \rightarrow_{\mathtt{Fp}} \lambda x.t'$, $t[x/u] \rightarrow_{\mathtt{Fp}} t'[x/u]$, $u[x/t] \rightarrow_{\mathtt{Fp}} u[x/t']$ and $t\llbracket x/u\rrbracket \rightarrow_{\mathtt{Fp}} t'\llbracket x/u\rrbracket$.

The **persistent** reduction relation $\rightarrow_{\mathtt{Pp}}$:

**Root non-labelling action:** If $t \mapsto_{\mathtt{gdB},\mathtt{j},\mathtt{p}} t'$ (where $\mapsto$ denotes root reduction), then $t \rightarrow_{\mathtt{Pp}} t'$.

**Closure by non-labelling contexts:** If $t \rightarrow_{\mathtt{Pp}} t'$, then $tu \rightarrow_{\mathtt{Pp}} t'u$, $ut \rightarrow_{\mathtt{Pp}} ut'$, $\lambda x.t \rightarrow_{\mathtt{Pp}} \lambda x.t'$, $t[x/u] \rightarrow_{\mathtt{Pp}} t'[x/u]$, $u[x/t] \rightarrow_{\mathtt{Pp}} u[x/t']$ and $t\llbracket x/u\rrbracket \rightarrow_{\mathtt{Pp}} t'\llbracket x/u\rrbracket$.

### 6.2 Well-Formed Labelled Terms

In order to prove that the $\lambda \mathtt{j_p/o}$-calculus enjoys PSN, according to Theorem 2 it is sufficient to show the **IE**-property. The reasoning for that is splitted in two steps: we first show that $u \in \mathcal{SN}_{\lambda \mathtt{j_p/o}}$ and $t\{x/u\}\overline{v}^1_n \in \mathcal{SN}_{\lambda \mathtt{j_p/o}}$ imply $t[\![x/u]\!]\overline{v}^1_n \in \mathcal{SN}_{\mathcal{J}_\mathtt{p}/\mathtt{O}}$ (Corollary 7), thereafter we prove that $t[\![x/u]\!]\overline{v}^1_n \in \mathcal{SN}_{\mathcal{J}_\mathtt{p}/\mathtt{O}}$ implies $t[x/u]\overline{v}^1_n \in \mathcal{SN}_{\lambda \mathtt{j_p/o}}$ (Corollary 8).

The first implication is much more difficult to prove, particularly because termination of the forgettable subsystem $\mathtt{F_p}$, proved using a strictly decreasing measure on labelled terms, is required. This measure is based on the assumption that all terms inside labelled jumps are $\lambda \mathtt{j_p/o}$-strongly normalising w.r.t. the environment in which they are evaluated. Moreover, this property of labelled jumps needs to be preserved by reduction and equivalence.

Unfortunately this is not enough, since labelled terms are not stable by reduction: the labelled term $y[\![y/x\ x]\!][x/\lambda z.zz]$ reduces to $y[\![y/(\lambda z.zz)\ \lambda z.zz]\!]$ which has a non-strongly normalising term inside a labelled jump, and thus it is not a labelled term according to our definition. Similarly the term $x[\![x/y]\!][y/z[\![z/v]\!]]$ reduces to $x[\![x/z[\![z/v]\!]]\!]$ which has a labelled jump inside another labelled jump, and thus it is not a labelled term.

We thus need labelled terms to be stable by equivalence and reduction. This can be done by defining a predicate of **well-formedness** on labelled terms such that $\mathbb{WF}(t)$ and $t \to_{\mathcal{J}_\mathtt{p}} t'$ imply $\mathbb{WF}(t')$. In order to formalize such a predicate we need some definitions. The notion of free variable contained in a labelled jump is particularly important.

The set of **labelled free variables** of $t \in \mathbb{T}_\mathtt{p}$ is given by:

$$\begin{array}{ll}
\mathbb{L}\mathtt{fv}(x) & := \emptyset \\
\mathbb{L}\mathtt{fv}(\lambda x.u) & := \mathbb{L}\mathtt{fv}(u) \setminus \{x\} \\
\mathbb{L}\mathtt{fv}(uv) & := \mathbb{L}\mathtt{fv}(u) \cup \mathbb{L}\mathtt{fv}(v) \\
\mathbb{L}\mathtt{fv}(u[x/v]) & := (\mathbb{L}\mathtt{fv}(u) \setminus \{x\}) \cup \mathbb{L}\mathtt{fv}(v) \\
\mathbb{L}\mathtt{fv}(u[\![x/v]\!]) & := (\mathbb{L}\mathtt{fv}(u) \setminus \{x\}) \cup \mathtt{fv}(v)
\end{array}$$

Note that $u \in \mathcal{T}$ implies $\mathbb{L}\mathtt{fv}(u) = \emptyset$. Also $\mathbb{L}\mathtt{fv}(t) \subseteq \mathtt{fv}(t)$.

We now formalise the notion ensuring that a labelled jump is strongly normalising with respect to labelled substitutions coming from the context.

A labelled term $t \in \mathbb{T}_\mathtt{p}$ is **SN-labelled** for a (meta-level) substitution $\gamma$ iff $\mathtt{SN}\mathbb{L}_\mathtt{p}(t,\gamma)$ holds:

$$\begin{array}{ll}
\mathtt{SN}\mathbb{L}_\mathtt{p}(x,\gamma) & := \mathtt{true} \\
\mathtt{SN}\mathbb{L}_\mathtt{p}(\lambda x.t,\gamma) & := \mathtt{SN}\mathbb{L}_\mathtt{p}(t,\gamma) \\
\mathtt{SN}\mathbb{L}_\mathtt{p}(tu,\gamma) & := \mathtt{SN}\mathbb{L}_\mathtt{p}(t,\gamma)\ \&\ \mathtt{SN}\mathbb{L}_\mathtt{p}(u,\gamma) \\
\mathtt{SN}\mathbb{L}_\mathtt{p}(t[x/u],\gamma) & := \mathtt{SN}\mathbb{L}_\mathtt{p}(t,\gamma)\ \&\ \mathtt{SN}\mathbb{L}_\mathtt{p}(u,\gamma) \\
\mathtt{SN}\mathbb{L}_\mathtt{p}(t[\![x/u]\!],\gamma) & := \mathtt{SN}\mathbb{L}_\mathtt{p}(t,\{x/u\}\gamma)\ \&\ u\gamma \in \mathcal{SN}_{\lambda \mathtt{j_p/o}}
\end{array}$$

Finally, a $\mathtt{p}$-labelled term $t$ is **$\mathtt{p}$-well-formed**, written $t \in \mathbb{WF}_\mathtt{p}$, iff

1. $\mathtt{SN}\mathbb{L}_\mathtt{p}(t,\mathtt{id})$
2. every subterm $u[y/v]$ or $\lambda y.u$ in $t$ verifies $y \notin \mathbb{L}\mathtt{fv}(u)$
3. $\mathtt{p} = \mathtt{b}$ implies subterms $u[\![y/v]\!] \in t$ verify $y \notin \mathbb{L}\mathtt{fv}(u)$.

Thus for example $t_0 = (xx)[\![x/y]\!][\![y/z]\!]$ is not $\mathtt{b}$-well-formed since $y$ is not a labelled free variable of $t_0$, whereas $t_0$ is $\mathtt{u}$-well-formed since $z \in \mathcal{SN}_{\lambda \mathtt{j}_\mathtt{u}/\mathtt{o}}$. Also, $t_1 = y[y/x][x/\lambda z.zz]$ is $\mathtt{b}$ and $\mathtt{u}$ well-formed but $t_2 = y[\![y/xx]\!][\![x/\lambda z.zz]\!]$ is not. More precisely, $x$ is a labelled free variable of $y[\![y/xx]\!]$ so that $t_2$ is not $\mathtt{b}$-well-formed, and $\mathtt{SN}\mathbb{L}_\mathtt{u}(t_2, \emptyset)$ does not hold (since $(\lambda z.zz)(\lambda z.zz) \notin \mathcal{SN}_{\lambda \mathtt{j}_\mathtt{u}/\mathtt{o}})$ hence $t_2$ is not $\mathtt{u}$-well-formed.

In order to show that well-formed terms are stable by equivalence and reduction we need the following lemmas:

**Lemma 18.** *Let $t \in \mathbb{WF}_\mathtt{p}$.*

1. *If $t_0 \equiv_\mathtt{0} t_1$, then $\mathbb{L}\mathtt{fv}(t_0) = \mathbb{L}\mathtt{fv}(t_1)$.*
2. *If $t_0 \to_{\mathcal{J}\mathtt{p}} t_1$, then $\mathbb{L}\mathtt{fv}(t_0) \supseteq \mathbb{L}\mathtt{fv}(t_1)$.*

*Proof.* By induction on $t$.

**Lemma 19.** *Let $t \in \mathbb{WF}_\mathtt{p}$.*

1. *If $\mathtt{SN}\mathbb{L}_\mathtt{p}(t, \gamma)$ and $\gamma \to^*_{\lambda \mathtt{j}_\mathtt{p}/\mathtt{o}} \gamma'$, then $\mathtt{SN}\mathbb{L}_\mathtt{p}(t, \gamma')$.*
2. *If $u \in \mathcal{T}$, then $\mathtt{SN}\mathbb{L}_\mathtt{p}(t\{x/u\}, \gamma) = \mathtt{SN}\mathbb{L}_\mathtt{p}(t, \{x/u\}\gamma)$.*

*Proof.*

1. By induction on $t$. Let $t = u[\![x/v]\!]$. We have $\mathtt{SN}\mathbb{L}_\mathtt{p}(u, \{x/v\}\gamma)$ and $v\gamma \in \mathcal{SN}_{\lambda \mathtt{j}_\mathtt{p}/\mathtt{o}}$. Since $v\gamma \to^*_{\lambda \mathtt{j}_\mathtt{p}/\mathtt{o}} v\gamma'$, then $v\gamma' \in \mathcal{SN}_{\lambda \mathtt{j}_\mathtt{p}/\mathtt{o}}$. Also $\{x/v\}\gamma \to^*_{\lambda \mathtt{j}_\mathtt{p}/\mathtt{o}} \{x/v\}\gamma'$ so that $\mathtt{SN}\mathbb{L}_\mathtt{p}(u, \{x/v\}\gamma')$ by the i.h. We thus conclude $\mathtt{SN}\mathbb{L}_\mathtt{p}(t, \gamma')$. All the other cases are straightforward.
2. By induction on $t$.
   - $t = x$. Then $\mathtt{SN}\mathbb{L}_\mathtt{p}(x\{x/u\}, \gamma) = \mathtt{SN}\mathbb{L}_\mathtt{p}(u, \gamma) = \mathtt{true} = \mathtt{SN}\mathbb{L}_\mathtt{p}(x, \{x/u\}\gamma)$.
   - $t = y \neq x$. Then $\mathtt{SN}\mathbb{L}_\mathtt{p}(y\{x/u\}, \gamma) = \mathtt{SN}\mathbb{L}_\mathtt{p}(y, \gamma) = \mathtt{true} = \mathtt{SN}\mathbb{L}_\mathtt{p}(y, \{x/u\}\gamma)$.
   - $t = t_1[\![y/t_2]\!]$. W.l.g. we can assume $y \notin \mathtt{fv}(u)$ and $y \neq x$. Then,

$$\begin{array}{ll}
\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1[\![y/t_2]\!]\{x/u\}, \gamma) & = \\
\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1\{x/u\}[\![y/t_2\{x/u\}]\!], \gamma) & = \\
\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1\{x/u\}, \{y/t_2\{x/u\}\}\gamma) \text{ and } t_2\{x/u\}\gamma \in \mathcal{SN}_{\lambda \mathtt{j}_\mathtt{p}/\mathtt{o}} & =_{i.h.} \\
\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1, \{x/u\}\{y/t_2\{x/u\}\}\gamma) \text{ and } t_2\{x/u\}\gamma \in \mathcal{SN}_{\lambda \mathtt{j}_\mathtt{p}/\mathtt{o}} & = \\
\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1, \{y/t_2\}\{x/u\}\gamma) \text{ and } t_2\{x/u\}\gamma \in \mathcal{SN}_{\lambda \mathtt{j}_\mathtt{p}/\mathtt{o}} & = \\
\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1[\![y/t_2]\!], \{x/u\}\gamma) &
\end{array}$$

   - All the other cases are straightforward.

**Lemma 20 (Stability of $\mathtt{SN}\mathbb{L}$).** *Let $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_0, \gamma)$. If $t_0 \equiv_\mathtt{0} t_1$ or $t_0 \to_{\mathcal{J}_\mathtt{p}} t_1$, then $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1, \gamma)$.*

*Proof.* By induction on the reduction relations.

- $\mathtt{CS}$: $t_0 = t[x/u][y/v] \equiv t[y/v][x/u] = t_1$ if $y \notin \mathtt{fv}(u)$ and $x \notin \mathtt{fv}(v)$. Then, $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_0, \gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{p}(t, \gamma)$ and $\mathtt{SN}\mathbb{L}_\mathtt{p}(u, \gamma)$ and $\mathtt{SN}\mathbb{L}_\mathtt{p}(v, \gamma)$ so that we conclude $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1, \gamma)$.
- $\underline{\mathtt{CS}}_1$: $t_0 = t[\![x/u]\!][y/v] \equiv t[y/v][\![x/u]\!] = t_1$ if $y \notin \mathtt{fv}(u)$ and $x \notin \mathtt{fv}(v)$. Then, $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_0, \gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{p}(t, \{x/u\}\gamma)$ and $\mathtt{SN}\mathbb{L}_\mathtt{p}(v, \gamma)$ and $u\gamma \in \mathcal{SN}_{\lambda \mathtt{j}_\mathtt{p}/\mathtt{o}}$. We also have
  $\mathtt{SN}\mathbb{L}_\mathtt{p}(v, \gamma) =_{L.\ 19:2} \mathtt{SN}\mathbb{L}_\mathtt{p}(v, \{x/u\}\gamma)$. We thus conclude $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1, \gamma)$.

- $\underline{\mathtt{CS}}_2$: $t_0 = t[\![x/u]\!][\![y/v]\!] \equiv t[\![y/v]\!][\![x/u]\!] = t_1$ if $y \notin \mathtt{fv}(u)$ and $x \notin \mathtt{fv}(v)$.
  Observe that the hypothesis implies $u\{y/v\}\gamma = u\gamma$ and $v\{x/u\}\gamma = v\gamma$ and $\{x/u\}\{y/v\}\gamma = \{y/v\}\{x/u\}\gamma$. Then, $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_0,\gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{p}(t,\{x/u\}\{y/v\}\gamma)$ and $u\gamma, v\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}}$. Thus we conclude also $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1,\gamma)$.
- $\sigma_1$: $t_0 = (\lambda y.t)[x/u] \equiv \lambda y.t[x/u] = t_1$ if $y \notin \mathtt{fv}(u)$.
  Then, $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_0,\gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{p}(t,\gamma)$ and $\mathtt{SN}\mathbb{L}_\mathtt{p}(u,\gamma)$ so that $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1,\gamma)$ is immediate.
- $\underline{\sigma}_1$: $t_0 = (\lambda y.t)[\![x/u]\!] \equiv \lambda y.t[\![x/u]\!] = t_1$ if $y \notin \mathtt{fv}(u)$.
  Then, $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_0,\gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{p}(t,\{x/u\}\gamma)$ and $u\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}}$ so that $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1,\gamma)$ is immediate.
- $\sigma_2$: $t_0 = (tv)[x/u] \equiv t[x/u]v = t_1$ if $x \notin \mathtt{fv}(v)$ and $x \in \mathtt{fv}(t)$.
  Then, $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_0,\gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{p}(t,\gamma)$ and $\mathtt{SN}\mathbb{L}_\mathtt{p}(v,\gamma)$ and $\mathtt{SN}\mathbb{L}_\mathtt{p}(u,\gamma)$ so that $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1,\gamma)$ is immediate.
- $\underline{\sigma}_2$: $t_0 = t[\![x/u]\!]v \equiv (tv)[\![x/u]\!] = t_1$ if $x \notin \mathtt{fv}(v)$.
  Then, $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_0,\gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{p}(t,\{x/u\}\gamma)$ and $\mathtt{SN}\mathbb{L}_\mathtt{p}(v,\{x/u\}\gamma)$ and $u\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}}$.
  We have $\mathtt{SN}\mathbb{L}_\mathtt{p}(v,\{x/u\}\gamma) =_{L.\ 19:2}\ \mathtt{SN}\mathbb{L}_\mathtt{p}(v,\gamma)$ so that we conclude $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1,\gamma)$.
- $\underline{\mathtt{w}}$: $t_0 = t[\![y/u]\!] \to t = t_1$, where $|t|_y = 0$.
  Then, $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_0,\gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{p}(t,\{y/u\}\gamma)$ and $u\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}}$.
  By Lemma 19:2 $\mathtt{SN}\mathbb{L}_\mathtt{p}(t,\gamma)$ so that we conclude.
- $\underline{\mathtt{d}}$: $t_0 = t[\![y/u]\!] \to t\{y/u\} = t_1$, where $|t|_y = 1$.
  Then, $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_0,\gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{p}(t,\{y/u\}\gamma)$ and $u\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}}$.
  By Lemma 19 $\mathtt{SN}\mathbb{L}_\mathtt{p}(t\{y/u\},\gamma)$ which concludes this case.
- $\underline{\mathtt{c}}$: $t_0 = t[\![x/u]\!] \to t_{[y]_x}[\![y/u]\!][\![x/u]\!] = t_1$, where $|t|_x \geq 2$ and $y$ is fresh.
  Then $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_0,\gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{p}(t,\{x/u\}\gamma)$ and $u\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}}$.
  On the other hand $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_1,\gamma) = \mathtt{SN}\mathbb{L}_\mathtt{p}(t_{[y]_x},\{y/u\}\{x/u\}\gamma)$ and $u\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{p}/\mathtt{o}}$.
  Since $\mathtt{SN}\mathbb{L}_\mathtt{p}(t,\{x/u\}\gamma) =_{L\ 19:2}\ \mathtt{SN}\mathbb{L}_\mathtt{p}(t_{[y]_x},\{y/u\}\{x/u\}\gamma)$ then we conclude.
- $\underline{\mathtt{ab}}$: $t_0 = (tv)[\![x/u]\!] \to tv[\![x/u]\!] = t_1$, where $x \in \mathtt{fv}(v)$ and $x \notin \mathtt{fv}(t)$.
  Then, $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_0,\gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{b}(t,\{x/u\}\gamma) =_{L.\ 19:2}\ \mathtt{SN}\mathbb{L}_\mathtt{b}(t,\gamma)$ and $\mathtt{SN}\mathbb{L}_\mathtt{b}(v,\{x/u\}\gamma)$ and $u\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{b}/\mathtt{o}}$. We thus conclude $\mathtt{SN}\mathbb{L}_\mathtt{b}(t_1,\gamma)$.
- $\underline{\mathtt{sb}}$: $t_0 = t[y/v][\![x/u]\!] \to t[y/v[\![x/u]\!]] = t_1$, where $x \in \mathtt{fv}(v)$ and $\mathtt{x} \notin \mathtt{fv}(t)$.
  Then, $\mathtt{SN}\mathbb{L}_\mathtt{b}(t_0,\gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{b}(t,\{x/u\}\gamma) =_{L.\ 19:2}\ \mathtt{SN}\mathbb{L}_\mathtt{b}(t,\gamma)$ and $\mathtt{SN}\mathbb{L}_\mathtt{b}(v,\{x/u\}\gamma)$ and $u\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{u}/\mathtt{o}}$. We thus conclude $\mathtt{SN}\mathbb{L}_\mathtt{b}(t_1,\gamma)$.
- $\underline{\mathtt{au}}$: $t_0 = tv[\![x/u]\!] \to (tv)[\![x/u]\!] = t_1$, where $x \in \mathtt{fv}(v)$ and $x \notin \mathtt{fv}(t)$.
  Then, $\mathtt{SN}\mathbb{L}_\mathtt{u}(t_0,\gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{u}(t,\gamma) =_{L.\ 19:2}\ \mathtt{SN}\mathbb{L}_\mathtt{u}(t,\{x/u\}\gamma)$ and $\mathtt{SN}\mathbb{L}_\mathtt{u}(v,\{x/u\}\gamma)$ and $u\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{u}/\mathtt{o}}$. We thus conclude $\mathtt{SN}\mathbb{L}_\mathtt{u}(t_1,\gamma)$.
- $\underline{\mathtt{su}}_1$: $t_0 = t[y/v[\![x/u]\!]] \to t[y/v][\![x/u]\!] = t_1$, where $x \in \mathtt{fv}(v)$.
  Then, $\mathtt{SN}\mathbb{L}_\mathtt{u}(t_0,\gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{u}(t,\gamma) =_{L.\ 19:2}\ \mathtt{SN}\mathbb{L}_\mathtt{u}(t,\{x/u\}\gamma)$ and $\mathtt{SN}\mathbb{L}_\mathtt{u}(v,\{x/u\}\gamma)$ and $u\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{u}/\mathtt{o}}$. We thus conclude $\mathtt{SN}\mathbb{L}_\mathtt{u}(t_1,\gamma)$.
- $\underline{\mathtt{su}}_2$: $t_0 = t[\![y/v[x/u]]\!] \to t[\![y/v]\!][\![x/u]\!] = t_1$, where $x \in \mathtt{fv}(v)$ and $x \notin \mathtt{fv}(t)$.
  Then, $\mathtt{SN}\mathbb{L}_\mathtt{u}(t_0,\gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{u}(t,\{y/v[x/u]\}\gamma)$ and $v[x/u]\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{u}/\mathtt{o}}$. To show $\mathtt{SN}\mathbb{L}_\mathtt{u}(t_1,\gamma)$ we need $u\gamma, v\{x/u\}\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{u}/\mathtt{o}}$ and $\mathtt{SN}\mathbb{L}_\mathtt{u}(t,\rho)$, for $\rho = \{y/v\}\{x/u\}\gamma$.
  Since $u\gamma \in v[x/u]\gamma$, then $u\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{u}/\mathtt{o}}$. Since $v[x/u]\gamma \to v\{x/u\}\gamma$, then $v\{x/u\}\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_\mathtt{u}/\mathtt{o}}$. Finally, $\{y/v[x/u]\}\gamma \to^* \{y/v\{x/u\}\}\gamma = \rho'$ so that $\mathtt{SN}\mathbb{L}_\mathtt{u}(t,\rho')$ holds by Lemma 19: 1. We conclude $\mathtt{SN}\mathbb{L}_\mathtt{u}(t_1,\gamma)$.
- $\mathtt{gdB}$: $t_0 = (\lambda x.t)\mathbb{L}u \to t[x/u]\mathbb{L} = t_1$. We can reason by induction on $\mathbb{L}$.
  If $\mathbb{L}$ is empty, then $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_0,\gamma)$ iff $\mathtt{SN}\mathbb{L}_\mathtt{p}(u,\gamma)$ and $\mathtt{SN}\mathbb{L}_\mathtt{p}(t,\gamma)$, which implies $\mathtt{SN}\mathbb{L}_\mathtt{p}(t_0,\gamma)$.

If $\mathbb{L} = [y/v]$, then it is straightforward. If $\mathbb{L} = [\![y/v]\!]$, then $\mathtt{SN}\mathbb{L}_{\mathtt{p}}(t_0, \gamma)$ iff $\mathtt{SN}\mathbb{L}_{\mathtt{p}}(u, \gamma)$ and $\mathtt{SN}\mathbb{L}_{\mathtt{p}}(t, \{y/v\}\gamma)$ and $v\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}}$. Since $y \notin \mathtt{fv}(u)$, then Lemma 19:2 gives $\mathtt{SN}\mathbb{L}_{\mathtt{p}}(u, \{y/v\}\gamma)$ so that we conclude $\mathtt{SN}\mathbb{L}_{\mathtt{p}}(t_1, \gamma)$.
If $\mathbb{L}$ has more than one substitution, the proof is straightforward by the i.h.

- The inductive cases. We only show the interesting cases. Let $t_0 = t[\![x/u]\!] \equiv t'[\![x/u]\!] = t_1$ (resp. $t_0 = t[\![x/u]\!] \to t'[\![x/u]\!] = t_1$). Then, $\mathtt{SN}\mathbb{L}_{\mathtt{p}}(t_0, \gamma)$ iff $\mathtt{SN}\mathbb{L}_{\mathtt{p}}(t, \{x/u\}\gamma)$ and $u\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}}$. The i.h. gives $\mathtt{SN}\mathbb{L}_{\mathtt{p}}(t', \{x/u\}\gamma)$ so that $\mathtt{SN}\mathbb{L}_{\mathtt{p}}(t_1, \gamma)$.
  Let $t_0 = t[\![x/u]\!] \equiv t[\![x/u']\!] = t_1$ or $t_0 = t[\![x/u]\!] \to t[\![x/u']\!] = t_1$. Then, $\mathtt{SN}\mathbb{L}_{\mathtt{p}}(t_0, \gamma)$ iff $\mathtt{SN}\mathbb{L}_{\mathtt{p}}(t, \{x/u\}\gamma)$ and $u\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}}$. We have $u\gamma \equiv u'\gamma$ (resp. $u\gamma \to^* u'\gamma$) so that $u'\gamma \in \mathcal{SN}_{\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}}$. Lemma 19:1 gives $\mathtt{SN}\mathbb{L}_{\mathtt{p}}(t, \{x/u'\}\gamma)$ so that we conclude $\mathtt{SN}\mathbb{L}_{\mathtt{p}}(t_1, \gamma)$.
  All the other cases are straightforward.

**Corollary 6.** *Let $t \in \mathbb{WF}_{\mathtt{p}}$. If $t \equiv_{\mathtt{O}} t'$ or $t \to_{\mathcal{J}_{\mathtt{p}}} t'$, then $t' \in \mathbb{WF}_{\mathtt{p}}$.*

The given corollary is essential in developing the termination proofs for the forgettable relations $\mathtt{F_b/O}$ and $\mathtt{F_u/O}$. More precisely, for each forgettable reduction $\mathtt{F_p/O}$, with $\mathtt{p} \in \{\mathtt{b}, \mathtt{u}\}$, we define a measure on $\mathtt{p}$-well-formed labelled terms which strictly decreases by $\mathtt{F_p/O}$-reduction.

**Lemma 21.** *The relation $\to_{\mathtt{F_b/O}}$ is terminating on $\mathtt{b}$ well-formed labelled terms.*

**Lemma 22.** *The relation $\to_{\mathtt{F_u/O}}$ is terminating on $\mathtt{u}$ well-formed labelled terms.*

We relegate the proofs of both Lemmas to the Appendix.

### 6.3 From Implicit to Explicit through Labelled

To show our first point, namely, that $u \in \mathcal{SN}_{\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}}$ and $t\{x/u\}\overline{v}^1_n \in \mathcal{SN}_{\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}}$ imply $t[\![x/u]\!]\overline{v}^1_n \in \mathcal{SN}_{\mathcal{J}_{\mathtt{p}}/\mathtt{O}}$, we now consider the following projection function $\mathbb{P}(\_)$ from labelled terms to terms, which also projects $\mathcal{J}_{\mathtt{p}}/\mathtt{O}$ into the reduction $\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}$:

$$\begin{array}{ll}
\mathbb{P}(x) & := x \\
\mathbb{P}(\lambda x.t) & := \lambda x.\mathbb{P}(t) \\
\mathbb{P}(tu) & := \mathbb{P}(t)\mathbb{P}(u) \\
\mathbb{P}(t[x/u]) & := \mathbb{P}(t)[x/\mathbb{P}(u)] \\
\mathbb{P}(t[\![x/u]\!]) & := \mathbb{P}(t)\{x/u\}
\end{array}$$

Note that $u \in \mathcal{T}$ implies $\mathbb{P}(u) = u$.

**Lemma 23.** *Let $t_0 \in \mathbb{T}_{\mathtt{p}}$. Then,*

1. *$t_0 \equiv_{\mathtt{O}} t_1$ implies $\mathbb{P}(t_0) \equiv_{\mathtt{o}} \mathbb{P}(t_1)$.*
2. *$t_0 \to_{\mathtt{Fp}} t_1$ implies $\mathbb{P}(t_0) \to^*_{\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}} \mathbb{P}(t_1)$.*
3. *$t_0 \to_{\mathtt{Pp}} t_1$ implies $\mathbb{P}(t_0) \to^+_{\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}} \mathbb{P}(t_1)$.*

*Proof.* By induction on labelled terms. The case $t_0 \to_{\underline{\mathtt{su}}_2} t_1$ uses Lemma 4.

**Lemma 24.** *Let* $t \in \mathbb{WF}_{\mathtt{p}}$. *If* $\mathbb{P}(t) \in \mathcal{SN}_{\lambda \mathtt{j_p/o}}$, *then* $t \in \mathcal{SN}_{\mathcal{J}_\mathtt{p}/\mathtt{0}}$.

*Proof.* Since $\rightarrow_{\mathcal{J}_\mathtt{p}} = \rightarrow_{\mathtt{F_p}} \cup \rightarrow_{\mathtt{P_p}}$ we show that $t \in \mathcal{SN}_{\mathtt{F_p} \cup \mathtt{P_p}/\mathtt{0}}$ by using Lemma 23 and termination of the forgettable relations (Lemmas 21 and 22).

Now let $\mathtt{p} \in \{\mathtt{b}, \mathtt{u}\}$ and consider $t, u, \overline{v}^1_n \in \mathcal{T}$ s.t. $u \in \mathcal{SN}_{\lambda \mathtt{j_p/o}}$. We immediately get $t[\![x/u]\!]\overline{v}^1_n \in \mathbb{WF}_\mathtt{p}$. Using $\mathbb{P}(t[\![x/u]\!]\overline{v}^1_n) = t\{x/u\}\overline{v}^1_n$ we thus conclude:

**Corollary 7.** *Let* $t, u, \overline{v}^1_n \in \mathcal{T}$. *If* $u \in \mathcal{SN}_{\lambda \mathtt{j_p/o}}$ & $t\{x/u\}\overline{v}^1_n \in \mathcal{SN}_{\lambda \mathtt{j_p/o}}$, *then* $t[\![x/u]\!]\overline{v}^1_n \in \mathcal{SN}_{\mathcal{J}_\mathtt{p}/\mathtt{0}}$.

The last point of our proof is to show that $t[\![x/u]\!]\overline{v}^1_n \in \mathcal{SN}_{\mathcal{J}_\mathtt{p}/\mathtt{0}}$ implies $t[x/u]\overline{v}^1_n \in \mathcal{SN}_{\lambda \mathtt{j_p/o}}$ by relating labelled terms and reductions to unlabelled terms and reductions. To do that, let us introduce an **unlabelling function on labelled terms**:

$$
\begin{array}{ll}
\mathtt{U}(x) & := x \\
\mathtt{U}(tu) & := \mathtt{U}(t)\mathtt{U}(u) \\
\mathtt{U}(\lambda x.t) & := \lambda x.\mathtt{U}(t) \\
\mathtt{U}(t[x/u]) & := \mathtt{U}(t)[x/\mathtt{U}(u)] \\
\mathtt{U}(t[\![x/u]\!]) & := \mathtt{U}(t)[x/u]
\end{array}
$$

Remark that $u \in \mathcal{T}$ implies $\mathtt{U}(u) = u$. Also, $\mathtt{fv}(t) = \mathtt{fv}(\mathtt{U}(t))$ and $\mathtt{U}(t\{x/u\}) = \mathtt{U}(t)\{x/\mathtt{U}(u)\}$.

**Lemma 25.** *If* $t \in \mathbb{WF}_\mathtt{p}$ *and* $\mathtt{U}(t) \rightarrow_{\lambda \mathtt{j_p/o}} u$, *then* $\exists\, v \in \mathbb{WF}_\mathtt{p}$ *s.t.* $t \rightarrow_{\mathcal{J}_\mathtt{p}/\mathtt{0}} v$ *and* $\mathtt{U}(v) = u$.

*Proof.* By induction on $\rightarrow_{\lambda \mathtt{j_p/o}}$ and case analysis. We only show the interesting cases of root equivalence/reduction.

1. The congruence $\equiv_\mathtt{o}$.
   - $t = u[x/v][\![y/w]\!]$ with $y \notin \mathtt{fv}(v)$ & $x \notin \mathtt{fv}(w)$ and

$$
\begin{array}{lll}
\mathtt{U}(u[x/v][\![y/w]\!]) & = & \\
\mathtt{U}(u)[x/\mathtt{U}(v)][y/w] & \equiv_{\mathtt{CS}} & \\
\mathtt{U}(u)[y/w][x/\mathtt{U}(v)] & = & \mathtt{U}(u[\![y/w]\!][x/v]) = t'_1
\end{array}
$$

   We then let $t_1 = u[\![y/w]\!][x/v]$ so that $\mathtt{U}(t_1) = t'_1$ and $t \equiv_{\underline{\mathtt{CS}}} t_1$.
   - $t = u[\![x/v]\!][y/w]$ with $y \notin \mathtt{fv}(v)$ & $x \notin \mathtt{fv}(w)$ and

$$
\begin{array}{lll}
\mathtt{U}(u[\![x/v]\!][y/w]) & = & \\
\mathtt{U}(u)[x/v][y/\mathtt{U}(w)] & \equiv_{\mathtt{CS}} & \\
\mathtt{U}(u)[y/\mathtt{U}(w)][x/v] & = & \mathtt{U}(u[y/w][\![x/v]\!]) = t'_1
\end{array}
$$

   We then let $t_1 = u[y/w][\![x/v]\!]$ so that $\mathtt{U}(t_1) = t'_1$ and $t \equiv_{\underline{\mathtt{CS}}} t_1$.
   - $t = u[\![x/v]\!][\![y/w]\!]$ with $y \notin \mathtt{fv}(v)$ & $x \notin \mathtt{fv}(w)$ and

$$
\begin{array}{lll}
\mathtt{U}(u[\![x/v]\!][\![y/w]\!]) & = & \\
\mathtt{U}(u)[x/v][y/w] & \equiv_{\mathtt{CS}} & \\
\mathtt{U}(u)[y/w][x/v] & = & \mathtt{U}(u[\![y/w]\!][\![x/v]\!]) = t'_1
\end{array}
$$

   We then let $t_1 = u[\![y/w]\!][\![x/v]\!]$ so that $\mathtt{U}(t_1) = t'_1$ and $t \equiv_{\underline{\mathtt{CS}}} t_1$.

- $t = \lambda y.u[\![x/v]\!]$ with $y \notin \mathtt{fv}(v)$ and

$$\begin{array}{ll} \mathtt{U}(\lambda y.u[\![x/v]\!]) & = \\ \lambda y.\mathtt{U}(u)[x/v] & \equiv_{\mathtt{SL}} \\ (\lambda y.\mathtt{U}(u))[x/v] = & \mathtt{U}((\lambda y.u)[\![x/v]\!]) = t'_1 \end{array}$$

We then let $t_1 = (\lambda y.u)[\![x/v]\!]$ so that $\mathtt{U}(t_1) = t'_1$ and $t \equiv_{\underline{\mathtt{SL}}} t_1$.
- $t = u[\![x/v]\!]w$ and

$$\begin{array}{ll} \mathtt{U}(u[\![x/v]\!]w) & = \\ \mathtt{U}(u)[x/v]\mathtt{U}(w) & \equiv_{\mathtt{SA_L}} \\ (\mathtt{U}(u)\mathtt{U}(w))[x/v] = & \mathtt{U}((uw)[\![x/v]\!]) = t'_1 \end{array}$$

We then let $t_1 = (uw)[\![x/v]\!]$ so that $\mathtt{U}(t_1) = t'_1$ and $t \equiv_{\underline{\mathtt{SA_L}}} t_1$.
- All the other cases are straightforward.

2. The reduction relation $\rightarrow_{\mathtt{j}}$.
- $t = u[\![x/v]\!]$ with $|u|_x = 0$ and

$$\begin{array}{l} \mathtt{U}(u[\![x/v]\!]) = \\ \mathtt{U}(u)[x/v] \ \rightarrow_{\mathtt{w}} \mathtt{U}(u) = t'_1 \end{array}$$

We then let $t_1 = u$ so that $\mathtt{U}(t_1) = t'_1$ and $t \rightarrow_{\underline{\mathtt{w}}} t_1$.
- $t = u[\![x/v]\!]$ with $|u|_x = 1$ and

$$\begin{array}{l} \mathtt{U}(u[\![x/v]\!]) = \\ \mathtt{U}(u)[x/v] \ \rightarrow_{\mathtt{d}} \mathtt{U}(u)\{x/v\} = \mathtt{U}(u\{x/v\}) = t'_1 \end{array}$$

We then let $t_1 = u\{x/v\}$ so that $\mathtt{U}(t_1) = t'_1$ and $t \rightarrow_{\underline{\mathtt{d}}} t_1$.
- $t = u[\![x/v]\!]$ with $|u|_x > 1$ and

$$\begin{array}{l} \mathtt{U}(u[\![x/v]\!]) = \\ \mathtt{U}(u)[x/v] \ \rightarrow_{\mathtt{c}} \mathtt{U}(u)_{[y]_x}[x/v][y/v] = \mathtt{U}(u_{[y]_x}[x/v][y/v]) = t'_1 \end{array}$$

We then let $t_1 = u_{[y]_x}[x/v][y/v]$ so that $\mathtt{U}(t_1) = t'_1$ and $t \rightarrow_{\underline{\mathtt{c}}} t_1$.

3. The reduction relations $\rightarrow_{\mathtt{b}}$ and $\rightarrow_{\mathtt{u}}$.
- $t = uw[\![x/v]\!]$ with $x \notin \mathtt{fv}(u)$ & $x \in \mathtt{fv}(w)$ and

$$\begin{array}{ll} \mathtt{U}(uw[\![x/v]\!]) & = \\ \mathtt{U}(u)\mathtt{U}(w)[x/v] & \rightarrow_{\mathtt{au}} \\ (\mathtt{U}(u)\mathtt{U}(w))[x/v] = & \mathtt{U}((uw)[\![x/v]\!]) = t'_1 \end{array}$$

We then let $t_1 = (uw)[\![x/v]\!]$ so that $\mathtt{U}(t_1) = t'_1$ and $t \rightarrow_{\underline{\mathtt{au}}} t_1$.
- $t = (uw)[\![x/v]\!]$ with $x \notin \mathtt{fv}(u)$ & $x \in \mathtt{fv}(w)$ and

$$\begin{array}{ll} \mathtt{U}((uw)[\![x/v]\!]) & = \\ (\mathtt{U}(u)\mathtt{U}(w))[x/v] & \rightarrow_{\mathtt{ab}} \\ \mathtt{U}(u)\mathtt{U}(w)[x/v] & = \quad \mathtt{U}(uw[\![x/v]\!]) = t'_1 \end{array}$$

We then let $t_1 = uw[\![x/v]\!]$ so that $\mathtt{U}(t_1) = t'_1$ and $t \rightarrow_{\underline{\mathtt{ab}}} t_1$.

– $t = u[y/w[\![x/v]\!]]$ with $x \notin \mathtt{fv}(u)$ & $x \in \mathtt{fv}(w)$ and

$$\begin{array}{l}\mathtt{U}(u[y/w[\![x/v]\!]]) \quad = \\ \mathtt{U}(u)[y/\mathtt{U}(w)[x/v]] \rightarrow_{\mathtt{su}} \\ \mathtt{U}(u)[y/\mathtt{U}(w)][x/v] = \mathtt{U}(u[y/w][\![x/v]\!]) = t'_1\end{array}$$

We then let $t_1 = u[y/w][\![x/v]\!]$ so that $\mathtt{U}(t_1) = t'_1$ and $t \rightarrow_{\underline{\mathtt{su}}_1} t_1$.

– $t = u[\![y/w[x/v]]\!]$ with $x \notin \mathtt{fv}(u)$ & $x \in \mathtt{fv}(w)$ and .

$$\begin{array}{l}\mathtt{U}(u[\![y/w[x/v]]\!]) = \\ \mathtt{U}(u)[y/w[x/v]] \rightarrow_{\mathtt{su}} \\ \mathtt{U}(u)[y/w][x/v] = \mathtt{U}(u[\![y/w]\!][\![x/v]\!]) = t'_1\end{array}$$

We then let $t_1 = u[\![y/w]\!][\![x/v]\!]$ so that $\mathtt{U}(t_1) = t'_1$ and $t \rightarrow_{\underline{\mathtt{su}}_2} t_1$.

– $t = u[\![y/w[\![x/v]\!]]\!]$ with $x \notin \mathtt{fv}(u)$ & $x \in \mathtt{fv}(w)$.
Since $t \in \mathbb{WF}_{\mathtt{p}}$, this case is not possible since $w[\![x/v]\!]$ is not a term.

– $t = u[y/w][\![x/v]\!]$ with $x \notin \mathtt{fv}(u)$ & $x \in \mathtt{fv}(w)$ and

$$\begin{array}{l}\mathtt{U}(u[y/w][\![x/v]\!]) \quad = \\ \mathtt{U}(u)[y/\mathtt{U}(w)][x/v] \rightarrow_{\mathtt{sb}} \\ \mathtt{U}(u)[y/\mathtt{U}(w)[x/v]] = \mathtt{U}(u[y/w[\![x/v]\!]]) = t'_1\end{array}$$

We then let $t_1 = u[y/w[\![x/v]\!]]$ so that $\mathtt{U}(t_1) = t'_1$ and $t \rightarrow_{\underline{\mathtt{sb}}} t_1$.

– $t = u[\![y/w]\!][x/v]$ with $x \notin \mathtt{fv}(u)$ & $x \in \mathtt{fv}(w)$. But $t \in \mathbb{WF}_{\mathtt{p}}$, so that $x \notin \mathbb{L}\mathtt{fv}(u[\![y/w]\!])$, which implies in particular $x \notin \mathtt{fv}(w)$. This case is not then possible.

– $t = u[\![y/w]\!][\![x/v]\!]$ with $x \notin \mathtt{fv}(u)$ & $x \in \mathtt{fv}(w)$. If $t \in \mathbb{WF}_{\mathtt{b}}$, then $x \notin \mathtt{fv}(w)$ as before. This case is not then possible.
If $t \in \mathbb{WF}_{\mathtt{u}}$, then $\rightarrow_{\mathtt{sb}}$ does not hold in the $\mathtt{u}$-system.

4. The reduction relation $\rightarrow_{\mathtt{gdB}}$.
Consider $t = (\lambda x.u)\mathbb{L}v$. Let $\mathtt{L}$ be the list containing all the unlabelling substitutions of the list $\mathbb{L}$. Then,

$$\begin{array}{ll}\mathtt{U}(t) & = \\ (\lambda x.\mathtt{U}(u))\mathtt{L}\mathtt{U}(v) & \rightarrow_{\mathtt{dB}} \\ \mathtt{U}(u)[x/\mathtt{U}(v)]\mathtt{L} = \mathtt{U}(u[x/v]\mathtt{L}) = t'_1 & \end{array}$$

We then let $t_1 = u[x/v]\mathbb{L}$ so that $\mathtt{U}(t_1) = t'_1$ and $t \rightarrow_{\mathtt{gdB}} t_1$.

5. All the other cases are straightforward.

**Lemma 26.** *Let $t \in \mathbb{WF}_{\mathtt{p}}$. If $t \in \mathcal{SN}_{\mathcal{J}_{\mathtt{p}}/\mathtt{O}}$, then $\mathtt{U}(t) \in \mathcal{SN}_{\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}}$.*

*Proof.* We prove $\mathtt{U}(t) \in \mathcal{SN}_{\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}}$ by induction on $\eta_{\mathcal{J}_{\mathtt{p}}/\mathtt{O}}(t)$. This is done by considering all the $\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}$-reducts of $\mathtt{U}(t)$ and using Lemma 25.

Now let $\mathtt{p} \in \{\mathtt{b},\mathtt{u}\}$ and consider $t, u, \overline{v}^1_n \in \mathcal{T}$ s.t. $u \in \mathcal{SN}_{\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}}$. We immediately get $t[\![x/u]\!]\overline{v}^1_n \in \mathbb{WF}_{\mathtt{p}}$. Using $\mathtt{U}(t[\![x/u]\!]\overline{v}^1_n) = t[x/u]\overline{v}^1_n$ we thus conclude:

**Corollary 8.** *Let $t, u, \overline{v}^1_n \in \mathcal{T}$. If $t[\![x/u]\!]\overline{v}^1_n \in \mathbb{WF}_{\mathtt{p}} \cap \mathcal{SN}_{\mathcal{J}_{\mathtt{p}}/\mathtt{O}}$, then $t[x/u]\overline{v}^1_n \in \mathcal{SN}_{\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}}$.*

From Corollaries 7 and 8 we get:

**Lemma 27 (IE for $\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}$).** *For $\mathtt{p} \in \{\mathtt{b},\mathtt{u}\}$, $\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}$ enjoys the* **IE** *property.*

Theorem 2 thus allows us to conclude with the main result of this section:

**Corollary 9 (PSN for $\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}$).** *For $\mathtt{p} \in \{\mathtt{b},\mathtt{u}\}$, $\lambda\mathtt{j}_{\mathtt{p}}/\mathtt{o}$ enjoys PSN.*

## 7 Conclusions

We have introduced the structural $\lambda$j-calculus, a concise but expressive $\lambda$-calculus with jumps. No prior knowledge of Linear Logic is necessary to understand $\lambda$j, despite their strong connection. We have established many different sanity properties for $\lambda$j such as confluence and PSN. We have used $\lambda$j as an operational framework to elaborate new characterisations of the well-known notions of full developments and L-developments, and to obtain the new, more powerful notion of XL-development. Finally, we have modularly added commutation of independent jumps, $\sigma$-equivalence and two kinds of propagations of jumps, while showing that PSN still holds.

As noted in Section 6, PSN for the $\lambda$j-calculus plus the constrained equivalences $\{\texttt{CS}, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is - at present - a challenging conjecture. Indeed, the merging of the two similar, yet different uses of $\{\sigma_3, \sigma_4\}$ that we study in this paper presents several non-trivial difficulties. A further in-depth re-elaboration of the labelling technique would be necessary, perhaps even the use of a completely different technique dealing with reduction modulo a set of equations.

An interesting research direction is the study of linear head reduction [2] for $\lambda$-calculus - which is closely connected to game semantics and abstract machines - whose formulation is not a strategy in the usual sense. Indeed, jumps and distance permit to reformulate linear head reduction as a strategy of $\lambda$j.

It would also be interesting to exploit distance and multiplicities in other frameworks for example when dealing with pattern matching, continuations or differential features.

## References


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## 8 Appendix

**Theorem 5 (Modular Abstract Strong Normalisation).** *Let* $\mathtt{A}_1$ *and* $\mathtt{A}_2$ *(resp.* $\mathcal{E}$*) be two reduction (resp. equivalence) relations on* $\mathtt{s}$. *Let* $\mathtt{A}$ *be a reduction relation on* $\mathtt{S}$ *and let consider a relation* $\mathcal{R} \subseteq \mathtt{s} \times \mathtt{S}$. *Suppose that forall* $u, v, U$

**(P0)** $u \mathrel{\mathcal{R}} U$ & $u \mathrel{\mathcal{E}} v$ *imply* $\exists V$ *s.t.* $v \mathrel{R} V$ & $U \ = V$.
**(P1)** $u \mathrel{\mathcal{R}} U$ & $u \mathrel{\mathtt{A}_1} v$ *imply* $\exists V$ *s.t.* $v \mathrel{\mathcal{R}} V$ & $U \mathrel{\mathtt{A}^*} V$.
**(P2)** $u \mathrel{\mathcal{R}} U$ & $u \mathrel{\mathtt{A}_2} v$ *imply* $\exists V$ *s.t.* $v \mathrel{\mathcal{R}} V$ & $U \mathrel{\mathtt{A}^+} V$.
**(P3)** *The relation* $\mathtt{A}_1$ *modulo* $\mathcal{E}$ *is well-founded.*

*Then,* $t \mathrel{\mathcal{R}} T$ & $T \in \mathcal{SN}_{\mathtt{A}}$ *imply* $t \in \mathcal{SN}_{(\mathtt{A}_1 \cup \mathtt{A}_2)/\mathcal{E}}$.

### 8.1 The Forgettable System Terminates

The termination proofs for $\to_{\mathtt{Fb}}$ and $\to_{\mathtt{Fu}}$ are not really parametric in $\mathtt{p}$, nonetheless they both make use of **potential multiplicities**, which are extended to labelled jumps by adding the following case to the notion given in Section 3.

$$\mathtt{M}_x(u[\![y/v]\!]) := \mathtt{M}_x(u) + \mathtt{max}(1, \mathtt{M}_y(u)) \cdot \mathtt{M}_x(v)$$

We first prove that the equivalence $\mathtt{O}$ and the propagations $\underline{\mathtt{p}}$ preserve potential multiplicities.

**Lemma 28.** *Let* $\mathtt{p} \in \{\mathtt{b}, \mathtt{u}\}$. *Let* $t_0 \in \mathbb{T}_{\mathtt{p}}$. *Then,*

- $t_0 \equiv_{\mathtt{o},\underline{\mathtt{o}}} t_1$ *implies* $\mathtt{M}_w(t) = \mathtt{M}_w(t')$.
- $t_0 \to_{\underline{\mathtt{p}}} t_1$ *implies* $\mathtt{M}_w(t) = \mathtt{M}_w(t')$.

*Proof.* As $\mathtt{M}_w(t)$ is defined in the same way for labelled and unlabelled substitutions it is sufficient to check just one of them. Moreover, we just show the property for two $\underline{\mathtt{p}}$-steps, the other cases being similar.

- $t[\![x/u]\!][\![y/v]\!] \equiv_{\underline{\mathtt{CS}}} t[\![y/v]\!][\![x/u]\!]$, with $y \notin \mathtt{fv}(u)$ & $x \notin \mathtt{fv}(v)$.
  Observe that $\mathtt{M}_w(t[\![x/u]\!]) = \mathtt{M}_w(t)$ if $w \notin \mathtt{fv}(u)$.

$$\begin{array}{ll}
\mathtt{M}_w(t[\![x/u]\!][\![y/v]\!]) & = \\
\mathtt{M}_w(t[\![x/u]\!]) + \mathtt{max}(1, \mathtt{M}_y(t[\![x/u]\!])) \cdot \mathtt{M}_w(v) & = \\
\mathtt{M}_w(t[\![x/u]\!]) + \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{M}_w(v) & = \\
\mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_w(t)) \cdot \mathtt{M}_w(u) + \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{M}_w(v) & = \\
\mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_w(t[\![y/v]\!])) \cdot \mathtt{M}_w(u) + \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{M}_w(v) & = \\
\mathtt{M}_w(t[\![y/v]\!]) + \mathtt{max}(1, \mathtt{M}_w(t[\![y/v]\!])) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t[\![y/v]\!][\![x/u]\!]) &
\end{array}$$

- $(\lambda y.t)[\![x/u]\!] \equiv_{\underline{\sigma_1}} \lambda y.t[\![x/u]\!]$, with $y \notin \mathtt{fv}(u)$.

$$\begin{array}{ll}
\mathtt{M}_w((\lambda y.t)[\![x/u]\!]) & = \\
\mathtt{M}_w(\lambda y.t) + \mathtt{max}(1, \mathtt{M}_x(\lambda y.t)) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t[\![x/u]\!]) & = \\
\mathtt{M}_w(\lambda y.t[\![x/u]\!]) &
\end{array}$$

- $(t\ v)[\![x/u]\!] \equiv_{\underline{\sigma_2}} t[\![x/u]\!]\ v$, with $x \notin \mathtt{fv}(v)$.

$$\begin{array}{ll}
\mathtt{M}_w((t\ v)[\![x/u]\!]) & = \\
\mathtt{M}_w(t\ v) + \mathtt{max}(1, \mathtt{M}_x(t\ v)) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t\ v) + \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t) + \mathtt{M}_w(v) + \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t[\![x/u]\!]) + \mathtt{M}_w(v) & = \\
\mathtt{M}_w(t[\![x/u]\!]\ v) &
\end{array}$$

– $(t\ v)[\![x/u]\!] \rightarrow_{\underline{\mathtt{ab}}} t\ v[\![x/u]\!]$, with $x \notin \mathtt{fv}(t)$ and $x \in \mathtt{fv}(v)$.

$$\begin{array}{ll}
\mathtt{M}_w((t\ v)[\![x/u]\!]) & = \\
\mathtt{M}_w(t\ v) + \mathtt{max}(1, \mathtt{M}_x(t\ v)) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t\ v) + \mathtt{max}(1, \mathtt{M}_x(v)) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t) + \mathtt{M}_w(v) + \mathtt{max}(1, \mathtt{M}_x(v)) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t) + \mathtt{M}_w(v[\![x/u]\!]) & = \mathtt{M}_w(t\ v[\![x/u]\!])
\end{array}$$

– $t[\![y/v[x/u]]\!] \rightarrow_{\underline{\mathtt{su}}_2} t[\![y/v]\!][\![x/u]\!]$, where $x \in \mathtt{fv}(v)$. First, let us show that $\mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{M}_x(v) = \mathtt{max}(1, \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{M}_x(v))$. If $y \in \mathtt{fv}(t)$ then both expression are equal to $\mathtt{M}_y(t) \cdot \mathtt{M}_x(v)$, as $x \in \mathtt{fv}(v)$, otherwise are both equal to $\mathtt{M}_x(v)$. Then,

$$\begin{array}{ll}
\mathtt{M}_w(t[\![y/v[x/u]]\!]) & = \\
\mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{M}_w(v[x/u]) & = \\
\mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_y(t)) \cdot (\mathtt{M}_w(v) + \mathtt{max}(1, \mathtt{M}_x(v)) \cdot \mathtt{M}_w(u)) & = \\
\mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_y(t)) \cdot (\mathtt{M}_w(v) + \mathtt{M}_x(v) \cdot \mathtt{M}_w(u)) & = \\
\mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{M}_w(v) + \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{M}_x(v) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t[\![y/v]\!]) + \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{M}_x(v) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t[\![y/v]\!]) + \mathtt{max}(1, \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{M}_x(v)) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t[\![y/v]\!]) + \mathtt{max}(1, \mathtt{M}_x(t) + \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{M}_x(v)) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t[\![y/v]\!]) + \mathtt{max}(1, \mathtt{M}_x(t[\![y/v]\!])) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t[\![y/v]\!][\![x/u]\!]) &
\end{array}$$

– The inductive cases are all straightforward.

To relate potential multiplicities and reductions we need two lemmas. The first is used for $\rightarrow_{\underline{\mathtt{c}}}$-steps and the second for $\rightarrow_{\underline{\mathtt{w}},\underline{\mathtt{d}}}$-steps.

**Lemma 29.** *Let $t \in \mathbb{T}_{\mathtt{p}}$ s.t. $|t|_x \geq 2$. Let $w \neq x, y$ and $x \neq y$. Then,*

1. $\mathtt{M}_w(t) = \mathtt{M}_w(t_{[y]_x})$
2. $\mathtt{M}_y(t) = \mathtt{M}_x(t_{[x]_y}) + \mathtt{M}_y(t_{[x]_y})$.

*Proof.* We first enrich the notation by writing $t_{[x]_y^k}$ if $t_{[x]_y}$ renames $k$ of occurrences of $y$ as $x$. We also admit (only for this proof) that $k = 0$ causes the renaming to be the identity. The statement now becomes: $\mathtt{M}_y(t) = \mathtt{M}_x(t_{[x]_y^k}) + \mathtt{M}_y(t_{[x]_y^k})$ for $0 \leq k \leq |t|_y$. The proof now proceeds by induction on $t$.

– For $t = z$ the statement is trivial.
– If $t = u\ v$ then there exist $k_1, k_2$ s. t. $t_{[x]_y^k} = u_{[x]_y^{k_1}}\ v_{[x]_y^{k_2}}$ and $k_1, k_2 \geq 0$, $k_1 + k_2 = k$, $k_1 \leq |u|_y$ and $k_2 \leq |v|_y$. We get

$$\begin{array}{ll}
\mathtt{M}_x(t_{[x]_y^k}) + \mathtt{M}_y(t_{[x]_y^k}) & = \\
\mathtt{M}_x(u_{[x]_y^{k_1}}\ v_{[x]_y^{k_2}}) + \mathtt{M}_y(u_{[x]_y^{k_1}}\ v_{[x]_y^{k_2}}) & = \\
\mathtt{M}_x(u_{[x]_y^{k_1}}) + \mathtt{M}_x(v_{[x]_y^{k_2}}) + \mathtt{M}_y(u_{[x]_y^{k_1}}) + \mathtt{M}_y(v_{[x]_y^{k_2}}) & = \\
\mathtt{M}_x(u_{[x]_y^{k_1}}) + \mathtt{M}_y(u_{[x]_y^{k_1}}) + \mathtt{M}_x(v_{[x]_y^{k_2}}) + \mathtt{M}_y(v_{[x]_y^{k_2}}) & =_{i.h.} \\
\mathtt{M}_y(u) + \mathtt{M}_y(v) & = \mathtt{M}_y(u\ v)
\end{array}$$

– If $t = \lambda x.u$ the i.h. is enough.
– If $t = u[z/v]$ then $t_{[x]_y^k} = u_{[x]_y^{k_1}}[z/v_{[x]_y^{k_2}}]$ for $k_1, k_2 \geq 0$, $k_1 + k_2 = k$, $k_1 \leq |u|_y$ and $k_2 \leq |v|_y$. Then,

$$\begin{array}{ll}
\mathtt{M}_x(t_{[x]_y}) + \mathtt{M}_y(t_{[x]_y}) & = \\
\mathtt{M}_x(u_{[x]_y^{k_1}}[z/v_{[x]_y^{k_2}}]) + \mathtt{M}_y(u_{[x]_y^{k_1}}[z/v_{[x]_y^{k_2}}]) & = \\
\mathtt{M}_x(u_{[x]_y^{k_1}}) + \mathtt{max}(1, \mathtt{M}_z(u_{[x]_y^{k_1}})) \cdot \mathtt{M}_x(v_{[x]_y^{k_2}}]) + \mathtt{M}_y(u_{[x]_y^{k_1}}) + \mathtt{max}(1, \mathtt{M}_z(u_{[x]_y^{k_1}})) \cdot \mathtt{M}_y(v_{[x]_y^{k_2}}) & =_{i.h.} \\
\mathtt{M}_x(u_{[x]_y^{k_1}}) + \mathtt{max}(1, \mathtt{M}_z(u)) \cdot \mathtt{M}_x(v_{[x]_y^{k_2}}]) + \mathtt{M}_y(u_{[x]_y^{k_1}}) + \mathtt{max}(1, \mathtt{M}_z(u)) \cdot \mathtt{M}_y(v_{[x]_y^{k_2}}) & = \\
\mathtt{M}_x(u_{[x]_y^{k_1}}) + \mathtt{M}_y(u_{[x]_y^{k_1}}) + \mathtt{max}(1, \mathtt{M}_z(u)) \cdot (\mathtt{M}_x(v_{[x]_y^{k_2}}]) + \mathtt{M}_y(v_{[x]_y^{k_2}})) & = \\
\mathtt{M}_y(u) + \mathtt{max}(1, \mathtt{M}_z(u)) \cdot \mathtt{M}_y(v) & = \\
\mathtt{M}_y(u[z/v]) & = \mathtt{M}_y(t)
\end{array}$$

– If $t = u[\![z/v]\!]$ then it is identical to the previous case.

**Lemma 30.** *Let $t \in \mathbb{T}_{\mathtt{p}}$ and $u \in \mathcal{T}$. Then,*

1. *If $|u|_w = 0$, then $\mathtt{M}_w(t\{x/u\}) = \mathtt{M}_w(t)$.*
2. *If $|t|_x = 1$, then $\mathtt{M}_w(t\{x/u\}) = \mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{M}_w(u)$*

*Proof.* The first statement follows from a straightforward induction on $t$. The second point follows by showing that $\mathtt{M}_w(t[\![x/u]\!]) = \mathtt{M}_w(t\{x/u\})$ if $|t|_x = 1$. We show this property by induction on $t$.

– If $t = x$

$$\begin{array}{ll}
\mathtt{M}_w(x[\![x/u]\!]) & = \\
\mathtt{M}_w(x) + \mathtt{max}(1, \mathtt{M}_x(x)) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(u) & = \mathtt{M}_w(x\{x/u\})
\end{array}$$

– If $t = \lambda y.t'$

$$\begin{array}{ll}
\mathtt{M}_w((\lambda y.t')[\![x/u]\!]) & =_{L.28} \\
\mathtt{M}_w(\lambda y.t'[\![x/u]\!]) & = \\
\mathtt{M}_w(t'[\![x/u]\!]) & =_{i.h.} \\
\mathtt{M}_w(t'\{x/u\}) & = \mathtt{M}_w((\lambda y.t')\{x/u\})
\end{array}$$

– If $t = v\ s$ with $x \notin s$ and $x \in v$.

$$\begin{array}{ll}
\mathtt{M}_w((v\ s)[\![x/u]\!]) & =_{L.28} \\
\mathtt{M}_w((v[\![x/u]\!])\ s) & = \\
\mathtt{M}_w(v[\![x/u]\!]) + \mathtt{M}_w(s) & =_{i.h.} \\
\mathtt{M}_w(v\{x/u\}) + \mathtt{M}_w(s) & = \\
\mathtt{M}_w((v\{x/u\})\ s) & = \mathtt{M}_w((v\ s)\{x/u\})
\end{array}$$

– If $t = v\ s$ with $x \in s$ and $x \notin v$.

$$\begin{array}{ll}
\mathtt{M}_w((v\ s)[\![x/u]\!]) & =_{L.28} \\
\mathtt{M}_w(v\ (s[\![x/u]\!])) & = \\
\mathtt{M}_w(v) + \mathtt{M}_w(s[\![x/u]\!]) & =_{i.h.} \\
\mathtt{M}_w(v) + \mathtt{M}_w(s\{x/u\}) & = \\
\mathtt{M}_w(v\ (s\{x/u\})) & = \mathtt{M}_w((v\ s)\{x/u\})
\end{array}$$

– If $t = v[y/s]$ then if $x \in \mathtt{fv}(v)$ we have

$$\begin{array}{lll}
\mathtt{M}_w(v[y/s][\![x/u]\!]) & =_{L.28} & \\
\mathtt{M}_w(v[\![x/u]\!][y/s]) & = & \\
\mathtt{M}_w(v[\![x/u]\!]) + \mathtt{max}(1, \mathtt{M}_y(v[\![x/u]\!])) \cdot \mathtt{M}_w(s) & =_{i.h.} & \\
\mathtt{M}_w(v\{x/u\}) + \mathtt{max}(1, \mathtt{M}_y(v\{x/u\})) \cdot \mathtt{M}_w(s) & = & \\
\mathtt{M}_w(v\{x/u\}[y/s]) & = & \mathtt{M}_w(v[y/s]\{x/u\})
\end{array}$$

Instead, if $x \in \mathtt{fv}(s)$ we have

$$\begin{array}{lll}
\mathtt{M}_w(v[y/s][\![x/u]\!]) & =_{L.28} & \\
\mathtt{M}_w(v[y/s[\![x/u]\!]]) & = & \\
\mathtt{M}_w(v) + \mathtt{max}(1, \mathtt{M}_y(v)) \cdot \mathtt{M}_w(s[\![x/u]\!]) & =_{i.h.} & \\
\mathtt{M}_w(v) + \mathtt{max}(1, \mathtt{M}_y(v)) \cdot \mathtt{M}_w(s\{x/u\}) & =_{i.h.} & \\
\mathtt{M}_w(v[y/s\{x/u\}]) & = & \mathtt{M}_w(v[y/s]\{x/u\})
\end{array}$$

– If $t = v[\![y/s]\!]$ then the proof is similar to the case $t = v[y/s]$.

The next lemmas show that potential multiplicities are not increased by $\to_{\underline{\mathtt{j}}}$-steps.

**Lemma 31.** *Let $t_0 \in \mathbb{T}_{\mathtt{p}}$. Then, $t_0 \to_{\underline{\mathtt{j}}} t_1$ implies $\mathtt{M}_w(t) \geq \mathtt{M}_w(t')$.*

*Proof.* By induction on $\to_{\underline{\mathtt{j}}}$.

– $t_0 = t[\![x/u]\!] \to_{\underline{\mathtt{w}}} t = t_1$ with $|t|_x = 0$. Then, $\mathtt{M}_w(t_0) = \mathtt{M}_w(t) = \mathtt{M}_w(t_1)$.
– $t_0 = t[\![x/u]\!] \to_{\underline{\mathtt{d}}} t\{x/u\} = t_1$ with $|t|_x = 1$. W.l.g. $x \neq w$. Then

$$\begin{array}{ll}
\mathtt{M}_w(t_0) & = \\
\mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{M}_w(u) & =_{L.\ 30:2} \\
\mathtt{M}_w(t\{x/u\}) & = \mathtt{M}_w(t_1)
\end{array}$$

– $t_0 = t[\![x/u]\!] \to_{\underline{\mathtt{c}}} t_{[y]_x}[\![x/u]\!][\![y/u]\!] = t_1$ with $|t|_x > 1$. If $w \in \mathtt{fv}(u)$, then

$$\begin{array}{ll}
\mathtt{M}_w(t_0) & = \\
\mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t) + \mathtt{M}_x(t) \cdot \mathtt{M}_w(u) & =_{L.\ 29} \\
\mathtt{M}_w(t_{[y]_x}) + \mathtt{M}_x(t_{[y]_x}) \cdot \mathtt{M}_w(u) + \mathtt{M}_y(t_{[y]_x}) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t_{[y]_x}) + \mathtt{max}(1, \mathtt{M}_x(t_{[y]_x})) \cdot \mathtt{M}_w(u) + \mathtt{M}_y(t_{[y]_x}) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t_{[y]_x}[\![x/u]\!]) + \mathtt{M}_y(t_{[y]_x}) \cdot \mathtt{M}_w(u) & =_{L.\ 30:1} \\
\mathtt{M}_w(t_{[y]_x}[\![x/u]\!]) + \mathtt{M}_y(t_{[y]_x}[\![x/u]\!]) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t_{[y]_x}[\![x/u]\!]) + \mathtt{max}(1, \mathtt{M}_y(t_{[y]_x}[\![x/u]\!])) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t_1) & =
\end{array}$$

– For the inductive cases, let $t_0 = t[\![x/u]\!] \to_{\mathtt{j}} t'[\![x/u]\!] = t_1$ where $t \to_{\mathtt{j}} t'$. The only interesting case is when $x \in \mathtt{fv}(t)$ and $x \notin \mathtt{fv}(t')$. Then

$$\begin{array}{ll}
\mathtt{M}_w(t_0) & = \\
\mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{M}_w(u) & \geq \\
\mathtt{M}_w(t) + \mathtt{M}_w(u) & \geq_{i.h.} \\
\mathtt{M}_w(t') + \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t') + \mathtt{max}(1, \mathtt{M}_x(t')) \cdot \mathtt{M}_w(u) & = \\
\mathtt{M}_w(t_1) &
\end{array}$$

The cases for $t_0 = t[x/u]$ is similar to the previous one when $t_0 \to_{\mathtt{j}} t_1$ because $t \to_{\mathtt{j}} t'$. Suppose instead that $t_0 = t[x/u] \to_{\mathtt{j}} t[x/u'] = t_1$ where $u \to_{\mathtt{j}} u'$. Then,

$$\begin{array}{ll} \mathtt{M}_w(t_0) & = \\ \mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{M}_w(u) & \geq_{i.h.} \\ \mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{M}_w(u') & = \\ \mathtt{M}_w(t_1) & \end{array}$$

The other cases are straightforward.

**Lemma 32.** *Let* $t_0 \in \mathbb{WF}_{\mathtt{p}}$ *s.t.* $w \notin \mathbb{L}\mathtt{fv}(t_0)$*. If* $t_0 \to_{\mathtt{Fp:i}} t_1$*, then* $\mathtt{M}_w(t_0) = \mathtt{M}_w(t_1)$*.*

*Proof.* By induction on $t_0 \to_{\mathtt{b}} t_1$. If $t_0 = t[\![x/u]\!] \to_{\mathtt{Fp:i}} t[\![x/u']\!]$ then $w \notin u$ by hypothesis so that also $w \notin \mathtt{fv}(u')$. Then, we get $\mathtt{M}_w(u) = \mathtt{M}_w(u') = 0$ and

$$\begin{array}{ll} \mathtt{M}_w(t[\![x/u]\!]) & = \\ \mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{M}_w(u) & = \\ \mathtt{M}_w(t) + \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{M}_w(u') & = \\ \mathtt{M}_w(t[\![x/u']\!]) & \end{array}$$

The inductive cases are all straightforward, using the i.h.

Potential multiplicities can be altered only by $\mathtt{w}$, $\underline{\mathtt{w}}$ and $\mathtt{gdB}$-steps. In the first two cases they can decrease, in the last one they can be both increased or decreased. Consider $t_0 = (\lambda y.x)z[z/w] \to_{\mathtt{gdB}} x[y/z[z/w]] = t_1$. Then $\mathtt{M}_w(t_0) = 1$, while $\mathtt{M}_w(t_1) = 0$. Instead for $t_2 = (\lambda y.yy)z[z/w] \to_{\mathtt{gdB}} yy[y/z[z/w]] = t_3$ we get $\mathtt{M}_w(t_2) = 1$ and $\mathtt{M}_w(t_3) = 2$.

### Termination of $\to_{\mathtt{Fb}}$

We now consider multisets of pairs of integers. We use $n \cdot \langle x, y \rangle$ to denote the pair $\langle x, n \cdot y \rangle$. The operation $n \cdot \langle x, y \rangle$ is extended to multisets in the following way: if $M$ is a multiset of pairs of integers then $n \cdot M$ is the multiset $[n \cdot \langle x, y \rangle \mid \langle x, y \rangle \in M]$. Moreover, to improve readability, we write $M \sqcup \langle x, y \rangle$ rather than $M \sqcup [\langle x, y \rangle]$.

The **boxing measure** of $t \in \mathbb{WF}_{\mathtt{b}}$ is a multiset of pairs of integers, written $\mathtt{dep}(t)$, and given by:

$$\begin{array}{ll} \mathtt{dep}(x) & := [\,] \\ \mathtt{dep}(\lambda x.t) & := \mathtt{dep}(t) \\ \mathtt{dep}(tu) & := \mathtt{dep}(t) \sqcup \mathtt{dep}(u) \\ \mathtt{dep}(t[x/u]) & := \mathtt{dep}(t) \sqcup \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{dep}(u) \\ \mathtt{dep}(t[\![x/u]\!]) & := \mathtt{dep}(t) \sqcup \langle \eta_{\lambda \mathtt{j_b/o}}(u), \mathtt{M}_x(t) \rangle \end{array}$$

Remark that for every $u \in \mathcal{T}$ we have $\mathtt{dep}(u) = [\,]$.

**Lemma 33.** *Let* $u \in \mathcal{T}$*,* $t \in \mathbb{WF}_{\mathtt{b}}$ *and* $x \notin \mathbb{L}\mathtt{fv}(t)$*. Then* $\mathtt{dep}(t) = \mathtt{dep}(t\{x/u\})$*.*

*Proof.* By induction on $t$.

- $t = x$. Then, $\mathtt{dep}(x) = [\,] = \mathtt{dep}(u) = \mathtt{dep}(x\{x/u\})$.

– $t = y \neq x$. Then, $\mathtt{dep}(y) = [\,] = \mathtt{dep}(y\{x/u\})$.
– $t = t_1[y/t_2]$. W.l.g. we assume $y \neq x$ and $y \notin \mathtt{fv}(u)$. Then,

$$\begin{array}{ll}
\mathtt{dep}(t) & = \\
\mathtt{dep}(t_1) \sqcup \mathtt{max}(1, \mathtt{M}_y(t_1)) \cdot \mathtt{dep}(t_2) & =_{i.h.} \\
\mathtt{dep}(t_1\{x/u\}) \sqcup \mathtt{max}(1, \mathtt{M}_y(t_1)) \cdot \mathtt{dep}(t_2\{x/u\}) & =_{L.\ 30:1} \\
\mathtt{dep}(t_1\{x/u\}) \sqcup \mathtt{max}(1, \mathtt{M}_y(t_1\{x/u\})) \cdot \mathtt{dep}(t_2\{x/u\}) & = \\
\mathtt{dep}(t\{x/u\}) &
\end{array}$$

– $t = t_1[\![y/t_2]\!]$. W.l.g. we assume $y \neq x$ and $y \notin \mathtt{fv}(u)$. By hypothesis we have $x \notin \mathtt{fv}(t_2)$. Then

$$\begin{array}{ll}
\mathtt{dep}(t) & = \\
\mathtt{dep}(t_1) \sqcup \langle \eta(t_2), \mathtt{M}_y(t_1) \rangle & =_{i.h.} \\
\mathtt{dep}(t_1\{x/u\}) \sqcup \langle \eta(t_2), \mathtt{M}_y(t_1) \rangle & =_{L.\ 30:1} \\
\mathtt{dep}(t_1\{x/u\}) + \langle \eta(t_2), \mathtt{M}_y(t_1\{x/u\}) \rangle & = \\
\mathtt{dep}(t_1\{x/u\}[\![y/t_2]\!]) & = \\
\mathtt{dep}(t\{x/u\}) &
\end{array}$$

– All the other cases are straightforward by the i.h.

**Lemma 34.** *If* $|t|_x \geq 2$*, then* $\mathtt{dep}(t) = \mathtt{dep}(t_{[y]_x})$.

*Proof.* By induction on $t$.

The next lemma give the exact relation between the boxing measure, the reductions and the equivalences.

**Lemma 35.** *Let* $t_0 \in \mathbb{WF}_{\mathtt{b}}$*. Then,*

1. *If* $t_0 \equiv_{\mathtt{o},\underline{\mathtt{o}}} t_1$ *, then* $\mathtt{dep}(t_0) = \mathtt{dep}(t_1)$*.*
2. *If* $t_0 \rightarrow_{\underline{\mathtt{p}}} t_1$ *, then* $\mathtt{dep}(t_0) = \mathtt{dep}(t_1)$*.*
3. *If* $t_0 \rightarrow_{\underline{\mathtt{j}}} t_1$*, then* $\mathtt{dep}(t_0) > \mathtt{dep}(t_1)$*.*
4. *If* $t_0 \rightarrow_{\mathtt{Fp:i}} t_1$*, then* $\mathtt{dep}(t_0) > \mathtt{dep}(t_1)$*.*

*Proof.* By induction on the relations. We only show the the interesting cases.

– $t_0 = (tv)[\![x/u]\!] \rightarrow_{\underline{\mathtt{au}}} tv[\![x/u]\!] = t_1$ with $x \notin \mathtt{fv}(t)$ and $x \in \mathtt{fv}(v)$. Then

$$\begin{array}{ll}
\mathtt{dep}((tv)[\![x/u]\!]) & = \\
\mathtt{dep}(t) \sqcup \mathtt{dep}(v) \sqcup \langle \eta(u), \mathtt{M}_x(v) \rangle & = \\
\mathtt{dep}(tv[\![x/u]\!]) &
\end{array}$$

– $t_0 = t[y/v][\![x/u]\!] \rightarrow_{\underline{\mathtt{Comp}}} t[y/v[\![x/u]\!]] = t_1$ with $x \notin \mathtt{fv}(t)$ and $x \in \mathtt{fv}(v)$. Then

$$\begin{array}{ll}
\mathtt{dep}(t[y/v][\![x/u]\!]) & = \\
\mathtt{dep}(t) \sqcup \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{dep}(v) \sqcup \langle \eta(u), \mathtt{M}_x(t[y/v]) \rangle & = \\
\mathtt{dep}(t) \sqcup \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{dep}(v) \sqcup \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \langle \eta(u), \mathtt{M}_x(v) \rangle & = \\
\mathtt{dep}(t) \sqcup \mathtt{max}(1, \mathtt{M}_y(t)) \cdot (\mathtt{dep}(v) \sqcup \langle \eta(u), \mathtt{M}_x(v) \rangle) & = \\
\mathtt{dep}(t) \sqcup \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{dep}(v[\![x/u]\!]) & = \\
\mathtt{dep}(t[y/v[\![x/u]\!]]) &
\end{array}$$

– $t_0 = t[\![x/u]\!] \to_{\underline{\mathtt{w}}} t = t_1$ with $|t|_x = 0$. Then

$$\begin{array}{ll} \mathtt{dep}(t[\![x/u]\!]) & = \\ \mathtt{dep}(t) \sqcup \langle \eta(u), \mathtt{M}_x(t)\rangle & > \\ \mathtt{dep}(t) & \end{array}$$

– $t_0 = t[\![x/u]\!] \to_{\underline{\mathtt{d}}} t\{x/u\} = t_1$ with $|t|_x = 1$.

$$\begin{array}{ll} \mathtt{dep}(t[\![x/u]\!]) & = \\ \mathtt{dep}(t) \sqcup \langle \eta(u), \mathtt{M}_x(t)\rangle & > \\ \mathtt{dep}(t) & =_{L.\ 33} \\ \mathtt{dep}(t\{x/u\}) & \end{array}$$

– $t_0 = t[\![x/u]\!] \to_{\underline{\mathtt{c}}} t_{[y]_x}[\![x/u]\!][\![y/u]\!] = t_1$ with $|t|_x > 1$.

$$\begin{array}{ll} \mathtt{dep}(t[\![x/u]\!]) & = \\ \mathtt{dep}(t) \sqcup \langle \eta(u), \mathtt{M}_x(t)\rangle & >_{L.\ 29} \\ \mathtt{dep}(t_{[y]_x}) \sqcup \langle \eta(u), \mathtt{M}_x(t_{[y]_x})\rangle \sqcup \langle \eta(u), \mathtt{M}_y(t_{[y]_x})\rangle & =_{L.\ 34} \\ \mathtt{dep}(t_{[y]_x}[\![x/u]\!][\![y/u]\!]) & \end{array}$$

– $t_0 = t[\![x/u]\!] \to_{\mathtt{Fp:i}} t[\![x/u']\!] = t_1$. We have $\mathtt{dep}(t_0) = \mathtt{dep}(t) \sqcup \langle \eta(u), \mathtt{M}_x(t)\rangle > \mathtt{dep}(t) \sqcup \langle \eta(u'), \mathtt{M}_x(t)\rangle$.
– $t_0 = t[\![x/u]\!] \to_{\underline{\mathtt{j}},\underline{\mathtt{p}},\mathtt{Fp:i}} t'[\![x/u]\!] = t_1$, where $t \to_{\underline{\mathtt{j}},\underline{\mathtt{p}},\mathtt{Fp:i}} t'$. Since $t_0 \in \mathbb{WF}_{\mathtt{Fp:i}}$, then the hypothesis gives $x \notin \mathbb{L}\mathtt{fv}(t)$. Lemmas 28, 31 and 32 then gives $\mathtt{M}_x(t) \geq \mathtt{M}_x(t')$. Since $\mathtt{dep}(t_0) = \mathtt{dep}(t) \sqcup \langle \eta(u), \mathtt{M}_x(t)\rangle$ and $\mathtt{dep}(t_1) = \mathtt{dep}(t') \sqcup \langle \eta(u), \mathtt{M}_x(t')\rangle$, then the property holds by the i.h.
– All the other cases are straightforward.

Now, we conclude with the main statement.

**Lemma 21.** The relation $\to_{\mathtt{Fb}}$ modulo $\mathtt{O}$ is terminating on $\mathtt{b}$ well-formed labelled terms.

*Proof.* Using the Modular Abstract Theorem 5, where $\mathtt{A}_1$ is $\{\underline{\mathtt{ab}}, \underline{\mathtt{sb}}\}$, $\mathtt{A}_2$ is $\{\underline{\mathtt{j}}, \mathtt{Fp:i}\}$, $\mathcal{E}$ is $\mathtt{O}$, $\mathtt{A}$ is the relation $>$ on $\mathbb{N}$ and $\mathcal{R}$ is given by $t\ \mathcal{R}\ T$ iff $\mathtt{dep}(t) = T$. Properties **P0**, **P1** and **P2** of the Theorem 5 are guaranteed by Lemma 35, Property **P3** (termination of $\mathtt{A}_1/\mathtt{O}$) is straightforward.

### Termination of $\to_{\mathtt{Fu}}$

To prove termination in the $\to_{\mathtt{Fu}}$ case one hopes that the reasoning done for the $\to_{\mathtt{Fb}}$ case may be somehow re-used. However, reduction *inside* labelled jumps and reduction *out of* labelled jumps are independent in $\mathtt{Fb}$ but not in $\mathtt{Fu}$. Consider the rule $\underline{\mathtt{su}}_2$, the source of all complications:

$$t_0 = t[\![y/v[x/u]]\!] \to_{\underline{\mathtt{su}}_2} t[\![y/v]\!][\![x/u]\!] = t_1 \text{ if } x \in \mathtt{fv}(v)$$

The status of the jump $[x/u]$ is changed by this rule, so that the possible $\mathtt{j}$-reductions involving $[x/u]$ from $t_0$ become *labelled* $\mathtt{j}$-reductions from $t_1$. Thus, *inside* and *out of* labelled reductions are no longer independent and need to be treated together. As in the $\mathtt{Fb}$ case, one observes that *length of reductions* inside labelled jumps decrease. Thus for example we have $\eta(v[x/u]) > \eta(u), \eta(v)$ in the previous rule $\underline{\mathtt{su}}_2$. This needs also to be combined with the multiplicity of the jump in order to handle the duplicating rule. However, the situation is not so simple: the $\underline{\mathtt{d}}$-rule, whose target can now be a variable inside a labelled jump, introduces a (new) problematic case. Let us see an example:

$$t[\![x/uy]\!][\![y/v]\!] \to_{\underline{\mathtt{d}}} t[\![x/uv]\!]$$

In general $\eta(uv)$ is not smaller than $\eta(uy)$, and can be even greater. Hence, the natural idea is to compose labelled jumps before the computation of its measure. Thus, coming back to the previous example, the weight of the left-hand side term is determined by $\eta(uy\{x/v\})$ and $\eta(v)$, while the weight of the right-hand side term is only given by $\eta(v)$.

Therefore, we define a measure which composes labelled jumps to compute $\eta$, it is defined using an environment which stores the composition of all the labelled jumps appearing in the context.

The **unboxing measure** of $t \in \mathbb{WF}_{\mathtt{u}}$, is given by $\mathtt{D}(t, \mathtt{id})$, where for any meta-level substitution $\gamma$, $\mathtt{D}(t, \gamma)$ is defined as follows:

$$\begin{array}{ll}
\mathtt{D}(x,\gamma) & := [\,] \\
\mathtt{D}(\lambda y.t,\gamma) & := \mathtt{D}(t,\gamma) \\
\mathtt{D}(tu,\gamma) & := \mathtt{D}(t,\gamma) \sqcup \mathtt{D}(u,\gamma) \\
\mathtt{D}(t[y/u],\gamma) & := \mathtt{D}(t,\gamma) \sqcup \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{D}(u,\gamma) \\
\mathtt{D}(t[\![y/u]\!],\gamma) & := \mathtt{D}(t,\{y/u\}\gamma) \sqcup \langle \eta_{\lambda\mathtt{j}_{\mathtt{u}}/\mathtt{o}}(u\gamma), \mathtt{M}_y(t)\rangle
\end{array}$$

Note that $u \in \mathcal{T}$ implies $\mathtt{D}(u,\gamma) = [\,]$. Some preliminaries are needed in order to relate the measure, the equivalence and the reductions.

**Lemma 36.** *Let* $t \in \mathbb{WF}_{\mathtt{u}}$.

1. *If* $\gamma \to^*_{\lambda\mathtt{j}_{\mathtt{u}}/\mathtt{o}} \gamma'$, *then* $\mathtt{D}(t,\gamma) \geq \mathtt{D}(t,\gamma')$.
2. *If* $u \in \mathcal{T}$, *then* $\mathtt{D}(t\{x/u\},\gamma) = \mathtt{D}(t,\{x/u\}\gamma)$.

*Proof.* By induction on $t$.

1. Let $t = u[\![x/v]\!]$. Then

$$\begin{array}{ll}
\mathtt{D}(u[\![x/v]\!],\gamma) & = \\
\mathtt{D}(u,\{x/u\}\gamma) \sqcup \langle \eta(v\gamma), \mathtt{M}_x(u)\rangle & \geq_{i.h.} \\
\mathtt{D}(u,\{x/u\}\gamma') \sqcup \langle \eta(v\gamma'), \mathtt{M}_x(u)\rangle & = \\
\mathtt{D}(u[\![x/v]\!],\gamma') &
\end{array}$$

   All the other cases are straightforward.
2. – $t = x$. Then $\mathtt{D}(x\{x/u\},\gamma) = \mathtt{D}(u,\gamma) = \mathtt{D}(x,\{x/u\}\gamma) = [\,]$.
   – $t = y \neq x$. Then $\mathtt{D}(y\{x/u\},\gamma) = \mathtt{D}(y,\gamma) = \mathtt{D}(y,\{x/u\}\gamma) = [\,]$.
   – $t = t_1[y/t_2]$. W.l.g. we can assume $y \notin \mathtt{fv}(u)$ and $y \neq x$. Then,

$$
\begin{array}{ll}
\mathtt{D}(t_1[y/t_2]\{x/u\},\gamma) & = \\
\mathtt{D}(t_1\{x/u\}[y/t_2\{x/u\}],\gamma) & = \\
\mathtt{D}(t_1\{x/u\},\gamma) + \mathtt{max}(1, \mathtt{M}_y(t_1\{x/u\})) \cdot \mathtt{D}(t_2\{x/u\},\gamma) & = \\
\mathtt{D}(t_1\{x/u\},\gamma) + \mathtt{max}(1, \mathtt{M}_y(t_1\{x/u\})) \cdot \mathtt{D}(t_2\{x/u\},\gamma) & =_{i.h.} \\
\mathtt{D}(t_1, \{x/u\}\gamma) + \mathtt{max}(1, \mathtt{M}_y(t_1\{x/u\})) \cdot \mathtt{D}(t_2, \{x/u\}\gamma) & =_{L.\ 30:1} \\
\mathtt{D}(t_1, \{x/u\}\gamma) + \mathtt{max}(1, \mathtt{M}_y(t_1)) \cdot \mathtt{D}(t_2, \{x/u\}\gamma) & = \\
\mathtt{D}(t_1[y/t_2], \{x/u\}\gamma) &
\end{array}
$$

- $t = t_1[\![y/t_2]\!]$. W.l.g. we can assume $y \notin \mathtt{fv}(u)$ and $y \neq x$. Then,

$$
\begin{array}{ll}
\mathtt{D}(t_1[\![y/t_2]\!]\{x/u\},\gamma) & = \\
\mathtt{D}(t_1\{x/u\}[\![y/t_2\{x/u\}]\!],\gamma) & = \\
\mathtt{D}(t_1\{x/u\}, \{y/t_2\{x/u\}\}\gamma) \sqcup \langle \eta(t_2\{x/u\}\gamma), \mathtt{M}_y(t_1\{x/u\})\rangle & =_{i.h.} \\
\mathtt{D}(t_1, \{x/u\}\{y/t_2\{x/u\}\}\gamma) \sqcup \langle \eta(t_2\{x/u\}\gamma), \mathtt{M}_y(t_1\{x/u\})\rangle & =_{L.\ 30} \\
\mathtt{D}(t_1, \{y/t_2\}\{x/u\}\gamma) \sqcup \langle \eta(t_2\{x/u\}\gamma), \mathtt{M}_y(t_1)\rangle & = \\
\mathtt{D}(t_1[\![y/t_2]\!], \{x/u\}\gamma) &
\end{array}
$$

- All the other cases are straightforward.

**Lemma 37.** *Let* $t \in \mathbb{WF}_{\mathtt{u}}$.

1. $t_0 \equiv_{\mathtt{O}} t_1$ *implies* $\mathtt{D}(t_0,\gamma) = \mathtt{D}(t_1,\gamma)$.
2. $t_0 \to_{\underline{\mathtt{au}},\underline{\mathtt{su}}_1} t_1$ *implies* $\mathtt{D}(t_0,\gamma) = \mathtt{D}(t_1,\gamma)$.
3. $t_0 \to_{\underline{\mathtt{j}},\underline{\mathtt{su}}_2} t_1$ *implies* $\mathtt{D}(t_0,\gamma) > \mathtt{D}(t_1,\gamma)$.

*Proof.* By induction on the relations.

1. The equivalence $\mathtt{O}$.
   - $\mathtt{CS}$: $t_0 = t[x/u][y/v] \equiv t[y/v][x/u] = t_1$ if $y \notin \mathtt{fv}(u)$ and $x \notin \mathtt{fv}(v)$. Observe that $y \notin \mathtt{fv}(u)$ implies $\mathtt{M}_y(t[x/u]) = \mathtt{M}_y(t) + \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{M}_y(u) = \mathtt{M}_y(t)$. Then,

$$
\begin{array}{ll}
\mathtt{D}(t_0,\gamma) & = \\
\mathtt{D}(t[x/u],\gamma) \sqcup \mathtt{max}(1, \mathtt{M}_y(t[x/u])) \cdot \mathtt{D}(v,\gamma) & = \\
\mathtt{D}(t,\gamma) \sqcup \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{D}(u,\gamma) \sqcup \mathtt{max}(1, \mathtt{M}_y(t[x/u])) \cdot \mathtt{D}(v,\gamma) & = \\
\mathtt{D}(t,\gamma) \sqcup \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{D}(u,\gamma) \sqcup \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{D}(v,\gamma) & = \\
\mathtt{D}(t[y/v],\gamma) \sqcup \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{D}(u,\gamma) & = \\
\mathtt{D}(t[y/v],\gamma) \sqcup \mathtt{max}(1, \mathtt{M}_x(t[y/v])) \cdot \mathtt{D}(u,\gamma) & = \\
\mathtt{D}(t_1,\gamma) &
\end{array}
$$

   - $\underline{\mathtt{CS}}_1$: $t_0 = t[\![x/u]\!][y/v] \equiv t[y/v][\![x/u]\!] = t_1$ if $y \notin \mathtt{fv}(u)$ and $x \notin \mathtt{fv}(v)$. As in the previous case we have $y \notin \mathtt{fv}(u)$ implies $\mathtt{M}_y(t[\![x/u]\!]) = \mathtt{M}_y(t) + \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{M}_y(u) = \mathtt{M}_y(t)$. Then,

$$
\begin{array}{ll}
\mathtt{D}(t_0,\gamma) & = \\
\mathtt{D}(t[\![x/u]\!],\gamma) \sqcup \mathtt{max}(1, \mathtt{M}_y(t[\![x/u]\!])) \cdot \mathtt{D}(v,\gamma) & = \\
\mathtt{D}(t[\![x/u]\!],\gamma) \sqcup \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{D}(v,\gamma) & = \\
\mathtt{D}(t, \{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t)\rangle \sqcup \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{D}(v,\gamma) & = \\
\mathtt{D}(t, \{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t[y/v])\rangle \sqcup \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{D}(v,\gamma) & = \\
\mathtt{D}(t, \{x/u\}\gamma) \sqcup \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{D}(v,\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t[y/v])\rangle & =_{L.36} \\
\mathtt{D}(t, \{x/u\}\gamma) \sqcup \mathtt{max}(1, \mathtt{M}_y(t)) \cdot \mathtt{D}(v, \{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t[y/v])\rangle & = \\
\mathtt{D}(t[y/v], \{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t[y/v])\rangle & = \\
\mathtt{D}(t_1,\gamma) &
\end{array}
$$

- $\underline{\mathtt{CS}}_2$: $t_0 = t[\![x/u]\!][\![y/v]\!] \equiv t[\![y/v]\!][\![x/u]\!] = t_1$ if $y \notin \mathtt{fv}(u)$ and $x \notin \mathtt{fv}(v)$. Observe that the hypotesis imply $u(\{y/v\}\gamma) = u\gamma$ and $v(\{x/u\}\gamma) = v\gamma$. Then,

$$\begin{array}{ll}
\mathtt{D}(t_0, \gamma) & = \\
\mathtt{D}(t[\![x/u]\!], \{y/v\}\gamma) \sqcup \langle \eta(v\gamma), \mathtt{M}_y(t[\![x/u]\!]) \rangle & = \\
\mathtt{D}(t, \{x/u\}\{y/v\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t) \rangle \sqcup \langle \eta(v\gamma), \mathtt{M}_y(t[\![x/u]\!]) \rangle & = \\
\mathtt{D}(t, \{x/u\}\{y/v\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t) \rangle \sqcup \langle \eta(v\gamma), \mathtt{M}_y(t) \rangle & = \\
\mathtt{D}(t, \{y/v\}\{x/u\}\gamma) \sqcup \langle \eta(v\gamma), \mathtt{M}_y(t) \rangle \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t) \rangle & = \\
\mathtt{D}(t[\![y/v]\!], \{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t) \rangle & = \\
\mathtt{D}(t[\![y/v]\!], \{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t[\![y/v]\!]) \rangle & = \\
\mathtt{D}(t_1, \gamma) &
\end{array}$$

- $\sigma_1$: $t_0 = (\lambda y.t)[x/u] \equiv \lambda y.t[x/u] = t_1$ if $y \notin \mathtt{fv}(u)$. Then,

$$\begin{array}{ll}
\mathtt{D}(t_0, \gamma) & = \\
\mathtt{D}(\lambda y.t, \gamma) \sqcup \mathtt{max}(1, \mathtt{M}_x(\lambda y.t)) \cdot \mathtt{D}(u, \gamma) & = \\
\mathtt{D}(t, \gamma) \sqcup \mathtt{max}(1, \mathtt{M}_x(\lambda y.t)) \cdot \mathtt{D}(u, \gamma) & = \\
\mathtt{D}(t, \gamma) \sqcup \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{D}(u, \gamma) & = \\
\mathtt{D}(t[x/u], \gamma) & = \\
\mathtt{D}(t_1, \gamma) &
\end{array}$$

- $\underline{\sigma}_1$: $t_0 = (\lambda y.t)[\![x/u]\!] \equiv \lambda y.t[\![x/u]\!] = t_1$ if $y \notin \mathtt{fv}(u)$.

$$\begin{array}{ll}
\mathtt{D}(t_0, \gamma) & = \\
\mathtt{D}(\lambda y.t, \{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(\lambda y.t) \rangle & = \\
\mathtt{D}(t, \{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(\lambda y.t) \rangle & = \\
\mathtt{D}(t, \{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t) \rangle & = \\
\mathtt{D}(t[\![x/u]\!], \gamma) & = \\
\mathtt{D}(t_1, \gamma) &
\end{array}$$

- $\sigma_2$: $t_0 = (tv)[x/u] \equiv t[x/u]v = t_1$ if $x \notin \mathtt{fv}(v)$. Then,

$$\begin{array}{ll}
\mathtt{D}(t_0, \gamma) & = \\
\mathtt{D}(tv, \gamma) \sqcup \mathtt{max}(1, \mathtt{M}_x(tv)) \cdot \mathtt{D}(u, \gamma) & = \\
\mathtt{D}(tv, \gamma) \sqcup \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{D}(u, \gamma) & = \\
\mathtt{D}(t, \gamma) \sqcup \mathtt{D}(v, \gamma) \sqcup \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{D}(u, \gamma) & = \\
\mathtt{D}(t, \gamma) \sqcup \mathtt{max}(1, \mathtt{M}_x(t)) \cdot \mathtt{D}(u, \gamma) \sqcup \mathtt{D}(v, \gamma) & = \\
\mathtt{D}(t[x/u], \gamma) \sqcup \mathtt{D}(v, \gamma) & = \\
\mathtt{D}(t_1, \gamma) &
\end{array}$$

- $\underline{\sigma}_2$: $t_0 = t[\![x/u]\!]v \equiv (tv)[\![x/u]\!] = t_1$ if $x \notin \mathtt{fv}(v)$. Then,

$$\begin{array}{ll}
\mathtt{D}(t_0, \gamma) & = \\
\mathtt{D}(tv, \{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(tv) \rangle & = \\
\mathtt{D}(t, \{x/u\}\gamma) \sqcup \mathtt{D}(v, \{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(tv) \rangle & = \\
\mathtt{D}(t, \{x/u\}\gamma) \sqcup \mathtt{D}(v, \{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t) \rangle & = \\
\mathtt{D}(t, \{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t) \rangle \sqcup \mathtt{D}(v, \{x/u\}\gamma) & = \\
\mathtt{D}(t[\![x/u]\!], \gamma) \sqcup \mathtt{D}(v, \{x/u\}\gamma) & = \\
\mathtt{D}(t[\![x/u]\!], \gamma) \sqcup \mathtt{D}(v, \gamma) & = \\
\mathtt{D}(t_1, \gamma) &
\end{array}$$

2. The reductions $\underline{\texttt{au}}$ and $\underline{\texttt{su}}_1$.

   – $\underline{\texttt{au}}$: $t_0 = tv[\![x/u]\!] \to (tv)[\![x/u]\!] = t_1$, where $x \in \texttt{fv}(v)$ and $x \notin \texttt{fv}(t)$. Then

$$\begin{array}{ll} \texttt{D}(t_0,\gamma) & = \\ \texttt{D}(t,\gamma) \sqcup \texttt{D}(v,\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(v)\rangle & = \\ \texttt{D}(t,\gamma) \sqcup \texttt{D}(v,\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(tv)\rangle & = \\ \texttt{D}(t_1,\gamma) & \end{array}$$

   – $\underline{\texttt{su}}_1$: $t_0 = t[y/v[\![x/u]\!]] \to t[y/v][\![x/u]\!] = t_1$, where $x \in \texttt{fv}(v)$ and $x \notin \texttt{fv}(t)$. Then

$$\begin{array}{ll} \texttt{D}(t_0,\gamma) & = \\ \texttt{D}(t,\gamma) \sqcup \texttt{max}(1,\texttt{M}_y(t)) \cdot \texttt{D}(v[\![x/u]\!],\gamma) & = \\ \texttt{D}(t,\gamma) \sqcup \texttt{max}(1,\texttt{M}_y(t)) \cdot (\texttt{D}(v,\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(v)\rangle) & = \\ \texttt{D}(t,\gamma) \sqcup \texttt{max}(1,\texttt{M}_y(t)) \cdot \texttt{D}(v,\{x/u\}\gamma) \sqcup \texttt{max}(1,\texttt{M}_y(t)) \cdot \langle \eta(u\gamma), \texttt{M}_x(v)\rangle & = \\ \texttt{D}(t,\gamma) \sqcup \texttt{max}(1,\texttt{M}_y(t)) \cdot \texttt{D}(v,\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \texttt{max}(1,\texttt{M}_y(t)) \cdot \texttt{M}_x(v)\rangle & = \\ \texttt{D}(t,\gamma) \sqcup \texttt{max}(1,\texttt{M}_y(t)) \cdot \texttt{D}(v,\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(t) + \texttt{max}(1,\texttt{M}_y(t)) \cdot \texttt{M}_x(v)\rangle & = \\ \texttt{D}(t,\gamma) \sqcup \texttt{max}(1,\texttt{M}_y(t)) \cdot \texttt{D}(v,\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(t[y/v])\rangle & =_{L.\ 36} \\ \texttt{D}(t,\{x/u\}\gamma) \sqcup \texttt{max}(1,\texttt{M}_y(t)) \cdot \texttt{D}(v,\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(t[y/v])\rangle & = \\ \texttt{D}(t[y/v],\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(t[y/v])\rangle & = \\ \texttt{D}(t_1,\gamma) & \end{array}$$

3. The reductions $\underline{\texttt{j}}$ and $\underline{\texttt{su}}_2$.

   – $\underline{\texttt{w}}$: $t_0 = t[\![y/u]\!] \to t = t_1$, with $|t|_y = 0$.

$$\begin{array}{lll} \texttt{D}(t[\![x/u]\!],\gamma) & = & \\ \texttt{D}(t,\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(t)\rangle & > & \\ \texttt{D}(t,\{x/u\}\gamma) & =_{L.\ 36:2} & \texttt{D}(t,\gamma) \end{array}$$

   – $\underline{\texttt{d}}$: $t_0 = t[\![x/u]\!] \to t\{x/u\} = t_1$, with $|t|_x = 1$.

$$\begin{array}{lll} \texttt{D}(t,\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(t)\rangle & > & \\ \texttt{D}(t,\{x/u\}\gamma) & =_{L.\ 36:2} & \texttt{D}(t\{x/u\},\gamma) \end{array}$$

   – $\underline{\texttt{c}}$: $t_0 = t[\![x/u]\!] \to t_{[y]_x}[\![y/u]\!][\![x/u]\!] = t_1$, with $|t|_x \geq 2$ and $y$ fresh. Then,

$$\begin{array}{ll} \texttt{D}(t_0,\gamma) & = \\ \texttt{D}(t,\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(t)\rangle & =_{L.\ 36:2} \\ \texttt{D}(t\{x/u\},\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(t)\rangle & = \\ \texttt{D}(t_{[y]_x}\{y/u\}\{x/u\},\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(t)\rangle & =_{L.\ 36:2} \\ \texttt{D}(t_{[y]_x},\{y/u\}\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(t)\rangle & > (1) \\ \texttt{D}(t_{[y]_x},\{y/u\}\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(t_{[y]_x})\rangle \sqcup \langle \eta(u\gamma), \texttt{M}_y(t_{[y]_x})\rangle & \\ \texttt{D}(t_{[y]_x}[\![y/u]\!],\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \texttt{M}_x(t_{[y]_x})\rangle & = \ \texttt{D}(t_1,\gamma) \end{array}$$

   Remark that (1) holds since $\texttt{M}_x(t_{[y]_x}), \texttt{M}_y(t_{[y]_x}) > 0$ by hypothesis and $\texttt{M}_x(t_{[y]_x}) + \texttt{M}_y(t_{[y]_x}) = \texttt{M}_x(t)$.

- $\underline{\mathtt{su}}_2$: $t_0 = t[\![y/v[x/u]]\!] \to t[\![y/v]\!][\![x/u]\!] = t_1$, with $x \in \mathtt{fv}(v)$ and $x \notin \mathtt{fv}(t)$. Then,

$$\begin{array}{ll}
\mathtt{D}(t_0,\gamma) & = \\
\mathtt{D}(t,\{y/v[x/u]\}\gamma) \sqcup \langle \eta(v[x/u]\gamma), \mathtt{M}_y(t)\rangle & >_{(1)} \\
\mathtt{D}(t,\{y/v[x/u]\}\gamma) \sqcup \langle \eta(v\{x/u\}\gamma), \mathtt{M}_y(t)\rangle \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t[\![y/v]\!])\rangle & \geq_{L.\ 36:1} \\
\mathtt{D}(t,\{y/v\{x/u\}\}\gamma) \sqcup \langle \eta(v\{x/u\}\gamma), \mathtt{M}_y(t)\rangle \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t[\![y/v]\!])\rangle & =_{L.\ 36:2} \\
\mathtt{D}(t,\{y/v\}\{x/u\}\gamma) \sqcup \langle \eta(v\{x/u\}\gamma), \mathtt{M}_y(t)\rangle \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t[\![y/v]\!])\rangle & = \\
\mathtt{D}(t[\![y/v]\!],\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t[\![y/v]\!])\rangle & = \ \mathtt{D}(t_1,\gamma)
\end{array}$$

Step (1) holds since $\eta(v[x/u]\gamma) > \eta(v\{x/u\}\gamma)$ and $\eta(v[x/u]\gamma) > \eta(u\gamma)$.

Now, for the inductive cases the only interesting case is when $t_0 = t[\![x/u]\!] \equiv$ (resp. $\to$) $t'[\![x/u]\!] = t_1$, where $t \equiv$ (resp. $\to$) $t'$.
If $t \equiv$ (resp. $\to_{\underline{\mathtt{au}},\underline{\mathtt{su}}_1}$) $t'$, we have

$$\begin{array}{ll}
\mathtt{D}(t_0,\gamma) & = \\
\mathtt{D}(t,\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t)\rangle & =_{i.h.} \\
\mathtt{D}(t',\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t)\rangle & =_{L.\ 28} \\
\mathtt{D}(t',\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t')\rangle & = \mathtt{D}(t_1,\gamma)
\end{array}$$

If $t \to_{\underline{\mathtt{j}},\underline{\mathtt{su}}_2} t'$, we have

$$\begin{array}{ll}
\mathtt{D}(t_0,\gamma) & = \\
\mathtt{D}(t,\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t)\rangle & >_{i.h.} \\
\mathtt{D}(t',\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t)\rangle & \geq_{L.\ 37} \\
\mathtt{D}(t',\{x/u\}\gamma) \sqcup \langle \eta(u\gamma), \mathtt{M}_x(t')\rangle & = \mathtt{D}(t_1,\gamma)
\end{array}$$

All the other cases are straightforward.

The next and last lemma proves that the measure with store decreases by $\to_{\mathtt{gdB}}$-steps. The second point of the lemma is technical, used to prove the first one.

**Lemma 38.** *Let $t_0 \in \mathbb{WF}_{\mathtt{u}}$ s.t. $t_0 \to_{\mathtt{Fp:i}} t_1$. Then*

1. $\mathtt{D}(t_0,\gamma) > \mathtt{D}(t_1,\gamma)$.
2. $\mathtt{D}(t_0,\rho \cup \{x/u\gamma\}) > \langle \eta(u\gamma), K\rangle$ $\forall K$ *and* $\forall x$ *s.t.* $\mathtt{M}_x(t_0) < \mathtt{M}_x(t_1)$.

*Proof.* By induction on $t_0$. Let us note $\gamma' = \rho \cup \{x/u\gamma\}$.

- $t_0 = z$ is not possible.
- $t_0 = v_0\ v_1$. Suppose $t_0 = v_0\ v_1 \to_{\mathtt{Fp:i}} v_0'\ v_1 = t_1$, where $v_0 \to_{\mathtt{Fp:i}} v_0'$. (the case $t = v_0\ v_1 \to_{\mathtt{Fp:i}} v_0\ v_1' = t'$, where $v_1 \to_{\mathtt{Fp:i}} v_1'$ being similar).
  1. We have
  $$\mathtt{D}(t_0,\gamma) = \mathtt{D}(v_0,\gamma) \sqcup \mathtt{D}(v_1,\gamma) >_{i.h.} \mathtt{D}(v_0',\gamma) \sqcup \mathtt{D}(v_1,\gamma) = \mathtt{D}(t_1,\gamma)$$
  2. We have $\mathtt{D}(t_0,\gamma') = \mathtt{D}(v_0,\gamma') \sqcup \mathtt{D}(v_1,\gamma')$. Also $\mathtt{M}_x(t_0) < \mathtt{M}_x(t_1)$ implies in particular $\mathtt{M}_x(v_0) < \mathtt{M}_x(v_0')$. The i.h. then states that $\mathtt{D}(v_0,\gamma')$ verifies the property, and so does also $\mathtt{D}(t_0,\gamma')$.
- $t_0 = v_0[y/v_1]$. Suppose $t_0 = v_0[y/v_1] \to_{\mathtt{Fp:i}} v_0'[y/v_1] = t_1$, where $v_0 \to_{\mathtt{Fp:i}} v_0'$..

1. We have

$$\begin{array}{l}\mathtt{D}(v_0[y/v_1],\gamma) = \\ \mathtt{D}(v_0,\gamma) \sqcup \mathtt{max}(1,\mathtt{M}_y(v_0)) \cdot \mathtt{D}(v_1,\gamma) >_{i.h.} \\ \mathtt{D}(v'_0,\gamma) \sqcup \mathtt{max}(1,\mathtt{M}_y(v_0)) \cdot \mathtt{D}(v_1,\gamma) = \\ \mathtt{D}(v_0[y/v'_1],\gamma)\end{array}$$

2. As for the preceding case.

The case $t_0 = v_0[y/v_1] \to_{\mathtt{Fp:i}} v_0[y/v'_1] = t_1$, where $v_1 \to_{\mathtt{Fp:i}} v'_1$ is similar.

– $t_0 = v_0[\![y/v_1]\!]$.

Suppose $t_0 = v_0[\![y/v_1]\!] \to_{\mathtt{Fp:i}} v_0[\![y/v'_1]\!] = t_1$, where $v_1 \to_{\lambda\mathtt{j_u/o}} v'_1$.

1. We have

$$\begin{array}{l}\mathtt{D}(v_0[\![y/v_1]\!],\gamma) = \\ \mathtt{D}(v_0,\{y/v_1\}\gamma) \sqcup \langle \eta(v_1\gamma),\mathtt{M}_y(v_0)\rangle \geq_{L.\ 36:1} \\ \mathtt{D}(v_0,\{y/v'_1\}\gamma) \sqcup \langle \eta(v_1\gamma),\mathtt{M}_y(v_0)\rangle > \\ \mathtt{D}(v_0,\{y/v'_1\}\gamma) \sqcup \langle \eta(v'_1\gamma),\mathtt{M}_y(v_0)\rangle = \mathtt{D}(v_0[\![y/v'_1]\!],\gamma)\end{array}$$

2. Let $\mathtt{M}_x(t_0) < \mathtt{M}_x(t_1)$. Then necessarily $0 \neq \mathtt{M}_x(v_1) < \mathtt{M}_x(v'_1)$. We have $\mathtt{D}(v_0[\![y/v_1]\!],\gamma') = \mathtt{D}(v_0,\{y/v_1\}\gamma') \sqcup \langle \eta(v_1\gamma'),\mathtt{M}_y(v_0)\rangle$.
Since $x \in \mathtt{fv}(v_1)$, then $v_1\gamma'$ contains $u\gamma$ and thus $\eta(v_1\gamma') \geq \eta(u\gamma)$. Moreover, $v_1$ is $\lambda\mathtt{j_u}/\mathtt{o}$-reducible so that $\eta(v_1\gamma') \geq \eta(u\gamma)+1$ and thus $\eta(v_1\gamma') > \eta(u\gamma)$. We thus conclude.

Suppose $t_0 = v_0[\![y/v_1]\!] \to_{\mathtt{Fp:i}} v'_0[\![y/v_1]\!] = t_1$, where $v_0 \to_{\mathtt{Fp:i}} v'_0$.

1. We have $\mathtt{D}(v_0[\![y/v_1]\!],\gamma) = \mathtt{D}(v_0,\{y/v_1\}\gamma) \sqcup \langle \eta(v_1\gamma),\mathtt{M}_y(v_0)\rangle$ and $\mathtt{D}(v'_0[\![y/v_1]\!],\gamma) = \mathtt{D}(v'_0,\{y/v_1\}\gamma) \sqcup \langle \eta(v_1\gamma),\mathtt{M}_y(v'_0)\rangle$. Also, $\mathtt{D}(v_0,\{y/v_1\}\gamma) >_{i.h.\ (1)} \mathtt{D}(v'_0,\{y/v_1\}\gamma)$.
If $\mathtt{M}_y(v_0) \geq \mathtt{M}_y(v'_0)$, then $\langle \eta(v_1\gamma),\mathtt{M}_y(v_0)\rangle \geq \langle \eta(v_1\gamma),\mathtt{M}_y(v'_0)\rangle$ and we conclude.
If $\mathtt{M}_y(v_0) < \mathtt{M}_y(v'_0)$, then $\mathtt{D}(v_0,\{y/v_1\}\gamma) >_{i.h.\ (2)} \langle \eta(v_1\gamma),\mathtt{M}_y(v'_0)\rangle$ and we also conclude.

– All the other cases are straightforward.

**Lemma 22.** The relation $\to_{\mathtt{Fu}}$ modulo $\mathtt{O}$ is terminating on $\mathtt{u}$ well-formed labelled terms.

*Proof.* Using the Modular Abstract Theorem 5, where $\mathtt{A}_1$ is $\{\underline{\mathtt{au}},\underline{\mathtt{su}}_1\}$, $\mathtt{A}_2$ is $\{\underline{\mathtt{j}},\underline{\mathtt{su}}_2,\mathtt{Fp}:\mathtt{i}\}$, $\mathcal{E}$ is $\mathtt{O}$, $\mathtt{A}$ is the relation $>$ on $\mathbb{N}$ and $\mathcal{R}$ is given by $t\ \mathcal{R}\ T$ iff $\mathtt{D}(t,[\ ]) = T$. Properties **P0**, **P1** and **P2** of the Theorem 5 are guaranteed by Lemmas 37 and 38 , Property **P3** (termination of $\mathtt{A}_1/\mathtt{O}$) is straightforward.