Modification of Coulomb law and energy levels of the hydrogen atom in a superstrong magnetic field

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26 November 2010. Revised 17 December 2010

We obtain the following analytical formula which describes the dependence of the electric potential of a point-like charge on the distance away from it in the direction of an external magnetic field B:
$\mathbf{\Phi}(z) = e/|z| \left[1 - \exp(-\sqrt{6m_e^2}\,|z|) + \exp(-\sqrt{(2/\pi)e^3B + 6m_e^2}\,|z|)\right]$.
The deviation from Coulomb's law becomes essential for $B > 3\pi B_{cr}/\alpha = 3\pi m_e^2/e^3 \approx 6 \cdot 10^{16}$ G. In such superstrong fields, electrons are ultra-relativistic except those which occupy the lowest Landau level (LLL) and which have the energy $\varepsilon_0^2 = m_e^2 + p_z^2$. The energy spectrum on which LLL splits in the presence of the atomic nucleus is found analytically. For $B > 3\pi B_{cr}/\alpha$ it substantially differs from the one obtained without accounting for the modification of the atomic potential.

1 Introduction

In the pioneering papers [1] an exponential modification of Coulomb's law by superstrong magnetic fields $B > m_e^2/e^3 = 137 B_{cr}$ was discovered ($B_{cr} \equiv m_e^2/e = 4.4 \cdot 10^{13} G = 4.4 \cdot 10^9 T$ is the so-called *critical* or *Schwinger* magnetic field [5]). It originates from the strong (linear) dependence of the photon polarization operator on the external B. The Coulomb potential gets modified at distances $1/m_e > r > 1/\sqrt{e^3 B} \equiv a_H/e$, where a_H is the Landau radius (the size of the ground state electron wave function in the direction transverse to the homogeneous magnetic field). In [1] the shape of the modified potential was determined numerically.

As found long ago in papers [2] the photon polarization operator in a strong magnetic fields $B > B_{cr}$ is dominated by electron and positron states belonging to the lowest Landau level (LLL) and it factorizes into transverse and longitudinal (with respect to the direction

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[5] All formulas are written in Gauss units, where $\alpha = e^2 = 1/137$ ($\hbar = c = 1$ is implied).

of B) parts. Its dependence on the longitudinal and time-like components of the momentum coincides with that in two-dimensional QED (with no B). In paper [3] a simple interpolating formula for the photon polarization operator in two-dimensional QED was found, which allows to find an analytical expression for the potential of a point-like electric charge in $D = 2$ QED. For light "electrons" propagating in the loop ($m < g$, where g is the electric charge in $D = 2$ QED), the electric potential gets screened at all distances larger than $1/(2g)$.

According to [3] the deviation of the interpolating formula from the exact expression for the polarization operator in $D = 2$ does not exceed 10%. In Section 2 we will show that for various values of the ratio g/m the accuracy of the analytical formula for the potential in $D = 2$ is always better than 4% for all distances z.

For the realistic case $D = 4$ only the asymptotics of the electric potential at large ($|z| \gg 1/m_e$) and small ($|z| \ll 1/m_e$) distances were found in [3]. In agreement with [1] the screening only takes place at short distances $|z| \ll 1/m_e$. In Section 3, using the interpolating formula for the polarization operator of [3], we will obtain an analytical expression for the dependence of the electric potential of a point-like charge on the distance to this charge in the longitudinal direction (parallel to the magnetic field B) and at zero transverse direction. This potential determines the energies of atomic electrons. The spectrum of electrons originating from LLL will be found in Section 4.

In [1] the ground state energy of a hydrogen atom in a superstrong magnetic field was found in the shallow-well approximation. The detailed analytical study and comparison with numerical results for the atomic levels in an external magnetic field performed in paper [4] clearly demonstrates that the shallow-well approximation used originally for this problem in textbook [5] has a very poor accuracy (better to say no accuracy at all). An algebraic expression for the energies of the ground and excited states was obtained in [4] which reproduces the results of numerical calculations with high accuracy. In Section 4 an analogous expression, but which now takes screening into account will be derived. It yields a value of the ground state energy in the limit of infinite B which is $E_0 = -1.7$ keV, which strongly differs from the result of the shallow-well approximation [1]: $E_0^{sw} = -4.0$ keV.

Excited states can be classified with respect to the change of sign of the z coordinate and divided into even and odd states. Odd states in the magnetic field $B \gtrsim m_e^2/e^3 \equiv B_{cr}/\alpha$ follow unperturbed Coulomb levels with very high accuracy: $E_{odd} = -me^4/(2n^2)$, $n = 1, 2, ...$ and the screening does not alter this result. Concerning even states, at finite B they considerably deviate from Coulomb levels. These deviations decrease with increasing B like $1/\log(B)$, and, in the limit of infinite B, the even and odd states become degenerate [6]. The screening changes this result qualitatively and lifts this degeneracy: the energies of even states remain considerably higher than those of corresponding odd states, even in the limit $B \to \infty$.

The dependence of the electron mass on the external magnetic field is discussed in Appendix.

We summarize our results in the Conclusion.

2 Coulomb potential modification in $D = 2$ QED

The fermionic part of the $D = 2$ QED Lagrangian is $L = \bar{\psi}(i\hat{\partial} - g\hat{A} - m)\psi$, where $\gamma_0 = \sigma_1, \gamma_1 = i\sigma_2$ are Pauli matrices. Inserting the photon polarization operator Π into the photon

propagator D leads to the following equations (see Fig. 1). The tree level Coulomb potential

$$\Phi(\vec{k}) \equiv A_0(\vec{k}) = \frac{4\pi g}{\vec{k}^2} \tag{1}$$

gets transformed into the geometrical series

$$\mathbf{\Phi} \equiv \mathbf{A}_0 = D_{00} + D_{00}\Pi_{00}D_{00} + ... \, , \tag{2}$$

and summing it we obtain

$$\mathbf{\Phi}(k) = -\frac{4\pi g}{k^2 + \Pi(k^2)} \, , \quad \Pi_{\mu\nu} \equiv \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi(k^2) \, , \tag{3}$$

where the expression for the photon polarization operator in $D = 2$ should be used. Instead of calculating the fermion loop we can take the expression for Π obtained in the dimensional regularization method [7], substitute $D = 2$ in it and divide it by two, because in two dimensions the traces of γ-matrices are proportional to 2 instead of 4:

$$\Pi(k^2) = 4g^2 \left[\frac{1}{\sqrt{t(1+t)}} \ln(\sqrt{1+t} + \sqrt{t}) - 1 \right] \equiv -4g^2 P(t) \tag{4}$$

where $t \equiv -k^2/4m^2$.

This approach resembles the one used in the calculation of the Uehling-Serber corrections to the Coulomb potential in [8]. However we resum the whole geometric series instead of considering 1-loop correction.

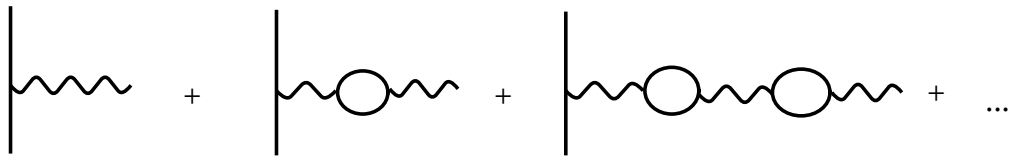

Fig. 1. *Modification of the Coulomb potential due to the dressing of the photon propagator.*

To obtain the electric potential $\mathbf{\Phi}(z)$ of a charge at rest we should set $k_\mu \equiv (k_0, k_\parallel) = (0, k_\parallel)$ (the notation $k_\parallel$ is convenient in the $D = 4$ case) and, then, to make the Fourier transformation:

$$\mathbf{\Phi}(z) = 4\pi g \int\limits_{-\infty}^{\infty} \frac{e^{ik_\parallel z} dk_\parallel / 2\pi}{k_\parallel^2 + 4g^2 P(k_\parallel^2/4m^2)} \, . \tag{5}$$

Finally, the potential energy of the charges $+g$ and $-g$ is

$$V(z) = -g\mathbf{\Phi}(z) \, . \tag{6}$$

We did not succeed to analytically perform the integration in (5) with $P(t)$ given by its exact expression (4); instead, an interpolating formula for $P(t)$ was suggested in [3]. The asymptotics of $P(t)$ are:

$$P(t) = \begin{cases} \frac{2}{3}t & , \quad t \ll 1 \\ 1 & , \quad t \gg 1 \, , \end{cases} \tag{7}$$

such that the following interpolating formula

$$\overline{P}(t) = \frac{2t}{3+2t} \tag{8}$$

has a correct behavior at small and large t. Substituting (8) in (5) and performing analytically the integration yields [3] [6]:

$$\begin{aligned}
\mathbf{\Phi}(z) &= 4\pi g \int\limits_{-\infty}^{\infty} \frac{e^{ik_\parallel z} dk_\parallel/2\pi}{k_\parallel^2 + 4g^2(k_\parallel^2/2m^2)/(3+k_\parallel^2/2m^2)} = \\
&= \frac{4\pi g}{1+2g^2/3m^2} \int\limits_{-\infty}^{\infty} \left[\frac{1}{k_\parallel^2} + \frac{2g^2/3m^2}{k_\parallel^2 + 6m^2 + 4g^2}\right] e^{ik_\parallel z} \frac{dk_\parallel}{2\pi} = \\
&= \frac{4\pi g}{1+2g^2/3m^2} \left[-\frac{1}{2}|z| + \frac{g^2/3m^2}{\sqrt{6m^2+4g^2}} \exp(-\sqrt{6m^2+4g^2}\ |z|)\right] \quad .
\end{aligned} \tag{9}$$

For heavy fermions ($m \gg g$) the potential is given by the tree level expression; radiative corrections are small:

$$\mathbf{\Phi}(z)\Big|_{\ m \gg g} = -2\pi g|z| \left(1 + O\left(\frac{g^2}{m^2}\right)\right) \quad . \tag{10}$$

In the case of light fermions ($m \ll g$) we get a much more interesting behavior:

$$\mathbf{\Phi}(z)\Big|_{\ m \ll g} = \begin{cases} \pi e^{-2g|z|} & , \quad z \ll \frac{1}{g}\ln\left(\frac{g}{m}\right) \\ -2\pi g \left(\frac{3m^2}{2g^2}\right)|z| & , \quad z \gg \frac{1}{g}\ln\left(\frac{g}{m}\right) \quad . \end{cases} \tag{11}$$

At short distances the potential has a Yukawa behavior, corresponding to the exchange of a massive vector particle. The fact that the photon remains massless for nonzero m follows from the behavior of the potential at large distances: a linear behavior corresponds to a massless exchange. However, the coupling constant differs from that at tree level by the small factor $(3m^2/2g^2)$: a screening of the tree level potential occurs at $|z| \gtrsim 1/(2g)$. Finally, for $m = 0$ at all z the potential is given by the first line of (11): the photon gets a mass $m_\gamma = 2g = 2\sqrt{\alpha} = e/\sqrt{\pi}$ (where, and only in this formula, e is the coupling constant in the Heaviside units usually used in Quantum Field Theory) – a well-known result in $D = 2$ QED with massless "electrons" (Schwinger model), where a nonzero mass for the gauge boson coexists with gauge invariance [9][10].

We now study the accuracy of the analytical formula for the potential (9) compared with the numerical calculation of the integral (5) in which the exact expression (4) for $P(t)$ is used. In order to normalize the potential universally for the different values of the ratio g/m we choose an arbitrary constant up to which the potential is defined in such a way that $\mathbf{\Phi}(z) = 0$ at $z = 0$. In Fig. 2 the exact ($V(z)$) and approximate ($\bar{V}(z)$) potential energies are plotted together with their asymptotics for $g = 0.5$ and $m = 0.1$. The straight line which corresponds to the potential asymptotic at $z \to \infty$ crosses the vertical axis at $V(0) \approx \pi g(1 - (3m/2g)^2) \approx 1.42$.

[6] We regularize the infrared divergence of the $\frac{1}{k_\parallel^2}$ integral making a subtraction at $z = 0$.

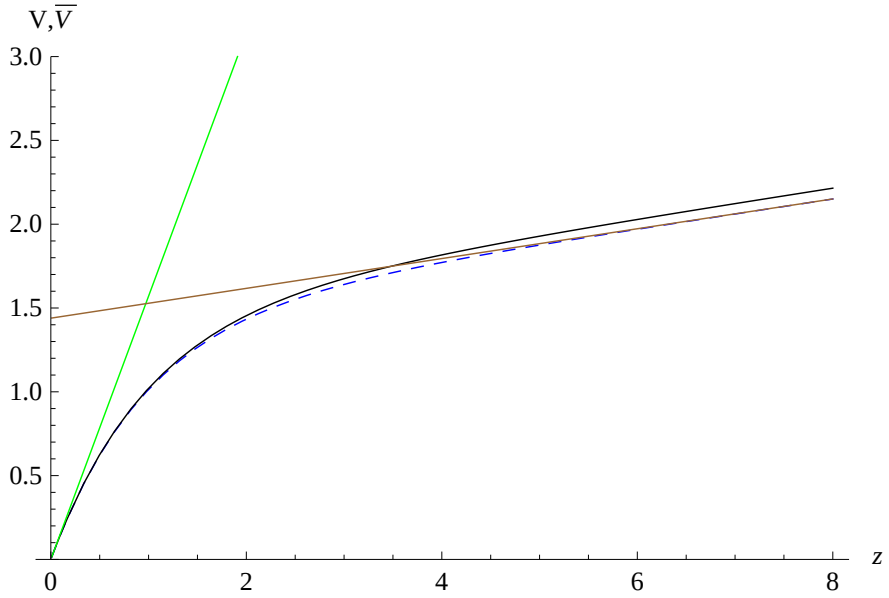

Fig. 2. *Potential energies of the charges $+g$ and $-g$ in D=2 for $g = 0.5$, $m = 0.1$. The black curve corresponds to P, the blue-dashed curve - to $\bar{P}$.*

We see that the accuracy of the interpolating formula is very good. In Fig. 3 the relative difference $(V - \bar{V})/V$ for the same values of g and m is shown.

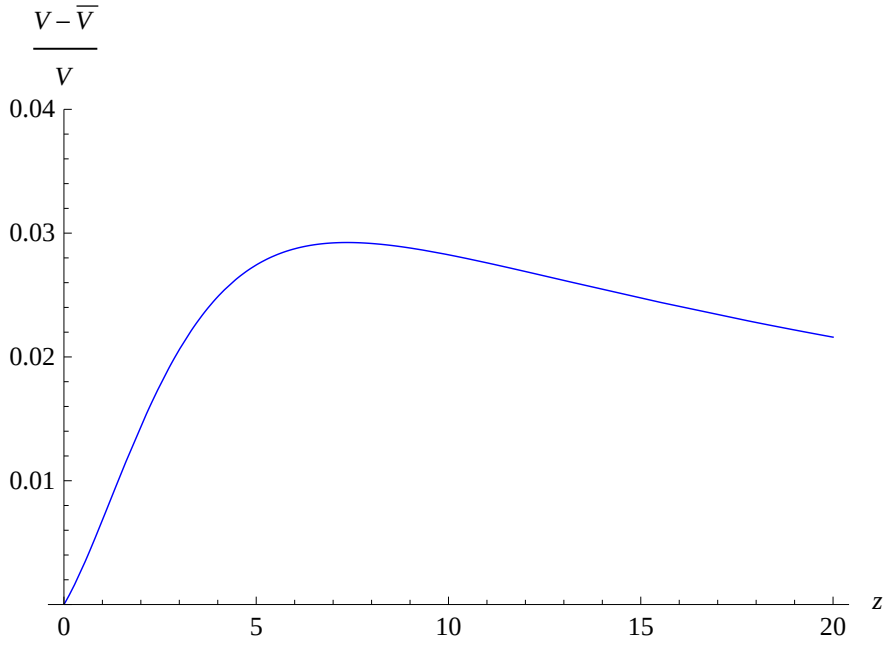

Fig. 3. *Relative difference of potential energies calculated with the exact and interpolating formulae for the polarization operator for $g = 0.5$, $m = 0.1$.*

The accuracy appears to be the worst at $z \approx 1/m$. We have checked that for various values of the ratio g/m, the relative accuracy of (9) is better than 4% at all z .

3 Screening of the Coulomb potential in $D = 4$ QED in a superstrong magnetic field

In order to write an expression for the electric potential of a point-like charge we need a formula for the photon polarization operator in an external magnetic field B. This quantity was studied in many papers; a detailed review and references can be found in [11]. In the calculation of the polarization operator, electron propagators in an external magnetic field are used. In the case of a strong field $B \gg B_{cr} = m_e^2/e$ the spacing of Landau levels is much larger than the electron mass. This is why the contribution of the electrons from LLL obeying the dispersion law $\varepsilon_0^2 = m_e^2 + p_z^2$ dominates in the electron propagator. A linear dependence of the polarization operator on the external field B follows. Finally, the electric potential writes [1, 3]:

$$\mathbf{\Phi}(k) = \frac{4\pi e}{k_\parallel^2 + k_\perp^2 + \frac{2e^3B}{\pi}\exp\left(-\frac{k_\perp^2}{2eB}\right) P\left(\frac{k_\parallel^2}{4m_e^2}\right)}\,, \tag{12}$$

where $k = (0, k_x, k_y, k_z)$, $k_\perp^2 = k_x^2 + k_y^2$, $k_z = k_\parallel$ and the magnetic field B is directed along the z axis. As stated in the introduction in (12) a quasi two-dimensional formula for the photon polarization operator occurs. It was obtained in [2]. Expression (12) is valid for strong magnetic fields $B \gg B_{cr} = m_e^2/e$ and for longitudinal momenta $k_\parallel^2 \ll eB$ (the motion of the virtual electrons contributing to the photon polarization operator Π occurs in one space and one time dimensions only for z larger than the Landau radius $a_H = 1/\sqrt{eB}$; for $z < a_H$ the motion is four dimensional).

Plugging in the Fourier transform of (12) the interpolating formula (8) for P introduced in Section 2 for the electric potential of a point-like charge along the direction of magnetic field we get:

$$\begin{aligned}
\mathbf{\Phi}(z) &= 4\pi e \int \frac{e^{ik_\parallel z} dk_\parallel d^2k_\perp/(2\pi)^3}{k_\parallel^2 + k_\perp^2 + \dfrac{2e^3B}{\pi}\exp(-k_\perp^2/(2eB))(k_\parallel^2/2m_e^2)/(3 + k_\parallel^2/2m_e^2)} = \\
&= \frac{2e}{2\pi|z|}\int\limits_0^\infty \cos(k_\parallel z)\ln\left[\frac{\Lambda^2 + k_\parallel^2 + \frac{2e^3B}{\pi}\frac{k_\parallel^2}{k_\parallel^2+6m_e^2}}{k_\parallel^2 + \frac{2e^3B}{\pi}\frac{k_\parallel^2}{k_\parallel^2+6m_e^2}}\right] d(k_\parallel|z|)\ ,
\end{aligned} \tag{13}$$

where Λ is an ultraviolet cutoff in $k_\perp$ and as we will see later the values $(k_\perp^2)_{max} \equiv \Lambda^2 \lesssim 2eB$ are essential. This condition allows us to replace the exponent in the denominator of the first line of (13) by one. This is why this exponent is absent in the second line of (13). Integrating by parts we get that, indeed, $\Lambda^2 \lesssim k_\parallel^2$ is important, and $k_\parallel^2 \ll eB$ for $z \gg a_H$.

So doing we obtain:

$$\begin{aligned}\mathbf{\Phi}(z) &= \frac{2e}{\pi|z|}\int\limits_0^\infty \frac{\sin(k_\parallel z)d(k_\parallel|z|)}{k_\parallel z\left(1+\frac{2e^3B/\pi}{k_\parallel^2+6m_e^2}\right)}\left[1+\frac{12e^3Bm_e^2}{\pi(k_\parallel^2+6m_e^2)^2}\right] = \qquad (14)\\ &= \frac{2e}{\pi|z|}\int\limits_0^\infty \frac{\sin(k_\parallel z)}{(k_\parallel z)}d(k_\parallel|z|)\left[1+\frac{6m_e^2}{k_\parallel^2+6m_e^2}-\frac{2e^3B/\pi+6m_e^2}{k_\parallel^2+2e^3B/\pi+6m_e^2}\right] = \\ &= \frac{2e}{\pi|z|}\left\{\frac{\pi}{2}+\frac{1}{2i}\int\limits_{-\infty}^\infty \frac{e^{ik_\parallel z}}{k_\parallel z}d(k_\parallel|z|)\left[\frac{6m_e^2}{k_\parallel^2+6m_e^2}-\frac{2e^3B/\pi+6m_e^2}{k_\parallel^2+2e^3B/\pi+6m_e^2}\right]\right\} \ .\end{aligned}$$

Performing the last integrals with the help of residues we finally obtain:

$$\mathbf{\Phi}(z) = \frac{e}{|z|}\left[1-e^{-\sqrt{6m_e^2}\,|z|}+e^{-\sqrt{(2/\pi)e^3B+6m_e^2}\,|z|}\right] \ . \qquad (15)$$

For magnetic fields $B \ll 3\pi m_e^2/e^3$ the potential is Coulomb up to small power suppressed terms:

$$\mathbf{\Phi}(z)\Big|_{\ e^3B \ll 3\pi m_e^2} = \frac{e}{|z|}\left[1+O\left(\frac{e^3B}{3\pi m_e^2}\right)\right] \qquad (16)$$

in full agreement with the $D=2$ case (see (10), where g^2 plays the role of e^3B). In the opposite case of superstrong magnetic fields $B \gg 3\pi m_e^2/e^3$ we get:

$$\mathbf{\Phi}(z) = \begin{cases} \frac{e}{|z|}e^{(-\sqrt{(2/\pi)e^3B}\,|z|)} & , \ |z| < \frac{1}{\sqrt{(2/\pi)e^3B}}\ln\left(\sqrt{\frac{e^3B}{3\pi m_e^2}}\right) \\ \frac{e}{|z|}(1-e^{(-\sqrt{6m_e^2}\,|z|)}) & , \ \frac{1}{m_e} > |z| > \frac{1}{\sqrt{(2/\pi)e^3B}}\ln\left(\sqrt{\frac{e^3B}{3\pi m_e^2}}\right) \\ \frac{e}{|z|} & , \ |z| > \frac{1}{m_e} \end{cases} \ , \qquad (17)$$

which also closely resembles $D=2$ case, (11), with the substitution $e^3B \to g^2$.

The expression for the potential energy of two opposite charges separated by the distance z along the magnetic field (the transverse separation is zero) is:

$$\bar{V}(z) = -e\mathbf{\Phi}(z) = -\frac{e^2}{|z|}\left[1-e^{-\sqrt{6m_e^2}\,|z|}+e^{-\sqrt{(2/\pi)e^3B+6m_e^2}\,|z|}\right] \ . \qquad (18)$$

In Fig. 4 the potential energy which follows from eq. (18) for a magnetic field $B = 10^{17}$ G is shown as well as its asymptotics at large distances:

$$\bar{V}(z) = -\frac{e^2}{|z|} \ , \ |z| > \frac{1}{m_e} \ , \qquad (19)$$

and at small distances:

$$\bar{V}(z) = -\frac{e^2}{|z|}e^{-\sqrt{(2/\pi)e^3B}\,|z|} \ , \ z < \frac{1}{\sqrt{(2/\pi)e^3B}}\ln\sqrt{\frac{e^3B}{3\pi m_e^2}} \ . \qquad (20)$$

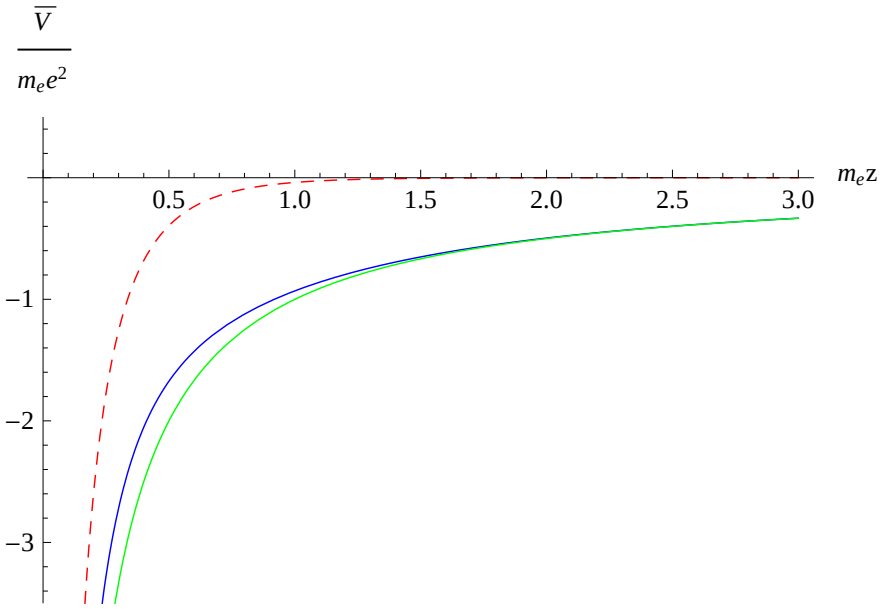

Fig. 4. *Modified Coulomb potential energy at* $B = 10^{17} G$ *(blue) and its long distance (green-pale) and short distance (red-dashed) asymptotics.*

In the following two figures (Fig. 5 and Fig. 6) the analytical formula (18) for the potential energy is compared with the result of a numerical integration using the exact formula for $P(k_\parallel^2/4m_e^2)$ given by equation (4). The same value of the magnetic field $B = 10^{17}$ G is used. We checked that for various values of B the agreement between analytical and numerical integrations is never worse than 3%.

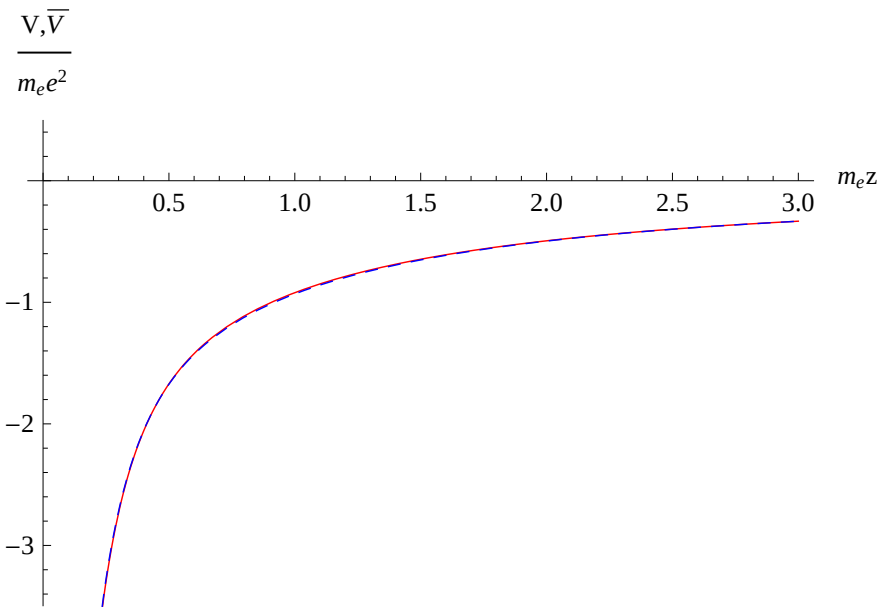

Fig. 5. *Modified Coulomb potential energy at* $B = 10^{17}\,G$. *Analytical formula (approximate* $\bar{P}$*) in blue-dashed versus numerical integration (exact* P*) in red (the two curves practically coincide).*

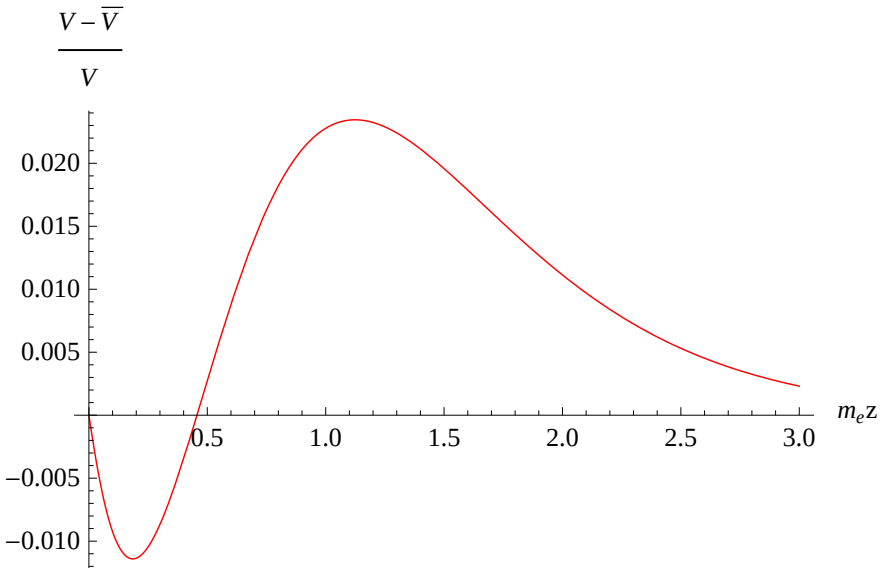

Fig. 6. *Relative accuracy of the analytical formula for the modified Coulomb potential energy at* $B = 10^{17}\,G$.

In the next section we will use the analytical expression for the electric potential energy (18) to determine the energies of a hydrogen atom in superstrong magnetic fields.

4 Energies of hydrogen atomic levels originating from LLL in superstrong magnetic fields

4.1 General considerations

We are interested in the spectrum of a hydrogen atom in a superstrong magnetic field B. In the absence of magnetic field the spatial size of the wave function of the ground state atomic electron is characterized by the Bohr radius $a_B = 1/(m_e e^2)$, its energy equals $E_0 = -m_e e^4/2 \equiv -Ry$, where Ry is the Rydberg constant. The transverse (with respect to B) size of the ground state of the electron wave function in an external magnetic field B is characterized by the Landau radius $a_H = 1/\sqrt{eB}$. The Larmour frequency of the electron precession is $\omega_L = eB/m_e$. For a magnetic field $B_a = e^3 m_e^2 = 2.35 \cdot 10^9$ G called "atomic magnetic field", these sizes and energies are close to each other: $a_B = a_H$, $E_0 \sim \omega_L$. We wish to study the spectrum of the hydrogen atom in magnetic fields much larger than B_a. In this case the motion of the electron is mainly controlled by the magnetic field: it makes many oscillations in this field before it makes one in the Coulomb field of the nucleus. This is the condition for applicability of the adiabatic approximation, used for this problem for the first time in [12].

The spectrum of a Dirac electron in a pure magnetic field is well known [13]; it admits a continuum of energy levels due to the free motion along the field:

$$\varepsilon_n^2 = m_e^2 + p_z^2 + (2n + 1 + \sigma_z)eB \ , \tag{21}$$

where $n = 0, 1, 2, ...$; $\sigma_z = \pm 1$ is the spin projection of the electron on z axis multiplied by two. For magnetic fields larger than $B_{cr} = m_e^2/e$, the electrons are relativistic with only one exception: electrons belonging to the lowest Landau level (LLL, $n = 0$, $\sigma_z = -1$) can be non-relativistic.

In what follows we will study the spectrum of electrons from LLL in the Coulomb field of the proton modified by the superstrong B . The solution, in cylindrical coordinates $(\vec{\rho}, z)$, of the Schrödinger equation for an electron in a magnetic field B which is homogeneous and constant in time, in the gauge in which $\vec{A} = \frac{1}{2}\ \vec{B} \times \vec{r}$ can be found in [14]. The electron energies are:

$$E_{p_z n_\rho m \sigma_z} = \left(n_\rho + \frac{|m| + m + 1 + \sigma_z}{2} \right) \frac{eB}{m_e} + \frac{p_z^2}{2m_e} \ , \tag{22}$$

where $n_\rho = 0, 1, 2, ...$ is the number of nodal surfaces, $m = 0, \pm 1, \pm 2, ...$ is the electron orbital momentum projection on the z axis (direction of the magnetic field) and $\sigma_z = \pm 1$. According to [14], the LLL wave functions are:

$$R_{0m}(\vec{\rho}) = \left[\pi (2a_H^2)^{1+|m|} (|m|!) \right]^{-1/2} \rho^{|m|} e^{\left(im\varphi - \rho^2/(4a_H^2) \right)} \ , \tag{23}$$

$$\rho = |\vec{\rho}|, \quad \int |R_{0m}(\vec{\rho})|^2 d^2\rho = 1 \ , \quad m = 0, -1, -2, ...$$

We should now take into account the electric potential of the atomic nucleus located at $\vec{\rho} = z = 0$. For $a_H \ll a_B$ the adiabatic approximation can be used and the wave function can be looked for in the following form:

$$\Psi_{n0m(-1)} = R_{0m}(\vec{\rho}) \chi_n(z) \ , \tag{24}$$

where $\chi_n(z)$ is the solution of a Schrödinger equation for an electron motion along the direction of the magnetic field:

$$\left[-\frac{1}{2m_e}\frac{d^2}{dz^2} + U_{eff}(z)\right]\chi_n(z) = E_n\chi_n(z) \quad . \tag{25}$$

Without screening the effective potential is given by the following formula [4]:

$$U_{eff}(z) = -e^2\int\frac{|R_{0m}(\vec{\rho})|^2}{\sqrt{\rho^2+z^2}}d^2\rho \quad , \tag{26}$$

which becomes the Coulomb potential for $|z| \gg a_H$

$$U_{eff}(z)\Big|_{\,z\,\gg\,a_H} = -\frac{e^2}{|z|} \tag{27}$$

and is regular at $z = 0$

$$U_{eff}(0) \sim -\frac{e^2}{|a_H|} \quad . \tag{28}$$

To take screening into account we must use (18) to modify (26) (see below). Since $U_{eff}(z) = U_{eff}(-z)$, the wave functions are odd or even under reflection $z \to -z$; the ground states (for $m = 0, -1, -2, ...$) are described by even wave functions.

In Fig.7 the different scales important in the consideration of the hydrogen atom in strong magnetic field are shown.

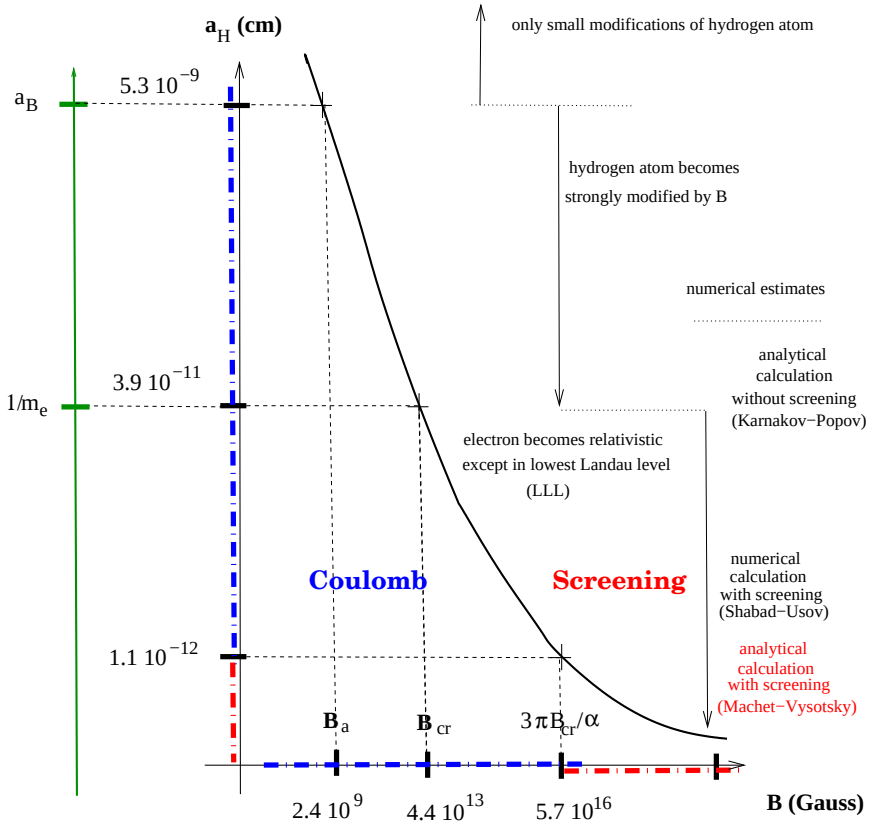

Fig. 7. *Landau radius a_H versus magnetic field B.*

4.2 The shallow-well approximation

This approximation was used in [1] for the calculation of the ground state energy of the hydrogen atom in a superstrong magnetic field, taking screening into account. The authors of [1] follow [5], where this approximation was used for calculating the ground state energy of hydrogen atom without screening. As pointed out in [4], the accuracy of the shallow-well approximation for the problem under consideration is very poor. Without taking the screening into account it can be estimated as:

$$\frac{\delta E}{E} = \left(\frac{1}{1 - \dfrac{\ln(\ln^2 H)}{\ln H}} \right)^2 - 1 \ , \tag{29}$$

where H is a magnetic field measured in units of the atomic field B_a, $H \equiv B/B_a = (a_B/a_H)^2$. Even for $B = B_{cr} = m_e^2/e = 4.41 \cdot 10^{13}$ G ($H \approx 2 \cdot 10^4$) we obtain $\delta E/E \approx 3.5$, while for $B = 10^{17}$ G ($H = 5 \cdot 10^7$) $\delta E/E \approx 1.4$.

However the formula for the ground state energy in a shallow-well approximation taken from [5] and used in [1]:

$$E^{sw}_{[5]} = -2m_e \left[\int_{a_H}^{a_B} U(z) dz \right]^2 \tag{30}$$

allows us to check with which accuracy the integral of our expression (18) reproduces the result obtained in [1], where the potential $A_0(z)$ found numerically was integrated. According to eq. (14) from P.R.L. paper [1]

$$E^{sw}_{\rm lim} = -2m_e\alpha^2 \times 73.8 = -4 \text{ keV} \ , \tag{31}$$

where $E_{\rm lim}$ is the limiting value of the ground state energy of a hydrogen atom at $B \to \infty$. Substituting (18) in (30) we get:

$$\begin{aligned} E^{sw}_{\rm analytical} &= -2m_e e^4 \left[\ln\left(\frac{a_B}{a_H}\right) + \ln(\sqrt{6m_e^2}\, a_H) - \ln(\sqrt{6m_e^2 + \frac{2}{\pi}e^3 B}\, a_H) \right]^2 = \\ &= -2m_e e^4 \ln^2 \sqrt{\frac{H}{1 + \dfrac{e^6}{3\pi} H}} \end{aligned} \tag{32}$$

and in the limit $H \gg 3\pi/e^6$

$$E^{sw}_{analytical}\Big|_{H\to\infty} = -2m_e e^4 \ln^2 \left(\sqrt{\frac{3\pi}{e^6}} \right) = -2m_e\alpha^2 \times 72.3 \ , \tag{33}$$

which coincides with (31) within 2% accuracy ($e^2 \equiv \alpha$ is used). It means that integrals over potential energy differ by 1% only.

Let us present the derivation of the formula for the ground state energy of a particle with mass μ undergoing a one-dimensional motion in a shallow-well potential [15]. One starts from the Schrödinger equation:

$$-\frac{1}{2\mu}\frac{d^2}{dz^2}\chi(z) + U(z)\chi(z) = E_0\chi(z) \ . \tag{34}$$

Neglecting E_0 in comparison with U and integrating (34) we get:

$$\chi'(a) = 2\mu \int_0^a U(x)\chi(x)dx \ , \tag{35}$$

where we assume that $U(x) = U(-x)$ (this is why $\chi(x)$ is an even function). The next assumptions are: 1. the finite range of the potential energy: $U(x) \neq 0$ for $a > x > -a$; 2. χ undergoes very small variations inside the well. Since outside the well $\chi(x) \sim e^{-\sqrt{2\mu|E_0|}\,x}$, we readily obtain:

$$|E_0| = 2\mu \left[\int_0^a U(x)dx\right]^2 \ . \tag{36}$$

For

$$\mu|U|a^2 \ll 1 \tag{37}$$

(condition for the potential to form a shallow well) we get that, indeed, $|E_0| \ll |U|$ and that the variation of χ inside the well is small, $\Delta\chi/\chi \sim \mu|U|a^2 \ll 1$.

Concerning the one-dimensional Coulomb potential, it satisfies the condition (37) only for $a \ll 1/(m_e e^2) \equiv a_B$, so (36) cannot be used. This explains why the accuracy of (30) is very poor.

4.3 The Karnakov–Popov equation [4]

It provides a several percent accuracy for the ground state energy for $H > 10^3$. It will be derived in this subsection and, in the next one, we will obtain the modification of this equation due to screening. The Karnakov-Popov (KP) equation determines the energies of the families of even states with given n_ρ and m. The starting point of its derivation is equation (25). Karnakov and Popov integrated it over z, from $z = 0$ till a value of z such that $a_H \ll z \ll a_B$, where the assumptions of a shallow well potential ($|U| \gg |E_0|$, $m_e|U|a^2 \ll 1$) can be used. Assuming $\chi(z) \approx 1$ we get:

$$\begin{aligned}
\chi'(z) &= -2m_e e^2 \int_0^z dx \int \frac{|R_{0m}(\vec{\rho})|^2}{\sqrt{\rho^2 + x^2}} d^2\rho = \\
&= -2m_e e^2 \int \ln\left(\frac{z}{\rho} + \sqrt{\frac{z^2}{\rho^2} + 1}\right) |R_{0m}(\vec{\rho})|^2 d^2\rho \ .
\end{aligned} \tag{38}$$

Since $R_{0m}(\vec{\rho})$ decreases exponentially for $\rho > a_H$ and since z is much larger than a_H, expanding the logarithm in powers of ρ/z, we can neglect the terms suppressed as $(\rho/z)^n$:

$$\ln\left(\frac{z}{\rho} + \sqrt{\frac{z^2}{\rho^2} + 1}\right) = \ln\left(2\frac{z}{\rho}\right) = \ln\left(\frac{z}{a_H}\right) + \ln\left(\frac{2a_H}{\rho}\right) \ , \tag{39}$$

$$\begin{aligned}
\chi'(z) &= -2m_e e^2 \left[\ln\left(\frac{z}{a_H}\right) + \int \ln\left(\frac{2a_H}{\rho}\right) |R_{0m}(\vec{\rho})|^2 d^2\rho\right] = \\
&= -2m_e e^2 \ln\left(\frac{z}{a_H}\right) + A_{|m|0} \ ,
\end{aligned} \tag{40}$$

in which

$$A_{|m|0} = -m_e e^2 \left[\ln(2) - \int\limits_0^\infty \ln(x) x^{|m|} e^{-x} \frac{dx}{|m|!} \right] \ , \tag{41}$$

and the expression (23) for $R_{0m}(\vec{\rho})$ was used. To calculate the last integral the following equality must be used:

$$\int\limits_0^\infty \ln(x) x^{|m|} e^{-x} dx = \Gamma'(|m|+1) = \Gamma(|m|+1)\psi(|m|+1) \ , \tag{42}$$

where ψ is the logarithmic derivative of the gamma function.

Finally we reproduce eq. (2.11) from [4]:

$$A_{|m|0} = -m_e e^2 [\ln(2) - \psi(1+|m|)] \ , \tag{43}$$

where $\psi(1) = -\gamma = -0.5772$, γ is Euler constant and $\psi(n+1) = -\gamma + \sum_{k=1}^{n}(1/k)$ for $n = 1, 2, 3, ...$

At $z \gg a_H$ the solution of the Schrödinger equation with the Coulomb potential which exponentially decreases at $z \gg a_B$ is given by the Whittaker function [4].

For the state with energy

$$E = -(m_e e^4/2)\lambda^2 \tag{44}$$

the logarithmic derivative of Whittaker function at $z \ll a_B$ is [4]:

$$\frac{W'}{W} = -m_e e^2 \left\{ 2\ln(z/a_B) + \lambda + 2\ln\lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) + 4\gamma + 2\ln 2 + O\Big((z/a_B)\ln\lambda\Big) \right\} \ . \tag{45}$$

From the condition $W'/W = \chi'$ we obtain an equation for the energies of the even bound states of the hydrogen atom originating from LLL ($n_\rho = 0$, $m = 0, -1, -2, ...$):

$$\begin{aligned} &2\ln\left(\frac{z}{a_H}\right) + \ln 2 - \psi(1+|m|) + O(a_H/z) = \\ &= 2\ln\left(\frac{z}{a_B}\right) + \lambda + 2\ln\lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) + 4\gamma + 2\ln 2 + O(z/a_B) \ , \end{aligned} \tag{46}$$

and for $a_B \gg a_H$, choosing $a_B \gg z \gg a_H$, we may neglect non-calculated terms. This leads to the KP equation ([4], (2.11, 2.13)):

$$\ln(H) = \lambda + 2\ln\lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) + \ln 2 + 4\gamma + \psi(1+|m|) \ , \tag{47}$$

where we recall that $H \equiv B/B_a = B/(e^3 m_e^2)$. We postpone the analysis of this eigenvalue equation till the next subsection, where its modification due to screening will be established.

The energies of the odd states are [4]:

$$E_{\rm odd} = -\frac{m_e e^4}{2n^2} + O\left(\frac{m_e^2 e^3}{B}\right) \ , \quad n = 1, 2, ... \ . \tag{48}$$

So, for superstrong magnetic fields $B \sim m_e^2/e^3$ the deviations of odd states from the Balmer series are negligible.

4.4 Equation for the energies of even states with screening

We must equate the logarithmic derivative of wave function obtained by integrating the Schrödinger equation at small distances with the logarithmic derivative of the Whittaker function. To do this we should choose z to be larger than the distance at which the Coulomb potential is screened. That is why the condition $z \gg a_H$ which was used in the case of the absence of screening is not enough. For $z_0 > 1.5/m_e$ the corrections coming from screening in (18) are only a few percent. The wave function at $z \gtrsim z_0$ is thus described by the Whittaker function with good accuracy.

To derive an analog of (47), it is necessary that χ stays practically constant from $z = 0$ up to the domain where the effective potential becomes Coulombian with high accuracy, and, accordingly, where the Whittaker function provides a very good approximation to the wave function. The characteristic distance along which the ground state wave function varies in the z direction is, according to the uncertainty relation:

$$z_{\rm char} \sim \frac{1}{\sqrt{2m_e(-E)}} \sim a_B/\lambda \ , \tag{49}$$

and it decreases when B increases since $\lambda \sim \ln B$ (see eq. (47)). However, screening helps: with the account of it λ goes to a finite value, $\lambda_\infty \approx 11$ at $B \to \infty$ (see below). For the size of a ground state wave function we get:

$$z_\infty \sim \frac{137}{11m_e} \approx \frac{12}{m_e} \ , \tag{50}$$

and for

$$1.5/m_e < z_0 < 3/m_e \tag{51}$$

the potential (18) is Coulombian with good accuracy while the variation of the wave function $[\chi(0) - \chi(z_0)]/\chi(0)$ is small.

When screening is taken into account an expression for effective potential (26) transforms into

$$\tilde{U}_{eff}(z) = -e^2 \int \frac{|R_{0m}(\vec{\rho})|^2}{\sqrt{\rho^2 + z^2}} d^2\rho \left[1 - e^{-\sqrt{6m_e^2}\, z} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}\, z}\right] \ , \tag{52}$$

see Fig. 8. On the same figure a simplified potential given by

$$U_{simpl}(z) = -e^2 \frac{1}{\sqrt{a_H^2 + z^2}} \left[1 - e^{-\sqrt{6m_e^2}\, z} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}\, z}\right] \ , \tag{53}$$

is shown.

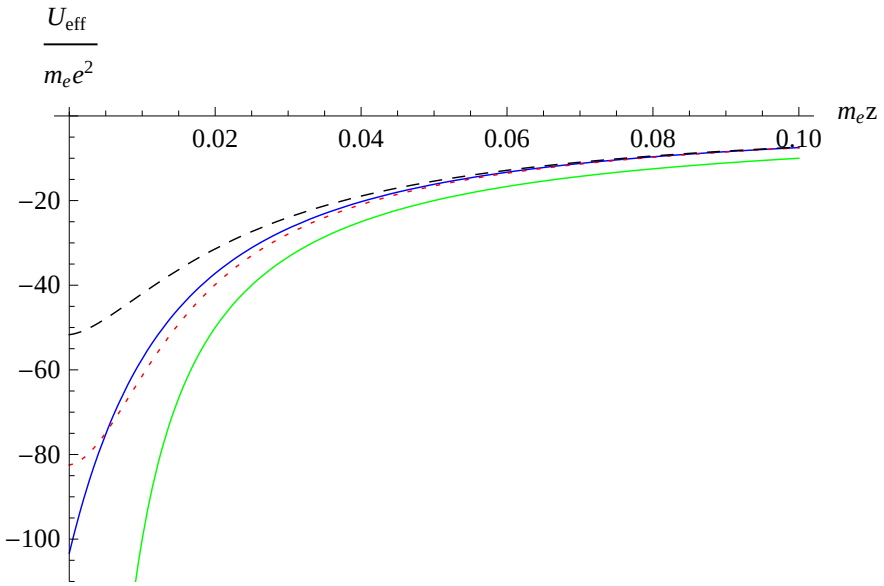

Fig. 8. *Effective potential with screening for $m = 0$ (blue) and $m = -1$ (black-dashed), (52); simplified potential (53) (red-dotted). The curves correspond to $B = 3\cdot 10^{17}$ G. Coulomb potential (green-pale) is also shown.*

The screening changes (38) into:

$$\tilde{\chi}'(z_0) = -2m_e e^2 \int_0^{z_0} dx \int \frac{|R_{0m}(\vec{\rho})|^2}{\sqrt{\rho^2+x^2}} \left[1 - e^{-\sqrt{6m_e^2}\,x} + \right.$$
$$\left. + e^{-\sqrt{(2/\pi)e^3B+6m_e^2}\,x}\right] d^2\rho \equiv \chi'(z_0) + \Delta\chi'(z_0) \quad . \tag{54}$$

Calculating integrals with exponents we get:

$$\int_0^{z_0} \frac{dx}{\sqrt{\rho^2+x^2}} e^{-ax} \equiv \int_0^{z_0 a} \frac{dy}{\sqrt{y^2+\rho^2a^2}} e^{-y} \approx \int_0^{\infty} \frac{dy}{\sqrt{y^2+\rho^2a^2}} e^{-y} \approx$$
$$\approx \ln\left(\frac{1}{\rho a}\right) + \text{const} \quad , \tag{55}$$

where, in the first approximate equality, we took into account that the integrand is exponentially damped for $y > 1$, and $z_0\sqrt{6m_e^2} \gtrsim 4$; in the second approximate equality we took into account that $\rho a \approx a_H\sqrt{6m_e^2 + (2/\pi)e^3B} \ll 1$ (because R_{0m} is exponentially damped for $\rho > a_H$ and, thus, only the values $\rho \sim a_H$ are important).

This leads to:

$$\Delta\chi'(z_0) = -2m_e e^2 \ln\sqrt{\frac{6m_e^2}{(2/\pi)e^3B + 6m_e^2}} \quad . \tag{56}$$

Equation (18) takes into account the modification of the Coulomb potential at zero transverse direction, $\rho = 0$. In (54) the modified potential at $\rho \neq 0$ should be used. We changed the factor $1/|z|$ occurring in eq. (18) into $1/\sqrt{\rho^2+x^2}$ in (54). However, according to numerical calculations of the electric potential in the direction transverse to that of the

magnetic field (at $z = 0$) made in [1], the screening occurs at $\rho \neq 0$ as well. To estimate the role of the " transverse screening", we substituted $\sqrt{\rho^2 + x^2}$ for x inside the exponents of (54). The corrections arising in eq. (55) turn out to be of the order of $\rho a \lesssim \sqrt{e^3 B/eB} = e$. The resulting correction to the ground state energy is negligible; the corrections to the energies of the excited states are small. In Fig. 9 we compare the effective potential given by (52) with the one obtained by substituting in the exponents of (52) $\sqrt{\rho^2 + z^2}$ for z for $m = 0$ and $m = -1$.

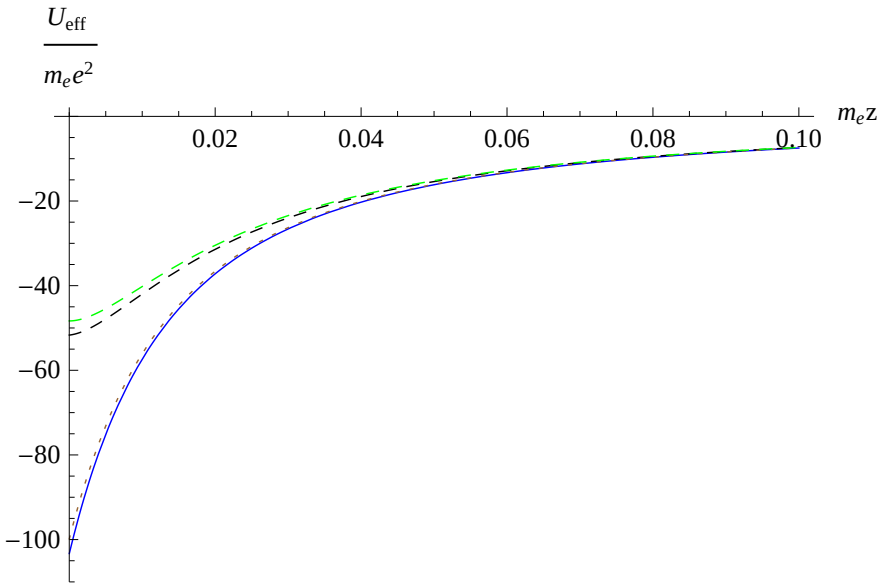

Fig. 9. *Effective potentials according to (52) with the replacement of z by $\sqrt{\rho^2 + z^2}$ in the exponents for $m = 0$ (brown-dotted) and $m = -1$ (green(pale)-dashed). For the comparison curves which directly correspond to (52) are also shown; $m = 0$ (blue) and $m = -1$ (black-dashed). The curves correspond to $B = 3 \cdot 10^{17}$ G.*

With the help of (56) we get the modified KP equation, which takes screening into account:

$$\ln\left(\frac{H}{1 + \dfrac{e^6}{3\pi} H}\right) = \lambda + 2\ln\lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) + \ln 2 + 4\gamma + \psi(1 + |m|) \ . \qquad (57)$$

The ground state has $m = 0$; for $H = 10^3$, its energy as given by (57) coincides with the result of the numerical solution of the Schrödinger equation at a few percent level.

For $H \geq 10^5$ the expansion $\psi(1 - \frac{1}{\lambda}) = \psi(1) - \frac{\pi^2}{6\lambda}$ has better than 1% accuracy [4]. Using this approximation, we obtain:

$$H = 10^5 \ ; \ B = 2.3 \cdot 10^{14} \text{ G} \ ; \ \lambda = 6.91 \ ; \ E_0 = -\frac{m_e e^4}{2}\lambda^2 = -0.65 \text{ keV} \ . \qquad (58)$$

The accuracy of the adiabatic approximation is characterized by the ratio $a_H/a_B \equiv \sqrt{B_a/B}$ which equals 0.0032 for $H = 10^5$.

The screening starts to be important at $H \sim 3\pi/e^6 \sim 10^7$. For example, $H = 10^7$ and $H = 1.7 \cdot 10^7$ correspond to the same ground state energy respectively without and with

screening. For $H \gg 10^7$, which corresponds to $B \gg 10^{16}$ G, the ground state energy reaches the limiting value

$$\lambda^{\rm lim}\Big|_{B\gg 10^{16}\ {\rm G}} \approx 11.2\ ,\quad E_0^{\rm lim} \approx -1.7\ {\rm keV}\ \ , \tag{59}$$

which differs from the result obtained in [1] by a factor 2.5. So, when screening is accounted for, the accuracy of the shallow-well approximation is never better than 250%.

For each value of m ($m = 0, -1, -2, ...$) eq. (57) gives a tower of even states, starting from the ground state, the energy of which is very large on the Rydberg scale. Let us rewrite (57) for the $m = 0$ tower:

$$\ln\left(\frac{H}{1+\frac{e^6}{3\pi}H}\right) = \lambda + 2\ln\lambda + 2\psi(1-\frac{1}{\lambda}) + \ln 2 + 3\gamma\ \ . \tag{60}$$

The ground state energy corresponds to a large value of λ, of the order of the left hand side of (60) as just discussed (see (58), (59)). $\psi(x)$ has poles at $x = 0, -1, -2, ...$ which correspond to $\lambda = 1, 1/2, 1/3, ...$ The Balmer series of energies $E_n = -m_e e^4/(2n^2)$, $n = 1, 2, ...$ that they form would correspond to an infinite value for the left hand side of (60). That it is always finite shows that even states get shifted from their Balmer values.

Let us determine the energy of the first excited even state. From the well-known expansion of the gamma function at small values of its argument we obtain the corresponding expansion for ψ:

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + O(\varepsilon)\ ,\quad \psi(-\varepsilon) = \frac{1}{\varepsilon} - \gamma\ \ . \tag{61}$$

Plugging in (60) $\lambda = 1 - \varepsilon$ we obtain:

$$\lambda = 1 - \frac{2}{\ln\left(\frac{H}{1+\frac{e^6}{3\pi}H}\right) - \ln 2 - \gamma + 1}\ \ . \tag{62}$$

Eq. (62) describes how the first excited even state (the second even state) moves towards its Balmer value $\lambda = 1$ with growing H. When $H \to \infty$ we obtain $\lambda^2_{(\infty)} = 0.78$ which determines the energy of the first excited even state in the limit of infinitely large B according to the formula $E = -(m_e e^4/2)\lambda^2$. The first odd state has the energy $E^1_{\rm odd} = -m_e e^4/2$.

For arbitrary n the analog of (62) looks like

$$\lambda_n = \frac{1}{n} - \frac{2/n^2}{\ln\left(\frac{H}{1+\frac{e^6}{3\pi}H}\right) - \ln 2 - \gamma + 1/n + 2\ln n - 2\sum_{k=1}^{n-1}\frac{1}{k}}\ \ , \tag{63}$$

where the last sum in the denominator is the harmonic number H_{n-1}.

A similar formula with no screening and in the limit $H \gg 1$ can be found in [16]; we however disagree concerning the factor 2 in the numerator of their eq. (11). An expression which coincides with the leading term in (63) for $H \gg 1$ without screening is contained in the book by Khriplovich "Teoreticheskii kaleidoskop", Novosibirsk, 2007, eq. (3.39).

The energy spectrum of the hydrogen atom in the limit of infinite magnetic field is shown in Fig. 10.

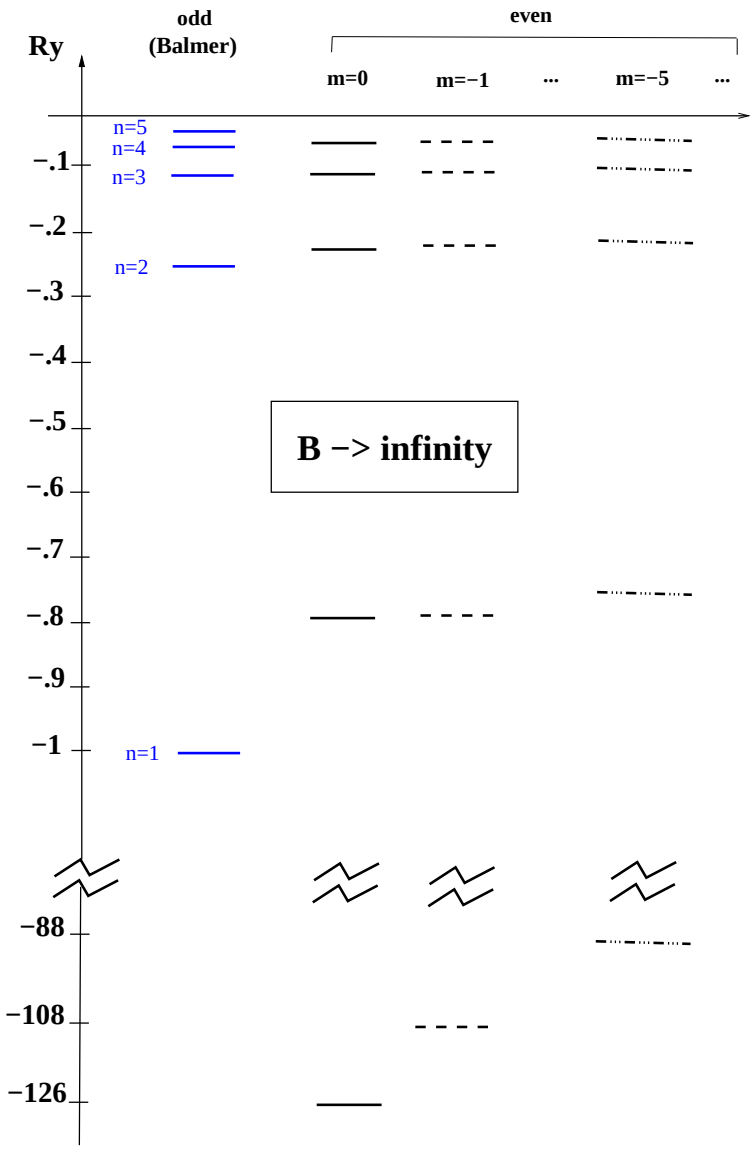

Fig. 10. *Spectrum of hydrogen levels in the limit of infinite magnetic field. Energies are given in rydberg units,* $Ry \equiv 13.6\ eV$.

5 Conclusion

An analytical formula for the Coulomb potential $\Phi(z)$ in a superstrong magnetic field has been derived. It reproduces the results of the numerical calculations made in [1] with good accuracy. Using it, an algebraic formula for the energy spectrum of the levels of a hydrogen atom originating from the lowest Landau level in a superstrong B has been obtained. The energies start to deviate from those obtained without taking the screening of the Coulomb potential into account [4] at $B \gtrsim 3\pi m_e^2/e^3 \approx 6 \cdot 10^{16}$ G which is much larger than the

largest magnetic field known to exist today, that of neutron stars called magnetars (where $B \approx 2 \cdot 10^{15}$ G may exist). Without the presented explicit calculations we would not know at which B substantial deviations of the energies of atomic levels from those obtained without taking screening into account take place.

Acknowledgments: We are grateful to V.S. Popov for important discussions and to S.I. Blinnikov, B.O. Kerbikov, L.B. Okun and A.V. Smilga for useful comments.

B.M. wants to thank ITEP for hospitality provided to him in June 2010 where this work was started; M.V. is grateful to LPTHE, where this work was finished, for hospitality provided to him in November 2010 and to grants RFBR 08-02-00494, NSh-4172.2010.2 and to the contract of the RF Ministry of Science and Education 02.740.11.5158 for partial support.

Appendix

Electron mass in a strong magnetic field

In the non-relativistic approximation, from (21) we obtain:

$$\varepsilon_n = m_e + \frac{p_z^2}{2m_e} + (n + \frac{1}{2})\frac{eB}{m_e} + \sigma_z \frac{eB}{2m_e} \quad , \tag{A1}$$

where the last term originates from the interaction of the electron magnetic moment with the magnetic field. Taking into account the anomalous magnetic moment of electron for the energy of the lowest Landau level ($n = 0$, $\sigma_z = -1$) we get:

$$\varepsilon_0 = m_e + \frac{p_z^2}{2m} - \frac{\alpha}{4\pi}\frac{eB}{m_e} \quad . \tag{A2}$$

This result is valid for a magnetic field $B \lesssim m_e^2/e$, while for $B \gg m_e^2/e$ the correction to the energy changes sign and the strong power dependence on B is replaced by a double logarithmic one [17]:

$$\varepsilon_0 = m_e + \frac{p_z^2}{2m_e} + \frac{\alpha}{4\pi} m_e \ln^2(eB/m_e^2) \quad . \tag{A3}$$

As a result, for the energy of the ground state of the hydrogen atom, we obtain:

$$E_0 = m_e \left[1 - \frac{e^4}{2}\lambda^2 + \frac{e^2}{4\pi}\ln^2\frac{eB}{m_e^2}\right] \quad . \tag{A4}$$

At $H \approx 10^5$ (see (58)) the last term in brackets compensates the second term and E_0 becomes bigger than m_e for a bigger H. However in the differences of the energies of atomic states originating from the LLL the universal correction (the last term in brackets in (A4)) cancels[7] and for the "experimentally observable" energies of transitions between these states, the formulas obtained in Section 4 should be used.

Concerning the double logarithmic term in (A3) it was noted in [18] that the appearance of an "effective photon mass" at fields $B \geq m_e^2/e^3$ leads to the transformation of the double logarithmic dependence on B into a single logarithmic dependence (concerning extraction and summation of higher order terms in $\alpha \ln B$ see also [19], [20]).

[7] We are grateful to V.S. Popov for this remark.

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