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Abstract. We consider the nonlinear Schrödinger equation on a compact manifold near an elliptic periodic geodesic. Using a geometric optics construction, we construct quasimodes to a nonlinear stationary problem which are highly localized near the periodic geodesic. We show the nonlinear Schrödinger evolution of such a quasimode remains localized near the geodesic, at least for short times.

1. Introduction

In this note, we study the Nonlinear Schrödinger Equation (NLS) on a compact, Riemannian manifold with a periodic elliptic (or stable) geodesic, which we define and discuss in more detail in Appendix A. Specifically, we study solutions to

$$\begin{cases} i\partial_t \psi = \Delta_g \psi + \sigma |\psi|^p \psi, \\ \psi(x,0) = \psi_0, \end{cases} \tag{1.1}$$

where Δ_g is the standard (negative semidefinite) Laplace-Beltrami operator and the solution is of the form

$$\psi(x,t) = e^{-i\lambda t} u(x).$$

To solve the resulting nonlinear elliptic equation, which can be analyzed using constrained variations, we will use Fermi coordinates to construct a nonlinear quasimode similar to the one presented in [Tho08b] for an arbitrary manifold with a periodic, elliptic orbit. To do so, we must analyze the metric geometry in a neighbourhood of a periodic orbit, for which we use the presentation in [Gra90]. For further references to construction of quasimodes along elliptic geodesics, see the seminal works Ralston [Ral82], Babič [Bab68], Guillemin-Weinstein [GW76], Cardoso-Popov [CP02], as well as the thorough survey on spectral theory by Zelditch [Zel08]. For experimental evidence of the existence of such Gaussian beam type solutions in nonlinear optics as solutions to nonlinear Maxwell's equations, see the recent paper by Schultheiss et al [SBS+10].

Henceforward, we assume that (M,g) is a compact Riemannian manifold of dimension $d \geq 2$ without boundary, and that it admits an elliptic periodic geodesic $\Gamma \subset M$. For $\beta > 0$, we introduce the notation

$$U_\beta = \{x \in M : \text{dist}_g(x,\Gamma) < \beta\}.$$

Let $L > 0$ be the period of Γ. Throughout the paper, each time we refer to the small parameter $h > 0$, this means that h takes the form

$$h = \frac{L}{2\pi N}, \tag{1.2}$$

for some large integer $N \in \mathbb{N}$.

Our first result is the existence, in the case of a smooth nonlinearity, of quasimodes which are highly concentrated near the elliptic periodic geodesic Γ. More precisely :

Theorem 1.1. *Let (M,g) be a compact Riemannian manifold of dimension $d \geq 2$, without boundary and which has an elliptic periodic geodesic. Let p be an even integer, let $s \geq 0$ and assume that*

$$p\big(\frac{d-1}{4} - s\big) < 1.$$

Then for $h \ll 1$ sufficiently small, and any $\delta > 0$, there exists $\varphi_h(x) \in H^\infty(M)$ satisfying

(i) *Frequency localization : For all $r \geq 0$, there exist C_1, C_2 independent of h such that*
$$C_1 h^{s-r} \leq \|\varphi_h\|_{H^r(M)} \leq C_2 h^{s-r}.$$
(ii) *Spatial localization near Γ :*
$$\|\varphi_h\|_{H^s(M \setminus U_{h^{1/2-\delta}})} = \mathcal{O}(h^\infty)\|\varphi_h\|_{H^s(M)}.$$
(iii) *Nonlinear quasimode : There exists $\lambda(h) \in \mathbb{R}$ so that φ_h satisfies the equation*
$$-\Delta_g \varphi_h = \lambda(h)\varphi_h - \sigma|\varphi_h|^p\varphi_h + \mathcal{O}(h^\infty).$$

The quasimode φ_h and the nonlinear eigenvalue $\lambda(h)$ will be constructed thanks to a WKB method, this will give a precise description of these objects (see Section 7.3). For instance, $\lambda(h)$ reads $\lambda(h) = h^{-2} - E_0 h^{-1} + o(h^{-1})$, for some $E_0 \in \mathbb{R}$.

As a consequence of Theorem 1.1, and under the same assumptions, we can state :

Theorem 1.2. *Consider the function φ_h given by Theorem 1.1. There exists $c_0 > 0$, such that if we denote by $T_h = c_0 h^{p(\frac{d-1}{4}-s)} \ln(\frac{1}{h})$, the solution u_h of the Cauchy problem* (1.1) *with initial condition φ_h satisfies*

$$\|u_h\|_{L^\infty([0,T_h];H^s(M\setminus U_{h^{1/2-\delta}}))} \leq C h^{(d+1)/4}.$$

To construct a quasimode with $\mathcal{O}(h^\infty)$ error, we need that the nonlinearity is of polynomial type, that is $p \in 2\mathbb{N}$. However, in the special case where $d = 2$ and $s = 0$, we are able to obtain a result for any $0 < p < 4$, which is a weaker version of Theorem 1.2.

Theorem 1.3. *Let (M,g) be a compact Riemannian surface without boundary which admits an elliptic periodic geodesic and let $0 < p < 4$. Then for $h \ll 1$ sufficiently small, and any $\delta, \epsilon > 0$, there exist $\varphi_h(x) \in H^\infty(M)$ and $\nu > 0$ satisfying*

(i) *Frequency localization : For all $r \geq 0$, there exist C_1, C_2 independent of h such that*
$$C_1 h^{-r} \leq \|\varphi_h\|_{H^r(M)} \leq C_2 h^{-r}.$$
(ii) *Spatial localization near Γ :*
$$\|\varphi_h\|_{L^2(M\setminus U_{h^{1/2-\delta}})} = \mathcal{O}(h^\infty)\|\varphi_h\|_{L^2(M)},$$

such that the corresponding solution u_h to (1.1) *satisfies*

$$\|u_h\|_{L^\infty([0,T_h];L^2(M\setminus U_{h^{1/2-\delta}}))} \leq C h^\nu,$$

for $T_h = h^p$ in the case $p \in (0,4)\backslash\{1\}$, and $T_h = h^{1+\epsilon}$ in the case $p = 1$.

Remark 1.1. *In our current notations, the $\dot{H}^1$ critical exponent for* (1.1) *is*

$$p = \frac{4}{d-2} = \frac{4}{n-1},$$

where n is the dimension of the geodesic normal hypersurface to Γ. The L^2 critical exponent is $p = 4/d$, and the $\dot{H}^{1/2}$ critical exponent is $p = 4/(d-1)$.

In general we construct highly concentrated solutions along an elliptic orbit, which is effectively a $d-1$ dimensional soliton. Since stable soliton solutions exist on $\mathbb{R}^d$ for monomial Schrödinger of the type in (1.1) *precisely for $p < \frac{4}{d}$, this numerology matches well that required for the expected local existence of stable nonlinear states on these lower dimensional manifolds.*

In addition, solving (1.1) *on compact manifolds with the power $p = 4/(d-2)$ plays an important role in the celebrated Yamabe problem, see* [SY94]*. The authors hope that the techniques here can give insight into the energy minimizers of such a problem related to the geometry.*

Remark 1.2. *In* [CM10]*, the existence of nonlinear bound states on hyperbolic space, $\mathbb{H}^d$, was explored. In that case, it was found that the geometry made compactness arguments at ∞ rather simple, hence the only driving force for the existence of a nonlinear bound state was the local behavior of the nonlinearity. Work in progress with the second author, third author, Michael Taylor and Jason Metcalfe is attempting to generalize this observation to any rank one symmetric space. In this note, we find nonlinear quasimodes based purely on the local geometry, where here the nonlinearity becomes a lower order correction, which is a nicely symmetric result.*

Remark 1.3. *In a private conversation with Nicolas Burq, he has pointed out that in settings where one assumes radial symmetry of the manifold, it is possible to construct exact nonlinear bound states.*

Notations. *In this paper c, C denote constants the value of which may change from line to line. We use the notations $a \sim b$, $a \lesssim b$ if $\frac{1}{C}b \le a \le Cb$, $a \le Cb$ respectively.*

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2. Outline of Proof

The proof of Theorem 1.2 is to solve approximately the associated stationary equation. That is, by separating variables in the t direction, we write

$$\psi(x,t) = e^{-i\lambda t} u(x),$$

from which we get the stationary equation

$$(\lambda + \Delta_g) u = \sigma |u|^p u.$$

For $h \ll 1$ of the form (1.2), the construction in the proof finds a family of functions

$$u_h(x) = h^{-(d-1)/4} g(h^{-1/2} x)$$

such that g is rapidly decaying away from Γ, $\mathcal{C}^\infty$, g is normalized in L^2, and

$$(\lambda + \Delta_g) u_h = \sigma |u_h|^p u_h + Q(u_h),$$

where $\lambda \sim h^{-2}$ and where the error $Q(u_h)$ is expressed by a truncation of an asymptotic series similar to that in [Tho08b] and is of lower order in h.

The point is that Theorem 1.2 (as well as Theorem 1.3) is an improvement over the trivial approximate solution. It is well known that there exist quasimodes for the linear equation localized near Γ of the form

$$v_h(x) = h^{-(d-1)/4} e^{is/h} f(s, h^{-1/2}x), \ (0 < h \ll 1),$$

with f a function rapidly decaying away from Γ, and s a parametrization around Γ, so that $v_h(x)$ satisfies

$$(\lambda + \Delta_g) v_h = \mathcal{O}(h^\infty) \|v_h\|$$

in any seminorm, see [Ral82]. Then

$$(\lambda + \Delta_g) v_h = \sigma |v_h|^p v_h + Q_2(v_h),$$

where the error $Q_2(v_\lambda) = |v_h|^p v_h$ satisfies

$$\|Q_2(v_h)\|_{\dot{H}^s} = \mathcal{O}(h^{-s-p(d-1)/4}).$$

3. A toy model

In this section we consider a toy model in two dimensions, and we give an idea of the proof of Theorem 1.3. For simplicity, we moreover assume that $2 \leq p < 4$. As it is a toy model, we will not dwell on error analysis, and instead make Taylor approximations at will without remarking on the error terms. Consider the manifold

$$M = \mathbb{R}_x / 2\pi\mathbb{Z} \times \mathbb{R}_\theta / 2\pi\mathbb{Z},$$

equipped with a metric of the form

$$ds^2 = dx^2 + A^2(x) d\theta^2,$$

where $A \in \mathcal{C}^\infty$ is a smooth function, $A \geq \epsilon > 0$ for some ϵ. From this metric, we get the volume form

$$d\mathrm{Vol} = A(x) dx d\theta,$$

and the Laplace-Beltrami operator acting on 0-forms

$$\Delta_g f = (\partial_x^2 + A^{-2}\partial_\theta^2 + A^{-1}A'\partial_x) f.$$

We observe that we can conjugate Δ_g by an isometry of metric spaces and separate variables so that spectral analysis of Δ_g is equivalent to a one-variable semiclassical problem with potential. That is, let $S : L^2(X, d\mathrm{Vol}) \to L^2(X, dxd\theta)$ be the isometry given by

$$Su(x, \theta) = A^{1/2}(x) u(x, \theta).$$

Then $\widetilde{\Delta} = S \Delta S^{-1}$ is essentially self-adjoint on $L^2(X, dxd\theta)$ with mild assumptions on A. A simple calculation gives

$$-\widetilde{\Delta} f = (-\partial_x^2 - A^{-2}(x)\partial_\theta^2 + V_1(x)) f,$$

where the potential

$$V_1(x) = \frac{1}{2} A'' A^{-1} - \frac{1}{4} (A')^2 A^{-2}.$$

We are interested in the nonlinear Schrödinger equation (1.1), so we make a separated Ansatz:

$$u_\lambda(t, x, \theta) = e^{-it\lambda} e^{ik\theta} \psi(x),$$

where $k \in \mathbb{Z}$ and ψ is to be determined (depending on both λ and k). Applying the Schrödinger operator (with $\widetilde{\Delta}$ replacing Δ) to u_λ yields the equation

$$(D_t + \widetilde{\Delta}) e^{it\lambda} e^{ik\theta} \psi(x) = (\lambda + \partial_x^2 - k^2 A^{-2}(x) + V_1(x)) e^{it\lambda} e^{ik\theta} \psi(x) = \sigma |\psi|^p e^{it\lambda} e^{ik\theta} \psi(x),$$

where we have used the standard notation $D = -i\partial$. We are interested in the behaviour of a solution or approximate solution near an elliptic periodic geodesic, which occurs at a maximum of the function A. For simplicity, let

$$A(x) = \sqrt{(1 + \cos^2(x))/2},$$

so that in a neighbourhood of $x = 0$, $A^2 \sim 1 - x^2$ and $A^{-2} \sim 1 + x^2$. The function $V_1(x) \sim$ const. in a neighbourhood of $x = 0$, so we will neglect V_1. If we assume $\psi(x)$ is localized near $x = 0$, we get the stationary reduced equation

$$(-\lambda + \partial_x^2 - k^2(1 + x^2))\psi = -\sigma|\psi|^p\psi.$$

Let $h = |k|^{-1}$ and use the rescaling operator $T\psi(x) = T_{h,0}\psi(x) = h^{-1/4}\psi(h^{-1/2}x)$ (see Lemma 4.5 below with $n = 1$) to conjugate:

$$T^{-1}(-\lambda + \partial_x^2 - k^2(1 + x^2))TT^{-1}\psi = T^{-1}(\sigma|\psi|^p\psi)$$

or

$$(-\lambda + h^{-1}\partial_x^2 - k^2(1 + hx^2))\varphi = \sigma h^{-p/4}|\varphi|^p\varphi,$$

where $\varphi = T^{-1}\psi$. Let us now multiply by h:

$$(-\partial_x^2 + x^2 - E)\varphi = \sigma h^q|\varphi|^p\varphi,$$

where

$$E := \frac{1 - \lambda h^2}{h}$$

and

$$q := 1 - p\frac{d-1}{4} = 1 - \frac{p}{4}.$$

Observe the range restriction on p is precisely so that

$$0 < q \leq 1/2.$$

We make a WKB type Ansatz, although in practice we will only take two terms (more is possible if the nonlinearity is smooth as in the context of Theorems 1.1 and 1.2):

$$\varphi = \varphi_0 + h^q\varphi_1, \;\; E = E_0 + h^q E_1.$$

The first two equations are

$$\begin{aligned} h^0: &\quad (-\partial_x^2 + x^2 - E_0)\varphi_0 = 0, \\ h^q: &\quad (-\partial_x^2 + x^2 - E_0 - h^q E_1)\varphi_1 = E_1\varphi_0 + \sigma|\varphi_0|^p\varphi_0. \end{aligned}$$

Observe we have included the $h^q E_1 \varphi_1$ term on the left hand side.

The first equation is easy:

$$\varphi_0(x) = e^{-x^2/2}, \;\; E_0 = 1.$$

For the second equation, we want to project away from φ_0 which is in the kernel of the operator on the left hand side. That is, choose E_1 satisfying

$$\langle E_1\varphi_0 + \sigma|\varphi_0|^p\varphi_0, \varphi_0\rangle = 0,$$

so that the right hand side is in $\varphi_0^\perp \subset L^2$. Then since the spectrum of the one-dimensional harmonic oscillator is simple (and of the form $(2m+1)$, $m \in \mathbb{Z}$), the operator $(-\partial_x^2 + x^2 - E_0 - h^q E_1)$ is invertible on $\varphi_0^\perp \subset L^2$ with inverse bounded by $(2 - h^q)^{-1}$. Hence for $h > 0$ sufficiently small, we can find $\varphi_1 \in L^2$ satisfying the second equation above (here we have used that φ_0 is Schwartz with bounded H^s norms). Further, since φ_0 is Schwartz and strictly positive, so is $|\varphi_0|^p\varphi_0$, so by propagation of singularities, φ_1 is also Schwartz. In particular, both φ_0 and φ_1 are rapidly decaying away from $x = 0$.

Let now $\psi(x) = T(\varphi_0(x) + h^q \varphi_1(x))$, and observe that by the above considerations, $\psi(x)$ is $\mathcal{O}(h^\infty)$ in any seminorm, outside an $h^{1/2-\delta}$ neighbourhood of $x = 0$. Let $u = e^{it\lambda} e^{ik\theta} \psi(x)$ so that $\|u\|_{L^2(dxd\theta)} \sim 1$, and u is $\mathcal{O}(h^\infty)$ outside an $h^{1/2-\delta}$ neighbourhood of $x = 0$. Furthermore, u satisfies the equation (again neglecting smaller terms)

$$\begin{aligned}(D_t + \widetilde{\Delta})u &= h^{-1} T h T^{-1} (-\lambda + \partial_x^2 - k^2 A^{-2}(x)) T T^{-1} u \\ &= h^{-1} T (\sigma h^q |T^{-1} u|^p T^{-1} u + \mathcal{O}(h |T^{-1} u|^{p+1}) \\ &= \sigma |u|^p u + Q,\end{aligned}$$

where Q satisfies the pointwise bound

$$Q = \mathcal{O}(h^q |u|^{p+1}).$$

We now let $\tilde{u}$ be the actual solution to (1.1) with the same initial profile:

$$\begin{cases} (D_t + \Delta)\tilde{u} = \sigma |\tilde{u}|^p \tilde{u} \\ \tilde{u}|_{t=0} = e^{ik\theta} \psi(x). \end{cases}$$

Set $T_h = h^p$, then with the Strichartz estimates of Burq-Gérard-Tzvetkov [BGT04] we prove (see Proposition 8.2) that there exists $\nu > 0$ so that

$$\|u - \tilde{u}\|_{L^\infty([0,T_h];L^2(M))} \leq C h^\nu,$$

and therefore we can compute:

$$\begin{aligned}\|\tilde{u}\|_{L^\infty([0,T_h];L^2(M \setminus U_{h^{1/2-\delta}}))} &\leq \|u\|_{L^\infty([0,T];L^2(M \setminus U_{h^{1/2-\delta}}))} + \|\tilde{u} - u\|_{L^\infty([0,T];L^2(M \setminus U_{h^{1/2-\delta}}))} \\ &= \mathcal{O}(h^\infty) + \mathcal{O}(h^\nu) = \mathcal{O}(h^\nu),\end{aligned}$$

which gives the result.

Remark 3.1. *It is very important to point out that the sources of additional error in this heuristic exposition have been ignored, and indeed, to apply a similar idea in the general case, a microlocal reduction to a tubular neighbourhood of* Γ *in cotangent space is employed. The function* φ_0 *is no longer so simple, and the nonlinearity* $|\varphi_0|^p \varphi_0$ *is no longer necessarily smooth. Because of this, the semiclassical wavefront sets are no longer necessarily compact, so a cutoff in frequency results in a fixed loss.*

Remark 3.2. *We remark that the Strichartz estimates from* [BGT04] *are sharp on the sphere* $\mathbb{S}^d$ *for a particular Strichartz pair, but this is not necessarily true on a generic Riemannian manifold. See* [BSS08] *for a thorough discussion of this fact.*

4. Preliminaries

4.1. **Symbol calculus on manifolds.** This section contains some basic definitions and results from semiclassical and microlocal analysis which we will be using throughout the paper. This is essentially standard, but we include it for completeness. The techniques presented have been established in multiple references, including but not limited to the previous works of the second author [Chr07, Chr08], Evans-Zworski [EZ07], Guillemin-Sternberg [GS77, GS10], Hörmander [Hör03, Hör05], Sjöstrand-Zworski [SZ02], Taylor [Tay81], and many more.

To begin we present results from [EZ07], Chapter 8 and Appendix E. Let X be a smooth, compact manifold. We will be operating on half-densities,

$$u(x)|dx|^{\frac{1}{2}} \in \mathcal{C}^\infty\left(X, \Omega_X^{\frac{1}{2}}\right),$$

with the informal change of variables formula

$$u(x)|dx|^{\frac{1}{2}} = v(y)|dy|^{\frac{1}{2}}, \text{ for } y = \kappa(x) \Leftrightarrow v(\kappa(x))|\kappa'(x)|^{\frac{1}{2}} = u(x),$$

where $|\kappa'(x)|$ has the canonical interpretation as the Jacobian $|\det(\partial\kappa)|$. By symbols on X we mean the set

$$\begin{aligned}&\mathcal{S}^{k,m}\left(T^*X, \Omega_{T^*X}^{\frac{1}{2}}\right) := \\ &= \left\{ a \in \mathcal{C}^\infty(T^*X \times (0,1], \Omega_{T^*X}^{\frac{1}{2}}) : \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)\right| \leq C_{\alpha\beta} h^{-m} \langle \xi \rangle^{k-|\beta|} \right\}.\end{aligned}$$

Essentially this is interpreted as saying that on each coordinate chart, U_α, of X, $a \equiv a_\alpha$ where is $a_\alpha \in \mathcal{S}^{k,m}$ in a standard symbol on $\mathbb{R}^d$. There is a corresponding class of pseudodifferential operators $\Psi_h^{k,m}(X, \Omega_X^{\frac{1}{2}})$ acting on half-densities defined by the local formula (Weyl calculus) in $\mathbb{R}^n$:[1]

$$\operatorname{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n} \int\int a\left(\frac{x+y}{2}, \xi; h\right) e^{i\langle x-y, \xi\rangle/h} u(y) dy d\xi.$$

We will occasionally use the shorthand notations $a^w := \operatorname{Op}_h^w(a)$ and $A := \operatorname{Op}_h^w(a)$ when there is no ambiguity in doing so.

We have the principal symbol map

$$\sigma_h : \Psi_h^{k,m}\left(X, \Omega_X^{\frac{1}{2}}\right) \to \mathcal{S}^{k,m} \Big/ \mathcal{S}^{k-1,m-1}\left(T^*X, \Omega_{T^*X}^{\frac{1}{2}}\right),$$

which gives the left inverse of Op_h^w in the sense that

$$\sigma_h \circ \operatorname{Op}_h^w : \mathcal{S}^{k,m} \to \mathcal{S}^{k,m}/\mathcal{S}^{k-1,m-1}$$

is the natural projection. Acting on half-densities in the Weyl calculus, the principal symbol is actually well-defined in $\mathcal{S}^{k,m}/\mathcal{S}^{k-2,m-2}$, that is, up to $\mathcal{O}(h^2)$ in h (see, for example [EZ07], Appendix E).

We will use the notion of semiclassical wave front sets for pseudodifferential operators on manifolds, see Hörmander [Hör05], [GS77]. If $a \in \mathcal{S}^{k,m}(T^*X, \Omega_{T^*X}^{\frac{1}{2}})$, we define the singular support or essential support for a:

$$\text{ess-supp}_h a \subset T^*X \bigsqcup \mathbb{S}^*X,$$

where $\mathbb{S}^*X = (T^*X \setminus \{0\})/\mathbb{R}_+$ is the cosphere bundle (quotient taken with respect to the usual multiplication in the fibers), and the union is disjoint. The $\text{ess-supp}_h a$ is defined using complements:

$$\begin{aligned}&\text{ess-supp}_h a := \\ &\quad \complement\left\{(x,\xi) \in T^*X : \exists \epsilon > 0, \ (\partial_x^\alpha \partial_\xi^\beta a)(x',\xi') = \mathcal{O}(h^\infty), \ d(x,x') + |\xi - \xi'| < \epsilon\right\} \\ &\quad \bigcup \complement\{(x,\xi) \in T^*X \setminus 0 : \exists \epsilon > 0, \ (\partial_x^\alpha \partial_\xi^\beta a)(x',\xi') = \mathcal{O}(h^\infty \langle\xi\rangle^{-\infty}), \\ &\qquad d(x,x') + 1/|\xi'| + |\xi/|\xi| - \xi'/|\xi'|| < \epsilon\}/\mathbb{R}_+.\end{aligned}$$

We then define the wave front set of a pseudodifferential operator $A \in \Psi_h^{k,m}(X, \Omega_X^{\frac{1}{2}})$:

$$\operatorname{WF}_h(A) := \text{ess-supp}_h(a), \ \text{ for } \ A = \operatorname{Op}_h^w(a).$$

[1]We use the semiclassical, or rescaled *unitary* Fourier transform throughout:

$$\mathcal{F}_h u(\xi) = (2\pi h)^{-d/2} \int e^{-i\langle x,\xi\rangle/h} u(x) dx.$$

If $u(h)$ is a family of distributional half-densities, $u \in \mathcal{C}^\infty((0,1]_h, \mathcal{D}'(X, \Omega_X^{\frac{1}{2}}))$, we say $u(h)$ is *h-tempered* if there is an N_0 so that $h^{N_0} u$ is bounded in $\mathcal{D}'(X, \Omega_X^{\frac{1}{2}})$. If $u = u(h)$ is an h-tempered family of distributions, we can define the semiclassical wave front set of u, again by complement:

$$\begin{aligned} \mathrm{WF}_h(u) := & \\ & \complement\{(x,\xi) : \exists A \in \Psi_h^{0,0}, \text{ with } \sigma_h(A)(x,\xi) \neq 0, \\ & \text{and } Au \in h^\infty \mathcal{C}^\infty((0,1]_h, \mathcal{C}^\infty(X, \Omega_X^{\frac{1}{2}}))\}. \end{aligned}$$

For $A = \mathrm{Op}_h^w(a)$ and $B = \mathrm{Op}_h^w(b)$, $a \in \mathcal{S}^{k,m}$, $b \in \mathcal{S}^{k',m'}$ we have the composition formula (see, for example, [DS99])

$$A \circ B = \mathrm{Op}_h^w(a \# b), \tag{4.1}$$

where

$$\mathcal{S}^{k+k',m+m'} \ni a\#b(x,\xi) := e^{\frac{ih}{2}\omega(Dx, D_\xi; D_y, D_\eta)} \left(a(x,\xi) b(y,\eta)\right)\Big|_{\substack{x=y \\ \xi=\eta}}, \tag{4.2}$$

with ω the standard symplectic form.

We record some useful Lemmas.

Lemma 4.1. *Suppose $(x_0, \xi_0) \notin \mathrm{WF}_h(u)$. Then $\forall b \in \mathcal{C}_c^\infty(T^*\mathbb{R}^n)$ with support sufficiently close to (x_0, ξ_0) we have*

$$b(x, hD)u = \mathcal{O}_{\mathcal{S}}(h^\infty).$$

Here $\mathcal{O}_{\mathcal{S}}(h^\infty)$ means $\mathcal{O}(h^\infty)$ in any Schwartz semi-norm. The proof of this Lemma follows similarly to that of Theorem 8.9 in [EZ07].

Theorem 4.2. (i) *Suppose $a \in \mathcal{S}(m)$ and $u(h)$ is h-tempered. Then*

$$\mathrm{WF}_h(a^w u) \subset \mathrm{WF}_h(u) \cap \textit{ess-supp}_h(a).$$

(ii) *If $a \in \mathcal{S}(m)$ is real-valued, then also*

$$\mathrm{WF}_h(u) \subset \mathrm{WF}_h(a^w u) \cup \overline{\complement\{\textit{ess-supp}_h a\}}.$$

Proof. Assertion 1 is straightforward. The proof of assertion 2 is standard, however we present it here so we can use it for the analogous result for the blown-up wavefront set.

We will show if $a(x_0,\xi_0) \neq 0$ and $a^w u = \mathcal{O}_{L^2}(h^\infty)$ then there exists b, $b(x_0,\xi_0) \neq 0$ so that $b^w u = \mathcal{O}_{L^2}(h^\infty)$. There exists a neighbourhood $U \ni (x_0, \xi_0)$, a real-valued function χ, and a positive number $\gamma > 0$ such that $\operatorname{supp} \chi \cap U = \emptyset$ and

$$|a + i\chi| \geq \gamma \text{ everywhere.}$$

Then $P = a^w + i\chi^w$ has an approximate left inverse c^w so that

$$c^w P = \mathrm{id} + R^w,$$

where $R^w = \mathcal{O}_{L^2 \to L^2}(h^\infty)$. Choose $b \in \mathcal{S}$ so that $\operatorname{supp}(b) \subset U$ and $b(x_0,\xi_0) \neq 0$. Then $b^w \chi^w = \mathcal{O}(h^\infty)$ as an operator on L^2. Hence

$$\begin{aligned} b^w u =& b^w c^w P u - b^w R^w u \\ =& b^w c^w a^w u + i b^w c^w \chi^w u - b^w R^w u \\ =& \mathcal{O}(h^\infty), \end{aligned}$$

where we have used the Weyl composition formula to conclude $\text{ess-supp}_h(c\#\chi) \cap \operatorname{supp} b = \emptyset$. $\square$

4.2. **Exotic symbol calculi.** Following ideas from [SZ02], since rescaling often means dealing with symbols with bad decay properties, we introduce weighted wave front sets as well. Let us first recall the non-classical symbol classes:

$$\begin{aligned} &\mathcal{S}^{k,m}_{\delta,\gamma}\left(T^*X, \Omega^{\frac{1}{2}}_{T^*X}\right) := \\ &= \quad \left\{a \in \mathcal{C}^\infty(T^*X \times (0,1], \Omega^{\frac{1}{2}}_{T^*X}) : \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)\right| \leq C_{\alpha\beta} h^{-\delta|\alpha|-\gamma|\beta|-m}\langle\xi\rangle^{k-|\beta|}\right\}. \end{aligned}$$

That is, symbols which lose δ powers of h upon differentiation in x and γ powers of h upon differentiation in ξ. Note the simplest way to achieve this is to take a symbol $a(x,\xi) \in \mathcal{S}$ and rescale $(x,\xi) \mapsto (h^{-\delta}x, h^{-\gamma}\xi)$, which then localizes on a scale $h^{\delta+\gamma}$ in phase space. We thus make the restriction that $0 \leq \delta, \gamma \leq 1$, $0 \leq \delta + \gamma \leq 1$, and to gain powers of h by integrations by parts, we usually also require $\delta + \gamma < 1$. We can define wavefront sets using $\mathcal{S}_{\delta,\gamma} = \mathcal{S}^{0,0}_{\delta,\gamma}$ symbols, but the localization of the wavefront sets is stronger.

Definition 4.3. *If $u(h)$ is h-tempered and $0 \leq \delta, \gamma \leq 1$, $0 \leq \delta + \gamma \leq 1$,*

$$\begin{aligned} \mathrm{WF}_{h,\delta,\gamma}(u) = &\complement\{(x_0,\xi_0) : \exists a \in \mathcal{S}_{\delta,\gamma} \cap \mathcal{C}^\infty_c,\ a(x_0,\xi_0) \neq 0, a(x,\xi) = \tilde{a}(h^{-\delta}x, h^{-\gamma}\xi) \\ &\textit{for some } \tilde{a} \in \mathcal{S} \textit{ and } a^w u = \mathcal{O}_{L^2}(h^\infty)\}. \end{aligned}$$

We have the following immediate corollary.

Corollary 4.4. *If $a \in \mathcal{S}_{\delta,\gamma}(m)$, $0 \leq \delta + \gamma < 1$, and a is real-valued, then*

$$\mathrm{WF}_{h,\delta,\gamma}(u) \subset \mathrm{WF}_{h,\delta,\gamma}(a^w u) \cup \overline{\complement\{\textit{ess-supp}_{\,h} a\}}.$$

The proof is exactly the same as in Theorem 4.2 only all symbols must scale the same, so they must be in $\mathcal{S}_{\delta,\gamma}$. In order to conclude the existence of approximate inverses, we need the restriction $\delta + \gamma < 1$, and the rescaling operators from §4.3 which can be used to reduce to the familiar $h^{-\delta'}$ calculus, where $\delta' = (\delta + \gamma)/2$, by replacing h with $h^{\delta-\gamma}$.

4.3. **Rescaling operators.** We would like to introduce h-dependent rescaling operators. The rescaling operators should be unitary with respect to natural Schrödinger energy norms, namely the homogeneous $\dot{H}^s$ spaces. Let us recall in $\mathbb{R}^n$, the $\dot{H}^s$ space is defined as the completion of $\mathcal{S}$ with respect to the topology induced by the inner product

$$\langle u, v\rangle_{\dot{H}^s} = \int_{\mathbb{R}^n} |\xi|^{2s} \hat{u}(\xi)\overline{\hat{v}}(\xi) d\xi,$$

where as usual $\hat{u}$ denotes the Fourier transform. The $\dot{H}^0$ norm is just the L^2 norm, and the $\dot{H}^1$ norm is

$$\|u\|_{\dot{H}^1} \simeq \||\nabla u|\|_{L^2}.$$

The purpose in taking the homogeneous norms instead of the usual Sobolev norms is to make the rescaling operators in the next Lemma unitary.

Lemma 4.5. *For any $s \in \mathbb{R}$, $h > 0$, the linear operator $T_{h,s}$ defined by*

$$T_{h,s}w(x) = h^{s/2-n/4} w(h^{-1/2}x)$$

is unitary on $\dot{H}^s(\mathbb{R}^n)$, and for any $r \in \mathbb{R}$,

$$\begin{aligned} \|T_{h,s}w\|_{\dot{H}^r} &= h^{(s-r)/2}\|w\|_{\dot{H}^r}, \\ \|T_{h,0}w\|_{\dot{H}^s} &= h^{-s/2}\|w\|_{\dot{H}^s}. \end{aligned}$$

Moreover, for any pseudodifferential operator $P(x,D)$ in the Weyl calculus,

$$T_{h,s}^{-1} P(x,D) T_{h,s} = P(h^{1/2}x, h^{-1/2}D).$$

Remark 4.1. *Observe that for this lemma, the usual assumption that h be small is not necessary.*

Proof. The proof is simple rescaling, but we include it here for the convenience of the reader. To check unitarity, we just change variables:

$$\begin{aligned}\langle u, T_{h,s}w\rangle_{\dot{H}^s} &= \left\langle |\xi|^s \hat{u}, |\xi|^s \widehat{T_{h,s}w}\right\rangle \\ &= h^{s/2+n/4}\int |\xi|^{2s}\hat{u}(\xi)\overline{\hat{w}(h^{1/2}\xi)}d\xi \\ &= h^{-n/4-s/2}\int |\xi|^{2s}\hat{u}(h^{-1/2}\xi)\overline{\hat{w}(\xi)}d\xi \\ &= h^{n/4-s/2}\int |\xi|^{2s}\widehat{u(h^{1/2}\cdot)}(\xi)\overline{\hat{w}(\xi)}d\xi \\ &= \left\langle T_{h,s}^{-1}u, w\right\rangle_{\dot{H}^s}.\end{aligned}$$

To check the conjugation property, we again compute

$$\begin{aligned}T_{h,s}^{-1}P(x,D)T_{h,s}w(x) &= T_{h,s}^{-1}(2\pi)^{-n}h^{s/2-n/4}\int e^{i\langle x-y,\xi\rangle}P(\frac{x+y}{2},\xi)w(h^{-1/2}y)dyd\xi \\ &= T_{h,s}^{-1}(2\pi)^{-n}h^{s/2+n/4}\int e^{i\left\langle x-h^{1/2}y,\xi\right\rangle}P(\frac{x+h^{1/2}y}{2},\xi)w(y)dyd\xi \\ &= T_{h,s}^{-1}(2\pi)^{-n}h^{s/2-n/4}\int e^{i\left\langle x-h^{1/2}y,h^{-1/2}\xi\right\rangle}P(\frac{x+h^{1/2}y}{2},h^{-1/2}\xi)w(y)dyd\xi \\ &= (2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}P(\frac{h^{1/2}(x+y)}{2},h^{-1/2}\xi)w(y)dyd\xi \\ &= P(h^{1/2}x,h^{-1/2}D)w(x).\end{aligned}$$

□

The purpose of using the rescaling operators $T_{h,s}$ is that if $u \in \dot{H}^s$ has h-wavefront set

$$\mathrm{WF}_{h,\delta,\gamma}(u) \subset \{|x| \leq \alpha(h) \text{ and } |\xi| \leq \beta(h)\},$$

where, according to the uncertainty principle, $\alpha(h)\beta(h) \geq h$, then $T_{h,s}u$ has h-wavefront set

$$\mathrm{WF}_{h,\delta-1/2,\gamma+1/2}(T_{h,s}u) \subset \{|x| \leq \alpha(h)h^{1/2} \text{ and } |\xi| \leq \beta(h)h^{-1/2}\},$$

provided, of course, that $\delta \geq 1/2$. To see this, we just observe that for any $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{2n})$, $\psi \equiv 0$ on $\{|x| \leq 1, |\xi| \leq 1\}$, we have

$$\psi(x/\alpha(h), D/\beta(h))u = \mathcal{O}(h^\infty)\|u\|_{L^2},$$

or any other semi-norm, and hence

$$\begin{aligned}\psi(h^{-1/2}x/\alpha(h), h^{1/2}D/\beta(h))T_{h,s}u &= T_{h,s}\psi(x/\alpha(h), D/\beta(h))T_{h,s}^{-1}T_{h,s}u \\ &= T_{h,s}\psi(x/\alpha(h), D/\beta(h))u \\ &= h^{s/2}\mathcal{O}(h^\infty)\|u\|_{L^2} \\ &= \mathcal{O}(h^\infty)\|u\|_{L^2}.\end{aligned}$$

Finally, we note that the symbol of the operator on the left-hand side is zero on the set

$$\{|x| \leq h^{1/2}\alpha(h), \text{ and } |\xi| \leq h^{-1/2}\beta(h)\},$$

and any such symbol in $\mathcal{S}_{\delta-1/2,\gamma+1/2}$, elliptic at a point outside this set, can be locally obtained in this fashion.

5. Geometry near an elliptic geodesic

Suppose Γ is a geodesic in a $n+1$ dimensional Riemannian manifold. Following [MP05, §2.1] (cf. [Gra90]) we fix an arclength parametrization $\gamma(t)$ of Γ, and a parallel orthonormal frame $E_1, \dots, E_n$ for the normal bundle $N\Gamma$ to Γ in M. This determines a coordinate system

$$x = (x_0, x_1, \dots, x_n) \mapsto \exp_{\gamma(x_0)}(x_1 E_1 + \dots x_n E_n) = F(x).$$

We write $x' = (x_1, \dots, x_n)$ and use indices $j, k, \ell \in \{1, \dots, n\}$, $\alpha, \beta, \delta \in \{0, \dots, n\}$. We also use $X_\alpha = F_*(\partial_{x_\alpha})$.

Note that $r(x) = \sqrt{x_1^2 + \dots x_n^2}$ is the geodesic distance from x to Γ and ∂_r is the unit normal to the geodesic tubes $\{x : d(x, \Gamma) = \text{cst}\}$. Let $p = F(x_0, 0)$, $q = F(x_0, x')$, and $r = r(q) = d(p, q)$ then we have [MP05, Proposition 2.1], which states

$$\begin{aligned} g_{jk}(q) &= \delta_{ij} + \frac{1}{3} g(R(X_s, X_j)X_\ell, X_k)_p x_s x_\ell + \mathcal{O}(r^3), \\ g_{0k}(q) &= \mathcal{O}(r^2), \\ g_{00}(q) &= 1 - g(R(X_k, X_0)X_0, X_\ell)_p x_k x_\ell + \mathcal{O}(r^3), \\ \Gamma^\delta_{\alpha\beta} &= \mathcal{O}(r), \\ \Gamma^k_{00} &= -\sum_{j=1}^n g(R(X_k, X_0)X_j, X_0)_p x_j + \mathcal{O}(r^2), \end{aligned} \tag{5.1}$$

where $\Gamma^\delta_{\alpha\beta} = \frac{1}{2} g^{\delta\eta}(X_\alpha g_{\eta\beta} + X_\beta g_{\alpha\eta} - X_\eta g_{\alpha\beta})$.

In these coordinates, the Laplacian has the form

$$\begin{aligned} \Delta_g &:= \frac{1}{\sqrt{\det g}} \operatorname{div}(\sqrt{\det g} g^{-1} \nabla) \\ &= g^{jk} X_j X_k - g^{jk} \Gamma^\ell_{jk} X_\ell \\ &= g^{00} X_0 X_0 + 2 g^{k0} X_k X_0 - g^{jk} \Gamma^0_{jk} X_0 + 2 g^{0j} \Gamma^k_{0j} X_k + \Delta_{\Gamma^\perp}, \end{aligned} \tag{5.2}$$

where $\Delta_{\Gamma^\perp}$ is the Laplacian in the directions transverse to Γ.

Denote the geodesic flow on SM, the unit tangent bundle, by φ_t. Let $\gamma(0) = p \in \Gamma$ and $\zeta = \gamma'(0) \in SM$. Associated to Γ is a periodic orbit $\varphi_t \zeta$ of the geodesic flow on SM. This orbit $\varphi_t \zeta$ is called *stable* if, whenever $\mathcal{V}$ is a tubular neighbourhood of $\varphi_t \zeta$, there is a neighbourhood $\mathcal{U}$ of ζ such that $\zeta' \in \mathcal{U}$ implies $\varphi_t \zeta' \subseteq \mathcal{V}$. Given a hypersurface Σ in SM containing ζ and transverse to $\varphi_t \zeta$, we can define a *Poincaré map* $\mathcal{P}$ near ζ, by assigning to each ζ' the next point on $\varphi_t(\zeta')$ that lies in Σ. Two Poincaré maps of $\varphi_t \zeta$ are locally conjugate and hence the eigenvalues of $d\mathcal{P}$ at ζ are invariants of the periodic orbit $\varphi_t \zeta$.

The Levi-Civita connection allows us to identify TTM with the sum of a horizontal space and a vertical space, each of which can be identified with TM. Thus we can choose Σ so that $T_\zeta \Sigma$ is equal to $E \oplus E$, where E is the orthogonal complement of ζ in $T_p M$. The linearized Poincaré map is then given by

$$\begin{aligned} E \oplus E &\xrightarrow{\ d\mathcal{P}\ } E \oplus E, \\ (V, W) &\longmapsto (J(\text{length}(\gamma)), \nabla_{X_0} J(\text{length}(\gamma))) \end{aligned} ,$$

where J is the unique Jacobi field along γ with $J(0) = V$ and $\nabla_{X_0} J(0) = W$, i.e., J solves

$$\begin{cases} \nabla_{X_0} \nabla_{X_0} J + R(J, X_0) X_0 = 0 \\ J(0) = V, \quad \nabla_{X_0} J(0) = W. \end{cases}$$

The linearized map $d\mathcal{P} = d\mathcal{P}|_\zeta$ preserves the symplectic form on $E \oplus E$,

$$\omega((V_1, W_1), (V_2, W_2)) = \langle V_1, W_2 \rangle - \langle W_1, V_2 \rangle,$$

and so its eigenvalues come in complex conjugate pairs. We say that Γ is a *non-degenerate elliptic closed geodesic* if the eigenvalues of $d\mathcal{P}$ have the form $\{e^{\pm i\alpha_j} : j = 1, \dots, n\}$ where each α_j is real and the set $\{\alpha_1, \dots, \alpha_n, \pi\}$ is linearly independent over $\mathbb{Q}$.

From (5.1), the Hessian of the function $g_{00}(q)$ as a function of x' is (minus) the transformation appearing in the Jacobi operator

$$E \ni V \mapsto R(V, X_0) X_0 \in E.$$

Notice that if V is a normalized eigenvector for this operator, with eigenvalue λ, then

$$\sec_p(X_0, V) = g_p(R(V, X_0)X_0, V) = \lambda g_p(V, V) = \lambda.$$

The very useful property

$$p \in \Gamma, V \in E \implies \sec_p(X_0, V) > 0, \tag{5.3}$$

holds for any elliptic closed geodesic.

Remark 5.1. *One can verify* (5.3) *by means of the Birkhoff normal form (see* [Zel08, §10.3]*). Indeed, if one of the sectional curvatures were negative, then the Birkhoff normal form of the linearized Poincare map (in T^*M) must have an eigenvalue off the unit circle. If so, then there is at least one nearby orbit which does not stay nearby, hence the periodic geodesic is not elliptic. Similarly, if one of the curvatures vanishes, then the linearized Poincare map has a zero eigenvalue, and hence the logs of the eigenvalues are not independent from π over the rationals (in other words, the Poincare map is degenerate, and not even symplectic).*

6. Compact Solitons: The nonlinear Ansatz

We are interested in finding quasimodes for the non-linear Schrödinger equation

$$-\Delta_g u = \lambda u - \sigma |u|^p u,$$

where $\sigma = \pm 1$ determines if we are in the focussing or defocussing case. We will construct u approximately solving this equation with u concentrated near Γ in a sense to be made precise below.

We take as Ansatz

$$F(x, h) = e^{ix_0/h} f(x, h), \quad h^{-1} = \frac{2\pi N}{L},$$

where $L > 0$ is the period of Γ, and assume for the time being that the function f is concentrated in an h-dependent neighbourhood of Γ. We are going to employ a semiclassical reduction, and we are interested in fast oscillations ($h \to 0$), so we assume $\mathrm{WF}_{h,1/2-\delta,0} f(x, h) \subset \{|x'| \le \epsilon h^{1/2-\delta}, |\xi'| \le \epsilon\}$ for some $\epsilon, \delta > 0$[2].

[2] The reason for the weaker concentration in frequency $|\xi'|$ is that the nonlinearity forces working with non-smooth functions, so some decay at infinity in frequency is lost.

The localization property of f as constructed later will be verified in Section 7. We compute from (5.2), with Δ the non-positive Laplacian,

$$\Delta F = e^{ix_0/h}\left[g^{00}\left(-\frac{1}{h^2}f + \frac{2i}{h}X_0 f + X_0X_0 f\right) + 2g^{k0}X_k\left(\frac{i}{h}f + X_0 f\right) - g^{kj}\Gamma^0_{kj}\left(\frac{i}{h}f + X_0 f\right) + 2g^{0j}\Gamma^k_{0j}X_k f + \Delta_{\Gamma^\perp} f\right]. \tag{6.1}$$

Remark 6.1. *One may initially be inclined to use the Ansatz of the original Gaussian beam from Ralston* [Ral82]*, which is*

$$e^{i\psi(x)/h}(a_0 + a_1 h + \cdots + a_N h^N),$$

the standard geometric optics quasimode construction. After all, Ralston is able to make very nice use of the geometry to construct a phase function of the form $i/h(x_0 + \frac{1}{2}x'B(x_0)x')$ *with* $\operatorname{Im} B(x_0) > 0$ *(for* $x_0 \neq 0$*, vanishing otherwise). In such a regime, however, the non-linear term in the Schrödinger equation* (1.1) *vanishes to infinite order in h. Thus while such a solution always exists, it fails to capture the effects of the nonlinearity that we are interested in.*

We analyze (6.1) by applying the operator $T^{-1}_{h,s}$ in the variables transversal to Γ. We normalize everything in the L^2 sense, so we take here $s = 0$. Let $z = h^{-1/2}x'$ and set $v(x_0, z, h) = T^{-1}_{h,0} f(x_0, z, h) = h^{n/4} f(x_0, h^{1/2}z, h)$. Notice that the distance to the geodesic $r = |x'|$ is scaled to $\rho = |z| = h^{-1/2}r$, as described above. In particular, now

$$\mathrm{WF}_{h,0,1/2} v \subset \{|z| \leq h^{-\delta}\epsilon,\ |\zeta| \leq h^{1/2}\epsilon\}, \tag{6.2}$$

if ζ is the (semiclassical) Fourier dual to z. We conjugate (6.1) to get

$$T^{-1}_{h,0}\Delta T_{h,0} T^{-1}_{h,0} F = e^{ix_0/h}\left[g^{00}\left(-\frac{1}{h^2}v + \frac{2i}{h}X_0 v + X_0X_0 v\right) + 2g^{k0}h^{-1/2}\partial_{z_k}\left(\frac{i}{h}v + X_0 v\right) - g^{kj}\Gamma^0_{kj}\left(\frac{i}{h}v + X_0 v\right) + 2g^{0j}\Gamma^k_{0j}h^{-1/2}\partial_{z_k} v + h^{-1}\Delta_{\Gamma^\perp} v\right],$$

where the metric components and Christoffel symbols are evaluated at $(x_0, h^{1/2}z)$. On the other hand, from (5.1), expanding in Taylor polynomials, we know that

$$\begin{aligned}
g^{00}(x) &= 1 + R_2(x) + R_3(x) + R_4(x) + \mathcal{O}(r^5)\\
&= 1 + hR_2(z) + h^{3/2}R_3(z) + h^2 R_4(z) + \mathcal{O}(h^{5/2}\rho^5)\\
g^{0k}(x) &= h\tilde{g}^{0k}_2(z) + \mathcal{O}(h^{3/2}\rho^3)\\
g^{jk}(x) &= \delta_{jk} + \mathcal{O}(h\rho^2)\\
\Gamma^0_{jk}(x) &= h^{1/2}\widetilde{\Gamma}^0_{jk1}(z) + h\widetilde{\Gamma}^0_{jk2}(z) + \mathcal{O}(h^{3/2}\rho^3)\\
\Gamma^k_{0j}(x) &= h^{1/2}\widetilde{\Gamma}^k_{0j1}(z) + \mathcal{O}(h\rho^2),
\end{aligned}$$

for some smooth functions R_ℓ, $\tilde{g}^{0k}_\ell$, $\widetilde{\Gamma}^\alpha_{jk\ell}$ homogeneous of degree ℓ respectively. Hence

$$T^{-1}_{h,0}\Delta T_{h,0} T^{-1}_{h,0} F = e^{ix_0/h}\left[-\frac{1}{h^2}v - \frac{1}{h}R_2(z)v - \frac{1}{h^{1/2}}R_3(z)v - R_4(z)v + \frac{2i}{h}X_0 v + \frac{2i}{h^{1/2}}\tilde{g}^{k0}_2(z)\partial_{z_k} v - \frac{i}{h^{1/2}}g^{jk}\widetilde{\Gamma}^0_{jk1}(z)v - ig^{jk}\widetilde{\Gamma}^0_{jk2}(z)v + \frac{1}{h}\Delta_{\Gamma^\perp} v\right] + Pv, \tag{6.3}$$

where P contains the remaining terms from the Taylor expansion. Let us record the following Lemma.

Lemma 6.1. *The operator P has the following expansion:*

$$\begin{aligned} P &= \mathcal{O}(h^{1/2}|z|^5) + X_0X_0 + \mathcal{O}(h|z|^2)h^{-1/2}\partial_{z_k}X_0 + \mathcal{O}(|z|^3)\partial_{z_k} + \mathcal{O}(h^{1/2}|z|^3) \\ &\quad + \mathcal{O}(h^{1/2}|z|)X_0 + \mathcal{O}(h^{3/2}|z|^3)h^{-1/2}\partial_{z_k}. \end{aligned}$$

In particular, if v satisfies (6.2), then

$$\|Pv\|_{L^2} \le C\|v\|_{L^2} + C\|X_0X_0v\|_{L^2} + Ch^{1/2-\delta}\|X_0v\|_{L^2}.$$

Remark 6.2. *We will show later that for the particular choice of v we construct, the operators X_0 and X_0^2 are bounded operators, so that the error Pv is bounded in L^2 by v (see Remark 7.3).*

For the purposes of exposition, let us then assume for now that the term Pv is bounded and proceed (this will be justified later). Applying $T_{h,0}^{-1}$ to the equation $-\Delta F = \lambda F - \sigma|F|^pF$ yields

$$\begin{aligned} -T_{h,0}^{-1}\Delta T_{h,0}T_{h,0}^{-1}F &= \lambda T_{h,0}^{-1}F - \sigma T_{h,0}^{-1}(|F|^pF) \\ &= \lambda T_{h,0}^{-1}F - \sigma h^{-pn/4}|T_{h,0}^{-1}F|^pT_{h,0}^{-1}F, \end{aligned}$$

so that multiplying (6.3) by $he^{-ix_0/h}$, we get

$$\begin{aligned} &(2iX_0 + \Delta_{\Gamma^\perp} - R_2(z))v \\ &\quad = \frac{1-h^2\lambda}{h}v + h^{1/2}\left(ig^{jk}\widetilde{\Gamma}^0_{jk1} - 2i\tilde{g}^{k0}(z)\partial_{z_k} + R_3(z)\right)v + h(R_4(z) + ig^{jk}\tilde{\Gamma}^0_{jk2})v \\ &\qquad + h^{1-pn/4}\sigma|v|^pv + he^{-ix_0/h}Pv, \end{aligned} \tag{6.4}$$

where P is the same as above.

Remark 6.3. *In order to ensure that the nonlinearity appears here as a lower order term, we require*

$$q := 1 - pn/4 > 0, \ \text{or } p < \frac{4}{n} = \frac{4}{d-1} \tag{6.5}$$

as stated in the theorems.

We want to think of the left hand side as similar to a time-dependent harmonic oscillator where x_0 plays the role of the time variable.

Let $q = 1 - pn/4$, $0 < q < 1$. We would like to assume that v and λ have expansions in h^q, however the spreading of wavefront sets due to the nonlinearity allows us to only take the first two terms when $0 < q \le 1/2$ and the first three terms otherwise.

Case 1: $0 < q \le 1/2$**.** Assume that v has a two-term expansion

$$v = v_0 + h^qv_1$$

and moreover that there exist E_k, $k = 0, 1$, satisfying

$$\frac{1-h^2\lambda}{h} = E_0 + h^qE_1 + \mathcal{O}(h^{2q}).$$

Since $q \le 1/2$, then the $\mathcal{O}(h^{1/2})$ term in (6.4) is of equal or lesser order than the nonlinear term, and substituting into (6.4) we get the following equations according to powers of h:

$$\begin{aligned} h^0: &\quad (2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0)v_0 = 0, \\ h^q: &\quad (2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0 - h^qE_1)v_1 = E_1v_0 + \sigma|v_0|^pv_0 + h^{1/2-q}Lv_0, \end{aligned} \tag{6.6}$$

where

$$Lv_0 = \left(ig^{jk}\widetilde{\Gamma}^0_{jk1} - 2i\tilde{g}^{k0}(z)\partial_{z_k} + R_3(z) + h^{1/2}R_4(z) + ih^{1/2}g^{jk}\tilde{\Gamma}^0_{jk2} + h^{1/2}e^{-ix_0/h}P\right)v_0.$$

We will show the error terms are $\mathcal{O}(h^{2q})$ in the appropriate H^s space. See §7.2.1.

Case 2: $1/2 < q < 1$**.** In the case $q > 1/2$, the $\mathcal{O}(h^{1/2})$ term becomes potentially larger than the nonlinearity, so we take three terms in the expansions of v and $E = \frac{1-h^2\lambda}{h}$:

$$v = v_0 + h^{1/2}v_1 + h^q v_2, \;\; E = E_0 + h^{1/2}E_1 + h^q E_2 + \mathcal{O}(h).$$

We then want to solve

$$\begin{aligned} h^0 : \quad & (2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0)v_0 = 0, \\ h^{1/2} : \quad & (2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0)v_1 = E_1 v_0 + Lv_0 \\ h^q : \quad & (2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0 - h^{1/2}E_1 - h^q E_2)v_2 = E_2 v_0 + h^{1-q}E_1 v_1 + h^{1/2}E_2 v_1 \\ & \qquad + \sigma|v_0|^p v_0 + h^{1-q}Lv_1. \end{aligned} \tag{6.7}$$

In this case we will show the error is $\mathcal{O}(h^{1/2+q})$ in the appropriate H^s space. See §7.2.2.

7. Quasimodes

We begin by approximately solving the h^0 equation by undoing our previous rescaling. That is, let $w_0(x_0, x, h) = T_{h.0}v_0(x_0, x, h) = h^{-n/4}v_0(x_0, h^{-1/2}x, h)$, and conjugate the h^0 equation by $T_{h,0}$ to get:

$$\begin{aligned} 0 &= T_{h,0}(2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0)T_{h,0}^{-1}T_{h,0}v_0 \\ &= (2iX_0 + h\Delta_{\Gamma^\perp} - h^{-1}R_2(x) - E_0)w_0, \end{aligned}$$

where now the coefficients in $\Delta_{\Gamma^\perp}$ are independent of h, and we have used the homogeneity of R_2 in the x variables. Multiplying by h, we have the following equation:

$$(2ihX_0 + h^2\Delta_{\Gamma^\perp} - R_2(z) - hE_0)w_0 = 0, \tag{7.1}$$

Hence, (7.1) is a semiclassical equation in a fixed neighbourhood of Γ with symbols in the h^0 calculus (i.e. no loss upon taking derivatives). The principal symbol of the operator in (7.1) is

$$p = \tau - |\zeta|^2_{\tilde{g}} - R_2(z),$$

where $\tilde{g}$ is the metric in the transversal directions to Γ, and τ is the dual variable to the x_0 direction. If we let $\tilde{\Gamma}$ be the (unit speed) lift of Γ to T^*M, and if $\exp(sH_p)$ is the Hamiltonian flow of p, then

$$\Gamma = \{\zeta = z = 0, x_0 = s \in \mathbb{R}/\mathbb{Z}\}.$$

Since, in the transversal directions, p is a negative definite quadratic form, the linearization of the Poincaré map S is easy to compute:

$$S = \exp(H_q),$$

where

$$q = -|\zeta|^2_{\tilde{g}} - R_2(z),$$

so that

$$H_q = -2a(x_0, z)^{j,k}\zeta_j\partial_{z_k} + 2b^{j,k}(x_0)z_j\partial_{\zeta_k},$$

where a and b are symmetric, positive definite matrices. Linearizing S about, say, x_0 and $z = \zeta = 0$ we get that $dS(0,0)$ has all eigenvalues on the unit circle, in complex conjugate pairs. That is, Γ is still a periodic elliptic orbit of the classical flow of p.

Since p is defined on a fixed scale, we can glue p together with an operator which is elliptic at infinity so that p is of real principal type so that we can apply Theorem A.1 in the appendix to construct linear quasimodes. Note that since we have quasi-eigenvalue of order $\mathcal{O}(h)$, Theorem A.1 implies the quasimodes are concentrated on a scale $|z| \le h^{1/2}$, $|\zeta| \le h^{1/2}$.

This is made precise in the following proposition.

Proposition 7.1. *There exists $w_0 \in L^2$, $\|w_0\|_{L^2} = 1$, and $E_0 = \mathcal{O}(1)$ such that*

$$(2ihX_0 + h^2\Delta_{\Gamma^\perp} - R_2(z) - hE_0)w_0 = e_0(h).$$

Here the error $e_0(h) = \mathcal{O}(h^\infty) \in L^2$ (or in any other seminorm), and w_0 has h-wavefront set sharply localized on Γ in the sense that if $\varphi \in \Psi^0_{1/2-\delta,1/2-\delta}$ is 1 near Γ, then for any $\delta > 0$, $\varphi w_0 = w_0 + \mathcal{O}(h^\infty)$, and if $\delta = 0$, $\|\varphi w_0\| \geq c_0 \|w_0\|$ for some positive c_0 depending on the support of φ.

Moreover, $w_0 \in H^\infty(M)$ and satisfies the estimate

$$\|w_0\|_{H^s(M)} = \mathcal{O}(h^{-s/2}).$$

Proof. The construction follows from Theorem A.1, and is well-known in other settings, see for instance [Ral82], [Bab68], and [CP02]. To get the sharp localization, apply Lemmas B.1 and B.2 from the appendix to get the localization on w_0. Once we know that w_0 is so localized, we can replace w_0 with φw_0, where $\varphi \in \Psi^0_{1/2-\delta}$ is as in the proposition. Then φw_0 satisfies

$$(2ihX_0 + h^2\Delta_{\Gamma^\perp} - R_2(z) - hE_0)\varphi w_0 = \tilde{e}_0(h),$$

where

$$\tilde{e}_0(h) = \varphi e_0(h) + [(2ihX_0 + h^2\Delta_{\Gamma^\perp} - R_2(z)), \varphi]w_0.$$

But now $\varphi e_0(h) = \mathcal{O}(h^\infty)$, while the commutator is supported outside of an $h^{1/2-\delta}$ neighbourhood of Γ, so by the localization of w_0 is $\mathcal{O}(h^\infty)$ and localized in a slightly larger set on the scale $h^{1/2-\delta}$. □

Now recalling $v_0 = T^{-1}_{h,0} w_0$, then v_0 satisfies

$$\begin{aligned}(2iX_0 + &\Delta_{\Gamma^\perp} - R_2(z) - E_0)v_0 \\ &= h^{-1}T^{-1}_{h,0}(2ihX_0 + h^2\Delta_{\Gamma^\perp} - R_2(z) - E_0)T_{h,0}v_0 \\ &= h^{-1}T^{-1}_{h,0}(2ihX_0 + h^2\Delta_{\Gamma^\perp} - R_2(z) - E_0)w_0 \\ &= h^{-1}T^{-1}_{h,0}e_0(h).\end{aligned}$$

The error $h^{-1}T^{-1}_{h,s}e_0(h) = \mathcal{O}(h^\infty)$ in any seminorm still, but the function v_0 is now localized on a scale $h^{-\delta}$ in space. That is,

$$\mathrm{WF}_{h,0,1-\delta} v_0 \subset \{|x| \leq \epsilon h^{-\delta}, |\xi| \leq \epsilon h^{1-\delta}\}.$$

7.1. **The inhomogeneous equation.** We are now in a position to solve the lower order inhomogeneous equations in (6.6). The quasimode v_0 has been constructed as a "Gaussian beam" (see [Ral82]); it is a harmonic oscillator eigenfunction extended in the x_0 direction by the *quantum monodromy operator* from [SZ02], which is defined in (7.4) below. From this construction, the boundedness of the error term Pv_0 as stated in Lemma 6.1. In what follows we construct v_1 in the case $0 \leq q \leq \frac{1}{2}$ and v_1, v_2 in the case $\frac{1}{2} < q < 1$ and show that a similar bound holds for $h^q Pv_1$ and $h^{\frac{1}{2}}Pv_1 + h^q Pv_2$ respectively.

We want to solve

$$(2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - \tilde{E}_0)v_1 = E_1 v_0 + G, \tag{7.2}$$

where $\tilde{E}_0 = E_0 + h^\eta E_1$ for some $\eta > 0$, E_0 and v_0 have been fixed by the previous construction, and $G = G(x_0, x)$ is a given function (periodic in x_0) with

$$\mathrm{WF}_{h,0,1/2} G \subset \{|x| \leq \epsilon h^{-\delta}, |\xi| \leq \epsilon h^{1/2}\}.$$

Note, the localization of G follows from the nonlinearity as well as geometric multipliers in the operator L, see Appendix E and (6.2).

Conjugating by $T_{h,0}$ as before, we get the equation

$$(2ihX_0 + h^2\Delta_{\Gamma^\perp} - R_2(x) - h\tilde{E}_0)w_1 = G_2, \tag{7.3}$$

where

$$G_2 = hT_{h,0}(E_1 v_0 + G).$$

We observe then that

$$\mathrm{WF}_{h,1/2-\delta,0} G_2 \subset \{|x| \leq \epsilon h^{1/2-\delta}, |\xi| \leq \epsilon\}.$$

Specifically, let $M(x_0)$ be the deformation family to the quantum monodromy operator defined as the solution to:

$$\begin{cases} (2ihX_0 + h^2\Delta_{\Gamma^\perp} - R_2(x) - h\tilde{E}_0)M(x_0) = 0, \\ M(x_0) = \mathrm{id}, \end{cases} \tag{7.4}$$

which exists microlocally in a neighbourhood of $\Gamma \subset T^*X$ (for further discussion see also Appendix A and references therein). By the Duhamel formula, we write

$$w_1 = M(x_0)w_{1,0} + \frac{i}{h}\int_0^{x_0} M(x_0)M(y_0)^* G_2(y_0,\cdot)dy_0.$$

We have to choose $w_{1,0}$ and E_1 (implicit in G_2) in such a fashion to make w_1 periodic in x_0; in other words, to solve the equation (approximately) around Γ. Let L be the primitive period of Γ, so that $x_0 = 0$ corresponds to $x_0 = L$. Then we require

$$w_1(L,\cdot) = w_1(0,\cdot),$$

or

$$w_{1,0} = M(L)w_{1,0} + \frac{i}{h}\int_0^{L} M(L)M(y_0)^* G_2(y_0,\cdot)dy_0.$$

In other words, we want to be able to invert the operator $(1 - M(L))$. The problem is that $w_{0,0} := w_0(0,\cdot) = T_{h,0}v_0(0,\cdot)$ is in the kernel of $(1 - M(L))$, so we need to choose E_1 in such a fashion to kill the contribution of G_2 in the direction of $w_{0,0}$.

Recall that

$$\begin{aligned} G_2 &= hT_{h,0}(E_1 v_0 + G) \\ &= h(E_1 w_0 + \tilde{G}), \end{aligned}$$

where

$$\tilde{G} = T_{h,0}G.$$

We want to solve microlocally

$$\begin{aligned} (1 - M(L))w_{1,0} &= \frac{i}{h}\int_0^L M(L)M(y_0)^*(h(E_1 w_0 + \tilde{G}))dy_0 \\ &= i\int_0^L M(L)M(y_0)^*(E_1 M(y_0)w_{0,0} + \tilde{G})dy_0 \\ &= iLM(L)E_1 w_{0,0} + i\int_0^L M(L)M(y_0)^*\tilde{G}dy_0. \end{aligned} \tag{7.5}$$

Let

$$E_1 = -\frac{1}{L}\left\langle \int_0^L M(y_0)^*\tilde{G}dy_0, w_{0,0} \right\rangle,$$

so that (7.5) is orthogonal to $w_{0,0}$.

If we denote

$$L^2_{w_{0,0}^\perp} = \{u \in L^2 : \langle u, w_{0,0} \rangle = 0\},$$

then by the nonresonance assumption (since $E_0 + h^\eta E_1$ is a small perturbation of E_0), and the fact that $M(L)$ is unitary on L^2, $(I - M(L))^{-1}$ is a bounded operator (see [Chr08])

$$(1 - M(L))^{-1} : L^2_{w_{0,0}^\perp} \to L^2_{w_{0,0}^\perp}.$$

Hence

$$w_{1,0} = (1 - M(L))^{-1} \frac{i}{h} \int_0^L M(L) M(y_0)^* G_2(y_0, \cdot) dy_0$$

satisfies

$$\|w_{1,0}\|_{L^2} \le Ch^{-1} \left\| \int_0^L M(y_0)^* G_2(y_0, \cdot) dy_0 \right\|_{L^2} \le CL^{1/2} h^{-1} \|G_2\|_{L^2(x_0) L^2}$$

and

$$w_{1,0} \in L^2_{w_{0,0}^\perp}.$$

Furthermore, we have the estimate

$$\|w_{1,0}\|_{\dot{H}^s} \le Ch^{-1} h^{-s/2} \left\| \int_0^L M(y_0)^* G_2(y_0, \cdot) dy_0 \right\|_{L^2} \le CL^{1/2} h^{-1-s/2} \|G_2\|_{L^2(x_0) L^2},$$

that is, the $\dot{H}^s$ norm is controlled, but not by the homogeneous Sobolev norm.

We have proved the following Proposition, which follows simply from tracing back the definitions.

Proposition 7.2. *Let v_0 be as constructed in the previous section, and let $G \in H^s$ for $s \ge 0$ sufficiently large satisfy*

$$\mathrm{WF}_{h,0,1/2} G \subset \{|x| \le \epsilon h^{-\delta}, |\xi| \le \epsilon h^{1/2}\}.$$

Then for any $\eta > 0$, there exists $v_1 \in L^2$ and $E_1 = \mathcal{O}(\|G\|_{L^2})$ such that

$$(2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0 - h^\eta E_1) v_1 = E_1 v_0 + G,$$

and moreover

$$\|v_1\|_{\dot{H}^s} \le C(\|G\|_{H^s} + \|v_0\|_{H^s}).$$

Remark 7.3. *We note here that by construction of v_1 we have implicitly microlocalized into a periodic tube in the x_0 variable. Using the fact that the Quantum Monodromy Operator is a microlocally unitary operator (see* [SZ02, Chr08]*), the bound*

$$\|X_0^j v_1\|_{L^2} \lesssim \|v_1\|_{L^2}$$

follows easily for any j, which is required for proof of Lemma 6.1.

7.2. **Construction of quasimodes in the context of Theorem 1.3.** We now have all the tools to construct the quasimodes which will be used to prove Theorem 1.3. Let $d \ge 2$ and $0 < p < 4/(d-1)$. As previously, denote by $q = 1 - p(d-1)/4$. The main results of this part will be stated in Propositions 7.1 and 7.2.

7.2.1. *The case* $0 < q \leq 1/2$, *i.e.* $\frac{2}{d-1} \leq p < \frac{4}{d-1}$. In this subsection, we see how to apply Proposition 7.2 in the case $0 < q \leq 1/2$. As described previously, in this case, the nonlinearity is the next largest term, and we have only one inhomogeneous equation so solve (see (6.6)).

According to Propostion E.2 and Corollary E.3 from Appendix E, if

$$G = \sigma |v_0|^p v_0 - h^{1/2-q} L v_0$$

is the nonlinear term on the right-hand side, then G is sharply localized in space but weakly localized in frequency. That is, if $\chi \in \mathcal{C}_c^\infty(T^*X)$ is equal to 1 in a neighbourhood of Γ, then for any $0 \leq \delta < 1/2$ and any $0 \leq \gamma \leq 1$,

$$\chi(h^\delta x, h^{1-\gamma} D_x) G(x_0, x) = G(x_0, x) + E,$$

where for any $0 \leq r \leq 3/2$,

$$\|E\|_{\dot{H}^r} \leq C h^{(1-\gamma)(3/2+p-r)}.$$

We are interested in the case where $\gamma = 1/2$, since in that case G is weakly concentrated in frequencies comparable to $h^{1/2}$, so by cutting off, satisfies the assumptions of Proposition 7.2. That is, take $\gamma = 1/2$, and replace G with $\tilde{G} = \chi(h^\delta x, h^{1-\gamma} D_x) G$, and apply Proposition 7.2 to get v_1 and E_1 satisfying

$$(2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0 - h^q E_1) v_1 = E_1 v_0 + \tilde{G},$$

or in other words

$$(2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0 - h^q E_1) v_1 = E_1 v_0 + \sigma |v_0|^p v_0 + h^{1/2-q} L v_0 + \tilde{Q}_1,$$

where

$$\|\tilde{Q}_1\|_{\dot{H}^r} \leq C h^{(1/2)(3/2+p-r)}.$$

Now letting $v = v_0 + h^q v_1$ and $E = E_0 + h^q E_1$, we have solved

$$\begin{aligned}(2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E) v &= h^q \sigma |v_0|^p v_0 + h^{1/2} L v_0 + h^q \tilde{Q}_1 \\ &= h^q \sigma |v|^p v + h^{1/2} L v + \tilde{Q}_2,\end{aligned}$$

where

$$\tilde{Q}_2 = h^q \tilde{Q}_1 - h^{1/2+q} L v_1 + \mathcal{O}(h^{2q} |v|^{p+1}).$$

The remainder $\tilde{Q}_2$ satisfies

$$\|\tilde{Q}_2\|_{\dot{H}^s} \leq C h^{-s/2 + \min\{2q, 1/2+q, q+3/4+p/2\}} = C h^{-s/2+2q},$$

since $q \leq 1/2$. Recalling the definitions, $\varphi = e^{ix_0/h} T_{h,0} v$ satisfies

(i)
$$\mathrm{WF}_{h,1/2-\delta,0}\, \varphi \subset \{|x| \leq \epsilon h^{1/2-\delta}, |\xi| \leq \epsilon\};$$

(ii)
$$\|\varphi\|_{L^2} \sim 1, \qquad \|D_{x_0}^\ell \varphi\|_{L^2} \sim h^{-\ell},$$

and
$$\|D_{x'}^\ell \varphi\|_{L^2} \leq C h^{-\ell/2};$$

(iii)
$$\begin{aligned}(\Delta + \lambda)\varphi &= h^{-1} e^{ix_0/h} T_{h,0} h e^{-ix_0/h} T_{h,0}^{-1} \Delta T_{h,0} T_{h,0}^{-1} \varphi \\ &= h^{-1} e^{ix_0/h} T_{h,0} (2iX_0 + \Delta_{\Gamma^\perp} - R_2 - E - h^{1/2} L) v \\ &= h^{-1} e^{ix_0/h} T_{h,0} (\sigma h^q |v|^p v + Q_2),\end{aligned}$$

or
$$(\Delta + \lambda)\varphi = \sigma |\varphi|^p \varphi + h^{\alpha(p)} Q,$$

where $Q = h^{-1} e^{ix_0/h} T_{h,0} \tilde{Q}_2$ satisfies $\|Q\|_{\dot{H}^s} \leq C h^{-s}$ and where

$$\alpha(p) := -1 + 2q = 1 - p\left(\frac{d-1}{2}\right). \tag{7.6}$$

We now sum up what we have proven in a proposition. Consider the objects we have just defined : $v = v_0 + h^q v_1$, $\varphi_h = e^{ix_0/h} T_{h,0} v$ and $\lambda(h) = h^{-2} - E_0 h^{-1} - E_1 h^{-1+q}$. Then we can state

Proposition 7.1. *Let $2/(d-1) \leq p < 4/(d-1)$ and $\alpha(p)$ be given by (7.6). Then the function φ_h satisfies the equation*

$$\big(\Delta + \lambda(h)\big)\varphi_h = \sigma|\varphi_h|^p \varphi_h + h^{\alpha(p)} Q(h),$$

where $Q(h)$ is an error term which satisfies $\|Q(h)\|_{\dot{H}^s} \lesssim h^{-s}$, for all $s \geq 0$.

7.2.2. *The case $1/2 < q \leq 1$, i.e. $0 \leq p < \frac{2}{d-1}$.* We again construct v_0 as a Gaussian beam using the quantum monodromy operator. We then set

$$G_1 = E_1 v_0 + L v_0,$$

which is smooth with compact wavefront set contained in the wavefront set of v_0, so no phase space cutoff is necessary to apply the inhomogeneous argument to get v_1 with wavefront set contained in the wavefront set of v_0.

Now let

$$G_2 = E_2 v_0 + h^{1-q} E_1 v_1 + h^{1/2} E_2 v_1 + \sigma |v_0|^p v_0 + h^{1-q} L v_1,$$

and solve for v_2 as in the previous subsection. This is possible since v_1 is orthogonal to v_0 by construction. We then have

$$(2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0 - h^q E_1) v_2 = E_2 v_0 + h^{1-q} E_1 v_1 + h^{1/2} E_2 v_1 + \sigma |v_0|^p v_0 + h^{1-q} L v_1 + \tilde{Q}_1,$$

where

$$\|\tilde{Q}_1\|_{\dot{H}^r} \leq C h^{(1/2)(3/2+p-r)}.$$

Letting $v = v_0 + h^{1/2} v_1 + h^q v_2$ and $E = E_0 + h^{1/2} E_1 + h^q E_2$, we have solved

$$\begin{aligned}(2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E) v &= h^q \sigma |v_0|^p v_0 + h^{1/2} L v_0 + h L v_1 + \tilde{Q}_1 \\ &= h^q \sigma |v|^p v + h^{1/2} L v + \tilde{Q}_2,\end{aligned}$$

where

$$\tilde{Q}_2 = h^q \tilde{Q}_1 - h^{1/2+q} L v_2 + \mathcal{O}(h^{1/2+q} |v|^{p+1}).$$

We now have the remainder estimate

$$\|\tilde{Q}_2\|_{\dot{H}^s} \leq C h^{-s/2 + \min\{1/2+q, q+3/4+p/2\}} = C h^{-s/2+1/2+q}.$$

Recalling the definitions, $\varphi := e^{ix_0/h} T_{h,0} v$ satisfies

(i)

$$\mathrm{WF}_{h,1/2-\delta,0}\, \varphi \subset \{|x| \leq \epsilon h^{1/2-\delta}, |\xi| \leq \epsilon\};$$

(ii)

$$\|\varphi\|_{L^2} \sim 1, \qquad \|D_{x_0}^\ell \varphi\|_{L^2} \sim h^{-\ell},$$

and

$$\|D_{x'}^\ell \varphi\|_{L^2} \leq C h^{-\ell/2};$$

(iii)

$$\begin{aligned}(\Delta+\lambda)\varphi &= h^{-1}e^{ix_0/h}T_{h_0}he^{-ix_0/h}T_{h_0}^{-1}\Delta T_{h_0}T_{h_0}^{-1}\varphi \\ &= h^{-1}e^{ix_0/h}T_{h_0}(2iX_0+\Delta_{\Gamma^\perp}-R_2-E-h^{1/2}L)v \\ &= h^{-1}e^{ix_0/h}T_{h_0}(\sigma h^q|v|^p v+\tilde{Q}_2),\end{aligned}$$

or

$$(\Delta+\lambda)\varphi = \sigma|\varphi|^p\varphi + h^{\alpha(p)}Q,$$

where $Q = h^{-1}e^{ix_0/h}T_{h_0}\tilde{Q}_2$ satisfies $\|Q\|_{\dot{H}^s} \le Ch^{-s}$ and where

$$\alpha(p) := -1/2+q = \frac{1}{2} - p\left(\frac{d-1}{4}\right). \tag{7.7}$$

Once again, by construction we have

$$\|X_0^j v_j\|_{L^2} \lesssim \|v_j\|_{L^2}$$

for $j = 0, 1, 2$.

Consider $v = v_0 + h^{\frac{1}{2}}v_1 + h^q v_2$, $\varphi_h = e^{ix_0/h}T_{h,0}v$ and $\lambda(h) = h^{-2} - E_0h^{-1} - E_1h^{-\frac{1}{2}} - E_2h^{-1+q}$ defined previously, then we have proven

Proposition 7.2. *Let $0 < p \le 2/(d-1)$ and $\alpha(p)$ be given by (7.7). Then the function φ_h satisfies the equation*

$$\big(\Delta+\lambda(h)\big)\varphi_h = \sigma|\varphi_h|^p\varphi_h + h^{\alpha(p)}Q(h),$$

where $Q(h)$ is an error term which satisfies $\|Q(h)\|_{\dot{H}^s} \lesssim h^{-s}$, for all $s \ge 0$.

7.3. **Higher order expansion and proof of Thereorem 1.1.**

7.3.1. *The two dimensional cubic equation.* We first deal with the simpler case $d = 2$, $p = 2$ and $s = 0$. As in the previous section, we define $q = 1 - p(d-1)/4$, thus for this choice we have $q = \frac{1}{2}$ allowing us to match powers of the asymptotic parameters in a canonical way. The general algorithm for any smooth nonlinearity arising when rescaling in the appropriate H^s space will follow similarly.

Using (6.4) and the Taylor expansions of the geometric components g^{ij} and Γ^i_{jk} for $i, j, k = 0, \dots, d$ in (6.3), we look for an asymptotic series solution of the form

$$v = v_0 + h^{\frac{1}{2}}v_1 + h^1 v_2 + \cdots + h^{m\frac{1}{2}}v_m + \cdots + h^{N\frac{1}{2}}v_N + \tilde{v},$$

for N sufficiently large.

Then, we have the following equations:

$$
\begin{aligned}
h^0 : \quad & (2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0)v_0 = 0, \\
h^{\frac{1}{2}} : \quad & (2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0)v_1 = E_1 v_0 + (i\delta_{jk}\tilde{\Gamma}^0_{jk1} - 2i\tilde{g}^{k0}\partial_{z_k} + R_3(z))v_0 + \sigma|v_0|^2 v_0, \\
h^1 : \quad & (2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0)v_2 = E_2 v_0 + E_1 v_1 + (i\delta_{jk}\tilde{\Gamma}^0_{jk1} - 2i\tilde{g}^{k0}\partial_{z_k} + R_3(z))v_1 \\
& +(i\delta_{jk}\tilde{\Gamma}^0_{jk2} + R_4)v_0 + \sigma(2|v_0|^2 v_1 + v_0^2 \bar{v}_1), \\
& \vdots \\
h^{\frac{m}{2}} : \quad & (2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0)v_m = \sum_{j=0}^{m-1} E_{m-j} v_j + \sigma\Big(\sum_{j,k,l=0}^{m-1} (c^m_{jkl} v_j v_k \bar{v}_l)\Big) \\
& + \sum_{j=0}^{m-1} (f^{\partial_{z_k}}_{j,m}(z)\partial_{z_k} + f^{X_0}_{j,m} X_0 + f^1_{j,m}(z))v_j, \\
& \vdots \quad , \\
h^{\frac{N}{2}} : \quad & (2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_0 - \sum_{j=1}^{N} h^{\frac{j}{2}} E_j)v_N = \sum_{j=0}^{N} E_{N-j} v_j + \sigma\Big(\sum_{j,k,l=0}^{N-1} (c^N_{jkl} v_j v_k \bar{v}_l)\Big) \\
& + \sum_{j=0}^{N-1} (f^{\partial_{z_k}}_{j,N}(z)\partial_{z_k} + f^{X_0}_{j,N} X_0 + f^1_{j,N}(z))v_j + P_N v,
\end{aligned}
$$

where

$$
\begin{aligned}
f^{\partial_{z_k}}_{j,m} &= \mathcal{O}_N(|z|^{m-j}), \\
f^{X_0}_{j,m} &= \mathcal{O}_N(|z|^{m-j}), \\
f^1_{j,m} &= \mathcal{O}_N(|z|^{m-j})
\end{aligned}
$$

for $j, m = 0, \dots, N$ and

$$
\begin{aligned}
P_N = \; & \mathcal{O}(h^{N/2-2}|z|^N) + X_0 X_0 + \mathcal{O}(h^{N/2}|z|^N)h^{-1/2}\partial_{z_k} X_0 \\
& +\mathcal{O}(h^{N/2-3/2}|z|^N)\partial_{z_k} + \mathcal{O}(h^{N/2-1}|z|^N) + \mathcal{O}(h^{N/2}|z|^N)X_0 + \mathcal{O}(h^{N/2}|z|^N)h^{-1/2}\partial_{z_k}.
\end{aligned}
$$

Note that all constants have implicit dependence upon N relating the number of terms in the expansion at each order. The expansion is valid provided first of all that

$$
\sum_{j=1}^{N} h^{\frac{j}{2}} E_j < E_0
$$

in order to justify the solvability of the $\mathcal{O}(h^{N/2})$ equation.

Remark 7.4. *We note here that in this expansion, the sign of σ can effect the sign and value of E_1, which will impact the remaining asymptotic expansion and in particular the order of quasimode expansion possible. It is possible that the focussing/defocussing problem enters in to the stability analysis of these quasimodes through this point.*

Applying Proposition 7.2 at each asymptotic order and bounds similar to those in Lemma 6.1 at order $h^{N/2}$ as in Section 7.2.1, we have by a simple calculation that v is a quasimode for the nonlinear elliptic equation with remainder Q_N such that

$$
\|Q_N\|_{\dot{H}^s} \le C_N h^{-s-1+N/2}.
$$

As a result, for sufficiently smooth nonlinearities, one is capable of constructing higher order asymptotic expansions and hence a quasimode of higher order accuracy.

7.3.2. *The general case.* Let $d \geq 2$, $p \in 2\mathbb{N}$ and $s \geq 0$. We define here $q_s = 1 + p(s - \frac{d-1}{4})$. Assume that $p(\frac{d-1}{4} - s) < 1$, or equivalently that $q_s > 0$. Firstly, write $w = h^s v$. Then v has to satisfy (6.4) but where the power in front of the nonlinearity is h^{q_s}. Hence, we can look for v and E of the form

$$v = \sum_{j,\ell=0}^{N} h^{j/2+\ell q_s}\, v_{j,\ell} + \tilde{v} \quad \text{and} \quad E = \sum_{j,\ell=0}^{N} h^{j/2+\ell q_s}\, E_{j,\ell} + \widetilde{E}.$$

Since $q_s > 0$, the nonlinearity does not affect the equation giving $v_{0,0}$, and we have

$$(2iX_0 + \Delta_{\Gamma^\perp} - R_2(z) - E_{0,0})v_{0,0} = 0,$$

which is the same equation as before. Then, using Taylor expansions, we write all the equations, similarly to the previous case, in powers of $h^{j/2+\ell q_s}$. Again, we can solve each equation and obtain bounds of the solutions and of the error terms. Moreover, it is clear that we can go as far as we want in the asymptotics, so that we can construct a $\mathcal{O}(h^\infty)$ quasimode, and this proves Theorem 1.1.

8. Error estimates

8.1. **The regular case.** In this section, we assume that p is an even integer.

Fix an integer $k > d/2$ (the fact that k is an integer is not necessary). We then define the semiclassical norm

$$\|f\|_{H^k_h} = \|\big(1 - h^2\Delta\big)^{k/2} f\|_{L^2(M)}.$$

In the previous section we have shown the following : Given $\alpha \in \mathbb{R}$, there exist two functions $\varphi_h \in H^k(M)$ and $Q(h) \in H^k(M)$ and $\lambda(h) \in \mathbb{R}$ so that

$$\big(\Delta + \lambda(h)\big)\varphi_h + \sigma|\varphi_h|^p\varphi_h + h^\alpha Q(h).$$

Moreover, microlocally the function φ_h takes the form

$$\varphi_h(\sigma, x', h) = h^{-\frac{d-1}{4}+s} e^{i\sigma/h} f(\sigma, h^{-1/2}x', h), \tag{8.1}$$

and we have $\|Q(h)\|_{H^k_h} \leq C$.

We set $u_{app}(t,\cdot) = e^{-it\lambda(h)}\varphi_h$. Then if we denote by $\widetilde{Q}(h) := e^{-it\lambda}Q(h)$, the following equation is satisfied

$$i\partial_t u_{\text{app}} - \Delta u_{\text{app}} = \sigma|u_{\text{app}}|^p u_{\text{app}} + h^\alpha \widetilde{Q}(h). \tag{8.2}$$

Proposition 8.1. *Let $s \geq 0$. Consider the function φ_h given by* (8.1)*. Let u be the solution of*

$$\begin{cases} i\partial_t u - \Delta u = \sigma|u|^p u, \\ u(0,\cdot) = \varphi_h. \end{cases} \tag{8.3}$$

Assume that $\alpha > \dfrac{d+1}{4} + s + p(-\dfrac{d-1}{4} + s)$. Then there exists $C > 0$ and $c_0 > 0$ independent of h so that

$$\|u - u_{app}\|_{L^\infty([0,T_h];H^s(M))} \leq C h^{(d+1)/4},$$

for $0 \leq T_h \leq c_0\, h^{p(\frac{d-1}{4}-s)} \ln(\frac{1}{h})$.

This result shows that u_{app} is a good approximation of u, provided that the quasimode φ_h has been computed at a sufficient order α.

Proof. Here we follow the main lines of [Tho08a, Corollary 3.3]. With the Leibniz rule and interpolation we check that for all $f \in H^k(M)$ and $g \in W^{k,\infty}(M)$

$$\|f\, g\|_{H_h^k} \lesssim \|f\|_{H_h^k} \|g\|_{L^\infty(M)} + \|f\|_{L^2(M)} \|\big(1 - h^2\Delta\big)^{k/2} g\|_{L^\infty(M)}. \tag{8.4}$$

Moreover, as $k > d/2$, for all $f_1, f_2 \in H^k(M)$

$$\|f_1\, f_2\|_{H_h^k} \lesssim h^{-d/2} \|f_1\|_{H_h^k} \|f_2\|_{H_h^k}. \tag{8.5}$$

Let u be the solution of (8.3) and define $w = u - u_{\text{app}}$. Then, by (8.2), w satisfies

$$\begin{cases} i\partial_t w - \Delta w = \sigma\big(|w + u_{\text{app}}|^p (w + u_{\text{app}}) - |u_{\text{app}}|^p u_{\text{app}}\big) - h^\alpha \widetilde{Q}(h) \\ w(0, x) = 0. \end{cases} \tag{8.6}$$

We expand the r.h.s. of (8.6), apply the operator $\big(1 - h^2\Delta\big)^{k/2}$ to the equation, and take the L^2- scalar product with $\big(1 - h^2\Delta\big)^{k/2} w$. Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \|w\|_{H_h^k} \lesssim \sum_{j=1}^{p+1} \|w^j\, u_{\text{app}}^{p+1-j}\|_{H_h^k} + h^\alpha. \tag{8.7}$$

We now have to estimate the terms $\|w^j\, u_{\text{app}}^{p+1-j}\|_{H_h^k}$, for $1 \leq j \leq p+1$. From (8.4) we deduce

$$\|w^j\, u_{\text{app}}^{p+1-j}\|_{H_h^k} \lesssim \|w^j\|_{H_h^k} \|u_{\text{app}}^{p+1-j}\|_{L^\infty(M)} + \|w^j\|_{L^2(M)} \|\big(1 - h^2\Delta\big)^{k/2} u_{\text{app}}^{p+1-j}\|_{L^\infty(M)}. \tag{8.8}$$

By (8.5), and as we have

$$\|u_{\text{app}}^{p+1-j}\|_{L^\infty(M)} \lesssim h^{(p+1-j)(-\frac{d-1}{4}+s)}, \quad \|\big(1 - h^2\Delta\big)^{k/2} u_{\text{app}}^{p+1-j}\|_{L^\infty(M)} \lesssim h^{(p+1-j)(-\frac{d-1}{4}+s)}, \tag{8.9}$$

thus inequality (8.8) yields

$$\|w^j\, u_{\text{app}}^{p+1-j}\|_{H_h^k} \lesssim h^{-d(j-1)/2} h^{(p+1-j)(-\frac{d-1}{4}+s)} \|w\|_{H_h^k}^j.$$

Therefore, from (8.7) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|w\|_{H_h^k} \lesssim h^{p(-\frac{d-1}{4}+s)} \|w\|_{H_h^k} + h^{-dp/2} \|w\|_{H_h^k}^{p+1} + h^\alpha.$$

Observe that $\|w(0)\|_{H_h^k} = 0$. Now, for times t so that

$$h^{-dp/2} \|w\|_{H_h^k}^{p+1} \lesssim h^{p(-\frac{d-1}{4}+s)} \|w\|_{H_h^k}, \tag{8.10}$$

i.e. $\|w\|_{H_h^k} \leq C h^{(d+1)/4+s}$, we can remove the nonlinear term in (8.8), and by the Gronwall Lemma,

$$\|w\|_{H_h^k} \leq C h^{\alpha - p(-\frac{d-1}{4}+s)} e^{C h^{p(-\frac{d-1}{4}+s)} t}. \tag{8.11}$$

If $c_0 > 0$ is small enough, and $t \leq c_0 h^{p(\frac{d-1}{4}-s)} \ln \frac{1}{h}$, then

$$C h^{\alpha - p(-\frac{d-1}{4}+s)} e^{C h^{p(-\frac{d-1}{4}+s)} t} \leq C h^{(d+1)/4+s},$$

so that inequality (8.10) is satisfied. By the usual bootstrap argument, we infer that for all

$$t \leq c_0 h^{p(\frac{d-1}{4}-s)} \ln \frac{1}{h}$$

we have

$$\|w\|_{H_h^k} \leq C h^{(d+1)/4+s}.$$

Finally, by interpolation we get $\|w\|_{H^s} \leq h^{-s} \|w\|_{H_h^k}$, hence the result. □

We are now ready to complete the proof of Theorem 1.2. Consider u_h, the exact solution to (1.1) with initial condition φ_h, then by the previous proposition and the description of u_{app}, we can write

$$\begin{aligned}\|u_h\|_{L^\infty([0,T_h];L^2(M\setminus U_{h^{1/2-\delta}}))} &\leq \|u_{app}\|_{L^\infty([0,T];L^2(M\setminus U_{h^{1/2-\delta}}))} + \|u_h - u_{app}\|_{L^\infty([0,T];L^2(M\setminus U_{h^{1/2-\delta}}))} \\ &= \mathcal{O}(h^\infty) + \mathcal{O}(h^{(d+1)/4}) = \mathcal{O}(h^{(d+1)/4}),\end{aligned} \tag{8.12}$$

which was the claim.

8.2. **The non regular case and $d = 2$.**

In this section we compute the error estimate in the case of a non smooth nonlinearity in dimension $d = 2$. Moreover we restrict ourselves to the case $s = 0$ in (8.1) (case of an L^2-normalized initial condition).

Proposition 8.2. *Let φ_h be the function given by* (8.1) *with $s = 0$. Let u be solution of*

$$\begin{cases} i\partial_t u - \Delta u = \sigma |u|^p u, \\ u(0,\cdot) = \varphi. \end{cases}$$

Let $\epsilon > 0$. For $p \in (0,4)\setminus\{1\}$, we set $T_h = h^p$, and in the case $p = 1$, $T_h = h^{1+\epsilon}$. Then there exists $C > 0$ and $\nu > 0$ independent of h so that

$$\|u - u_{app}\|_{L^\infty([0,T_h];L^2(M))} \leq C h^\nu.$$

Remark 8.3. *Note the difference between the results of Propositions* (8.1) *(when $s = 0$ and $d = 2$) and* (8.2)*. In the first case, we have T_h of order $h^{p/4}$, which is better than $T_h \sim h^p$ obtained in the second result. However, in this latter result, there is no restrictive condition on the size of the error term in the equation.*

Proof. First, we follow the strategy of Burq-Gérard-Tzvetkov [BGT04, Section 3.] Let $0 < p < 4$, choose $r > \max(p, 2)$ and take $1 - \dfrac{1}{r} < s < 1$ (there will be an additional constraint on s in the sequel). Then take q so that $\frac{1}{r} + \frac{1}{q} = \frac{1}{2}$ and $s_1 = s - \frac{1}{r}$. For $T > 0$ define the space

$$Y^s = \mathcal{C}\big([0,T]; H^s(M)\big) \cap L^r\big([0,T]; W^{s_1,q}(M)\big),$$

which is endowed with the norm

$$\|u\|_{Y^s} = \max_{0\leq t\leq T} \|u(t)\|_{H^s} + \|(1-\Delta)^{s_1/2} u\|_{L^r([0,T];L^q)}.$$

By the Sobolev embeddings, we have $Y^s \subset L^r\big([0,T], L^\infty\big)$. Now, define $w = u - u_{app}$. Then w satisfies the equation

$$\begin{cases} i\partial_t w - \Delta w = \sigma\big(|w + u_{\text{app}}|^p (w + u_{\text{app}}) - |u_{\text{app}}|^p u_{\text{app}}\big) - h^{\alpha(p)} \widetilde{Q}(h), \\ w(0,x) = 0, \end{cases} \tag{8.13}$$

with $\alpha(p) = 1 - p/2$ when $2 \leq p \leq 4$ and $\alpha(p) = 1/2 - p/4$ when $0 \leq p \leq 2$ (see (7.6) and (7.7)) and $\|\widetilde{Q}(h)\|_{H^s} \leq C h^{-s}$.

• Case $0 < p < 1$. In [CFH, Estimate (2.25)], Cazenave, Fang and Han prove that for all $0 \leq s < 1$

$$\big\||w + u_{app}|^p (w + u_{app}) - |u_{app}|^p u_{app}\big\|_{H^s} \leq C\|u_{app}\|_{H^s} \|w\|^p_{L^\infty} + C\|w\|_{H^s}\big(\|u_{app}\|^p_{L^\infty} + \|w\|^p_{L^\infty}\big). \tag{8.14}$$

Indeed, in [CFH], the estimate is not stated exactly with these indices, but the proof still holds true. Moreover, in [CFH], (8.14) is proved for $x \in \mathbb{R}^d$, but the inequality can be adapted to the case of a compact manifold thanks to a partition of unity argument.

Assume that w satisfies the equation

$$i\partial_t w - \Delta w = F, \quad w(0,x) = 0,$$

then with the Strichartz estimates of [BGT04], the estimate $\|w\|_{Y^s} \leq C\|F\|_{L^1_T H^s}$ holds true. Thus, with the notation $\gamma = 1 - \dfrac{p}{r}$, with (8.13) and (8.14) we have

$$\begin{aligned}
\|w\|_{Y^s} &\leq C\int_0^T \|u_{app}\|_{H^s}\|w\|^p_{L^\infty} + C\int_0^T \|w\|_{H^s}\big(\|u_{app}\|^p_{L^\infty} + \|w\|^p_{L^\infty}\big) + CTh^{\alpha(p)}\|\widetilde{Q}\|_{L^\infty_T H^s} \\
&\leq CT^\gamma\|u_{app}\|_{L^\infty_T H^s}\|w\|^p_{Y^s} + C\|w\|_{Y^s}\big(T\|u_{app}\|^p_{L^\infty_T L^\infty} + T^\gamma\|w\|^p_{Y^s}\big) + CTh^{\alpha(p)}\|\widetilde{Q}\|_{L^\infty_T H^s} \\
&\leq CT^\gamma h^{-s}\|w\|^p_{Y^s} + C\|w\|_{Y^s}\big(Th^{-p/4} + T^\gamma\|w\|^p_{Y^s}\big) + CTh^{-s+1/2-p/4}. \qquad (8.15)
\end{aligned}$$

Similarly, we obtain

$$\|w\|_{L^\infty_T L^2} \leq CT^\gamma\|w\|^p_{Y^s} + C\|w\|_{L^\infty_T L^2}\big(T\|u_{app}\|^p_{L^\infty_T L^\infty} + T^\gamma\|w\|^p_{Y^s}\big) + CTh^{1/2-p/4}. \qquad (8.16)$$

Therefore, if we define the semiclassical norm $\|\ \ \|_{Y^s_h}$ by

$$\|u\|_{Y^s_h} = h^{-s}\|u\|_{L^\infty_T L^2} + \|u\|_{Y^s}, \qquad (8.17)$$

thanks to (8.15) and (8.16) we infer

$$\|w\|_{Y^s_h} \leq CT^\gamma h^{-s}\|w\|^p_{Y^s_h} + C\|w\|_{Y^s_h}\big(Th^{-p/4} + T^\gamma\|w\|^p_{Y^s_h}\big) + CTh^{-s+1/2-p/4}. \qquad (8.18)$$

Next we use the inequality $ab \leq \dfrac{1}{p_1}\epsilon^{p_1}a^{p_1} + \dfrac{1}{p_2}\epsilon^{-p_2}b^{p_2}$ which holds for $a, b, \epsilon > 0$ and $\dfrac{1}{p_1} + \dfrac{1}{p_2} = 1$. With a suitable choice of ϵ and p_1 (here we use that $0 < p \leq 1$) we get

$$CT^\gamma h^{-s}\|w\|^p_{Y^s_h} \leq \frac{1}{2}\|w\|_{Y^s_h} + C(T^\gamma h^{-s})^{1/(1-p)}. \qquad (8.19)$$

Now, re-inject (8.19) into (8.18) and obtain

$$\|w\|_{Y^s_h} \leq C\|w\|_{Y^s_h}\big(Th^{-p/4} + T^\gamma\|w\|^p_{Y^s_h}\big) + C(T^\gamma h^{-s})^{1/(1-p)} + CTh^{-s+1/2-p/4}. \qquad (8.20)$$

We now perform a bootstrap argument : Fix $\epsilon > 0$ and set $T_h = h^p$. Fix $1 - \dfrac{1}{r} < s < 1 - \dfrac{p}{r}$. Then it is possible to pick $\nu > 0$ small enough so that $\gamma = 1 - \dfrac{p}{r} > s + \dfrac{\nu(1-p)}{p}$. Assume that

$$\|w\|_{Y^s_h} \leq h^{-s+\nu}. \qquad (8.21)$$

Then

$$T_h h^{-p/4} + T_h^\gamma\|w\|^p_{Y^s_h} \leq h^{3p/4} + h^{p(\gamma-s+\nu)},$$

which tends to 0 with h, thanks to the assumption made on γ. Hence, for $h > 0$ small enough, with (8.20) we get

$$\|w\|_{Y^s_h} \leq C(T_h^\gamma h^{-s})^{1/(1-p)} + CT_h h^{-s+1/2-p/4} \leq Ch^{(p\gamma-s)/(1-p)} + Ch^{-s+3p/4+1/2}.$$

Finally, observe that $-s + 3p/4 + 1/2 > -s + \nu$, and the assumption $\gamma > s + \dfrac{\nu(1-p)}{p}$ is equivalent to $(p\gamma - s)/(1-p) > -s + \nu$. Hence for $h > 0$ small enough, we recover $\|w\|_{Y^s_h} \leq \dfrac{1}{2}h^{-s+\nu}$, and by the usual bootstrap argument, the condition (8.21) holds for $T_h = h^p$. Now we can deduce the bound $\|u\|_{L^\infty_T L^2} \leq h^s\|w\|_{Y^s_h} \leq h^\nu$, which was the claim.

• Case $1 < p < 4$. Here we have, by [CFH, Estimate (2.25)], for all $0 \leq s < 1$

$$\big\| |w + u_{app}|^p (w + u_{app}) - |u_{app}|^p u_{app} \big\|_{H^s} \leq$$
$$C\|u_{app}\|_{H^s}\big(\|u_{app}\|^{p-1}_{L^\infty} + \|w\|^{p-1}_{L^\infty}\big)\|w\|_{L^\infty} + C\|w\|_{H^s}\big(\|u_{app}\|^p_{L^\infty} + \|w\|^p_{L^\infty}\big).$$

With the same arguments as for (8.15) we get, with $\widetilde{\gamma} = 1 - 1/r$

$$\begin{aligned} \|w\|_{Y^s} &\leq CT^{\widetilde{\gamma}}\|u_{app}\|_{L^\infty_T H^s}\|u_{app}\|^{p-1}_{L^\infty_T L^\infty}\|w\|_{Y^s} + CT^\gamma\|u_{app}\|_{L^\infty_T H^s}\|w\|^p_{Y^s} + \\ &\quad + C\big(T\|u_{app}\|^p_{L^\infty_T L^\infty} + T^\gamma\|w\|^p_{Y^s}\big)\|w\|_{Y^s} + CTh^{\alpha(p)}\|Q\|_{L^\infty_T H^s} \\ &\leq C\big(T^{\widetilde{\gamma}}h^{-s-(p-1)/4} + Th^{-p/4}\big)\|w\|_{Y^s} + CT^\gamma h^{-s}\|w\|^p_{Y^s} + CT^\gamma\|w\|^{p+1}_{Y^s} + CTh^{-s+\alpha(p)}, \end{aligned}$$

with $\alpha(p) = 1 - p/2$ when $2 \leq p \leq 4$ and $\alpha(p) = 1/2 - p/4$ when $1 \leq p \leq 2$ (see (7.6) and (7.7)). Then, by the same manner, we get the following a priori estimate with the semiclassical norm $\| \ \|_{Y^s_h}$ (recall definition (8.17))

$$\|w\|_{Y^s_h} \leq C\big(T^{\widetilde{\gamma}}h^{-s-(p-1)/4} + Th^{-p/4}\big)\|w\|_{Y^s_h} + CT^\gamma h^{-s}\|w\|^p_{Y^s_h} + CT^\gamma\|w\|^{p+1}_{Y^s_h} + CTh^{-s+\alpha(p)}. \tag{8.22}$$

We now perform the bootstrap : Let $r > \max(2, p)$ (there will be an additional constraint on r). Fix $1 - \dfrac{1}{r} < s < 1$ and set $T_h = h^p$. Then if r is large enough (recall that $\widetilde{\gamma} = 1 - 1/r$), the term $T_h^{\widetilde{\gamma}}h^{-s-(p-1)/4} + T_h h^{-p/4}$ tends to 0 with h, therefore if $h > 0$ is small enough, from (8.22) we deduce that

$$\|w\|_{Y^s_h} \leq CT_h^\gamma h^{-s}\|w\|^p_{Y^s_h} + CT_h^\gamma\|w\|^{p+1}_{Y^s_h} + CT_h h^{-s+\alpha(p)}. \tag{8.23}$$

Choose $0 < \nu < p + \alpha(p)$. As previously we assume that

$$\|w\|_{Y^s_h} \leq h^{-s+\nu}. \tag{8.24}$$

Then with (8.23) we get

$$\|w\|_{Y^s_h} \leq Ch^{\gamma p - s + p(-s+\nu)} + Ch^{\gamma p + (p+1)(-s+\nu)} + Ch^{p-s+\alpha(p)}.$$

Next when $r > 0$ is large enough (and under the assumption $0 < \nu < p + \alpha(p)$), we have $\gamma p - s + p(-s+\nu) > -s+\nu$, $\gamma p + (p+1)(-s+\nu) > -s+\nu$ and $p - s + \alpha(p) > -s+\nu$. To see this, observe that $\gamma \longrightarrow 1$ and $s \longrightarrow 1$ when $r \longrightarrow +\infty$. Therefore for $h > 0$ small enough, we recover $\|w\|_{Y^s_h} \leq \dfrac{1}{2}h^{-s+\nu}$, hence the condition (8.24) holds for $T_h = h^p$, and similarly to the previous part, we deduce that $\|u\|_{L^\infty_T L^2} \leq h^s\|w\|_{Y^s_h} \leq h^\nu$.

• Case $p = 1$. By (8.22) we have

$$\|w\|_{Y^s_h} \leq C\big(T^{1-\frac{1}{r}}h^{-s} + Th^{-1/4}\big)\|w\|_{Y^s_h} + CT^{1-\frac{1}{r}}\|w\|^2_{Y^s_h} + CTh^{-s+\frac{1}{4}}.$$

here we set $T_h = h^{1+\epsilon}$ with $\epsilon > 0$, and we perform the same argument as in the previous case. □

Thanks to this proposition and the same argument as (8.12), we can conclude the proof of Theorem 1.3.

Appendix A. Quasimodes for linear equations near Elliptic Orbits

In this section, we state, without proof, a theorem on existence of quasimodes near elliptic periodic orbits of the Hamiltonian flow. A proof can be found in [Chr08].

Let X be a smooth, compact manifold, $\dim X = n$, and suppose $P \in \Psi^{k,0}(X)$, $k \geq 1$, be a semiclassical pseudodifferential operator of real principal type which is semiclassically elliptic outside a compact subset of T^*X. Let $\Phi_t = \exp tH_p$ be the classical flow of p and assume there is a closed *elliptic* orbit $\gamma \subset \{p = 0\}$.

That γ is elliptic means if $N \subset \{p = 0\}$ is a Poincaré section for γ and $S : N \to S(N)$ is the Poincaré map, then $dS(0,0)$ has eigenvalues all of modulus 1. We will also need the following non-resonance assumption:

$$\left\{ \begin{array}{l} \text{if } e^{\pm i\alpha_1}, e^{\pm i\alpha_2}, \ldots, e^{\pm i\alpha_k} \text{ are eigenvalues of } dS(0,0), \text{ then} \\ \alpha_1, \alpha_2, \ldots, \alpha_k \text{ are independent over } \pi\mathbb{Z}. \end{array} \right. \tag{A.1}$$

Under these assumptions, it is well known that there is a family of elliptic closed orbits $\gamma_z \subset \{p = z\}$ for z near 0, with $\gamma_0 = \gamma$. In this work we consider the following eigenvalue problem for z in a neighbourhood of $z = 0$:

$$\left\{ \begin{array}{l} (P - z)u = 0; \\ \|u\|_{L^2(X)} = 1. \end{array} \right. \tag{A.2}$$

We prove the following Theorem.

Theorem A.1. *For each $m \in \mathbb{Z}$, $m > 1$, and each $c_0 > 0$ sufficiently small, there is a finite, distinct family of values*

$$\{z_j\}_{j=1}^{N(h)} \subset [-c_0 h^{1/m}, c_0 h^{1/m}]$$

and a family of quasimodes $\{u_j\} = \{u_j(h)\}$ with

$$\mathrm{WF}_h u_j = \gamma_{z_j},$$

satisfying

$$\left\{ \begin{array}{l} (P - z_j)u_j = \mathcal{O}(h^\infty)\|u_j\|_{L^2(X)}; \\ \|u_j\|_{L^2(X)} = 1. \end{array} \right. \tag{A.3}$$

Further, for each $m \in \mathbb{Z}$, $m > 1$, there is a constant $C = C(c_0, 1/m)$ such that

$$C^{-1} h^{-n(1-1/m)} \leq N(h) \leq C h^{-n}. \tag{A.4}$$

Remark A.1. *The proof is essentially to construct quasimodes on the Poincaré section as eigenfunctions for the semiclassical harmonic oscillator, and then to propagate them around the orbit Γ. This shows that the quasimodes have the localization property as in Lemmas B.1 and B.2.*

Appendix B. Some commutator estimates

In this section, we prove two results which we have used in the above computations. Specifically, we have constructed approximate solutions to the homogeneous and inhomogeneous equations associated to a semiclassical operator of the form

$$Q = hD_t - P(t, x, hD_x),$$

where P is a "time-dependent" harmonic oscillator,

$$P(t, x, hD_x) = -h^2 \Delta(t) + V,$$

where

$$V = b^{ij}(t) x_i x_j$$

is a positive definite quadratic form. Both $\Delta(t)$ and V have t-periodic coefficients of period T (the same as Γ), and we seek periodic solutions to equations

$$Qv = Ev, \ \ (Q - E)v = f,$$

where E is an eigenvalue to be determined and f is periodic in t. The constant-coefficient semiclassical harmonic oscillator is well-known to have eigenfunctions of a semiclassically scaled Hermite polynomial times a semiclassical Gaussian. These eigenfunctions, for small eigenvalues, have semiclassical wavefront set at $(0,0)$. Moreover, for any $\epsilon, \delta > 0$, if $|(x, \xi)| \geq \epsilon h^{1/2-\delta}$, these eigenfunctions are $\mathcal{O}(h^\infty)$ in the

Schwartz space. The purpose of this section is to prove that for the periodic orbit case, similar localization occurs.

Lemma B.1. *Suppose v solves*

$$\begin{cases} Qv = Ev + \mathcal{O}(h^\infty)\|v\|, \ E = \mathcal{O}(h), \\ \|v\| = 1, \\ v(0) = v(T) = v_0, \end{cases}$$

where v_0 satisfies the localization property:

$$\begin{cases} \forall \delta, \epsilon > 0, \exists \psi \in \mathcal{C}_c^\infty(\mathbb{R}), \psi \equiv 1 \ \text{on } \{|(x,\xi)| \leq \epsilon h^{1/2-\delta}\}, \\ \ \text{with} \operatorname{supp} \psi \subset \{|(x,\xi)| \leq 2\epsilon h^{1/2-\delta}\}, \ \text{we have} \\ \operatorname{Op}_h(\psi) v_0 = v_0 + \mathcal{O}(h^\infty)\|v_0\|. \end{cases}$$

Then there exists $\epsilon_1 > 0$, $\epsilon_1 \to 0$ as $\epsilon \to 0$, such that for all $0 \leq t \leq T$, if $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\psi(r) \equiv 1$ for $\{|r| \leq 1\}$ with $\operatorname{supp} \psi(r) \subset \{|r| \leq 2\}$, then

$$\operatorname{Op}_h(\psi(p(t,x,\xi)/(\epsilon_1^2 h^{1-2\delta})))v(t) = v(t) + \mathcal{O}(h^\infty)\|v(t)\|.$$

Proof. Let $\psi(r) \equiv 1$ for $|r| \leq 1$. Then for any $\epsilon, \delta > 0$,

$$\operatorname{Op}_h(\psi(p(0,x,\xi)/(\epsilon^2 h^{1-2\delta})))v_0 = v_0 + \mathcal{O}(h^\infty),$$

since $p(0,x,\xi)$ is comparable to $|(x,\xi)|^2$. Let $I(t)$ be the forward propagator for P:

$$\begin{cases} (hD_t + P(t,x,hD))I(t) = 0, \\ I(0) = \operatorname{id}_{L^2 \to L^2}. \end{cases}$$

Let $\Psi = \operatorname{Op}_h(\psi(p(0,x,\xi)/(\epsilon^2 h^{1-2\delta})))$, and set

$$\Gamma(t) = I(t)\Psi I(t)^{-1}.$$

We have $hD_t\Gamma = [P(t,x,hD), \Gamma(t,x,hD)]$, and by Egorov's theorem, $\mathrm{WF}_h\Gamma$ is contained in the flowout by $\exp(tH_p)$ of $\{p(0,x,\xi) \leq 2\epsilon^2 h^{1-2\delta}\}$. Now the flowout of $\exp(tH_p)$ no longer preserves the level set of p because p depends on t, however, if $(x(t),\xi(t))$ is an integral curve, then

$$\frac{d}{dt}p(t,x(t),\xi(t)) = p_t(t,x(t),\xi(t)),$$

so that there is a constant $C > 0$, independent of h so that

$$-Cp \leq \frac{d}{dt}p(t,x(t),\xi(t)) \leq Cp,$$

by the homogeneity of p for (x,ξ) in a neighbourhood of $(0,0)$. Hence there is a constant c_0 so that

$$c_0^{-1}p(0,x(0),\xi(0)) \leq p(t,x(t),\xi(t)) \leq c_0 p(0,x(0),\xi(0))$$

on the flowout for $0 \leq t \leq T$. Hence a neighbourhood of $(0,0)$ of order $h^{1/2-\delta}$ stays of the same order, although the size of $\epsilon > 0$ may increase.

We have yet to show the asserted identity property acting on v. But for this, we simply note that $v(t) = I(t)v_0$, and $I(t)^* = I(t)^{-1}$ to write

$$\Gamma v(t) = I(t)\Psi v_0 = I(t)(v_0 + \mathcal{O}(h^\infty)_{L^2}) = v(t) + I(t)\mathcal{O}(h^\infty)_{L^2},$$

and since $I(t)$ is unitary, we have proved the Lemma.

□

We now know that modes and quasimodes are concentrated on a scale of $h^{1/2-\delta}$ for any $\delta > 0$. We can measure the distance to Γ in the transversal direction at the point t on Γ using $p(t,x,\xi)$, so it is convenient to have cutoffs which in a sense depend only on p, and more specifically nearly commute with $P(t,x,hD)$.

Lemma B.2. *Let $P(t,x,hD)$ be as above and fix $N > 0$. Fix $\epsilon > 0$ sufficiently small and fix $0 \leq a < b < \epsilon$, and suppose $\varphi^0 \in \mathcal{C}_c^\infty(\mathbb{R})$ has support $\operatorname{supp}\varphi^0 \subset [a,b]$. Then for each $\delta > 0$ and for each $1 \leq j \leq N$, there exist symbols $\varphi^j \in \mathcal{S}_{1/2-\delta}$ such that*

i *If $\tilde{\varphi}^0(t,x,\xi,h) = \varphi^0(p(t,x,\xi)/h^{1-2\delta})$, then $\tilde{\varphi}^0 \in \mathcal{S}_{1/2-\delta}$ and $\psi^0 = \mathrm{Op}_h(\tilde{\varphi}^0)$ satisfies*
$$[P, \psi^0] = \mathcal{O}(h^{1/2+\delta})_{L^2 \to L^2},$$
ii $\operatorname{supp}\varphi^j \subset \{ah^{1/2-\delta} \leq p \leq bh^{1/2-\delta}\} \cap \operatorname{supp}\nabla\tilde{\psi}^0$, *for each $1 \leq j \leq N$, and*
iii *if $\psi^j = \mathrm{Op}_h(\varphi^j)$, then*
$$\psi = \psi^0 + \sum_{j=1}^N h^{j(1/2+\delta)}\psi^j$$
satisfies
$$[P, \psi] = h^{N(1/2+\delta)}R,$$
with $R : L^2 \to L^2$ bounded with compact h-wavefront set.

Proof. The proof is relatively standard, however the addition of the periodic "boundary conditions" in t adds a small difficulty, so we reproduce the basic argument here.

The principal symbol of P is
$$p(t,x,\xi) = a^{ij}(t,x)\xi_i\xi_j + b^{ij}(t)x_ix_j,$$
with both a^{ij}, b^{ij} positive definite matrices. By the homogeneity of the quadratic forms, we clearly have $\tilde{\varphi}^0 \in \mathcal{S}_{1/2-\delta}$. The symbol calculus then implies the assertion (i), once we observe that $\tilde{\psi}^0$ cuts off to a compact region, and hence p is bounded there.

We now proceed to construct the φ^j satisfying (ii)-(iii). For notation simplicity, denote
$$A = \{ah^{1/2-\delta} \leq p \leq bh^{1/2-\delta}\} \cap \operatorname{supp}\nabla\tilde{\psi}^0 \subset T^*M.$$
From (i) and the symbol calculus, we have
$$[P, \psi^0] = \frac{h}{i}\mathrm{Op}_h(\{p, \tilde{\varphi}^0\}) + R_0,$$
where $R_0 = h^{1+2\delta}\mathrm{Op}_h(r_0) + R_{0,N}$, where $r_0 \in \mathcal{S}_{1/2-\delta}$ with $\operatorname{supp} r_0 \subset A$, and $R_{0,N} = \mathcal{O}(h^{N(1/2+\delta)})_{L^2\to L^2}$ has compact h-wavefront set. We now compute for an arbitrary choice of $\psi^j \in \mathcal{S}_{1/2-\delta}$ with compact support:
$$[P, h^{j(1/2+\delta)}\psi^j] = h^{j(1/2+\delta)}\frac{h}{i}\mathrm{Op}_h(\{p, \varphi^j\}) + h^{j(1/2+\delta)}R_j,$$
where $R_j = h^{1+2\delta}\mathrm{Op}_h(r_j) + R_{j,N}$. As in the case of φ^0, here $r_j \in \mathcal{S}_{1/2-\delta}$ has support contained in $\operatorname{supp}\nabla\varphi^j$ and $R_{j,N} = \mathcal{O}_{L^2\to L^2}(h^{N(1/2+\delta)})$ has compact h-wavefront set.

We observe that $\{p, \varphi^j\}$ has a prefactor of $h^{-1/2+\delta}$ since $\varphi^j \in \mathcal{S}_{1/2-\delta}$ but $p \in \mathcal{S}_0$, so we want to construct φ^j so that $h^{j(1/2+\delta)}h\{p, \varphi^j\}/i$ cancels the term of order $h^{(j+1)(1/2+\delta)}$. Assume for $1 \leq k \leq j-1$ we have found φ^k satisfying (ii) such that
$$ih^{1/2-\delta}\{p, \varphi^k\} - r_{k-1} = \mathcal{O}(h^{1/2-\delta})$$
with the r_{k-1} and the error in $\mathcal{S}_{1/2-\delta}$ with support in A. Then if $\Gamma^{j-1} = \sum_{k=0}^{j-1} h^{k(1/2+\delta)}\psi^k$ satisfies
$$[P, \Gamma^{j-1}] = h^{(j-1)(1/2+\delta)}\tilde{R}_{j-1},$$

where $\tilde{R}_{j-1} = h^{1+2\delta}\mathrm{Op}_h(\tilde{r}_{j-1}) + R_{j-1,N}$ with $r_{j-1} \in \mathcal{S}_{1/2-\delta}$, $\mathrm{supp}\, r_{j-1} \subset A$, and $R_{j-1,N} = \mathcal{O}(h^{N(1/2+\delta)})$ with compact h-wavefront set. Then we want to solve

$$ih^{1/2-\delta}\{p, \varphi^j\} - r_{j-1} = \mathcal{O}(h^{1/2-\delta}).$$

If $r_{j-1,0}$ is the principal symbol of r_{j-1}, we apply the Frobenius theorem to find such a φ^j. The support properties follow from the assumed support properties on r_{j-1} and the observation that we can always multiply φ^j by a function of p to ensure it is supported in A, at the expense of another compactly supported error of order $h^{1/2+\delta}$. The symbolic properties follow from the assumed symbolic properties of r_{j-1}.

□

Appendix C. Real principal type

Let us quickly show that our local principal symbol near Γ can be glued into a symbol of real principal type. Let us recall that a symbol p is of real principal type if the principal symbol is real valued, smooth, has compact level sets, is elliptic outside a compact set, and $dp \neq 0$ on $\{p = 0\}$. Our local model near Γ is of the form

$$p_1(t, x, \xi) = -\tau + a^{ij}(t, x)\xi_i\xi_j + b^{ij}(t)x_i x_j,$$

a^{ij} and b^{ij} are positive definite symmetric matrices, $t \in \mathbb{S}^1$ and τ is the dual variable to t. Let $p_0 = a^{ij}(t, x)\xi_i\xi_j + b^{ij}(t)x_i x_j$, which is a suitable measure of the distance squared to Γ. Fix $\delta > 0$, and let $\chi(r) \in \mathcal{C}_c^\infty(\mathbb{R})$ be equal to 1 for $|r| \leq \delta$, and $\chi(r) \equiv 0$ for $|r| \geq 2\delta$. Let $q = \tau^2 + p_0$, which is elliptic outside a compact set, and set

$$p_2 = \chi(p_0)p_1 + (1 - \chi(p_0))q.$$

The function p_2 satisfies all the requirements of real principal type, once we show that $dp_2 \neq 0$ on $\{p_2 = 0\}$. For this, first note that

$$\{p_2 = 0\} \subset (\{\tau = p_0\} \cap \{p_0 \leq 2\delta\}) \cup (\{q = 0\} \cap \{p_0 \geq \delta\}).$$

The latter set is empty, but the first is not. We observe that

$$dp_2 = (1 + \chi'(p_0)(-\tau - \tau^2))dp_0 + (2(1 - \chi)\tau - \chi)d\tau,$$

and that $dp_0 = 0$ only if $x = \xi = 0$. Now χ is a non-increasing function of p_0, so on $\{\tau = p_0\} \cap \{p_0 \leq 2\delta\}$, $\chi'(p_0)(-\tau - \tau^2) \geq 0$. Hence if $(x, \xi) \neq (0, 0)$, $dp_2 \neq 0$. If $x = \xi = 0$, we have $p_0 = 0$, so on $\{\tau = p_0\} \cap \{p_0 \leq 2\delta\}$, $\tau = 0$ also. But

$$(2(1 - \chi)\tau - \chi)d\tau \neq 0$$

for τ in a neighbourhood of 0, which shows $dp_2 \neq 0$ on $\{p_2 = 0\}$.

Appendix D. Rescaled wavefront sets: an example

In this section, we provide for the reader's convenience an example where one encounters the rescaled wavefront sets. Let us consider the quantum harmonic oscillator

$$P = (hD_x)^2 + x^2.$$

The eigenfunctions $Pu_j = E_j u_j$, are semiclassical Hermite polynomials times semiclassical Gaussians. If $u_0(x) = c_0 h^{-1/4} e^{-x^2/2h}$ is the (L^2-normalized) semiclassical Gaussian, and if $\chi(x) \in \mathcal{C}_c^\infty(\mathbb{R})$ is equal to 1 near 0, then clearly

$$\|(1 - \chi(h^{-\delta}x))u_0\| = \mathcal{O}(h^\infty)$$

in any seminorm, provided $\delta < 1/2$. On the other hand, a simple computation shows that the semiclassical Fourier transform of u_0

$$\mathcal{F}_h(u_0)(\xi) = c_0' h^{-1/4} e^{-\xi^2/2h},$$

so that $\mathcal{F}_h(u_0)$ has the same localization in ξ as u_0 did in x. In other words, for any $0 \leq \delta < 1/2$,

$$\mathrm{WF}_{h,\delta,\gamma}(u_0) = \{(0,0)\}.$$

A similar argument implies the same is true for any u_j provided $E_j = \mathcal{O}(h)$.

On the other hand, let us see how to use Corollary 4.4 to prove the same localization. If $E_j = \mathcal{O}(h)$, then the principal of $P - E_j$ is $p = \xi^2 + x^2$. By homogeneity, we can rescale

$$p = h^{2\delta}((h^{-\delta}\xi)^2 + (h^{-\delta}x)^2),$$

which is a symbol in $\mathcal{S}_\delta^{2,-2\delta}$. As this symbol is $\mathcal{O}(h^\infty)$ only at $(x,\xi) = (0,0)$, we get the same result as above.

Appendix E. Harmonic oscillator eigenfunctions

Let $P_0 = -d^2/dx^2 + x^2$ be the one-dimensional quantum harmonic oscillator operator. The theory of the eigenfunctions of P_0 is well established, however we need several important facts recalled here.

Proposition E.1. *Each eigenfunction u_n, $n = 0, 1, 2, \ldots$, of P_0 is a polynomial times a Gaussian:*

$$u_n(x) = H_n(x)e^{-x^2/2},$$

with eigenvalue $\lambda_n = 2n + 1$.

The u_n can be taken to be real-valued, they form a complete orthonormal basis of $L^2(\mathbb{R})$, and the zeros of u_n are simple.

Proof. The proof as usual is by creation and annihiliation operators. Let

$$A_\pm = D_x \pm ix,$$

so that $A_\pm = A_\mp^*$ and

$$A_+A_- = P_0 - 1 = A_-A_+ - 2.$$

A computation shows

$$A_-e^{-x^2/2} = 0,$$

so that

$$A_+A_-e^{-x^2/2} = 0 = (P_0 - 1)e^{-x^2/2}.$$

The other eigenfunctions are constructed with the creation operator:

$$u_n = A_+^n e^{-x^2/2},$$

and a simple computation shows the u_n is a polynomial times the Gaussian $e^{-x^2/2}$. The creation operator $A_+ = -i(\partial_x - x)$ is $-i$ times a real-valued operator, so the polynomials can be taken to be real-valued. By induction, we have

$$\begin{aligned} P_0 u_n &= (A_+A_- + 1)u_n \\ &= A_+(A_-A_+^n)u_0 + u_n \\ &= A_+(P_0 + 1)A_+^{n-1}u_0 + u_n \\ &= A_+(2(n-1) + 2)u_{n-1} + u_n \\ &= (2n + 1)u_n. \end{aligned}$$

A simple computation shows the eigenfunctions $\{u_n\}$ form a complete orthnormal set.

To show the zeros are simple, we again assume for induction that u_n has only simple zeros, and suppose $u_{n+1}(x_0) = 0$. Then

$$\begin{aligned} A_- u_{n+1} &= A_- A_+ u_n \\ &= (P_0 + 1) u_n \\ &= (2n+2) u_n, \end{aligned}$$

so

$$u'_{n+1}(x) + x u_{n+1}(x) = i(2n+2) u_n(x),$$

and if $u_{n+1}(x_0) = u'_{n+1}(x_0) = 0$, then x_0 is a zero of u_n as well. Differentiating again, and using the fact that u_{n+1} is an eigenfunction, we get

$$\begin{aligned} i(2n+2) u'_n(x) &= u''_{n+1}(x) + u_{n+1}(x) + x u'_{n+1}(x) \\ &= -(2n+2) u_{n+1}(x) + x^2 u_{n+1}(x) + x u'_{n+1}(x), \end{aligned}$$

so if $u_{n+1}(x_0) = u'_{n+1}(x_0) = 0$, then $u'_n(x_0) = 0$ as well, which contradicts the induction hypothesis. Hence $u_{n+1}(x)$ has only simple zeros.

□

We are interested in these properties of the quantum harmonic oscillator eigenfunctions because, for the nonlinear problem studied in this note, we will take a non-smooth function of these eigenfunctions, and we want to understand the singularities. Let $p > 0$, let $u_n(x)$ be an eigenfunction of the quantum harmonic oscillator, and set

$$v(x) = |u_n(x)|^p u_n(x).$$

Proposition E.2. *The function $v(x)$ is rapidly decaying, $v(x) \in \mathcal{C}^1 \cap H^1$, and*

$$\hat{v}(\xi) \in {\mathcal{S}_{cl}}^{-2-p},$$

where ${\mathcal{S}_{cl}}^{-2-p}$ is the space of classical symbols of order $-2-p$.

In particular, we are interested in semiclassical rescaling, and to what extent v is localized in phase space. The function v is not smooth, so it does not have compact semiclassical wavefront set, but because of the symbolic assertion in the previous proposition, there is some decay at infinity, as described in the next corollary.

Corollary E.3. *For any $\delta > 0$, the function $v(x)$ satisfies:*

$$\|v(x)\|_{H^1(B(0,h^{-\delta})^{\complement})} = \mathcal{O}(h^\infty).$$

Moreover, for any $0 \leq \gamma \leq 1$ and $0 \leq s \leq 3/2$, the semiclassical Fourier transform satisfies

$$\| |\xi/h|^s \mathcal{F}_h v \|_{L^2(B(0,h^\gamma)^{\complement})} \leq C h^{(1-\gamma)(3/2+p-s)}.$$

In particular, if $\chi(x,\xi) \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ is 1 in a neighbourhood of $(0,0)$, then

$$\chi(h^\delta x, h^{1-\gamma} D_x) v = v + E,$$

where for any $0 \leq s \leq 3/2$,

$$\|E\|_{\dot{H}^s} \leq C h^{(1-\gamma)(3/2+p-s)}.$$

Proof of Proposition E.2. Each zero of u_n is a simple zero, so we can write

$$v(x) = \sum_{l=0}^{n} v_l(x),$$

with $v_0 \in \mathcal{S}$, and for $1 \leq l \leq n$, v_l has compact support containing a single zero of u_n. If x_l is a zero of u_n contained in the support of v_l, then

$$v_l(x + x_l) \sim |x|^p x$$

near $x = 0$, and $v_l(x + x_l)$ is smooth and compactly supported away from from $x = 0$. Then $v_l(x + x_l)$ is conormal at $x = 0$, which implies $\hat{v}_l$ is a symbol in class ${\mathcal{S}_{\mathrm{cl}}}^{-2-p}$. Summing in l, and using that $\hat{v}_0 \in {\mathcal{S}_{\mathrm{cl}}}^{-\infty}$, we get $\hat{v} \in {\mathcal{S}_{\mathrm{cl}}}^{-2-p}$ as claimed.

□

References

[Bab68] V. M. Babič. Eigenfunctions which are concentrated in the neighborhood of a closed geodesic. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 9:15–63, 1968.

[BGT04] Nicolas Burq, Patrick Gérard, and Nikolay Tzvetkov. Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. *Amer. J. Math.*, 126(3):569–605, 2004.

[BSS08] Matthew D. Blair, Hart F. Smith, and Christopher D. Sogge. On Strichartz estimates for Schrödinger operators in compact manifolds with boundary. *Proc. Amer. Math. Soc.*, 136(1):247–256 (electronic), 2008.

[CFH] Thierry Cazenave, Daoyuan Fang, and Zheng Han. Continuous dependence for nls in fractional order spaces. *arXiv:1006.2745*.

[Chr07] Hans Christianson. Semiclassical non-concentration near hyperbolic orbits. *J. Funct. Anal.*, 246(2):145–195, 2007.

[Chr08] Hans Christianson. Quantum monodromy and non-concentration near a closed semi-hyperbolic orbit. *preprint*, 2008.

[CM10] Hans Christianson and Jeremy L. Marzuola. Existence and stability of solitons for the nonlinear Schrödinger equation on hyperbolic space. *Nonlinearity*, 23(1):89–106, 2010.

[CP02] Fernando Cardoso and Georgi Popov. Quasimodes with exponentially small errors associated with elliptic periodic rays. *Asymptot. Anal.*, 30(3-4):217–247, 2002.

[DS99] Mouez Dimassi and Johannes Sjöstrand. *Spectral asymptotics in the semi-classical limit*, volume 268 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1999.

[EZ07] Lawrence C. Evans and Maciej Zworski. *Lectures on Semiclassical Analysis*. Online Edition, 2007.

[Gra90] Alfred Gray. *Tubes*. Advanced Book Program. Addison-Wesley Publishing Company, Redwood City, CA, 1990.

[GS77] Victor Guillemin and Shlomo Sternberg. *Geometric asymptotics*. American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 14.

[GS10] Victor Guillemin and Shlomo Sternberg. *Semiclassical analysis*. on-line lecture notes, 2010. http://www-math.mit.edu/ vwg/semiclassGuilleminSternberg.pdf.

[GW76] Victor Guillemin and Alan Weinstein. Eigenvalues associated with a closed geodesic. *Bull. Amer. Math. Soc.*, 82(1):92–94, 1976.

[Hör03] Lars Hörmander. *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[Hör05] Lars Hörmander. *The analysis of linear partial differential operators. II.* Classics in Mathematics. Springer-Verlag, Berlin, 2005. Differential operators with constant coefficients, Reprint of the 1983 original.

[MP05] Rafe Mazzeo and Frank Pacard. Foliations by constant mean curvature tubes. *Comm. Anal. Geom.*, 13(4):633–670, 2005.

[Ral82] James Ralston. Gaussian beams and the propagation of singularities. In *Studies in partial differential equations*, volume 23 of *MAA Stud. Math.*, pages 206–248. Math. Assoc. America, Washington, DC, 1982.

[SBS+10] Vincent H. Schultheiss, Sascha Batz, Alex Szameit, Felix Dreisow, Stefan Nolte, Andreas Tünnermann, Stefano Longhi, and Ulf Peschel. Optics in curved space. *Phys. Rev. Lett.*, 105(14):143901, 2010.

[SY94] R. Schoen and S.-T. Yau. *Lectures on differential geometry*. Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994. Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu, Translated from the Chinese by Ding and S. Y. Cheng, Preface translated from the Chinese by Kaising Tso.

[SZ02] Johannes Sjöstrand and Maciej Zworski. Quantum monodromy and semi-classical trace formulae. *J. Math. Pures Appl. (9)*, 81(1):1–33, 2002.

[Tay81] Michael E. Taylor. *Pseudodifferential operators*, volume 34 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1981.

[Tho08a] Laurent Thomann. Instabilities for supercritical Schrödinger equations in analytic manifolds. *J. Differential Equations*, 245(1):249–280, 2008.

[Tho08b] Laurent Thomann. The WKB method and geometric instability for nonlinear Schrödinger equations on surfaces. *Bull. Soc. Math. France*, 136(2):167–193, 2008.

[Zel08] Steve Zelditch. Local and global analysis of eigenfunctions on Riemannian manifolds. In *Handbook of geometric analysis. No. 1*, volume 7 of *Adv. Lect. Math. (ALM)*, pages 545–658. Int. Press, Somerville, MA, 2008.

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