Problèmes d'inclusions couplées : Éclatement, algorithmes et applications

Luis M. Briceno-Arias

THÈSE DE L'UNIVERSITÉ PIERRE ET MARIE CURIE – PARIS VI

Spécialité :

MATHÉMATIQUES APPLIQUÉES

Présentée par :

Luis M. BRICEÑO-ARIAS

pour l'obtention du titre de

DOCTEUR DE L'UNIVERSITÉ PIERRE ET MARIE CURIE – PARIS VI

Sujet :

PROBLÈMES D'INCLUSIONS COUPLÉES : ÉCLATEMENT, ALGORITHMES ET APPLICATIONS

Soutenue le 27 mai 2011 devant le jury composé de :

Felipe ÁLVAREZ	*Rapporteur*
Hédy ATTOUCH	*Examinateur*
Jean-Bernard BAILLON	*Examinateur*
Patrick L. COMBETTES	*Directeur de thèse*
Roberto COMINETTI	*Examinateur*
Alain HARAUX	*Examinateur*
Sylvain SORIN	*Président*
Lionel THIBAULT	*Rapporteur*

Laboratoire Jacques-Louis Lions – UMR 7598

Remerciements

Je veux remercier tout d'abord mon directeur de thèse, Patrick L. Combettes, pour sa patience, dévouement et disponibilité pendant ces années. J'apprécie beaucoup d'avoir appris de lui l'importance d'un travail bien fait et je le remercie pour tous les précieux conseils qu'il m'a donné tout au long de ce travail. Je veux exprimer ma profonde gratitude et ma sincère reconnaissance pour l'effort et l'enthousiasme qu'il a fourni.

Je suis très reconnaissant envers Felipe Álvarez et Lionel Thibault d'avoir accepté d'être rapporteur de ma thèse, et particulièrement honoré que ce dernier ait accepté d'assister à ma soutenance. Je veux aussi remercier Hédy Attouch, Jean-Bernard Baillon, Roberto Cominetti, Alain Haraux et Sylvain Sorin de me faire l'honneur de faire partie du jury. De plus, je veux exprimer ma sincère gratitude à H. Attouch, R. Cominetti et S. Sorin pour tout le soutien qu'ils m'ont apporté avant et pendant mon séjour en France, ce qui a été indispensable aussi bien d'un point de vue académique que humain.

Je veux remercier aussi les membres du Laboratoire Jacques-Louis Lions et de l'Équipe Combinatoire et Optimisation de m'avoir accueilli et de m'avoir donné les outils nécessaires pour le développement de cette thèse. En particulier, je veux remercier mes copains du bureau, qui ont coloré les moments de travail.

Je ne trouverai jamais les mots pour remercier ma famille pour leur constant soutien et affection. *Agradezco a mis padres, Marina y Luis, y hermanas, Nury y Any, por todo el cariño y apoyo durante todos estos años en que, aunque hemos estado lejos espacialmente, nunca lo hemos estado en el ámbito afectivo. Me otorga una alegría desbordante el tener a mis padres en este momento y poder compartir este logro con ellos.*

Quiero agradecer también a mi otra familia, mis amigos. Un gigantezco abrazo de agradecimiento para los que han estado siempre compartiendo conmigo los momentos buenos y malos y que han sido un gran apoyo en el desarrollo de este trabajo : Mario, Carolina, Emilita, Negro, Claudia, Nicolás, Feña, Ratita, Giorgio, André, César, Ale, Salvador, Miguel, Kamila, Miquel, David, Juan y Bárbara. A mes amis, qui m'ont montré la musique, l'amitié et la sympathie dans ses versions françaises : Maël, Fanny, Gabi, Caroline et Sophie.

Pero como un pie en Europa no es suficiente para permitirme estar erguido, quiero agradecer a mi pie en Chile. Agradezco a Marco y Sebastián, con quienes, pese a la distancia, seguimos disfrutando los momentos dulces y amargos de la vida. Junto con ellos, agradezco a Carmen, Javi, Antonio, Pablo, Lautaro, Tucapel, Matías, Andrés, Vero, María Paz, Pancho, Paula, Nico, José y a todos quienes me han dado su cariño y confianza todos estos años.

Finalmente, quiero agradecer a quien me ha aguantado todo el estrés pre y post soutenance y ha sido un pilar, un sustento y un hombro para descansar. Un beso y un abrazo enormes para Blanca, con quien tengo la suerte y el agrado de compartir todos mis momentos.

A todos ellos, mi familia en el sentido amplio y profundo, muchas gracias.

Paris, mai 2011.

A mi familia…

Table des matières

Résumé

Problèmes d'Inclusions Couplées : Éclatement, Algorithmes et Applications

Cette thèse est consacrée à la résolution de problèmes d'analyse non linéaire multivoque dans lesquels plusieurs variables interagissent. Le problème générique est modélisé par une inclusion vis-à-vis d'une somme d'opérateurs monotones sur un espace hilbertien produit. Notre objectif est de concevoir des nouveaux algorithmes pour résoudre ce problème sous divers jeux d'hypothèses sur les opérateurs impliqués et d'étudier le comportement asymptotique des méthodes élaborées. Une propriété commune aux algorithmes est le fait qu'ils procèdent par éclatement en ceci que les opérateurs monotones et, le cas échéant, les opérateurs linéaires constitutifs du modèle agissent indépendamment au sein de chaque itération. Nous abordons en particulier le cas où les opérateurs monotones sont des sous-différentiels de fonctions convexes, ce qui débouche sur de nouveaux algorithmes de minimisation. Les méthodes proposées unifient et dépassent largement l'état de l'art. Elles sont appliquées aux inclusions monotones composites en dualité, aux problèmes d'équilibre, au traitement du signal et de l'image, à la théorie des jeux, à la théorie du trafic, aux équations d'évolution, aux problèmes de meilleure approximation et à la décomposition de domaine dans les équations aux dérivées partielles.

Mots-clés : analyse convexe, décomposition d'images, décomposition de domaine, équilibre de Nash, inclusions d'évolution, inclusions monotones en dualité, opérateur maximalement monotone, problème d'équilibre, problème de point fixe, reconstruction d'images, restauration d'images, théorie du signal, théorie du trafic.

Resumen

Problemas de Inclusiones Acopladas : Separación, Algoritmos y Aplicaciones

Esta tesis está dedicada a la resolución de problemas de análisis no lineal multievaluado en los cuales varias variables interactúan. El problema general es modelado a través de una inclusión que involucra una suma de operadores monótonos en un espacio de Hilbert producto. Nuestro objetivo es concebir nuevos algoritmos para resolver este problema bajo diversos conjuntos de hipótesis sobre los operadores implicados y estudiar el comportamiento asintótico de los métodos elaborados. Una propiedad común a los algoritmos es que actuan por separación en el hecho que los operadores monótonos y, si los hay, los operadores lineales constituyentes del modelo actúan independientemente en cada iteración. En particular abordamos el caso donde los operadores monótonos son subdiferenciales de funciones convexas, lo que da lugar a nuevos algoritmos de minimización. Los métodos propuestos unifican y superan ampliamente el estado del arte. Estos son aplicados a inclusiones monótonas en dualidad, a problemas de equilibrio, al tratamiento de señales e imágenes, a la teoría de juegos, a la teoría de tráfico, a las ecuaciones de evolución, a problemas de mejor aproximación y a la decomposición de dominio en ecuaciones de derivadas parciales.

Palabras clave : análisis convexo, descomposición de imágenes, descomposición de dominio, equilibrio de Nash, inclusiones de evolución, inclusiones monótonas en dualidad, operador maximalmente monótono, problemas de equilibrio, problema de punto fijo, reconstrucción de imágenes, restauración de imágenes, teoría de señales, teoría de tráfico.

Abstract

Coupled Inclusions Problems : Splitting, Algorithms, and Applications

This thesis is devoted to solving problems in set-valued nonlinear analysis in which several variables interact. The generic problem is modeled by an inclusion involving a sum of monotone operators in a product Hilbert space. Our objective is to design new algorithms for solving this problem under various sets of hypotheses on the underlying operators, and to study the asymptotic behavior of the resulting methods. A common property of the algorithms is the fact that they proceed by splitting in that the monotone operators and, if any, the linear operators present in the model act independently at each iteration. In particular, we address the case when the monotone operators are subdifferentials of convex functions, which leads to new minimization algorithms. The proposed methods unify and significantly extend the state-of-the art. They are applied to monotone inclusions in duality, to equilibrium problems, to signal and image processing, to game theory, to traffic theory, to evolution inclusions, to best approximation, and to domain decomposition in partial differential equations.

Key words : composite fixed point problems, convex analysis, domain decomposition, equilibrium problems, evolution inclusions, image decomposition, image reconstruction, image restoration, maximally monotone operators, monotone inclusions in duality, Nash equilibrium, signal theory, traffic theory.

Notations et Glossaire

Les notations suivantes seront utilisées dans toute la thèse. De plus, nous rappelons certaines définitions de base en analyse convexe.

Notations générales

- $\mathcal{H}, \mathcal{H}_1, \ldots, \mathcal{H}_m, \mathcal{G}, \mathcal{G}_1, \ldots, \mathcal{G}_p$: Espaces de Hilbert réels.
- $\langle \cdot \mid \cdot \rangle$: Produit scalaire des espaces $\mathcal{H}$, $\mathcal{H}_1, \ldots \mathcal{H}_m$ et $\mathcal{G}, \mathcal{G}_1, \ldots, \mathcal{G}_p$.
- $2^{\mathcal{H}}$: Ensemble des parties de $\mathcal{H}$.
- $\boldsymbol{\mathcal{H}} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$: Somme hilbertienne directe.
- $\langle\langle \cdot \mid \cdot \rangle\rangle \colon ((x_i)_{1\leq i\leq m}, (y_i)_{1\leq i\leq m}) \mapsto \sum_{i=1}^m \langle x_i \mid y_i \rangle$: Produit scalaire de l'espace $\boldsymbol{\mathcal{H}}$.
- $\Gamma_0(\mathcal{H})$: Famille de fonctions convexes, propres et semi-continues inférieurement de $\mathcal{H}$ dans $]-\infty, +\infty]$.
- $\mathcal{B}\left(\mathcal{H}, \mathcal{G}\right)$: Espace d'opérateurs linéaires et bornés de $\mathcal{H}$ dans $\mathcal{G}$.
- L^* : Adjoint de l'opérateur $L \in \mathcal{B}\left(\mathcal{H}, \mathcal{G}\right)$.
- $\to$: Convergence forte.
- $\rightharpoonup$: Convergence faible.
- $\overline{\lim}\, x_n$: Limite supérieure de la suite $(x_n)_{n\in\mathbb{N}}$ de $\mathcal{H}$.
- $\underline{\lim}\, x_n$: Limite inférieure de la suite $(x_n)_{n\in\mathbb{N}}$ de $\mathcal{H}$.

Soit C un sous-ensemble non vide de $\mathcal{H}$.

- $\iota_C \colon x \mapsto \begin{cases} 0, & \text{si } x \in C\,; \\ +\infty, & \text{si } x \notin C \end{cases}$: Fonction indicatrice de C.
- $d_C \colon x \mapsto \inf_{y\in C} \|x - y\|$: Fonction distance à C associée à la norme $\|\cdot\| = \sqrt{\langle \cdot \mid \cdot \rangle}$.
- $\sigma_C \colon x \mapsto \sup_{y\in C} \langle x \mid y \rangle$: Fonction d'appui de C.
- P_C : Projecteur sur le sous-ensemble convexe fermé non vide C de $\mathcal{H}$.
- $N_C \colon x \mapsto \begin{cases} \big\{u \in \mathcal{H} \mid (\forall y \in C) \;\; \langle u \mid y - x \rangle \leq 0\big\} & \text{si } x \in C \\ \varnothing & \text{sinon} \end{cases}$: Opérateur cône normal à C.
- $C^{\perp}$: Orthogonal d'une partie C de $\mathcal{H}$.

Notations et définitions relatives à un opérateur multivoque $A\colon \mathcal{H} \to 2^{\mathcal{H}}$

- $\operatorname{dom} A = \big\{x \in \mathcal{H} \mid Ax \neq \varnothing\big\}$: Domaine de A.
- $\operatorname{gr} A = \big\{(x,u) \in \mathcal{H}^2 \mid u \in Ax\big\}$: Graphe de A.
- $A^{-1}\colon \mathcal{H} \to 2^{\mathcal{H}}\colon u \mapsto \big\{x \in \mathcal{H} \mid u \in Ax\big\}$: Inverse de A.
- $\operatorname{Fix} A = \big\{x \in \mathcal{H} \mid x \in Ax\big\}$: Points fixes de A.
- $\operatorname{zer} A = \big\{x \in \mathcal{H} \mid 0 \in Ax\big\}$: Zéros de A.
- $\operatorname{ran} A = \big\{u \in \mathcal{H} \mid (\exists x \in \mathcal{H})\ u \in Ax\big\}$: Image de A.
- $J_A = (\operatorname{Id} + A)^{-1}$: Résolvante de A.
- $R_A = 2J_A - \operatorname{Id}$: Opérateur de réflexion de A.
- ${}^{\gamma}A = (\operatorname{Id} - J_{\gamma A})/\gamma$: Approximation de Yosida de A d'indice $\gamma \in \left]0, +\infty\right[$.
- A est monotone :
$$(\forall (x,u) \in \operatorname{gr} A)(\forall (y,v) \in \operatorname{gr} A) \quad \langle x - y \mid u - v\rangle \geq 0.$$
- A est maximalement monotone :
$$(\forall (x,u) \in \mathcal{H} \oplus \mathcal{H}) \quad \Big((x,u) \in \operatorname{gr} A \Leftrightarrow (\forall (y,v) \in \operatorname{gr} A) \quad \langle x - y \mid u - v\rangle \geq 0\Big).$$
- A est demirégulier en $x \in \operatorname{dom} A$:
$$(\forall ((x_n,u_n))_{n\in\mathbb{N}} \in (\operatorname{gr} A)^{\mathbb{N}})(\forall u \in Ax) \quad \begin{cases} x_n \rightharpoonup x \\ u_n \to u \end{cases} \Rightarrow \ x_n \to x.$$

Quelques définitions relatives à un opérateur univoque $T\colon \mathcal{H} \to \mathcal{H}$

- L'ensemble des points fixes de T :
$$\operatorname{Fix} T = \big\{x \in \mathcal{H} \mid Tx = x\big\}.$$
- T est lipschitzien de constante $\chi \in \left]0, +\infty\right[$ (ou T est χ–lipschitzien) :
$$(\forall (x,y) \in \mathcal{H}^2) \quad \|Tx - Ty\| \leq \chi\|x - y\|.$$
- T est une contraction stricte : T est χ–lipschitzien avec $\chi \in \left]0, 1\right[$.
- T est une contraction : T est χ–lipschitzien avec $\chi \in \left]0, 1\right]$.
- T est une contraction α–moyennée, avec $\alpha \in \left]0, 1\right[$: T vérifie l'une des conditions équivalentes suivantes :
  (a) $(\forall (x,y) \in \mathcal{H}^2) \quad \|Tx - Ty\|^2 + (1 - 2\alpha)\|x - y\|^2 \leq 2(1 - \alpha)\,\langle x - y \mid Tx - Ty\rangle$
  (b) $(\forall (x,y) \in \mathcal{H}^2) \quad \|Tx - Ty\|^2 \leq \|x - y\|^2 - \dfrac{1 - \alpha}{\alpha}\|(\operatorname{Id} - T)x - (\operatorname{Id} - T)y\|^2.$
- T est une contraction ferme : T est une contraction $1/2$–moyennée.
- T est β–cocoercif, où $\beta \in \left]0, +\infty\right[$: βT est une contraction ferme.

Notations relatives à une fonction $\varphi \in \Gamma_0(\mathcal{H})$

- Domaine de φ
$$\operatorname{dom}\varphi = \big\{x \in \mathcal{H} \mid \varphi(x) < +\infty\big\}.$$

- Ensemble des minimiseurs de φ
$$\operatorname{Argmin}\varphi.$$

- Le minimiseur de φ en cas d'unicité
$$\operatorname{argmin}\varphi(\mathcal{H}) \quad \text{ou} \quad \underset{y\in\mathcal{H}}{\operatorname{argmin}}\ \varphi(y).$$

- Conjuguée de φ
$$\varphi^*\colon u \mapsto \sup_{x\in\mathcal{H}}\big(\langle x \mid u\rangle - \varphi(x)\big).$$

- Enveloppe de Moreau d'indice $\gamma \in \left]0,+\infty\right[$ de φ
$${}^{\gamma}\varphi\colon x \mapsto \inf_{y\in\mathcal{H}}\left(\varphi(y) + \frac{1}{2\gamma}\|x-y\|^2\right).$$

- Le sous-différentiel de φ en $x \in \operatorname{dom}\varphi$
$$\partial\varphi(x) = \big\{u \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \langle y - x \mid u\rangle + \varphi(x) \leq \varphi(y)\big\}.$$

- L'opérateur proximal de φ
$$\operatorname{prox}_{\varphi}\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto \underset{y\in\mathcal{H}}{\operatorname{argmin}}\ \left(\varphi(y) + \frac{1}{2}\|x-y\|^2\right).$$

- La section inférieure de φ hauteur $\eta \in \mathbb{R}$
$$\operatorname{lev}_{\leq\eta}\varphi = \big\{x \in \mathcal{H} \mid \varphi(x) \leq \eta\big\}.$$

Chapitre 1

Introduction

1.1 Présentation générale et objectifs

Cette thèse est consacrée au problème d'analyse non linéaire multivoque suivant.

Problème 1.1 Soient m et q des entiers strictement positifs. Soient $(\mathcal{H}_i)_{1\leq i\leq m}$ des espaces de Hilbert réels et, pour tout $j \in \{1,\ldots,q\}$, soit $\boldsymbol{A}_j \colon \mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_m \to 2^{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_m}$ un opérateur maximalement monotone. Le problème est de trouver un zéro de $\boldsymbol{A}_1+\cdots+\boldsymbol{A}_q$, i.e.,

$$\text{trouver} \quad x_1 \in \mathcal{H}_1,\ldots,x_m \in \mathcal{H}_m \quad \text{tels que} \quad (0,\ldots,0) \in \sum_{j=1}^{q} \boldsymbol{A}_j(x_1,\ldots,x_m). \quad (1.1)$$

Comme nous le verrons, la formulation ci-dessus permet de modéliser une vaste classe de problèmes en analyse non-linéaire où plusieurs variables interagissent. En particulier, entrent dans ce cadre des problèmes de traitement du signal [6, 7, 8, 9, 23, 30, 31, 40, 48, 77, 78], de la théorie du trafic [18, 19, 39, 59, 60, 66, 68], d'économie [32, 33, 37, 47, 57], de la théorie des jeux [2, 4, 55, 60, 76], d'admissibilité [12, 25, 28], de point selle [63], de meilleure approximation [10, 11, 16, 23, 34], de décomposition de domaine dans les équations aux dérivées partielles [62, 70, 74], d'équations aux dérivées partielles [38, 41, 51, 79] et d'inclusions d'évolution [5, 42, 69].

Plusieurs algorithmes pour résoudre le Problème 1.1 existent dans la littérature dans le cas où $m = 1$; ils seront décrits dans la Section 1.2. Cependant, pour le cas $m \geq 2$, seulement des cas très particuliers du Problème 1.1 peuvent être résolus. Le cadre plus général étudié dans la littérature est le cas où $m = 2$, $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $q = 2$, $\boldsymbol{A}_1 \colon (x_1,x_2) \mapsto A_1x_1 \times A_2x_2$, où A_1 et A_2 sont des opérateurs maximalement monotones dans $\mathcal{H}$, et $\boldsymbol{A}_2 \colon (x_1, x_2) \mapsto (x_1 - x_2, x_2 - x_1)$. Dans ce cas, le Problème 1.1 s'écrit

$$\text{trouver} \quad x_1 \in \mathcal{H} \text{ et } x_2 \in \mathcal{H} \quad \text{tels que} \quad \begin{cases} 0 \in A_1 x_1 + x_1 - x_2 \\ 0 \in A_2 x_2 + x_2 - x_1. \end{cases} \tag{1.2}$$

Ce problème a été résolu dans [17] à l'aide d'une méthode inspirée de l'algorithme des projections alternées [58] et de certains développements ultérieurs [1, 24, 34]. La méthode alterne le calcul des résolvantes de A_1 et A_2, et ne semble pas extensible au cas de plus de deux composantes qui appartiennent aux espaces hilbertiens différents ou de plus de deux opérateurs.

La principale motivation de cette thèse est de fournir des méthodes de résolution du Problème 1.1 sous divers jeux d'hypothèses et de les appliquer à certains problèmes concrets d'analyse non-linéaire. Les méthodes que nous proposons procèdent par éclatement en ceci que chaque opérateur présent dans le modèle est utilisé individuellement à chaque itération de l'algorithme. Historiquement, les méthodes d'éclatement d'opérateurs trouvent leurs racines dans certaines méthodes de résolution de problèmes non linéaires, par exemple [52, 53, 61].

Nous ne nous proposons pas d'examiner les questions d'existence ou d'unicité des solutions du Problème 1.1 puisque ces questions sont déjà amplement traitées dans la littérature [5, 15, 64, 65, 79].

Dans le Problème 1.1, nous étudierons avec attention le cas où $q = 2$, i.e.,

$$\text{trouver} \quad x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m \quad \text{tels que} \\ (0, \ldots, 0) \in \boldsymbol{A}_1(x_1, \ldots, x_m) + \boldsymbol{A}_2(x_1, \ldots, x_m). \tag{1.3}$$

En particulier, nous examinerons le cas où $\boldsymbol{A}_1$ est séparable, c'est-à-dire,

$$\boldsymbol{A}_1 \colon (x_i)_{1 \leq i \leq m} \mapsto A_1 x_1 \times \cdots \times A_m x_m, \tag{1.4}$$

où $(A_i)_{1 \leq i \leq m}$ sont des opérateurs maximalement monotones et $\boldsymbol{A}_2$ est univoque. Nous nous pencherons notamment sur le cas où, pour tout $i \in \{1, \ldots, m\}$, A_i est le sous-différentiel d'une fonction dans $\Gamma_0(\mathcal{H}_i)$ et $\boldsymbol{A}_2$ est le gradient d'une fonction dans $\Gamma_0(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m)$, ce que nous permettra d'étudier certains problèmes de minimisation. D'autres méthodes seront proposées lorsque $\boldsymbol{A}_1$ n'est pas séparable et $\boldsymbol{A}_2$ est multivoque et nous étudierons attentivement des cas particuliers pour résoudre des problèmes de minimisation liés à ce cadre. Des algorithmes seront proposées pour le cas général où $q \geq 2$ opérateurs interviennent et nous illustrerons la flexibilité de toutes ces méthodes à travers des applications à la théorie des jeux, au traitement d'images, à la décomposition de domaine dans les équations aux dérivées partielles, à la théorie du trafic, aux inclusions d'évolution couplées et d'autres problèmes.

De plus, nous nous pencherons sur une application du problème (1.3) aux inclusions primale

$$\text{trouver} \quad x \in \mathcal{H} \quad \text{tel que} \quad z \in Ax + L^*(B(Lx - r)) \tag{1.5}$$

et duale

$$\text{trouver} \quad v \in \mathcal{G} \quad \text{tel que} \quad -r \in -L(A^{-1}(z - L^*v)) + B^{-1}(v), \tag{1.6}$$

où $\mathcal{H}$ et $\mathcal{G}$ sont des espaces hilbertiens, $z \in \mathcal{H}$, $r \in \mathcal{G}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ et $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ sont des opérateurs maximalement monotones, et $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$. Nous proposerons une méthode pour trouver simultanément des solutions de (1.5) et (1.6) à partir d'un algorithme qui résout le problème (1.3) dans le cas $m = 2$ pour un choix approprié des opérateurs $\boldsymbol{A}_1$ et $\boldsymbol{A}_2$.

Par ailleurs, nous développerons une méthode pour résoudre des problèmes de point fixe composites sous contraintes. La méthode, avec un choix approprié des opérateurs concernés, généralisera en particulier l'approche étudiée dans [75] pour la résolution de certaines inclusions monotones.

1.2 Algorithmes d'éclatement d'opérateurs existants

La plupart des méthodes proposées dans la thèse sont basées sur des formulations originales dans des espaces produits adéquats d'algorithmes d'éclatement d'opérateurs existants. Dans cette section, nous passons en revue ces algorithmes, qui permettent de résoudre le Problème 1.1 en présence d'une seule composante ($m = 1$).

1.2.1 Méthode explicite-implicite

Cet algorithme permet de résoudre le problème suivant.

Problème 1.2 Soit $\beta \in \,]0, +\infty[$, soit $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ un opérateur maximalement monotone et soit $B\colon \mathcal{H} \to \mathcal{H}$ un opérateur β–cocoercif. Le problème est de

$$\text{trouver } x \in \mathcal{H} \text{ tel que } 0 \in Ax + Bx. \tag{1.7}$$

Notons que le Problème 1.2 correspond au cas particulier du Problème 1.1 où $m = 1$, $q = 2$, $\boldsymbol{A}_1 = A$ et $\boldsymbol{A}_2 = B$, avec l'hypothèse additionnelle de cocoercivité sur $\boldsymbol{A}_2$.

Dans la suite nous énonçons une version de la méthode explicite-implicite développée dans [26] pour résoudre le Problème 1.2 (voir [3, 26] et leurs bibliographies pour l'historique).

Algorithme 1.3 Soient $(a_n)_{n\in\mathbb{N}}$ et $(b_n)_{n\in\mathbb{N}}$ des suites dans $\mathcal{H}$. On génère une suite $(x_n)_{n\in\mathbb{N}}$

comme suit.

$$
\begin{array}{l}
\text{Initialisation} \\
\left\lfloor
\begin{array}{l}
\varepsilon \in \left]0, \min\{1,\beta\}\right[\\
x_0 \in \mathcal{H}
\end{array}
\right. \\
\text{Pour } n = 0,1,\ldots \\
\left\lfloor
\begin{array}{l}
\gamma_n \in [\varepsilon, 2\beta - \varepsilon] \\
\lambda_n \in [\varepsilon, 1] \\
z_n = x_n - \gamma_n(Bx_n + b_n) \\
y_n = J_{\gamma_n A} z_n + a_n \\
x_{n+1} = x_n + \lambda_n(y_n - x_n).
\end{array}
\right.
\end{array}
\tag{1.8}
$$

Proposition 1.4 ([26, Corollary 6.5]) *On suppose que* $\operatorname{zer}(A+B) \neq \varnothing$ *et que*

$$
\sum_{n\in\mathbb{N}} \|a_n\| < +\infty \quad \textit{et} \quad \sum_{n\in\mathbb{N}} \|b_n\| < +\infty. \tag{1.9}
$$

Alors la suite $(x_n)_{n\in\mathbb{N}}$ *engendrée par l'Algorithme 1.3 converge faiblement vers un zéro de* $A+B$.

1.2.2 Méthode explicite-implicite-explicite

Cette méthode permet de résoudre un problème plus général que le Problème 1.2 en permettant que l'opérateur B soit lipschitzien au lieu d'être cocoercif.

Problème 1.5 Soit $\beta \in \left]0,+\infty\right[$, soient $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ et $B\colon \mathcal{H} \to \mathcal{H}$ deux opérateurs maximalement monotones et supposons que B soit β^{-1}–lipschitzien. Le problème est de

$$
\text{trouver } x \in \mathcal{H} \text{ tel que } 0 \in Ax + Bx. \tag{1.10}
$$

Notons que le Problème 1.5 correspond au cas particulier du Problème 1.1 où $m = 1$, $q = 2$, $\boldsymbol{A}_1 = A$ et $\boldsymbol{A}_2 = B$.

L'algorithme proposé dans [75] résout un problème plus général que le Problème 1.5 en considérant de plus une contrainte dans (1.10) et un opérateur B dont le domaine n'est pas nécessairement tout l'espace.

La méthode explicite-implicite-explicite trouve ses racines dans des méthodes de résolution d'inéquations variationnelles [43, 45, 46, 49, 50]. Nous présentons ici le cas sans contrainte de l'algorithme proposé dans [75] pour résoudre le Problème 1.5. Nous reviendrons au cas général au Chapitre 8.

Algorithme 1.6 On génère une suite $(x_n)_{n\in\mathbb{N}}$ comme suit.

$$
\begin{array}{l}
\text{Initialisation} \\
\left\lfloor
\begin{array}{l}
\varepsilon \in \left]0, \beta/2\right[\\
x_0 \in \mathcal{H}
\end{array}
\right. \\
\text{Pour } n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
\gamma_n \in [\varepsilon, \beta - \varepsilon] \\
y_n = x_n - \gamma_n B x_n \\
p_n = J_{\gamma_n A} y_n \\
q_n = p_n - \gamma_n B p_n \\
x_{n+1} = x_n - y_n + q_n.
\end{array}
\right.
\end{array}
\tag{1.11}
$$

Le résultat suivant est une conséquence de [75, Theorem 3.4(b)].

Proposition 1.7 *On suppose que* $\operatorname{zer}(A+B) \neq \varnothing$*. Alors la suite* $(x_n)_{n\in\mathbb{N}}$ *engendrée par l'Algorithme 1.6 converge faiblement vers un zéro de* $A+B$*.*

1.2.3 Méthode de Douglas-Rachford

On ici considère un problème plus général que le Problème 1.5, en supposant simplement que B soit multivoque et maximalement monotone.

Problème 1.8 Soient $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ et $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ deux opérateurs maximalement monotones. Le problème est de

$$
\text{trouver } x \in \mathcal{H} \text{ tel que } 0 \in Ax + Bx. \tag{1.12}
$$

Notons que le Problème 1.8 correspond au cas particulier du Problème 1.1 où $m = 1$, $q = 2$, $\boldsymbol{A}_1 = A$ et $\boldsymbol{A}_2 = B$.

Cette méthode a été proposée initialement dans [54] (voir aussi [26, 27, 36] et leur bibliographies). Nous énonçons à présent la version de l'algorithme de Douglas-Rachford donnée dans [27].

Algorithme 1.9 Soient $(a_n)_{n\in\mathbb{N}}$ et $(b_n)_{n\in\mathbb{N}}$ des suites dans $\mathcal{H}$. On génère des suites

$(x_n)_{n\in\mathbb{N}}$ et $(y_n)_{n\in\mathbb{N}}$ comme suit.

$$
\begin{array}{l}
\text{Initialisation} \\
\left\lfloor
\begin{array}{l}
\gamma \in]0,+\infty[\\
x_0 \in \mathcal{H}
\end{array}
\right. \\
\text{Pour } n = 0,1,\ldots \\
\left\lfloor
\begin{array}{l}
\lambda_n \in]0,2[\\
y_n = J_{\gamma B}x_n + b_n \\
z_n = J_{\gamma A}(2y_n - x_n) + a_n \\
x_{n+1} = x_n + \lambda_n(z_n - y_n).
\end{array}
\right.
\end{array}
\tag{1.13}
$$

Proposition 1.10 ([27, Theorem 2.1]) *Soient $(x_n)_{n\in\mathbb{N}}$ et $(y_n)_{n\in\mathbb{N}}$ les suites engendrées par l'Algorithme 1.9. Supposons que $\operatorname{zer}(A+B) \neq \varnothing$, et que*

$$
\sum_{n\in\mathbb{N}} \lambda_n(\|a_n\| + \|b_n\|) < +\infty \quad \textit{et} \quad \sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty. \tag{1.14}
$$

Alors nous avons les résultats suivants.

(i) $x_n \rightharpoonup x \in \mathcal{H}$ *et* $J_{\gamma B}x \in \operatorname{zer}(A+B)$.

(ii) *Supposons que $J_{\gamma B}$ soit faiblement séquentiellement continu et que $b_n \rightharpoonup 0$. Alors la suite $(y_n)_{n\in\mathbb{N}}$ converge faiblement vers un zéro de $A+B$.*

(iii) *Supposons que $A = N_D$, où D est un sous-espace affine fermé de $\mathcal{H}$. Alors la suite $(J_{\gamma A}x_n)_{n\in\mathbb{N}}$ converge faiblement vers un zéro de $A+B$.*

(iv) *Supposons que $A = N_D$, où D est un sous-espace vectoriel fermé de $\mathcal{H}$. Alors la suite $(J_{\gamma A}y_n)_{n\in\mathbb{N}}$ converge faiblement vers un zéro de $A+B$.*

Remarque 1.11 Dans [73] il a été très récemment montré que la suite $(y_n)_{n\in\mathbb{N}}$ engendrée par l'Algorithme 1.9 converge faiblement vers un zéro de $A+B$ avec un processus d'erreurs $(b_n)_{n\in\mathbb{N}}$ moins général et pour le cas où $\lambda_n \equiv 1$ (voir aussi [15, Theorem 25.6] pour le cas correspondant à (1.14) sans erreurs).

1.2.4 Méthode parallèle basée sur Douglas-Rachford

Nous décrivons ici une méthode pour trouver un zéro de la somme de $q \geq 2$ opérateurs maximalement monotones.

Problème 1.12 Soit $q \geq 2$ un entier, soient $(\omega_j)_{1\leq j\leq q}$ des nombres réels dans $]0,1[$ tels que $\sum_{j=1}^q \omega_j = 1$ et soient $(A_j)_{1\leq j\leq q}$ des opérateurs maximalement monotones de $\mathcal{H}$ vers $2^{\mathcal{H}}$. Le problème est de

$$
\text{trouver } x \in \mathcal{H} \text{ tel que } 0 \in \sum_{j=1}^q \omega_j A_j x. \tag{1.15}
$$

Notons que le Problème 1.12 correspond au cas particulier du Problème 1.1 où $m = 1$ et, pour tout $j \in \{1, \dots, q\}$, $\boldsymbol{A}_j = \omega_j A_j$.

Dans la suite nous énonçons une méthode développée dans [27] pour résoudre le Problème 1.12. La méthode est basée sur l'algorithme de Douglas-Rachford appliqué dans un espace produit et tolère des erreurs dans le calcul des résolvantes des opérateurs $(A_j)_{1 \leq j \leq q}$ (voir [27] pour le lien avec l'algorithme parallèle proposé dans [72], qui est basé sur la méthode des inverses partiels).

Algorithme 1.13 Pour tout $j \in \{1, \dots, q\}$, soit $(a_{j,n})_{n \in \mathbb{N}}$ une suite dans $\mathcal{H}$. On génère une suite $(p_n)_{n \in \mathbb{N}}$ comme suit.

$$
\begin{array}{l}
\text{Initialisation} \\
\left\lfloor
\begin{array}{l}
\gamma \in \left]0, +\infty\right[\\
\text{Pour } j = 1, \dots, q \\
\lfloor\, x_{j,0} \in \mathcal{H}
\end{array}
\right. \\
\text{Pour } n = 0, 1, \dots \\
\left\lfloor
\begin{array}{l}
p_n = \sum_{j=1}^{r} \omega_j x_{j,n} \\
\text{Pour } j = 1, \dots, q \\
\lfloor\, y_{j,n} = J_{\gamma A_j} x_{j,n} + a_{j,n} \\
z_n = \sum_{j=1}^{q} \omega_j y_{j,n} \\
\lambda_n \in \left]0, 2\right[\\
\text{Pour } j = 1, \dots, q \\
\lfloor\, x_{j,n+1} = x_{j,n} + \lambda_n (2 z_n - p_n - y_{j,n}).
\end{array}
\right.
\end{array}
\tag{1.16}
$$

Proposition 1.14 ([27, Theorem 2.5]) *Supposons que* $\operatorname{zer}(\sum_{j=1}^{q} \omega_j A_j) \neq \varnothing$*, et que*

$$
\max_{1 \leq j \leq q} \sum_{n \in \mathbb{N}} \lambda_n \|a_{j,n}\| < +\infty \quad \textit{et} \quad \sum_{n \in \mathbb{N}} \lambda_n (2 - \lambda_n) = +\infty. \tag{1.17}
$$

Alors la suite $(p_n)_{n \in \mathbb{N}}$ *engendrée par l'Algorithme 1.13 converge faiblement vers un zéro de* $\sum_{j=1}^{q} \omega_j A_j$.

1.2.5 Méthode parallèle de type Dykstra

Cette méthode résout le problème fortement monotone suivant.

Problème 1.15 Soit $q \geq 2$ un entier, soit $z \in \mathcal{H}$, soient $(A_j)_{1 \leq j \leq q-1}$ des opérateurs maximalement monotones de $\mathcal{H}$ vers $2^{\mathcal{H}}$, et soient $(\omega_j)_{1 \leq j \leq q-1}$ des nombres réels dans $]0, 1[$ tels que $\sum_{j=1}^{q-1} \omega_j = 1$. Le problème est de

$$
\text{trouver } x \in \mathcal{H} \text{ tel que } z \in x + \sum_{j=1}^{q-1} \omega_j A_j x. \tag{1.18}
$$

Notons que le Problème 1.15 correspond au cas particulier du Problème 1.1 où $m = 1$, pour tout $j \in \{1, \dots, q-1\}$, $\boldsymbol{A}_j = \omega_j A_j$ et $\boldsymbol{A}_q \colon x \mapsto x - z$. Dès que l'opérateur $\boldsymbol{A}_q$ est fortement monotone, nous pouvons assurer l'existence et l'unicité de la solution du Problème 1.15 : la solution unique est $x = J_B z$, où $B = \sum_{j=1}^{q-1} \omega_j A_j$.

Dans la suite nous énonçons la méthode proposée dans [27] qui tolère des erreurs dans le calcul des résolvantes des opérateurs $(A_j)_{1 \le j \le q-1}$. La méthode généralise la méthode de Dykstra parallèle pour trouver un point dans l'intersection de convexes [20, 35] (voir [13, 27] et leur références pour l'historique).

Algorithme 1.16 Pour tout $j \in \{1, \dots, q-1\}$, soit $(a_{j,n})_{n \in \mathbb{N}}$ une suite dans $\mathcal{H}$. On génère une suite $(x_n)_{n \in \mathbb{N}}$ comme suit.

$$
\begin{array}{l}
\text{Initialisation} \\
\left\lfloor
\begin{array}{l}
x_0 = z \\
\text{Pour } j = 1, \dots, q-1 \\
\lfloor \; z_{j,0} = x_0
\end{array}
\right. \\
\text{Pour } n = 0, 1, \dots \\
\left\lfloor
\begin{array}{l}
\text{Pour } j = 1, \dots, q-1 \\
\lfloor \; y_{j,n} = J_{A_j} z_{j,n} + a_{i,n} \\
x_{n+1} = \displaystyle\sum_{j=1}^{q-1} \omega_j y_{j,n} \\
\text{Pour } j = 1, \dots, q-1 \\
\lfloor \; z_{j,n+1} = x_{n+1} + z_{i,n} - y_{i,n}.
\end{array}
\right.
\end{array}
\tag{1.19}
$$

Proposition 1.17 ([27, Theorem 3.3]) *On suppose que $z \in \operatorname{ran}(\mathrm{Id} + \sum_{j=1}^{q-1} \omega_j A_j)$ et que, pour tout $j \in \{1, \dots, q-1\}$, $\sum_{n \in \mathbb{N}} \|a_{j,n}\| < +\infty$. Alors la suite $(x_n)_{n \in \mathbb{N}}$ engendrée par l'Algorithme 1.16 converge fortement vers $J_B z$, où $B = \sum_{j=1}^{q-1} \omega_j A_j$.*

1.3 Organisation de la thèse

Au Chapitre 2 nous étudions le cas particulier du Problème 1.1 où $q = 2$, $\boldsymbol{A}_1 \colon (x_i)_{1 \le i \le m} \mapsto \times_{i=1}^m A_i x_i$, $(A_i)_{1 \le i \le m}$ sont des opérateurs maximalement monotones et $\boldsymbol{A}_2 \colon (x_i)_{1 \le i \le m} \mapsto (B_i(x_1, \dots, x_m))_{1 \le i \le m}$ est cocoercif. Dans ce cas, le Problème 1.1 donne

$$
\text{trouver} \quad x_1 \in \mathcal{H}_1, \dots, x_m \in \mathcal{H}_m \quad \text{tels que} \quad
\begin{cases}
0 \in A_1 x_1 + B_1(x_1, \dots, x_m) \\
\quad \vdots \\
0 \in A_m x_m + B_m(x_1, \dots, x_m)
\end{cases}
\tag{1.20}
$$

et nous proposons de le résoudre avec une méthode basée sur l'algorithme d'éclatement explicite-implicite décrit dans la Section 1.2.1. Cet algorithme est appliqué ensuite à la résolution d'inclusions d'évolution couplées et de problèmes de minimisation. Parmi ces derniers, nous examinons des problèmes de meilleure approximation et des problèmes de la théorie du trafic.

Au Chapitre 3 nous étudions plus en détail le problème de minimisation traité dans (1.20), qui est obtenu en supposant que, pour tout $i \in \{1, \ldots, m\}$, A_i est le sous-différentiel d'une fonction $f_i \in \Gamma_0(\mathcal{H}_i)$ et $(x_i)_{1 \leq i \leq m} \mapsto (B_i(x_1, \ldots, x_m))_{1 \leq i \leq m}$ est le gradient d'une fonction convexe $\boldsymbol{g} \in \Gamma_0(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m)$. Dans ce cas nous obtenons le problème

$$\underset{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m}{\text{minimiser}} \quad \sum_{i=1}^{m} f_i(x_i) + \boldsymbol{g}(x_1, \ldots, x_m). \tag{1.21}$$

Nous étudions des structures particulières de $\boldsymbol{g}$ qui fournissent la cocoercivité de $\nabla \boldsymbol{g}$. Plus précisément, nous supposons que

$$\boldsymbol{g} \colon (x_i)_{1 \leq i \leq m} \mapsto \sum_{k=1}^{p} \varphi_k\bigg(\sum_{i=1}^{m} L_{ki} x_i\bigg), \tag{1.22}$$

où, pour tout $k \in \{1, \ldots, p\}$, $\varphi_k \in \Gamma_0(\mathcal{G}_k)$ est une fonction différentiable avec gradient lipschitzien et, pour tout $i \in \{1, \ldots, m\}$, $L_{ki} \in \mathcal{B}(\mathcal{H}_i, \mathcal{G}_k)$. Dans ce cas, nous utilisons un résultat de convergence d'un algorithme proposé au Chapitre 2 pour le cas de problèmes de minimisation. Ensuite, cet algorithme est perfectionné pour le cas où les fonctions $(\varphi_k)_{1 \leq k \leq p}$ sont quadratiques et, en particulier, pour le cas où, pour tout $k \in \{1, \ldots, p\}$ et $i \in \{1, \ldots, m\}$, $L_{ki} = \xi_{ki} \,\mathrm{Id}$. Nous utilisons ces algorithmes pour résoudre divers problèmes de traitement du signal (décomposition, synthèse, représentation).

Au Chapitre 4 nous proposons différentes méthodes issues des algorithmes d'éclatement décrits dans la Section 1.2 pour résoudre le Problème 1.1 dans chacun des cas suivants.

(i) $q = 2$ et $\boldsymbol{A}_2$ est cocoercif.

(ii) $q = 2$ et $\boldsymbol{A}_2$ est lipschitzien.

(iii) $q = 2$.

(iv) $q \geq 3$ et $\boldsymbol{A}_q = \mathrm{Id} - \boldsymbol{z}$, avec $\boldsymbol{z} \in \boldsymbol{\mathcal{H}}$.

(v) $q \geq 3$.

Contrairement à l'algorithme proposé au Chapitre 2, la méthode pour résoudre le cas (i) permet de considérer un opérateur $\boldsymbol{A}_1$ non séparable. Dans la Section 4.3 nous appliquons les algorithmes pour résoudre (i)–(v) aux cas de problèmes de minimisation. De cette manière nous résolvons le Problème 1.1 dans chacun des cas suivants.

(vi) $q = 2$, $\boldsymbol{f}_1$ dans $\Gamma_0(\boldsymbol{\mathcal{H}})$, $\boldsymbol{f}_2 \colon \boldsymbol{\mathcal{H}} \to \mathbb{R}$ convexe différentiable avec gradient lipschitzien, $\boldsymbol{A}_1 = \partial \boldsymbol{f}_1$ et $\boldsymbol{A}_2 = \nabla \boldsymbol{f}_2$.

(vii) $q = 2$, $\boldsymbol{f}_1$ et $\boldsymbol{f}_2$ dans $\Gamma_0(\boldsymbol{\mathcal{H}})$, $\boldsymbol{A}_1 = \partial \boldsymbol{f}_1$ et $\boldsymbol{A}_2 = \partial \boldsymbol{f}_2$.

(viii) $q \geq 3$, pour tout $j \in \{1, \ldots, q-1\}$, $\boldsymbol{f}_j$ dans $\Gamma_0(\boldsymbol{\mathcal{H}})$, $\boldsymbol{A}_j = \partial \boldsymbol{f}_j$ et $\boldsymbol{A}_q = \mathrm{Id} - \boldsymbol{z}$, avec $\boldsymbol{z} \in \boldsymbol{\mathcal{H}}$.

(ix) $q \geq 3$ et, pour tout $j \in \{1, \ldots, q\}$, $\boldsymbol{f}_j$ dans $\Gamma_0(\boldsymbol{\mathcal{H}})$ et $\boldsymbol{A}_j = \partial \boldsymbol{f}_j$.

Contrairement au Chapitre 3, la méthode pour résoudre le cas (vi) du Problème 1.1 permet de considérer une fonction $\boldsymbol{f}_1$ non séparable, c'est-à-dire, qui n'est pas de la forme $(x_i)_{1 \leq i \leq m} \mapsto \sum_{i=1}^{m} f_i(x_i)$ (voir (1.21)). Vu que tous les algorithmes utilisent des opérateurs proximaux, une partie de la Section 4.3 est consacrée au calcul explicite des opérateurs proximaux de certaines fonctions utiles. Le chapitre se conclut avec des expériences numériques visant à évaluer les algorithmes sur des problèmes de traitement d'images.

Au Chapitre 5 nous nous intéressons à la résolution simultanée des inclusions primale (1.5) et duale (1.6). Dans la littérature, plusieurs méthodes résolvent (1.5), mais dans des conditions très restrictives. En particulier, si B est univoque et cocoercif, alors $L^*(B(L \cdot -r))$ est cocoercif et la méthode explicite-implicite résout (1.5). Par ailleurs, si B est lipschitzien, alors $L^*(B(L \cdot -r))$ est lipschitzien et (1.5) peut être résolu par la méthode explicite-implicite-explicite (voir Section 1.2). Cependant, le cas général de (1.5) où B est multivoque et monotone ne peut être résolu par aucune des méthodes présentées dans la Section 1.2, sauf sous des hypothèses fortes imposées à L [15, Theorem 23.23]. Notre stratégie est de résoudre simultanément (1.5) et (1.6) via l'inclusion

$$\text{trouver} \quad (x, v) \in \mathcal{H} \times \mathcal{G} \quad \text{tel que} \quad (0,0) \in \boldsymbol{M}(x, v) + \boldsymbol{S}(x, v), \tag{1.23}$$

où $\boldsymbol{M} \colon (x, v) \mapsto (-z + Ax) \times (r + B^{-1}v)$ et $\boldsymbol{S} \colon (x, v) \mapsto (L^*v, -Lx)$. Nous montrons que, pour toute solution $(x, v) \in \mathcal{H} \times \mathcal{G}$ de (1.23), x est solution de (1.5) et v est solution de (1.6). Vu que $\boldsymbol{M}$ et $\boldsymbol{S}$ sont maximalement monotones, l'inclusion (1.23) correspond au cas particulier du Problème 1.1, où $m = 2$, $q = 2$, $\mathcal{H}_1 = \mathcal{H}$, $\mathcal{H}_2 = \mathcal{G}$, $\boldsymbol{A}_1 = \boldsymbol{M}$ et $\boldsymbol{A}_2 = \boldsymbol{S}$. De plus, le caractère lipschitzien de $\boldsymbol{S}$ nous suggère l'emploi de la méthode explicite-implicite-explicite de la Section 1.2.2 pour résoudre le problème (1.23), ce qui permet de trouver une solution primale-duale de (1.5)–(1.6). Ce nouveau formalisme de décomposition monotone+anti-adjoint fournit un outil puissant de modélisation, que nous appliquons ensuite aux problèmes de minimisation.

Le Chapitre 6 est consacré aux problèmes de décomposition de domaine dans les équations aux dérivées partielles. Un des objectifs principaux de ce problème est la résolution d'équations aux dérivées partielles et des problèmes de frontière associés par la décomposition du domaine original en des sous-domaines simples, dans le cas où ils ne se chevauchent pas (voir [23, 25]). Le problème général est modélisé par un problème de minimisation où les variables sont les solutions restreintes à chaque sous-domaine. La fonction objectif prend en compte des potentiels d'énergie associés aux équations aux dérivées partielles définies dans chaque sous-domaine et des fonctions convexes et semi-continues inférieurement qui modélisent les contraintes sur les sauts aux interfaces.

Puisque nous permettons à ces dernières de prendre la valeur $+\infty$, nous pouvons modéliser des contraintes dures sur les sauts aux interfaces, en particulier, des conditions de continuité ou de transmission unilatérale. Ce problème de minimisation multi-variable est résolu par la méthode primale-duale proposée au Chapitre 5 appliquée dans un espace produit, ce qui débouche sur une méthode parallèle qui converge vers une solution primale-duale du problème. La solution du problème dual représente les tensions aux interfaces.

Le Chapitre 7 est dédié à la construction d'équilibres de Nash de jeux sans potentiel à m joueurs. Nous supposons que l'espace de stratégies de chaque joueur $i \in \{1, \ldots, m\}$ est un espace hilbertien réel $\mathcal{H}_i$ et que la fonction de pénalité peut s'exprimer comme la somme d'une fonction convexe qui est commune à tous les joueurs et d'une fonction individuelle qui est différentiable. Le problème est de

$$\text{trouver } x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m \text{ tels que}$$
$$\begin{cases} x_1 \in \underset{x \in \mathcal{H}_1}{\operatorname{Argmin}} \; \boldsymbol{f}(x, x_2, \ldots, x_m) + \boldsymbol{g}_1(x, x_2, \ldots, x_m) \\ \quad \vdots \\ x_m \in \underset{x \in \mathcal{H}_m}{\operatorname{Argmin}} \; \boldsymbol{f}(x_1, \ldots, x_{m-1}, x) + \boldsymbol{g}_m(x_1, \ldots, x_{m-1}, x), \end{cases} \tag{1.24}$$

où $\boldsymbol{f} \colon \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \to \left]-\infty, +\infty\right]$ est une fonction propre, semi-continue inférieurement et convexe et, pour tout $i \in \{1, \ldots, m\}$, $\boldsymbol{g}_i \colon \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \to \left]-\infty, +\infty\right]$ est une fonction différentiable par rapport à la i-ème variable. Vu que la fonction $\boldsymbol{f}$ peut prendre la valeur $+\infty$, cette formulation permet de modéliser des contraintes communes à tous les joueurs.

Dans le cas particulier où $\boldsymbol{g}_i \equiv \boldsymbol{g}$, où $\boldsymbol{g}$ est convexe et différentiable avec gradient lipschitzien, (1.24) revient à trouver des équilibres de Nash d'un jeu de potentiel [55]. Plusieurs techniques sont disponibles pour résoudre de tels problèmes [15, Chapter 27]. Nous nous intéressons au cadre sans potentiel, où les fonctions $(\boldsymbol{g}_i)_{1 \leq i \leq m}$ peuvent être différentes. Notre stratégie est de résoudre l'inclusion monotone

$$\text{trouver} \quad x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m \quad \text{tels que} \quad (0, \ldots, 0) \in \partial \boldsymbol{f}(x_1, \ldots, x_m) + \boldsymbol{B}(x_1, \ldots, x_m), \tag{1.25}$$

où $\boldsymbol{B} \colon (x_i)_{1 \leq i \leq m} \mapsto (\nabla_i \, \boldsymbol{g}_i(x_1, \ldots, x_m))_{1 \leq i \leq m}$ et, pour tout $i \in \{1, \ldots, m\}$, ∇_i est la dérivée partielle par rapport à la i-ème composante. Nous montrons que toute solution de (1.25) est une solution du problème (1.24), et ensuite nous appliquons des méthodes de résolution d'inclusions monotones au problème (1.25). Enfin, nous appliquons les algorithmes obtenus aux jeux à somme nulle, aux équilibres de Nash généralisés et aux problèmes de proximation cyclique.

Au Chapitre 8 nous nous intéressons au problème de point fixe composite

$$\text{trouver} \quad x \in S \quad \text{tel que} \quad (\forall n \in \mathbb{N}) \quad T_n R_n x = x, \tag{1.26}$$

où, pour tout $n \in \mathbb{N}$, $T_n \colon \mathcal{H} \to \mathcal{H}$ est une contraction ferme, $R_n \colon \operatorname{dom} R_n \subset \mathcal{H} \to \mathcal{H}$ est une pseudo contraction telle que $(\mathrm{Id} - R_n)$ est un opérateur lipschitzien de constante dans $]0,1[$, et S est un sous-ensemble fermé et convexe de l'espace hilbertien $\mathcal{H}$. Dans la littérature il existe des méthodes qui résolvent (1.26) dans certains scénarios restreints. Si $S = \mathcal{H}$ et $R_n \equiv \mathrm{Id}$, des méthodes peuvent être trouvées dans [21, 26]. Par ailleurs, si $S = \mathcal{H}$, $T_n \equiv \mathrm{Id}$, et $R_n \equiv R$, où R est une pseudo contraction lipschitzienne d'un ensemble convexe dans lui même, des algorithmes sont proposés dans [22, 44, 67, 80]. Vu que l'opérateur obtenu de la composition d'une contraction ferme avec une pseudo contraction lipschitzienne n'est pas une pseudo contraction en général, le problème (1.26) ne peut pas être résolu par les méthodes précédentes. Nous proposons une méthode qui, à chaque itération $n \in \mathbb{N}$, considère des calculs explicites des opérateurs R_n, T_n et R_n, suivis d'une approximation extérieure de la contrainte S. Cette approximation est obtenue par la projection sur un demi-espace affine qui contient S. La méthode tolère des erreurs de calcul dans l'évaluation des opérateurs $(T_n)_{n\in\mathbb{N}}$ et $(R_n)_{n\in\mathbb{N}}$.

Ensuite, nous appliquons la méthode à deux cas particuliers : les inclusions monotones avec contraintes convexes et les problèmes d'équilibre avec contraintes convexes. L'application aux inclusions monotones permet de généraliser la méthode proposée dans [75] en considérant un nombre fini de contraintes et des erreurs dans le calcul des opérateurs impliqués. De plus, la méthode proposée évite le calcul de la projection sur l'ensemble des contraintes en projetant sur un demi-espace qui les contient, ce qui simplifie le calcul en général. D'autre part, l'application aux problèmes d'équilibre fournit une méthode qui généralise les approches dans [29, 56].

Nous présentons dans le Chapitre 9 notre bilan et les questions que nous envisageons d'aborder dans nos travaux futurs.

1.4 Contributions principales

Les contributions principales de cette thèse sont les suivantes.

- La conception et l'étude asymptotique d'une grande classe de méthodes pour résoudre des problèmes multicomposantes issus du Problème 1.1. Nous avons vu dans la Section 1.1 que la littérature permet de résoudre des cas très particuliers du Problème 1.1, qui sont restreints au cas $m = 2$ et $\mathcal{H}_1 = \mathcal{H}_2$. Nous proposons des méthodes parallèles qui résolvent le cas $m \geq 2$ avec, de surcroît, des espaces hilbertiens $(\mathcal{H}_i)_{1\leq i\leq m}$ différents. Des algorithmes pour résoudre des cas particuliers du Problème 1.1 sont présentés dans les Chapitres 2, 3, 4 et 5.

- La conception et l'étude asymptotique d'une méthode pour résoudre simultanément le problème primal (1.5) et le problème dual (1.6). Nous avons vu dans la Section 1.3 que dans la littérature il existe certaines méthodes pour résoudre

(1.5) sous des hypothèses très restrictives. Dans le Chapitre 5 nous proposons un nouveau formalisme de décomposition *monotone+anti-adjoint* pour résoudre un problème formulé sur l'espace produit primal-dual dont les solutions sont celles de (1.5)–(1.6).

- L'étude des conditions suffisantes pour obtenir la convergence forte à partir de la notion de *demiregularité* que nous introduisons dans le Chapitre 2. Cette notion est appliquée dans les Chapitres 2 et 5.

- La conception et l'étude asymptotique d'une méthode pour résoudre des problèmes de point fixe de type (1.26) dans le Chapitre 8. Dans la Section 1.3 nous avons vu que dans la littérature il n'existe des méthodes de résolution de (1.26) que dans certains cas très particuliers. Nous généralisons en particulier les méthodes de [29, 56, 75].

- Le développement de nouvelles méthodes pour résoudre des problèmes du traitement d'image et de signal. Le Chapitre 3 est consacré à des techniques variationnelles pour résoudre problèmes de décomposition, reconstruction et restauration de signaux multicomposantes. En particulier, nous avons résolu des problèmes de décomposition de signal en $m \geq 2$ composantes, de synthèse de signal et de représentation multitrame des signaux. Contrairement aux problèmes de minimisation considérés au Chapitre 3, dans la Section 4.3 nous considérons des fonctions convexes qui ne sont pas nécessairement lisses. Cette section est dédiée aux images multicomposantes où nous examinons en particulier des problèmes de restauration stéréoscopique, de débruitage d'images multispectrales et de décomposition d'images en composantes texture et structure.

- La conception d'une méthode primale-duale pour résoudre des problèmes de décomposition de domaine dans les équations aux dérivées partielles, en considérant $m \geq 2$ sous-domaines disjoints et des contraintes dures aux interfaces (Chapitre 6). La convergence de cette méthode est établie sous des hypothèses très faibles en comparaison avec les méthodes existantes.

- La première utilisation systématique des méthodes d'éclatement d'opérateurs pour la construction d'équilibres de Nash dans le cas de jeux sans potentiel (Chapitre 7).

- Le développement de nouvelles méthodes de résolution de problèmes de théorie du trafic, d'inclusions d'évolution et de meilleure approximation (Chapitre 2).

1.5 Bibliographie

[1] F. Acker et M. A. Prestel, Convergence d'un schéma de minimisation alternée, *Ann. Fac. Sci. Toulouse V. Sér. Math.*, vol. 2, pp. 1–9, 1980.

[2] H. Attouch, J. Bolte, P. Redont, and A. Soubeyran, Alternating proximal algorithms for weakly coupled convex minimization problems – Applications to dynamical games and PDE's, *J. Convex Anal.,* vol. 15, pp. 485–506, 2008.

[3] H. Attouch, L. M. Briceño-Arias, and P. L. Combettes, A parallel splitting method for coupled monotone inclusions, *SIAM J. Control Optim.*, vol. 48, pp. 3246–3270, 2010.

[4] H. Attouch, P. Redont, and A. Soubeyran, A new class of alternating proximal minimization algorithms with costs-to-move, *SIAM J. Optim.,* vol. 18, pp. 1061–1081, 2007.

[5] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis,* Birkhäuser, Boston, MA, 1990.

[6] J.-F. Aujol, G. Aubert, L. Blanc-Féraud, and A. Chambolle, Image decomposition into a bounded variation component and an oscillating component, *J. Math. Imaging Vision*, vol. 22, pp. 71–88, 2005.

[7] J.-F. Aujol and A. Chambolle, Dual norms and image decomposition models, *Int. J. Comput. Vis.*, vol. 63, pp. 85–104, 2005.

[8] J.-F. Aujol, G. Gilboa, T. Chan, and S. Osher, Structure-texture image decomposition – Modeling, algorithms, and parameter selection, *Int. J. Comput. Vis.*, vol. 67, pp. 111–136, 2006.

[9] J.-F. Aujol and S. H. Kang, Color image decomposition and restoration, *J. Vis. Commun. Image Represent.*, vol. 17, pp. 916–928, 2006.

[10] H. H. Bauschke and J. M. Borwein, On the convergence of von Neumann's alternating projection algorithm for two sets, *Set-Valued Anal.,* vol. 1, pp. 185–212, 1993.

[11] H. H. Bauschke and J. M. Borwein, Dykstra's alternating projection algorithm for two sets, *J. Approx. Theory*, vol. 79, pp. 418–443, 1994.

[12] H. H. Bauschke and J. M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Rev.*, vol. 38, pp. 367–426, 1996.

[13] H. H. Bauschke and P. L. Combettes, A Dykstra-like algorithm for two monotone operators, *Pac. J. Optim.*, vol. 4, pp. 383–391, 2008.

[14] H. H. Bauschke and P. L. Combettes, The Baillon-Haddad theorem revisited, *J. Convex Anal.*, vol. 17, pp. 781–787, 2010.

[15] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, 2011.

[16] H. H. Bauschke, P. L. Combettes, and D. R. Luke, Finding best approximation pairs relative to two closed convex sets in Hilbert spaces, *J. Approx. Theory*, vol. 127, pp. 178–192, 2004.

[17] H. H. Bauschke, P. L. Combettes, and S. Reich, The asymptotic behavior of the composition of two resolvents, *Nonlinear Anal.*, vol. 60, pp. 283–301, 2005.

[18] M. Beckmann, C. McGuire, and C. Winsten, *Studies in Economics of Transportation,* Yale University Press, New Haven, 1956.

[19] D. P. Bertsekas and E. M. Gafni, Projection methods for variational inequalities with application to the traffic assignment problem, *Math. Programming Stud.*, vol. 17, pp. 139–159, 1982.

[20] J. P. Boyle and R. L. Dykstra, A method for finding projections onto the intersection of convex sets in Hilbert spaces, *Lecture Notes in Statistics*, vol. 37, pp. 28–47, 1986.

[21] F. E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces, *Math. Z.*, vol. 100, pp. 201–225, 1967.

[22] R. E. Bruck Jr., A strongly convergent iterative solution of $0 \in U(x)$ for a maximal monotone operator U in Hilbert space, *J. Math. Anal. Appl.*, vol. 48, pp. 114–126, 1974.

[23] C. L. Byrne, *Signal Processing – A Mathematical Approach,* A. K. Peters, Wellesley, MA, 2005.

[24] P. L. Combettes, The foundations of set theoretic estimation, *Proc. IEEE*, vol. 81, pp 182–208, 1993.

[25] P. L. Combettes, Hilbertian convex feasibility problem : Convergence of projection methods, *Appl. Math. Optim.,* vol. 35, pp. 311–330, 1997.

[26] P. L. Combettes, Solving monotone inclusions via compositions of nonexpansive averaged operators, *Optimization*, vol. 53, pp. 475–504, 2004.

[27] P. L. Combettes, Iterative construction of the resolvent of a sum of maximal monotone operators, *J. Convex Anal.*, vol. 16, pp. 727–748, 2009.

[28] P. L. Combettes and P. Bondon, Hard-constrained inconsistent signal feasibility problems, *IEEE Trans. Signal Process.,* vol. 47, pp. 2460–2468, 1999.

[29] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.*, vol. 6, pp. 117–136, 2005.

[30] P. L. Combettes and J.-C. Pesquet, Proximal splitting methods in signal processing, in : H. H. Bauschke, R. S. Burachik, P. L. Combettes, V. Elser, D. R. Luke, and H. Wolkowicz (Eds.), *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, pp. 185–212, Springer-Verlag, New York, 2011.

[31] P. L. Combettes and V. R. Wajs, Signal recovery by proximal forward-backward splitting, *Multiscale Model. Simul.*, vol. 4, pp. 1168–1200, 2005.

[32] S. C. Dafermos, Traffic equilibrium and variational inequalities, *Transportation Sci.*, vol. 14, pp. 42–54, 1980.

[33] S. C. Dafermos and S. C. McKelvey, Partitionable variational inequalities with applications to network and economic equilibria, *J. Optim. Theory Appl.*, vol. 73, pp. 243–268, 1992.

[34] F. Deutsch, *Best Approximation in Inner Product Spaces*, Springer-Verlag, New York, 2001.

[35] R. Dykstra, An algorithm for restricted least squares regression, *J. Amer. Statist. Assoc.*, vol. 78, pp. 837–842, 1983.

[36] J. Eckstein and D. Bertsekas, On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators, *Math. Programming*, vol. 55, pp. 293–318, 1992.

[37] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems I/II*, Springer-Verlag, New York, 2003.

[38] M. Fortin and R. Glowinski (eds.), *Augmented Lagrangian Methods : Applications to the Numerical Solution of Boundary-Value Problems*, North-Holland, Amsterdam, 1983.

[39] M. Fukushima, The primal Douglas-Rachford splitting algorithm for a class of monotone mappings with applications to the traffic equilibrium problem, *Math. Programming,* vol. 72, pp. 1–15, 1996.

[40] M. Goldburg and R. J. Marks II, Signal synthesis in the presence of an inconsistent set of constraints, *IEEE Trans. Circuits and Systems*, vol. 32, pp. 647–663, 1985.

[41] R. Glowinski and P. L. Tallec (eds.), *Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics*, SIAM, Philadelphia, 1989.

[42] A. Haraux, *Nonlinear Evolution Equations : Global Behavior of Solutions*, Lecture Notes in Math., vol. 841, Springer-Verlag, New York, 1981.

[43] B. S. He, A new method for a class of linear variational inequalities, *Math. Programming*, vol. 66, pp. 137–144, 1994.

[44] S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, vol. 44, pp. 147–150, 1974.

[45] A. N. Iusem, An iterative algorithm for the variational inequality problem, *Mat. Apl. Comput.*, vol. 13, pp. 103–114, 1994.

[46] A. N. Iusem and B. F. Svaiter, A variant of Korpelevich's method for variational inequalities with a new search strategy, *Optimization*, vol. 42, pp. 309–321, 1997.

[47] A. Jofré, R. T. Rockafellar, and R. J.-B. Wets, Variational inequalities and economic equilibrium, *Math. Oper. Res.*, vol. 32, pp. 32–50, 2007.

[48] M. Kang, Generalized multichannel image deconvolution approach and its applications, *Opt. Eng.*, vol. 37, pp. 2953–2964, 1998.

[49] E. N. Khobotov, A modification of the extragradient method for solving variational inequalities and some optimization problems, *Comput. Math. Math. Phys.*, vol. 27, pp. 1462–1473, 1987.

[50] G. M. Korpelevič, An extragradient method for finding saddle points and for other problems, *Èkonom. i Mat. Metody*, vol. 12, pp. 747–756, 1976.

[51] J.-L. Lions and G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.*, vol. 20, pp. 493–519, 1967.

[52] J.-L. Lions and R. Temam, Une méthode d'éclatement des opérateurs et des contraintes en calcul des variations, *C. R. Acad. Sci. Paris Sér. A,* vol. 263, pp. 563–565, 1966.

[53] J.-L. Lions and R. Temam, Éclatement et décentralisation en calcul des variations, *Lecture Notes in Mathematics,* vol. 132, pp. 196–217, 1970.

[54] P.-L. Lions and B. Mercier, Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Numer. Anal.*, vol. 16, pp. 964–979, 1979.

[55] D. Monderer and L. S. Shapley, Potential games, *Games Econom. Behav.*, vol. 14, pp. 124–143, 1996.

[56] A. Moudafi, Mixed equilibrium problems : sensitivity analysis and algorithmic aspect, *Comput. Math. Appl.*, vol. 44, pp. 1099–1108, 2002.

[57] A. Nagurney, *Network Economics – A Variational Inequality Approach*, Kluwer Academic Publishers Group, Dordrecht, 1993.

[58] J. von Neumann, *Functional Operators. II. The Geometry of Orthogonal Spaces*, Annals of Mathematics Studies no. 22, Princeton University Press, Princeton, NJ, 1950.

[59] J.-S. Pang, Asymmetric variational inequality problems over product sets – applications and iterative methods, *Math. Programming*, vol. 31, pp. 206–219, 1985.

[60] M. Patriksson, *The Traffic Assignment Problem : Models and Methods,* Utrecht, The Netherlands, 1994.

[61] G. Pierra, Éclatement de contraintes en parallèle pour la minimisation d'une forme quadratique, *Lecture Notes in Computer Science,* vol. 41, pp. 200–218, 1976.

[62] A. Quarteroni and A. Valli, *Domain Decomposition Methods for Partial Differential Equations,* Oxford University Press, New York, 1999.

[63] R. T. Rockafellar, Monotone operators associated with saddle-functions and minimax problems, in : F. E. Browder (ed.), *Nonlinear Functional Analysis, Part 1*, Proc. Sympos. Pure Math., vol. 18, pp. 241–250, Amer. Math. Soc., Providence, RI, 1970.

[64] R. T. Rockafellar, Local boundedness of nonlinear monotone operators, *Michigan Math. J.*, vol. 16, pp. 397–407, 1969.

[65] R. T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM J. Control Optimization*, vol. 14, pp. 877–898, 1976.

[66] R. T. Rockafellar, *Network Flows and Monotropic Optimization,* Wiley, New York, 1984.

[67] J. Schu, Approximating fixed points of Lipschitzian pseudocontractive mappings, *Houston J. Math.*, vol. 19, pp. 107–115, 1993.

[68] Y. Sheffi, *Urban Transportation Networks : Equilibrium Analysis with Mathematical Programming Methods,* Prentice-Hall, Englewood Cliffs, NJ, 1985.

[69] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Mathematical Surveys and Monographs, vol. 49, Amer. Math. Soc., Providence, RI, 1997.

[70] B. F. Smith, P. E. Bjørstad, and W. D. Gropp, *Domain Decomposition – Parallel Multilevel Methods for Elliptic Partial Differential Equations*, Cambridge University Press, Cambridge, 1996.

[71] M. V. Solodov and B. F. Svaiter, A hybrid approximate extragradient-proximal point algorithm using the enlargement of a maximal monotone operator, *Set-Valued Anal.*, vol. 7, pp. 323–345, 1999.

[72] J. E. Spingarn, Partial inverse of a monotone operator, *Appl. Math. Optim.*, vol. 10, pp. 247–265, 1983.

[73] B. F. Svaiter, On weak convergence of the Douglas-Rachford method, *SIAM J. Control Optim.*, vol. 49, pp. 280–287, 2011.

[74] A. Toselli and O. Widlund, *Domain Decomposition Methods – Algorithms and Theory,* Springer-Verlag, Berlin, Germany, 2005.

[75] P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, *SIAM J. Control Optim.*, vol. 38, pp. 431–446, 2000.

[76] M. Voorneveld, Best-response potential games, *Econom. Lett.,* vol. 66, pp. 289–295, 2000.

[77] Y.-W. Wen, M. K. Ng, and W.-K. Ching, Iterative algorithms based on decoupling of deblurring and denoising for image restoration, *SIAM J. Sci. Comput.*, vol. 30, pp. 2655–2674, 2008.

[78] D. C. Youla and V. Velasco, Extensions of a result on the synthesis of signals in the presence of inconsistent constraints, *IEEE Trans. Circuits and Systems*, vol. 33, pp. 465–468, 1986.

[79] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/B– Nonlinear Monotone Operators*, Springer-Verlag, New York, 1990.

[80] H. Zhou, Convergence theorems of fixed points for Lipschitz pseudo-contractions in Hilbert spaces, *J. Math. Anal. Appl.*, vol. 343, pp. 546–556, 2008.

Chapitre 2

Résolution d'inclusions monotones couplées par un opérateur cocoercif

2.1 Description et résultats principaux

Dans ce chapitre nous nous intéressons au problème suivant.

Problème 2.1 Pour tout $i \in \{1,\ldots,m\}$, soit $A_i\colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$ un opérateur maximalement monotone et soit $B_i\colon \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \to \mathcal{H}_i$. Supposons qu'il existe $\beta \in \left]0,+\infty\right[$ tel que

$$(\forall (x_1,\ldots,x_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m)(\forall (y_1,\ldots,y_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m)$$
$$\sum_{i=1}^m \langle x_i - y_i \mid B_i(x_1,\ldots,x_m) - B_i(y_1,\ldots,y_m)\rangle \geq \beta \sum_{i=1}^m \big\|B_i(x_1,\ldots,x_m) - B_i(y_1,\ldots,y_m)\big\|^2. \quad (2.1)$$

Le problème est de

$$\text{trouver } x_1 \in \mathcal{H}_1,\ldots,x_m \in \mathcal{H}_m \text{ tels que } \begin{cases} 0 \in A_1x_1 + B_1(x_1,\ldots,x_m) \\ \quad\vdots \\ 0 \in A_mx_m + B_m(x_1,\ldots,x_m). \end{cases} \quad (2.2)$$

Ce problème correspond au cas particulier du Problème 1.1 où $r = 2$, $\boldsymbol{A}_1 = A_1 \times \cdots \times A_m$ et $\boldsymbol{A}_2\colon (x_i)_{1\leq i\leq m} \mapsto (B_i(x_1,\ldots,x_m))_{1\leq i\leq m}$. Nous remarquons que la condition (2.1) est équivalente à la propriété de β-cocoercivité de $\boldsymbol{A}_2$, ce qui implique qu'il est maximalement monotone.

Dans (2.2), pour tout $i \in \{1,\ldots,m\}$, l'opérateur A_i modélise des propriétés inhérentes de la variable x_i, tandis que les opérateurs $(B_i)_{1\leq i\leq m}$ modélisent les interactions entre ces variables. Cette structure du Problème 2.1 permet la modélisation d'une grande classe de problèmes, incluant en autres des problèmes de théorie des jeux [3, 4, 37], de théorie du trafic [14, 37], de meilleure approximation [8, 11, 18, 29], d'inclusions d'évolution [5, 30, 42].

Pour résoudre le Problème 2.1 nous proposons l'algorithme suivant.

Algorithme 2.2 On génère des suites $(x_{1,n})_{n\in\mathbb{N}}$,. . .,$(x_{m,n})_{n\in\mathbb{N}}$ comme suit.

$$
\begin{array}{l}
\text{Initialisation}\\
\left\lfloor
\begin{array}{l}
\varepsilon \in \left]0, \min\{1,\beta\}\right[\\
\text{Pour } i = 1,\ldots,m\\
\lfloor\; x_{i,0} \in \mathcal{H}_i
\end{array}
\right.\\
\text{Pour } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\gamma_n \in [\varepsilon, 2\beta-\varepsilon]\\
\lambda_n \in [0, 1-\varepsilon]\\
\text{Pour } i = 1,\ldots,m\\
\left\lfloor
\begin{array}{l}
\lambda_{i,n} \in \left[0,1\right[\\
z_{i,n} = x_{i,n} - \gamma_n\big(B_{i,n}(x_{1,n},\ldots,x_{m,n}) + b_{i,n}\big)\\
y_{i,n} = J_{\gamma_n A_{i,n}} z_{i,n} + a_{1,n}\\
x_{i,n+1} = \lambda_{i,n} x_{i,n} + (1-\lambda_{i,n}) y_{i,n},
\end{array}
\right.
\end{array}
\right.
\end{array}
\tag{2.3}
$$

où, pour tout $i \in \{1,\ldots,m\}$, les conditions suivantes sont satisfaites.

(i) $(A_{i,n})_{n\in\mathbb{N}}$ sont des opérateurs maximalement monotones de $\mathcal{H}_i$ vers $2^{\mathcal{H}_i}$ tels que

$$(\forall \rho \in \left]0,+\infty\right[) \quad \sum_{n\in\mathbb{N}} \sup_{\|y\|\leq\rho} \|J_{\gamma_n A_{i,n}} y - J_{\gamma_n A_i} y\| < +\infty. \tag{2.4}$$

(ii) $(B_{i,n})_{n\in\mathbb{N}}$ sont des opérateurs de $\mathcal{H}_1 \times \cdots \times \mathcal{H}_m$ vers $\mathcal{H}_i$ tels que

 (a) les opérateurs $(B_{i,n} - B_i)_{n\in\mathbb{N}}$ sont lipschitziens avec des constantes respectives $(\kappa_{i,n})_{n\in\mathbb{N}}$ dans $\left]0,+\infty\right[$ qui satisfont $\sum_{n\in\mathbb{N}} \kappa_{i,n} < +\infty$;

 (b) il existe $\boldsymbol{z} \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m$, qui ne dépend pas de i, tel que $(\forall n \in \mathbb{N})\ B_{i,n}\boldsymbol{z} = B_i\boldsymbol{z}$.

(iii) $(a_{i,n})_{n\in\mathbb{N}}$ et $(b_{i,n})_{n\in\mathbb{N}}$ sont des suites dans $\mathcal{H}_i$ qui satisfont $\sum_{n\in\mathbb{N}} \|a_{i,n}\| < +\infty$ et $\sum_{n\in\mathbb{N}} \|b_{i,n}\| < +\infty$.

(iv) $(\lambda_{i,n})_{n\in\mathbb{N}}$ est une suite dans $\left[0,1\right[$ telle que $\sum_{n\in\mathbb{N}} |\lambda_{i,n} - \lambda_n| < +\infty$.

Pour tout $i \in \{1,\ldots,m\}$ et $n \in \mathbb{N}$, l'Algorithme 2.2 tolère des opérateurs perturbés $A_{i,n}$ et $B_{i,n}$ qui doivent être proches, au sens des conditions (i) et (ii), de A_i et B_i, respectivement. De plus, cette méthode tolère aussi des erreurs dans le calcul des opérateurs impliqués et une relaxation à chaque étape, sous les conditions (iii) et (iv), respectivement. Nous montrons le résultat de convergence suivant.

Théorème 2.3 *Soient $(x_{1,n})_{n\in\mathbb{N}}, \ldots, (x_{m,n})_{n\in\mathbb{N}}$ les suites générées par l'Algorithme 2.2 et supposons que le Problème 2.1 admette au moins une solution. Alors, pour tout $i \in \{1,\ldots,m\}$, il existe $x_i \in \mathcal{H}_i$ tel que $x_{i,n} \rightharpoonup x_i$. De plus, $(x_i)_{1\leq i\leq m}$ est une solution du Problème 2.1.*

De plus, nous montrons que la convergence forte dans le Théorème 2.3 est assurée lorsque, pour tout $i \in \{1,\ldots,m\}$, A_i est demirégulier dans $\mathcal{H}_i$, ou $(B_i)_{1\leq i\leq m}$ est demirégulier dans $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$, ou l'ensemble des solutions a un intérieur non vide.

Ensuite, nous illustrons des applications de (2.2) aux équations d'évolution couplées et aux problèmes de minimisation. Cette dernière permet de résoudre certains problèmes liés à la théorie du trafic et aux problèmes de meilleure approximation. En ce qui concerne les équations d'évolution, nous considérons le cas particulier de (2.2) où, pour tout $i \in \{1,\ldots,m\}$, H_i est un espace hilbertien réel, $\mathcal{H}_i = L^2([0,T],\mathsf{H}_i)$ et

$$A_i\colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$$
$$x \mapsto \begin{cases} \Big\{u\in\mathcal{H}_i \;\Big|\; u(t)\in x'(t)+\partial\mathsf{f}_i(x(t)) \;\text{ p.p. sur }]0,T[\Big\}, & \text{si } x\in\mathcal{W}_i;\\ \varnothing, & \text{sinon.}\end{cases} \tag{2.5}$$

On note

$$\mathcal{W}_i = \big\{x\in\mathcal{C}([0,T];\mathsf{H}_i)\cap W^{1,2}([0,T];\mathsf{H}_i) \;\mid\; x(T)=x(0)\big\}, \tag{2.6}$$

où $\mathcal{C}([0,T];\mathsf{H}_i)$ est la famille des fonctions continues de $[0,T]$ vers H_i et $W^{1,2}([0,T];\mathsf{H}_i)$ est la famille des fonctions de $[0,T]$ vers H_i de carré intégrable, dont la dérivée faible est une fonction de carré intégrable. De plus, si nous considérons, pour tout $i\in\{1,\ldots,m\}$ et $(x_1,\ldots,x_m)\in\mathcal{H}_1\times\cdots\times\mathcal{H}_m$,

$$B_i(x_1,\ldots,x_m)\colon [0,T]\to\mathsf{H}_i\colon t\mapsto\mathsf{B}_i(x_1(t),\ldots,x_m(t)), \tag{2.7}$$

où $\mathsf{B}_i\colon \mathsf{H}_1\times\cdots\times\mathsf{H}_m\to\mathsf{H}_i$ est un opérateur tel que les $(\mathsf{B}_i)_{1\leq i\leq m}$ satisfont (2.1) dans $\mathsf{H}_1\times\cdots\times\mathsf{H}_m$ avec $\beta\in\,]0,+\infty[$, le Problème 2.1 donne le système d'inclusions suivant.

Problème 2.4

Trouver $x_1\in\mathcal{W}_1,\ldots,x_m\in\mathcal{W}_m$ telles que, pour tout $i\in\{1,\ldots,m\}$,

$$0\in x_i'(t)+\partial\mathsf{f}_i(x_i(t))+\mathsf{B}_i(x_1(t),\ldots,x_m(t)) \;\text{ p.p. sur }]0,T[\,. \tag{2.8}$$

Dans ce problème nous résolvons m inclusions d'évolution du type $0\in x_i'(t)+\partial\mathsf{f}_i(x_i(t))$, avec $i\in\{1,\ldots,m\}$, en considérant des interactions entre les fonctions $(x_i)_{1\leq i\leq m}$ représentées par les opérateurs $(\mathsf{B}_i)_{1\leq i\leq m}$.

D'autre part, pour l'application aux problèmes de minimisation du Problème 2.1, nous considérons que, pour tout $i\in\{1,\ldots,m\}$,

$$A_i=\partial f_i \quad\text{et}\quad B_i\colon (x_i)_{1\leq i\leq m}\mapsto\sum_{k=1}^{p}L_{ki}^*\nabla\varphi_k\bigg(\sum_{j=1}^{m}L_{kj}x_j\bigg), \tag{2.9}$$

où $f_i \in \Gamma_0(\mathcal{H}_i)$ et, pour tout $k \in \{1,\ldots,p\}$, $\mathcal{G}_k$ est un espace hilbertien réel, $L_{ki} \in \mathcal{B}(\mathcal{H}_i,\mathcal{G}_k)$, $\tau_k \in \,]0,+\infty[$ et $\varphi_k \in \Gamma_0(\mathcal{G}_k)$ est différentiable avec gradient τ_k-lipschitzien. Dans ce cas, (2.2) se réduit au problème de minimisation suivant.

Problème 2.5

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\text{minimiser}} \quad \sum_{i=1}^m f_i(x_i) + \sum_{k=1}^p \varphi_k\bigg(\sum_{i=1}^m L_{ki}x_i\bigg). \tag{2.10}$$

Le premier terme de la fonction objectif dans (2.10) permet de représenter des caractéristiques individuelles de chaque variable x_i, alors que le deuxième terme représente les interactions qui existent parmi les variables.

En appliquant l'Algorithme 2.2 à ces cas particuliers nous tirons des algorithmes qui résolvent le Problèmes 2.4 et 2.5.

Enfin, nous présentons certaines applications du Problème 2.5 à la théorie du trafic et aux problèmes de meilleure approximation.

2.2 Article en anglais

A PARALLEL SPLITTING METHOD FOR COUPLED MONOTONE INCLUSIONS [1]

Abstract : A parallel splitting method is proposed for solving systems of coupled monotone inclusions in Hilbert spaces and its convergence is established under the assumption that solutions exist. Unlike existing alternating algorithms, which are limited to two variables and linear coupling, our parallel method can handle an arbitrary number of variables as well as nonlinear coupling schemes. The breadth and flexibility of the proposed framework is illustrated through applications in the areas of evolution inclusions, variational problems, best approximation, and network flows.

2.2.1 Problem statement

This paper is concerned with the numerical solution of systems of coupled monotone inclusions in Hilbert spaces. A simple instance of this problem is to

$$\text{find } x_1 \in \mathcal{H},\ x_2 \in \mathcal{H} \quad \text{such that} \quad \begin{cases} 0 \in A_1x_1 + x_1 - x_2 \\ 0 \in A_2x_2 - x_1 + x_2, \end{cases} \tag{2.11}$$

1. H. Attouch, L. M. Briceño-Arias, and P. L. Combettes, A parallel splitting method for coupled monotone inclusions, *SIAM Journal on Control and Optimization*, vol. 48, pp. 3246–3270, 2010.

where $(\mathcal{H}, \|\cdot\|)$ is a real Hilbert space, and where A_1 and A_2 are maximal monotone operators acting on $\mathcal{H}$. This formulation arises in various areas of nonlinear analysis [12]. For example, if $A_1 = \partial f_1$ and $A_2 = \partial f_2$ are the subdifferentials of proper lower semicontinuous convex functions f_1 and f_2 from $\mathcal{H}$ to $]-\infty, +\infty]$, (2.11) is equivalent to

$$\underset{x_1 \in \mathcal{H},\, x_2 \in \mathcal{H}}{\text{minimize}} \quad f_1(x_1) + f_2(x_2) + \frac{1}{2}\|x_1 - x_2\|^2. \tag{2.12}$$

This joint minimization problem, which was first investigated in [1], models problems in disciplines such as the cognitive sciences [4], image processing [26], and signal processing [28] (see also the references therein for further applications in mechanics, filter design, and dynamical games). In particular, if f_1 and f_2 are the indicator functions of closed convex subsets C_1 and C_2 of $\mathcal{H}$, (2.12) reduces to the classical best approximation pair problem [8, 11, 18, 29]

$$\underset{x_1 \in C_1,\, x_2 \in C_2}{\text{minimize}} \quad \|x_1 - x_2\|. \tag{2.13}$$

On the numerical side, a simple algorithm is available to solve (2.11), namely,

$$x_{1,0} \in \mathcal{H} \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \begin{cases} x_{2,n} & = (\mathrm{Id} + A_2)^{-1} x_{1,n} \\ x_{1,n+1} & = (\mathrm{Id} + A_1)^{-1} x_{2,n}. \end{cases} \tag{2.14}$$

This alternating resolvent method produces sequences $(x_{1,n})_{n\in\mathbb{N}}$ and $(x_{2,n})_{n\in\mathbb{N}}$ that converge weakly to points x_1 and x_2, respectively, such that (x_1, x_2) solves (2.11) if solutions exist [12, Theorem 3.3]. In [3], the variational formulation (2.12) was extended to

$$\underset{x_1 \in \mathcal{H}_1,\, x_2 \in \mathcal{H}_2}{\text{minimize}} \quad f_1(x_1) + f_2(x_2) + \frac{1}{2}\|L_1 x_1 - L_2 x_2\|_{\mathcal{G}}^2, \tag{2.15}$$

where $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{G}$ are Hilbert spaces, $f_1 \colon \mathcal{H}_1 \to]-\infty, +\infty]$ and $f_2 \colon \mathcal{H}_2 \to]-\infty, +\infty]$ are proper lower semicontinuous convex functions, and $L_1 \colon \mathcal{H}_1 \to \mathcal{G}$ and $L_2 \colon \mathcal{H}_2 \to \mathcal{G}$ are linear and bounded. This problem was solved in [3] via an inertial alternating minimization procedure first proposed in [4] for (2.12).

The above problems and their solution algorithms are limited to two variables which, in addition, must be linearly coupled. These are serious restrictions since models featuring more than two variables and/or nonlinear coupling schemes arise naturally in applications. The purpose of this paper is to address simultaneously these restrictions by proposing a parallel algorithm for solving systems of monotone inclusions involving an arbitrary number of variables and nonlinear coupling. The breadth and flexibility of this framework will be illustrated through applications in the areas of evolution inclusions, variational problems, best approximation, and network flows.

We now state our problem formulation and our standing assumptions.

Problem 2.6 Let $(\mathcal{H}_i)_{1\leq i\leq m}$ be real Hilbert spaces, where $m \geq 2$. For every $i \in \{1,\ldots,m\}$, let $A_i\colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be maximal monotone and let $B_i\colon \mathcal{H}_1 \times\cdots\times \mathcal{H}_m \to \mathcal{H}_i$. It is assumed that there exists $\beta \in \left]0,+\infty\right[$ such that

$$
\begin{aligned}
&(\forall(x_1,\ldots,x_m) \in \mathcal{H}_1\times\cdots\times\mathcal{H}_m)(\forall(y_1,\ldots,y_m) \in \mathcal{H}_1\times\cdots\times\mathcal{H}_m)\\
&\quad\sum_{i=1}^m \langle x_i - y_i \mid B_i(x_1,\ldots,x_m) - B_i(y_1,\ldots,y_m)\rangle\\
&\qquad\qquad\qquad \geq \beta\sum_{i=1}^m \big\|B_i(x_1,\ldots,x_m) - B_i(y_1,\ldots,y_m)\big\|^2. \qquad (2.16)
\end{aligned}
$$

The problem is to

$$
\text{find } x_1 \in \mathcal{H}_1,\ldots,x_m \in \mathcal{H}_m \quad \text{such that} \quad
\begin{cases}
0 \in A_1x_1 + B_1(x_1,\ldots,x_m)\\
\quad\vdots\\
0 \in A_mx_m + B_m(x_1,\ldots,x_m),
\end{cases} \qquad (2.17)
$$

under the assumption that such points exist.

In abstract terms, the system of inclusions in (2.17) models an equilibrium involving m variables in different Hilbert spaces. The ith inclusion in this system is a perturbation of the basic inclusion $0 \in A_ix_i$ by addition of the coupling term $B_i(x_1,\ldots,x_m)$. Our analysis captures various linear and nonlinear coupling schemes. If

$$
(\forall i \in \{1,\ldots,m\}) \quad \mathcal{H}_i = \mathcal{H} \quad \text{and} \quad (\forall x \in \mathcal{H}) \quad B_i(x,\ldots,x) = 0, \qquad (2.18)
$$

then Problem 2.6 is a relaxation of the standard problem [20, 33] of finding a common zero of the operators $(A_i)_{1\leq i\leq m}$, i.e., of solving the inclusion $0 \in \bigcap_{i=1}^m A_ix$. In particular, if $m = 2$, $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $B_1 = -B_2\colon (x_1,x_2) \mapsto x_1 - x_2$, and $\beta = 1/2$, then Problem 2.6 reverts to (2.11). On the other hand, if $m = 2$, $A_1 = \partial f_1$, $A_2 = \partial f_2$, $B_1\colon (x_1,x_2) \mapsto L_1^*(L_1x_1 - L_2x_2)$, $B_2\colon (x_1,x_2) \mapsto -L_2^*(L_1x_1 - L_2x_2)$, and $\beta = (\|L_1\|^2 + \|L_2\|^2)^{-1}$, then Problem 2.6 reverts to (2.15). Generally speaking, (2.17) covers coupled problems involving minimizations, variational inequalities, saddle points, or evolution inclusions, depending on the type of the maximal monotone operators $(A_i)_{1\leq i\leq m}$.

The paper is organized as follows. In Section 2.2.2, we present our algorithm for solving Problem 2.6 and prove its convergence. Applications to systems of evolution inclusions are treated in Section 2.2.3. Finally, Section 2.2.4 is devoted to variational formulations deriving from Problem 2.6 and features applications to best approximation and network flows.

Notation. Throughout, $\mathcal{H}$ and $(\mathcal{H}_i)_{1\leq i\leq m}$ are real Hilbert spaces. For convenience, their scalar products are all denoted by $\langle\cdot\mid\cdot\rangle$ and the associated norms by $\|\cdot\|$. The symbols $\rightharpoonup$ and $\to$ denote, respectively, weak and strong convergence, Id denotes the identity operator, and L^* denotes the adjoint of a bounded linear operator L. The indicator function of a subset C of $\mathcal{H}$ is

$$\iota_C\colon x \mapsto \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{if } x \notin C, \end{cases} \tag{2.19}$$

and the distance from $x \in \mathcal{H}$ to C is $d_C(x) = \inf_{y\in C} \|x-y\|$; if C is nonempty closed and convex, the projection of x onto C is the unique point $P_C x$ in C such that $\|x - P_C x\| = d_C(x)$. We denote by $\Gamma_0(\mathcal{H})$ the class of lower semicontinuous convex functions $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ which are proper in the sense that $\operatorname{dom} f = \big\{x \in \mathcal{H} \mid f(x) < +\infty\big\} \neq \varnothing$. The subdifferential of $f \in \Gamma_0(\mathcal{H})$ is the maximal monotone operator

$$\partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{u \in \mathcal{H} \mid (\forall y \in \mathcal{H}) \;\; \langle y - x \mid u\rangle + f(x) \leq f(y)\big\}. \tag{2.20}$$

We denote by $\operatorname{gr} A = \big\{(x,u) \in \mathcal{H}\times\mathcal{H} \mid u \in Ax\big\}$ the graph of a set-valued operator $A\colon \mathcal{H} \to 2^{\mathcal{H}}$, by $\operatorname{dom} A = \big\{x \in \mathcal{H} \mid Ax \neq \varnothing\big\}$ its domain, and by $J_A = (\mathrm{Id} + A)^{-1}$ its resolvent. If A is monotone, then J_A is single-valued and nonexpansive and, furthermore, if A is maximal monotone, then $\operatorname{dom} J_A = \mathcal{H}$. For complements and further background on convex analysis and monotone operator theory, see [5, 15, 44, 46, 48].

2.2.2 Algorithm

Let us start with a characterization of the solutions to Problem 2.6.

Proposition 2.7 *Let $(x_i)_{1\leq i\leq m} \in \mathcal{H}_1\times\cdots\times\mathcal{H}_m$, let $(\lambda_i)_{1\leq i\leq m} \in \left[0,1\right[^m$, and let $\gamma \in \left]0,+\infty\right[$. Then $(x_i)_{1\leq i\leq m}$ solves Problem 2.6 if and only if*

$$(\forall i \in \{1,\ldots,m\}) \quad x_i = \lambda_i x_i + (1-\lambda_i) J_{\gamma A_i}\big(x_i - \gamma B_i(x_1,\ldots,x_m)\big). \tag{2.21}$$

Proof. Let $i \in \{1,\ldots,m\}$. Then, since B_i is single-valued,

$$\begin{aligned} 0 \in A_i x_i + B_i(x_1,\ldots,x_m) \;&\Leftrightarrow\; x_i - \gamma B_i(x_1,\ldots,x_m) \in x_i + \gamma A_i x_i \\ &\Leftrightarrow\; x_i = J_{\gamma A_i}\big(x_i - \gamma B_i(x_1,\ldots,x_m)\big) \\ &\Leftrightarrow\; x_i = x_i + (1-\lambda_i)\big(J_{\gamma A_i}\big(x_i - \gamma B_i(x_1,\ldots,x_m)\big) - x_i\big), \end{aligned} \tag{2.22}$$

and we obtain (2.21). □

The above characterization suggests the following algorithm, which constructs m sequences $((x_{i,n})_{n\in\mathbb{N}})_{1\leq i\leq m}$. Recall that β is the constant appearing in (2.16).

Algorithm 2.8 Fix $\varepsilon \in \left]0, \min\{1,\beta\}\right[$, $(\gamma_n)_{n\in\mathbb{N}}$ in $[\varepsilon, 2\beta-\varepsilon]$, $(\lambda_n)_{n\in\mathbb{N}}$ in $[0,1-\varepsilon]$, and $(x_{i,0})_{1\leq i\leq m} \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m$. Set, for every $n \in \mathbb{N}$,

$$\begin{cases} x_{1,n+1} = \lambda_{1,n}x_{1,n} + (1-\lambda_{1,n})\Big(J_{\gamma_n A_{1,n}}\big(x_{1,n}-\gamma_n(B_{1,n}(x_{1,n},\ldots,x_{m,n})+b_{1,n})\big)+a_{1,n}\Big)\\ \quad\vdots\\ x_{m,n+1} = \lambda_{m,n}x_{m,n} + (1-\lambda_{m,n})\Big(J_{\gamma_n A_{m,n}}\big(x_{m,n}-\gamma_n(B_{m,n}(x_{1,n},\ldots,x_{m,n})+b_{m,n})\big)+a_{m,n}\Big), \end{cases} \tag{2.23}$$

where, for every $i \in \{1,\ldots,m\}$, the following hold.

(i) $(A_{i,n})_{n\in\mathbb{N}}$ are maximal monotone operators from $\mathcal{H}_i$ to $2^{\mathcal{H}_i}$ such that

$$(\forall \rho \in \left]0,+\infty\right[) \quad \sum_{n\in\mathbb{N}} \sup_{\|y\|\leq\rho} \|J_{\gamma_n A_{i,n}}y - J_{\gamma_n A_i}y\| < +\infty. \tag{2.24}$$

(ii) $(B_{i,n})_{n\in\mathbb{N}}$ are operators from $\mathcal{H}_1 \times \cdots \times \mathcal{H}_m$ to $\mathcal{H}_i$ such that

- (a) the operators $(B_{i,n}-B_i)_{n\in\mathbb{N}}$ are Lipschitz-continuous with respective constants $(\kappa_{i,n})_{n\in\mathbb{N}}$ in $\left]0,+\infty\right[$ satisfying $\sum_{n\in\mathbb{N}} \kappa_{i,n} < +\infty$; and
- (b) there exists $\boldsymbol{z} \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m$, independent of i, such that $(\forall n \in \mathbb{N})$ $B_{i,n}\boldsymbol{z} = B_i\boldsymbol{z}$.

(iii) $(a_{i,n})_{n\in\mathbb{N}}$ and $(b_{i,n})_{n\in\mathbb{N}}$ are sequences in $\mathcal{H}_i$ such that $\sum_{n\in\mathbb{N}} \|a_{i,n}\| < +\infty$ and $\sum_{n\in\mathbb{N}} \|b_{i,n}\| < +\infty$.

(iv) $(\lambda_{i,n})_{n\in\mathbb{N}}$ is a sequence in $\left[0,1\right[$ such that $\sum_{n\in\mathbb{N}} |\lambda_{i,n}-\lambda_n| < +\infty$.

Conditions (i) and (ii) describe the types of approximations to the original operators $(A_i)_{1\leq i\leq m}$ and $(B_i)_{1\leq i\leq m}$ which can be utilized. Condition (iii) quantifies the tolerance which is allowed in the implementation of these approximations (see [25, 31, 32] for specific examples), while (iv) quantifies that allowed in the departure from the global relaxation scheme. The parallel nature of Algorithm 2.8 stems from the fact that the m evaluations of the resolvent operators in (2.23) can be performed independently and, therefore, simultaneously on concurrent processors.

Our asymptotic analysis of Algorithm 2.8 will be based on Theorem 2.14 below on the convergence of the forward-backward algorithm. First, we need to introduce the notion of demiregularity. This notion captures various properties typically used to establish the strong convergence of dynamical systems, e.g., compactness [18], bounded compactness [8, 21, 22], uniform monotonicity [22, 24, 48], uniform convexity [26, 29, 34, 46], compactness of resolvents [30], and demicompactness [38, 47]. In the case of at most single-valued operators, demiregularity captures standard regularity properties used in nonlinear analysis [48, Definition 27.1].

Définition 2.9 An operator $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is demiregular at $y \in \operatorname{dom} A$ if, for every sequence $((y_n, v_n))_{n\in\mathbb{N}}$ in $\operatorname{gr} A$ and every $v \in Ay$, we have

$$\begin{cases} y_n \rightharpoonup y \\ v_n \to v \end{cases} \quad \Rightarrow \quad y_n \to y. \tag{2.25}$$

Proposition 2.10 *Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$, let $y \in \operatorname{dom} A$, and let $\mathcal{M}$ be the class of nondecreasing functions from $[0, +\infty[$ to $[0, +\infty]$ that vanish only at 0. Suppose that one of the following holds.*

(i) *A is uniformly monotone at y, i.e., there exists $\phi \in \mathcal{M}$ such that*

$$(\forall v \in Ay)(\forall (x, u) \in \operatorname{gr} A) \quad \langle x - y \mid u - v\rangle \geq \phi(\|x - y\|). \tag{2.26}$$

(ii) *A is uniformly monotone, i.e., there exists $\phi \in \mathcal{M}$ such that (2.26) holds for every $y \in \operatorname{dom} A$.*

(iii) *A is strongly monotone, i.e., there exists $\rho \in]0, +\infty[$ such that $A - \rho\operatorname{Id}$ is monotone.*

(iv) *$A = \partial f$, where $f \in \Gamma_0(\mathcal{H})$ is uniformly convex at y [46, Section 3.4], i.e., there exists $\phi \in \mathcal{M}$ such that*

$$\begin{aligned}&(\forall \alpha \in]0, 1[)(\forall x \in \operatorname{dom} f)\\ &\qquad f\big(\alpha x + (1 - \alpha)y\big) + \alpha(1 - \alpha)\phi(\|x - y\|) \leq \alpha f(x) + (1 - \alpha)f(y).\end{aligned} \tag{2.27}$$

(v) *$A = \partial f$, where $f \in \Gamma_0(\mathcal{H})$ is uniformly convex, i.e., there exists $\phi \in \mathcal{M}$ such that (2.27) holds for every $y \in \operatorname{dom} f$.*

(vi) *$A = \partial f$, where $f \in \Gamma_0(\mathcal{H})$ is strongly convex, i.e., there exists $\rho \in]0, +\infty[$ such that $f - \rho\|\cdot\|^2/2$ is convex.*

(vii) *$A = \partial f$, where $f \in \Gamma_0(\mathcal{H})$ and the lower level sets of f are boundedly compact.*

(viii) *J_A is compact, i.e., for every bounded set $C \subset \mathcal{H}$, the closure of $J_A(C)$ is compact.*

(ix) *$\operatorname{dom} A$ is boundedly relatively compact, i.e., the intersection of its closure with every closed ball is compact.*

(x) *$\mathcal{H}$ is finite-dimensional.*

(xi) *$A\colon \mathcal{H} \to \mathcal{H}$ is single-valued with a single-valued continuous inverse.*

(xii) *A is single-valued on $\operatorname{dom} A$ and $\operatorname{Id} - A$ demicompact [38], [47, Section 10.4], i.e., for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $\operatorname{dom} A$ such that $(Ax_n)_{n\in\mathbb{N}}$ converges strongly, $(x_n)_{n\in\mathbb{N}}$ admits a strong cluster point.*

Then A is demiregular at y.

Proof. Let $((y_n, v_n))_{n\in\mathbb{N}}$ be a sequence in $\operatorname{gr} A$ and let $v \in Ay$ be such that $y_n \rightharpoonup y$ and $v_n \to v$. We must show that $y_n \to y$.

(i) : By (2.26), there exists $\phi \in \mathcal{M}$ such that $(\forall n \in \mathbb{N})$ $\langle y_n - y \mid v_n - v\rangle \geq \phi(\|y_n - y\|)$. However, since $y_n \rightharpoonup y$ and $v_n \to v$, we have $\langle y_n - y \mid v_n - v\rangle \to 0$. Therefore, appealing to the properties of ϕ, we conclude that $\|y_n - y\| \to 0$.

(ii)⇒(i) : Clear.

(iii)⇒(ii) : Indeed, A is uniformly monotone with $\phi\colon t \mapsto \rho t^2$.

(iv)⇒(i) : See [46, Section 3.4].

(v)⇒(iv) : Clear.

(vi)⇒(v) : Indeed, f is uniformly convex with $\phi\colon t \mapsto \rho t^2/2$.

(vii) : Since $\langle y_n - y \mid v_n\rangle \to 0$, there exists $\rho \in \,]0, +\infty[$ such that

$$\sup_{n\in\mathbb{N}} \langle y_n - y \mid v_n\rangle \leq \rho. \tag{2.28}$$

Hence, since $y \in \operatorname{dom} \partial f \subset \operatorname{dom} f$, (2.20) yields

$$(\forall n \in \mathbb{N}) \quad f(y_n) \leq f(y) + \langle y_n - y \mid v_n\rangle \leq f(y) + \rho < +\infty, \tag{2.29}$$

which shows that $(y_n)_{n\in\mathbb{N}}$ lies in a lower level set of f. Since $(y_n)_{n\in\mathbb{N}}$ is bounded, it therefore lies in a compact set. However, since weak convergence and strong convergence coincide for sequences in compact sets, we conclude that $y_n \to y$.

(viii) : We have $(\forall n \in \mathbb{N})$ $(y_n, v_n) \in \operatorname{gr} A \Rightarrow (v_n + y_n) - y_n \in Ay_n \Rightarrow y_n = J_A(v_n + y_n)$. Since $(v_n + y_n)_{n\in\mathbb{N}}$ converge weakly, it lies in a bounded set C. Thus, $(y_n)_{n\in\mathbb{N}}$ lies in $J_A(C)$, which has compact closure. Hence $y_n \rightharpoonup y \Rightarrow y_n \to y$.

(ix)⇒(viii) : Let $C \subset \mathcal{H}$ be bounded. Then $J_A(C) \subset J_A(\mathcal{H}) = \operatorname{dom} A$ and, by nonexpansivity of J_A, $J_A(C)$ is bounded. Altogether, $J_A(C)$ has compact closure.

(x)⇒(ix) : Clear.

(xi) : Since $Ay_n = v_n \to v = Ay$, we have $y_n = A^{-1}v_n \to A^{-1}v = y$.

(xii) : Since $(y_n)_{n\in\mathbb{N}}$ converges weakly, it is bounded. In addition, $(Ay_n)_{n\in\mathbb{N}} = (v_n)_{n\in\mathbb{N}}$ converges strongly. Hence, by demicompactness of $\operatorname{Id} - A$, $(y_n)_{n\in\mathbb{N}}$ has a strong cluster point x and, since $y_n \rightharpoonup y$, we must have $x = y$. Now suppose that $y_n \not\to y$. Then, there exist $\varepsilon \in \,]0, +\infty[$ and a subsequence $(y_{k_n})_{n\in\mathbb{N}}$ such that

$$(\forall n \in \mathbb{N}) \quad \|y_{k_n} - y\| \geq \varepsilon. \tag{2.30}$$

However, since $y_{k_n} \rightharpoonup y$ and $(Ay_{k_n})_{n\in\mathbb{N}}$ converges strongly, arguing as above, we can extract a further subsequence $(y_{l_{k_n}})_{n\in\mathbb{N}}$ such that $y_{l_{k_n}} \to y$, which contradicts (2.30). Therefore, $y_n \to y$. □

Next, we recall the notion of cocoercivity.

Définition 2.11 Let $\chi \in \,]0, +\infty[$. An operator $B\colon \mathcal{H} \to \mathcal{H}$ is χ-cocoercive if χB is firmly nonexpansive, i.e.,

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x - y \mid Bx - By\rangle \geq \chi \|Bx - By\|^2. \tag{2.31}$$

Firmly nonexpansive operators include resolvents of maximal monotone operators, proximity operators, and projectors onto nonempty closed convex sets. In addition, the Yosida approximation of a maximal monotone operator of index χ is χ-cocoercive [2] (further examples of cocoercive operators can be found in [49]). It is clear from (2.31) that, if B is χ-cocoercive, then it is χ^{-1}-Lipschitz continuous. The next lemma, which provides a converse implication, supplies us with another important instance of cocoercive operator (see also [27]).

Lemma 2.12 [7, Corollaire 10] *Let $\varphi\colon \mathcal{H} \to \mathbb{R}$ be a differentiable convex function and let $\tau \in \,]0, +\infty[$. Suppose that $\nabla\varphi$ is τ-Lipschitz continuous. Then $\nabla\varphi$ is τ^{-1}-cocoercive.*

We shall also use the following fact.

Lemma 2.13 [22, Lemma 2.3] *Let $\chi \in \,]0, +\infty[$, let $B\colon \mathcal{H} \to \mathcal{H}$ be a χ-cocoercive operator, and let $\gamma \in \,]0, 2\chi[$. Then $\mathrm{Id} - \gamma B$ is nonexpansive.*

We are now ready to record some convergence properties of the forward-backward algorithm, which are of interest in their own right. The forward-backward algorithm finds its roots in the projected gradient method [34] and certain methods for solving variational inequalities [6, 16, 35, 43] (see also the bibliography of [22] for more recent developments).

Theorem 2.14 *Let $(\boldsymbol{\mathcal{H}}, |||\cdot|||)$ be a real Hilbert space, let $\chi \in \,]0, +\infty[$, let $\boldsymbol{A}\colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}$ be a maximal monotone operator, and let $\boldsymbol{B}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}$ be a χ-cocoercive operator such that*

$$\boldsymbol{Z} = (\boldsymbol{A} + \boldsymbol{B})^{-1}(\boldsymbol{0}) \neq \varnothing. \tag{2.32}$$

Fix $\varepsilon \in \,]0, \min\{1, \chi\}[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 2\chi - \varepsilon]$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[0, 1-\varepsilon]$, and let $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$ be sequences in $\boldsymbol{\mathcal{H}}$ such that $\sum_{n\in\mathbb{N}} |||\boldsymbol{a}_n||| < +\infty$ and $\sum_{n\in\mathbb{N}} |||\boldsymbol{b}_n||| < +\infty$. Finally, fix $\boldsymbol{x}_0 \in \boldsymbol{\mathcal{H}}$ and, for every $n \in \mathbb{N}$, set

$$\boldsymbol{x}_{n+1} = \lambda_n \boldsymbol{x}_n + (1 - \lambda_n)\big(J_{\gamma_n \boldsymbol{A}}(\boldsymbol{x}_n - \gamma_n(\boldsymbol{B}\boldsymbol{x}_n + \boldsymbol{b}_n)) + \boldsymbol{a}_n\big). \tag{2.33}$$

Then the following hold for some $\boldsymbol{x} \in \boldsymbol{Z}$.

(i) $\boldsymbol{x}_n \rightharpoonup \boldsymbol{x}$.

(ii) $\boldsymbol{B}\boldsymbol{x}_n \to \boldsymbol{B}\boldsymbol{x}$.

(iii) $\boldsymbol{x}_n - J_{\gamma_n \boldsymbol{A}}(\boldsymbol{x}_n - \gamma_n \boldsymbol{B}\boldsymbol{x}_n) \to \boldsymbol{0}$.

(iv) *Suppose that one of the following is satisfied.*

(a) $\boldsymbol{A}$ *is demiregular at* $\boldsymbol{x}$ *(see Proposition 2.10 for special cases).*

(b) $\boldsymbol{B}$ *is demiregular at* $\boldsymbol{x}$ *(see Proposition 2.10 for special cases).*

(c) $\operatorname{int} \boldsymbol{Z} \neq \varnothing$.

Then $\boldsymbol{x}_n \to \boldsymbol{x}$.

Proof. For every $n \in \mathbb{N}$, set

$$\boldsymbol{T}_{1,n} = J_{\gamma_n \boldsymbol{A}},\ \boldsymbol{T}_{2,n} = \mathbf{Id} - \gamma_n \boldsymbol{B},$$
$$\boldsymbol{e}_{1,n} = \boldsymbol{a}_n,\ \boldsymbol{e}_{2,n} = -\gamma_n \boldsymbol{b}_n,\ \mu_n = 1 - \lambda_n,\ \beta_{1,n} = 2,\ \text{and}\ \beta_{2,n} = \frac{2\chi}{\gamma_n}. \tag{2.34}$$

Then $\sum_{n\in\mathbb{N}} \mu_n |||\boldsymbol{e}_{1,n}||| < +\infty$, $\sum_{n\in\mathbb{N}} \mu_n |||\boldsymbol{e}_{2,n}||| < +\infty$, and, by [22, Equation (6.5)], $\boldsymbol{Z} = \bigcap_{n\in\mathbb{N}} \operatorname{Fix} \boldsymbol{T}_{1,n}\boldsymbol{T}_{2,n}$. Moreover, as seen in [22, Section 6], $(1-\beta_{1,n})\mathbf{Id} + \beta_{1,n}\boldsymbol{T}_{1,n}$ and $(1-\beta_{2,n})\mathbf{Id} + \beta_{2,n}\boldsymbol{T}_{2,n}$ are nonexpansive, and (2.33) can be rewritten as

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \mu_n\big(\boldsymbol{T}_{1,n}\big(\boldsymbol{T}_{2,n}\boldsymbol{x}_n + \boldsymbol{e}_{2,n}\big) + \boldsymbol{e}_{1,n} - \boldsymbol{x}_n\big), \tag{2.35}$$

which is precisely the iteration governing [22, Algorithm 1.2], where $m = 2$.

(i) : [22, Corollary 6.5].

(ii)&(iii) : We derive from (2.35), [22, Remark 3.4], and our assumptions on $(\lambda_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{N}}$ that $(\mathbf{Id}-\boldsymbol{T}_{2,n})\boldsymbol{x}_n-(\mathbf{Id}-\boldsymbol{T}_{2,n})\boldsymbol{x} \to \mathbf{0}$ and, in turn, that $\boldsymbol{B}\boldsymbol{x}_n \to \boldsymbol{B}\boldsymbol{x}$. Likewise, [22, Remark 3.4] yields $\boldsymbol{x}_n-\boldsymbol{T}_{1,n}\boldsymbol{T}_{2,n}\boldsymbol{x}_n \to \mathbf{0}$ and, therefore, $\boldsymbol{x}_n-J_{\gamma_n\boldsymbol{A}}(\boldsymbol{x}_n-\gamma_n\boldsymbol{B}\boldsymbol{x}_n) \to \mathbf{0}$.

(iv)(a) : Set $\boldsymbol{v} = -\boldsymbol{B}\boldsymbol{x}$ and

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \boldsymbol{y}_n = J_{\gamma_n\boldsymbol{A}}\big(\boldsymbol{x}_n - \gamma_n\boldsymbol{B}\boldsymbol{x}_n\big) \\ \boldsymbol{v}_n = \gamma_n^{-1}(\boldsymbol{x}_n - \boldsymbol{y}_n) - \boldsymbol{B}\boldsymbol{x}_n. \end{cases} \tag{2.36}$$

On the one hand, we have $\boldsymbol{v} = -\boldsymbol{B}\boldsymbol{x} \in \boldsymbol{A}\boldsymbol{x}$ and $(\forall n \in \mathbb{N})$ $(\boldsymbol{y}_n, \boldsymbol{v}_n) \in \operatorname{gr}\boldsymbol{A}$. On the other hand, we derive from (i) and (iii) that $\boldsymbol{y}_n \rightharpoonup \boldsymbol{x}$. Furthermore, since

$$(\forall n \in \mathbb{N}) \ \ |||\boldsymbol{v}_n - \boldsymbol{v}||| \leq \frac{|||\boldsymbol{x}_n - \boldsymbol{y}_n|||}{\gamma_n} + |||\boldsymbol{B}\boldsymbol{x}_n - \boldsymbol{B}\boldsymbol{x}|||, \tag{2.37}$$

it follows from (ii), (iii), and the condition $\inf_{n\in\mathbb{N}} \gamma_n > 0$ that $\boldsymbol{v}_n \to \boldsymbol{v}$. It then results from Definition 2.9 that $\boldsymbol{y}_n \to \boldsymbol{x}$ and, in turn, from (iii) that $\boldsymbol{x}_n \to \boldsymbol{x}$.

(iv)(b) : Set $\boldsymbol{v} = \boldsymbol{B}\boldsymbol{x}$ and $(\forall n \in \mathbb{N})$ $\boldsymbol{v}_n = \boldsymbol{B}\boldsymbol{x}_n$. Then (i) yields $\boldsymbol{x}_n \rightharpoonup \boldsymbol{x}$ and (ii) yields $\boldsymbol{v}_n \to \boldsymbol{v}$. It thus follows from Definition 2.9 that $\boldsymbol{x}_n \to \boldsymbol{x}$.

(iv)(c) : This follows from (i) and [22, Theorem 3.3(i) & Lemma 2.8(iv)]. ☐

The main results of this section are the following theorems. Let us start with weak convergence.

Theorem 2.15 *Let $((x_{i,n})_{n\in\mathbb{N}})_{1\leq i\leq m}$ be sequences generated by Algorithm 2.8. Then, for every $i \in \{1,\ldots,m\}$, $(x_{i,n})_{n\in\mathbb{N}}$ converges weakly to a point $x_i \in \mathcal{H}_i$, and $(x_i)_{1\leq i\leq m}$ is a solution to Problem 2.6.*

Proof. Throughout the proof, a generic element $\boldsymbol{x}$ in the Cartesian product $\mathcal{H}_1\times\cdots\times\mathcal{H}_m$ will be expressed in terms of its components as $\boldsymbol{x} = (x_i)_{1\leq i\leq m}$. We shall show that our algorithmic setting reduces to the situation described in Theorem 2.14(i) in the Hilbert direct sum $\boldsymbol{\mathcal{H}} = \mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_m$ obtained by endowing $\mathcal{H}_1\times\cdots\times\mathcal{H}_m$ with the scalar product

$$\langle\langle\cdot \mid \cdot\rangle\rangle\colon (\boldsymbol{x},\boldsymbol{y})\mapsto\sum_{i=1}^m\langle x_i \mid y_i\rangle\,, \tag{2.38}$$

with associated norm

$$|||\cdot|||\colon \boldsymbol{x}\mapsto\sqrt{\sum_{i=1}^m\|x_i\|^2}. \tag{2.39}$$

To this end, we shall show that the iterations (2.23) can be cast in the form of (2.33). First, define

$$\boldsymbol{A}\colon\boldsymbol{\mathcal{H}}\to 2^{\boldsymbol{\mathcal{H}}}\colon\boldsymbol{x}\mapsto\bigtimes_{i=1}^m A_ix_i\quad\text{and}\quad(\forall n\in\mathbb{N})\quad\boldsymbol{A}_n\colon\boldsymbol{\mathcal{H}}\to 2^{\boldsymbol{\mathcal{H}}}\colon\boldsymbol{x}\mapsto\bigtimes_{i=1}^m A_{i,n}x_i. \tag{2.40}$$

It follows from the maximal monotonicity of the operators $(A_i)_{1\leq i\leq m}$, condition (i) in Algorithm 2.8, (2.38), and (2.40) that

$$\boldsymbol{A}\text{ and }(\boldsymbol{A}_n)_{n\in\mathbb{N}}\text{ are maximal monotone,} \tag{2.41}$$

with resolvents

$$J_{\boldsymbol{A}}\colon\boldsymbol{\mathcal{H}}\to\boldsymbol{\mathcal{H}}\colon\boldsymbol{x}\mapsto(J_{A_i}x_i)_{1\leq i\leq m}\text{ and }(\forall n\in\mathbb{N})\ J_{\boldsymbol{A}_n}\colon\boldsymbol{\mathcal{H}}\to\boldsymbol{\mathcal{H}}\colon\boldsymbol{x}\mapsto(J_{A_{i,n}}x_i)_{1\leq i\leq m}, \tag{2.42}$$

respectively. Moreover, for every $\rho\in\,]0,+\infty[$, we derive from (2.39), (2.42), and condition (i) in Algorithm 2.8 that

$$\begin{aligned}\sum_{n\in\mathbb{N}}\sup_{|||\boldsymbol{y}|||\leq\rho}|||J_{\gamma_n\boldsymbol{A}_n}\boldsymbol{y}-J_{\gamma_n\boldsymbol{A}}\boldsymbol{y}||| &= \sum_{n\in\mathbb{N}}\sup_{|||\boldsymbol{y}|||\leq\rho}\sqrt{\sum_{i=1}^m\|J_{\gamma_nA_{i,n}}y_i-J_{\gamma_nA_i}y_i\|^2}\\ &\leq\sum_{n\in\mathbb{N}}\sup_{|||\boldsymbol{y}|||\leq\rho}\sum_{i=1}^m\|J_{\gamma_nA_{i,n}}y_i-J_{\gamma_nA_i}y_i\|\\ &\leq\sum_{i=1}^m\sum_{n\in\mathbb{N}}\sup_{\|y_i\|\leq\rho}\|J_{\gamma_nA_{i,n}}y_i-J_{\gamma_nA_i}y_i\|\\ &<+\infty.\end{aligned} \tag{2.43}$$

Now define

$$\boldsymbol{B}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{x} \mapsto (B_i\boldsymbol{x})_{1\leq i\leq m} \text{ and } (\forall n\in\mathbb{N})\ \boldsymbol{B}_n\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{x} \mapsto (B_{i,n}\boldsymbol{x})_{1\leq i\leq m}. \tag{2.44}$$

Then (2.17) is equivalent to

$$\text{find } \boldsymbol{x} \in \boldsymbol{Z} = (\boldsymbol{A}+\boldsymbol{B})^{-1}(\boldsymbol{0}). \tag{2.45}$$

Moreover, in the light of (2.38), (2.39), and (2.44), (2.16) becomes

$$(\forall \boldsymbol{x}\in\boldsymbol{\mathcal{H}})(\forall \boldsymbol{y}\in\boldsymbol{\mathcal{H}})\quad \langle\langle \boldsymbol{x}-\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{x}-\boldsymbol{B}\boldsymbol{y}\rangle\rangle \geq \beta |||\boldsymbol{B}\boldsymbol{x}-\boldsymbol{B}\boldsymbol{y}|||^2. \tag{2.46}$$

In other words, $\boldsymbol{B}$ is β-cocoercive. Next, let $n\in\mathbb{N}$ and set

$$\boldsymbol{c}_n = (a_{i,n})_{1\leq i\leq m} \quad\text{and}\quad \boldsymbol{d}_n = (b_{i,n})_{1\leq i\leq m}. \tag{2.47}$$

We deduce from (2.39) and condition (iii) in Algorithm 2.8 that

$$\sum_{k\in\mathbb{N}} |||\boldsymbol{c}_k||| \leq \sum_{k\in\mathbb{N}} \sqrt{\sum_{i=1}^m \|a_{i,k}\|^2} \leq \sum_{i=1}^m \sum_{k\in\mathbb{N}} \|a_{i,k}\| < +\infty \tag{2.48}$$

and, likewise, that

$$\sum_{k\in\mathbb{N}} |||\boldsymbol{d}_k||| < +\infty. \tag{2.49}$$

Now set

$$\boldsymbol{x}_n = (x_{i,n})_{1\leq i\leq m} \quad\text{and}\quad \boldsymbol{\Lambda}_n\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{x} \mapsto (\lambda_{i,n}x_i)_{1\leq i\leq m}. \tag{2.50}$$

It follows from (2.39) and condition (iv) in Algorithm 2.8 that

$$|||\boldsymbol{\Lambda}_n||| = \max_{1\leq i\leq m} \lambda_{i,n} \leq 1 \quad\text{and}\quad |||\mathbf{Id}-\boldsymbol{\Lambda}_n||| = 1 - \min_{1\leq i\leq m} \lambda_{i,n} \leq 1. \tag{2.51}$$

Hence,

$$|||\boldsymbol{\Lambda}_n||| + |||\mathbf{Id}-\boldsymbol{\Lambda}_n||| = 1 + \max_{1\leq i\leq m} (\lambda_{i,n}-\lambda_n) - \min_{1\leq i\leq m} (\lambda_{i,n}-\lambda_n) \leq 1+\tau_n, \tag{2.52}$$

where

$$\tau_n = 2\max_{1\leq i\leq m} |\lambda_{i,n}-\lambda_n|. \tag{2.53}$$

We observe that, by virtue of condition (iv) in Algorithm 2.8,

$$\sum_{k\in\mathbb{N}} \tau_k = 2\sum_{k\in\mathbb{N}} \max_{1\leq i\leq m} |\lambda_{i,k}-\lambda_k| \leq 2\sum_{i=1}^m \sum_{k\in\mathbb{N}} |\lambda_{i,k}-\lambda_k| < +\infty. \tag{2.54}$$

Moreover, in view of (2.42), (2.44), (2.47), and (2.50), the iterations (2.23) are equivalent to

$$\boldsymbol{x}_{n+1} = \boldsymbol{\Lambda}_n \boldsymbol{x}_n + (\mathbf{Id} - \boldsymbol{\Lambda}_n)\big(J_{\gamma_n \boldsymbol{A}_n}\big(\boldsymbol{x}_n - \gamma_n(\boldsymbol{B}_n \boldsymbol{x}_n + \boldsymbol{d}_n)\big) + \boldsymbol{c}_n\big). \tag{2.55}$$

Now define

$$\boldsymbol{D}_n = \boldsymbol{B}_n - \boldsymbol{B}. \tag{2.56}$$

It follows from condition (ii)(a) in Algorithm 2.8, (2.39), and (2.44) that $\boldsymbol{D}_n$ is Lipschitz continuous with constant $\kappa_n = \sqrt{\sum_{i=1}^m \kappa_{i,n}^2}$ and that

$$\sum_{k\in\mathbb{N}} \kappa_k = \sum_{k\in\mathbb{N}} \sqrt{\sum_{i=1}^m \kappa_{i,k}^2} \leq \sum_{i=1}^m \sum_{k\in\mathbb{N}} \kappa_{i,k} < +\infty. \tag{2.57}$$

Furthermore, set

$$\boldsymbol{b}_n = \boldsymbol{D}_n \boldsymbol{x}_n + \boldsymbol{d}_n \tag{2.58}$$

and let $\boldsymbol{x} \in \boldsymbol{Z}$. Then

$$\begin{aligned} |||\boldsymbol{b}_n||| &\leq |||\boldsymbol{D}_n \boldsymbol{x}_n||| + |||\boldsymbol{d}_n||| \\ &\leq |||\boldsymbol{D}_n \boldsymbol{x}_n - \boldsymbol{D}_n \boldsymbol{x}||| + |||\boldsymbol{D}_n \boldsymbol{x} - \boldsymbol{D}_n \boldsymbol{z}||| + |||\boldsymbol{d}_n||| \\ &\leq \kappa_n(|||\boldsymbol{x}_n - \boldsymbol{x}||| + |||\boldsymbol{x} - \boldsymbol{z}|||) + |||\boldsymbol{d}_n|||, \end{aligned} \tag{2.59}$$

where $\boldsymbol{z}$ is provided by assumption (ii)(b) in Algorithm 2.8. We now set

$$\boldsymbol{T}_n = \mathbf{Id} - \gamma_n \boldsymbol{B} \quad \text{and} \quad \boldsymbol{e}_n = J_{\gamma_n \boldsymbol{A}_n}\big(\boldsymbol{T}_n \boldsymbol{x}\big) - \boldsymbol{x}. \tag{2.60}$$

On the one hand, the inequality $\sup_{k\in\mathbb{N}} \gamma_k \leq 2\beta$ yields

$$|||\boldsymbol{T}_n \boldsymbol{x}||| \leq \rho, \quad \text{where} \quad \rho = |||\boldsymbol{x}||| + 2\beta|||\boldsymbol{B}\boldsymbol{x}|||. \tag{2.61}$$

On the other hand, since $\boldsymbol{x}$ is a solution to Problem 2.6, Proposition 2.7, (2.42), and (2.44) supply

$$\boldsymbol{x} = J_{\gamma_n \boldsymbol{A}}(\boldsymbol{T}_n \boldsymbol{x}). \tag{2.62}$$

Therefore, (2.60), (2.61), and (2.43) imply that

$$\sum_{k\in\mathbb{N}} |||\boldsymbol{e}_k||| = \sum_{k\in\mathbb{N}} |||J_{\gamma_k \boldsymbol{A}_k}(\boldsymbol{T}_k \boldsymbol{x}) - \boldsymbol{x}||| = \sum_{k\in\mathbb{N}} |||J_{\gamma_k \boldsymbol{A}_k}(\boldsymbol{T}_k \boldsymbol{x}) - J_{\gamma_k \boldsymbol{A}}(\boldsymbol{T}_k \boldsymbol{x})||| < +\infty. \tag{2.63}$$

In addition, (2.56), (2.58), and (2.60) yield

$$J_{\gamma_n \boldsymbol{A}_n}\big(\boldsymbol{x}_n - \gamma_n(\boldsymbol{B}_n \boldsymbol{x}_n + \boldsymbol{d}_n)\big) - \boldsymbol{x} = J_{\gamma_n \boldsymbol{A}_n}\big(\boldsymbol{T}_n \boldsymbol{x}_n - \gamma_n \boldsymbol{b}_n\big) - J_{\gamma_n \boldsymbol{A}_n}(\boldsymbol{T}_n \boldsymbol{x}) + \boldsymbol{e}_n. \tag{2.64}$$

Since $J_{\gamma_n \boldsymbol{A}}$ and, by Lemma 2.13, $\boldsymbol{T}_n$ are nonexpansive, we derive from (2.64) and (2.59) that

$$\begin{aligned} |||J_{\gamma_n \boldsymbol{A}_n}\big(\boldsymbol{x}_n - \gamma_n(\boldsymbol{B}_n\boldsymbol{x}_n + \boldsymbol{d}_n)\big) - \boldsymbol{x}||| &\leq |||J_{\gamma_n \boldsymbol{A}_n}\big(\boldsymbol{T}_n\boldsymbol{x}_n - \gamma_n\boldsymbol{b}_n\big) - J_{\gamma_n \boldsymbol{A}_n}(\boldsymbol{T}_n\boldsymbol{x})||| + |||\boldsymbol{e}_n||| \\ &\leq |||\boldsymbol{T}_n\boldsymbol{x}_n - \gamma_n\boldsymbol{b}_n - \boldsymbol{T}_n\boldsymbol{x}||| + |||\boldsymbol{e}_n||| \\ &\leq |||\boldsymbol{x}_n - \boldsymbol{x}||| + \gamma_n|||\boldsymbol{b}_n||| + |||\boldsymbol{e}_n||| \\ &\leq |||\boldsymbol{x}_n - \boldsymbol{x}||| + 2\beta|||\boldsymbol{b}_n||| + |||\boldsymbol{e}_n||| \\ &\leq (1 + 2\beta\kappa_n)|||\boldsymbol{x}_n - \boldsymbol{x}||| + 2\beta\kappa_n|||\boldsymbol{x} - \boldsymbol{z}||| \\ &\quad + 2\beta|||\boldsymbol{d}_n||| + |||\boldsymbol{e}_n|||. \end{aligned} \tag{2.65}$$

Thus, it results from (2.55), (2.65), (2.52), and (2.51) that

$$\begin{aligned} |||\boldsymbol{x}_{n+1} - \boldsymbol{x}||| &= |||\boldsymbol{\Lambda}_n(\boldsymbol{x}_n - \boldsymbol{x}) + (\mathbf{Id} - \boldsymbol{\Lambda}_n)\big(J_{\gamma_n \boldsymbol{A}_n}\big(\boldsymbol{x}_n - \gamma_n(\boldsymbol{B}_n\boldsymbol{x}_n + \boldsymbol{d}_n)\big) - \boldsymbol{x} + \boldsymbol{c}_n\big)||| \\ &\leq |||\boldsymbol{\Lambda}_n|||\,|||\boldsymbol{x}_n - \boldsymbol{x}||| + |||\mathbf{Id} - \boldsymbol{\Lambda}_n|||\,|||\boldsymbol{c}_n||| \\ &\quad + |||\mathbf{Id} - \boldsymbol{\Lambda}_n|||\,|||J_{\gamma_n \boldsymbol{A}_n}\big(\boldsymbol{x}_n - \gamma_n(\boldsymbol{B}_n\boldsymbol{x}_n + \boldsymbol{d}_n)\big) - \boldsymbol{x}||| \\ &\leq |||\boldsymbol{\Lambda}_n|||\,|||\boldsymbol{x}_n - \boldsymbol{x}||| + |||\mathbf{Id} - \boldsymbol{\Lambda}_n|||\,|||\boldsymbol{c}_n||| \\ &\quad + |||\mathbf{Id} - \boldsymbol{\Lambda}_n|||\,\big((1 + 2\beta\kappa_n)|||\boldsymbol{x}_n - \boldsymbol{x}||| + 2\beta\kappa_n|||\boldsymbol{x} - \boldsymbol{z}||| \\ &\quad + 2\beta|||\boldsymbol{d}_n||| + |||\boldsymbol{e}_n|||\big) \\ &\leq \big(|||\boldsymbol{\Lambda}_n||| + |||\mathbf{Id} - \boldsymbol{\Lambda}_n|||\big)|||\boldsymbol{x}_n - \boldsymbol{x}||| + |||\mathbf{Id} - \boldsymbol{\Lambda}_n|||\,\big(|||\boldsymbol{c}_n||| + 2\beta\kappa_n|||\boldsymbol{x}_n - \boldsymbol{x}||| \\ &\quad + 2\beta\kappa_n|||\boldsymbol{x} - \boldsymbol{z}||| + 2\beta|||\boldsymbol{d}_n||| + |||\boldsymbol{e}_n|||\big) \\ &\leq (1 + \tau_n)|||\boldsymbol{x}_n - \boldsymbol{x}||| + |||\boldsymbol{c}_n||| + 2\beta\kappa_n|||\boldsymbol{x}_n - \boldsymbol{x}||| \\ &\quad + 2\beta\kappa_n|||\boldsymbol{x} - \boldsymbol{z}||| + 2\beta|||\boldsymbol{d}_n||| + |||\boldsymbol{e}_n||| \\ &\leq (1 + \alpha_n)|||\boldsymbol{x}_n - \boldsymbol{x}||| + \delta_n, \end{aligned} \tag{2.66}$$

where

$$\alpha_n = \tau_n + 2\beta\kappa_n \quad \text{and} \quad \delta_n = |||\boldsymbol{c}_n||| + 2\beta\kappa_n|||\boldsymbol{x} - \boldsymbol{z}||| + 2\beta|||\boldsymbol{d}_n||| + |||\boldsymbol{e}_n|||. \tag{2.67}$$

In turn, it follows from (2.54), (2.57), (2.48), (2.49), and (2.63) that $\sum_{k\in\mathbb{N}} \alpha_k < +\infty$ and $\sum_{k\in\mathbb{N}} \delta_k < +\infty$. Thus, (2.66) and [39, Lemma 2.2.2] yield

$$\sup_{k\in\mathbb{N}} |||\boldsymbol{x}_k - \boldsymbol{x}||| < +\infty \tag{2.68}$$

and, using (2.57) and (2.49), we derive from (2.59) that

$$\sum_{k\in\mathbb{N}} |||\boldsymbol{b}_k||| < +\infty. \tag{2.69}$$

In view of (2.58), (2.56), and (2.60), (2.55) is equivalent to

$$\boldsymbol{x}_{n+1} = \boldsymbol{\Lambda}_n\boldsymbol{x}_n + (\mathbf{Id} - \boldsymbol{\Lambda}_n)\big(J_{\gamma_n \boldsymbol{A}}(\boldsymbol{T}_n\boldsymbol{x}_n - \gamma_n\boldsymbol{b}_n) + \boldsymbol{h}_n\big), \tag{2.70}$$

where

$$\boldsymbol{h}_n = J_{\gamma_n \boldsymbol{A}_n}(\boldsymbol{T}_n \boldsymbol{x}_n - \gamma_n \boldsymbol{b}_n) - J_{\gamma_n \boldsymbol{A}}(\boldsymbol{T}_n \boldsymbol{x}_n - \gamma_n \boldsymbol{b}_n) + \boldsymbol{c}_n. \tag{2.71}$$

Now set $\mu = \sup_{k\in\mathbb{N}} |||\boldsymbol{x}_k - \boldsymbol{x}||| + \rho + 2\beta \sup_{k\in\mathbb{N}} |||\boldsymbol{b}_k|||$. Then it follows from (2.68) and (2.69) that $\mu < +\infty$. Moreover, we deduce from the nonexpansivity of $\boldsymbol{T}_n$ and (2.61) that

$$\begin{aligned} |||\boldsymbol{T}_n \boldsymbol{x}_n - \gamma_n \boldsymbol{b}_n||| &\leq |||\boldsymbol{T}_n \boldsymbol{x}_n - \boldsymbol{T}_n \boldsymbol{x}||| + |||\boldsymbol{T}_n \boldsymbol{x}||| + 2\beta |||\boldsymbol{b}_n||| \\ &\leq |||\boldsymbol{x}_n - \boldsymbol{x}||| + \rho + 2\beta |||\boldsymbol{b}_n||| \\ &\leq \mu. \end{aligned} \tag{2.72}$$

Hence, appealing to (2.43) and (2.48), we infer from (2.71) that

$$\sum_{k\in\mathbb{N}} |||\boldsymbol{h}_k||| < +\infty. \tag{2.73}$$

Note that, upon introducing

$$\boldsymbol{a}_n = \boldsymbol{h}_n + \frac{1}{1-\lambda_n}(\boldsymbol{\Lambda}_n - \lambda_n \mathbf{Id})\big(\boldsymbol{x}_n - J_{\gamma_n \boldsymbol{A}}(\boldsymbol{T}_n \boldsymbol{x}_n - \gamma_n \boldsymbol{b}_n) - \boldsymbol{h}_n\big) \tag{2.74}$$

and using (2.60), we can rewrite (2.70) in the form of (2.33), namely,

$$\boldsymbol{x}_{n+1} = \lambda_n \boldsymbol{x}_n + (1-\lambda_n)\big(J_{\gamma_n \boldsymbol{A}}(\boldsymbol{x}_n - \gamma_n(\boldsymbol{B}\boldsymbol{x}_n + \boldsymbol{b}_n)) + \boldsymbol{a}_n\big). \tag{2.75}$$

On the other hand, using (2.62) and the nonexpansivity of $J_{\gamma_n \boldsymbol{A}}$ and $\boldsymbol{T}_n$, we get

$$\begin{aligned} |||\boldsymbol{x}_n - J_{\gamma_n \boldsymbol{A}}(\boldsymbol{T}_n \boldsymbol{x}_n - \gamma_n \boldsymbol{b}_n) - \boldsymbol{h}_n||| &\leq |||\boldsymbol{x}_n - \boldsymbol{x}||| + |||J_{\gamma_n \boldsymbol{A}}(\boldsymbol{T}_n \boldsymbol{x}) - J_{\gamma_n \boldsymbol{A}}(\boldsymbol{T}_n \boldsymbol{x}_n - \gamma_n \boldsymbol{b}_n)||| \\ &\quad + |||\boldsymbol{h}_n||| \\ &\leq 2|||\boldsymbol{x}_n - \boldsymbol{x}||| + 2\beta |||\boldsymbol{b}_n||| + |||\boldsymbol{h}_n|||. \end{aligned} \tag{2.76}$$

Therefore, we derive from (2.68), (2.69), and (2.73) that

$$\nu = \sup_{k\in\mathbb{N}} |||\boldsymbol{x}_k - J_{\gamma_k \boldsymbol{A}}(\boldsymbol{T}_k \boldsymbol{x}_k - \gamma_k \boldsymbol{b}_k) - \boldsymbol{h}_k||| < +\infty, \tag{2.77}$$

and hence, from (2.74), (2.50) and the inequality $\lambda_n \leq 1-\varepsilon$, that

$$\begin{aligned} |||\boldsymbol{a}_n||| &\leq |||\boldsymbol{h}_n||| + \frac{1}{1-\lambda_n} |||\boldsymbol{\Lambda}_n - \lambda_n \mathbf{Id}|||\ |||\boldsymbol{x}_n - J_{\gamma_n \boldsymbol{A}}(\boldsymbol{T}_n \boldsymbol{x}_n - \gamma_n \boldsymbol{b}_n) - \boldsymbol{h}_n||| \\ &\leq |||\boldsymbol{h}_n||| + \frac{\nu}{\varepsilon} \max_{1\leq i\leq m} |\lambda_{i,n} - \lambda_n|. \end{aligned} \tag{2.78}$$

Thus, using (2.73) and arguing as in (2.54), we get

$$\sum_{k\in\mathbb{N}} |||\boldsymbol{a}_k||| < +\infty. \tag{2.79}$$

However, Theorem 2.14(i) asserts that, with properties (2.41), (2.46), (2.69), (2.79), and under the hypotheses on $(\gamma_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ stated in Algorithm 2.8, the sequence $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ generated by (2.75) converges weakly to a point in $\boldsymbol{Z}$. Since (2.75) is equivalent to (2.23) and (2.45) is equivalent to (2.17), the proof is complete. □

We conclude this section with the following theorem, in which we describe instances of strong convergence derived from Theorem 2.14.

Theorem 2.16 *Let $((x_{i,n})_{n\in\mathbb{N}})_{1\leq i\leq m}$ and $(x_i)_{1\leq i\leq m}$ be as in Theorem 2.15. Then the following hold.*

(i) *Suppose that, for some $i\in\{1,\ldots,m\}$, A_i is demiregular at x_i (see Proposition 2.10 for special cases). Then $x_{i,n}\to x_i$.*

(ii) *Suppose that the operator $(y_j)_{1\leq j\leq m}\mapsto\big(B_i(y_j)_{1\leq j\leq m}\big)_{1\leq i\leq m}$ is demiregular at $(x_i)_{1\leq i\leq m}$ (see Proposition 2.10 for special cases). Then, for every $i\in\{1,\ldots,m\}$, $x_{i,n}\to x_i$.*

(iii) *Suppose that the set of solutions to Problem 2.6 has a nonempty interior. Then, for every $i\in\{1,\ldots,m\}$, $x_{i,n}\to x_i$.*

Proof. We use the same product space setting and notation as in the proof of Theorem 2.15. In particular, we set $\boldsymbol{x}=(x_1,\ldots,x_m)$ and $\boldsymbol{\mathcal{H}}=\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_m$, and we define

$$\boldsymbol{A}\colon\boldsymbol{\mathcal{H}}\to 2^{\boldsymbol{\mathcal{H}}}\colon\boldsymbol{y}\mapsto\bigtimes_{i=1}^m A_iy_i\quad\text{and}\quad\boldsymbol{B}\colon\boldsymbol{\mathcal{H}}\to\boldsymbol{\mathcal{H}}\colon\boldsymbol{y}\mapsto(B_i\boldsymbol{y})_{1\leq i\leq m}.\tag{2.80}$$

As seen in the proof of Theorem 2.15, the convergence properties of $(\boldsymbol{x}_n)_{n\in\mathbb{N}}=((x_{i,n})_{1\leq i\leq m})_{n\in\mathbb{N}}$ follow from those listed in Theorem 2.14 and applied to the operators defined in (2.80) ; moreover, the set of solutions to Problem 2.6 is $\boldsymbol{Z}=(\boldsymbol{A}+\boldsymbol{B})^{-1}(\boldsymbol{0})$.

(i) : Set $v_i=-B_i(x_1,\ldots,x_m)$ and

$$(\forall n\in\mathbb{N})\qquad\begin{cases}y_{i,n}=J_{\gamma_nA_i}\big(x_{i,n}-\gamma_nB_i(x_{1,n},\ldots,x_{m,n})\big)\\ v_{i,n}=\gamma_n^{-1}(x_{i,n}-y_{i,n})-B_i(x_{1,n},\ldots,x_{m,n}).\end{cases}\tag{2.81}$$

We first derive from (2.17) that

$$v_i=-B_i(x_1,\ldots,x_m)\in A_ix_i.\tag{2.82}$$

Moreover, it follows from Theorem 2.14(i) that

$$x_{i,n}\rightharpoonup x_i,\tag{2.83}$$

from Theorem 2.14(ii) that

$$\|B_i(x_{1,n},\ldots,x_{m,n})-B_i(x_1,\ldots,x_m)\|=\|B_i\boldsymbol{x}_n-B_i\boldsymbol{x}\|\leq|||\boldsymbol{B}\boldsymbol{x}_n-\boldsymbol{B}\boldsymbol{x}|||\to 0,\tag{2.84}$$

and from Theorem 2.14(iii) and (2.42) that

$$\|x_{i,n} - y_{i,n}\| \leq |||\boldsymbol{x}_n - J_{\gamma_n \boldsymbol{A}}(\boldsymbol{x}_n - \gamma_n \boldsymbol{B}\boldsymbol{x}_n)||| \to 0. \tag{2.85}$$

Combining (2.83) and (2.85), we obtain

$$y_{i,n} \rightharpoonup x_i. \tag{2.86}$$

Next, we derive from (2.81) that

$$(\forall n \in \mathbb{N}) \quad (y_{i,n}, v_{i,n}) \in \operatorname{gr} A_i \tag{2.87}$$

and that

$$(\forall n \in \mathbb{N}) \ \ \|v_{i,n} - v_i\| \leq \frac{\|x_{i,n} - y_{i,n}\|}{\gamma_n} + \|B_i(x_{1,n}, \ldots, x_{m,n}) - B_i(x_1, \ldots, x_m)\|. \tag{2.88}$$

Hence, it follows from (2.85), the condition $\inf_{n\in\mathbb{N}} \gamma_n > 0$, and (2.84), that

$$v_{i,n} \to v_i. \tag{2.89}$$

Altogether, (2.82), (2.86), (2.87), (2.89), and Definition 2.9 yield $y_{i,n} \to x_i$. In turn, appealing to (2.85), we conclude that $x_{i,n} \to x_i$.

(ii) : This follows Theorem 2.14(iv)(b).

(iii) : This follows Theorem 2.14(iv)(c). □

2.2.3 Coupling evolution inclusions

Evolution inclusions arise in various fields of applied mathematics [30, 42]. In this section, we address the problem of solving systems of coupled evolution inclusions with periodicity conditions.

Let us recall some standard notation [15, 48]. Fix $T \in \left]0, +\infty\right[$ and $p \in \left[1, +\infty\right[$. Then $\mathcal{D}(]0, T[)$ is the set of infinitely differentiable functions from $]0, T[$ to $\mathbb{R}$ with compact support in $]0, T[$. Given a real Hilbert space H, $\mathcal{C}([0, T]; \mathsf{H})$ is the space of continuous functions from $[0, T]$ to H and $L^p([0, T]; \mathsf{H})$ is the space of classes of equivalences of Borel measurable functions $x\colon [0, T] \to \mathsf{H}$ such that $\int_0^T \|x(t)\|_{\mathsf{H}}^p dt < +\infty$. $L^2([0, T]; \mathsf{H})$ is a Hilbert space with scalar product $(x, y) \mapsto \int_0^T \langle x(t) \mid y(t)\rangle_{\mathsf{H}}\, dt$. Now take x and y in $L^1([0, T]; \mathsf{H})$. Then y is the weak derivative of x if $\int_0^T \phi(t)y(t)dt = -\int_0^T (d\phi(t)/dt)x(t)dt$ for every $\phi \in \mathcal{D}(]0, T[)$, in which case we use the notation $y = x'$. Moreover,

$$W^{1,2}([0, T]; \mathsf{H}) = \big\{x \in L^2([0, T]; \mathsf{H}) \ \big|\ x' \in L^2([0, T]; \mathsf{H})\big\}, \tag{2.90}$$

equipped with the scalar product $(x, y) \mapsto \int_0^T \langle x(t) \mid y(t)\rangle_{\mathsf{H}}\, dt + \int_0^T \langle x'(t) \mid y'(t)\rangle_{\mathsf{H}}\, dt$, is a Hilbert space.

Problem 2.17 Let $(\mathsf{H}_i)_{1\leq i\leq m}$ be real Hilbert spaces and let $T \in \left]0,+\infty\right[$. For every $i \in \{1,\ldots,m\}$, set

$$\mathcal{W}_i = \big\{x \in \mathcal{C}([0,T];\mathsf{H}_i) \cap W^{1,2}([0,T];\mathsf{H}_i) \;\big|\; x(T) = x(0)\big\}, \tag{2.91}$$

let $\mathsf{f}_i \in \Gamma_0(\mathsf{H}_i)$, and let $\mathsf{B}_i \colon \mathsf{H}_1 \times \cdots \times \mathsf{H}_m \to \mathsf{H}_i$. It is assumed that there exists $\beta \in \left]0,+\infty\right[$ such that

$$\begin{aligned}(\forall(\mathsf{x}_1,\ldots,\mathsf{x}_m) \in \mathsf{H}_1 \times \cdots \times \mathsf{H}_m)(\forall(\mathsf{y}_1,\ldots,\mathsf{y}_m) \in \mathsf{H}_1 \times \cdots \times \mathsf{H}_m)\\ \sum_{i=1}^m \langle \mathsf{x}_i - \mathsf{y}_i \mid \mathsf{B}_i(\mathsf{x}_1,\ldots,\mathsf{x}_m) - \mathsf{B}_i(\mathsf{y}_1,\ldots,\mathsf{y}_m)\rangle_{\mathsf{H}_i}\\ \geq \beta \sum_{i=1}^m \big\|\mathsf{B}_i(\mathsf{x}_1,\ldots,\mathsf{x}_m) - \mathsf{B}_i(\mathsf{y}_1,\ldots,\mathsf{y}_m)\big\|^2_{\mathsf{H}_i}.\end{aligned} \tag{2.92}$$

The problem is to

$$\begin{aligned}&\text{find } x_1 \in \mathcal{W}_1,\ldots,x_m \in \mathcal{W}_m \text{ such that}\\ &\quad(\forall i \in \{1,\ldots,m\}) \quad 0 \in x_i'(t) + \partial\mathsf{f}_i(x_i(t)) + \mathsf{B}_i(x_1(t),\ldots,x_m(t)) \;\text{ a.e. on } \left]0,T\right[,\end{aligned} \tag{2.93}$$

under the assumption that such functions exist.

Algorithm 2.18 Fix $\varepsilon \in \left]0,\min\{1,\beta\}\right[$, $(\gamma_n)_{n\in\mathbb{N}}$ in $[\varepsilon, 2\beta-\varepsilon]$, and $(\lambda_n)_{n\in\mathbb{N}}$ in $[0,1-\varepsilon]$. Let, for every $n \in \mathbb{N}$ and every $i \in \{1,\ldots,m\}$, $y_{i,n}$ be the unique solution in $\mathcal{W}_i$ to the inclusion

$$\begin{aligned}\frac{x_{i,n}(t) - y_{i,n}(t)}{\gamma_n} - \big(\mathsf{B}_i(x_{1,n}(t),\ldots,x_{m,n}(t)) + b_{i,n}(t)\big)\\ \in y_{i,n}'(t) + \partial\mathsf{f}_i(y_{i,n}(t)) + e_{i,n}(t) \;\text{ a.e. on } \left]0,T\right[\end{aligned} \tag{2.94}$$

and set

$$x_{i,n+1} = \lambda_{i,n}x_{i,n} + (1-\lambda_{i,n})y_{i,n} \tag{2.95}$$

where, for every $i \in \{1,\ldots,m\}$, the following hold.

(i) $x_{i,0} \in W^{1,2}([0,T];\mathsf{H}_i)$.

(ii) $(b_{i,n})_{n\in\mathbb{N}}$ and $(e_{i,n})_{n\in\mathbb{N}}$ are sequences in $L^2([0,T];\mathsf{H}_i)$ such that

$$\sum_{n\in\mathbb{N}} \sqrt{\int_0^T \|b_{i,n}(t)\|^2_{\mathsf{H}_i} dt} < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} \sqrt{\int_0^T \|e_{i,n}(t)\|^2_{\mathsf{H}_i} dt} < +\infty. \tag{2.96}$$

(iii) $(\lambda_{i,n})_{n\in\mathbb{N}}$ is a sequence in $\left[0,1\right[$ such that $\sum_{n\in\mathbb{N}} |\lambda_{i,n} - \lambda_n| < +\infty$.

In (2.94), $b_{i,n}(t)$ models the error tolerated in computing $\mathsf{B}_i(x_{1,n}(t),\ldots,x_{m,n}(t))$, while $e_{i,n}(t)$ models the error tolerated in solving the inclusion with respect to $\partial\mathsf{f}_i(y_{i,n}(t))$.

We now examine the weak convergence properties of Algorithm 2.18 (strong convergence conditions can be derived from Theorem 2.16).

Theorem 2.19 *Let $((x_{i,n})_{n\in\mathbb{N}})_{1\leq i\leq m}$ be sequences generated by Algorithm 2.18. Then, for every $i\in\{1,\ldots,m\}$, $(x_{i,n})_{n\in\mathbb{N}}$ converges weakly in $W^{1,2}([0,T];\mathsf{H}_i)$ to a point $x_i\in\mathcal{W}_i$, and $(x_i)_{1\leq i\leq m}$ is a solution to Problem 2.17.*

Proof. For every $i\in\{1,\ldots,m\}$, set $\mathcal{H}_i=L^2([0,T];\mathsf{H}_i)$ and

$$A_i\colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$$
$$x \mapsto \begin{cases} \Big\{u\in\mathcal{H}_i \;\Big|\; u(t)\in x'(t)+\partial\mathsf{f}_i(x(t)) \text{ a.e. in }]0,T[\Big\}, & \text{if } x\in\mathcal{W}_i;\\ \varnothing, & \text{otherwise.}\end{cases} \tag{2.97}$$

Let us first show that the operators $(A_i)_{1\leq i\leq m}$ are maximal monotone. For this purpose, let $i\in\{1,\ldots,m\}$, and take $(x,u)\in\operatorname{gr}A_i$ and $(y,v)\in\operatorname{gr}A_i$. It follows from (2.97) that, almost everywhere on $]0,T[$, $u(t)-x'(t)\in\partial\mathsf{f}_i(x(t))$ and $v(t)-y'(t)\in\partial\mathsf{f}_i(y(t))$. Therefore, by monotonicity of $\partial\mathsf{f}_i$, we have

$$\int_0^T \big\langle x(t)-y(t)\mid \big(u(t)-x'(t)\big)-\big(v(t)-y'(t)\big)\big\rangle_{\mathsf{H}_i}\,dt\geq 0. \tag{2.98}$$

Hence,

$$\begin{aligned}\langle x-y\mid u-v\rangle &= \int_0^T\langle x(t)-y(t)\mid u(t)-v(t)\rangle_{\mathsf{H}_i}\,dt\\ &= \int_0^T\big\langle x(t)-y(t)\mid \big(u(t)-x'(t)\big)-\big(v(t)-y'(t)\big)\big\rangle_{\mathsf{H}_i}\,dt\\ &\quad+\int_0^T\langle x(t)-y(t)\mid x'(t)-y'(t)\rangle_{\mathsf{H}_i}\,dt\\ &\geq \frac{1}{2}\int_0^T\frac{d\,\|x(t)-y(t)\|_{\mathsf{H}_i}^2}{dt}dt\\ &= \frac{1}{2}\big(\|x(T)-y(T)\|_{\mathsf{H}_i}^2-\|x(0)-y(0)\|_{\mathsf{H}_i}^2\big)\\ &= 0.\end{aligned} \tag{2.99}$$

Thus, A_i is monotone. To prove maximality, set $\mathsf{g}_i=(1/2)\|\cdot\|_{\mathsf{H}_i}^2+\mathsf{f}_i$. Then $\mathsf{g}_i\in\Gamma_0(\mathsf{H}_i)$ and $\partial\mathsf{g}_i=\operatorname{Id}+\partial\mathsf{f}_i$. Moreover, since $\mathsf{f}_i\in\Gamma_0(\mathsf{H}_i)$, it follows from the Fenchel-Moreau theorem that it is minorized by a continuous affine functional, say $\mathsf{f}_i\geq\langle\cdot\mid\mathsf{v}\rangle_{\mathsf{H}_i}+\eta$ for some

$\mathsf{v} \in \mathsf{H}_i$ and $\eta \in \mathbb{R}$. Now, let $\mathsf{y} \in \operatorname{dom} \mathsf{f}_i = \operatorname{dom} \mathsf{g}_i$ and take $(\mathsf{x}, \mathsf{u}) \in \operatorname{gr} \partial \mathsf{g}_i$. Then (2.20) and Cauchy-Schwarz imply the coercivity property

$$\begin{aligned}\frac{\langle \mathsf{x} - \mathsf{y} \mid \mathsf{u} \rangle_{\mathsf{H}_i}}{\|\mathsf{x}\|_{\mathsf{H}_i}} &\geq \frac{\mathsf{g}_i(\mathsf{x}) - \mathsf{g}_i(\mathsf{y})}{\|\mathsf{x}\|_{\mathsf{H}_i}} \\ &= \frac{\|\mathsf{x}\|_{\mathsf{H}_i}}{2} + \frac{\mathsf{f}_i(\mathsf{x}) - \mathsf{g}_i(\mathsf{y})}{\|\mathsf{x}\|_{\mathsf{H}_i}} \\ &\geq \frac{\|\mathsf{x}\|_{\mathsf{H}_i}}{2} - \|\mathsf{v}\|_{\mathsf{H}_i} + \frac{\eta - \mathsf{g}_i(\mathsf{y})}{\|\mathsf{x}\|_{\mathsf{H}_i}} \\ &\to +\infty \quad \text{as} \quad \|\mathsf{x}\|_{\mathsf{H}_i} \to +\infty. \end{aligned} \tag{2.100}$$

Therefore, [15, Corollaire 3.4] asserts that for every $w \in \mathcal{H}_i$ there exists $z \in \mathcal{W}_i$ such that

$$w(t) \in z'(t) + \partial \mathsf{g}_i(z(t)) = z'(t) + z(t) + \partial \mathsf{f}_i(z(t)) \ \text{ a.e. on }]0, T[\,, \tag{2.101}$$

i.e., by (2.97), such that $w - z \in A_i z$. This shows that the range of $\operatorname{Id} + A_i$ is $\mathcal{H}_i$ and hence, by Minty's theorem [5, Theorem 3.5.8], that A_i is maximal monotone.

Next, for every $i \in \{1, \ldots, m\}$ and every $(x_1, \ldots, x_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m$, define almost everywhere

$$B_i(x_1, \ldots, x_m) \colon [0, T] \to \mathsf{H}_i \colon t \mapsto \mathsf{B}_i(x_1(t), \ldots, x_m(t)). \tag{2.102}$$

Now let $(x_1, \ldots, x_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m$ and set $(\forall i \in \{1, \ldots, m\})\ \mathsf{b}_i = \mathsf{B}_i(0, \ldots, 0)$. Then it follows from (2.92) and Cauchy-Schwarz that, almost everywhere on $[0, T]$,

$$\begin{aligned}\beta \sum_{j=1}^m \|\mathsf{B}_j(x_1(t), \ldots, x_m(t)) - \mathsf{b}_j\|^2_{\mathsf{H}_j} &\leq \sum_{j=1}^m \langle x_j(t) - 0 \mid \mathsf{B}_j(x_1(t), \ldots, x_m(t)) - \mathsf{b}_j \rangle_{\mathsf{H}_j} \\ &\leq \sum_{j=1}^m \|x_j(t)\|_{\mathsf{H}_j} \|\mathsf{B}_j(x_1(t), \ldots, x_m(t)) - \mathsf{b}_j\|_{\mathsf{H}_j} \\ &\leq \sqrt{\sum_{j=1}^m \|x_j(t)\|^2_{\mathsf{H}_j}} \sqrt{\sum_{j=1}^m \|\mathsf{B}_j(x_1(t), \ldots, x_m(t)) - \mathsf{b}_j\|^2_{\mathsf{H}_j}}. \end{aligned} \tag{2.103}$$

Therefore, for every $i \in \{1, \ldots, m\}$,

$$\begin{aligned}\|B_i(x_1, \ldots, x_m)(t)\|^2_{\mathsf{H}_i} &\leq 2\big(\|\mathsf{b}_i\|^2_{\mathsf{H}_i} + \|B_i(x_1, \ldots, x_m)(t) - \mathsf{b}_i\|^2_{\mathsf{H}_i}\big) \\ &\leq 2\Bigg(\|\mathsf{b}_i\|^2_{\mathsf{H}_i} + \sum_{j=1}^m \|\mathsf{B}_j(x_1(t), \ldots, x_m(t)) - \mathsf{b}_j\|^2_{\mathsf{H}_j}\Bigg) \\ &\leq 2\Bigg(\|\mathsf{b}_i\|^2_{\mathsf{H}_i} + \frac{1}{\beta^2} \sum_{j=1}^m \|x_j(t)\|^2_{\mathsf{H}_j}\Bigg) \ \text{ a.e. on }]0, T[\,, \end{aligned} \tag{2.104}$$

which yields

$$\int_0^T \|B_i(x_1,\ldots,x_m)(t)\|^2_{\mathsf{H}_i} dt \leq 2T\|\mathsf{b}_i\|^2_{\mathsf{H}_i} + \frac{2}{\beta^2}\sum_{j=1}^m \|x_j\|^2, \tag{2.105}$$

so that we can now claim that $B_i\colon \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \to L^2([0,T];\mathsf{H}_i) = \mathcal{H}_i$. In addition, upon integrating, we derive from (2.92) and (2.102) that, for every $(y_1,\ldots,y_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m$,

$$\sum_{i=1}^m \langle x_i - y_i \mid B_i(x_1,\ldots,x_m) - B_i(y_1,\ldots,y_m)\rangle \\ \geq \beta \sum_{i=1}^m \big\|B_i(x_1,\ldots,x_m) - B_i(y_1,\ldots,y_m)\big\|^2. \tag{2.106}$$

We have thus established (2.16).

Let us now make the connection between Algorithm 2.18 and Algorithm 2.8. For every $n \in \mathbb{N}$ and every $i \in \{1,\ldots,m\}$, it follows from (2.94), (2.97), (2.102), and the maximal monotonicity of A_i that $y_{i,n}$ is uniquely defined and can be expressed as

$$y_{i,n} = J_{\gamma_n A_i}\Big(x_{i,n} - \gamma_n\big(B_i(x_{1,n},\ldots,x_{m,n}) + b_{i,n}\big)\Big) + a_{i,n}\,, \tag{2.107}$$

where

$$a_{i,n} = J_{\gamma_n A_i}\Big(-\gamma_n e_{i,n} + x_{i,n} - \gamma_n\big(B_i(x_{1,n},\ldots,x_{m,n}) + b_{i,n}\big)\Big) \\ - J_{\gamma_n A_i}\Big(x_{i,n} - \gamma_n\big(B_i(x_{1,n},\ldots,x_{m,n}) + b_{i,n}\big)\Big). \tag{2.108}$$

We therefore deduce from (2.95) that

$$x_{i,n+1} = \lambda_{i,n}x_{i,n} + (1-\lambda_{i,n})\Big(J_{\gamma_n A_i}\Big(x_{i,n} - \gamma_n\big(B_i(x_{1,n},\ldots,x_{m,n}) + b_{i,n}\big)\Big) + a_{i,n}\Big). \tag{2.109}$$

Thus, (2.109) derives from (2.23) with $A_{i,n} \equiv A_i$ and $B_{i,n} \equiv B_i$. On the other hand, for every $i \in \{1,\ldots,m\}$, by nonexpansivity of the operators $(J_{\gamma_n A_i})_{n\in\mathbb{N}}$, we deduce from (2.108) and (2.96) that

$$\sum_{n\in\mathbb{N}} \|a_{i,n}\| \leq \sum_{n\in\mathbb{N}} \gamma_n\|e_{i,n}\| \leq 2\beta\sum_{n\in\mathbb{N}} \|e_{i,n}\| < +\infty. \tag{2.110}$$

As a result, all the hypotheses of Algorithm 2.8 are satisfied and hence Theorem 2.15 asserts that, for every $i \in \{1,\ldots,m\}$, $(x_{i,n})_{n\in\mathbb{N}}$ converges weakly in $\mathcal{H}_i = L^2([0,T];\mathsf{H}_i)$ to a point x_i, and $(x_i)_{1\leq i\leq m}$ satisfies

$$(\forall i \in \{1,\ldots,m\}) \quad 0 \in A_i x_i + B_i(x_1,\ldots,x_m). \tag{2.111}$$

Accordingly,

$$\sigma = \max_{1 \le i \le m} \sup_{n \in \mathbb{N}} \|x_{i,n}\| < +\infty \tag{2.112}$$

and $(\forall i \in \{1, \dots, m\})$ $x_i \in \operatorname{dom} A_i \subset \mathcal{W}_i$. Moreover since, in view of (2.97) and (2.102), (2.111) reduces to (2.93), $(x_i)_{1 \le i \le m}$ is a solution to Problem 2.17.

To complete the proof, let $i \in \{1, \dots, m\}$. To show that $(x_{i,n})_{n \in \mathbb{N}}$ converges weakly to x_i in $W^{1,2}([0,T]; \mathsf{H}_i)$, it remains to show that $(x'_{i,n})_{n \in \mathbb{N}}$ converges weakly to x'_i in $L^2([0,T]; \mathsf{H}_i)$. We first observe that $(x_{i,n})_{n \in \mathbb{N}}$ lies in $W^{1,2}([0,T]; \mathsf{H}_i)$. Indeed, it follows from (2.97) that

$$(\forall n \in \mathbb{N})(\forall z \in \mathcal{H}_i) \quad J_{\gamma_n A_i} z \in \operatorname{dom}(\gamma_n A_i) \subset \mathcal{W}_i \subset W^{1,2}([0,T]; \mathsf{H}_i). \tag{2.113}$$

As a result, we deduce from (2.108) that $(a_{i,n})_{n \in \mathbb{N}}$ lies in $W^{1,2}([0,T]; \mathsf{H}_i)$. On the other hand, by construction, $(y_{i,n})_{n \in \mathbb{N}}$ lies in $\mathcal{W}_i \subset W^{1,2}([0,T]; \mathsf{H}_i)$. In view of (2.95) and (i) in Algorithm 2.18, $(x_{i,n})_{n \in \mathbb{N}}$ is therefore in $W^{1,2}([0,T]; \mathsf{H}_i)$. Next, let us show that $(x'_{i,n})_{n \in \mathbb{N}}$ is bounded in $L^2([0,T]; \mathsf{H}_i)$. To this end, let $n \in \mathbb{N}$ and set

$$\begin{aligned} w_{i,n}(t) = \frac{x_{i,n}(t) - y_{i,n}(t)}{\gamma_n} - \mathsf{B}_i(x_{1,n}(t), \dots, x_{m,n}(t)) - b_{i,n}(t) \\ - y'_{i,n}(t) - e_{i,n}(t) \ \text{ a.e. on } \left]0,T\right[. \end{aligned} \tag{2.114}$$

Then we derive from (2.94) that

$$w_{i,n}(t) \in \partial \mathsf{f}_i(y_{i,n}(t)) \ \text{ a.e. on } \left]0,T\right[. \tag{2.115}$$

Hence, since $w_{i,n} \in \mathcal{H}_i$, it follows from [15, Lemme 3.3] that

$$\frac{d(\mathsf{f}_i \circ y_{i,n})(t)}{dt} = \left\langle w_{i,n}(t) \mid y'_{i,n}(t) \right\rangle_{\mathsf{H}_i} \ \text{ a.e. on } \left]0,T\right[. \tag{2.116}$$

On the other hand, since $y_{i,n} \in \mathcal{W}_i$, we have $y_{i,n}(T) = y_{i,n}(0)$. Therefore

$$\begin{aligned} \int_0^T \left\langle w_{i,n}(t) \mid y'_{i,n}(t) \right\rangle_{\mathsf{H}_i} dt &= \int_0^T \frac{d(\mathsf{f}_i \circ y_{i,n})(t)}{dt} dt \\ &= \mathsf{f}_i(y_{i,n}(T)) - \mathsf{f}_i(y_{i,n}(0)) \\ &= 0 \end{aligned} \tag{2.117}$$

and, furthermore,

$$\begin{aligned} \int_0^T \left\langle y_{i,n}(t) \mid y'_{i,n}(t) \right\rangle_{\mathsf{H}_i} dt &= \frac{1}{2} \int_0^T \frac{d\|y_{i,n}(t)\|^2_{\mathsf{H}_i}}{dt} dt \\ &= \frac{\|y_{i,n}(T)\|^2_{\mathsf{H}_i} - \|y_{i,n}(0)\|^2_{\mathsf{H}_i}}{2} \\ &= 0. \end{aligned} \tag{2.118}$$

We deduce from (2.117), (2.114), and (2.118) that

$$0 = \int_0^T \left\langle \frac{x_{i,n}(t)}{\gamma_n} \,\middle|\, y'_{i,n}(t) \right\rangle_{\mathsf{H}_i} dt - \int_0^T \left\langle \mathsf{B}_i(x_{1,n}(t), \ldots, x_{m,n}(t)) \mid y'_{i,n}(t) \right\rangle_{\mathsf{H}_i} dt$$
$$- \int_0^T \left\langle b_{i,n}(t) \mid y'_{i,n}(t) \right\rangle_{\mathsf{H}_i} dt - \int_0^T \|y'_{i,n}(t)\|^2_{\mathsf{H}_i} dt - \int_0^T \left\langle e_{i,n}(t) \mid y'_{i,n}(t) \right\rangle_{\mathsf{H}_i} dt. \tag{2.119}$$

Thus, using Cauchy-Schwarz, the inequality $\gamma_n \geq \varepsilon$, and (2.102), we obtain

$$\|y'_{i,n}\|^2 \leq \left(\frac{1}{\varepsilon}\|x_{i,n}\| + \|B_i(x_{1,n}, \ldots, x_{m,n})\| + \|b_{i,n}\| + \|e_{i,n}\| \right) \|y'_{i,n}\|. \tag{2.120}$$

In turn, it follows from (2.95) that

$$\|x'_{i,n+1}\| \leq \lambda_{i,n}\|x'_{i,n}\|$$
$$+ (1 - \lambda_{i,n}) \left(\frac{1}{\varepsilon}\|x_{i,n}\| + \|B_i(x_{1,n}, \ldots, x_{m,n})\| + \|b_{i,n}\| + \|e_{i,n}\| \right). \tag{2.121}$$

On the other hand, arguing as in (2.105), we derive from (2.112) that

$$\|B_i(x_{1,n}, \ldots, x_{m,n})\| \leq \sqrt{2T\|\mathsf{b}_i\|^2_{\mathsf{H}_i} + \frac{2m\sigma^2}{\beta^2}} \leq \sqrt{2T}\|\mathsf{b}_i\|_{\mathsf{H}_i} + \sqrt{2m}\,\frac{\sigma}{\beta}. \tag{2.122}$$

Hence, using (ii) in Algorithm 2.18, we derive by induction from (2.121) that

$$\|x'_{i,n}\| \leq \max\left\{ \|x'_{i,0}\|, \frac{\sigma}{\varepsilon} + \sqrt{2T}\|\mathsf{b}_i\|_{\mathsf{H}_i} + \sqrt{2m}\,\frac{\sigma}{\beta} + \sup_{k\in\mathbb{N}} \left(\|b_{i,k}\| + \|e_{i,k}\| \right) \right\}. \tag{2.123}$$

This shows the boundedness of $(x'_{i,n})_{n\in\mathbb{N}}$ in $L^2([0,T];\mathsf{H}_i)$. Now let z be the weak limit in $L^2([0,T];\mathsf{H}_i)$ of an arbitrary weakly convergent subsequence of $(x'_{i,n})_{n\in\mathbb{N}}$. Since $(x_{i,n})_{n\in\mathbb{N}}$ converges weakly in $L^2([0,T];\mathsf{H}_i)$ to x_i, it therefore follows from [48, Proposition 23.19] that $z = x'_i$. In turn, this shows that $(x'_{i,n})_{n\in\mathbb{N}}$ converges weakly in $L^2([0,T];\mathsf{H}_i)$ to x'_i. □

2.2.4 The variational case

We study a special case of Problem 2.6 which yields a variational formulation that extends (2.15).

Recall that, for every $f \in \Gamma_0(\mathcal{H})$ and every $x \in \mathcal{H}$, the function $y \mapsto f(y) + \|x-y\|^2/2$ admits a unique minimizer, which is denoted by $\mathrm{prox}_f\, x$. The proximity operator thus defined can be expressed as $\mathrm{prox}_f = J_{\partial f}$ [36].

Problem 2.20 Let $(\mathcal{H}_i)_{1\leq i\leq m}$ and $(\mathcal{G}_k)_{1\leq k\leq p}$ be real Hilbert spaces. For every $i \in \{1,\ldots,m\}$, let $f_i \in \Gamma_0(\mathcal{H}_i)$ and, for every $k \in \{1,\ldots,p\}$, let $\tau_k \in \left]0,+\infty\right[$, let $\varphi_k \colon \mathcal{G}_k \to \mathbb{R}$ be a differentiable convex function with a τ_k-Lipschitz-continuous gradient, and let $L_{ki} \colon \mathcal{H}_i \to \mathcal{G}_k$ be linear and bounded. It is assumed that $\min_{1\leq k\leq p} \sum_{i=1}^m \|L_{ki}\|^2 > 0$. The problem is to

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\text{minimize}} \quad \sum_{i=1}^m f_i(x_i) + \sum_{k=1}^p \varphi_k\Bigg(\sum_{i=1}^m L_{ki}x_i\Bigg), \tag{2.124}$$

under the assumption that solutions exist.

Algorithm 2.21 Set

$$\beta = \frac{1}{p \displaystyle\max_{1\leq k\leq p} \tau_k \sum_{i=1}^m \|L_{ki}\|^2}. \tag{2.125}$$

Fix $\varepsilon \in \left]0,\min\{1,\beta\}\right[$, $(\gamma_n)_{n\in\mathbb{N}}$ in $[\varepsilon, 2\beta-\varepsilon]$, $(\lambda_n)_{n\in\mathbb{N}}$ in $[0,1-\varepsilon]$, and $(x_{i,0})_{1\leq i\leq m} \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m$. Set, for every $n \in \mathbb{N}$,

$$\begin{cases} x_{1,n+1} = \lambda_{1,n}x_{1,n} + \\ \qquad (1-\lambda_{1,n})\Bigg(\operatorname{prox}_{\gamma_n f_{1,n}}\Bigg(x_{1,n} - \gamma_n\Bigg(\displaystyle\sum_{k=1}^p L_{k1}^*\nabla\varphi_k\Bigg(\sum_{j=1}^m L_{kj}x_{j,n}\Bigg) + b_{1,n}\Bigg)\Bigg) + a_{1,n}\Bigg), \\ \qquad \vdots \\ x_{m,n+1} = \lambda_{m,n}x_{m,n} + \\ \qquad (1-\lambda_{m,n})\Bigg(\operatorname{prox}_{\gamma_n f_{m,n}}\Bigg(x_{m,n} - \gamma_n\Bigg(\displaystyle\sum_{k=1}^p L_{km}^*\nabla\varphi_k\Bigg(\sum_{j=1}^m L_{kj}x_{j,n}\Bigg) + b_{m,n}\Bigg)\Bigg) + a_{m,n}\Bigg), \end{cases} \tag{2.126}$$

where, for every $i \in \{1,\ldots,m\}$, the following hold.

(i) $(f_{i,n})_{n\in\mathbb{N}}$ are functions in $\Gamma_0(\mathcal{H}_i)$ such that

$$(\forall \rho \in \left]0,+\infty\right[) \quad \sum_{n\in\mathbb{N}} \sup_{\|y\|\leq\rho} \|\operatorname{prox}_{\gamma_n f_{i,n}} y - \operatorname{prox}_{\gamma_n f_i} y\| < +\infty. \tag{2.127}$$

(ii) $(a_{i,n})_{n\in\mathbb{N}}$ and $(b_{i,n})_{n\in\mathbb{N}}$ are sequences in $\mathcal{H}_i$ such that $\sum_{n\in\mathbb{N}} \|a_{i,n}\| < +\infty$ and $\sum_{n\in\mathbb{N}} \|b_{i,n}\| < +\infty$.

(iii) $(\lambda_{i,n})_{n\in\mathbb{N}}$ is a sequence in $[0,1[$ such that $\sum_{n\in\mathbb{N}} |\lambda_{i,n} - \lambda_n| < +\infty$.

We now turn our attention to the asymptotic behavior of Algorithm 2.21 (strong convergence conditions can be derived from Theorem 2.16).

Theorem 2.22 *Let $((x_{i,n})_{n\in\mathbb{N}})_{1\leq i\leq m}$ be sequences generated by Algorithm 2.21. Then, for every $i \in \{1,\dots,m\}$, $(x_{i,n})_{n\in\mathbb{N}}$ converges weakly to a point $x_i \in \mathcal{H}_i$, and $(x_i)_{1\leq i\leq m}$ is a solution to Problem 2.20.*

Proof. Problem 2.20 is a special case of Problem 2.6 where, for every $i \in \{1,\dots,m\}$,

$$A_i = \partial f_i \quad \text{and} \quad B_i \colon (x_j)_{1\leq j\leq m} \mapsto \sum_{k=1}^{p} L_{ki}^* \nabla\varphi_k\bigg(\sum_{j=1}^{m} L_{kj}x_j\bigg). \tag{2.128}$$

Indeed, define $\boldsymbol{\mathcal{H}}$ as in the proof of Theorem 2.15 and set

$$\boldsymbol{f} \colon \boldsymbol{\mathcal{H}} \to \left]-\infty,+\infty\right] : (x_i)_{1\leq i\leq m} \mapsto \sum_{i=1}^{m} f_i(x_i) \tag{2.129}$$

and

$$\boldsymbol{g} \colon \boldsymbol{\mathcal{H}} \to \mathbb{R} \colon (x_i)_{1\leq i\leq m} \mapsto \sum_{k=1}^{p} \varphi_k\bigg(\sum_{i=1}^{m} L_{ki}x_i\bigg). \tag{2.130}$$

Then $\boldsymbol{f}$ and $\boldsymbol{g}$ are in $\Gamma_0(\boldsymbol{\mathcal{H}})$ and it follows from Fermat's rule and elementary subdifferential calculus that, for every $(x_1,\dots,x_m) \in \boldsymbol{\mathcal{H}}$,

$$\begin{aligned}
(x_1,\dots,x_m) \text{ solves (2.124)} \quad &\Leftrightarrow \quad (0,\dots,0) \in \partial(\boldsymbol{f}+\boldsymbol{g})(x_1,\dots,x_m)\\
&\Leftrightarrow \quad (0,\dots,0) \in \partial\boldsymbol{f}(x_1,\dots,x_m) + \nabla\boldsymbol{g}(x_1,\dots,x_m)\\
&\Leftrightarrow \quad (\forall i \in \{1,\dots,m\}) \;\; 0 \in \partial f_i(x_i) + \sum_{k=1}^{p} L_{ki}^*\nabla\varphi_k\bigg(\sum_{j=1}^{m} L_{kj}x_j\bigg)\\
&\Leftrightarrow \quad (\forall i \in \{1,\dots,m\}) \;\; 0 \in A_ix_i + B_i(x_1,\dots,x_m).
\end{aligned} \tag{2.131}$$

Next, let us show that the family $(B_i)_{1\leq i\leq m}$ in (2.128) satisfies (2.16) with β as in (2.125). First, Lemma 2.12 asserts that, for every $k \in \{1,\dots,p\}$, $\nabla\varphi_k$ is τ_k^{-1}-cocoercive. Hence, for every $(x_1,\dots,x_m) \in \boldsymbol{\mathcal{H}}$ and every $(y_1,\dots,y_m) \in \boldsymbol{\mathcal{H}}$, it follows from (2.128), (2.125), and the convexity of $\|\cdot\|^2$ that

$$
\begin{aligned}
&\sum_{i=1}^m \langle x_i - y_i \mid B_i(x_1,\ldots,x_m) - B_i(y_1,\ldots,y_m)\rangle \\
&\quad= \sum_{i=1}^m \sum_{k=1}^p \Big\langle x_i - y_i \,\Big|\, L_{ki}^*\Big(\nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}x_j\Big) - \nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}y_j\Big)\Big)\Big\rangle \\
&\quad= \sum_{i=1}^m \sum_{k=1}^p \Big\langle L_{ki}(x_i - y_i) \,\Big|\, \nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}x_j\Big) - \nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}y_j\Big)\Big\rangle \\
&\quad= \sum_{k=1}^p \Big\langle \sum_{i=1}^m L_{ki}x_i - \sum_{i=1}^m L_{ki}y_i \,\Big|\, \nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}x_j\Big) - \nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}y_j\Big)\Big\rangle \\
&\quad\geq \sum_{k=1}^p \frac{1}{\tau_k}\Big\|\nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}x_j\Big) - \nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}y_j\Big)\Big\|^2 \\
&\quad= \sum_{k=1}^p \frac{1}{\tau_k \sum_{i=1}^m \|L_{ki}\|^2} \sum_{i=1}^m \|L_{ki}\|^2 \Big\|\nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}x_j\Big) - \nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}y_j\Big)\Big\|^2 \\
&\quad\geq p\beta \sum_{k=1}^p \sum_{i=1}^m \|L_{ki}\|^2 \Big\|\nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}x_j\Big) - \nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}y_j\Big)\Big\|^2 \\
&\quad\geq \beta \sum_{i=1}^m p \sum_{k=1}^p \Big\|L_{ki}^*\Big(\nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}x_j\Big) - \nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}y_j\Big)\Big)\Big\|^2 \\
&\quad\geq \beta \sum_{i=1}^m \Big\|\sum_{k=1}^p L_{ki}^*\nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}x_j\Big) - \sum_{k=1}^p L_{ki}^*\nabla\varphi_k\Big(\sum_{j=1}^m L_{kj}y_j\Big)\Big\|^2.
\end{aligned}
\tag{2.132}
$$

This shows that (2.16) holds. Furthermore, upon setting

$$
(\forall i \in \{1,\ldots,m\})(\forall n \in \mathbb{N}) \quad A_{i,n} = \partial f_{i,n} \quad \text{and} \quad B_{i,n} = B_i, \tag{2.133}
$$

we deduce from (2.127) that Algorithm 2.21 is a particular case of Algorithm 2.8. Altogether, Theorem 2.22 follows from Theorem 2.15. ☐

Here are a couple of applications of Problem 2.20.

Example 2.23 (network flows) Consider a network with M links indexed by $j \in \{1,\ldots,M\}$ and N paths indexed by $l \in \{1,\ldots,N\}$, linking a subset of Q origin-destination node pairs indexed by $k \in \{1,\ldots,Q\}$. There are m types of users indexed by $i \in \{1,\ldots,m\}$ transiting on the network. For every $i \in \{1,\ldots,m\}$ and $l \in \{1,\ldots,N\}$, let $\xi_{il} \in \mathbb{R}$ be the flux of user i on path l and let $x_i = (\xi_{il})_{1\leq l\leq N}$ be the flow associated with user i. A standard problem in traffic theory is to find a Wardrop equilibrium [45] of the network, i.e., flows $(x_i)_{1\leq i\leq m}$ such that the costs in all paths actually used are equal and less than those a single user would face on any unused path. Such an equilibrium can be obtained by solving the variational problem [13, 37, 41]

$$\underset{x_1\in C_1,\ldots,x_m\in C_m}{\text{minimize}} \quad \sum_{j=1}^{M}\int_0^{h_j(x_1,\ldots,x_m)}\phi_j(h)dh, \tag{2.134}$$

where $\phi_j\colon\mathbb{R}\to[0,+\infty[$ is a strictly increasing τ-Lipschitz continuous function modeling the cost of transiting on link j and $h_j(x_1,\ldots,x_m)$ is the total flow through link j, which can be expressed as $h_j(x_1,\ldots,x_m)=\sum_{i=1}^m(Lx_i)^\top e_j$, where e_j is the jth canonical basis vector of $\mathbb{R}^M$ and L is an $M\times N$ binary matrix with jlth entry equal to 1 or 0, according as link j belongs to path l or not. Furthermore, each closed and convex constraint set C_i in (2.134) is defined as $C_i=\big\{(\eta_l)_{1\leq l\leq N}\in[0,+\infty[^N \mid (\forall k\in\{1,\ldots,Q\})\ \sum_{l\in N_k}\eta_l=\delta_{ik}\big\}$, where $\varnothing\neq N_k\subset\{1,\ldots,N\}$ is the set of paths linking the pair k and $\delta_{ik}\in[0,+\infty[$ is the flow of user i that must transit from the origin to the destination of pair k (for more details on network flows, see [40, 41]). Upon setting

$$\varphi_1\colon\mathbb{R}^M\to\mathbb{R}\colon(\nu_j)_{1\leq j\leq M}\mapsto\sum_{j=1}^{M}\int_0^{\nu_j}\phi_j(h)dh, \tag{2.135}$$

problem (2.134) can be written as

$$\underset{x_1\in\mathbb{R}^N,\ldots,x_m\in\mathbb{R}^N}{\text{minimize}} \quad \sum_{i=1}^{m}\iota_{C_i}(x_i)+\varphi_1\bigg(\sum_{i=1}^{m}Lx_i\bigg). \tag{2.136}$$

Since φ_1 is strictly convex and differentiable with a τ-Lipschitz-continuous gradient, (2.136) is a particular instance of Problem 2.20 with $p=1$, $\mathcal{G}_1=\mathbb{R}^M$ and $(\forall i\in\{1,\ldots,m\})$ $\mathcal{H}_i=\mathbb{R}^N$, $f_i=\iota_{C_i}$, and $L_{1i}=L$. Accordingly, Theorem 2.22 asserts that (2.136) can be solved by Algorithm 2.21 which, with the choice of parameters $\gamma_n\equiv\gamma\in\,]0,2/\tau[$, $\lambda_{i,n}\equiv0$, $\lambda_n\equiv0$, $a_{i,n}\equiv0$, and $b_{i,n}\equiv0$, yields

$$(\forall i\in\{1,\ldots,m\})\quad x_{i,n+1}=P_{C_i}\Big(x_{i,n}-\gamma L^\top\big(\phi_1(\rho_{1,n}),\ldots,\phi_M(\rho_{M,n})\big)\Big), \tag{2.137}$$

where $(\rho_{1,n},\ldots,\rho_{M,n})=\sum_{j=1}^m Lx_{j,n}$. In the special case when $m=1$ the algorithm described in (2.137) is proposed in [14]. Let us note that, as an alternative to (2.135), we can consider the function

$$\varphi_1\colon\mathbb{R}^M\to\mathbb{R}\colon(\nu_j)_{1\leq j\leq M}\mapsto\sum_{j=1}^{M}\nu_j\phi_j(\nu_j), \tag{2.138}$$

under suitable assumptions on $(\phi_j)_{1\leq j\leq M}$. In this case, (2.136) reduces to the problem of finding the social optimum in the network [41], that is

$$\underset{x_1\in C_1,\ldots,x_m\in C_m}{\text{minimize}} \quad \sum_{j=1}^{M}h_j(x_1,\ldots,x_m)\phi_j\big(h_j(x_1,\ldots,x_m)\big), \tag{2.139}$$

which can also be solved with Algorithm 2.21.

Example 2.24 (best approximation) The convex feasibility problem is to find a point in the intersection of closed convex subsets $(C_i)_{1\leq i\leq m}$ of a real Hilbert space $\mathcal{H}$ [10, 21]. This problem arises in many applications in engineering and the physical sciences [17, 19]. In many instances, the intersection of the sets $(C_i)_{1\leq i\leq m}$ may turn out to be empty and a relaxation of this problem in the presence of a hard constraint represented by C_1 is to [23]

$$\underset{x_1\in C_1}{\text{minimize}}\ \frac{1}{2}\sum_{i=2}^{m}\omega_i d_{C_i}^2(x_1), \tag{2.140}$$

where $(\omega_i)_{2\leq i\leq m}$ are strictly positive weights such that $\max_{2\leq i\leq m}\omega_i = 1$. We assume that this problem admits at least one solution, as is the case when one of the sets in $(C_i)_{1\leq i\leq m}$ is bounded [23, Proposition 4]. Since, for every $i\in\{2,\ldots,m\}$ and every $x_1\in C_1$, $d_{C_i}^2(x_1)=\min_{x_i\in C_i}\|x_1-x_i\|^2$, (2.140) can be reformulated as

$$\underset{x_1\in C_1,\ldots,x_m\in C_m}{\text{minimize}}\ \frac{1}{2}\sum_{k=1}^{m-1}\omega_{k+1}\|x_1-x_{k+1}\|^2. \tag{2.141}$$

This is a special instance of Problem 2.20 with $p=m-1$ and, for every $i\in\{1,\ldots,m\}$, $f_i=\iota_{C_i}$ and

$$(\forall k\in\{1,\ldots,m-1\})\quad \varphi_k=\frac{\omega_{k+1}}{2}\|\cdot\|^2\quad\text{and}\quad L_{ki}=\begin{cases}\mathrm{Id}, & \text{if } i=1;\\ -\mathrm{Id}, & \text{if } i=k+1;\\ 0, & \text{otherwise.}\end{cases} \tag{2.142}$$

We can derive from Algorithm 2.21 an algorithm which, by Theorem 2.22, generates orbits that are guaranteed to converge weakly to a solution to (2.141). Indeed, in this case, (2.125) yields $\beta=1/(2(m-1))$. For example, upon setting $\gamma_n\equiv\gamma\in\,]0,1/(m-1)[$, $\lambda_n\equiv 0$, $\lambda_{i,n}\equiv 0$, $a_{i,n}\equiv 0$, $b_{i,n}\equiv 0$, and $f_{i,n}=\iota_{C_i}$ for simplicity, Algorithm 2.21 becomes

$$\begin{cases} x_{1,n+1}=P_{C_1}\big((1-\gamma\sum_{i=2}^m\omega_i)x_{1,n}+\gamma\sum_{i=2}^m\omega_i x_{i,n}\big)\\ (\forall i\in\{2,\ldots,m\})\quad x_{i,n+1}=P_{C_i}\big(\gamma\omega_i x_{1,n}+(1-\gamma\omega_i)x_{i,n}\big).\end{cases} \tag{2.143}$$

In the particular case when $m=2$ and $\gamma=1/2$, then $\omega_2=1$, (2.141) is equivalent to finding a best approximation pair relative to (C_1,C_2) [9, 11], and (2.143) reduces to

$$\begin{cases} x_{1,n+1}=P_{C_1}\big((x_{1,n}+x_{2,n})/2\big)\\ x_{2,n+1}=P_{C_2}\big((x_{1,n}+x_{2,n})/2\big).\end{cases} \tag{2.144}$$

2.3 Bibliographie

[1] F. Acker and M. A. Prestel, Convergence d'un schéma de minimisation alternée, *Ann. Fac. Sci. Toulouse V. Sér. Math.*, vol. 2, pp. 1–9, 1980.

[2] H. Attouch, *Variational Convergence for Functions and Operators,* Pitman, Boston, MA, 1984.

[3] H. Attouch, J. Bolte, P. Redont, and A. Soubeyran, Alternating proximal algorithms for weakly coupled convex minimization problems – Applications to dynamical games and PDE's, *J. Convex Anal.,* vol. 15, pp. 485–506, 2008.

[4] H. Attouch, P. Redont, and A. Soubeyran, A new class of alternating proximal minimization algorithms with costs-to-move, *SIAM J. Optim.,* vol. 18, pp. 1061–1081, 2007.

[5] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis,* Birkhäuser, Boston, MA, 1990.

[6] A. Auslender, *Problèmes de Minimax via l'Analyse Convexe et les Inégalités Variationnelles : Théorie et Algorithmes,* Lecture Notes in Econom. and Math. Systems, vol. 77, Springer-Verlag, New York, 1972.

[7] J.-B. Baillon and G. Haddad, Quelques propriétés des opérateurs angle-bornés et n-cycliquement monotones, *Israel J. Math.,* vol. 26, pp. 137–150, 1977.

[8] H. H. Bauschke and J. M. Borwein, On the convergence of von Neumann's alternating projection algorithm for two sets, *Set-Valued Anal.,* vol. 1, pp. 185–212, 1993.

[9] H. H. Bauschke and J. M. Borwein, Dykstra's alternating projection algorithm for two sets, *J. Approx. Theory,* vol. 79, pp. 418–443, 1994.

[10] H. H. Bauschke and J. M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Rev.,* vol. 38, pp. 367–426, 1996.

[11] H. H. Bauschke, P. L. Combettes, and D. R. Luke, Finding best approximation pairs relative to two closed convex sets in Hilbert spaces, *J. Approx. Theory,* vol. 127, pp. 178–192, 2004.

[12] H. H. Bauschke, P. L. Combettes, and S. Reich, The asymptotic behavior of the composition of two resolvents, *Nonlinear Anal.*, vol. 60, pp. 283–301, 2005.

[13] M. Beckmann, C. McGuire, and C. Winsten, *Studies in Economics of Transportation,* Yale University Press, New Haven, CT, 1956.

[14] D. P. Bertsekas and E. M. Gafni, Projection methods for variational inequalities with application to the traffic assignment problem, *Math. Program. Stud.*, vol. 17, pp. 139–159, 1982.

[15] H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert,* North-Holland/Elsevier, New York, 1973.

[16] H. Brézis and M. Sibony, Méthodes d'approximation et d'itération pour les opérateurs monotones, *Arch. Ration. Mech. Anal.,* vol. 28, pp. 59–82, 1967/1968.

[17] Y. Censor and S. A. Zenios, *Parallel Optimization : Theory, Algorithms, and Applications,* Oxford University Press, New York, 1997.

[18] W. Cheney and A. A. Goldstein, Proximity maps for convex sets, *Proc. Amer. Math. Soc.,* vol. 10, pp. 448–450, 1959.

[19] P. L. Combettes, The foundations of set theoretic estimation, *Proc. IEEE,* vol. 81, pp. 182–208, 1993.

[20] P. L. Combettes, Construction d'un point fixe commun à une famille de contractions fermes, *C. R. Acad. Sci. Paris Sér. I Math.,* vol. 320, pp. 1385–1390, 1995.

[21] P. L. Combettes, Hilbertian convex feasibility problem : Convergence of projection methods, *Appl. Math. Optim.,* vol. 35, pp. 311–330, 1997.

[22] P. L. Combettes, Solving monotone inclusions via compositions of nonexpansive averaged operators, *Optimization,* vol. 53, pp. 475–504, 2004.

[23] P. L. Combettes and P. Bondon, Hard-constrained inconsistent signal feasibility problems, *IEEE Trans. Signal Process.,* vol. 47, pp. 2460–2468, 1999.

[24] P. L. Combettes and S. A. Hirstoaga, Approximating curves for nonexpansive and monotone operators, *J. Convex Anal.,* vol. 13, pp. 633–646, 2006.

[25] P. L. Combettes and T. Pennanen, Proximal methods for cohypomonotone operators, *SIAM J. Control Optim.,* vol. 43, pp. 731–742, 2004.

[26] P. L. Combettes and V. R. Wajs, Signal recovery by proximal forward-backward splitting, *Multiscale Model. Simul.,* vol. 4, pp. 1168–1200, 2005.

[27] J. C. Dunn, Convexity, monotonicity, and gradient processes in Hilbert space, *J. Math. Anal. Appl.,* vol. 53, pp. 145–158, 1976.

[28] M. Goldburg and R. J. Marks II, Signal synthesis in the presence of an inconsistent set of constraints, *IEEE Trans. Circuits Syst.,* vol. 32, pp. 647–663, 1985.

[29] L. G. Gubin, B. T. Polyak, and E. V. Raik, The method of projections for finding the common point of convex sets, *U.S.S.R. Comput. Math. Math. Phys.*, vol. 7, pp. 1–24, 1967.

[30] A. Haraux, *Nonlinear Evolution Equations : Global Behavior of Solutions,* Lecture Notes in Math., vol. 841, Springer-Verlag, New York, 1981.

[31] B. He, L.-Z. Liao, and S. Wang, Self-adaptive operator splitting methods for monotone variational inequalities, *Numer. Math.,* vol. 94, pp. 715–737, 2003.

[32] A. Kaplan and R. Tichatschke, Proximal point approach and approximation of variational inequalities, *SIAM J. Control Optim.,* vol. 39, pp. 1136–1159, 2000.

[33] N. Lehdili and B. Lemaire, The barycentric proximal method, *Comm. Appl. Nonlinear Anal.,* vol. 6, pp. 29–47, 1999.

[34] E. S. Levitin and B. T. Polyak, Constrained minimization methods, *U.S.S.R. Comput. Math. Math. Phys.*, vol. 6, pp. 1–50, 1966.

[35] B. Mercier, *Topics in Finite Element Solution of Elliptic Problems* Lectures on Mathematics, no. 63, Tata Institute of Fundamental Research, Bombay, 1979.

[36] J.-J. Moreau, Proximité et dualité dans un espace hilbertien, *Bull. Soc. Math. France,* vol. 93, pp. 273–299, 1965.

[37] M. Patriksson, *The Traffic Assignment Problem : Models and Methods,* Utrecht, The Netherlands, 1994.

[38] W. V. Petryshyn, Construction of fixed points of demicompact mappings in Hilbert space, *J. Math. Anal. Appl.,* vol. 14, pp. 276–284, 1966.

[39] B. T. Polyak, *Introduction to Optimization,* Optimization Software Inc., New York, 1987.

[40] R. T. Rockafellar, *Network Flows and Monotropic Optimization,* Wiley, New York, 1984.

[41] Y. Sheffi, *Urban Transportation Networks : Equilibrium Analysis with Mathematical Programming Methods,* Prentice-Hall, Englewood Cliffs, NJ, 1985.

[42] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Mathematical Surveys and Monographs, vol. 49, Amer. Math. Soc., Providence, RI, 1997.

[43] M. Sibony, Méthodes itératives pour les équations et inéquations aux dérivées partielles non linéaires de type monotone, *Calcolo,* vol. 7, pp. 65–183, 1970.

[44] S. Simons, *From Hahn-Banach to Monotonicity,* Lecture Notes in Math. 1693, Springer-Verlag, New York, 2008.

[45] J. G. Wardrop, Some theoretical aspects of road traffic research, *Proc. Inst. Civil Eng. II,* vol. 1, pp. 325–378, 1952.

[46] C. Zălinescu, *Convex Analysis in General Vector Spaces,* World Scientific, River Edge, NJ, 2002.

[47] E. Zeidler, *Nonlinear Functional Analysis and Its Applications I*, Springer-Verlag, New York, 1993.

[48] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II*, Springer-Verlag, New York, 1990.

[49] D. L. Zhu and P. Marcotte, Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities, *SIAM J. Optim.,* vol. 6, pp. 714–726, 1996.

Chapitre 3

Méthodes variationnelles pour la décomposition, la reconstruction et la restauration de signaux multicomposantes avec couplage lisse

3.1 Description et résultats principaux

Dans ce chapitre nous nous intéressons à la résolution de problèmes de traitement de signaux multicomposantes sur des espaces hilbertiens, plus précisément à la restauration et la décomposition de signaux. Dans les problèmes de restauration, nous cherchons un signal original $x \in \mathcal{H}$ à partir d'un signal dégradé $z \in \mathcal{G}$, où $\mathcal{H}$ et $\mathcal{G}$ sont des espaces hilbertiens réels. Nous supposons que z résulte de la dégradation de x par un opérateur linéaire borné $L\colon \mathcal{H} \to \mathcal{G}$ (par exemple un flou) et d'un bruit additif $w \in \mathcal{G}$, c'est-à-dire $z = Lx + w$. Des formulations variationnelles ont été proposées dans [32, 44] pour résoudre ce problème en dimension finie. Par ailleurs, dans les problèmes de décomposition nous cherchons un nombre fini des composantes $x_1, \ldots, x_m$ d'un signal original $x \in \mathcal{H}$ à partir d'une observation bruitée $z \in \mathcal{H}$. Nous supposons que $x = x_1 + \cdots + x_m$ et que le bruit $w \in \mathcal{H}$ est additif, c'est-à-dire $z = x_1 + \cdots + x_m + w$. Dans le cas où $m = 2$, des formulations variationnelles et des méthodes convergentes ont été présentées dans [5, 23] (voir aussi [3]), dans [6] une méthode avec propriétés modestes de convergence a été proposée pour résoudre le cas où $m = 3$ et dans [15] une méthode sans résultat de convergence est proposée pour résoudre le cas où $m = 4$.

Nous nous intéressons à une formulation générale qui permet de traiter les deux problèmes simultanément. Dans ce but, supposons que $p \geq 2$ signaux $(z_1, \ldots, z_p) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_p$ soient observés, où, pour tout $k \in \{1, \ldots, p\}$, $\mathcal{G}_k$ est un espace hilbertien et z_k est une observation bruitée qui provient d'un signal original $y_k \in \mathcal{G}_k$. De plus, nous supposons que y_k est un mélange linéaire des composantes $x_1, \ldots, x_m$, lesquelles appartiennent à des espaces de Hilbert réels $\mathcal{H}_1, \ldots, \mathcal{H}_m$, respectivement. Plus précisément,

$$(\forall k \in \{1, \ldots, p\}) \quad z_k = y_k + w_k = L_{k1}x_1 + \cdots + L_{km}x_m + w_k, \tag{3.1}$$

où, pour tout $i \in \{1, \ldots, m\}$, $L_{ki} \in \mathcal{B}(\mathcal{H}_i, \mathcal{G}_k)$ et $w_k \in \mathcal{G}_k$ représente le bruit. De plus, supposons que, pour tout $i \in \{1, \ldots, m\}$, la fonction $f_i \in \Gamma_0(\mathcal{H}_i)$ modélise des caractéristiques intrinsèques de la composante x_i. Les composantes $x_1, \ldots, x_m$ peuvent alors être obtenues comme solution du problème d'optimisation

$$\underset{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m}{\text{minimiser}} \quad \sum_{i=1}^{m} f_i(x_i) + \frac{1}{2} \sum_{k=1}^{p} \left\| z_k - \sum_{i=1}^{m} L_{ki}x_i \right\|^2. \tag{3.2}$$

D'une manière générale, si l'on considère que le bruit n'est pas additif, les composantes peuvent être obtenues comme solution du problème suivant.

Problème 3.1 Pour tout $i \in \{1, \ldots, m\}$, soit $f_i \in \Gamma_0(\mathcal{H}_i)$ et pour tout $k \in \{1, \ldots, p\}$ soit $L_{ki} \in \mathcal{B}(\mathcal{H}_i, \mathcal{G}_k)$, soit $\tau_k \in \left]0, +\infty\right[$, et soit $\varphi_k \colon \mathcal{G}_k \to \mathbb{R}$ une fonction convexe et différentiable avec un gradient τ_k-lipschitzien. Le problème est de

$$\underset{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m}{\text{minimiser}} \quad \sum_{i=1}^{m} f_i(x_i) + \sum_{k=1}^{p} \varphi_k\bigg(\sum_{i=1}^{m} L_{ki}x_i\bigg). \tag{3.3}$$

Dans le Problème 3.1, les fonctions $(\varphi_k)_{1 \leq k \leq p}$ peuvent agir comme des fonctions de pénalisation reflétant les propriétés connues sur le bruit. Par exemple, si pour tout $k \in \{1, \ldots, p\}$, $\varphi_k = \|z_k - \cdot\|^2/2$, (3.3) donne (3.2).

Notons que, en utilisant des outils basiques d'analyse convexe, cette formulation est un cas particulier du Problème 1.1 avec $q = 2$ et

$$\begin{cases} \boldsymbol{A}_1 = \partial f_1 \times \cdots \times \partial f_m \\ \boldsymbol{A}_2 \colon (x_i)_{1 \leq i \leq m} \mapsto \left(\displaystyle\sum_{k=1}^{p} L_{ki}^* \nabla \varphi_k\bigg(\sum_{i=1}^{m} L_{ki}x_i\bigg)\right)_{1 \leq i \leq m}. \end{cases} \tag{3.4}$$

Le Problème 3.1 a été résolu par l'algorithme que nous avons proposé dans le Chapitre 2, qui nous rappelons ci-dessous.

Algorithme 3.2 On pose

$$\beta_1 = \frac{1}{p \displaystyle\max_{1 \leq k \leq p} \tau_k \sum_{i=1}^{m} \|L_{ki}\|^2}, \tag{3.5}$$

pour tout $i \in \{1,\ldots,m\}$, soient $(a_{i,n})_{n\in\mathbb{N}}$ et $(b_{i,n})_{n\in\mathbb{N}}$ des suites dans $\mathcal{H}_i$ telles que $\sum_{n\in\mathbb{N}} \|a_{i,n}\| < +\infty$ et $\sum_{n\in\mathbb{N}} \|b_{i,n}\| < +\infty$. On génère des suites $(x_{1,n})_{n\in\mathbb{N}}$, ..., $(x_{m,n})_{n\in\mathbb{N}}$ comme suit.

$$
\begin{array}{l}
\text{Initialisation} \\
\left\lfloor \begin{array}{l}
\varepsilon \in \left]0, \min\{1,\beta_1\}\right[\\
\text{Pour } i = 1,\ldots,m \\
\left\lfloor x_{i,0} \in \mathcal{H}_i \right.
\end{array} \right. \\
\text{Pour } n = 0,1,\ldots \\
\left\lfloor \begin{array}{l}
\gamma_n \in [\varepsilon, 2\beta_1 - \varepsilon] \\
\lambda_n \in [\varepsilon, 1] \\
\text{Pour } i = 1,\ldots,m \\
\left\lfloor \begin{array}{l}
y_{i,n} = \displaystyle\sum_{k=1}^{p} L_{ki}^* \nabla\varphi_k \left(\sum_{j=1}^{m} L_{kj} x_{j,n} \right) + b_{i,n} \\
z_{i,n} = \operatorname{prox}_{\gamma_n f_i}(x_{i,n} - \gamma_n y_{i,n}) + a_{i,n} \\
x_{i,n+1} = x_{i,n} + \lambda_n (z_{i,n} - x_{i,n}).
\end{array} \right.
\end{array} \right.
\end{array}
\tag{3.6}
$$

Le résultat de convergence démontré dans le Chapitre 2 est le suivant.

Théorème 3.3 [4, Theorem 4.3] *Soient $(x_{1,n})_{n\in\mathbb{N}}$, ..., $(x_{m,n})_{n\in\mathbb{N}}$ les suites générées par l'Algorithme 3.2 et supposons que le Problème 3.1 admette au moins une solution. Alors, pour tout $i \in \{1,\ldots,m\}$, $x_{i,n} \rightharpoonup x_i \in \mathcal{H}_i$ et $(x_i)_{1\le i\le m}$ est une solution du Problème 3.1.*

Ce chapitre est consacré à la résolution de certains cas particuliers importants du Problème 3.1 d'une manière efficace en modifiant l'Algorithme 3.2. En particulier, si nous considérons que, pour tout $k \in \{1,\ldots,p\}$, $\varphi_k = \|z_k - \cdot\|^2/2$, l'Algorithme 3.2 peut être décrit par (on prend $\lambda_n \equiv 1$ par simplicité)

$$
x_{i,n+1} = \operatorname{prox}_{\gamma_n f_i} \left(x_{i,n} + \gamma_n \left(\sum_{k=1}^{p} L_{ki}^* \left(z_k - \sum_{j=1}^{m} L_{kj} x_{j,n} \right) + b_{i,n} \right) \right) + a_{i,n}, \tag{3.7}
$$

et les suites ainsi générées convergent vers une solution de (3.2) si $(\gamma_n)_{n\in\mathbb{N}}$ est dans $]0, 2\beta_1[$. De plus, nous montrons que ce résultat reste vrai si la suite $(\gamma_n)_{n\in\mathbb{N}}$ est dans $]0, 2\beta_2[$, avec

$$
\beta_2 = \frac{1}{\displaystyle\sum_{k=1}^{p} \sum_{i=1}^{m} \|L_{ki}\|^2}. \tag{3.8}
$$

Vu que $\beta_2 \ge \beta_1$, on peut donc élargir l'intervalle dans lequel se trouve la suite $(\gamma_n)_{n\in\mathbb{N}}$.

D'autre part, si nous considérons le cas particulier de (3.2) où, pour tout $k \in \{1, \ldots, p\}$ et $i \in \{1, \ldots, m\}$, $L_{ki} = \xi_{ki} \operatorname{Id}$ avec $\xi_{ki} \in \mathbb{R}$, (3.2) se réduit à

$$\underset{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m}{\text{minimiser}} \quad \sum_{i=1}^{m} f_i(x_i) + \frac{1}{2} \sum_{k=1}^{p} \left\| z_k - \sum_{i=1}^{m} \xi_{ki} x_i \right\|^2 \tag{3.9}$$

et (3.7) donne

$$x_{i,n+1} = \operatorname{prox}_{\gamma_n f_i} \left(x_{i,n} + \gamma_n \sum_{k=1}^{p} \xi_{ki} \left(z_k - \sum_{j=1}^{m} \xi_{kj} x_{j,n} \right) \right) + a_{i,n}. \tag{3.10}$$

Nous montrons que le même résultat de convergence reste vrai en prenant la suite $(\gamma_n)_{n \in \mathbb{N}}$ dans $]0, 2\beta_3[$, où

$$\beta_3 = \frac{1}{\lambda_{\max}} \tag{3.11}$$

et $\lambda_{\max}$ est la plus grande valeur propre de la matrice $\Xi^\top \Xi$ avec $\Xi = [\xi_{ki}] \in \mathbb{R}^{p \times m}$. Puisque $\beta_3 \geq \beta_2$, ceci permet d'élargir encore plus l'intervalle dans lequel se trouve la suite $(\gamma_n)_{n \in \mathbb{N}}$.

Les algorithmes proposés sont appliqués aux problèmes de décomposition du signal, de synthèse du signal, de représentation du signal multitrame et aux problèmes de minimisation avec des contraintes pénalisées. Notons dans (3.6), (3.7) et (3.10) que le calcul d'opérateurs proximaux est un élément-clé dans la mise en œuvre des algorithmes. Pour cette raison nous fournissons plusieurs calculs explicites d'opérateurs proximaux que nous utilisons dans les applications mentionnées.

3.2 Article en anglais

CONVEX VARIATIONAL FORMULATION WITH SMOOTH COUPLING FOR MULTICOMPONENT SIGNAL DECOMPOSITION AND RECOVERY [1]

Abstract : A convex variational formulation is proposed to solve multicomponent signal processing problems in Hilbert spaces. The cost function consists of a separable term, in which each component is modeled through its own potential, and of a coupling term, in which constraints on linear transformations of the components are penalized with smooth functionals. An algorithm with guaranteed weak convergence to a solution to the problem is provided. Various multicomponent signal decomposition and recovery applications are discussed.

1. L. M. Briceño-Arias and P. L. Combettes, Convex variational formulation with smooth coupling for multicomponent signal decomposition and recovery, *Numerical Mathematics : Theory, Methods, and Applications*, vol. 2, pp. 485–508, 2009.

3.2.1 Problem statement

The processing of multicomponent signals has become increasingly important due, on the one hand, to the development of new imaging modalities and sensing devices, and, on the other hand, to the introduction of sophisticated mathematical models to represent complex signals. It is for instance required in applications dealing with the recovery of multichannel signals [8, 33, 34, 40], which arise in particular in color imaging and in the multi- and hyperspectral imaging techniques used in astronomy and in satellite imaging. Another important instance of multicomponent processing is found in signal decomposition problems, e.g., [2, 5, 6, 7, 15, 43, 44]. In such problems, the ideal signal is viewed as a mixture of elementary components that need to be identified individually.

Mathematically, a multicomponent signal can be viewed as an m-tuple $(x_i)_{1\le i\le m}$, where each component x_i lies in a real Hilbert space $\mathcal{H}_i$. A generic convex variational formulation for solving multicomponent signal recovery or decomposition problems is

$$\underset{x_1\in\mathcal{H}_1,\ldots,\,x_m\in\mathcal{H}_m}{\text{minimize}} \quad \Phi(x_1,\ldots,x_m), \tag{3.12}$$

where $\Phi\colon \mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_m \to \left]-\infty,+\infty\right]$ is a convex cost function. At this level of generality, however, no algorithm exists to solve (3.12) reliably in the sense that it produces m sequences $(x_{1,n})_{n\in\mathbb{N}}$, ..., $(x_{m,n})_{n\in\mathbb{N}}$ converging (weakly or strongly) to points x_1, ..., x_m, respectively, such that $(x_i)_{1\le i\le m}$ minimizes Φ. Let us recall that, even in the elementary case when $m=2$ and $\mathcal{H}_1=\mathcal{H}_2=\mathbb{R}$, the basic Gauss-Seidel alternating minimization algorithm does not possess this property [28]. In this paper, we consider the following, more structured version of (3.12).

Problem 3.4 Let $m\ge 2$ and $p\ge 1$ be integers, let $(\mathcal{H}_i)_{1\le i\le m}$ and $(\mathcal{G}_k)_{1\le k\le p}$ be real Hilbert spaces, and let $(\tau_k)_{1\le k\le p}$ be in $\left]0,+\infty\right[$. For every $i\in\{1,\ldots,m\}$, let $f_i\colon\mathcal{H}_i\to\left]-\infty,+\infty\right]$ be a proper lower semicontinuous convex function and, for every $k\in\{1,\ldots,p\}$, let $\varphi_k\colon\mathcal{G}_k\to\mathbb{R}$ be convex and differentiable with a τ_k–Lipschitz continuous gradient, and let $L_{ki}\colon\mathcal{H}_i\to\mathcal{G}_k$ be linear and bounded. It is assumed that $\min_{1\le k\le p}\sum_{i=1}^m\|L_{ki}\|^2>0$. The problem is to

$$\underset{x_1\in\mathcal{H}_1,\ldots,\,x_m\in\mathcal{H}_m}{\text{minimize}} \quad \sum_{i=1}^m f_i(x_i)+\sum_{k=1}^p\varphi_k\Bigg(\sum_{i=1}^m L_{ki}x_i\Bigg), \tag{3.13}$$

under the assumption that solutions exist.

Let us note that (3.13) is a particular case of (3.12), in which Φ is decomposed in two terms, namely

$$\Phi(x_1,\ldots,x_m)=\underbrace{\sum_{i=1}^m f_i(x_i)}_{\text{separable term}}+\underbrace{\sum_{k=1}^p\varphi_k\Bigg(\sum_{i=1}^m L_{ki}x_i\Bigg)}_{\text{coupling term}}. \tag{3.14}$$

Each function f_i in the separable term promotes an intrinsic property of the ith component x_i of the signal. On the other hand, the coupling term models p interactions between the m components $(x_i)_{1\leq i\leq m}$. An elementary interaction is associated with a potential φ_k acting on a linear transformation $\sum_{i=1}^m L_{ki}x_i$ of the components. The coupling is smooth in the sense that the function φ_k is differentiable with a Lipschitz gradient. As will be seen in subsequent sections, Problem 3.4 not only captures existing formulations for which reliable solution methods are not available, but it also allows us to investigate a wide range of new problems. In addition, it can be solved reliably by the following proximal algorithm recently developed in [4] (the definition of the proximity operator $\operatorname{prox}_{f_i}$ of a convex function $f_i\colon \mathcal{H}_i \to]-\infty,+\infty]$ is given in Section 3.2.2.2).

Algorithm 3.5 Set

$$\beta_1 = \frac{1}{p \max\limits_{1\leq k\leq p} \tau_k \sum\limits_{i=1}^m \|L_{ki}\|^2}, \tag{3.15}$$

and fix ε in $]0,\min\{1,\beta_1\}[$, $(\lambda_n)_{n\in\mathbb{N}}$ in $[\varepsilon,1]$, $(\gamma_n)_{n\in\mathbb{N}}$ in $[\varepsilon,2\beta_1-\varepsilon]$, and $(x_{i,0})_{1\leq i\leq m}$ in $\mathcal{H}_1\times\cdots\times\mathcal{H}_m$. For every $i\in\{1,\ldots,m\}$ set, for every $n\in\mathbb{N}$,

$$x_{i,n+1} = x_{i,n} + \lambda_n\Bigg(\operatorname{prox}_{\gamma_n f_i}\Bigg(x_{i,n} - \gamma_n\Bigg(\sum_{k=1}^p L_{ki}^*\nabla\varphi_k\Bigg(\sum_{j=1}^m L_{kj}x_{j,n}\Bigg) + b_{i,n}\Bigg)\Bigg) + a_{i,n} - x_{i,n}\Bigg), \tag{3.16}$$

where $(a_{i,n})_{n\in\mathbb{N}}$ and $(b_{i,n})_{n\in\mathbb{N}}$ are sequences in $\mathcal{H}_i$ such that

$$\sum_{n\in\mathbb{N}}\|a_{i,n}\| < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}}\|b_{i,n}\| < +\infty. \tag{3.17}$$

Algorithm 3.5 generates m sequences $(x_{1,n})_{n\in\mathbb{N}}$, ..., $(x_{m,n})_{n\in\mathbb{N}}$ in parallel. It also tolerates errors $a_{i,n}$ and $b_{i,n}$ in the implementation of the proximity operator and of the gradients, respectively. Its convergence to a solution to Problem 3.4 is guaranteed by the following theorem. Let us stress that, although some algorithms are available for specific instances of Problem 3.4 with $m=2$ (see [1], [3], [10], and [23, Section 4.4]), no method with such convergence properties seems to be available in the literature in the general setting we consider here.

Theorem 3.6 [4, Theorem 4.3] *Let $(x_{1,n})_{n\in\mathbb{N}}$, ..., $(x_{m,n})_{n\in\mathbb{N}}$ be sequences generated by Algorithm 3.5. Then, for every $i\in\{1,\ldots,m\}$, $(x_{i,n})_{n\in\mathbb{N}}$ converges weakly to a point $x_i\in\mathcal{H}_i$, and $(x_i)_{1\leq i\leq m}$ is a solution to Problem 3.4.*

The paper is organized as follows. In Section 3.2.2, we introduce our notation and recall some important definitions and properties from convex analysis, and discuss proximity operators. In Section 3.2.3, we study the particular case when the coupling functions are Moreau envelopes and address specific cases. Section 3.2.4 is devoted to problems in which the coupling functions are quadratic. In Section 3.2.5, the focus is placed on coupling terms involving linear combinations of the components. Finally, Section 3.2.6 is devoted to an application to multiframe signal representation.

3.2.2 Notation and background

Throughout the paper, $\mathcal{H}$ and $(\mathcal{H}_i)_{1\leq i\leq m}$ are real Hilbert spaces. Their scalar products are denoted by $\langle\cdot\mid\cdot\rangle$ and the associated norms by $\|\cdot\|$. Moreover, Id denotes the identity operator and $B(x;\rho)$ the closed ball of center $x\in\mathcal{H}$ and radius $\rho\in\left]0,+\infty\right[$. In this section, we recall some useful definitions and facts from convex analysis [31, 36, 46] and provide background and new results on proximity operators.

3.2.2.1 Convex analysis

We denote by $\Gamma_0(\mathcal{H})$ the class of lower semicontinuous convex functions $\varphi\colon\mathcal{H}\to\left]-\infty,+\infty\right]$ which are proper in the sense that $\operatorname{dom}\varphi=\big\{x\in\mathcal{H}\mid\varphi(x)<+\infty\big\}\neq\varnothing$.

Let $\varphi\in\Gamma_0(\mathcal{H})$. The set of minimizers of φ is denoted by $\operatorname{Argmin}\varphi$ and, if φ has a unique minimizer, this minimizer is denoted by $\operatorname{argmin}_{x\in\mathcal{H}}\varphi(x)$. The conjugate of φ is the function $\varphi^*\in\Gamma_0(\mathcal{H})$ defined by

$$\varphi^*\colon\mathcal{H}\to\left]-\infty,+\infty\right]\colon u\mapsto\sup_{x\in\mathcal{H}}\langle x\mid u\rangle-\varphi(x)\tag{3.18}$$

and the subdifferential of φ is the set-valued operator

$$\partial\varphi\colon\mathcal{H}\to 2^{\mathcal{H}}\colon x\mapsto\Big\{u\in\mathcal{H}\;\Big|\;(\forall y\in\mathcal{H})\quad\langle y-x\mid u\rangle+\varphi(x)\leq\varphi(y)\Big\}.\tag{3.19}$$

The Fenchel-Moreau theorem states that

$$\varphi^{**}=\varphi.\tag{3.20}$$

In addition,

$$(\forall x\in\mathcal{H})(\forall u\in\mathcal{H})\quad\begin{cases}\varphi(x)+\varphi^*(u)\geq\langle x\mid u\rangle\\ \varphi(x)+\varphi^*(u)=\langle x\mid u\rangle\;\Leftrightarrow\;u\in\partial\varphi(x).\end{cases}\tag{3.21}$$

The next lemma follows directly from [17, Corollary 3.5].

Lemma 3.7 *Let $g\colon \mathcal{H} \to [0,+\infty[$ be a continuous convex function and let $\phi \in \Gamma_0(\mathbb{R})$. Suppose that ϕ is increasing on $[0,+\infty[$ and that there exists a point $z \in \mathcal{H}$ such that $g(z) \in \operatorname{int}\operatorname{dom}\phi$. Then, for every $x \in \mathcal{H}$,*

$$\partial(\phi\circ g)(x) = \bigcup_{\nu\in\partial\phi(g(x))\cap[0,+\infty[} \nu\,\partial g(x). \tag{3.22}$$

Now, let C be a nonempty closed convex subset of $\mathcal{H}$. The indicator function of C is

$$\iota_C\colon x \mapsto \begin{cases} 0, & \text{if } x \in C;\\ +\infty, & \text{if } x \notin C, \end{cases} \tag{3.23}$$

the normal cone operator of C is

$$N_C = \partial\iota_C\colon x \mapsto \begin{cases} \big\{u \in \mathcal{H} \mid (\forall y \in C)\ \ \langle y-x \mid u\rangle \le 0\big\}, & \text{if } x \in C;\\ \varnothing, & \text{otherwise,} \end{cases} \tag{3.24}$$

the support function of C is

$$\sigma_C = \iota_C^*\colon \mathcal{H} \to \left]-\infty,+\infty\right] : u \mapsto \sup_{x\in C} \langle x \mid u\rangle\,, \tag{3.25}$$

and the distance from $x \in \mathcal{H}$ to C is $d_C(x) = \inf_{y\in C}\|x-y\|$. For every $x \in \mathcal{H}$, there exists a unique point $P_Cx \in C$ such that $d_C(x) = \|x - P_Cx\|$; P_Cx is called the projection of x onto C and it is characterized by

$$(\forall p \in \mathcal{H})\quad p = P_Cx \quad\Leftrightarrow\quad x-p \in N_Cp. \tag{3.26}$$

We have

$$(\forall x \in \mathcal{H})\quad \partial d_C(x) = \begin{cases} \left\{\dfrac{x-P_Cx}{d_C(x)}\right\}, & \text{if} \quad x \in \mathcal{H}\setminus C;\\ N_Cx \cap B(0;1), & \text{if} \quad x \in C. \end{cases} \tag{3.27}$$

Lemma 3.8 *Let C be a nonempty convex closed subset of $\mathcal{H}$, let $\phi\colon \mathbb{R} \to \left]-\infty,+\infty\right]$ be increasing on $[0,+\infty[$ and even, and set $\varphi = \phi\circ d_C$. Then $\varphi^* = \sigma_C + \phi^*\circ\|\cdot\|$.*

Proof. Set, for every $\eta \in [0,+\infty[$, $D_\eta = \big\{z \in \mathcal{H} \mid \|z\| = \eta\big\}$. For every $x \in \mathcal{H}$, since $\inf_{y\in C}\|x-y\| = \|x-P_Cx\|$ and since ϕ is increasing on $[0,+\infty[$, we have

$$(\forall z \in C)\quad \inf_{y\in C}\phi(\|x-y\|) \le \phi(\|x-P_Cx\|) = \phi\Big(\inf_{y\in C}\|x-y\|\Big) \le \phi(\|x-z\|), \tag{3.28}$$

which implies that $\inf_{y\in C}\phi(\|x-y\|) = \phi(\inf_{y\in C}\|x-y\|)$. Hence, since $\mathcal{H} = \bigcup_{\eta\in[0,+\infty[} D_\eta$ and since ϕ is even, we have

$$
\begin{aligned}
(\forall u \in \mathcal{H}) \quad \varphi^*(u) &= \sup_{x\in\mathcal{H}} \langle x \mid u\rangle - \phi\Big(\inf_{y\in C}\|x-y\|\Big) \\
&= \sup_{x\in\mathcal{H}} \langle x \mid u\rangle - \inf_{y\in C}\phi(\|x-y\|) \\
&= \sup_{y\in C}\sup_{x\in\mathcal{H}} \langle x \mid u\rangle - \phi(\|x-y\|) \\
&= \sup_{y\in C} \langle y \mid u\rangle + \sup_{z\in\mathcal{H}} \langle z \mid u\rangle - \phi(\|z\|) \\
&= \sup_{y\in C} \langle y \mid u\rangle + \sup_{\eta\in[0,+\infty[}\ \sup_{z\in D_\eta} \langle z \mid u\rangle - \phi(\eta) \\
&= \sup_{y\in C} \langle y \mid u\rangle + \sup_{\eta\in[0,+\infty[} \eta\,\|u\| - \phi(\eta) \\
&= \sup_{y\in C} \langle y \mid u\rangle + \sup_{\eta\in\mathbb{R}} \eta\,\|u\| - \phi(\eta) \\
&= \sigma_C(u) + \phi^*(\|u\|),
\end{aligned}
\tag{3.29}
$$

which completes the proof. □

Lemma 3.9 *Let* $\phi \in \Gamma_0(\mathbb{R})$ *be such that* $0 \in \operatorname{int}\operatorname{dom}\phi$*, let* $\xi \in\,]0,+\infty[\, \cap \operatorname{dom}\partial\phi$*, and let* $\nu \in \partial\phi(\xi)$*. Then*

$$
\max \partial\phi(0) \leq \nu. \tag{3.30}
$$

Proof. Since $0 \in \operatorname{int}\operatorname{dom}\phi$, $\partial\phi(0)$ is a nonempty compact set [36, p. 215 and Theorem 23.4]. Moreover, (3.19) yields

$$
(\forall \mu \in \partial\phi(0)) \quad
\begin{cases}
(\xi - 0)\mu + \phi(0) \leq \phi(\xi) \\
(0 - \xi)\nu + \phi(\xi) \leq \phi(0).
\end{cases}
\tag{3.31}
$$

Adding these inequalities results in

$$
(\forall \mu \in \partial\phi(0)) \quad \mu\xi \leq \nu\xi, \tag{3.32}
$$

from which we deduce (3.30). □

3.2.2.2 Proximity operators

For a detailed account of the theory of proximity operators, see [31, 23] and the classical paper [35].

Let $\varphi \in \Gamma_0(\mathcal{H})$ and let $\gamma \in\,]0,+\infty[$. The Moreau envelope of index γ of φ is the continuous convex function

$$
{}^{\gamma}\varphi\colon \mathcal{H} \to \mathbb{R}\colon x \mapsto \inf_{y\in\mathcal{H}} \varphi(y) + \frac{1}{2\gamma}\|x-y\|^2. \tag{3.33}
$$

For every $x \in \mathcal{H}$, the infimum in (3.33) is achieved at a unique point denoted by $\operatorname{prox}_{\gamma\varphi} x$, which is characterized by the inclusion

$$(\forall p \in \mathcal{H}) \quad p = \operatorname{prox}_{\gamma\varphi} x \quad \Leftrightarrow \quad x - p \in \gamma\partial\varphi(p). \tag{3.34}$$

The proximity operator of φ is defined as

$$\operatorname{prox}_{\varphi} \colon \mathcal{H} \to \mathcal{H} \colon x \mapsto \operatorname{argmin}_{y\in\mathcal{H}} \varphi(y) + \frac{1}{2}\|x - y\|^2. \tag{3.35}$$

The Moreau envelope ${}^{\gamma}\varphi$ satisfies

$${}^{\gamma}\varphi \leq \varphi \quad \text{and} \quad \operatorname{Argmin} {}^{\gamma}\varphi = \operatorname{Argmin} \varphi. \tag{3.36}$$

Moreover, it is Fréchet differentiable and

$$\nabla\, {}^{\gamma}\varphi = \frac{1}{\gamma}(\operatorname{Id} - \operatorname{prox}_{\gamma\varphi}) = \operatorname{prox}_{\varphi^*/\gamma}(\cdot/\gamma) \;\text{ is } 1/\gamma\text{–Lipschitz continuous.} \tag{3.37}$$

Lemma 3.10 [19, Proposition 11] *Let $\mathcal{G}$ be a real Hilbert space, let $\psi \in \Gamma_0(\mathcal{G})$, let $L \colon \mathcal{H} \to \mathcal{G}$ be linear and bounded, and set $\varphi = \psi \circ L$. Suppose that $L \circ L^* = \kappa \operatorname{Id}$, for some $\kappa \in \left]0, +\infty\right[$. Then $\varphi \in \Gamma_0(\mathcal{H})$ and*

$$\operatorname{prox}_{\varphi} = \operatorname{Id} + \frac{1}{\kappa} L^* \circ (\operatorname{prox}_{\kappa\psi} - \operatorname{Id}) \circ L. \tag{3.38}$$

If C is a nonempty closed and convex subset C of $\mathcal{H}$, we have

$$\operatorname{prox}_{\gamma\iota_C} = P_C. \tag{3.39}$$

Closed-form expressions for the proximity operators of various functions can be found in [16, 19, 20, 21, 23, 35]. We now derive new examples, some of which will be used in Section 3.2.3.3.

Proposition 3.11 *Let C be a nonempty closed convex subset of $\mathcal{H}$, let $\phi \in \Gamma_0(\mathbb{R})$ be even, and set $\varphi = \phi \circ d_C$. Then $\varphi \in \Gamma_0(\mathcal{H})$. Moreover, $\operatorname{prox}_{\varphi} = P_C$ if $\phi = \iota_{\{0\}} + \eta$ for some $\eta \in \mathbb{R}$ and, otherwise, for every $x \in \mathcal{H}$,*

$$\operatorname{prox}_{\varphi} x = \begin{cases} x + \dfrac{\operatorname{prox}_{\phi^*} d_C(x)}{d_C(x)}(P_C x - x), & \text{if } d_C(x) > \max \partial\phi(0); \\ P_C x, & \text{if } x \notin C \text{ and } d_C(x) \leq \max\partial\phi(0); \\ x, & \text{if } x \in C. \end{cases} \tag{3.40}$$

Proof. If $\phi = \iota_{\{0\}} + \eta$ for some $\eta \in \mathbb{R}$, then $\varphi = \iota_C + \eta$, which implies that $\varphi \in \Gamma_0(\mathcal{H})$ and that $\operatorname{prox}_{\varphi} = P_C$. Now assume that $\phi \neq \iota_{\{0\}} + \eta$ with $\eta \in \mathbb{R}$. Since ϕ is even, convex, and proper, we have $0 \in \operatorname{int}\operatorname{dom}\phi$ and it follows that

$$(\forall z \in C) \quad d_C(z) = 0 \in \operatorname{int}\operatorname{dom}\phi. \tag{3.41}$$

Thus, $\varnothing \neq C \subset \operatorname{dom}\varphi$, which shows that φ is proper. Next, since d_C is continuous and ϕ is lower semicontinuous, φ is lower semicontinuous. Moreover, since ϕ is convex and even, it is increasing on $[0,+\infty[$ and, by convexity of d_C, we deduce that φ is convex. Altogether $\varphi \in \Gamma_0(\mathcal{H})$.

Now, let $x \in \mathcal{H}$ and set $p = \operatorname{prox}_\varphi x$. We derive from (3.34) that

$$x - p \in \partial(\phi \circ d_C)(p). \tag{3.42}$$

Therefore, in view of (3.41), taking $g = d_C$ in Lemma 3.7 yields

$$x - p \in \bigcup_{\nu \in \partial\phi(d_C(p)) \cap [0,+\infty[} \nu\, \partial d_C(p). \tag{3.43}$$

We examine two alternatives.

(a) $p \in C$: In this case, $d_C(p) = 0$ and, from (3.27), $\partial d_C(p) = N_C p \cap B(0;1)$. Hence, (3.43) asserts that there exists $\nu \in \partial\phi(0) \cap [0,+\infty[$ such that

$$x - p \in N_C p \cap B(0;\nu). \tag{3.44}$$

Using (3.26), we first deduce that

$$p = P_C x. \tag{3.45}$$

In addition,

$$d_C(x) = \|x - P_C x\| = \|x - p\| \leq \nu \leq \max \partial\phi(0). \tag{3.46}$$

(b) $p \notin C$: In this case, $d_C(p) > 0$ and (3.27) yields $\partial d_C(p) = \{(p - P_C p)/d_C(p)\}$. Hence, (3.43) implies that there exists $\nu \in \partial\phi(d_C(p)) \cap [0,+\infty[$ such that

$$x - p = \nu \frac{p - P_C p}{d_C(p)}, \tag{3.47}$$

which can be written equivalently as

$$x - P_C p = \frac{\nu + d_C(p)}{d_C(p)}(p - P_C p). \tag{3.48}$$

Since (3.26) asserts that $p - P_C p \in N_C(P_C p)$, (3.48) yields $x - P_C p \in N_C(P_C p)$ and, therefore,

$$P_C x = P_C p. \tag{3.49}$$

Consequently, (3.48) is equivalent to

$$x - P_C x = \frac{\nu + d_C(p)}{d_C(p)}(p - P_C p). \tag{3.50}$$

In turn, upon applying the norm, we obtain

$$d_C(x) = \nu + d_C(p). \tag{3.51}$$

Since $\nu \in \partial\phi(d_C(p))$, we deduce from (3.51) that

$$d_C(x) - d_C(p) \in \partial\phi(d_C(p)), \tag{3.52}$$

which yields $d_C(p) = \operatorname{prox}_\phi d_C(x)$ by (3.34). Thus, it follows from (3.51) and (3.37) that

$$\nu = \operatorname{prox}_{\phi^*}(d_C(x)). \tag{3.53}$$

On the other hand, it follows from (3.50) and (3.51) that

$$\frac{p - P_C p}{d_C(p)} = \frac{x - P_C x}{\nu + d_C(p)} = \frac{x - P_C x}{d_C(x)}. \tag{3.54}$$

Hence, using (3.53) and (3.54), we deduce from (3.47) that

$$p = x + \frac{\operatorname{prox}_{\phi^*}(d_C(x))}{d_C(x)}(P_C x - x). \tag{3.55}$$

In view of (3.45) and (3.55), it remains to show that

$$p \in C \quad \Leftrightarrow \quad d_C(x) \leq \max \partial\phi(0). \tag{3.56}$$

To this end, we first observe that (3.46) yields $p \in C \Rightarrow d_C(x) \leq \max \partial\phi(0)$. For the reverse implication, suppose that $d_C(x) \leq \max \partial\phi(0)$ and that $p \notin C$. Then, we deduce from Lemma 3.9 and (3.51) that

$$\max \partial\phi(0) + d_C(p) \leq \nu + d_C(p) = d_C(x) \leq \max \partial\phi(0), \tag{3.57}$$

which implies that $d_C(p) = 0$ and therefore that $p \in \overline{C} = C$, which contradicts our assumption. □

Proposition 3.12 *Let C be a nonempty closed convex subset of $\mathcal{H}$, let $\phi \in \Gamma_0(\mathbb{R})$ be even and nonconstant, and set $\varphi = \sigma_C + \phi \circ \|\cdot\|$. Then $\varphi \in \Gamma_0(\mathcal{H})$ and, for every $x \in \mathcal{H}$,*

$$\operatorname{prox}_\varphi x = \begin{cases} \dfrac{\operatorname{prox}_\phi d_C(x)}{d_C(x)}(x - P_C x), & \text{if } d_C(x) > \max \operatorname{Argmin} \phi; \\ x - P_C x, & \text{if } x \notin C \text{ and } d_C(x) \leq \max \operatorname{Argmin} \phi; \\ 0, & \text{if } x \in C. \end{cases} \tag{3.58}$$

Proof. Set $\psi = \phi^* \circ d_C$. Since ϕ is an even function in $\Gamma_0(\mathbb{R})$, ϕ^* is likewise. Hence, it follows from Proposition 3.11 that $\psi \in \Gamma_0(\mathcal{H})$. Using the facts that $\partial\phi^*(0) = \operatorname{Argmin}\phi$, that ϕ^* is not of the form $\iota_{\{0\}} + \eta$ with $\eta \in \mathbb{R}$, and that, by (3.20), $\phi^{**} = \phi$, we also derive from Proposition 3.11 that, for every $x \in \mathcal{H}$,

$$\operatorname{prox}_\psi x = \begin{cases} x + \dfrac{\operatorname{prox}_\phi d_C(x)}{d_C(x)}(P_C x - x), & \text{if } d_C(x) > \max \operatorname{Argmin}\phi; \\ P_C x, & \text{if } x \notin C \text{ and } d_C(x) \leq \max \operatorname{Argmin}\phi; \\ x, & \text{if } x \in C. \end{cases} \tag{3.59}$$

On the other hand, Lemma 3.8 yields $\psi^* = \sigma_C + \phi^{**} \circ \|\cdot\| = \sigma_C + \phi \circ \|\cdot\| = \varphi$. Hence, it follows from (3.37) (with $\gamma = 1$) that $\operatorname{prox}_\varphi = \operatorname{prox}_{\psi^*} = \operatorname{Id} - \operatorname{prox}_\psi$. In view of (3.59), we thus obtain (3.58). □

Proposition 3.13 *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{G}$ and let $z \in \mathcal{G}$. Let $\phi \in \Gamma_0(\mathbb{R})$ be even and not of the form $\phi = \iota_{\{0\}} + \eta$ with $\eta \in \mathbb{R}$, let $L\colon \mathcal{H} \to \mathcal{G}$ be a bounded linear operator such that $L \circ L^* = \kappa \operatorname{Id}$ for some $\kappa \in \left]0, +\infty\right[$, and set*

$$\varphi\colon \mathcal{H} \to \left]-\infty, +\infty\right] : x \mapsto \phi(d_C(Lx - z)). \tag{3.60}$$

Then $\varphi \in \Gamma_0(\mathcal{H})$ and, for every $x \in \mathcal{H}$,

$$\operatorname{prox}_\varphi x = \begin{cases} x + \dfrac{\operatorname{prox}_{(\kappa\phi)^*} d_C(Lx - z)}{\kappa\, d_C(Lx - z)} L^*\big(P_C(Lx - z) + z - Lx\big), \\ \qquad\qquad \text{if } d_C(Lx - z) > \kappa \max \partial\phi(0); \\ x + \dfrac{1}{\kappa} L^*\big(P_C(Lx - z) + z - Lx\big), \\ \qquad\qquad \text{if } Lx - z \notin C \text{ and } d_C(Lx - z) \leq \kappa \max \partial\phi(0); \\ x, \qquad\qquad \text{if } Lx - z \in C. \end{cases} \tag{3.61}$$

Proof. Set $g = \phi \circ d_C$. It follows from Proposition 3.11 that $g \in \Gamma_0(\mathcal{G})$ and that, for every $y \in \mathcal{G}$,

$$\operatorname{prox}_{\kappa g} y = \begin{cases} y + \dfrac{\operatorname{prox}_{(\kappa\phi)^*} d_C(y)}{d_C(y)}(P_C y - y), & \text{if } d_C(y) > \kappa \max \partial\phi(0); \\ P_C y, & \text{if } d_C(y) \leq \kappa \max \partial\phi(0). \end{cases} \tag{3.62}$$

We also observe that, since $\varphi = g \circ (L \cdot - z)$ and L is linear and continuous, $\varphi \in \Gamma_0(\mathcal{H})$. Now take $x \in \mathcal{H}$ and set $p = \operatorname{prox}_\varphi x$. Using (3.34), the identity $L \circ L^* = \kappa \operatorname{Id}$, and elementary subdifferential calculus, we obtain

$$\begin{aligned} p = \operatorname{prox}_\varphi x \quad &\Leftrightarrow \quad x - p \in \partial\varphi(p) = L^*\partial g(Lp - z) \\ &\Leftrightarrow \quad (x - \kappa^{-1}L^*z) - (p - \kappa^{-1}L^*z) \in L^*\partial g(L(p - \kappa^{-1}L^*z)) \\ &\Leftrightarrow \quad (x - \kappa^{-1}L^*z) - (p - \kappa^{-1}L^*z) \in \partial(g \circ L)(p - \kappa^{-1}L^*z) \\ &\Leftrightarrow \quad p - \kappa^{-1}L^*z = \operatorname{prox}_{g\circ L}(x - \kappa^{-1}L^*z). \end{aligned} \tag{3.63}$$

Hence, by Lemma 3.10,

$$\begin{aligned} p &= \kappa^{-1}L^*z + (x - \kappa^{-1}L^*z) + \kappa^{-1}L^*\big(\operatorname{prox}_{\kappa g}(L(x-\kappa^{-1}L^*z)) - L(x-\kappa^{-1}L^*z)\big) \\ &= x + \kappa^{-1}L^*\big(\operatorname{prox}_{\kappa g}(Lx - z) + z - Lx\big). \end{aligned} \tag{3.64}$$

Upon combining (3.64) and (3.62) we obtain (3.61). □

3.2.3 Coupling with Moreau envelopes

In this section we interpret Problem 3.4 as a relaxation of a problem with a nonsmooth coupling term.

3.2.3.1 Problem formulation

As seen in (3.37), the Moreau envelope of index $\rho_k \in \,]0,+\infty[$ of a function $g_k \in \Gamma_0(\mathcal{G})$ is a convex function which is $1/\rho_k$–Lipschitz differentiable everywhere. We can therefore set

$$(\forall k \in \{1,\ldots,p\}) \quad \varphi_k = {}^{\rho_k}g_k \tag{3.65}$$

in Problem 3.4 to obtain the following formulation.

Problem 3.14 Let $(\mathcal{H}_i)_{1\le i\le m}$ and $(\mathcal{G}_k)_{1\le k\le p}$ be real Hilbert spaces, and let $(\rho_k)_{1\le k\le p}$ be in $]0,+\infty[$. For every $i \in \{1,\ldots,m\}$ and $k \in \{1,\ldots,p\}$, let $f_i \in \Gamma_0(\mathcal{H}_i)$, let $g_k \in \Gamma_0(\mathcal{G}_k)$, and let $L_{ki}\colon \mathcal{H}_i \to \mathcal{G}_k$ be linear and bounded. It is assumed that

$$\min_{1\le k\le p} \sum_{i=1}^m \|L_{ki}\|^2 > 0. \tag{3.66}$$

The problem is to

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\text{minimize}} \quad \sum_{i=1}^m f_i(x_i) + \sum_{k=1}^p {}^{\rho_k}g_k\Bigg(\sum_{i=1}^m L_{ki}x_i\Bigg), \tag{3.67}$$

under the assumption that solutions exist.

The functions $({}^{\rho_k}g_k)_{1\le k\le p}$ are approximations to the functions $(g_k)_{1\le k\le p}$ in the sense of (3.36). Thus, (3.67) can be regarded as a relaxation of the problem

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\text{minimize}} \quad \sum_{i=1}^m f_i(x_i) + \sum_{k=1}^p g_k\Bigg(\sum_{i=1}^m L_{ki}x_i\Bigg). \tag{3.68}$$

Since this problem involves not necessarily smooth coupling functions $(g_k)_{1\leq k\leq p}$, it will in general be harder to solve than (3.67) and, in some cases, it may not possess any solution while (3.67) does (see [18] for an illustration of the latter situation).

In view of (3.65) and (3.37), the specialization of Algorithm 3.5 to Problem 3.14 assumes the following form.

Algorithm 3.15 Set

$$\beta_1 = \frac{1}{p}\min_{1\leq k\leq p} \frac{\rho_k}{\sum_{i=1}^m \|L_{ki}\|^2}, \tag{3.69}$$

and fix ε in $]0, \min\{1, \beta_1\}[$, $(\lambda_n)_{n\in\mathbb{N}}$ in $[\varepsilon, 1]$, $(\gamma_n)_{n\in\mathbb{N}}$ in $[\varepsilon, 2\beta_1 - \varepsilon]$, and $(x_{i,0})_{1\leq i\leq m}$ in $\mathcal{H}_1 \times \cdots \times \mathcal{H}_m$. For every $i \in \{1, \ldots, m\}$ set, for every $n \in \mathbb{N}$,

$$x_{i,n+1} = x_{i,n} + \lambda_n\Bigg(\operatorname{prox}_{\gamma_n f_i}\Bigg(x_{i,n} + \gamma_n\Bigg(\sum_{k=1}^p L_{ki}^*\Bigg(\frac{\operatorname{prox}_{\rho_k g_k} - \mathrm{Id}}{\rho_k}\Bigg)\Bigg(\sum_{j=1}^m L_{kj}x_{j,n}\Bigg) + b_{i,n}\Bigg)\Bigg) + a_{i,n} - x_{i,n}\Bigg), \tag{3.70}$$

where $(a_{i,n})_{n\in\mathbb{N}}$ and $(b_{i,n})_{n\in\mathbb{N}}$ are sequences in $\mathcal{H}_i$ such that

$$\sum_{n\in\mathbb{N}} \|a_{i,n}\| < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} \|b_{i,n}\| < +\infty. \tag{3.71}$$

We obtain the weak convergence of this algorithm as a direct application of Theorem 3.6.

Corollary 3.16 *Let $(x_{1,n})_{n\in\mathbb{N}}$, ..., $(x_{m,n})_{n\in\mathbb{N}}$ be sequences generated by Algorithm 3.15. Then, for every $i \in \{1, \ldots, m\}$, $(x_{i,n})_{n\in\mathbb{N}}$ converges weakly to a point $x_i \in \mathcal{H}_i$, and $(x_i)_{1\leq i\leq m}$ is a solution to Problem 3.14.*

Remark 3.17 In the particular case of $m = 1$ variable, Problem 3.14 reduces to [23, Problem 4.1], which was itself shown in [23, Section 4] to cover several signal decomposition and recovery problems.

3.2.3.2 Relaxation of problems with hard coupling

As a first application of the results of Section 3.2.3.1, we consider problems in which hard constraints on p linear mixtures of the signals are available. More precisely, the constraints are of the form

$$(\forall k \in \{1, \ldots, p\}) \quad \sum_{i=1}^m L_{ki}x_i \in D_k, \tag{3.72}$$

where each D_k is a nonempty closed convex subset of $\mathcal{G}_k$ and, for every $i \in \{1, \ldots, m\}$, L_{ki} is a bounded linear operator from $\mathcal{H}_i$ to $\mathcal{G}_k$. In our setting, this leads to the hard-coupled problem

$$\underset{\substack{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m \\ \sum_{i=1}^m L_{1i} x_i \in D_1, \ldots, \sum_{i=1}^m L_{pi} x_i \in D_p}}{\text{minimize}} \quad \sum_{i=1}^m f_i(x_i), \tag{3.73}$$

which amounts to setting

$$(\forall k \in \{1, \ldots, p\}) \quad g_k = \iota_{D_k} \tag{3.74}$$

in (3.68). Let us note that, due to inaccuracies in the definition of the sets $(D_k)_{1 \le k \le p}$ [18, 29], (3.73) may be infeasible in the sense that $\bigcap_{k=1}^p D_k = \varnothing$. On the other hand, the approximate problem (3.67), which becomes

$$\underset{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m}{\text{minimize}} \quad \sum_{i=1}^m f_i(x_i) + \sum_{k=1}^p \frac{1}{2\rho_k} d_{D_k}^2 \left(\sum_{i=1}^m L_{ki} x_i \right), \tag{3.75}$$

will admit solutions under mild assumptions [23, Proposition 3.1(i)]. Moreover, using (3.39), the iteration (3.70) in Algorithm 3.15 reduces to (we set $a_{i,n} \equiv 0$, $b_{i,n} \equiv 0$, and $\lambda_n \equiv 1$ for simplicity)

$$x_{i,n+1} = \operatorname{prox}_{\gamma_n f_i} \left(x_{i,n} + \gamma_n \sum_{k=1}^p L_{ki}^* \left(\frac{P_{D_k} - \mathrm{Id}}{\rho_k} \right) \left(\sum_{j=1}^m L_{kj} x_{j,n} \right) \right). \tag{3.76}$$

As an illustration of the construction of the sets $(D_k)_{1 \le k \le p}$, let us consider the problem of finding m sources $(x_i)_{1 \le i \le m}$ from the noisy observation of p mixtures

$$(\forall k \in \{1, \ldots, p\}) \quad z_k = \sum_{i=1}^m L_{ki} x_i + w_k, \tag{3.77}$$

where $w_k \in \mathcal{G}_k$ represents the noise corrupting the kth measurement. As discussed in [22, 39], a wide range of probabilistic *a priori* information on the kth noise process can be translated into constraints of the form $z_k - \sum_{i=1}^m L_{ki} x_i \in E_k$, where E_k is a closed convex subset of $\mathcal{G}_k$. This corresponds to (3.72), where $D_k = z_k - E_k$. For instance, if a statistical bound η_k is available on the energy of the kth noise process, we obtain $D_k = B(z_k; \sqrt{\eta_k})$.

3.2.3.3 Relaxation of problems with hard constraints and hard coupling

We place ourselves in the same setting as in Section 3.2.3.2 and make the additional assumption that hard constraints are available for each signal, namely

$$(\forall i \in \{1, \ldots, m\}) \quad x_i \in C_i, \tag{3.78}$$

where each C_i is a nonempty closed convex subset of $\mathcal{H}_i$. In this context, (3.73) coincides with the feasibility problem

$$\text{Find} \;\; x_1 \in C_1, \dots, x_m \in C_m \;\; \text{such that} \;\; \sum_{i=1}^m L_{1i} x_i \in D_1, \dots, \sum_{i=1}^m L_{pi} x_i \in D_p. \tag{3.79}$$

Let us relax the p constraints $\sum_{i=1}^m L_{ki} x_i \in D_k$ as in (3.75) and the m constraints in (3.78) by penalizing the distances to the sets via functions

$$(\forall i \in \{1, \dots, m\}) \quad f_i = \phi_i \circ d_{C_i}, \tag{3.80}$$

where $(\phi_i)_{1 \le i \le m}$ are nonzero even functions in $\Gamma_0(\mathbb{R}) \smallsetminus \{\iota_{\{0\}}\}$ such that $\phi_i(0) \equiv 0$. Thus, (3.75) becomes

$$\underset{x_1 \in \mathcal{H}_1, \dots, x_m \in \mathcal{H}_m}{\text{minimize}} \quad \sum_{i=1}^m \phi_i(d_{C_i}(x_i)) + \sum_{k=1}^p \frac{1}{2\rho_k} d_{D_k}^2 \left(\sum_{i=1}^m L_{ki} x_i \right), \tag{3.81}$$

which is our relaxation of (3.79). Corollary 3.16 asserts that this problem can be solved via Algorithm 3.15 where, by virtue of Proposition 3.11, (3.70) reduces to (we set $a_{i,n} \equiv 0$, $b_{i,n} \equiv 0$, and $\lambda_n \equiv 1$ for simplicity)

$$\begin{cases} y_{i,n} = x_{i,n} + \gamma_n \displaystyle\sum_{k=1}^p L_{ki}^* \left(\frac{P_{D_k} - \mathrm{Id}}{\rho_k} \right) \left(\sum_{j=1}^m L_{kj} x_{j,n} \right) \\ x_{i,n+1} = \begin{cases} y_{i,n} + \dfrac{\operatorname{prox}_{(\gamma_n \phi_i)^*} d_{C_i}(y_{i,n})}{d_{C_i}(y_{i,n})} (P_{C_i} y_{i,n} - y_{i,n}), \\ \qquad\qquad \text{if} \;\; d_{C_i}(y_{i,n}) > \gamma_n \max \partial\phi_i(0); \\ P_{C_i} y_{i,n}, \qquad \text{if} \;\; y_{i,n} \notin C_i \;\text{and}\; d_{C_i}(y_{i,n}) \le \gamma_n \max \partial\phi_i(0); \\ y_{i,n}, \qquad \text{if} \;\; y_{i,n} \in C_i. \end{cases} \end{cases} \tag{3.82}$$

3.2.4 Quadratic coupling

In this section, we study Problem 3.4 when the coupling functions $(\varphi_k)_{1 \le k \le p}$ are of the form

$$(\forall k \in \{1, \dots, p\}) \quad \varphi_k = \frac{1}{2} \| z_k - \cdot \|^2, \quad \text{where} \quad z_k \in \mathcal{G}_k. \tag{3.83}$$

3.2.4.1 Problem formulation

We first restate Problem 3.4 under assumption (3.83).

Problem 3.18 Let $(\mathcal{H}_i)_{1\leq i\leq m}$ and $(\mathcal{G}_k)_{1\leq k\leq p}$ be real Hilbert spaces. For every $i \in \{1,\ldots,m\}$, let $f_i \in \Gamma_0(\mathcal{H}_i)$ and, for every $k \in \{1,\ldots,p\}$, let $z_k \in \mathcal{G}_k$ and let $L_{ki}\colon \mathcal{H}_i \to \mathcal{G}_k$ be linear and bounded. It is assumed that $\min_{1\leq k\leq p} \sum_{i=1}^m \|L_{ki}\|^2 > 0$. The problem is to

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\text{minimize}} \quad \sum_{i=1}^m f_i(x_i) + \frac{1}{2}\sum_{k=1}^p \left\| z_k - \sum_{i=1}^m L_{ki}x_i \right\|^2, \tag{3.84}$$

under the assumption that solutions exist.

Here is a variant of Algorithm 3.5 for solving Problem 3.18.

Algorithm 3.19 Set

$$\beta_2 = \frac{1}{\displaystyle\sum_{k=1}^p \sum_{i=1}^m \|L_{ki}\|^2}, \tag{3.85}$$

and fix ε in $]0, \min\{1,\beta_2\}[$, $(\lambda_n)_{n\in\mathbb{N}}$ in $[\varepsilon,1]$ $(\gamma_n)_{n\in\mathbb{N}}$ in $[\varepsilon, 2\beta_2-\varepsilon]$, and $(x_{i,0})_{1\leq i\leq m}$ in $\mathcal{H}_1 \times \cdots \times \mathcal{H}_m$. For every $i \in \{1,\ldots,m\}$ set, for every $n \in \mathbb{N}$,

$$x_{i,n+1} = x_{i,n} + \lambda_n\Bigg(\operatorname{prox}_{\gamma_n f_i}\Bigg(x_{i,n} + \gamma_n\Bigg(\sum_{k=1}^p L_{ki}^*\Bigg(z_k - \sum_{j=1}^m L_{kj}x_{j,n}\Bigg) + b_{i,n}\Bigg)\Bigg) + a_{i,n} - x_{i,n}\Bigg), \tag{3.86}$$

where $(a_{i,n})_{n\in\mathbb{N}}$ and $(b_{i,n})_{n\in\mathbb{N}}$ are sequences in $\mathcal{H}_i$ such that

$$\sum_{n\in\mathbb{N}} \|a_{i,n}\| < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} \|b_{i,n}\| < +\infty. \tag{3.87}$$

Remark 3.20 The Lipschitz constant of each $\nabla\varphi_k$ is $\tau_k = 1$. Hence, the bound β_1 of (3.15) is $\beta_1 = 1/\big(p \max_{1\leq k\leq p} \sum_{i=1}^m \|L_{ki}\|^2\big)$. If we used this bound in (3.85), we could derive at once the convergence of Algorithm 3.19 from Theorem 3.6. However, we use the bound β_2 of (3.85), which is better than the general bound β_1 since

$$\beta_1 = \left(p \max_{1\leq k\leq p} \sum_{i=1}^m \|L_{ki}\|^2\right)^{-1} \leq \left(\sum_{k=1}^p \sum_{i=1}^m \|L_{ki}\|^2\right)^{-1} = \beta_2. \tag{3.88}$$

Theorem 3.21 *Let $(x_{1,n})_{n\in\mathbb{N}}$, ..., $(x_{m,n})_{n\in\mathbb{N}}$ be sequences generated by Algorithm 3.19. Then, for every $i \in \{1,\ldots,m\}$, $(x_{i,n})_{n\in\mathbb{N}}$ converges weakly to a point $x_i \in \mathcal{H}_i$, and $(x_i)_{1\leq i\leq m}$ is a solution to Problem 3.18.*

Proof. We set $\boldsymbol{\mathcal{H}} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$, i.e., $\boldsymbol{\mathcal{H}}$ is the real Hilbert space obtained by endowing $\mathcal{H}_1 \times \cdots \times \mathcal{H}_m$ with the scalar product $\langle\langle \cdot \mid \cdot \rangle\rangle\colon (\boldsymbol{x},\boldsymbol{y}) \mapsto \sum_{i=1}^m \langle x_i \mid y_i\rangle$, with associated norm $|||\cdot|||\colon \boldsymbol{x} \mapsto \sqrt{\sum_{i=1}^m \|x_i\|^2}$, where $\boldsymbol{x} = (x_i)_{1\leq i\leq m}$ denotes a generic element in $\boldsymbol{\mathcal{H}}$. We also introduce

$$\boldsymbol{g}\colon \boldsymbol{\mathcal{H}} \to \mathbb{R}\colon \boldsymbol{x} \mapsto \frac{1}{2}\sum_{k=1}^{p}\left\| z_k - \sum_{j=1}^{m} L_{kj}x_j\right\|^2 \tag{3.89}$$

and, for every $i \in \{1,\ldots,m\}$, we let B_i be the gradient of $\boldsymbol{g}$ with respect to the ith variable. Thus, $\nabla\boldsymbol{g} = (B_i)_{1\leq i\leq m}$, where

$$B_i\colon \boldsymbol{\mathcal{H}} \to \mathcal{H}_i\colon \boldsymbol{x} \mapsto \sum_{k=1}^{p} L_{ki}^*\left(\sum_{j=1}^{m} L_{kj}x_j - z_k\right). \tag{3.90}$$

Now take $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\boldsymbol{\mathcal{H}}$. Since, in view of (3.83), (3.86) is a special case of (3.16), proceeding as in the proof of [4, Theorem 4.3], to reach the announced conclusion it is enough to show that

$$\sum_{i=1}^{m}\langle B_i(\boldsymbol{x}) - B_i(\boldsymbol{y}) \mid x_i - y_i\rangle \geq \beta_2 \sum_{i=1}^{m}\left\|B_i(\boldsymbol{x}) - B_i(\boldsymbol{y})\right\|^2 \tag{3.91}$$

or, equivalently, that $\langle\langle \nabla\boldsymbol{g}(\boldsymbol{x}) - \nabla\boldsymbol{g}(\boldsymbol{y}) \mid \boldsymbol{x}-\boldsymbol{y}\rangle\rangle \geq \beta_2|||\nabla\boldsymbol{g}(\boldsymbol{x}) - \nabla\boldsymbol{g}(\boldsymbol{y})|||^2$. Since $\boldsymbol{g}$ is convex, it follows from the Baillon-Haddad theorem [9, Corollary 10] that this inequality is equivalent to $|||\nabla\boldsymbol{g}(\boldsymbol{x}) - \nabla\boldsymbol{g}(\boldsymbol{y})||| \leq |||\boldsymbol{x}-\boldsymbol{y}|||/\beta_2$, i.e., to

$$\sum_{i=1}^{m}\left\|B_i(\boldsymbol{x}) - B_i(\boldsymbol{y})\right\|^2 \leq |||\boldsymbol{x}-\boldsymbol{y}|||^2/\beta_2^2. \tag{3.92}$$

For every $i \in \{1,\ldots,m\}$, (3.90) and the Cauchy-Schwarz inequality imply that

$$\begin{aligned}
\left\|B_i(\boldsymbol{x}) - B_i(\boldsymbol{y})\right\|^2 &= \left\|\sum_{j=1}^{m}\sum_{k=1}^{p} L_{ki}^*L_{kj}(x_j - y_j)\right\|^2\\
&\leq \left(\sum_{j=1}^{m}\sum_{k=1}^{p}\|L_{ki}\|\,\|L_{kj}\|\,\|x_j - y_j\|\right)^2\\
&= \left(\sum_{k=1}^{p}\|L_{ki}\|\sum_{j=1}^{m}\|L_{kj}\|\,\|x_j - y_j\|\right)^2\\
&\leq \left(\sum_{k=1}^{p}\|L_{ki}\|^2\right)\sum_{k=1}^{p}\left(\sum_{j=1}^{m}\|L_{kj}\|\,\|x_j - y_j\|\right)^2\\
&\leq \left(\sum_{k=1}^{p}\|L_{ki}\|^2\right)\sum_{k=1}^{p}\left(\sum_{j=1}^{m}\|L_{kj}\|^2\right)\left(\sum_{j=1}^{m}\|x_j - y_j\|^2\right)\\
&= \left(\sum_{k=1}^{p}\|L_{ki}\|^2\right)|||\boldsymbol{x}-\boldsymbol{y}|||^2/\beta_2.
\end{aligned} \tag{3.93}$$

Hence,

$$\sum_{i=1}^{m} \left\|B_i(\boldsymbol{x}) - B_i(\boldsymbol{y})\right\|^2 \leq \left(\sum_{i=1}^{m}\sum_{k=1}^{p} \|L_{ki}\|^2\right) |||\boldsymbol{x}-\boldsymbol{y}|||^2/\beta_2 = |||\boldsymbol{x}-\boldsymbol{y}|||^2/\beta_2^2, \quad (3.94)$$

which yields (3.92). □

3.2.4.2 Split feasibility problems

Suppose that $m = p+1$. For every $k \in \{1,\ldots,p\}$, set $z_k = 0$, $\mathcal{G}_k = \mathcal{H}_{k+1}$, and, for every $i \in \{2,\ldots,m\}$,

$$L_{ki} = \begin{cases} -\operatorname{Id}, & \text{if } i = k+1; \\ 0, & \text{otherwise.} \end{cases} \quad (3.95)$$

Then (3.84) becomes

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\text{minimize}} \quad \sum_{i=1}^{m} f_i(x_i) + \frac{1}{2}\sum_{k=1}^{m-1} \|L_{k1}x_1 - x_{k+1}\|^2. \quad (3.96)$$

Setting errors to zero and $\lambda_n \equiv 1$ for simplicity, the updating rule (3.86) in Algorithm 3.19 reduces to

$$\begin{cases} x_{1,n+1} = \operatorname{prox}_{\gamma_n f_1}\left(x_{1,n} - \gamma_n \sum_{k=1}^{m-1} L_{k1}^*\big(L_{k1}x_{1,n} - x_{k+1,n}\big)\right), \\ x_{i,n+1} = \operatorname{prox}_{\gamma_n f_i}\Big((1-\gamma_n)x_{i,n} + \gamma_n L_{i-1,1}x_{1,n}\Big), \quad \text{for} \quad 2 \leq i \leq m. \end{cases} \quad (3.97)$$

In particular, if each f_i in (3.96) is the indicator function of a nonempty closed convex set $C_i \subset \mathcal{H}_1$, we obtain

$$\underset{x_1\in C_1,\ldots,x_m\in C_m}{\text{minimize}} \quad \frac{1}{2}\sum_{k=1}^{m-1} \|L_{k1}x_1 - x_{k+1}\|^2, \quad (3.98)$$

which can be regarded as a relaxation of the split feasibility problem

$$\text{find} \quad x_1 \in C_1 \quad \text{such that} \quad L_{11}x_1 \in C_2,\ L_{21}x_1 \in C_3,\ \ldots,\ L_{m-1,1}x_1 \in C_m. \quad (3.99)$$

For $m = 2$, this type of problem was introduced in [13] and further studied in [11, 14, 23].

In [42] a problem similar to (3.96) is investigated in the case when $m = 2$, in which the linear operator depends on the partial derivatives of one component.

3.2.5 Strong coupling

An important instance of Problem 3.18 occurs when the linear mixtures describing the interactions between the components in (3.84) reduce to linear combinations. Such a coupling is referred to as strong.

3.2.5.1 Problem formulation

Problem 3.22 For every $i \in \{1, \ldots, m\}$, let $f_i \in \Gamma_0(\mathcal{H})$ and, for every $k \in \{1, \ldots, p\}$, let $\xi_{ki} \in \mathbb{R}$ and let $z_k \in \mathcal{H}$. It is assumed that $\min_{1 \le k \le p} \sum_{i=1}^m |\xi_{ki}| > 0$. The problem is to

$$\underset{x_1 \in \mathcal{H}, \ldots, x_m \in \mathcal{H}}{\text{minimize}} \quad \sum_{i=1}^m f_i(x_i) + \frac{1}{2} \sum_{k=1}^p \left\| z_k - \sum_{i=1}^m \xi_{ki} x_i \right\|^2, \tag{3.100}$$

under the assumption that solutions exist.

To solve this problem, we propose the following variant of Algorithm 3.19, which features a better bound than (3.85).

Algorithm 3.23 Set $\Xi = [\xi_{ki}] \in \mathbb{R}^{p \times m}$, $\Delta = \Xi^\top \Xi = [\delta_{ij}]$, and

$$\beta_3 = \frac{1}{\lambda_{\max}}, \tag{3.101}$$

where $\lambda_{\max}$ is the largest eigenvalue of Δ. Fix ε in $]0, \min\{1, \beta_3\}[$, $(\lambda_n)_{n \in \mathbb{N}}$ in $[\varepsilon, 1]$, $(\gamma_n)_{n \in \mathbb{N}}$ in $[\varepsilon, 2\beta_3 - \varepsilon]$, and $(x_{i,0})_{1 \le i \le m}$ in $\mathcal{H}^m$. For every $i \in \{1, \ldots, m\}$ set, for every $n \in \mathbb{N}$,

$$x_{i,n+1} = x_{i,n} + \lambda_n \left(\operatorname{prox}_{\gamma_n f_i} \left(x_{i,n} + \gamma_n \sum_{k=1}^p \xi_{ki} \left(z_k - \sum_{j=1}^m \xi_{kj} x_{j,n} \right) \right) + a_{i,n} - x_{i,n} \right), \tag{3.102}$$

where $(a_{i,n})_{n \in \mathbb{N}}$ is a sequence in $\mathcal{H}$ such that $\sum_{n \in \mathbb{N}} \|a_{i,n}\| < +\infty$.

Remark 3.24 Using standard matrix norm inequalities, we obtain

$$\lambda_{\max} \le \sum_{k=1}^p \sum_{i=1}^m |\xi_{ki}|^2 \le p \max_{1 \le k \le p} \sum_{i=1}^m |\xi_{ki}|^2, \tag{3.103}$$

which yields

$$\beta_3 = \frac{1}{\lambda_{\max}} \ge \beta_2 = \frac{1}{\displaystyle\sum_{k=1}^p \sum_{i=1}^m |\xi_{ki}|^2} \ge \beta_1 = \frac{1}{p \displaystyle\max_{1 \le k \le p} \sum_{i=1}^m |\xi_{ki}|^2}. \tag{3.104}$$

In other words, in the problem under consideration, the bound of (3.101) is better than that of (3.85), which is itself better than that of (3.15).

Theorem 3.25 *Let $(x_{1,n})_{n\in\mathbb{N}}$, ..., $(x_{m,n})_{n\in\mathbb{N}}$ be sequences generated by Algorithm 3.23. Then, for every $i \in \{1,\dots,m\}$, $(x_{i,n})_{n\in\mathbb{N}}$ converges weakly to a point $x_i \in \mathcal{H}$, and $(x_i)_{1\le i\le m}$ is a solution to Problem 3.22.*

Proof. Define $\boldsymbol{\mathcal{H}}$ as in the proof of Theorem 3.21 (with $\mathcal{H}_i \equiv \mathcal{H}$), set

$$\boldsymbol{g}\colon \boldsymbol{\mathcal{H}} \to \mathbb{R}\colon \boldsymbol{x} \mapsto \frac{1}{2}\sum_{k=1}^{p}\left\|z_k - \sum_{i=1}^{m}\xi_{ki}x_i\right\|^2, \tag{3.105}$$

and introduce the bounded linear operator

$$\boldsymbol{B}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{x} \mapsto \left(\sum_{k=1}^{p}\sum_{j=1}^{m}\xi_{ki}\xi_{kj}x_j\right)_{1\le i\le m}. \tag{3.106}$$

As in the proof of Theorem 3.21, it is sufficient to prove that ∇g is β_3^{-1}–Lipschitz continuous.

Since Δ is a real $m\times m$ symmetric matrix, there exists an orthogonal matrix $\Pi = [\pi_{ij}] \in \mathbb{R}^{m\times m}$ such that $\Delta = \Pi\Lambda\Pi^\top$, where Λ is the diagonal matrix the diagonal entries of which are the eigenvalues $(\lambda_i)_{1\le i\le m}$ of Δ. Now set $\boldsymbol{D}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{x} \mapsto \big(\lambda_i x_i\big)_{1\le i\le m}$ and $\boldsymbol{U}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{x} \mapsto \big(\sum_{j=1}^{m}\pi_{ij}x_j\big)_{1\le i\le m}$. Then $\boldsymbol{U}$ is unitary and

$$|||\boldsymbol{B}|||^2 = |||\boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^*|||^2 = |||\boldsymbol{D}|||^2 = \sup_{|||\boldsymbol{x}|||\le 1}\sum_{i=1}^{m}\lambda_i^2\|x_i\|^2 = \lambda_{\max}^2. \tag{3.107}$$

Hence, for every $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\boldsymbol{\mathcal{H}}$, we have

$$|||\nabla\boldsymbol{g}(\boldsymbol{x}) - \nabla\boldsymbol{g}(\boldsymbol{y})|||^2 = |||\boldsymbol{B}\boldsymbol{x} - \boldsymbol{B}\boldsymbol{y}|||^2 \le \lambda_{\max}^2|||\boldsymbol{x}-\boldsymbol{y}|||^2, \tag{3.108}$$

which implies that $\nabla\boldsymbol{g}$ is β_3^{-1}–Lipschitz continuous and completes the proof. □

3.2.5.2 Signal decomposition

Suppose that an ideal signal $\overline{x} \in \mathcal{H}$ can be decomposed as

$$\overline{x} = \sum_{i=1}^{m}\overline{x}_i, \quad \text{where} \quad (\forall i \in \{1,\dots,m\}) \quad \overline{x}_i \in \mathcal{H}. \tag{3.109}$$

A common problem is to recover the components $(\overline{x}_i)_{1\le i\le m}$ from some measurement z of $\overline{x}$ and some prior information. Assuming that the prior information on each component $\overline{x}_i$ is promoted by a potential $f_i \in \Gamma_0(\mathcal{H})$ and using a least-squares data fitting term leads to the variational problem

$$\underset{x_1\in\mathcal{H},\dots,x_m\in\mathcal{H}}{\text{minimize}} \quad \sum_{i=1}^{m}f_i(x_i) + \frac{1}{2}\left\|z - \sum_{i=1}^{m}x_i\right\|^2. \tag{3.110}$$

Instances of this problem have been considered in [5, 7, 23, 41, 42] for $m = 2$, in [6, 27] for $m = 3$, and in [15] for $m = 4$.

We observe that (3.110) is a special case of (3.100), where $p = 1$, $z_1 = z$, and, for every $i \in \{1, \ldots, m\}$, $\xi_{1i} = 1$. Thus, $\Xi = [1 \cdots 1] \in \mathbb{R}^{1\times m}$ and $\beta_3 = 1/m$ in Algorithm 3.23. Moreover, the updating rule (3.102) now assumes the form

$$x_{i,n+1} = x_{i,n} + \lambda_n\Bigg(\operatorname{prox}_{\gamma_n f_i}\Bigg(x_{i,n} + \gamma_n\Bigg(z - \sum_{j=1}^{m} x_{j,n}\Bigg)\Bigg) + a_{i,n} - x_{i,n}\Bigg). \tag{3.111}$$

The weak convergence of the m sequences so generated to a solution to (3.110) is guaranteed by Theorem 3.25. For $m = 2$, an alternative weakly convergent method is proposed in [23, Section 4.4], which subsumes that of [5] (see also the alternative method of [3]). However, for $m > 2$, no weakly convergent algorithm seems to be available in the literature. Thus, in [15], a model of the form (3.110) with $m = 4$ component is investigated but no convergence proof is furnished for the proposed cyclic minimization algorithm ; in [6], a model with $m = 3$ components is investigated in $\mathcal{H} = \mathbb{R}^N$ and a coordinate descent algorithm with modest convergence properties is utilized.

For the sake of illustration, consider the case when $m = 3$. Then (3.101) yields $\beta = 1/3$. Taking for simplicity $\gamma_n \equiv 1/2$, $\lambda_n \equiv 1$, and, for every $i \in \{1, 2, 3\}$, $a_{i,n} \equiv 0$, (3.111) leads to the simple parallel scheme

$$\begin{cases} x_{1,n+1} = \operatorname{prox}_{f_1/2}\big((z + x_{1,n} - x_{2,n} - x_{3,n})/2\big) \\ x_{2,n+1} = \operatorname{prox}_{f_2/2}\big((z - x_{1,n} + x_{2,n} - x_{3,n})/2\big) \\ x_{3,n+1} = \operatorname{prox}_{f_3/2}\big((z - x_{1,n} - x_{2,n} + x_{3,n})/2\big). \end{cases} \tag{3.112}$$

On the other hand, if $m = 2$, (3.110) becomes

$$\underset{x_1\in\mathcal{H},\, x_2\in\mathcal{H}}{\text{minimize}} \;\; f_1(x_1) + f_2(x_2) + \frac{1}{2}\|z - x_1 - x_2\|^2. \tag{3.113}$$

This problem is studied in [23], where an alternating algorithm is proposed which converges weakly to a solution to (3.113). In particular, if we take f_1 to be the indicator function of a nonempty closed convex set $C_1 \subset \mathcal{H}$ and $f_2 = \sigma_{C_2}$ to be the support function of a nonempty closed convex set $C_2 \subset \mathcal{H}$, (3.113) becomes

$$\underset{x_1\in C_1,\, x_2\in\mathcal{H}}{\text{minimize}} \;\; \sigma_{C_2}(x_2) + \frac{1}{2}\|z - x_1 - x_2\|^2. \tag{3.114}$$

This problem is studied in [5].

The role of each potential f_i in (3.110) is to promote certain known properties of the component $\overline{x}_i$. For instance, if some properties of the coefficients $(\langle \overline{x}_i \mid e_{ik}\rangle)_{k\in\mathbb{N}}$ of the decomposition of $\overline{x}_i$ in an orthonormal basis $(e_{ik})_{k\in\mathbb{N}}$ of $\mathcal{H}$ are available, we can take (see [16, 20, 25] for specific choices of the potentials $(\phi_{ik})_{k\in\mathbb{N}}$)

$$f_i\colon \mathcal{H} \to \left]-\infty,+\infty\right] : x_i \mapsto \sum_{k\in\mathbb{N}} \phi_{ik}\big(\langle x_i \mid e_{ik}\rangle\big), \tag{3.115}$$

where, for every $k \in \mathbb{N}$, $\phi_{ik} \in \Gamma_0(\mathbb{R})$ satisfies $\phi_{ik} \geq \phi_{ik}(0) = 0$. If we adopt this model for each component in (3.110), we obtain

$$\underset{x_1\in\mathcal{H},\ldots,x_m\in\mathcal{H}}{\text{minimize}} \quad \sum_{i=1}^{m}\sum_{k\in\mathbb{N}} \phi_{ik}\big(\langle x_i \mid e_{ik}\rangle\big) + \frac{1}{2}\left\| z - \sum_{i=1}^{m} x_i \right\|^2. \tag{3.116}$$

In addition, we derive from (3.115) and [23, Example 2.19] that (3.111) reduces to (we set $\lambda_n \equiv 1$ and $a_{i,n} \equiv 0$ for simplicity)

$$x_{i,n+1} = \sum_{k\in\mathbb{N}} \text{prox}_{\gamma_n\phi_{ik}}\left(\langle x_{i,n} \mid e_{ik}\rangle + \gamma_n\left(\langle z \mid e_{ik}\rangle - \sum_{j=1}^{m}\langle x_{j,n} \mid e_{ik}\rangle\right)\right)e_{ik}. \tag{3.117}$$

3.2.5.3 Signal synthesis

Let $p = m(m-1)/2$ be the cardinality of the set $\mathbb{K} = \big\{(i,j) \in \{1,\ldots,m\}^2 \mid j > i\big\}$. For every $k = (k_1,k_2) \in \mathbb{K}$ set $z_k = 0$ and

$$\xi_{ki} = \begin{cases} 1, & \text{if} \quad i = k_1; \\ -1, & \text{if} \quad i = k_2; \\ 0, & \text{otherwise.} \end{cases} \tag{3.118}$$

With this scenario, Problem 3.22 features pairwise quadratic couplings, which yields

$$\underset{x_1\in\mathcal{H},\ldots,x_m\in\mathcal{H}}{\text{minimize}} \quad \sum_{i=1}^{m} f_i(x_i) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=i+1}^{m} \|x_i - x_j\|^2. \tag{3.119}$$

For instance, when $m = 2$ and f_1 and f_2 are the indicator functions of nonempty closed convex sets C_1 and C_2 in $\mathcal{H}$, we obtain the classical problem

$$\underset{x_1\in C_1,x_2\in C_2}{\text{minimize}} \quad \|x_1 - x_2\|^2, \tag{3.120}$$

which has been studied in [29, 45]. Another instance of (3.119) with $m = 2$, is that obtained by taking $f_2\colon x \mapsto \|y - Lx\|^2/2$, where L is a bounded linear operator from $\mathcal{H}$ to a real Hilbert space $\mathcal{G}$ and $y \in \mathcal{G}$. In this case (3.119) becomes

$$\underset{x_1\in\mathcal{H},x_2\in\mathcal{H}}{\text{minimize}} \quad f_1(x_1) + \frac{1}{2}\|y - Lx_2\|^2 + \frac{1}{2}\|x_1 - x_2\|^2. \tag{3.121}$$

This formulation arises in the image restoration problems of [32, 44] for specific choices of f_1 in finite dimensional spaces.

Since the matrix $\Delta = \Xi^\top \Xi = m\,\mathrm{Id} - 1 \cdot 1^\top$ has largest eigenvalue $\lambda_{\max} = m$, we have $\beta_3 = 1/m$ in Algorithm 3.23. In addition, (3.102) becomes

$$x_{i,n+1} = x_{i,n} + \lambda_n \Bigg(\mathrm{prox}_{\gamma_n f_i} \Bigg((1 - m\gamma_n) x_{i,n} + \gamma_n \sum_{j=1}^{m} x_{j,n} \Bigg) + a_{i,n} - x_{i,n} \Bigg). \tag{3.122}$$

In particular, upon setting $\gamma_n \equiv 1/m$, $\lambda_n \equiv 1$, and $a_{i,n} \equiv 0$, we obtain the parallel method

$$\begin{cases} x_{1,n+1} = \mathrm{prox}_{f_1/m} \left(\dfrac{1}{m} \displaystyle\sum_{j=1}^{m} x_{j,n} \right), \\ \quad\quad \vdots \\ x_{m,n+1} = \mathrm{prox}_{f_m/m} \left(\dfrac{1}{m} \displaystyle\sum_{j=1}^{m} x_{j,n} \right). \end{cases} \tag{3.123}$$

3.2.6 Application to multiframe signal representation

This section is devoted to an application to multiframe signal processing in a real Hilbert space $\mathcal{G}$. Recall that a sequence $(e_k)_{k\in\mathbb{N}}$ in $\mathcal{G}$ is a frame if there exist constants μ and ν in $]0, +\infty[$ such that [24, 30, 38]

$$(\forall y \in \mathcal{G}) \quad \mu \|y\|^2 \leq \sum_{k\in\mathbb{N}} \big| \langle y \mid e_k \rangle \big|^2 \leq \nu \|y\|^2. \tag{3.124}$$

The associated frame operator is the injective bounded linear operator

$$F \colon \mathcal{G} \to \ell^2(\mathbb{N}) \colon y \mapsto \big(\langle y \mid e_k \rangle \big)_{k\in\mathbb{N}}, \tag{3.125}$$

and its adjoint is the surjective bounded linear operator

$$F^* \colon \ell^2(\mathbb{N}) \to \mathcal{G} \colon (\eta_k)_{k\in\mathbb{N}} \mapsto \sum_{k\in\mathbb{N}} \eta_k e_k. \tag{3.126}$$

Frames extend the notion of orthonormal bases and they have been used in a number of variational signal processing problems due to their ability to efficiently capture a wide range signal features, e.g., [16, 12, 26]. We consider a variational formulation which exploits information on the frame representation of each signal component. In the case of $m = 1$ component, a similar setting is considered in [16].

Problem 3.26 Set $\mathcal{H} = \ell^2(\mathbb{R})$ and let $\varphi \in \Gamma_0(\mathcal{G})$ be a τ–Lipschitz differentiable function, for some $\tau \in \left]0, +\infty\right[$. For every $i \in \{1, \ldots, m\}$, let $(e_{ik})_{k\in\mathbb{N}}$ be a frame of $\mathcal{G}$ with associated frame operator F_i and, for every $k \in \mathbb{N}$, let $\phi_{ik} \in \Gamma_0(\mathbb{R})$ be such that $\phi_{ik} \geq \phi_{ik}(0) = 0$. The problem is to

$$\underset{(\eta_{1k})_{k\in\mathbb{N}}\in\mathcal{H},\ldots,(\eta_{mk})_{k\in\mathbb{N}}\in\mathcal{H}}{\text{minimize}} \quad \sum_{i=1}^{m}\sum_{k\in\mathbb{N}} \phi_{ik}(\eta_{ik}) + \varphi\bigg(\sum_{i=1}^{m}\sum_{k\in\mathbb{N}} \eta_{ik} e_{ik}\bigg), \tag{3.127}$$

under the assumption that solutions exist.

Algorithm 3.27 Set $\mathcal{H} = \ell^2(\mathbb{R})$ and

$$\beta_4 = \frac{1}{\tau \sum_{i=1}^{m} \nu_i}, \tag{3.128}$$

where, for every $i \in \{1, \ldots, m\}$, $\nu_i \in \left]0, +\infty\right[$ is the upper frame constant of $(e_{ik})_{k\in\mathbb{N}}$ (see (3.124)). Fix ε in $\left]0, \min\{1, \beta_4\}\right[$, $(\lambda_n)_{n\in\mathbb{N}}$ in $[\varepsilon, 1]$, $(\gamma_n)_{n\in\mathbb{N}}$ in $[\varepsilon, 2\beta_4 - \varepsilon]$, and let $(\eta_{1k,0})_{k\in\mathbb{N}}$, ..., $(\eta_{mk,0})_{k\in\mathbb{N}}$ be sequences in $\mathcal{H}$. For every $i \in \{1, \ldots, m\}$ set, for every $n \in \mathbb{N}$,

$$(\forall k \in \mathbb{N}) \quad \eta_{ik,n+1} = \eta_{ik,n} + \lambda_n\Bigg(\operatorname{prox}_{\gamma_n\phi_{ik}}\Bigg(\eta_{ik,n} - \gamma_n\Bigg(\Bigg\langle \nabla\varphi\bigg(\sum_{j=1}^{m}\sum_{k\in\mathbb{N}} \eta_{jk,n} e_{jk}\bigg) \,\Bigg|\, e_{ik}\Bigg\rangle + \beta_{ik,n}\Bigg)\Bigg) + \alpha_{ik,n} - \eta_{ik,n}\Bigg), \tag{3.129}$$

where $(\alpha_{ik,n})_{(k,n)\in\mathbb{N}^2}$ and $(\beta_{ik,n})_{(k,n)\in\mathbb{N}^2}$ are real sequences such that

$$\sum_{n\in\mathbb{N}} \sqrt{\sum_{k\in\mathbb{N}} |\alpha_{ik,n}|^2} < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} \sqrt{\sum_{k\in\mathbb{N}} |\beta_{ik,n}|^2} < +\infty. \tag{3.130}$$

Remark 3.28 In some cases, it may be possible to obtain a sharper bound than (3.128) ; see [16, Remark 5.3].

Corollary 3.29 *Let $\big((\eta_{1k,n})_{k\in\mathbb{N}}\big)_{n\in\mathbb{N}}$, ..., $\big((\eta_{mk,n})_{k\in\mathbb{N}}\big)_{n\in\mathbb{N}}$ be sequences generated by Algorithm 3.27. Then, for every $i \in \{1, \ldots, m\}$ and $k \in \mathbb{N}$, $(\eta_{ik,n})_{n\in\mathbb{N}}$ converges to a point $\eta_{ik} \in \mathbb{R}$, and $\big((\eta_{ik})_{k\in\mathbb{N}}\big)_{1\leq i\leq m}$ is a solution to Problem 3.26.*

Proof. Problem 3.26 is a particular case of Problem 3.4 in which $p = 1$, $\varphi_1 = \varphi$, and for every $i \in \{1, \ldots, m\}$, $\mathcal{H}_i = \mathcal{H} = \ell^2(\mathbb{R})$, $f_i \colon \mathcal{H} \to \left]-\infty, +\infty\right] : (\eta_k)_{k\in\mathbb{N}} \mapsto \sum_{k\in\mathbb{N}} \phi_{ik}(\eta_k)$, and $L_{1i} = F_i^*$. In addition, we derive from (3.128), (3.15), and (3.124) that $\beta_4 \leq \beta_1$. Finally, using [23, Example 2.19], we deduce that Algorithm 3.27 is a particular case of Algorithm 3.5. The result therefore follows from Theorem 3.6. ☐

We conclude with a specific instance of Problem 3.26.

Example 3.30 Let $\mathbb{K}$ be a real Hilbert space, let $z \in \mathbb{K}$, and let $L\colon \mathcal{G} \to \mathbb{K}$ be linear and bounded. Set $\varphi = \|z - L\cdot\|^2/2$ and, for every $i \in \{1,\ldots,m\}$ and $k \in \mathbb{N}$, set $\phi_{ik} = w_{ik}|\cdot|^{p_i}$, where $p_i \in [1,2]$ and $w_{ik} \in \,]0,+\infty[$. Then (3.127) becomes

$$\underset{(\eta_{1k})_{k\in\mathbb{N}}\in\mathcal{H},\ldots,(\eta_{mk})_{k\in\mathbb{N}}\in\mathcal{H}}{\text{minimize}} \quad \sum_{i=1}^{m}\sum_{k\in\mathbb{N}} w_{ik}|\eta_{ik}|^{p_i} + \frac{1}{2}\left\| z - L\left(\sum_{i=1}^{m}\sum_{k\in\mathbb{N}} \eta_{ik}e_{ik}\right)\right\|^2. \tag{3.131}$$

This problem is studied in [37].

3.3 Bibliographie

[1] F. Acker et M. A. Prestel, Convergence d'un schéma de minimisation alternée, *Ann. Fac. Sci. Toulouse V. Sér. Math.*, vol. 2, pp. 1–9, 1980.

[2] S. Anthoine, E. Pierpaoli, and I. Daubechies, Deux méthodes de déconvolution et séparation simultanées ; application à la reconstruction des amas de galaxies, *Traitement Signal*, vol. 23, pp. 439–447, 2006.

[3] H. Attouch, J. Bolte, P. Redont, and A. Soubeyran, Alternating proximal algorithms for weakly coupled convex minimization problems. Applications to dynamical games and PDE's, *J. Convex Anal.*, vol. 15, pp. 485–506, 2008.

[4] H. Attouch, L. M. Briceño-Arias, and P. L. Combettes, A parallel splitting method for coupled monotone inclusions, *SIAM J. Control Optim.*, vol. 48, pp. 3246–3270, 2010.

[5] J.-F. Aujol, G. Aubert, L. Blanc-Féraud, and A. Chambolle, Image decomposition into a bounded variation component and an oscillating component, *J. Math. Imaging Vision*, vol. 22, pp. 71–88, 2005.

[6] J.-F. Aujol and A. Chambolle, Dual norms and image decomposition models, *Int. J. Comput. Vis.*, vol. 63, pp. 85–104, 2005.

[7] J.-F. Aujol, G. Gilboa, T. Chan, and S. Osher, Structure-texture image decomposition – Modeling, algorithms, and parameter selection, *Int. J. Comput. Vis.*, vol. 67, pp. 111–136, 2006.

[8] J.-F. Aujol and S. H. Kang, Color image decomposition and restoration, *J. Vis. Commun. Image Represent.*, vol. 17, pp. 916–928, 2006.

[9] J.-B. Baillon et G. Haddad, Quelques propriétés des opérateurs angle-bornés et n-cycliquement monotones, *Israel J. Math.*, vol. 26, pp. 137–150, 1977.

[10] H. H. Bauschke, P. L. Combettes, and D. Noll, Joint minimization with alternating Bregman proximity operators, *Pacific J. Optim.*, vol. 2, pp. 401–424, 2006.

[11] C. L. Byrne, *Signal Processing – A Mathematical Approach,* A. K. Peters, Wellesley, MA, 2005.

[12] J.-F. Cai, R. H. Chan, L. Shen, and Z. Shen, Convergence analysis of tight framelet approach for missing data recovery, *Adv. Comput. Math.*, vol. 31, pp. 87–113, 2009.

[13] Y. Censor and T. Elfving, A multiprojection algorithm using Bregman projections in a product space, *Numer. Algorithms*, vol. 8, pp. 221–239, 1994.

[14] Y. Censor and S. A. Zenios, *Parallel Optimization : Theory, Algorithms and Applications,* Oxford University Press, New York, 1997.

[15] T. F. Chan, S. Esedoglu, and F. E. Park, Image decomposition combining staircase reduction and texture extraction, *J. Vis. Commun. Image Represent.*, vol. 18, pp. 464–486, 2007.

[16] C. Chaux, P. L. Combettes, J.-C. Pesquet, and V. R. Wajs, A variational formulation for frame-based inverse problems, *Inverse Problems*, vol. 23, pp. 1495–1518, 2007.

[17] C. Combari, M. Laghdir, and L. Thibault, A note on subdifferentials of convex composite functionals, *Arch. Math. (Basel)*, vol. 67, pp. 239–252, 1996.

[18] P. L. Combettes and P. Bondon, Hard-constrained inconsistent signal feasibility problems, *IEEE Trans. Signal Process.,* vol. 47, pp. 2460–2468, 1999.

[19] P. L. Combettes and J.-C. Pesquet, A Douglas-Rachford splitting approach to nonsmooth convex variational signal recovery, *IEEE J. Selected Topics Signal Process.*, vol. 1, pp. 564–574, 2007.

[20] P. L. Combettes and J.-C. Pesquet, Proximal thresholding algorithm for minimization over orthonormal bases, *SIAM J. Optim.,* vol. 18, pp. 1351–1376, 2007.

[21] P. L. Combettes and J.-C. Pesquet, A proximal decomposition method for solving convex variational inverse problems, *Inverse Problems*, vol. 24, Article ID 065014, 27 pp, 2008.

[22] P. L. Combettes and H. J. Trussell, The use of the noise properties in set theoretic estimation, *IEEE Trans. Signal Process.*, vol. 39 pp. 1630–1641, 1991.

[23] P. L. Combettes and V. R. Wajs, Signal recovery by proximal forward-backward splitting, *Multiscale Model. Simul.*, vol. 4, pp. 1168–1200, 2005.

[24] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, PA, 1992.

[25] I. Daubechies, M. Defrise, and C. De Mol, An iterative thresholding algorithm for linear inverse problems with a sparsity constraint, *Comm. Pure Appl. Math.*, vol. 57, pp. 1413–1457, 2004.

[26] S. Durand and M. Nikolova, Denoising of frame coefficients using ℓ^1 data-fidelity term and edge-preserving regularization, *Multiscale Model. Simul.*, vol. 6, pp. 547–576, 2007.

[27] J. Gilles, Noisy image decomposition : a new structure, texture and noise model based on local adaptivity, *J. Math. Imaging Vision*, vol. 28, pp. 285–295, 2007.

[28] R. Glowinski, J.-L. Lions et R. Trémolières, *Analyse Numérique des Inéquations Variationnelles*, Dunod, Paris, 1976.

[29] M. Goldburg and R. J. Marks II, Signal synthesis in the presence of an inconsistent set of constraints, *IEEE Trans. Circuits and Systems*, vol. 32, pp. 647–663, 1985.

[30] C. E. Heil and D. F. Walnut, Continuous and discrete wavelet transforms, *SIAM Rev.*, vol. 31, pp. 628–666, 1989.

[31] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms*, Springer-Verlag, New York, 1993.

[32] Y. Huang, M. K. Ng, and Y.-W. Wen, A fast total variation minimization method for image restoration, *Multiscale Model. Simul.*, vol. 7, pp. 774–795, 2008.

[33] M. Kang, Generalized multichannel image deconvolution approach and its applications, *Opt. Eng.*, vol. 37, pp. 2953–2964, 1998.

[34] A. Katsaggelos, K. Lay, and N. Galatsanos, A general framework for frequency domain multi-channel signal processing, *IEEE Trans. Image Process.*, vol. 2, pp. 417–420, 1993.

[35] J.-J. Moreau, Proximité et dualité dans un espace hilbertien, *Bull. Soc. Math. France*, vol. 93, pp. 273–299, 1965.

[36] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.

[37] G. Teschke, Multi-frame representations in linear inverse problems with mixed multi-constraints, *Appl. Comput. Harmon. Anal.*, vol. 22, pp. 43–60, 2007.

[38] R. Tolimieri and M. An, *Time-Frequency Representations,* Birkhäuser, Boston, MA, 1998.

[39] H. J. Trussell and M. R. Civanlar, The feasible solution in signal restoration, *IEEE Trans. Acoust. Speech Signal Process.,* vol. 32, pp. 201–212, 1984.

[40] D. Tschumperlé and R. Deriche, Diffusion PDEs on vector-valued images, *Signal Process. Mag.*, vol. 19, pp. 16–25, 2002.

[41] L. A. Vese and S. J. Osher, Modeling textures with total variation minimization and oscillating patterns in image processing, *J. Sci. Comput.,* vol. 19, pp. 553–572, 2003.

[42] L. A. Vese and S. J. Osher, Image denoising and decomposition with total variation minimization and oscillatory functions, *J. Math. Imaging Vision*, vol. 20, pp. 7–18, 2004.

[43] Y. Wang, J. Yang, W. Yin, and Y. Zhang, A new alternating minimization algorithm for total variation image reconstruction, *SIAM J. Imaging Sci.*, vol. 1, pp. 248–272, 2008.

[44] Y.-W. Wen, M. K. Ng, and W.-K. Ching, Iterative algorithms based on decoupling of deblurring and denoising for image restoration, *SIAM J. Sci. Comput.*, vol. 30, pp. 2655–2674, 2008.

[45] D. C. Youla and V. Velasco, Extensions of a result on the synthesis of signals in the presence of inconsistent constraints, *IEEE Trans. Circuits and Systems*, vol. 33, pp. 465–468, 1986.

[46] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, River Edge, NJ, 2002.

Chapitre 4

Résolution d'inclusions monotones multicomposantes

4.1 Introduction

Ce chapitre est consacré à la résolution du Problème 1.1 sous des hypothèses plus générales que celles utilisées dans le Chapitre 2. Nous fournissons aussi certaines méthodes pour résoudre de problèmes de minimisation sous des conditions moins contraignantes que celles utilisées dans le Chapitre 3. Nous rappelons en premier lieu le Problème 1.1.

Problème 4.1 Soient m et q des entiers strictement positifs et, pour tout $j \in \{1, \ldots, q\}$, soit $\boldsymbol{A}_j \colon \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \to 2^{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m}$ un opérateur maximalement monotone. Le problème est de

$$\text{trouver} \quad x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m \quad \text{tels que} \quad (0, \ldots, 0) \in \sum_{j=1}^{q} \boldsymbol{A}_j(x_1, \ldots, x_m). \tag{4.1}$$

Nous avons vu dans la Section 1.2 du Chapitre 1 qu'il existe des méthodes pour résoudre le Problème 4.1 dans le cas monocomposante ($m = 1$). D'autre part, dans le Chapitre 2 nous avons proposé une méthode pour résoudre le Problème 4.1 avec $q = 2$, $\boldsymbol{A}_1 \colon (x_i)_{1 \leq i \leq m} \mapsto A_1 x_1 \times \cdots \times A_m x_m$, où, pour tout $i \in \{1, \ldots, m\}$, $A_i \colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$ est un opérateur maximalement monotone, et $\boldsymbol{A}_2$ est cocoercif. Dans ce chapitre nous concevons des méthodes pour résoudre le Problème 4.1 sous d'autres hypothèses sur les opérateurs $(\boldsymbol{A}_j)_{1 \leq j \leq q}$. Ces algorithmes sont des applications des méthodes énoncées au Chapitre 1 sur la somme hilbertienne directe $\boldsymbol{\mathcal{H}} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$.

Dans la Section 4.2 nous traitons tout d'abord le cas où $(\boldsymbol{A}_j)_{1 \leq j \leq q}$ sont des opérateurs maximalement monotones. Dans la Section 4.3 nous étudions plus en détail le cas de problèmes de minimisation, qui entrent dans le cadre où les opérateurs $(\boldsymbol{A}_j)_{1 \leq j \leq q}$

sont des sous-différentiels des fonctions convexes. Finalement, nous illustrons la flexibilité des algorithmes par des applications au traitement d'images.

4.2 Algorithmes d'éclatement multicomposantes

Nous présentons plusieurs méthodes pour résoudre le Problème 4.1 sous divers jeux d'hypothèses sur les opérateurs $(\boldsymbol{A}_j)_{1\leq j\leq q}$. Nous présentons tout d'abord trois algorithmes qui résolvent des cas particuliers du Problème 4.1 lorsque $q = 2$. Ensuite, deux algorithmes seront étudiés pour traiter le cas général.

4.2.1 Méthode explicite-implicite

Prenons $q = 2$ et supposons que $\boldsymbol{A}_2$ est univoque et cocoercif. Dans ce cas le Problème 4.1 se réduit au problème suivant.

Problème 4.2 Soit $\boldsymbol{A}_1 \colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}$ un opérateur maximalement monotone et soit $\boldsymbol{A}_2 \colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}$ un opérateur β–cocoercif avec $\beta \in \left]0,+\infty\right[$. Le problème est de

$$\text{trouver} \quad x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m \quad \text{tels que} \quad (0,\ldots,0) \in \boldsymbol{A}_1(x_1,\ldots,x_m) + \boldsymbol{A}_2(x_1,\ldots,x_m). \tag{4.2}$$

Le cas où $\boldsymbol{A}_1$ est séparable, c'est-à-dire, $\boldsymbol{A}_1 \colon (x_i)_{1\leq i\leq m} \mapsto A_1x_1 \times \cdots \times A_mx_m$, où, pour tout $i \in \{1,\ldots,m\}$, $A_i \colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$ est un opérateur maximalement monotone, a été étudié dans le Chapitre 2. Dans la proposition suivante un algorithme est décrit pour résoudre le cas général.

Proposition 4.3 *Pour tout $i \in \{1,\ldots,m\}$, soient $(a_{i,n})_{n\in\mathbb{N}}$ et $(b_{i,n})_{n\in\mathbb{N}}$ des suites dans $\mathcal{H}_i$ telles que $\sum_{n\in\mathbb{N}}\|a_{i,n}\| < +\infty$ et $\sum_{n\in\mathbb{N}}\|b_{i,n}\| < +\infty$. On génère des suites $(x_{1,n})_{n\in\mathbb{N}}, \ldots, (x_{m,n})_{n\in\mathbb{N}}$ comme suit.*

Initialisation
$$\left\lfloor \begin{array}{l} \varepsilon \in \left]0,\min\{1,\beta\}\right[\\ \text{Pour } i = 1,\ldots,m \\ \lfloor\; x_{i,0} \in \mathcal{H}_i \end{array}\right. \tag{4.3}$$

Pour $n = 0, 1, \dots$

$$\left\lfloor \begin{array}{l} (y_{1,n}, \dots, y_{m,n}) = \boldsymbol{A}_2(x_{1,n}, \dots, x_{m,n}) + (b_{1,n}, \dots, b_{m,n}) \\ \gamma_n \in [\varepsilon, 2\beta - \varepsilon] \\ (z_{1,n}, \dots, z_{m,n}) = J_{\gamma_n \boldsymbol{A}_1}(x_{1,n} - \gamma_n y_{1,n}, \dots, x_{m,n} - \gamma_n y_{m,n}) + (a_{1,n}, \dots, a_{m,n}) \\ \lambda_n \in [\varepsilon, 1] \\ \text{Pour } i = 1, \dots, m \\ \lfloor \; x_{i,n+1} = x_{i,n} + \lambda_n (z_{i,n} - x_{i,n}). \end{array} \right.$$

De plus, on suppose que $\mathrm{zer}(\boldsymbol{A}_1 + \boldsymbol{A}_2) \neq \varnothing$*. Alors, pour tout* $i \in \{1, \dots, m\}$*,* $x_{i,n} \rightharpoonup x_i \in \mathcal{H}_i$ *et* $(x_1, \dots, x_m) \in \mathrm{zer}(\boldsymbol{A}_1 + \boldsymbol{A}_2)$*.*

Démonstration. Il suffit d'appliquer la Proposition 1.4 du Chapitre 1 dans $\boldsymbol{\mathcal{H}}$. □

Remarque 4.4 La convergence forte peut être assurée sous certaines conditions présentées dans [17, Remark 6.6].

4.2.2 Méthode explicite-implicite-explicite

Prenons $q = 2$ dans le Problème 4.1, mais, contrairement au Problème 4.2, supposons que l'opérateur $\boldsymbol{A}_2$ soit lipschitzien au lieu d'être cocoercif. Dans ce cas nous obtenons le problème suivant.

Problème 4.5 Soit $\boldsymbol{A}_1 \colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}$ et $\boldsymbol{A}_2 \colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}$ deux opérateurs maximalement monotones tels que $\boldsymbol{A}_1 + \boldsymbol{A}_2$ soit maximalement monotone. Soit $\beta \in \left]0, +\infty\right[$ et supposons que $\boldsymbol{A}_2$ est β^{-1}–lipschitzien. Le problème est de

$$\text{trouver} \quad x_1 \in \mathcal{H}_1, \dots, x_m \in \mathcal{H}_m \quad \text{tels que} \\ (0, \dots, 0) \in \boldsymbol{A}_1(x_1, \dots, x_m) + \boldsymbol{A}_2(x_1, \dots, x_m). \quad (4.4)$$

Proposition 4.6 *On génère des suites* $(x_{1,n})_{n\in\mathbb{N}}$ *,...,* $(x_{m,n})_{n\in\mathbb{N}}$ *comme suit.*

Initialisation

$$\left\lfloor \begin{array}{l} \varepsilon \in \left]0, \beta/2\right[\\ \text{Pour } i = 1, \dots, m \\ \lfloor \; x_{i,0} \in \mathcal{H}_i \end{array} \right.$$

Pour $n = 0, 1, \dots$

$$\left\lfloor \begin{array}{l} \gamma_n \in [\varepsilon, \beta - \varepsilon] \\ (y_{1,n}, \dots, y_{m,n}) = (x_{1,n}, \dots, x_{m,n}) - \gamma_n \boldsymbol{A}_2(y_{1,n}, \dots, y_{m,n}) \\ (p_{1,n}, \dots, p_{m,n}) = J_{\gamma_n A}(y_{1,n}, \dots, y_{m,n}) \\ (q_{1,n}, \dots, q_{m,n}) = (p_{1,n}, \dots, p_{m,n}) - \gamma_n \boldsymbol{A}_2(p_{1,n}, \dots, p_{m,n}) \\ \text{Pour } i = 1, \dots, m \\ \lfloor \; x_{i,n+1} = x_{i,n} - y_{i,n} + q_{i,n}. \end{array} \right. \quad (4.5)$$

De plus, on suppose que $\operatorname{zer}(\boldsymbol{A}_1+\boldsymbol{A}_2) \neq \varnothing$*. Alors, pour tout* $i \in \{1,\dots,m\}$*,* $x_{i,n} \rightharpoonup x_i \in \mathcal{H}_i$ *et* $(x_1,\dots,x_m) \in \operatorname{zer}(\boldsymbol{A}_1+\boldsymbol{A}_2)$.

Démonstration. Il suffit d'appliquer la Proposition 1.7 du Chapitre 1 dans $\boldsymbol{\mathcal{H}}$. □

Remarque 4.7 Des conditions suffisantes pour que la somme de deux opérateurs maximalement monotones soit maximalement monotone peuvent être trouvées dans [9, Corollary 24.4].

4.2.3 Méthode de Douglas-Rachford

Nous relaxons ici les hypothèses sur $\boldsymbol{A}_2$ afin de considérer des opérateurs multivoques monotones quelconques. Nous supposons que la résolvante $J_{\boldsymbol{A}_2}$ est calculable à une erreur sommable près. En prenant $q=2$ dans le Problème 4.1 on obtient le problème suivant.

Problème 4.8 Soient $\boldsymbol{A}_1\colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}$ et $\boldsymbol{A}_2\colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}$ deux opérateurs maximalement monotones. Le problème est de

$$\text{trouver} \quad x_1 \in \mathcal{H}_1,\dots,x_m \in \mathcal{H}_m \quad \text{tels que} \quad (0,\dots,0) \in \boldsymbol{A}_1(x_1,\dots,x_m)+\boldsymbol{A}_2(x_1,\dots,x_m). \tag{4.6}$$

Proposition 4.9 *Pour tout* $i \in \{1,\dots,m\}$*, soient* $(a_{i,n})_{n\in\mathbb{N}}$ *et* $(b_{i,n})_{n\in\mathbb{N}}$ *des suites dans* $\mathcal{H}_i$ *telles que* $\sum_{n\in\mathbb{N}}\|a_{i,n}\| < +\infty$ *et* $\sum_{n\in\mathbb{N}}\|b_{i,n}\| < +\infty$*. On génère des suites* $(x_{1,n})_{n\in\mathbb{N}}$ *,...,* $(x_{m,n})_{n\in\mathbb{N}}$ *comme suit.*

$$
\begin{array}{l}
\text{Initialisation}\\
\left\lfloor
\begin{array}{l}
\varepsilon \in \left]0,1\right[,\ \ \gamma \in \left]0,+\infty\right[\\
\text{Pour } i=1,\dots,m\\
\left\lfloor\ x_{i,0} \in \mathcal{H}_i\right.
\end{array}
\right.\\
\text{Pour } n=0,1,\dots\\
\left\lfloor
\begin{array}{l}
(y_{1,n},\dots,y_{m,n}) = J_{\gamma\boldsymbol{A}_2}(x_{1,n},\dots,x_{m,n}) + (b_{1,n},\dots,b_{m,n})\\
(z_{1,n},\dots,z_{m,n}) = J_{\gamma\boldsymbol{A}_1}(2y_{1,n}-x_{1,n},\dots,2y_{m,n}-x_{m,n}) + (a_{1,n},\dots,a_{m,n})\\
\lambda_n \in [\varepsilon,2-\varepsilon]\\
\text{Pour } i=1,\dots,m\\
\left\lfloor\ x_{i,n+1} = x_{i,n} + \lambda_n(z_{i,n}-y_{i,n}).\right.
\end{array}
\right.
\end{array}
$$

De plus, on suppose que $\operatorname{zer}(\boldsymbol{A}_1+\boldsymbol{A}_2) \neq \varnothing$*. Alors, pour tout* $i \in \{1,\dots,m\}$*,* $x_{i,n} \rightharpoonup x_i \in \mathcal{H}_i$ *et* $J_{\gamma\boldsymbol{A}_2}(x_1,\dots,x_m) \in \operatorname{zer}(\boldsymbol{A}_1+\boldsymbol{A}_2)$.

Démonstration. Il suffit d'appliquer la Proposition 1.10 du Chapitre 1 dans $\boldsymbol{\mathcal{H}}$. □

Remarque 4.10 Des résultats de convergence plus détaillés peuvent être tirés de la Proposition 1.10 du Chapitre 1.

4.2.4 Méthode parallèle basée sur Douglas-Rachford

Nous présentons ici un algorithme qui résout le Problème 4.1 en supposant seulement qu'il existe une solution. Le coût de mise en œuvre dépend du calcul des résolvantes des opérateurs $(\boldsymbol{A}_j)_{1\leq j\leq q}$.

Proposition 4.11 *Pour tout $i \in \{1,\ldots,m\}$ et $j \in \{1,\ldots,q\}$, soit $(a_{i,j,n})_{n\in\mathbb{N}}$ une suite dans $\mathcal{H}_i$ telle que $\sum_{n\in\mathbb{N}} \|a_{i,j,n}\| < +\infty$. On génère des suites $(x_{1,n})_{n\in\mathbb{N}}, \ldots, (x_{m,n})_{n\in\mathbb{N}}$ comme suit.*

$$
\begin{array}{l}
\text{Initialisation} \\
\left\lfloor
\begin{array}{l}
\varepsilon \in \left]0,1\right[, \quad \gamma \in \left]0,+\infty\right[\\
(\omega_j)_{1\leq j\leq q} \subset \left]0,1\right] \text{ et } \displaystyle\sum_{j=1}^{q} \omega_j = 1 \\
\text{Pour } i = 1,\ldots,m \\
\left\lfloor
\begin{array}{l}
\text{Pour } j = 1,\ldots,q \\
\lfloor\; y_{i,j,0} \in \mathcal{H}_i \\
x_{i,0} = \displaystyle\sum_{j=1}^{q} \omega_j y_{i,j,0}
\end{array}
\right.
\end{array}
\right. \\
\text{Pour } n = 0,1,\ldots \\
\left\lfloor
\begin{array}{l}
\text{Pour } j = 1,\ldots,q \\
\lfloor\; (z_{1,j,n},\ldots,z_{m,j,n}) = J_{\gamma \boldsymbol{A}_j/\omega_j}(y_{1,j,n},\ldots,y_{m,j,n}) + (a_{1,j,n},\ldots,a_{m,j,n}) \\
\text{Pour } i = 1,\ldots,m \\
\left\lfloor
\begin{array}{l}
s_{i,n} = \displaystyle\sum_{j=1}^{q} \omega_j z_{i,j,n} \\
\lambda_n \in [\varepsilon, 2-\varepsilon] \\
\text{Pour } j = 1,\ldots,q \\
\lfloor\; y_{i,j,n+1} = y_{i,j,n} + \lambda_n(2s_{i,n} - x_{i,n} - z_{i,j,n}) \\
x_{i,n+1} = x_{i,n} + \lambda_n(s_{i,n} - x_{i,n}).
\end{array}
\right.
\end{array}
\right.
\end{array}
$$

De plus, on suppose que $\operatorname{zer}(\sum_{j=1}^{q} \boldsymbol{A}_j) \neq \varnothing$. Alors, pour tout $i \in \{1,\ldots,m\}$, $(x_{i,n})_{n\in\mathbb{N}}$ converge faiblement vers un point $x_i \in \mathcal{H}_i$ et $(x_1,\ldots,x_m) \in \operatorname{zer}(\sum_{j=1}^{q} \boldsymbol{A}_j)$.

Démonstration. Il suffit d'appliquer la Proposition 1.14 dans $\boldsymbol{\mathcal{H}}$ lorsque, pour tout $j \in \{1,\ldots,q\}$, $A_j = \boldsymbol{A}_j/\omega_j$. □

4.2.5 Méthode parallèle de type Dykstra

Nous proposons ici une méthode alternative à l'algorithme proposé dans la Proposition 4.11 pour résoudre le Problème 4.1 dans le cas particulier où $q \geq 2$ et $\boldsymbol{A}_q = \mathbf{Id} - \boldsymbol{z}$ avec $\boldsymbol{z} \in \boldsymbol{\mathcal{H}}$.

Problème 4.12 Soit $q \geq 2$ un entier, soient $(\boldsymbol{A}_j)_{1 \leq j \leq q-1}$ des opérateurs maximalement monotones dans $\boldsymbol{\mathcal{H}}$ et soit $\boldsymbol{z} = (z_i)_{1 \leq i \leq m} \in \boldsymbol{\mathcal{H}}$. Le problème est de

$$\text{trouver } x_1 \in \mathcal{H}_1, \dots, x_m \in \mathcal{H}_m \quad \text{tels que}$$
$$(z_1, \dots, z_m) \in (x_1, \dots, x_m) + \sum_{j=1}^{q-1} \boldsymbol{A}_j(x_1, \dots, x_m). \tag{4.7}$$

Notons que l'opérateur $\boldsymbol{A}_q = \mathbf{Id} - \boldsymbol{z}$ est fortement monotone. Ceci implique que le Problème 4.12 admet une solution unique $(x_i)_{1 \leq i \leq m}$ qui satisfait $(x_1, \dots, x_m) = J_{\boldsymbol{B}}(z_1, \dots, z_m)$, où $\boldsymbol{B} = \sum_{j=1}^{q-1} \boldsymbol{A}_j$.

Proposition 4.13 *Pour tout $i \in \{1, \dots, m\}$ et $j \in \{1, \dots, q\}$, soit $(a_{i,j,n})_{n \in \mathbb{N}}$ une suite dans $\mathcal{H}_i$ telle que $\sum_{n \in \mathbb{N}} \|a_{i,j,n}\| < +\infty$. On génère des suites $(x_{1,n})_{n \in \mathbb{N}}, \dots, (x_{m,n})_{n \in \mathbb{N}}$ comme suit.*

$$
\begin{array}{l}
\text{Initialisation} \\
\left\lfloor
\begin{array}{l}
(\omega_j)_{1 \leq j \leq q-1} \subset \,]0,1] \text{ et } \displaystyle\sum_{j=1}^{q-1} \omega_j = 1 \\
\text{Pour } i = 1, \dots, m \\
\left\lfloor
\begin{array}{l}
x_{i,0} = z_i \\
\text{Pour } j = 1, \dots, q-1 \\
\lfloor \; z_{i,j,0} = x_{i,0}
\end{array}
\right.
\end{array}
\right. \\
\text{Pour } n = 0, 1, \dots \\
\left\lfloor
\begin{array}{l}
\text{Pour } j = 1, \dots, q-1 \\
\lfloor \; (y_{1,j,n}, \dots, y_{m,j,n}) = J_{\boldsymbol{A}_j/\omega_j}(z_{1,j,n}, \dots, z_{m,j,n}) + (a_{1,j,n}, \dots, a_{m,j,n}) \\
\text{Pour } i = 1, \dots, m \\
\left\lfloor
\begin{array}{l}
x_{i,n+1} = \displaystyle\sum_{j=1}^{q-1} \omega_j y_{i,j,n} \\
\text{Pour } j = 1, \dots, q-1 \\
\lfloor \; z_{i,j,n+1} = x_{i,n+1} + z_{i,j,n} - y_{i,j,n}.
\end{array}
\right.
\end{array}
\right.
\end{array}
\tag{4.8}
$$

De plus, on suppose que $(z_1, \dots, z_m) \in \operatorname{ran}(\mathbf{Id} + \sum_{j=1}^{q-1} \boldsymbol{A}_j)$. Alors, pour tout $i \in \{1, \dots, m\}$, $x_{i,n} \to x_i \in \mathcal{H}_i$ et $(x_1, \dots, x_m) = J_{\boldsymbol{B}}(z_1, \dots, z_m)$, où $\boldsymbol{B} = \sum_{j=1}^{q-1} \boldsymbol{A}_j$.

Démonstration. Il suffit d'appliquer la Proposition 1.17 du Chapitre 1 dans $\boldsymbol{\mathcal{H}}$ lorsque, pour tout $j \in \{1, \dots, q\}$, $A_j = \boldsymbol{A}_j/\omega_j$. □

4.3 Application à la restauration et reconstruction variationnelle d'images multicomposantes

Dans cette section, nous étudions en détail des méthodes pour résoudre des problèmes de minimisation issues des méthodes multicomposantes énoncées dans la Section 4.2. Nous examinons aussi des applications au traitement d'images. Plus précisément, nous réalisons des expériences numériques sur la restauration d'images stéréoscopiques, le débruitage d'images multispectrales et la décomposition d'images en composantes structure–texture.

4.3.1 Description et résultats principaux

L'article qui suit est consacré au problème suivant.

Problème 4.14 Soit $q \geq 2$ un entier et soient $(\boldsymbol{f}_j)_{1\leq j\leq q}$ des fonctions dans $\Gamma_0(\boldsymbol{\mathcal{H}})$. Le problème est de

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\text{minimiser}} \quad \sum_{j=1}^{q} \boldsymbol{f}_j(x_1,\ldots,x_m). \tag{4.9}$$

Notons que une solution du Problème 4.14 peut être trouvée en résolvant le Problème 1.1 avec $\boldsymbol{A}_1 = \partial\boldsymbol{f}_1,\ldots,\boldsymbol{A}_m = \partial\boldsymbol{f}_m$. Sous certaines conditions de qualification (voir [9, Theorem 16.37]) ces problèmes sont équivalents.

Le Problème 4.14 modélise des problèmes de restauration et reconstruction d'images et de signaux où la solution idéale est représentée par m composantes x_1, $\ldots,x_m$ qui appartiennent aux espaces hilbertiens $\mathcal{H}_1,\ldots,\mathcal{H}_m$, respectivement.

Dans le cas monocomposante ($m=1$), le Problème 4.14 a été étudié dans [19, 21, 23], où plusieurs méthodes proximales ont été proposées (des applications peuvent être trouvées dans [22] et sa bibliographie). Si $m\geq 2$, $q=2$, $\boldsymbol{f}_1\colon (x_i)_{1\leq i\leq m}\mapsto \sum_{i=1}^m \varphi_i(x_i)$, où $(\varphi_i)_{1\leq i\leq m}$ sont des fonctions convexes, propres et semi-continues inférieurement, et $\boldsymbol{f}_2$ est différentiable avec gradient lipschitzien, le Problème 4.14 a été résolu dans le Chapitre 3 et plusieurs applications au traitement d'images peuvent être trouvées dans [5, 6, 7, 23, 28, 29, 43, 44, 48].

Nous étudions cas multicomposante général et nous proposons plusieurs algorithmes sous divers jeux d'hypothèses sur les fonctions impliquées dans le Problème 4.14. Nous traitons les problèmes suivants.

Problème 4.15 Soit $\boldsymbol{f}_1\in\Gamma_0(\boldsymbol{\mathcal{H}})$ et soit $\boldsymbol{f}_2\colon\boldsymbol{\mathcal{H}}\to\mathbb{R}$ une fonction convexe différentiable telle que son gradient est β^{-1}–lipschitzien avec $\beta\in\,]0,+\infty[$. Le problème est de

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\text{minimiser}} \quad \boldsymbol{f}_1(x_1,\ldots,x_m)+\boldsymbol{f}_2(x_1,\ldots,x_m). \tag{4.10}$$

Problème 4.16 Soient $\boldsymbol{f}_1$ et $\boldsymbol{f}_2$ deux fonctions dans $\Gamma_0(\boldsymbol{\mathcal{H}})$ telles que

$$(0, \ldots, 0) \in \operatorname{sri}(\operatorname{dom} \boldsymbol{f}_1 - \operatorname{dom} \boldsymbol{f}_2). \tag{4.11}$$

Le problème est de

$$\underset{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m}{\text{minimiser}} \quad \boldsymbol{f}_1(x_1, \ldots, x_m) + \boldsymbol{f}_2(x_1, \ldots, x_m). \tag{4.12}$$

Problème 4.17 Soit $q \geq 2$ un entier et soient $(\boldsymbol{f}_j)_{1 \leq j \leq q}$ des fonctions dans $\Gamma_0(\boldsymbol{\mathcal{H}})$ telles que

$$(0, \ldots, 0) \in \operatorname{sri}(\boldsymbol{D} - \operatorname{dom} \boldsymbol{f}_1 \times \cdots \times \operatorname{dom} \boldsymbol{f}_q), \tag{4.13}$$

où $\boldsymbol{D} = \big\{(\boldsymbol{x}, \ldots, \boldsymbol{x}) \mid \boldsymbol{x} \in \boldsymbol{\mathcal{H}}\big\}$. Le problème est de

$$\underset{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m}{\text{minimiser}} \quad \sum_{j=1}^{q} \boldsymbol{f}_j(x_1, \ldots, x_m). \tag{4.14}$$

Problème 4.18 Soit $q \geq 2$ un entier et soient $(\boldsymbol{f}_j)_{1 \leq j \leq q-1}$ des fonctions dans $\Gamma_0(\boldsymbol{\mathcal{H}})$ telles que

$$\bigcap_{j=1}^{q-1} \operatorname{dom} \boldsymbol{f}_j \neq \varnothing \tag{4.15}$$

et soit $\boldsymbol{z} = (z_i)_{1 \leq i \leq m} \in \boldsymbol{\mathcal{H}}$. Le problème est de

$$\underset{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m}{\text{minimiser}} \quad \sum_{j=1}^{q-1} \boldsymbol{f}_j(x_1, \ldots, x_m) + \frac{q-1}{2} \sum_{i=1}^{m} \|x_i - z_i\|^2. \tag{4.16}$$

Tous les problèmes décrits ci-dessus sont résolus par des méthodes classiques appliquées sur l'espace produit $\boldsymbol{\mathcal{H}}$. Ainsi, sur l'espace produit $\boldsymbol{\mathcal{H}}$, l'algorithme explicite-implicite dans [23] résout le Problème 4.15, l'algorithme de Douglas-Rachford dans [19] résout le Problème 4.16, l'algorithme d'éclatement parallèle dans [21] résout le Problème 4.17 et un algorithme de Dykstra dans [18] résout le Problème 4.18. Contrairement aux algorithmes conçus dans [4, 11], la méthode que nous proposons ici pour résoudre le Problème 4.15 permet de considérer une fonction $\boldsymbol{f}_1$ qui ne soit pas séparable, c'est-à-dire, qui ne soit pas de la forme $(x_i)_{1 \leq i \leq m} \mapsto \sum_{i=1}^{m} f_i(x_i)$. Nous énonçons ci-dessous les théorèmes principaux, pour lesquels la notation suivante sera utile : pour deux suites $(\boldsymbol{x}_n)_{n \in \mathbb{N}}$ et $(\boldsymbol{y}_n)_{n \in \mathbb{N}}$ dans $\boldsymbol{\mathcal{H}}$,

$$\Big[(\forall n \in \mathbb{N}) \ \ \boldsymbol{x}_n \approx \boldsymbol{y}_n\Big] \ \Leftrightarrow \ \sum_{n \in \mathbb{N}} \|\boldsymbol{x}_n - \boldsymbol{y}_n\| < +\infty. \tag{4.17}$$

Théorème 4.19 *On génère des suites* $(x_{1,n})_{n\in\mathbb{N}}$, …, $(x_{m,n})_{n\in\mathbb{N}}$ *comme suit.*

$$
\begin{array}{l}
\text{Initialisation}\\
\left\lfloor\begin{array}{l}
\varepsilon \in \left]0,\min\{1,\beta\}\right[\\
\text{Pour } i=1,\ldots,m\\
\lfloor\, x_{i,0}\in\mathcal{H}_i
\end{array}\right.\\
\text{Pour } n=0,1,\ldots\\
\left\lfloor\begin{array}{l}
(y_{i,n})_{1\le i\le m}\approx \nabla \boldsymbol{f}_2(x_{i,n})_{1\le i\le m}\\
\gamma_n\in[\varepsilon,2\beta-\varepsilon]\\
(z_{i,n})_{1\le i\le m}\approx \operatorname{prox}_{\gamma_n \boldsymbol{f}_1}(x_{i,n}-\gamma_n y_{i,n})_{1\le i\le m}\\
\lambda_n\in[\varepsilon,1]\\
\text{Pour } i=1,\ldots,m\\
\lfloor\, x_{i,n+1}=x_{i,n}+\lambda_n(z_{i,n}-x_{i,n})
\end{array}\right.
\end{array}
$$

De plus, on suppose que le Problème 4.15 admet au moins une solution. Alors, pour tout $i\in\{1,\ldots,m\}$, $x_{i,n}\rightharpoonup x_i\in\mathcal{H}_i$ *et* $(x_i)_{1\le i\le m}$ *est une solution du Problème 4.15.*

Théorème 4.20 *On génère des suites* $(x_{1,n})_{n\in\mathbb{N}}$, …, $(x_{m,n})_{n\in\mathbb{N}}$ *comme suit.*

$$
\begin{array}{l}
\text{Initialisation}\\
\left\lfloor\begin{array}{l}
\varepsilon \in \left]0,1\right[,\ \ \gamma\in\left]0,+\infty\right[\\
\text{Pour } i=1,\ldots,m\\
\lfloor\, x_{i,0}\in\mathcal{H}_i
\end{array}\right.\\
\text{Pour } n=0,1,\ldots\\
\left\lfloor\begin{array}{l}
(y_{i,n})_{1\le i\le m}\approx \operatorname{prox}_{\gamma \boldsymbol{f}_2}(x_{i,n})_{1\le i\le m}\\
(z_{i,n})_{1\le i\le m}\approx \operatorname{prox}_{\gamma \boldsymbol{f}_1}(2y_{i,n}-x_{i,n})_{1\le i\le m}\\
\lambda_n\in[\varepsilon,2-\varepsilon]\\
\text{Pour } i=1,\ldots,m\\
\lfloor\ \ x_{i,n+1}=x_{i,n}+\lambda_n(z_{i,n}-y_{i,n}).
\end{array}\right.
\end{array}
$$

De plus, on suppose que le Problème 4.16 admet au moins une solution. Alors, pour tout $i\in\{1,\ldots,m\}$, $x_{i,n}\rightharpoonup x_i\in\mathcal{H}_i$ *et* $(x_i)_{1\le i\le m}$ *est une solution du Problème 4.16.*

Théorème 4.21 *On génère des suites* $(x_{1,n})_{n\in\mathbb{N}}$, ..., $(x_{m,n})_{n\in\mathbb{N}}$ *comme suit.*

$$
\begin{array}{l}
\text{Initialisation}\\
\left\lfloor
\begin{array}{l}
\varepsilon \in \,]0,1[\,, \quad \gamma \in \,]0,+\infty[\\
\{\omega_j\}_{1\leq j\leq q} \subset \,]0,1] \text{ et } \displaystyle\sum_{j=1}^{q} \omega_j = 1\\
\text{Pour } i = 1,\ldots,m\\
\left\lfloor
\begin{array}{l}
\text{Pour } j = 1,\ldots,q\\
\lfloor\; y_{i,j,0} \in \mathcal{H}_i\\
x_{i,0} = \displaystyle\sum_{j=1}^{q} \omega_j y_{i,j,0}
\end{array}
\right.
\end{array}
\right.\\
\text{Pour } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{Pour } j = 1,\ldots,q\\
\left\lfloor\; (z_{i,j,n})_{1\leq i\leq m} \approx \operatorname{prox}_{\gamma \boldsymbol{f}_j/\omega_j}(y_{i,j,n})_{1\leq i\leq m}\right.\\
\text{Pour } i = 1,\ldots,m\\
\left\lfloor
\begin{array}{l}
s_{i,n} = \displaystyle\sum_{j=1}^{q} \omega_j z_{i,j,n}\\
\lambda_n \in [\varepsilon, 2-\varepsilon]\\
\text{Pour } j = 1,\ldots,q\\
\lfloor\; y_{i,j,n+1} = y_{i,j,n} + \lambda_n(2s_{i,n} - x_{i,n} - z_{i,j,n})\\
x_{i,n+1} = x_{i,n} + \lambda_n(s_{i,n} - x_{i,n})
\end{array}
\right.
\end{array}
\right.
\end{array}
$$

De plus, on suppose que le Problème 4.17 admet au moins une solution. Alors, pour tout $i \in \{1,\ldots,m\}$, $x_{i,n} \rightharpoonup x_i \in \mathcal{H}_i$ *et* $(x_i)_{1\leq i\leq m}$ *est une solution du Problème 4.17.*

Théorème 4.22 *On génère des suites* $(x_{1,n})_{n\in\mathbb{N}}$, ..., $(x_{m,n})_{n\in\mathbb{N}}$ *comme suit.*

$$
\begin{array}{l}
\text{Initialisation}\\
\left\lfloor
\begin{array}{l}
\text{Pour } i = 1,\ldots,m\\
\left\lfloor
\begin{array}{l}
x_{i,0} = z_i\\
\text{Pour } j = 1,\ldots,q-1\\
\lfloor\; y_{i,j,0} = x_{i,0}
\end{array}
\right.
\end{array}
\right.\\
\text{Pour } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{Pour } j = 1,\ldots,q-1\\
\left\lfloor\; (z_{i,j,n})_{1\leq i\leq m} = \operatorname{prox}_{\boldsymbol{f}_j}(y_{i,j,n})_{1\leq i\leq m}\right.\\
\text{Pour } i = 1,\ldots,m\\
\left\lfloor
\begin{array}{l}
x_{i,n+1} = \dfrac{1}{q-1}\displaystyle\sum_{j=1}^{q-1} z_{i,j,n}\\
\text{Pour } j = 1,\ldots,q-1\\
\lfloor\; y_{i,j,n+1} = x_{i,n+1} + y_{i,j,n} - z_{i,j,n}.
\end{array}
\right.
\end{array}
\right.
\end{array}
\tag{4.18}
$$

Alors, pour tout $i \in \{1,\ldots,m\}$, $x_{i,n} \rightharpoonup x_i \in \mathcal{H}_i$ *et* $(x_i)_{1\leq i\leq m}$ *est une solution du Problème 4.18.*

De surcroît, une partie de cet article est destinée au calcul des opérateurs proximaux multicomposantes, ce qui enrichit la diversité des applications qui peuvent être traitées avec ces techniques. Les résultats principaux sont les suivants.

Proposition 4.23 *Supposons que les espaces* $(\mathcal{H}_i)_{1\leq i\leq m}$ *aient la même dimension et que, pour tout* $i \in \{1,\ldots,m\}$, $(e_{i,k})_{k\in\mathbb{K}}$ *soit une base orthonormale de* $\mathcal{H}_i$. *De plus, soient* $(\boldsymbol{\phi}_k)_{k\in\mathbb{K}}$ *des fonctions dans* $\Gamma_0(\mathbb{R}^m)$ *et supposons que l'une des conditions suivantes soit satisfaite.*

(i) *Pour tout* $i \in \{1,\ldots,m\}$, *la dimension de* $\mathcal{H}_i$ *est infinie et, pour tout* $k \in \mathbb{K}$, $\boldsymbol{\phi}_k \geq \boldsymbol{\phi}_k(\mathbf{0}) = 0$.

(ii) *Pour tout* $i \in \{1,\ldots,m\}$, *la dimension de* $\mathcal{H}_i$ *est finie.*

Posons

$$\boldsymbol{f}\colon \boldsymbol{\mathcal{H}} \to \left]-\infty,+\infty\right] : \boldsymbol{x} \mapsto \sum_{k\in\mathbb{K}} \boldsymbol{\phi}_k\big(\langle x_1 \mid e_{1,k}\rangle,\ldots,\langle x_m \mid e_{m,k}\rangle\big). \tag{4.19}$$

Alors $\boldsymbol{f} \in \Gamma_0(\boldsymbol{\mathcal{H}})$ *et*

$$(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{H}}) \quad \operatorname{prox}_{\boldsymbol{f}} \boldsymbol{x} = \left(\sum_{k\in\mathbb{K}} \pi_{1,k} e_{1,k},\ldots,\sum_{k\in\mathbb{K}} \pi_{m,k} e_{m,k}\right), \tag{4.20}$$

où

$$(\forall k \in \mathbb{K}) \quad (\pi_{1,k},\ldots,\pi_{m,k}) = \operatorname{prox}_{\boldsymbol{\phi}_k}\big(\langle x_1 \mid e_{1,k}\rangle,\ldots,\langle x_m \mid e_{m,k}\rangle\big). \tag{4.21}$$

Proposition 4.24 *Pour tout* $j \in \{1,\ldots,M\}$, *soit* $\mathcal{G}_j$ *un espace hilbertien réel, soit* $\varphi_j \in \Gamma_0(\mathcal{G}_j)$ *et, pour tout* $i \in \{1,\ldots,m\}$, *soit* $L_{ji} \in \mathcal{B}(\mathcal{H}_i,\mathcal{G}_j)$. *Posons*

$$\boldsymbol{f}\colon \boldsymbol{\mathcal{H}} \to \left]-\infty,+\infty\right] : \boldsymbol{x} \mapsto \sum_{j=1}^{M} \varphi_j\left(\sum_{i=1}^{m} L_{ji} x_i\right) \tag{4.22}$$

et supposons que, pour tout $j \in \{1,\ldots,M\}$, *il existe* $\alpha_j \in \left]0,+\infty\right[$ *tel que*

$$(\forall k \in \{1,\ldots,M\}) \qquad \sum_{i=1}^{m} L_{ji} \circ L_{ki}^* = \begin{cases} \alpha_j \operatorname{Id}, & \textit{si } j = k;\\ 0, & \textit{autrement}. \end{cases} \tag{4.23}$$

Alors $\boldsymbol{f} \in \Gamma_0(\boldsymbol{\mathcal{H}})$ *et*

$$(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{H}}) \quad \operatorname{prox}_{\boldsymbol{f}} \boldsymbol{x} = (p_1,\ldots,p_m) \tag{4.24}$$

où, pour tout $i \in \{1,\ldots,m\}$,

$$p_i = x_i + \sum_{j=1}^{M} \alpha_j^{-1} L_{ji}^* \operatorname{prox}_{\alpha_j\varphi_j}\left(\sum_{k=1}^{m} L_{jk} x_k\right) - \sum_{j=1}^{M} \alpha_j^{-1} L_{ji}^*\left(\sum_{k=1}^{m} L_{jk} x_k\right). \tag{4.25}$$

Proposition 4.25 *Supposons que la dimension de* $\mathcal{G}$ *soit* $K < +\infty$*, soient* $(\phi_k)_{1 \le k \le K}$ *des fonctions dans* $\Gamma_0(\mathbb{R})$*, et soit* $(e_k)_{1 \le k \le K}$ *une base orthonormale de* $\mathcal{G}$*. Pour tout* $i \in \{1, \dots, m\}$*, soit* $L_i \in \mathcal{B}(\mathcal{H}_i, \mathcal{G})$ *et supposons qu'il existe* $\{\alpha_k\}_{1 \le k \le K} \subset \left]0, +\infty\right[$ *tels que*

$$(\forall y \in \mathcal{G}) \quad \sum_{i=1}^{m} L_i L_i^* y = \sum_{k=1}^{K} \alpha_k \langle y \mid e_k \rangle e_k. \tag{4.26}$$

Posons

$$\boldsymbol{f} \colon \boldsymbol{\mathcal{H}} \to \left]-\infty, +\infty\right] : \boldsymbol{x} \mapsto \sum_{k=1}^{K} \phi_k \left(\left\langle \sum_{j=1}^{m} L_j x_j \,\middle|\, e_k \right\rangle \right) \tag{4.27}$$

et

$$(\forall k \in \{1 \dots, K\}) \quad \pi_k = \frac{1}{\alpha_k} \operatorname{prox}_{\alpha_k \phi_k} \left(\left\langle \sum_{j=1}^{m} L_j x_j \,\middle|\, e_k \right\rangle \right). \tag{4.28}$$

Alors $\boldsymbol{f} \in \Gamma_0(\boldsymbol{\mathcal{H}})$ *et, pour tout* $\boldsymbol{x} \in \boldsymbol{\mathcal{H}}$*,* $\operatorname{prox}_{\boldsymbol{f}} \boldsymbol{x} = (p_i)_{1 \le i \le m}$ *où*

$$(\forall i \in \{1, \dots, m\}) \quad p_i = x_i + L_i^* \sum_{k=1}^{K} \left(\pi_k - \frac{1}{\alpha_k} \sum_{j=1}^{m} \langle L_j x_j \mid e_k \rangle \right) e_k. \tag{4.29}$$

Enfin, la dernière section de l'article est consacrée aux applications numériques au traitement de l'image. En particulier, à l'aide des Propositions 4.23, 4.24 et 4.25, nous adaptons les algorithmes proposés pour résoudre les Problèmes 4.15, 4.16, 4.17 et 4.18 à la restauration d'images stéréoscopiques, le débruitage d'images multispectrales et la décomposition d'images en composantes structure-texture.

4.3.2 Article en anglais

PROXIMAL ALGORITHMS FOR MULTICOMPONENT IMAGE RECOVERY PROBLEMS [1]

Abstract : In recent years, proximal splitting algorithms have been applied to various monocomponent signal and image recovery problems. In this paper, we address the case of multicomponent problems. We first provide closed form expressions for several important multicomponent proximity operators and then derive extensions of existing proximal algorithms to the multicomponent setting. These results are applied to stereoscopic image recovery, multispectral image denoising, and image decomposition into texture and geometry components.

1. L. M. Briceño-Arias, P. L. Combettes, J.-C. Pesquet, and N. Pustelnik, Proximal algorithms for multicomponent image recovery problems, *Journal of Mathematical Imaging and Vision*, DOI : 10.1007/s10851-010-0243-1, 2011.

4.3.2.1 Problem statement

In this paper, we consider signal and image recovery problems in which the ideal solution is represented by m components x_1, ..., x_m lying, respectively, in real Hilbert spaces $\mathcal{H}_1$, ..., $\mathcal{H}_m$. Such problems arise in many areas ranging from color and hyper-spectral imaging to multichannel signal processing and geometry/texture image decomposition [2, 5, 6, 7, 8, 13, 25, 27, 31, 32, 42, 45, 48]. Oftentimes, multicomponent signal/image processing tasks can be formulated as variational problems of the form

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\text{minimize}} \quad \Phi(x_1,\ldots,x_m), \tag{4.30}$$

where Φ is a convex function modeling the available information on the m components, their interactions, and, possibly, the data acquisition process.

The abstract convex minimization problem (4.30) is usually too generic to be solved directly and it must be formulated in a more structured fashion to be amenable to efficient numerical solution. To this end, Φ can be decomposed as a sum of p functions that can be handled individually more easily. This leads to the following model, which will be the focus of the paper.

Problem 4.26 Let $(\mathcal{H}_i)_{1\le i\le m}$ be real Hilbert spaces, and let $(f_k)_{1\le k\le p}$ be proper lower semicontinuous convex functions from the direct Hilbert sum $\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_m$ to $]-\infty,+\infty]$. The problem is to

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\text{minimize}} \quad \sum_{k=1}^{p} f_k(x_1,\ldots,x_m), \tag{4.31}$$

under the assumption that solutions exist.

In the case of univariate ($m=1$) signal processing problems, proximal methods have been successfully used to solve (4.31) ; see [19, 21, 23] for basic work, and [22] and the references therein for a variety of applications. It is therefore natural to ask whether these methods can be extended to the multivariate case. Initial work in this direction was carried out in [11] in the special instance when $m=2$, f_1 is a separable sum (i.e., $f_1\colon (x_i)_{1\le i\le m}\mapsto \sum_{i=1}^{m}\varphi_i(x_i)$), and f_2 is differentiable on $\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_m$ with a Lipschitz continuous gradient (this setting also covers formulations found in [5, 6, 7, 23, 28, 29, 43, 44, 48]). The objective of our paper is to address the general case and to present several proximal algorithms with guaranteed convergence to a solution to Problem 4.26 under suitable assumptions.

The paper is organized as follows. In section 4.3.2.2, the main notation used in the paper is introduced. Proximity operators will be an essential ingredient in the multicomponent algorithms proposed in the paper. They are briefly reviewed in section 4.3.2.3, where we also provide new results concerning multicomponent proximity operators. In section 4.3.2.4, we describe proximal splitting algorithms which are pertinent for solving Problem 4.26. Finally, in section 4.3.2.5, we illustrate the effectiveness of the proposed algorithms in three multicomponent imaging examples.

4.3.2.2 Notation

Throughout, $\mathcal{H}$, $\mathcal{G}$, and $(\mathcal{H}_i)_{1\leq i\leq m}$ are real Hilbert spaces. For convenience, their scalar products are all denoted by $\langle\cdot\mid\cdot\rangle$, the associated norms by $\|\cdot\|$, and their identity operators are all denoted by Id. It will be convenient to denote by $\boldsymbol{x}=(x_i)_{1\leq i\leq m}$ a generic element in $\mathcal{H}_1\times\cdots\times\mathcal{H}_m$, and by $\boldsymbol{\mathcal{H}}$ the direct Hilbert sum $\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_m$, i.e., the product space $\mathcal{H}_1\times\cdots\times\mathcal{H}_m$ equipped with the usual vector space structure and the scalar product

$$(\boldsymbol{x},\boldsymbol{y})\mapsto\sum_{i=1}^{m}\langle x_i\mid y_i\rangle\,. \tag{4.32}$$

The space of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$ is denoted by $\mathcal{B}\,(\mathcal{H},\mathcal{G})$. Moreover, $\Gamma_0(\mathcal{H})$ denotes the class of lower semicontinuous convex functions $\varphi\colon\mathcal{H}\to\left]-\infty,+\infty\right]$ which are proper in the sense that

$$\operatorname{dom}\varphi=\big\{x\in\mathcal{H}\;\big|\;\varphi(x)<+\infty\big\}\neq\varnothing. \tag{4.33}$$

Let C and D be nonempty convex subsets of $\mathcal{H}$. The indicator function of C is

$$\iota_C\colon x\mapsto\begin{cases}0, & \text{if } x\in C;\\ +\infty, & \text{if } x\notin C.\end{cases} \tag{4.34}$$

If C is closed, for every $x\in\mathcal{H}$, there exists a unique point $P_Cx\in C$ such that $\|x-P_Cx\|=\inf_{y\in C}\|x-y\|$; P_Cx is called the projection of x onto C. We say that 0 lies in the strong relative interior of C, in symbol, $0\in\operatorname{sri}C$, if $\bigcup_{\lambda>0}\lambda C=\overline{\operatorname{span}}\,C$. In particular, if we set $C-D=\big\{x-y\;\big|\;(x,y)\in C\times D\big\}$, the inclusion $0\in\operatorname{sri}(C-D)$ holds in each of the following cases :

- $C-D$ is a closed vector subspace.
- $0\in\operatorname{int}(C-D)$.
- $C\cap\operatorname{int}D\neq\varnothing$.
- $\mathcal{H}$ is finite dimensional and $(\operatorname{ri}C)\cap(\operatorname{ri}D)\neq\varnothing$, where $\operatorname{ri}C$ denotes the relative interior of C, i.e., its interior relative to its affine hull.

General background on convex analysis will be found in [9, 49].

4.3.2.3 Proximity operators

4.3.2.3.1 Definition and properties For a detailed account of the theory of proximity operators, see [9] and the classical paper [35].

Let $\varphi \in \Gamma_0(\mathcal{H})$. For every $x \in \mathcal{H}$, the function

$$y \mapsto \varphi(y) + \frac{1}{2}\|x - y\|^2 \tag{4.35}$$

has a unique minimizer, which is denoted by $\operatorname{prox}_\varphi x$ and characterized by the variational inequality

$$(\forall p \in \mathcal{H}) \quad p = \operatorname{prox}_\varphi x \quad \Leftrightarrow \quad (\forall y \in \mathcal{H}) \quad \langle y - p \mid x - p\rangle + \varphi(p) \leq \varphi(y). \tag{4.36}$$

The proximity operator $\operatorname{prox}_\varphi$ of φ thus defined is nonexpansive, i.e.,

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \|\operatorname{prox}_\varphi x - \operatorname{prox}_\varphi y\| \leq \|x - y\|. \tag{4.37}$$

Example 4.27 Let C be a nonempty closed convex subset of $\mathcal{H}$. Then $\operatorname{prox}_{\iota_C} = P_C$.

Other closed-form expressions for the proximity operators can be found in [3, 11, 15, 19, 20, 23, 35].

Lemma 4.28 [19, Proposition 11] *Let $\psi \in \Gamma_0(\mathcal{G})$, let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, and set $\varphi = \psi \circ L$. Suppose that $L \circ L^* = \alpha \operatorname{Id}$, for some $\alpha \in \,]0, +\infty[$. Then $\varphi \in \Gamma_0(\mathcal{H})$ and*

$$\operatorname{prox}_\varphi = \operatorname{Id} + \frac{1}{\alpha} L^* \circ (\operatorname{prox}_{\alpha\psi} - \operatorname{Id}) \circ L. \tag{4.38}$$

4.3.2.3.2 Multicomponent proximity operators The computation of proximity operators in the Hilbert direct sum $\boldsymbol{\mathcal{H}}$ will play a fundamental role in the next sections. Below, we provide some important situations in which this computation is explicit.

Proposition 4.29 *Suppose that, for every $i \in \{1, \ldots, m\}$, $(e_{i,k})_{k\in\mathbb{K}}$ is an orthonormal basis of $\mathcal{H}_i$. Furthermore, let $(\phi_k)_{k\in\mathbb{K}}$ be functions in $\Gamma_0(\mathbb{R}^m)$ and suppose that one of the following holds.*

(i) *For every $i \in \{1, \ldots, m\}$, $\mathcal{H}_i$ is infinite dimensional and, for every $k \in \mathbb{K}$, $\phi_k \geq \phi_k(\mathbf{0}) = 0$.*

(ii) *For every $i \in \{1, \ldots, m\}$, $\mathcal{H}_i$ is finite dimensional.*

Set

$$f \colon \boldsymbol{\mathcal{H}} \to \,]-\infty, +\infty] : \boldsymbol{x} \mapsto \sum_{k\in\mathbb{K}} \phi_k\big(\langle x_1 \mid e_{1,k}\rangle, \ldots, \langle x_m \mid e_{m,k}\rangle \big). \tag{4.39}$$

Then $f \in \Gamma_0(\boldsymbol{\mathcal{H}})$ and, for every $\boldsymbol{x} \in \boldsymbol{\mathcal{H}}$,

$$\operatorname{prox}_f \boldsymbol{x} = \left(\sum_{k\in\mathbb{K}} \pi_{1,k} e_{1,k}, \ldots, \sum_{k\in\mathbb{K}} \pi_{m,k} e_{m,k} \right), \tag{4.40}$$

where

$$(\forall k \in \mathbb{K}) \quad (\pi_{1,k}, \ldots, \pi_{m,k}) = \operatorname{prox}_{\phi_k}\big(\langle x_1 \mid e_{1,k}\rangle, \ldots, \langle x_m \mid e_{m,k}\rangle \big). \tag{4.41}$$

Proof. Let us set, for every $k \in \mathbb{K}$,

$$\psi_k\colon \boldsymbol{\mathcal{H}} \to \left]-\infty,+\infty\right] : \boldsymbol{x} \mapsto \phi_k\big(\langle x_1 \mid e_{1,k}\rangle, \ldots, \langle x_m \mid e_{m,k}\rangle\big). \tag{4.42}$$

Then our assumptions imply that the functions $(\psi_k)_{k\in\mathbb{K}}$ are in $\Gamma_0(\boldsymbol{\mathcal{H}})$. Under assumption (i), assuming without loss of generality that $\mathbb{K} = \mathbb{N}$, we can write $f = \sup_{K\in\mathbb{K}} \sum_{k=0}^{K} \psi_k$. Since lower semicontinuity and convexity are preserved under finite sums and taking suprema, it follows that f is lower semicontinuous and convex. In addition, since $f(\mathbf{0}) = 0$, we obtain $f \in \Gamma_0(\boldsymbol{\mathcal{H}})$. On the other hand, under assumption (ii), the sum in (4.39) is finite and our assumptions imply at once that $f \in \Gamma_0(\boldsymbol{\mathcal{H}})$.

Now let $\boldsymbol{x} \in \boldsymbol{\mathcal{H}}$ and denote the Euclidean norm on $\mathbb{R}^m$ by $|\cdot|$. Set

$$(\forall i \in \{1,\ldots,m\})(\forall k \in \mathbb{K}) \quad \xi_{i,k} = \langle x_i \mid e_{i,k}\rangle\,. \tag{4.43}$$

Moreover, for every $i \in \{1,\ldots,m\}$, let $y_i \in \mathcal{H}_i$ and set $(\eta_{i,k})_{k\in\mathbb{K}} = (\langle y_i \mid e_{i,k}\rangle)_{k\in\mathbb{K}}$. We derive from (4.41) and (4.36) that, for every $k \in \mathbb{K}$,

$$\sum_{i=1}^{m} (\eta_{i,k} - \pi_{i,k})(\xi_{i,k} - \pi_{i,k}) + \phi_k(\pi_{1,k},\ldots,\pi_{m,k}) \le \phi_k(\eta_{1,k},\ldots,\eta_{m,k}). \tag{4.44}$$

Let us first assume that (i) holds. For every $k \in \mathbb{K}$ observe that, since $\mathbf{0}$ is a minimizer of ϕ_k, (4.36) yields $\operatorname{prox}_{\phi_k}\mathbf{0} = \mathbf{0}$. Hence, using (4.41), (4.43), (4.37), and Parseval's identity, we obtain

$$\begin{aligned}
\sum_{k\in\mathbb{K}}\sum_{i=1}^{m} |\pi_{i,k}|^2 &= \sum_{k\in\mathbb{K}} |(\pi_{1,k},\ldots,\pi_{m,k})|^2 \\
&= \sum_{k\in\mathbb{K}} |\operatorname{prox}_{\phi_k}(\xi_{1,k},\ldots,\xi_{m,k})|^2 \\
&= \sum_{k\in\mathbb{K}} |\operatorname{prox}_{\phi_k}(\xi_{1,k},\ldots,\xi_{m,k}) - \operatorname{prox}_{\phi_k}\mathbf{0}|^2 \\
&\le \sum_{k\in\mathbb{K}} |(\xi_{1,k},\ldots,\xi_{m,k}) - \mathbf{0}|^2 \\
&= \sum_{k\in\mathbb{K}} |(\xi_{1,k},\ldots,\xi_{m,k})|^2 \\
&= \sum_{k\in\mathbb{K}}\sum_{i=1}^{m} |\xi_{i,k}|^2 \\
&= \sum_{i=1}^{m} \|x_i\|^2.
\end{aligned} \tag{4.45}$$

Therefore, $(\forall i \in \{1,\ldots,m\})\ \sum_{k\in\mathbb{K}} |\pi_{i,k}|^2 < +\infty$. Consequently, we can define

$$(\forall i \in \{1,\ldots,m\}) \quad z_i = \sum_{k\in\mathbb{K}} \pi_{i,k} e_{i,k}, \tag{4.46}$$

which is equally well defined under assumption (ii) as $\mathbb{K}$ is then finite. It remains to show that $(z_i)_{1\leq i\leq m} = \operatorname{prox}_f(x_1,\ldots,x_m)$. Summing over k in (4.44) yields

$$\sum_{k\in\mathbb{K}}\sum_{i=1}^{m}(\eta_{i,k}-\pi_{i,k})(\xi_{i,k}-\pi_{i,k}) + \sum_{k\in\mathbb{K}}\phi_k(\pi_{1,k},\ldots,\pi_{m,k}) \leq \sum_{k\in\mathbb{K}}\phi_k(\eta_{1,k},\ldots,\eta_{m,k}) \tag{4.47}$$

and, therefore,

$$\sum_{i=1}^{m}\langle y_i - z_i \mid x_i - z_i\rangle + g(z_1,\ldots,z_m) \leq g(y_1,\ldots,y_m). \tag{4.48}$$

In view of (4.36), the proof is complete. □

Proposition 4.30 *For every $j\in\{1,\ldots,q\}$, let $\mathcal{G}_j$ be a real Hilbert space, let $\varphi_j\in\Gamma_0(\mathcal{G}_j)$, and, for every $i\in\{1,\ldots,m\}$, let $L_{j,i}\in\mathcal{B}(\mathcal{H}_i,\mathcal{G}_j)$. Set*

$$f\colon \boldsymbol{\mathcal{H}} \to \left]-\infty,+\infty\right] : \boldsymbol{x} \mapsto \sum_{j=1}^{q}\varphi_j\Bigg(\sum_{i=1}^{m}L_{j,i}x_i\Bigg) \tag{4.49}$$

and suppose that, for every $j\in\{1,\ldots,q\}$, there exists $\alpha_j\in\left]0,+\infty\right[$ such that

$$(\forall k\in\{1,\ldots,q\})\quad \sum_{i=1}^{m}L_{j,i}\circ L_{k,i}^* = \begin{cases}\alpha_j\,\mathrm{Id}, & \text{if } j=k;\\ 0, & \text{otherwise}.\end{cases} \tag{4.50}$$

Then $f\in\Gamma_0(\boldsymbol{\mathcal{H}})$ and, for every $\boldsymbol{x}\in\boldsymbol{\mathcal{H}}$,

$$\operatorname{prox}_f \boldsymbol{x} = (p_1,\ldots,p_m) \tag{4.51}$$

where, for every $i\in\{1,\ldots,m\}$,

$$p_i = x_i + \sum_{j=1}^{q}\alpha_j^{-1}L_{j,i}^*\operatorname{prox}_{\alpha_j\varphi_j}\left(\sum_{k=1}^{m}L_{j,k}x_k\right) - \sum_{j=1}^{q}\alpha_j^{-1}L_{j,i}^*\sum_{k=1}^{m}L_{j,k}x_k. \tag{4.52}$$

Proof. Let us denote by $\boldsymbol{\mathcal{G}}$ the product space $\mathcal{G}_1\times\cdots\times\mathcal{G}_q$ equipped with the usual vector space structure and the scalar product

$$(\boldsymbol{y},\boldsymbol{z}) \mapsto \sum_{j=1}^{q}\alpha_j^{-1}\langle y_j \mid z_j\rangle\,. \tag{4.53}$$

We can write $f = g\circ\boldsymbol{L}$, where

$$g\colon \boldsymbol{\mathcal{G}} \to \left]-\infty,+\infty\right] : \boldsymbol{y} \mapsto \sum_{j=1}^{q}\varphi_j(y_j) \tag{4.54}$$

and $\boldsymbol{L} \in \mathcal{B}(\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{G}})$ is defined by

$$\boldsymbol{L} \colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{G}} \colon \boldsymbol{x} \mapsto \left(\sum_{i=1}^{m} L_{1,i} x_i, \ldots, \sum_{i=1}^{m} L_{q,i} x_i \right). \tag{4.55}$$

It follows from (4.53) that, for every $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{\mathcal{H}} \times \boldsymbol{\mathcal{G}}$,

$$\langle \boldsymbol{L}\boldsymbol{x} \mid \boldsymbol{y} \rangle = \sum_{j=1}^{q} \alpha_j^{-1} \left\langle \sum_{i=1}^{m} L_{j,i} x_i \;\middle|\; y_j \right\rangle = \sum_{j=1}^{q} \alpha_j^{-1} \sum_{i=1}^{m} \langle x_i \mid L_{j,i}^* y_j \rangle = \sum_{i=1}^{m} \left\langle x_i \;\middle|\; \sum_{j=1}^{q} \alpha_j^{-1} L_{j,i}^* y_j \right\rangle, \tag{4.56}$$

from which we deduce that the adjoint of $\boldsymbol{L}$ is

$$\boldsymbol{L}^* \colon \boldsymbol{\mathcal{G}} \to \boldsymbol{\mathcal{H}} \colon \boldsymbol{y} \mapsto \left(\sum_{j=1}^{q} \alpha_j^{-1} L_{j,1}^* y_j, \ldots, \sum_{j=1}^{q} \alpha_j^{-1} L_{j,m}^* y_j \right). \tag{4.57}$$

We then get from (4.50) that $\boldsymbol{L} \circ \boldsymbol{L}^* = \mathrm{Id}$. Hence, Lemma 4.28 implies that $f = g \circ \boldsymbol{L} \in \Gamma_0(\boldsymbol{\mathcal{H}})$ and that

$$\operatorname{prox}_{g \circ \boldsymbol{L}} = \mathrm{Id} + \boldsymbol{L}^* \circ (\operatorname{prox}_g - \mathrm{Id}) \circ \boldsymbol{L}. \tag{4.58}$$

In addition, it follows from (4.54) and (4.53) that, for every $\boldsymbol{y} \in \boldsymbol{\mathcal{G}}$,

$$\operatorname{prox}_g \boldsymbol{y} = (\operatorname{prox}_{\alpha_1 \varphi_1} y_1, \ldots, \operatorname{prox}_{\alpha_q \varphi_q} y_q). \tag{4.59}$$

Altogether, (4.55), (4.57), (4.58), and (4.59) yield (4.51)–(4.52). □

Corollary 4.31 *Let $\varphi \in \Gamma_0(\mathcal{G})$ and, for every $i \in \{1, \ldots, m\}$, let $L_i \in \mathcal{B}(\mathcal{H}_i, \mathcal{G})$. Set*

$$f \colon \boldsymbol{\mathcal{H}} \to \left]-\infty, +\infty\right] \colon \boldsymbol{x} \mapsto \varphi\left(\sum_{i=1}^{m} L_i x_i \right) \tag{4.60}$$

and suppose that there exists $\alpha \in \left]0, +\infty\right[$ such that

$$\sum_{i=1}^{m} L_i \circ L_i^* = \alpha \,\mathrm{Id}\,. \tag{4.61}$$

Then $f \in \Gamma_0(\boldsymbol{\mathcal{H}})$ and, for every $\boldsymbol{x} \in \boldsymbol{\mathcal{H}}$, $\operatorname{prox}_f \boldsymbol{x} = (p_1, \ldots, p_m)$ where, for every $i \in \{1, \ldots, m\}$,

$$p_i = x_i + \alpha^{-1} L_i^* \operatorname{prox}_{\alpha\varphi}\left(\sum_{k=1}^{m} L_k x_k \right) - \alpha^{-1} L_i^* \sum_{k=1}^{m} L_k x_k. \tag{4.62}$$

Proof. Set $q = 1$ in Proposition 4.30. □

Proposition 4.32 *Suppose that $\mathcal{G}$ has finite dimension K, let $(\phi_k)_{1\leq k\leq K}$ be functions in $\Gamma_0(\mathbb{R})$, and let $(e_k)_{1\leq k\leq K}$ be an orthonormal basis of $\mathcal{G}$. For every $i \in \{1,\ldots,m\}$, let $L_i \in \mathcal{B}(\mathcal{H}_i, \mathcal{G})$ and suppose that there exists $\{\alpha_k\}_{1\leq k\leq K} \subset]0,+\infty[$ such that*

$$(\forall y \in \mathcal{G}) \quad \sum_{i=1}^{m} L_i L_i^* y = \sum_{k=1}^{K} \alpha_k \langle y \mid e_k \rangle e_k. \tag{4.63}$$

Set

$$f\colon \boldsymbol{\mathcal{H}} \to]-\infty,+\infty] : \boldsymbol{x} \mapsto \sum_{k=1}^{K} \phi_k\left(\left\langle \sum_{j=1}^{m} L_j x_j \,\middle|\, e_k \right\rangle\right) \tag{4.64}$$

and, for every $k \in \{1\ldots,K\}$,

$$\pi_k = \frac{1}{\alpha_k} \operatorname{prox}_{\alpha_k\phi_k}\left(\left\langle \sum_{j=1}^{m} L_j x_j \,\middle|\, e_k \right\rangle\right). \tag{4.65}$$

Then $f \in \Gamma_0(\boldsymbol{\mathcal{H}})$ and, for every $\boldsymbol{x} \in \boldsymbol{\mathcal{H}}$, $\operatorname{prox}_f \boldsymbol{x} = (p_i)_{1\leq i\leq m}$ where, for every $i \in \{1,\ldots,m\}$,

$$p_i = x_i + L_i^* \sum_{k=1}^{K} \left(\pi_k - \frac{1}{\alpha_k}\sum_{j=1}^{m} \langle L_j x_j \mid e_k\rangle\right) e_k. \tag{4.66}$$

Proof. For every $j \in \{1,\ldots,K\}$, set $\mathcal{G}_j = \mathbb{R}$, $\varphi_j = \phi_j$, and

$$(\forall i \in \{1,\ldots,m\}) \quad L_{j,i}\colon \mathcal{H}_i \to \mathcal{G}_j : x \mapsto \langle L_i x \mid e_j \rangle, \tag{4.67}$$

hence

$$L_{j,i}^*\colon \mathcal{G}_j \to \mathcal{H}_i\colon \xi \mapsto \xi\, L_i^* e_j. \tag{4.68}$$

Thus, for every j and k in $\{1,\ldots,K\}$ and every $\xi \in \mathbb{R}$, we derive from (4.63) that

$$\begin{aligned}\sum_{i=1}^{m} L_{j,i} L_{k,i}^* \xi &= \sum_{i=1}^{m} L_{j,i} \xi\, L_i^* e_k \\ &= \xi \sum_{i=1}^{m} \langle (L_i L_i^*) e_k \mid e_j \rangle \\ &= \xi \left\langle \sum_{l=1}^{K} \alpha_l \langle e_k \mid e_l\rangle e_l \,\middle|\, e_j \right\rangle \\ &= \xi \sum_{l=1}^{K} \alpha_l \langle e_k \mid e_l \rangle \langle e_l \mid e_j \rangle.\end{aligned} \tag{4.69}$$

Therefore, for every $j \in \{1,\ldots,K\}$, (4.50) is satisfied. In turn, Proposition 4.30 with $q = K$ guarantees that $f \in \Gamma_0(\boldsymbol{\mathcal{H}})$, and (4.52) reduces to (4.66). □

4.3.2.4 Multicomponent proximal algorithms

We present several algorithms for solving Problem 4.26 under various assumptions on the functions involved. Most of these algorithms are tolerant to errors in the computation of proximal points and gradients. To quantify the amount of error which is tolerated, it will be convenient to use the following notation : given two sequences $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$ in $\boldsymbol{\mathcal{H}}$,

$$\Big[(\forall n \in \mathbb{N})\ \ \boldsymbol{x}_n \approx \boldsymbol{y}_n\Big] \ \Leftrightarrow \ \sum_{n\in\mathbb{N}} \|\boldsymbol{x}_n - \boldsymbol{y}_n\| < +\infty. \tag{4.70}$$

4.3.2.4.1 Forward-backward splitting

Problem 4.33 In Problem 4.26, suppose that $p = 2$ and that f_2 is differentiable on $\boldsymbol{\mathcal{H}}$ with a β–Lipschitz continuous gradient for some $\beta \in\,]0, +\infty[$. Hence, the problem is to

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\text{minimize}} \quad f_1(x_1, \ldots, x_m) + f_2(x_1, \ldots, x_m), \tag{4.71}$$

under the assumption that solutions exist.

The particular case when f_1 is a separable sum and f_2 involves a linear mixture of the variables was investigated in [11]. The following result addresses the general case ; it implicitly assumes that the proximity operator of f_1 can be computed to within a quantifiable error.

Theorem 4.34 *Let $(x_{1,n})_{n\in\mathbb{N}}$, ..., $(x_{m,n})_{n\in\mathbb{N}}$ be sequences generated by the following routine.*

Initialization
$$\left\lfloor \begin{array}{l} \varepsilon \in\,]0, \min\{1, 1/\beta\}[\\ \text{For } i = 1, \ldots, m \\ \lfloor\ x_{i,0} \in \mathcal{H}_i \end{array} \right.$$
For $n = 0, 1, \ldots$
$$\left\lfloor \begin{array}{l} (y_{i,n})_{1\le i\le m} \approx \nabla f_2(x_{i,n})_{1\le i\le m} \\ \gamma_n \in [\varepsilon, (2/\beta) - \varepsilon] \\ (u_{i,n})_{1\le i\le m} \approx \operatorname{prox}_{\gamma_n f_1}(x_{i,n} - \gamma_n y_{i,n})_{1\le i\le m} \\ \lambda_n \in [\varepsilon, 1] \\ \text{For } i = 1, \ldots, m \\ \lfloor\ x_{i,n+1} = x_{i,n} + \lambda_n(u_{i,n} - x_{i,n}) \end{array} \right.$$

Then, for every $i \in \{1, \ldots, m\}$, $(x_{i,n})_{n\in\mathbb{N}}$ converges weakly to a point $x_i \in \mathcal{H}_i$. Moreover, $(x_i)_{1\le i\le m}$ is a solution to Problem 4.33.

Proof. Apply [23, Theorem 3.4(i)] in $\boldsymbol{\mathcal{H}}$ and use (4.32). □

Remark 4.35

(i) Multicomponent version of variants of the above forward-backward algorithm such as the Nesterov-like first-order methods [10, 36, 47] can be obtained by similar reformulations in $\boldsymbol{\mathcal{H}}$. However, for these methods, convergence of the iterates $((x_{i,n})_{1\leq i\leq m})_{n\in\mathbb{N}}$ to a solution to Problem 4.33 is not guaranteed, even in a finite-dimensional setting.

(ii) Strong convergence conditions in Theorem 4.34 can be derived from [23, Theorem 3.4(iv)].

4.3.2.4.2 Douglas-Rachford splitting

In this section, we relax the assumption of smoothness on f_2 and assume that its proximity operator is implementable to within a quantifiable error.

Problem 4.36 In Problem 4.26, suppose that $p = 2$ and that

$$\mathbf{0} \in \operatorname{sri}(\operatorname{dom} f_1 - \operatorname{dom} f_2). \tag{4.72}$$

Hence, the problem is to

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\text{minimize}} \quad f_1(x_1,\ldots,x_m) + f_2(x_1,\ldots,x_m), \tag{4.73}$$

under the assumption that solutions exist.

Theorem 4.37 *Let $(x_{1,n})_{n\in\mathbb{N}}$, ..., $(x_{m,n})_{n\in\mathbb{N}}$ be sequences generated by the following routine.*

$$
\begin{array}{l}
\text{Initialization}\\
\left\lfloor
\begin{array}{l}
\varepsilon\in\,]0,1[\,,\quad \gamma\in\,]0,+\infty[\\
\text{For } i=1,\ldots,m\\
\lfloor\ x_{i,0}\in\mathcal{H}_i
\end{array}
\right.\\
\text{For } n=0,1,\ldots\\
\left\lfloor
\begin{array}{l}
(y_{i,n})_{1\leq i\leq m}\approx \operatorname{prox}_{\gamma f_2}(x_{i,n})_{1\leq i\leq m}\\
(u_{i,n})_{1\leq i\leq m}\approx \operatorname{prox}_{\gamma f_1}(2y_{i,n}-x_{i,n})_{1\leq i\leq m}\\
\lambda_n\in[\varepsilon,2-\varepsilon]\\
\text{For } i=1,\ldots,m\\
\lfloor\ x_{i,n+1}=x_{i,n}+\lambda_n(u_{i,n}-y_{i,n}).
\end{array}
\right.
\end{array}
$$

Then, for every $i\in\{1,\ldots,m\}$, $(x_{i,n})_{n\in\mathbb{N}}$ converges weakly to a point $x_i\in\mathcal{H}_i$. Moreover, $\operatorname{prox}_{\gamma f_2}(x_1,\ldots,x_m)$ is a solution to Problem 4.36 and $((y_{i,n})_{1\leq i\leq m})_{n\in\mathbb{N}}$ converges weakly to $\operatorname{prox}_{\gamma f_2}(x_1,\ldots,x_m)$.

Proof. For the first two claims, apply [19, Theorem 20] in $\boldsymbol{\mathcal{H}}$ and use (4.32). For the weak convergence claim, combine the results of [9, Corollary 27.4] and [41]. □

Remark 4.38

(i) Strong convergence conditions in Theorem 4.37 can be derived from [18, Theorem 2.1(ii)].

(ii) If $\boldsymbol{\mathcal{H}}$ is finite dimensional, the qualification condition (4.72) reduces to

$$(\operatorname{ri}\operatorname{dom} f_1) \cap (\operatorname{ri}\operatorname{dom} f_2) \neq \varnothing. \tag{4.74}$$

4.3.2.4.3 Parallel proximal algorithm (PPXA) The algorithm presented in this section aims at solving Problem 4.26 under minimal technical assumptions. Its cost of implementation depends on the ease of (approximate) computation of the individual proximity operators.

Problem 4.39 In Problem 4.26, suppose that

$$\mathbf{0} \in \operatorname{sri}(\boldsymbol{D} - \operatorname{dom} f_1 \times \cdots \times \operatorname{dom} f_p) \tag{4.75}$$

where $\boldsymbol{D} = \big\{(\boldsymbol{x}, \ldots, \boldsymbol{x}) \mid \boldsymbol{x} \in \boldsymbol{\mathcal{H}}\big\}$. The problem is to

$$\underset{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m}{\text{minimize}} \quad \sum_{k=1}^{p} f_k(x_1, \ldots, x_m), \tag{4.76}$$

under the assumption that solutions exist.

In [1], a particular instance of Problem 4.39 in finite dimensional spaces is considered. It is approached via the alternating direction method of multipliers. The algorithm used below is an application of the PPXA algorithm proposed in [21] that allows us to address the general case.

Theorem 4.40 *Let* $(x_{1,n})_{n\in\mathbb{N}}$, ..., $(x_{m,n})_{n\in\mathbb{N}}$ *be sequences generated by the following routine.*

Initialization

$$\left\lfloor \begin{array}{l} \varepsilon \in \left]0,1\right[, \quad \gamma \in \left]0,+\infty\right[\\ \{\omega_k\}_{1\le k\le p} \subset \left]0,1\right] \text{ and } \displaystyle\sum_{k=1}^{p} \omega_k = 1 \\ \text{For } i = 1,\ldots,m \\ \left\lfloor \begin{array}{l} \text{For } k = 1,\ldots,p \\ \lfloor\, y_{i,k,0} \in \mathcal{H}_i \\ x_{i,0} = \displaystyle\sum_{k=1}^{p} \omega_k y_{i,k,0} \end{array} \right. \end{array} \right.$$

$$
\begin{array}{l}
\text{For } n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
\text{For } k = 1, \ldots, p \\
\left\lfloor (u_{i,k,n})_{1 \le i \le m} \approx \operatorname{prox}_{\gamma f_k/\omega_k} (y_{i,k,n})_{1 \le i \le m} \right. \\
\text{For } i = 1, \ldots, m \\
\left\lfloor
\begin{array}{l}
s_{i,n} = \displaystyle\sum_{k=1}^{p} \omega_k u_{i,k,n} \\
\lambda_n \in [\varepsilon, 2 - \varepsilon] \\
\text{For } k = 1, \ldots, p \\
\left\lfloor y_{i,k,n+1} = y_{i,k,n} + \lambda_n (2 s_{i,n} - x_{i,n} - u_{i,k,n}) \right. \\
x_{i,n+1} = x_{i,n} + \lambda_n (s_{i,n} - x_{i,n})
\end{array}
\right.
\end{array}
\right.
\end{array}
$$

Then, for every $i \in \{1, \ldots, m\}$, $(x_{i,n})_{n \in \mathbb{N}}$ converges weakly to a point $x_i \in \mathcal{H}_i$. Moreover, $(x_1, \ldots, x_m)$ is a solution to Problem 4.39.

Proof. Apply [21, Theorem 3.4] in $\boldsymbol{\mathcal{H}}$ and use (4.32). □

Remark 4.41 Suppose that $\boldsymbol{\mathcal{H}}$ is finite dimensional and that

$$
\bigcap_{k=1}^{p} \operatorname{ri} \operatorname{dom} f_k \neq \varnothing. \tag{4.77}
$$

Then it follows from [21, Proposition 3.6(vi)] that the qualification condition (4.75) is satisfied.

4.3.2.4.4 Dykstra-like splitting

We consider instances of Problem 4.26 in which f_p is a simple quadratic function.

Problem 4.42 In Problem 4.26, suppose that $p \ge 3$, that

$$
\bigcap_{k=1}^{p-1} \operatorname{dom} f_k \neq \varnothing, \tag{4.78}
$$

and that $f_p \colon \boldsymbol{x} \mapsto (p-1) \sum_{i=1}^{m} \|x_i - z_i\|^2/2$, where $\boldsymbol{z} \in \boldsymbol{\mathcal{H}}$. Hence, the problem is to

$$
\underset{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m}{\text{minimize}} \quad \sum_{k=1}^{p-1} f_k(x_1, \ldots, x_m) + \frac{p-1}{2} \sum_{i=1}^{m} \|x_i - z_i\|^2. \tag{4.79}
$$

Set $f = \sum_{k=1}^{p-1} f_k/(p-1)$. Then it follows from (4.78) that $f \in \Gamma_0(\boldsymbol{\mathcal{H}})$. Hence, in view of (4.32), Problem 4.42 admits a unique solution, namely $\operatorname{prox}_f \boldsymbol{z}$.

Theorem 4.43 *Let $(x_{1,n})_{n\in\mathbb{N}}$, ..., $(x_{m,n})_{n\in\mathbb{N}}$ be sequences generated by the following routine.*

$$
\begin{array}{l}
\text{Initialization}\\
\left\lfloor
\begin{array}{l}
\text{For } i = 1,\ldots,m\\
\left\lfloor
\begin{array}{l}
x_{i,0} = z_i\\
\text{For } k = 1,\ldots,p-1\\
\lfloor\; y_{i,k,0} = x_{i,0}
\end{array}
\right.
\end{array}
\right.\\
\text{For } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{For } k = 1,\ldots,p-1\\
\lfloor\; (u_{i,k,n})_{1\le i\le m} = \operatorname{prox}_{f_k}(y_{i,k,n})_{1\le i\le m}\\
\text{For } i = 1,\ldots,m\\
\left\lfloor
\begin{array}{l}
x_{i,n+1} = \dfrac{1}{p-1}\displaystyle\sum_{k=1}^{p-1} u_{i,k,n}\\
\text{For } k = 1,\ldots,p-1\\
\lfloor\; y_{i,k,n+1} = x_{i,n+1} + y_{i,k,n} - u_{i,k,n}.
\end{array}
\right.
\end{array}
\right.
\end{array}
\tag{4.80}
$$

Then, for every $i \in \{1,\ldots,m\}$, $(x_{i,n})_{n\in\mathbb{N}}$ converges strongly to a point $x_i \in \mathcal{H}_i$. Moreover, $(x_1,\ldots,x_m)$ is a solution to Problem 4.42.

Proof. Apply [18, Theorem 4.2] in $\boldsymbol{\mathcal{H}}$ and use (4.32). □

Remark 4.44 Suppose that (4.78) is replaced by the stronger condition (4.75) (applied to the functions $(f_k)_{1\le k\le p-1}$). Then it follows from [18, Theorem 3.3] that the conclusion of the above theorem remains valid if the proximity operators are implemented approximately in (4.80).

4.3.2.5 Applications to image decomposition and recovery

In this section, we apply the algorithms proposed in Section 4.3.2.4 to stereoscopic image restoration, multispectral imaging, and image decomposition.

4.3.2.5.1 Stereoscopic image restoration

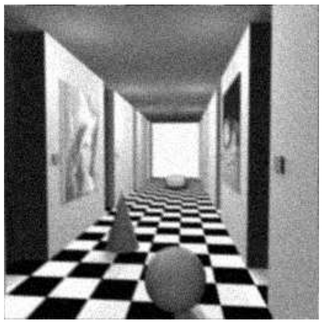
Problem formulation. We consider the problem of restoring a pair of N-pixel stereoscopic images $\overline{x}_1 \in \mathbb{R}^N$ and $\overline{x}_2 \in \mathbb{R}^N$, which correspond to the left and the right views of the same scene. For a given value of the disparity field, the disparity compensation process between the two images is modeled as

$$\overline{x}_1 = D\overline{x}_2 + v, \tag{4.81}$$

Original left $\overline{x}_1$

Original right $\overline{x}_2$

Degraded left z_1
SNR = 12.9 dB – SSIM = 0.39



Degraded right z_2
SNR = 18.0 dB – SSIM = 0.56

Restored left x_1 with $\vartheta = 0$
SNR = 15.5 dB – SSIM = 0.58

Restored right x_2 with $\vartheta = 0$
SNR = 19.3 dB – SSIM = 0.73

Restored left x_1 with $\vartheta = 1.6 \times 10^{-3}$
SNR = 17.8 dB – SSIM = 0.79

Restored right x_2 with $\vartheta = 1.6 \times 10^{-3}$
SNR = 19.7 dB – SSIM = 0.83

FIGURE 4.1 – Stereoscopic image restoration.

where the matrix D is in $\mathbb{R}^{N\times N}$ [40] and where v stands for modeling errors. The observations consist of degraded versions

$$z_1 = L_1\overline{x}_1 + w_1 \quad \text{and} \quad z_2 = L_2\overline{x}_2 + w_2 \tag{4.82}$$

of $\overline{x}_1$ and $\overline{x}_2$, respectively, where the matrices $L_1 \in \mathbb{R}^{N\times N}$ and $L_2 \in \mathbb{R}^{N\times N}$ model the data acquisition process, and where w_1 and w_2 are mutually independent Gaussian noise vectors with independent components which are $\mathcal{N}(0,\sigma_1^2)$– and $\mathcal{N}(0,\sigma_2^2)$–distributed, respectively. In addition, it is assumed that the decompositions of $\overline{x}_1$ and $\overline{x}_2$ in orthonormal bases $(e_{1,k})_{1\leq k\leq N}$ and $(e_{2,k})_{1\leq k\leq N}$, respectively, of $\mathbb{R}^N$ are sparse. For every $k \in \{1,\dots,N\}$, functions $\phi_{1,k} \in \Gamma_0(\mathbb{R})$ and $\phi_{2,k} \in \Gamma_0(\mathbb{R})$ are used to promote the sparsity of the decompositions [20, 24]. The following variational formulation is consistent with the above hypotheses and models.

Problem 4.45 Let $\vartheta \in [0,+\infty[$. The objective is to

$$\underset{x_1\in\mathbb{R}^N,\, x_2\in\mathbb{R}^N}{\text{minimize}} \; \sum_{k=1}^{N} \phi_{1,k}(\langle x_1 \mid e_{1,k}\rangle) + \sum_{k=1}^{N} \phi_{2,k}(\langle x_2 \mid e_{2,k}\rangle) \\ + \frac{1}{2\sigma_1^2}\|L_1x_1 - z_1\|^2 + \frac{1}{2\sigma_2^2}\|L_2x_2 - z_2\|^2 + \frac{\vartheta}{2}\|x_1 - Dx_2\|^2. \tag{4.83}$$

We can formulate Problem 4.45 as an instance of Problem 4.26 with $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}^N$ and $m = 2$ functions, namely

$$f_1\colon (x_1,x_2) \mapsto \sum_{k=1}^{N} \phi_{1,k}(\langle x_1 \mid e_{1,k}\rangle) + \sum_{k=1}^{N} \phi_{2,k}(\langle x_2 \mid e_{2,k}\rangle) \tag{4.84}$$

and

$$f_2\colon (x_1,x_2) \mapsto \frac{1}{2\sigma_1^2}\|L_1x_1 - z_1\|^2 + \frac{1}{2\sigma_2^2}\|L_2x_2 - z_2\|^2 + \frac{\vartheta}{2}\|x_1 - Dx_2\|^2. \tag{4.85}$$

Proposition 4.46 *Let x_1 and x_2 be arbitrary vectors in $\mathbb{R}^N$. Then f_2 is differentiable at (x_1,x_2) and*

$$\nabla f_2(x_1,x_2) = \Big(\frac{1}{\sigma_1^2}L_1^\top(L_1x_1 - z_1) + \vartheta(x_1 - Dx_2), \frac{1}{\sigma_2^2}L_2^\top(L_2x_2 - z_2) + \vartheta D^\top(Dx_2 - x_1)\Big). \tag{4.86}$$

Moreover, ∇f_2 is β-Lipschitz continuous, with

$$\beta = \max\{\sigma_1^{-2}\|L_1\|^2, \sigma_2^{-2}\|L_2\|^2\} + \vartheta(1 + \|D\|^2). \tag{4.87}$$

Proof. The expression (4.86) follows from straightforward calculus. Now set

$$\boldsymbol{L} = \begin{bmatrix} \sigma_1^{-1}L_1 & [0] \\ [0] & \sigma_2^{-1}L_2 \end{bmatrix} \text{ and } \boldsymbol{M} = \sqrt{\vartheta}\begin{bmatrix} \mathrm{I} & -D \\ [0] & [0] \end{bmatrix}. \tag{4.88}$$

Then, using matrix notation, we can write

$$\nabla f_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (\boldsymbol{L}^\top \boldsymbol{L} + \boldsymbol{M}^\top \boldsymbol{M}) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \boldsymbol{L}^\top \begin{bmatrix} \sigma_1^{-1} z_1 \\ \sigma_2^{-1} z_2 \end{bmatrix}. \tag{4.89}$$

Hence, a Lipschitz constant of ∇f_2 is $\|\boldsymbol{L}^\top \boldsymbol{L} + \boldsymbol{M}^\top \boldsymbol{M}\|$, where $\|\cdot\|$ denotes the spectral norm. To obtain a tractable bound, we observe that

$$\begin{aligned} \|\boldsymbol{L}^\top \boldsymbol{L} + \boldsymbol{M}^\top \boldsymbol{M}\| &\le \|\boldsymbol{L}^\top \boldsymbol{L}\| + \|\boldsymbol{M}^\top \boldsymbol{M}\| \\ &= \|\boldsymbol{L}\|^2 + \|\boldsymbol{M}\|^2 \\ &= \|\boldsymbol{L}\|^2 + \vartheta(1 + \|D\|^2). \end{aligned} \tag{4.90}$$

Now set $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$. Then

$$\begin{aligned} \|\boldsymbol{L}\boldsymbol{x}\|^2 &= \sigma_1^{-2}\|L_1 x_1\|^2 + \sigma_2^{-2}\|L_2 x_2\|^2 \\ &\le \sigma_1^{-2}\|L_1\|^2\|x_1\|^2 + \sigma_2^{-2}\|L_2\|^2\|x_2\|^2 \\ &\le \max\{\sigma_1^{-2}\|L_1\|^2, \sigma_2^{-2}\|L_2\|^2\}\|\boldsymbol{x}\|^2. \end{aligned} \tag{4.91}$$

Hence, $\|\boldsymbol{L}\|^2 \le \max\{\sigma_1^{-2}\|L_1\|^2, \sigma_2^{-2}\|L_2\|^2\}$ and (4.90) yields $\|\boldsymbol{L}^\top \boldsymbol{L} + \boldsymbol{M}^\top \boldsymbol{M}\| \le \beta$. □

In view of Proposition 4.46, Problem 4.45 can be solved by the forward-backward algorithm (see Theorem 4.34).

Numerical experiments. Experimental results are displayed in Figure 4.1 for stereoscopic images of size 256×256 ($N = 256^2$). In this example, L_1 and L_2 are periodic convolution operators with motion kernel blur of sizes 7×7 and 3×3, respectively. This kind of blur was considered in a related context in [38]. A white Gaussian noise is added corresponding to a blurred signal-to-noise-ratio (BSNR) of 21.6 dB for z_1 and 21.8 dB for z_2 (the BSNR is defined as $10\log_{10}\big(\|L_i\overline{x}_i\|^2/(N\sigma_i^2)\big)$). In addition, $(e_{1,k})_{1\le k\le N}$ and $(e_{2,k})_{1\le k\le N}$ are symmlet wavelet orthonormal bases (length 6) over 2 resolution levels. For every $k \in \{1,\ldots,N\}$, $\phi_{1,k} = \mu_{1,k}|\cdot|^{p_{1,k}}$ and $\phi_{2,k} = \mu_{2,k}|\cdot|^{p_{2,k}}$, where $\{\mu_{1,k}, \mu_{2,k}\} \subset]0,+\infty[$ and $\{p_{1,k}, p_{2,k}\} \subset [1,+\infty[$.

The operators $(\operatorname{prox}_{\phi_{1,k}})_{1\le k\le N}$ and $(\operatorname{prox}_{\phi_{2,k}})_{1\le k\le N}$ can be calculated explicitly [15, Examples 4.2 and 4.4]. The proximity operator of f_1 can thus be deduced from Proposition 4.29, the separability of this function, and [23, Lemma 2.8 and 2.9]. For every $k \in \{1,\ldots,N\}$, the values of $\mu_{1,k}$, $\mu_{2,k}$, $p_{1,k}$, and $p_{2,k}$ are chosen using a maximum likelihood approach in a subband-adaptive manner with $p_{1,k}$ and $p_{2,k}$ in $\{1, 4/3, 3/2, 2\}$. The value of ϑ is selected so as to maximize the signal-to-noise-ratio (SNR). The SNR between an image y and the original image $\overline{y}$ is defined as $20\log_{10}(\|\overline{y}\|/\|y-\overline{y}\|)$. In our experiments we also propose to compare the restored images in terms of structural similarity (SSIM) [46]. The SSIM takes on values from -1 to 1. The value 1 is achieved for two identical images. The disparity map has been estimated by using the method described in [34]. Note that the existence of a solution to Problem 4.45 is secured by the fact that $f_1 + f_2$ is a coercive function in $\Gamma_0(\mathbb{R}^N \oplus \mathbb{R}^N)$ [23, Propositions 3.1(i) and

5.15(i)]. Thus, Problem 4.45 is a special case of Problem 4.33. In this context, setting $\lambda_n \equiv 1$, the forward-backward algorithm assumes the following form.

Initialization

$$
\left\lfloor
\begin{array}{l}
\sigma_1 = \sigma_2 = 12 \\
\vartheta = 0 \quad \text{or} \quad \vartheta = 1.6 \times 10^{-3} \\
\gamma = 1.9/\big(\max\{\sigma_1^{-2}\|L_1\|^2, \sigma_2^{-2}\|L_2\|^2\} + \vartheta(1 + \|D\|^2)\big) \\
x_{1,0} = z_1 \\
x_{2,0} = z_2
\end{array}
\right.
$$

For $n = 0, 1, \ldots$

$$
\left\lfloor
\begin{array}{l}
y_{1,n} = \sigma_1^{-2} L_1^\top (L_1 x_{1,n} - z_1) + \vartheta(x_{1,n} - D x_{2,n}) \\
y_{2,n} = \sigma_2^{-2} L_2^\top (L_2 x_{2,n} - z_2) - \vartheta D^\top (x_{1,n} - D x_{2,n}) \\
x_{1,n+1} = \displaystyle\sum_{k=1}^{N} \big(\operatorname{prox}_{\gamma \phi_{1,k}} \langle x_{1,n} - \gamma y_{1,n} \mid e_{1,k} \rangle \big) e_{1,k} \\
x_{2,n+1} = \displaystyle\sum_{k=1}^{N} \big(\operatorname{prox}_{\gamma \phi_{2,k}} \langle x_{2,n} - \gamma y_{2,n} \mid e_{2,k} \rangle \big) e_{2,k}
\end{array}
\right.
$$

When $\vartheta = 0$, there is no coupling between the left and right views (images in the third row of Figure 4.1). As can be observed in Figure 4.1, the coupling term leads to a significant improvement of the restoration, especially for the most degraded image (bottom-left image).

Using $\vartheta = 1.6 \times 10^{-3}$, we compare the forward-backward algorithm of Theorem 4.34 (implemented with $\lambda_n \equiv 1$ and $\gamma_n \equiv 1.99/\beta$) to a multicomponent version of the Beck-Teboulle algorithm [10] and a multicomponent version of the Nesterov algorithm [37]. Although, contrary to the forward-backward algorithm, the Beck-Teboulle and Nesterov algorithms do not insure convergence of the iterates, they are known to provide a theoretically optimal convergence rate for the objective function. However, in this example, their performance appear to be quite comparable on that score (see Figure 4.2).

4.3.2.5.2 Multispectral image denoising

Problem formulation. A common multispectral image processing problem is to denoise m images $(\overline{y}_i)_{1 \le i \le m}$ in $\mathbb{R}^N$ from noisy observations $(z_i)_{1 \le i \le m}$ given by

$$(\forall i \in \{1, \ldots, m\}) \quad z_i = \overline{y}_i + w_i, \tag{4.92}$$

where $(w_i)_{1 \le i \le m}$ are realizations of mutually independent zero-mean white Gaussian noise processes with respective variances $(\sigma_i^2)_{1 \le i \le m}$. Early methods for multispectral image recovery are described in [30]. A tutorial on wavelet-based multispectral denoising can be found in [14].

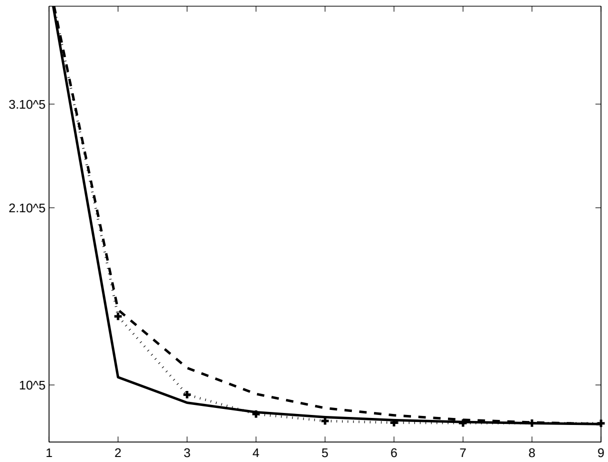


FIGURE 4.2 – Convergence of the objective function in Problem 4.45 for the forward-backward algorithm (solid line), the Nesterov algorithm (dotted line), and the Beck-Teboulle algorithm (dashed line) versus iteration number.

To solve this denoising problem, we assume that, for every $i \in \{1, \ldots, m\}$, $\overline{y}_i$ satisfies some constraint represented by a nonempty closed convex set $C_i \subset \mathbb{R}^N$, and that it admits a sparse decomposition in an orthonormal basis $(e_{i,k})_{1 \leq k \leq N}$ of $\mathbb{R}^N$. In addition, similarities between the images are promoted by penalizing a distance between their components in some orthonormal basis $(b_k)_{1 \leq k \leq N}$ of $\mathbb{R}^N$. These considerations lead to the variational problem

$$\underset{y_1 \in C_1, \ldots, y_m \in C_m}{\text{minimize}} \sum_{i=1}^{m} \frac{1}{2\sigma_i^2} \|y_i - z_i\|^2 + \sum_{i=1}^{m} \sum_{k=1}^{N} \widetilde{\mu}_{i,k} |\langle y_i \mid e_{i,k} \rangle| + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \widetilde{\vartheta}_{i,j} \sum_{k=1}^{N} |\langle y_i - y_j \mid b_k \rangle| \tag{4.93}$$

where, for every $i \in \{1, \ldots, m\}$, $\{\widetilde{\mu}_{i,k}\}_{1 \leq k \leq N} \subset \left]0, +\infty\right[$ and $\{\widetilde{\vartheta}_{i,j}\}_{i+1 \leq j \leq m} \subset \left]0, +\infty\right[$. After appropriate rescaling of the variables, this problem can be reformulated as follows.

Problem 4.47 For every $i \in \{1, \ldots, m\}$, let $\{\mu_{i,k}\}_{1 \leq k \leq N} \subset \left]0, +\infty\right[$ and $\{\vartheta_{i,j}\}_{i+1 \leq j \leq m} \subset \left]0, +\infty\right[$. The objective is to

$$\underset{x_1 \in \mathbb{R}^N, \ldots, x_m \in \mathbb{R}^N}{\text{minimize}} \frac{p-1}{2} \sum_{i=1}^{m} \|x_i - \sigma_i^{-1} z_i\|^2 + \sum_{i=1}^{m} \sum_{k=1}^{N} \mu_{i,k} \sigma_i |\langle x_i \mid e_{i,k} \rangle| + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \vartheta_{i,j} \sum_{k=1}^{N} |\langle \sigma_i x_i - \sigma_j x_j \mid b_k \rangle| + \sum_{i=1}^{m} \iota_{C_i}(\sigma_i x_i). \tag{4.94}$$

To cast this problem in the format of Problem 4.26, let us define

$$J = \big\{(i,j) \in \mathbb{N}^2 \mid 1 \le i \le m-1,\, i+1 \le j \le m\big\} \tag{4.95}$$

and

$$\mathrm{i}\colon J \to \big\{1,\ldots,m(m-1)/2\big\}\colon (i,j) \mapsto m(i-1) - i(i+1)/2 + j. \tag{4.96}$$

Moreover, let us set $p = m(m-1)/2 + 3$ and

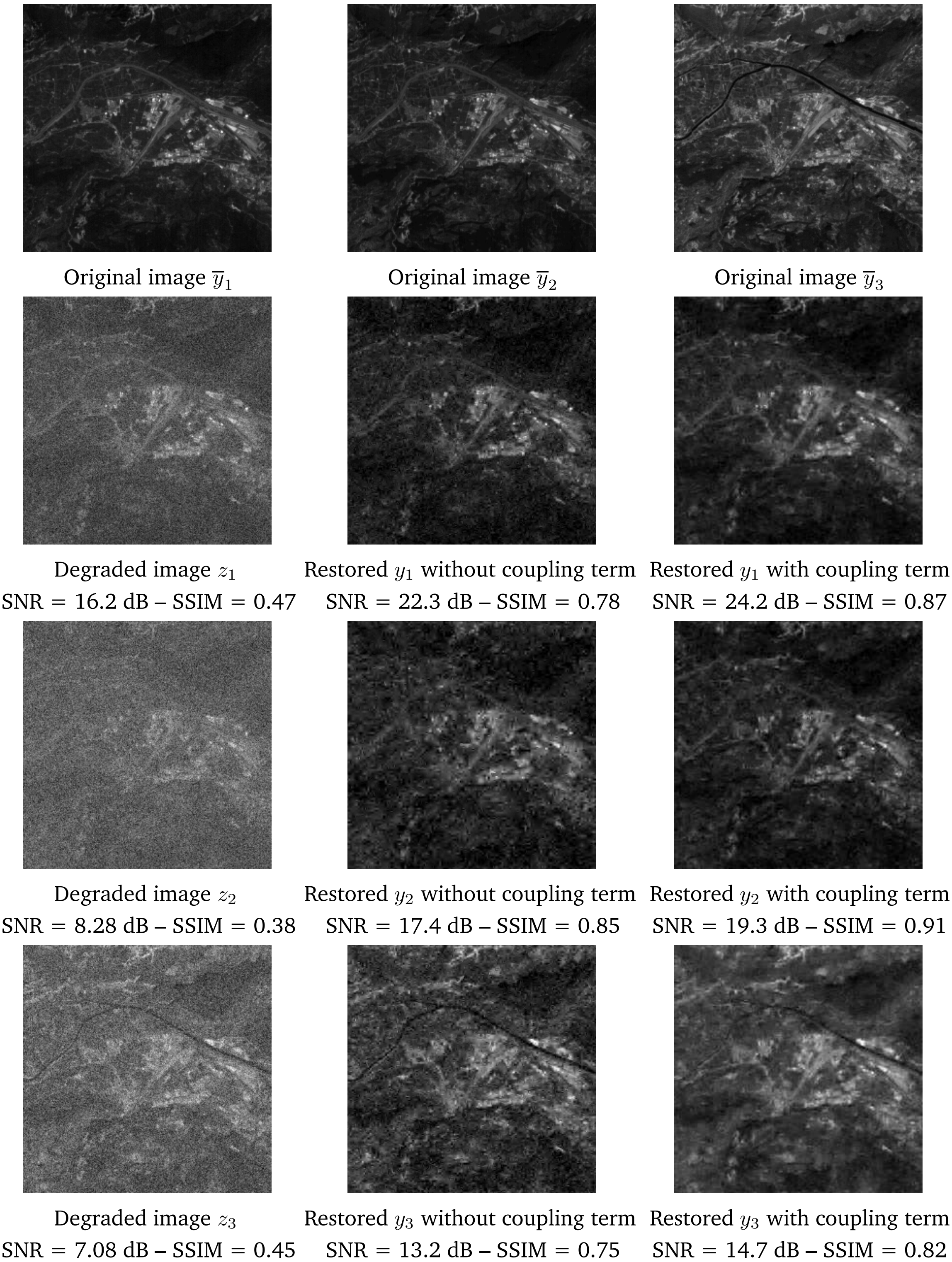
$$\begin{cases} (\forall (i,j) \in J) \quad f_{\mathrm{i}(i,j)}\colon (x_1,\ldots,x_m) \mapsto \vartheta_{i,j} \displaystyle\sum_{k=1}^{N} |\left\langle \sigma_i x_i - \sigma_j x_j \mid b_k \right\rangle| \\ f_{p-2}\colon (x_1,\ldots,x_m) \mapsto \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{N} \mu_{i,k}\sigma_i |\left\langle x_i \mid e_{i,k}\right\rangle| \\ f_{p-1}\colon (x_1,\ldots,x_m) \mapsto \displaystyle\sum_{i=1}^{m} \iota_{C_i}(\sigma_i x_i) \\ f_p\colon (x_1,\ldots,x_m) \mapsto \dfrac{p-1}{2}\displaystyle\sum_{i=1}^{m} \|x_i - \sigma_i^{-1} z_i\|^2. \end{cases} \tag{4.97}$$

Note that, for every $k \in \{1,\ldots,p-2\}$, $\operatorname{dom} f_k = (\mathbb{R}^N)^m$ and $\operatorname{dom} f_{p-1} = \sigma_1^{-1}C_1 \times \cdots \times \sigma_m^{-1}C_m$. Hence, since the sets $(C_i)_{1\le i\le m}$ are nonempty, (4.78) holds and Problem 4.47 can be solved by the Dykstra-like algorithm presented in Theorem 4.43, with $\mathcal{H}_1 = \cdots = \mathcal{H}_m = \mathbb{R}^N$. An explicit form of the proximity operators of the functions $(f_k)_{1\le k\le m(m-1)/2}$ can be deduced from Proposition 4.32. Indeed, for every $(i,j) \in J$, we can set in this proposition $\mathcal{H}_1 = \cdots = \mathcal{H}_m = \mathcal{G} = \mathbb{R}^N$, $(\forall k \in \{1,\ldots,N\})$ and $\phi_k = \vartheta_{i,j}|\cdot|$, and define the matrices $(L_\ell)_{1\le \ell\le m}$ in $\mathbb{R}^{N\times N}$ as

$$(\forall \ell \in \{1,\ldots,m\}) \quad L_\ell = \begin{cases} \sigma_\ell \,\mathrm{I}\,, & \text{if } \ell = i; \\ -\sigma_\ell \,\mathrm{I}\,, & \text{if } \ell = j; \\ 0 & \text{otherwise.} \end{cases} \tag{4.98}$$

Finally, the proximity operator of f_{p-2} can be derived from Proposition 4.29 combined with the separability of this function, [23, Lemma 2.8 and 2.9], and [15, Example 4.2]. The proximity operator of f_{p-1} is provided in Example 4.27.

Numerical experiments. Figure 4.3 shows the results obtained on a multispectral image of size 256×256 ($N = 256^2$) with 3 channels ($m = 3$) and pixel values in the range $[0, 255]$. These images are corrupted by white Gaussian noises with standard deviations $\sigma_1 = 11$, $\sigma_2 = 12$, and $\sigma_3 = 13$ (the corresponding SNR values are indicated in Figure 4.3). On the other hand, $(b_k)_{1\le k\le N}$ is the Haar orthonormal wavelet basis on 3 resolution levels and, for every $i \in \{1,\ldots,m\}$, $(e_{i,k})_{1\le k\le N}$ are symmlet orthonormal wavelet bases (length 6) on 3 resolution levels. The values of the regularization parameters $((\mu_{i,k})_{1\le i\le 3})_{1\le k\le N}$ (chosen subband-adaptive by a maximum likelihood approach), and



Original image $\overline{y}_1$ | Original image $\overline{y}_2$ | Original image $\overline{y}_3$

Degraded image z_1 SNR = 16.2 dB – SSIM = 0.47 | Restored y_1 without coupling term SNR = 22.3 dB – SSIM = 0.78 | Restored y_1 with coupling term SNR = 24.2 dB – SSIM = 0.87

Degraded image z_2 SNR = 8.28 dB – SSIM = 0.38 | Restored y_2 without coupling term SNR = 17.4 dB – SSIM = 0.85 | Restored y_2 with coupling term SNR = 19.3 dB – SSIM = 0.91

Degraded image z_3 SNR = 7.08 dB – SSIM = 0.45 | Restored y_3 without coupling term SNR = 13.2 dB – SSIM = 0.75 | Restored y_3 with coupling term SNR = 14.7 dB – SSIM = 0.82

FIGURE 4.3 – Multispectral restoration.

of the coupling parameters $\vartheta_{1,2}$, $\vartheta_{1,3}$, and $\vartheta_{2,3}$ are selected so as to maximize the SNR. For every $i \in \{1, \ldots, m\}$, $C_i = [0, 255]^N$ models the constraint on the range of pixel values. The resulting Dykstra-like algorithm is described below.

Initialization

$$
\left\lfloor
\begin{array}{l}
\sigma_1 = 11\ ;\ \sigma_2 = 12\ ;\ \sigma_3 = 13 \\
y_{1,1,0} = \ldots = y_{1,5,0} = x_{1,0} = z_1 \\
y_{2,1,0} = \ldots = y_{2,5,0} = x_{2,0} = z_2 \\
y_{3,1,0} = \ldots = y_{3,5,0} = x_{3,0} = z_3 \\
\alpha_{1,2} = \sigma_1^2 + \sigma_2^2 \\
\alpha_{1,3} = \sigma_1^2 + \sigma_3^2 \\
\alpha_{2,3} = \sigma_2^2 + \sigma_3^2
\end{array}
\right.
$$

For $n = 0, 1, \ldots$

$$
\left|
\begin{array}{l}
u_{1,1,n} = y_{1,1,n} + \alpha_{1,2}^{-1}\sigma_1 \sum_{k=1}^N \Big(\operatorname{prox}_{\alpha_{1,2}\vartheta_{1,2}|\cdot|} \langle \sigma_1 y_{1,1,n} - \sigma_2 y_{2,1,n} \mid b_k \rangle \\
\qquad\qquad + \langle \sigma_1 y_{1,1,n} - \sigma_2 y_{2,1,n} \mid b_k \rangle \Big) b_k \\
u_{2,1,n} = y_{2,1,n} - \alpha_{1,2}^{-1}\sigma_2 \sum_{k=1}^N \Big(\operatorname{prox}_{\alpha_{1,2}\vartheta_{1,2}|\cdot|} \langle \sigma_1 y_{1,1,n} - \sigma_2 y_{2,1,n} \mid b_k \rangle \\
\qquad\qquad + \langle \sigma_1 y_{1,1,n} - \sigma_2 y_{2,1,n} \mid b_k \rangle \Big) b_k \\
u_{3,1,n} = y_{3,1,n} \\
u_{1,2,n} = y_{1,2,n} + \alpha_{1,3}^{-1}\sigma_1 \sum_{k=1}^N \Big(\operatorname{prox}_{\alpha_{1,3}\vartheta_{1,3}|\cdot|} \langle \sigma_1 y_{1,2,n} - \sigma_3 y_{3,2,n} \mid b_k \rangle \\
\qquad\qquad + \langle \sigma_1 y_{1,2,n} - \sigma_3 y_{3,2,n} \mid b_k \rangle \Big) b_k \\
u_{2,2,n} = y_{2,2,n} \\
u_{3,2,n} = y_{3,2,n} - \alpha_{1,3}^{-1}\sigma_3 \sum_{k=1}^N \Big(\operatorname{prox}_{\alpha_{1,3}\vartheta_{1,3}|\cdot|} \langle \sigma_1 y_{1,2,n} - \sigma_3 y_{3,2,n} \mid b_k \rangle \\
\qquad\qquad + \langle \sigma_1 y_{1,2,n} - \sigma_3 y_{3,2,n} \mid b_k \rangle \Big) b_k \\
u_{1,3,n} = y_{1,3,n} \\
u_{2,3,n} = y_{2,3,n} + \alpha_{2,3}^{-1}\sigma_2 \sum_{k=1}^N \Big(\operatorname{prox}_{\alpha_{2,3}\vartheta_{2,3}|\cdot|} \langle \sigma_2 y_{2,3,n} - \sigma_3 y_{3,3,n} \mid b_k \rangle \\
\qquad\qquad + \langle \sigma_2 y_{2,3,n} - \sigma_3 y_{3,3,n} \mid b_k \rangle \Big) b_k \\
u_{3,3,n} = y_{3,3,n} - \alpha_{2,3}^{-1}\sigma_3 \sum_{k=1}^N \Big(\operatorname{prox}_{\alpha_{2,3}\vartheta_{2,3}|\cdot|} \langle \sigma_2 y_{2,3,n} - \sigma_3 y_{3,3,n} \mid b_k \rangle \\
\qquad\qquad + \langle \sigma_2 y_{2,3,n} - \sigma_3 y_{3,3,n} \mid b_k \rangle \Big) b_k \\
\vdots
\end{array}
\right.
$$

$$
\begin{array}{l}
\vdots \\
u_{1,4,n} = \sum_{k=1}^{N} \big(\operatorname{prox}_{\mu_{1,k}\sigma_1|\cdot|} \langle y_{1,4,n} \mid e_{1,k} \rangle \big) e_{1,k} \\
u_{2,4,n} = \sum_{k=1}^{N} \big(\operatorname{prox}_{\mu_{2,k}\sigma_2|\cdot|} \langle y_{2,4,n} \mid e_{2,k} \rangle \big) e_{2,k} \\
u_{3,4,n} = \sum_{k=1}^{N} \big(\operatorname{prox}_{\mu_{3,k}\sigma_3|\cdot|} \langle y_{3,4,n} \mid e_{3,k} \rangle \big) e_{3,k} \\
\\
u_{1,5,n} = P_{C_1}(\sigma_1\, y_{1,5,n}) \\
u_{2,5,n} = P_{C_2}(\sigma_2\, y_{2,5,n}) \\
u_{3,5,n} = P_{C_3}(\sigma_3\, y_{3,5,n}) \\
\\
x_{1,n+1} = (u_{1,1,n} + u_{1,2,n} + u_{1,3,n} + u_{1,4,n} + u_{1,5,n})/5 \\
x_{2,n+1} = (u_{2,1,n} + u_{2,2,n} + u_{2,3,n} + u_{2,4,n} + u_{2,5,n})/5 \\
x_{3,n+1} = (u_{3,1,n} + u_{3,2,n} + u_{3,3,n} + u_{3,4,n} + u_{3,5,n})/5 \\
\\
\text{For } j = 1, \ldots, 5 \\
\left\lfloor
\begin{array}{l}
y_{1,j,n+1} = x_{1,n+1} + y_{1,j,n} - u_{1,j,n} \\
y_{2,j,n+1} = x_{2,n+1} + y_{2,j,n} - u_{2,j,n} \\
y_{3,j,n+1} = x_{3,n+1} + y_{3,j,n} - u_{3,j,n}
\end{array}
\right.
\end{array}
$$

It can be observed from the images displayed on the second and third columns of Figure 4.3 that the introduction of the coupling term has a significant influence on denoising performance. Moreover, in our experiments, we observed that better results were obtained when different bases $(b_k)_{1\le k\le N}$, $(e_{1,k})_{1\le k\le N}$, ..., $(e_{m,k})_{1\le k\le N}$ were employed.

It turns out that, in this particular problem, an alternative solution method is PPXA (see Theorem 4.40) applied to the minimization of the sum of the $m(m-1)/2+2$ functions f_1, f_2, ..., f_{p-2}, and $f_{p-1} + f_p$ defined in (4.97). The proximity operator of the latter is given by [23, Lemma 2.6(i)]. Indeed, the qualification condition (see (4.77)) is satisfied since $\operatorname{dom} f_1 = \cdots = \operatorname{dom} f_{p-2} = (\mathbb{R}^N)^m$ and, $(\forall i \in \{1,\ldots,m\})$ $\operatorname{int} C_i =]0,255[\neq \varnothing$. The choice of the PPXA parameters has been optimized empirically for speed of convergence and set to $\lambda_n \equiv 1.3$, $\gamma = 1$, and $\omega_1 = \cdots = \omega_{p-1} = 1/(p-1)$. In Figure 4.4, $\|\boldsymbol{x}_n - \boldsymbol{x}_\infty\|/\|\boldsymbol{x}_0 - \boldsymbol{x}_\infty\|$ is plotted as a function of computation time, where $(\boldsymbol{x}_n)_{n\in\mathbb{N}} = \big((x_{1,n}, x_{2,n}, x_{3,n})\big)_{n\in\mathbb{N}}$ is the sequence generated by an algorithm and $\boldsymbol{x}_\infty$ is the unique solution to Problem 4.47. In our experiments, 500 iterations were used to produce this solution.

4.3.2.5.3 Structure-texture image decomposition An important problem in image processing is to decompose an image into elementary structures. In the context of denoising, this decomposition was investigated in [39] with a total variation potential. In [33],

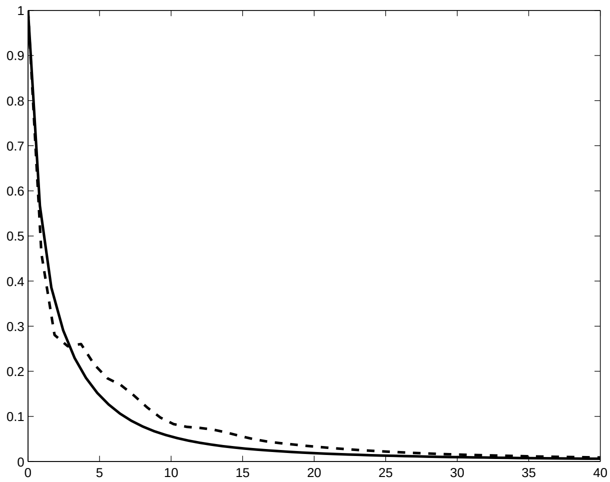


FIGURE 4.4 – Problem 4.47 : Convergence profiles of the Dykstra-like algorithm (solid line) and of PPXA (dashed line) versus computation time in seconds.

a different potential was used to better penalize strongly oscillating components. The resulting variational problem is not straightforward. Numerical methods were proposed in [5, 43] and experiments were performed for image denoising and analysis problems based on a geometry-texture decomposition. Another challenging problem is to extract meaningful components from a blurred and noise-corrupted image. In the presence of additive Gaussian noise, a decomposition into geometry and texture components is proposed in [2, 25]. The method developed in the present paper, will make it possible to consider general (not necessarily additive and Gaussian) noise models and arbitrary linear degradation operators. We consider a simple geometry-texture decomposition from a degraded observation.

Problem formulation. In this experiment, the observed image $z \in \mathbb{R}^N$ is obtained by multiplying the original image $\overline{x} \in \mathbb{R}^N$ with a matrix $T \in \mathbb{R}^{N \times N}$, which models a blur, and corrupting $T\overline{x}$ by a Poisson noise with scaling parameter α. It is assumed that

$$T \text{ has its entries in } [0, +\infty[\text{ and each of its rows is nonzero.} \tag{4.99}$$

The inverse problem we address is to obtain the decomposition of $\overline{x}$ into the sum of a geometry and a texture component, say

$$\overline{x} = R_1(\overline{x}_1) + R_2(\overline{x}_2), \tag{4.100}$$

where $R_1 \colon \mathbb{R}^{N_1} \mapsto \mathbb{R}^N$ and $R_2 \colon \mathbb{R}^{N_2} \mapsto \mathbb{R}^N$ are known operators. The vectors $\overline{x}_1 \in \mathbb{R}^{N_1}$ and $\overline{x}_2 \in \mathbb{R}^{N_2}$ to be estimated parameterize, respectively, the geometry and the texture components.

We consider a simple instance of (4.100) involving a linear mixture : $N_1 = N$, $R_1 \colon x_1 \mapsto x_1$, and $R_2 \colon x_2 \mapsto F^\top x_2$, where $F^\top \in \mathbb{R}^{N \times K}$ is a linear tight frame synthesis operator. In other words, the information regarding the texture component pertains to the coefficients $\overline{x}_2$ of its decomposition in the frame. The tightness condition implies that

$$F^\top F = \nu \,\mathrm{Id}, \text{ for some } \nu \in \left]0, +\infty\right[. \tag{4.101}$$

Thus, the original image is decomposed as $\overline{x} = \overline{x}_1 + F^\top \overline{x}_2$. It is known a priori that $\overline{x} \in C_1 \cap C_2$, where

$$C_1 = [0, 255]^N \tag{4.102}$$

models the constraint on the range of pixel values, and

$$C_2 = \left\{ x \in \mathbb{R}^N \;\middle|\; \widehat{x} = (\eta_k)_{1\le k\le N},\ \sum_{k\in\mathbb{I}} |\eta_k|^2 \le \delta \right\}, \tag{4.103}$$

for some $\delta \in \left]0, +\infty\right[$, models an energy constraint on the 2-D DFT $\widehat{x}$ of the original image in some low frequency band $\mathbb{I} \subset \{1, \ldots, N\}$. In addition, to limit the total variation [12] of the geometrical component, the potential $x \mapsto \mathrm{tv}(Hx, Vx)$ is used, with

$$\mathrm{tv}\colon \big((\eta_k)_{1\le k\le N}, (\zeta_k)_{1\le k\le N}\big) \mapsto \chi \sum_{k=1}^{N} \sqrt{|\eta_k|^2 + |\zeta_k|^2}, \tag{4.104}$$

where $H \in \mathbb{R}^{N\times N}$ and $V \in \mathbb{R}^{N\times N}$ are matrix representations of the horizontal and vertical discrete differentiation operators, respectively, and where $\chi \in \left]0, +\infty\right[$. Furthermore, to promote sparsity in the frame of the texture component of the image, the potential

$$h\colon (\eta_k)_{1\le k\le K} \mapsto \sum_{k=1}^{K} \tau_k |\eta_k| \tag{4.105}$$

is introduced, where $\{\tau_k\}_{1\le k\le K} \subset \left]0, +\infty\right[$. Finally, as a data fidelity term well adapted to Poisson noise, we employ the generalized Kullback-Leibler divergence with a scaling parameter $\alpha \in \left]0, +\infty\right[$. Upon setting $z = (\zeta_k)_{1\le k\le N}$, this leads to the function

$$g\colon (\xi_k)_{1\le k\le N} \mapsto \sum_{k=1}^{N} \phi_k(\xi_k), \tag{4.106}$$

where, for every $k \in \{1, \ldots, K\}$,

$$\begin{aligned} \phi_k\colon \mathbb{R} &\to \left]-\infty, +\infty\right] \\ \xi &\mapsto \begin{cases} -\zeta_k \ln(\xi) + \alpha\xi, & \text{if } \zeta_k \ge 0 \text{ and } \xi > 0; \\ 0, & \text{if } \xi = 0; \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned} \tag{4.107}$$

Altogether, the variational problem is to

$$\underset{\substack{x_1\in\mathbb{R}^N,\, x_2\in\mathbb{R}^K\\ x_1+F^\top x_2\in C_1 x_1+F^\top x_2\in C_2}}{\text{minimize}} \quad \mathrm{tv}(Hx_1, Vx_1) + h(x_2) + g(Tx_1 + TF^\top x_2). \tag{4.108}$$

This problem is a particular case of (4.31) with $m = 2$, $p = 4$, and

$$\begin{cases} f_1\colon (x_1, x_2) \mapsto \mathrm{tv}(Hx_1, Vx_1) + h(x_2),\\ f_2\colon (x_1, x_2) \mapsto g(Tx_1 + TF^\top x_2),\\ f_3\colon (x_1, x_2) \mapsto \iota_{C_1}(x_1 + F^\top x_2),\\ f_4\colon (x_1, x_2) \mapsto \iota_{C_2}(x_1 + F^\top x_2). \end{cases} \tag{4.109}$$

However, since the operators $(\mathrm{prox}_{f_i})_{1\leq i\leq 4}$ are not easily implementable, we cannot apply directly Theorems 4.34, 4.37, or 4.40. To circumvent this difficulty, a strategy is to decompose (4.108) into an equivalent problem by introducing auxiliary variables.

A first equivalent problem to (4.108) is

$$\underset{\substack{x_1,x_2,x_3,x_4,x_5,x_6\\ x_3=x_1+F^\top x_2\\ x_3\in C_1\cap C_2\\ x_4=Tx_3\\ x_5=Hx_1,\, x_6=Vx_1}}{\text{minimize}} \mathrm{tv}(x_5, x_6) + h(x_2) + g(x_4), \tag{4.110}$$

where we have introduced the auxiliary variables $(x_3, x_4, x_5, x_6) \in \mathbb{R}^N \oplus \mathbb{R}^N \oplus \mathbb{R}^N \oplus \mathbb{R}^N$. Problem (4.110) is a particular case of (4.31) with $m = 6$, $p = 3$, and

$$\begin{cases} f_1\colon (x_1, \ldots, x_6) \mapsto h(x_2) + \iota_{C_1}(x_3) + g(x_4) + \mathrm{tv}(x_5, x_6),\\ f_2\colon (x_1, \ldots, x_6) \mapsto \iota_{C_2}(x_3),\\ f_3\colon (x_1, \ldots, x_6) \mapsto \iota_{\{0\}}(x_1 + F^\top x_2 - x_3) + \iota_{\{0\}}(Tx_3 - x_4) + \\ \qquad\qquad\qquad \iota_{\{0\}}(Hx_1 - x_5) + \iota_{\{0\}}(Vx_1 - x_6). \end{cases} \tag{4.111}$$

In this formulation, the rôle of f_3 is to impose the constraints $x_1 + F^\top x_2 = x_3$, $Tx_3 = x_4$, $Hx_1 = x_5$, and $Vx_1 = x_6$. As seen in Example 4.27, $\mathrm{prox}_{\iota_{C_1}} = P_{C_1}$ and $\mathrm{prox}_{\iota_{C_2}} = P_{C_2}$. On the other hand, the proximity operators of tv, h, and g can be obtained from [21, Proposition 2.8(i)], [23, Example 2.16], and [19, Example 30], respectively. In turn, since f_1 is separable, its proximity operator follows straightforwardly componentwise. Now set

$$\boldsymbol{L}_1 = \begin{bmatrix} \mathrm{I} & F^\top & -\mathrm{I} & [0] & [0] & [0]\\ [0] & [0] & T & -\mathrm{I} & [0] & [0]\\ H & [0] & [0] & [0] & -\mathrm{I} & [0]\\ V & [0] & [0] & [0] & [0] & -\mathrm{I} \end{bmatrix}. \tag{4.112}$$

It follows from (4.111) and (4.112) that $f_3 = \iota_{\ker \boldsymbol{L}_1}$, where $\ker \boldsymbol{L}_1 = \big\{\boldsymbol{x} \in \boldsymbol{\mathcal{H}} \mid \boldsymbol{L}_1\boldsymbol{x} = \boldsymbol{0}\big\}$. Hence, by Example 4.27 and [26, Chapter 8],

$$\mathrm{prox}_{f_3} = P_{\ker \boldsymbol{L}_1} = \mathrm{I} - \boldsymbol{L}_1^\top\big(\boldsymbol{L}_1\boldsymbol{L}_1^\top\big)^{-1}\boldsymbol{L}_1. \tag{4.113}$$

Under the assumption that the matrices T, H, and V are block-circulant with circulant blocks, they are diagonalized by the 2-D DFT. Hence, combining (4.113), (4.112), and (4.101) we deduce that prox_{f_3} is computable explicitly. On the other hand, it follows from (4.111), (4.105), (4.102), (4.106), (4.104), (4.103), and (4.112) that

$$\begin{cases} \text{ri dom } f_1 & = \mathbb{R}^N \times \mathbb{R}^K \times \text{int } C_1 \times]0,+\infty[^N \times \mathbb{R}^N \times \mathbb{R}^N \\ \text{ri dom } f_2 & = \mathbb{R}^N \times \mathbb{R}^K \times \text{int } C_2 \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \\ \text{ri dom } f_3 & = \ker \boldsymbol{L}_1. \end{cases} \tag{4.114}$$

Hence, qualification condition (4.77) reduces to

$$(\exists\,(x_1,\ldots,x_6)\in \ker \boldsymbol{L}_1) \quad \begin{cases} x_3 \in \text{int } C_1 \cap \text{int } C_2 \\ x_4 \in]0,+\infty[^N, \end{cases} \tag{4.115}$$

which is equivalent to

$$(\exists(x_1,\ldots,x_6)\in \mathbb{R}^N\times\mathbb{R}^K\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N) \qquad \begin{cases} x_1 + F^\top x_2 = x_3 \in \text{int } C_1 \cap \text{int } C_2, \\ Tx_3 = x_4 \in]0,+\infty[^N \\ Hx_1 = x_5 \\ Vx_1 = x_6. \end{cases} \tag{4.116}$$

This condition is satisfied if

$$T(\text{int}(C_1\cap C_2)) \cap]0,+\infty[^N \neq \varnothing. \tag{4.117}$$

Indeed, let $y \in T(\text{int}(C_1\cap C_2)) \cap]0,+\infty[^N$. Then there exists $x \in \text{int}(C_1\cap C_2)$ such that $Tx = y \in]0,+\infty[^N$. Hence, for every $x_2 \in \mathbb{R}^K$, if we set $x_3 = x$, $x_4 = y = Tx_3$, $x_1 = x_3 - F^\top x_2$, $x_5 = Hx_1$, and $Vx_1 = x_6$, (4.116) is seen to hold. Since (4.102) and (4.103) yield $\text{int}(C_1\cap C_2)\neq\varnothing$, we deduce from (4.99) that (4.117) (and hence (4.77)) is satisfied. Thus, (4.110) can be solved by PPXA (see Theorem 4.40 and Remark 4.41).

Another equivalent formulation of (4.108) is

$$\underset{\substack{x_1,x_2,x_3,x_4,x_5,x_6,x_7\\ x_3=x_1+F^\top x_2\\ x_4=Tx_3\\ x_5=Hx_1,\ x_6=Vx_1\\ x_7=x_3\\ x_3\in C_1,\ x_7\in C_2}}{\text{minimize}} \text{tv}(x_5,x_6)+h(x_2)+g(x_4), \tag{4.118}$$

where the additional auxiliary variable $x_7 \in \mathbb{R}^N$ has been introduced. Problem (4.118) is a particular case of (4.31) with $m = 7$, $p = 2$, and

$$\begin{cases} f_1\colon (x_1,\ldots,x_7)\mapsto h(x_2)+\iota_{C_1}(x_3)+g(x_4)+\text{tv}(x_5,x_6)+\iota_{C_2}(x_7) \\ f_2\colon (x_1,\ldots,x_7)\mapsto \iota_{\{0\}}(x_1+F^\top x_2-x_3)+\iota_{\{0\}}(Tx_3-x_4)+\iota_{\{0\}}(Hx_1-x_5) \\ \qquad\qquad +\iota_{\{0\}}(Vx_1-x_6)+\iota_{\{0\}}(x_3-x_7). \end{cases} \tag{4.119}$$

As previously observed, since the proximity operators of tv, h, g, ι_{C_1}, and ι_{C_2} are easily computable, so is prox_{f_1}. Furthermore, if we set

$$\boldsymbol{L}_2 = \begin{bmatrix} \mathrm{I} & F^\top & -\mathrm{I} & [0] & [0] & [0] & [0] \\ [0] & [0] & T & -\mathrm{I} & [0] & [0] & [0] \\ H & [0] & [0] & [0] & -\mathrm{I} & [0] & [0] \\ V & [0] & [0] & [0] & [0] & -\mathrm{I} & [0] \\ [0] & [0] & \mathrm{I} & [0] & [0] & [0] & -\mathrm{I} \end{bmatrix}, \tag{4.120}$$

it can be deduced from (4.119) that the proximity operator of $f_2 = \iota_{\ker \boldsymbol{L}_2}$ can be computed like that of $\iota_{\ker \boldsymbol{L}_1}$. We derive from (4.119), (4.105), (4.102), (4.106), (4.104), (4.103), and (4.120) that

$$\begin{cases} \mathrm{ri\,dom}\, f_1 & = \mathbb{R}^N \times \mathbb{R}^K \times \mathrm{int}\, C_1 \times \,]0,+\infty[^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathrm{int}\, C_2 \\ \mathrm{ri\,dom}\, f_2 & = \ker \boldsymbol{L}_2. \end{cases} \tag{4.121}$$

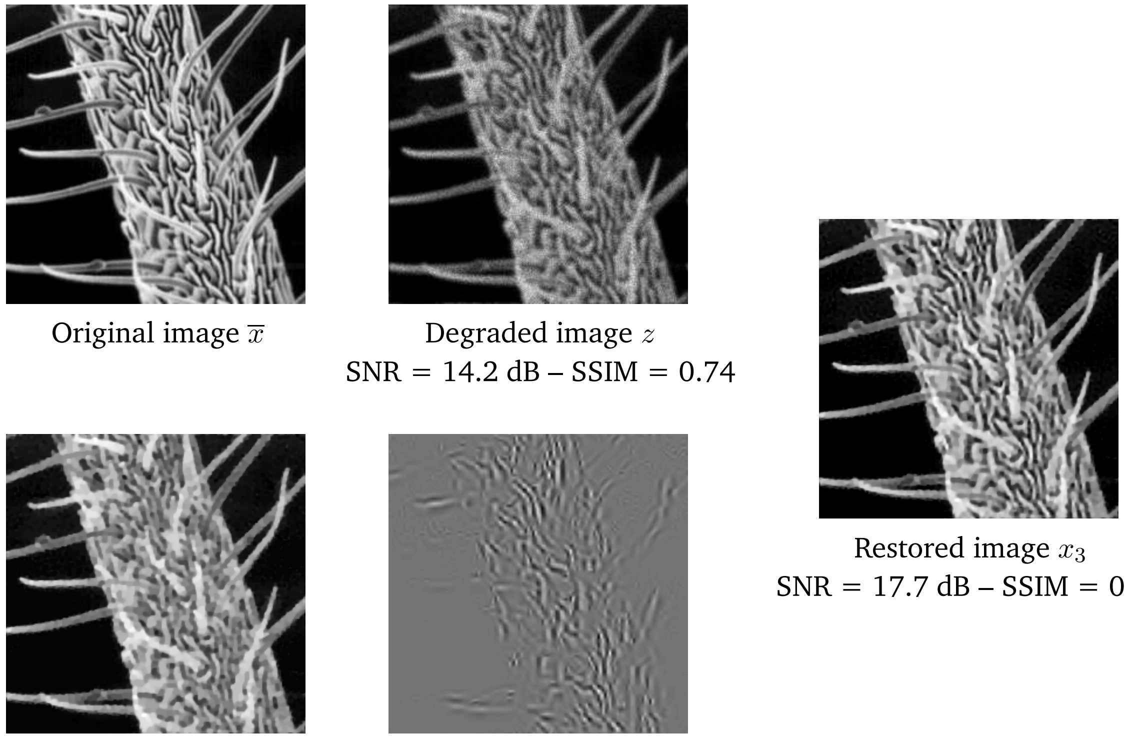
Hence, arguing as above, (4.74) reduces to (4.117), which is seen to be satisfied. This shows that (4.118) can be solved by the Douglas-Rachford algorithm (see Theorem 4.37 and Remark 4.38(ii)).

Numerical experiments. Figure 4.5 shows the results of the decomposition into geometry and texture components of an electron microscopy image of size 512×512 ($N = 512^2$) degraded by a Gaussian blur of size 5×5 and Poisson noise with scaling parameter $\alpha = 0.5$. The parameter χ of (4.104) and the parameters $(\tau_k)_{1 \le k \le K}$ of (4.105) are selected so as to maximize the SNR. The matrix F is a tight frame version of the dual-tree transform proposed in [16] using symmlet of length 6 applied over 3 resolution levels ($\nu = 2$ and $K = 2N$). The same discrete gradient matrices H and V as in [5] are used. We aim at comparing the PPXA and Douglas-Rachford algorithms in the image decomposition problem under consideration. In both algorithms we set $\lambda_n \equiv 1$.

In this context, setting $\omega_1 = \omega_2 = \omega_3 = 1/3$, PPXA assumes the following form.

Initialization

$$\left\lfloor \begin{array}{l} \gamma = 100 \\ (y_{1,1,0}, \ldots, y_{6,1,0}) = (z, F^\top z, z, z, z, z) \\ (y_{1,2,0}, \ldots, y_{6,2,0}) = (z, F^\top z, z, z, z, z) \\ (y_{1,3,0}, \ldots, y_{6,3,0}) = (z, F^\top z, z, z, z, z) \\ \text{For } i = 1, \ldots, 6 \\ \lfloor \; x_{i,0} = (y_{i,1,0} + y_{i,2,0} + y_{i,3,0})/3 \end{array} \right.$$



Original image $\overline{x}$

Degraded image z
SNR = 14.2 dB – SSIM = 0.74

Restored image x_3
SNR = 17.7 dB – SSIM = 0.86

Geometry component x_1

Texture component $F^\top x_2$

FIGURE 4.5 – Decomposition and restoration results.

For $n = 0, 1, \ldots$

$$
\left\lfloor
\begin{array}{l}
u_{1,1,n} = y_{1,1,n} \\
u_{2,1,n} = \operatorname{prox}_{3\gamma h}(y_{2,1,n}) \\
u_{3,1,n} = P_{C_1}(y_{3,1,n}) \\
u_{4,1,n} = \operatorname{prox}_{3\gamma g}(y_{4,1,n}) \\
(u_{5,1,n}, u_{6,1,n}) = \operatorname{prox}_{3\gamma\psi}(y_{5,1,n}, y_{6,1,n}) \\
(u_{1,2,n}, u_{2,2,n}) = (y_{1,2,n}, y_{2,2,n}) \\
u_{3,2,n} = P_{C_2}(y_{3,2,n}) \\
(u_{4,2,n}, u_{5,2,n}, u_{6,2,n}) = (y_{4,2,n}, y_{5,2,n}, y_{6,2,n}) \\
(u_{1,3,n}, \ldots, u_{6,3,n}) = P_{\ker \boldsymbol{L}_1}(y_{1,3,n}, \ldots, y_{6,3,n}) \\
\text{For } i = 1, \ldots, 6 \\
\left\lfloor
\begin{array}{l}
s_{i,n} = (1/3) \displaystyle\sum_{k=1}^{3} u_{i,k,n} \\
y_{i,1,n+1} = y_{i,1,n} + 2s_{i,n} - x_{i,n} - u_{i,1,n} \\
y_{i,2,n+1} = y_{i,2,n} + 2s_{i,n} - x_{i,n} - u_{i,2,n} \\
y_{i,3,n+1} = y_{i,3,n} + 2s_{i,n} - x_{i,n} - u_{i,3,n} \\
x_{i,n+1} = x_{i,n} + s_{i,n} - x_{i,n}
\end{array}
\right.
\end{array}
\right.
$$

On the other hand, the Douglas-Rachford algorithm reduces to the following.

Initialization

$$\left\lfloor \begin{array}{l} \gamma = 100 \\ (x_{1,0}, \ldots, x_{7,0}) = (z, F^\top z, z, z, z, z, z) \end{array} \right.$$

For $n = 0, 1, \ldots$

$$\left\lfloor \begin{array}{l} y_{1,n} = x_{1,n} \\ y_{2,n} = \operatorname{prox}_{\gamma h}(x_{2,n}) \\ y_{3,n} = P_{C_1}(x_{3,n}) \\ y_{4,n} = \operatorname{prox}_{\gamma g}(x_{4,n}) \\ (y_{5,n}, y_{6,n}) = \operatorname{prox}_{\gamma \psi}(x_{5,n}, x_{6,n}) \\ y_{7,n} = P_{C_2}(x_{7,n}) \\ (u_{1,n}, \ldots, u_{7,n}) = P_{\ker \boldsymbol{L}_2}\big(2(y_{1,n}, \ldots, y_{7,n}) - (x_{1,n}, \ldots, x_{7,n})\big) \\ \text{For } i = 1, \ldots, 7 \\ \lfloor\, x_{i,n+1} = x_{i,n} + u_{i,n} - y_{i,n} \end{array} \right.$$

In Figure 4.6, the value of $\|\boldsymbol{y}_n - \boldsymbol{y}_\infty\|/\|\boldsymbol{y}_0 - \boldsymbol{y}_\infty\|$ for the sequence $(\boldsymbol{y}_n)_{n\in\mathbb{N}} = \big((y_{1,n}, \ldots, y_{7,n})\big)_{n\in\mathbb{N}}$ of Theorem 4.37 and $\|\boldsymbol{x}_n - \boldsymbol{x}_\infty\|/\|\boldsymbol{x}_0 - \boldsymbol{x}_\infty\|$ for the sequence $(\boldsymbol{x}_n)_{n\in\mathbb{N}} = \big((x_{1,n}, \ldots, x_{6,n})\big)_{n\in\mathbb{N}}$ of Theorem 4.40 (where $\boldsymbol{y}_\infty$ and $\boldsymbol{x}_\infty$ denote the respective limits) are plotted as a function of the computation time in seconds. In our experiments, 1000 iterations were used to produce a solution.

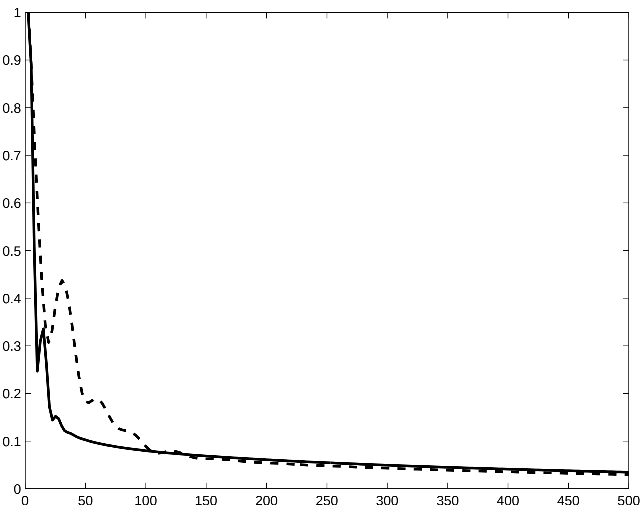


FIGURE 4.6 – Convergence profiles of the Douglas-Rachford algorithm (solid line) and PPXA (dashed line) versus computation time in seconds.

4.3.2.6 Conclusion

In this paper, the proximal formalism has been applied to multicomponent signal/image processing. Expressions of new proximity operators in product spaces have been derived. The proposed multicomponent framework has been illustrated through three different applications : stereocospy, multispectral imagery, and decomposition into geometry and texture components. Another field of application in which these techniques could be useful is the processing of color images. The proposed proximal formalism can also be used to derive algorithms for complex signal and image processing by regarding a complex signal as a signal with $m = 2$ real components, namely its real and imaginary parts.

4.4 Bibliographie

[1] M. V. Afonso, J. M. Bioucas-Dias, and M. A. T. Figueiredo, An augmented Lagrangian approach to the constrained optimization formulation of imaging inverse problems, *IEEE Trans. Image Process.*, 2010, doi : 10.1109/TIP.2010.2076294.

[2] S. Anthoine, E. Pierpaoli, and I. Daubechies, Deux méthodes de déconvolution et séparation simultanées ; application à la reconstruction des amas de galaxies, *Traitement Signal*, vol. 23, pp. 439–447, 2006.

[3] A. Antoniadis and J. Fan, Regularization of wavelet approximations, *J. Amer. Statist. Assoc.*, vol. 96, pp. 939–967, 2001.

[4] H. Attouch, L. M. Briceño-Arias, and P. L. Combettes, A parallel splitting method for coupled monotone inclusions, *SIAM J. Control Optim.*, vol. 48, pp. 3246–3270, 2010.

[5] J.-F. Aujol, G. Aubert, L. Blanc-Féraud, and A. Chambolle, Image decomposition into a bounded variation component and an oscillating component, *J. Math. Imag. Vis.*, vol. 22, pp. 71–88, 2005.

[6] J.-F. Aujol and A. Chambolle, Dual norms and image decomposition models, *Int. J. Comput. Vis.*, vol. 63, pp. 85–104, 2005.

[7] J.-F. Aujol, G. Gilboa, T. Chan, and S. J. Osher, Structure-texture image decomposition – Modeling, algorithms, and parameter selection, *Int. J. Comput. Vis.*, vol. 67, pp. 111–136, 2006.

[8] J.-F. Aujol and S. H. Kang, Color image decomposition and restoration, *J. Vis. Comm. Image Represent*, vol. 17, pp. 916–928, 2006.

[9] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, 2011.

[10] A. Beck and M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, *SIAM J. Imaging Sci.*, vol. 2, pp. 183–202, 2009.

[11] L. M. Briceño-Arias and P. L. Combettes, Convex variational formulation with smooth coupling for multicomponent signal decomposition and recovery, *Numer. Math. Theory Methods Appl.*, vol. 2, pp. 485–508, 2009.

[12] A. Chambolle, An algorithm for total variation minimization and applications, *J. Math. Imaging Vision*, vol. 20, pp. 89–97, 2004.

[13] T. F. Chan, S. Esedoglu, and F. E. Park, Image decomposition combining staircase reduction and texture extraction, *J. Vis. Comm. Image Represent.*, vol. 18, pp. 464–486, 2007.

[14] C. Chaux, A. Benazza-Benyahia, J.-C. Pesquet, and L. Duval, Wavelet transform for the denoising of multivariate images, in : C. Collet, J. Chanussot, K. Chehdi (Eds.), *Multivariate Image Processing*, pp. 203–237, Wiley, New York, 2010.

[15] C. Chaux, P. L. Combettes, J.-C. Pesquet, and V. R. Wajs, A variational formulation for frame-based inverse problems, *Inverse Problems*, vol. 23, pp. 1495–1518, 2007.

[16] C. Chaux, L. Duval, and J.-C. Pesquet, Image analysis using a dual-tree M-band wavelet transform, *IEEE Trans. Image Process.*, vol. 15, pp. 2397–2412, 2006.

[17] P. L. Combettes, Solving monotone inclusions via compositions of nonexpansive averaged operators, *Optimization*, vol. 53, pp. 475–504, 2004.

[18] P. L. Combettes, Iterative construction of the resolvent of a sum of maximal monotone operators, *J. Convex Anal.*, vol. 16, pp. 727–748, 2009.

[19] P. L. Combettes and J.-C. Pesquet, A Douglas-Rachford splitting approach to nonsmooth convex variational signal recovery, *IEEE J. Selected Topics Signal Process.*, vol. 1, pp. 564–574, 2007.

[20] P. L. Combettes and J.-C. Pesquet, Proximal thresholding algorithm for minimization over orthonormal bases, *SIAM J. Optim.*, vol. 18, pp. 1351–1376, 2007.

[21] P. L. Combettes and J.-C. Pesquet, A proximal decomposition method for solving convex variational inverse problems, *Inverse Problems*, vol. 24, ID 065014, 2008.

[22] P. L. Combettes and J.-C. Pesquet, Proximal splitting methods in signal processing, in : H. H. Bauschke, R. Burachik, P. L. Combettes, V. Elser, D. R. Luke, H. Wolkowicz (Eds.), *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, pp. 185–212, Springer-Verlag, New York, 2010.

[23] P. L. Combettes and V. R. Wajs, Signal recovery by proximal forward-backward splitting, *Multiscale Model. Simul.*, vol. 4, pp. 1168–1200, 2005.

[24] I. Daubechies, M. Defrise, and C. De Mol, An iterative thresholding algorithm for linear inverse problems with a sparsity constraint, *Comm. Pure Appl. Math.*, vol. 57, pp. 1413–1457, 2004.

[25] I. Daubechies and G. Teschke, Variational image restoration by means of wavelets : simultaneous decomposition, deblurring and denoising, *Appl. Comp. Harm. Anal.*, vol. 19, pp. 1–16, 2005.

[26] F. Deutsch, *Best Approximation in Inner Product Spaces*, Springer-Verlag, New York, 2001.

[27] M. Elad, J.-L. Starck, D. Donoho, and P. Querre, Simultaneous cartoon and texture image inpainting using morphological component analysis (MCA), *Appl. Comp. Harm. Analysis*, vol. 19, pp. 340–358, 2005.

[28] M. Goldburg and R. J. Marks II, Signal synthesis in the presence of an inconsistent set of constraints, *IEEE Trans. Circ. Syst.*, vol. 32, pp. 647–663, 1985.

[29] Y. Huang, M. K. Ng, and Y.-W. Wen, A fast total variation minimization method for image restoration, *Multiscale Model. Simul.*, vol. 7, pp. 774–795, 2008.

[30] B. R. Hunt and O. Kübler, Karhunen-Loève multispectral image restoration, part I : theory, *IEEE Trans. Acoust. Speech Signal Process.*, vol. 32, pp. 592–600, 1984.

[31] M. Kang, Generalized multichannel image deconvolution approach and its applications, *Opt. Eng.*, vol. 37, pp. 2953–2964, 1998.

[32] A. Katsaggelos, K. Lay, and N. Galatsanos, A general framework for frequency domain multi-channel signal processing, *IEEE Trans. Image Process.*, vol. 2, pp. 417–420, 1993.

[33] Y. Meyer, *Oscillating Patterns in Image Processing and Nonlinear Evolution Equations*, AMS, Providence, RI, 2001.

[34] W. Miled, J.-C. Pesquet, and M. Parent, A convex optimization approach for depth estimation under illumination variation, *IEEE Trans. Image Process.*, vol. 18, pp. 813–830, 2009.

[35] J.-J. Moreau, Proximité et dualité dans un espace hilbertien, *Bull. Soc. Math. France*, vol. 93, pp. 273–299, 1965.

[36] Yu. Nesterov, A method of solving a convex programming problem with convergence rate $O(1/k^2)$, *Soviet Math. Dokl.*, vol. 27, pp. 372–376, 1983.

[37] Yu. Nesterov, Primal-dual subgradient methods for convex problems, *Math. Program.*, vol. 120, pp. 221–259, 2009.

[38] M. Pedone and J. Heikkilä, Blur and contrast invariant fast stereo matching, *Lecture Notes in Comput. Sci.*, vol. 5259, pp. 883–890, 2008.

[39] L. I. Rudin, S. J. Osher, and E. Fatemi, Nonlinear total variation based noise removal algorithms, *Phys. D.*, vol. 60, pp. 259–268, 1992.

[40] D. Scharstein and R. Szeliski, A taxonomy and evaluation of dense two-frame stereo correspondence algorithms, *Int. J. Comput. Vis.*, vol. 47, pp. 7–42, 2002.

[41] B. F. Svaiter, On weak convergence of the Douglas-Rachford method, *SIAM J. Control Optim.*, vol. 49, pp. 280–287, 2011.

[42] D. Tschumperlé and R. Deriche, Diffusion PDEs on vector-valued images, *IEEE Signal Process. Mag.*, vol. 19, pp. 16–25, 2002.

[43] L. A. Vese and S. J. Osher, Modeling textures with total variation minimization and oscillating patterns in image processing, *J. Sci. Comput.*, vol. 19, pp. 553–572, 2003.

[44] L. A. Vese and S. J. Osher, Image denoising and decomposition with total variation minimization and oscillatory functions, *J. Math. Imag. Vis.*, vol. 20, pp. 7–18, 2004.

[45] Y. Wang, J. Yang, W. Yin, and Y. Zhang, A new alternating minimization algorithm for total variation image reconstruction, *SIAM J. Imaging Sci.*, vol. 1, pp. 248–272, 2008.

[46] Z. Wang, A. C. Bovik, H. R. Sheikh, and E. P. Simoncelli, Image quality assessment : from error visibility to structural similarity, *IEEE Trans. Image Process.*, vol. 13, pp. 600–612, 2004.

[47] P. Weiss, G. Aubert, and L. Blanc-Féraud, Efficient schemes for total variation minimization under constraints in image processing, *SIAM J. Sci. Comput.*, vol. 31, pp. 2047–2080, 2009.

[48] Y.-W. Wen, M. K. Ng, and W.-K. Ching, Iterative algorithms based on decoupling of deblurring and denoising for image restoration, *SIAM J. Sci. Comput.*, vol. 30, pp. 2655–2674, 2008.

[49] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, River Edge, NJ, 2002.

Chapitre 5

Résolution d'inclusions monotones composites en dualité

5.1 Description et résultats principaux

Dans ce chapitre on présente un nouveau formalisme de décomposition monotone +anti-adjoint pour résoudre des inclusions monotones composites en dualité. On s'intéresse au problème suivant.

Problème 5.1 Soient $\mathcal{H}$ et $\mathcal{G}$ deux espaces hilbertiens réels, soient $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ et $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ deux opérateurs maximalement monotones, soit $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, soit $z \in \mathcal{H}$ et soit $r \in \mathcal{G}$. Le problème est de résoudre l'inclusion primale

$$\text{trouver } x \in \mathcal{H} \quad \text{tel que} \quad z \in Ax + L^*(B(Lx - r)) \tag{5.1}$$

avec l'inclusion duale

$$\text{trouver } v \in \mathcal{G} \quad \text{tel que} \quad -r \in -L(A^{-1}(z - L^*v)) + B^{-1}v. \tag{5.2}$$

Nous appelons $\mathcal{P}$ l'ensemble de solutions de (5.1) et $\mathcal{D}$ l'ensemble de solutions de (5.2).

Dans la littérature, plusieurs méthodes résolvent (5.1), mais dans des conditions très restrictives. En particulier, si B est univoque et cocoercif, alors $L^*(B(L\cdot - r))$ est cocoercif et la méthode explicite-implicite résout (5.1). Par ailleurs, si B est lipschitzien, alors $L^*(B(L\cdot - r))$ est lipschitzien et (5.1) peut être résolu par la méthode explicite-implicite-explicite (voir Section 1.2). Cependant, le cas général de (5.1) où B est multivoque et monotone ne peut être résolu par aucune des méthodes présentées dans la Section 1.2, sauf sous des hypothèses fortes imposées à L [6, Proposition 23.23]. Notre stratégie est de résoudre simultanément (5.1) et (5.2) via l'inclusion auxiliaire suivante.

Problème 5.2 Sous les hypothèses du Problème 5.1, posons $\mathcal{K} = \mathcal{H} \oplus \mathcal{G}$ et

$$\boldsymbol{M}\colon \boldsymbol{\mathcal{K}} \to 2^{\boldsymbol{\mathcal{K}}}\colon (x,v) \mapsto (-z+Ax)\times(r+B^{-1}v) \quad \text{et} \quad \boldsymbol{S}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}\colon (x,v) \mapsto (L^*v, -Lx). \tag{5.3}$$

Le problème est de

$$\text{trouver } \boldsymbol{x} \in \boldsymbol{\mathcal{K}} \quad \text{tel que} \quad 0 \in \boldsymbol{M}\boldsymbol{x} + \boldsymbol{S}\boldsymbol{x}. \tag{5.4}$$

Notons que le Problème 5.2 correspond au cas particulier du Problème 1.1 où $m = 2$, $\boldsymbol{A}_1 = \boldsymbol{M}$ et $\boldsymbol{A}_2 = \boldsymbol{S}$.

Nous montrons dans l'article de la Section 5.2.2 que $\operatorname{zer}(\boldsymbol{M} + \boldsymbol{S})$ est un sous-ensemble fermé et convexe de $\mathcal{P} \times \mathcal{D}$, que $\boldsymbol{M}$ est maximalement monotone et que $\boldsymbol{S} \in \mathcal{B}(\boldsymbol{\mathcal{K}})$ est anti-adjoint (voir la Proposition 5.19 (i) et la Proposition 5.18). Alors, vu que toute solution du Problème 5.2 résout le Problème 5.1, la méthode primale-duale introduite dans cet article pour résoudre le Problème 5.1 découle d'une méthode pour résoudre le Problème 5.2. Ce nouveau formalisme de décomposition monotone+anti-adjoint fournit un outil puissant de modélisation.

Le Problème 5.2 est résolu par l'algorithme décrit dans le Théorème 5.3 ci-dessous. Il est inspiré par la méthode explicite-implicite-explicite présentée dans la Section 1.2.2 mais qui tolère des erreurs de calcul à chaque évaluation des opérateurs concernés. Nous démontrons la convergence faible de cette méthode vers un zéro de la somme de deux opérateurs maximalement monotones, où l'un est univoque et lipschitzien.

Théorème 5.3 *Soit $\boldsymbol{\mathcal{H}}$ un espace hilbertien réel, soit $\boldsymbol{A}\colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}$ un opérateur maximalement monotone et soit $\boldsymbol{B}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}$ un opérateur monotone. Supposons que $\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B}) \neq \varnothing$ et que $\boldsymbol{B}$ soit β-lipschitzien avec $\beta \in \left]0,+\infty\right[$. Soient $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$ et $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ des suites dans $\boldsymbol{\mathcal{H}}$ telles que $\sum_{n\in\mathbb{N}} \|\boldsymbol{a}_n\| < +\infty$, $\sum_{n\in\mathbb{N}} \|\boldsymbol{b}_n\| < +\infty$ et $\sum_{n\in\mathbb{N}} \|\boldsymbol{c}_n\| < +\infty$, soit $\boldsymbol{x}_0 \in \boldsymbol{\mathcal{H}}$, soit $\varepsilon \in \left]0, 1/(\beta+1)\right[$ et soit $(\gamma_n)_{n\in\mathbb{N}}$ une suite dans $[\varepsilon, (1-\varepsilon)/\beta]$. Des suites $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{p}_n)_{n\in\mathbb{N}}$ et $(\boldsymbol{q}_n)_{n\in\mathbb{N}}$ sont générées comme suit.*

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \boldsymbol{y}_n = \boldsymbol{x}_n - \gamma_n(\boldsymbol{B}\boldsymbol{x}_n + \boldsymbol{a}_n) \\ \boldsymbol{p}_n = J_{\gamma_n \boldsymbol{A}}\, \boldsymbol{y}_n + \boldsymbol{b}_n \\ \boldsymbol{q}_n = \boldsymbol{p}_n - \gamma_n(\boldsymbol{B}\boldsymbol{p}_n + \boldsymbol{c}_n) \\ \boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n. \end{array} \right. \tag{5.5}$$

Alors pour quelque $\overline{\boldsymbol{x}} \in \operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$ nous avons ce qui suit.

(i) $\sum_{n\in\mathbb{N}} \|\boldsymbol{x}_n - \boldsymbol{p}_n\|^2 < +\infty$ *et* $\sum_{n\in\mathbb{N}} \|\boldsymbol{y}_n - \boldsymbol{q}_n\|^2 < +\infty$.

(ii) $\boldsymbol{x}_n \rightharpoonup \overline{\boldsymbol{x}}$ *et* $\boldsymbol{p}_n \rightharpoonup \overline{\boldsymbol{x}}$.

(iii) *Supposons que l'une des conditions suivantes soit satisfaite.*

(a) $\boldsymbol{A}+\boldsymbol{B}$ *est demirégulier en* $\overline{\boldsymbol{x}}$.

(b) $\boldsymbol{A}$ *ou* $\boldsymbol{B}$ *est uniformément monotone en* $\overline{\boldsymbol{x}}$*.*

(c) $\operatorname{int}\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})\neq\varnothing$.

Alors $\boldsymbol{x}_n\to\overline{\boldsymbol{x}}$ *et* $\boldsymbol{p}_n\to\overline{\boldsymbol{x}}$*.*

En appliquant la routine (5.5) au Problème 5.2 nous obtenons du Théorème 5.3 le résultat de convergence suivant.

Théorème 5.4 *Dans le Problème 5.1, supposons que* $L\neq 0$ *et que*

$$z\in\operatorname{ran}\big(A+L^*\circ B\circ(L\cdot -r)\big). \tag{5.6}$$

Soient $(a_{1,n})_{n\in\mathbb{N}}$*,* $(b_{1,n})_{n\in\mathbb{N}}$ *et* $(c_{1,n})_{n\in\mathbb{N}}$ *des suites absolument sommables dans* $\mathcal{H}$*, et soient* $(a_{2,n})_{n\in\mathbb{N}}$*,* $(b_{2,n})_{n\in\mathbb{N}}$ *et* $(c_{2,n})_{n\in\mathbb{N}}$ *des suites absolument sommables dans* $\mathcal{G}$*. De plus, soit* $x_0\in\mathcal{H}$*, soit* $v_0\in\mathcal{G}$*, soit* $\varepsilon\in\left]0,1/(\|L\|+1)\right[$ *et soit* $(\gamma_n)_{n\in\mathbb{N}}$ *une suite dans* $[\varepsilon,(1-\varepsilon)/\|L\|\,]$*. Des suites* $(x_n)_{n\in\mathbb{N}}$*,* $(v_n)_{n\in\mathbb{N}}$*,* $(p_{1,n})_{n\in\mathbb{N}}$ *et* $(p_{2,n})_{n\in\mathbb{N}}$ *sont générées comme suit.*

$$(\forall n\in\mathbb{N})\quad \left\lfloor\begin{array}{l} y_{1,n}=x_n-\gamma_n(L^*v_n+a_{1,n})\\ y_{2,n}=v_n+\gamma_n(Lx_n+a_{2,n})\\ p_{1,n}=J_{\gamma_nA}(y_{1,n}+\gamma_nz)+b_{1,n}\\ p_{2,n}=J_{\gamma_nB^{-1}}(y_{2,n}-\gamma_nr)+b_{2,n}\\ q_{1,n}=p_{1,n}-\gamma_n(L^*p_{2,n}+c_{1,n})\\ q_{2,n}=p_{2,n}+\gamma_n(Lp_{1,n}+c_{2,n})\\ x_{n+1}=x_n-y_{1,n}+q_{1,n}\\ v_{n+1}=v_n-y_{2,n}+q_{2,n}. \end{array}\right. \tag{5.7}$$

Alors pour quelque solution $\overline{x}$ *de* (5.1) *et quelque solution* $\overline{v}$ *de* (5.2) *telles que* $z-L^*\overline{v}\in A\overline{x}$ *et* $\overline{v}\in B(L\overline{x}-r)$*, nous avons ce qui suit.*

(i) $x_n-p_{1,n}\to 0$ *et* $v_n-p_{2,n}\to 0$.

(ii) $x_n\rightharpoonup\overline{x}$*,* $p_{1,n}\rightharpoonup\overline{x}$*,* $v_n\rightharpoonup\overline{v}$ *et* $p_{2,n}\rightharpoonup\overline{v}$*.*

(iii) *Supposons que* A *soit uniformément monotone en* $\overline{x}$*. Alors* $x_n\to\overline{x}$ *et* $p_{1,n}\to\overline{x}$*.*

(iv) *Supposons que* B^{-1} *soit uniformément monotone en* $\overline{v}$*. Alors* $v_n\to\overline{v}$ *et* $p_{2,n}\to\overline{v}$*.*

Cette méthode a des avantages numériques par rapport à la méthode de Douglas-Rachford appliquée au Problème 5.2, où il faut calculer $(\operatorname{Id}+LL^*)^{-1}$ et $(\operatorname{Id}+L^*L)^{-1}$, ce qui est souvent une tâche difficile.

Dans le cas particulier où A est fortement monotone et B est cocoercif, le Théorème 5.4 garantit les convergences fortes des suites primale et duale, mais de plus, nous pouvons aussi obtenir la convergence linéaire dans le cas où les erreurs sont nulles. En effet, vu que l'opérateur $\boldsymbol{M}$ défini dans (5.3) est fortement monotone et que l'algorithme dans (5.7) est un cas particulier de la méthode dans [40] lorsque les erreurs sont nulles, la convergence linéaire est assurée dans [40, Theorem 3.4(c)]. Cependant, puisque l'opérateur B est cocoercif, $L^*(B(L\cdot - r))$ l'est aussi et alors l'algorithme explicite-implicite décrit dans la Section 1.2.1 peut résoudre l'inclusion primale (5.1) sans avoir besoin d'une dernière étape explicite comme dans (5.3). De plus, vu que A est fortement monotone, la convergence forte est aussi garantie.

Les deux résultats suivants sont des conséquences directes du Théorème 5.4 dans le cas où $L = \mathrm{Id}$, $A = A_1$, $B = A_2$, $r = 0$ et $z = 0$, et dans le cas où $A = 0$, $r = 0$ et $z = 0$, respectivement.

Corollaire 5.5 *Soient $A_1 \colon \mathcal{H} \to 2^{\mathcal{H}}$ et $A_2 \colon \mathcal{H} \to 2^{\mathcal{H}}$ des opérateurs maximalement monotones tels que $\mathrm{zer}(A_1 + A_2) \neq \varnothing$. Soient $(b_{1,n})_{n\in\mathbb{N}}$ et $(b_{2,n})_{n\in\mathbb{N}}$ des suites absolument sommables dans $\mathcal{H}$, soit x_0 et v_0 dans $\mathcal{H}$, soit $\varepsilon \in \,]0, 1/2[$ et soit $(\gamma_n)_{n\in\mathbb{N}}$ une suite dans $[\varepsilon, 1-\varepsilon]$. Des suites $(x_n)_{n\in\mathbb{N}}$ et $(v_n)_{n\in\mathbb{N}}$ sont générées comme suit.*

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} p_{1,n} = J_{\gamma_n A_1}(x_n - \gamma_n v_n) + b_{1,n} \\ p_{2,n} = J_{\gamma_n A_2^{-1}}(v_n + \gamma_n x_n) + b_{2,n} \\ x_{n+1} = p_{1,n} + \gamma_n(v_n - p_{2,n}) \\ v_{n+1} = p_{2,n} + \gamma_n(p_{1,n} - x_n). \end{array} \right. \tag{5.8}$$

Alors pour quelque $\overline{x} \in \mathrm{zer}(A_1 + A_2)$ et quelque $\overline{v} \in \mathrm{zer}(-A_1^{-1} \circ (-\mathrm{Id}) + A_2^{-1})$ tels que $-\overline{v} \in A_1\overline{x}$ et $\overline{v} \in A_2\overline{x}$, nous avons ce qui suit.

(i) $x_n \rightharpoonup \overline{x}$ *et* $v_n \rightharpoonup \overline{v}$.

(ii) *Supposons que A_1 soit uniformément monotone en $\overline{x}$. Alors $x_n \to \overline{x}$.*

(iii) *Supposons que A_2^{-1} soit uniformément monotone en $\overline{v}$. Alors $v_n \to \overline{v}$.*

Corollaire 5.6 *Dans le Problème 5.1, supposons que $L \neq 0$ et que $\mathrm{zer}(L^*BL) \neq \varnothing$. Soient $(a_{1,n})_{n\in\mathbb{N}}$ et $(c_{1,n})_{n\in\mathbb{N}}$ des suites absolument sommables dans $\mathcal{H}$, et soient $(a_{2,n})_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ et $(c_{2,n})_{n\in\mathbb{N}}$ des suites absolument sommables dans $\mathcal{G}$. Soit $x_0 \in \mathcal{H}$, soit $v_0 \in \mathcal{G}$, soit $\varepsilon \in \,]0, 1/(\|L\|+1)[$ et soit $(\gamma_n)_{n\in\mathbb{N}}$ une suite dans $[\varepsilon, (1-\varepsilon)/\|L\|\,]$. Des suites $(x_n)_{n\in\mathbb{N}}$ et $(v_n)_{n\in\mathbb{N}}$ sont générées comme suit.*

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} s_n = \gamma_n(L^*v_n + a_{1,n}) \\ y_n = v_n + \gamma_n(Lx_n + a_{2,n}) \\ p_n = J_{\gamma_n B^{-1}} y_n + b_n \\ x_{n+1} = x_n - \gamma_n(L^*p_n + c_{1,n}) \\ v_{n+1} = p_n - \gamma_n(Ls_n + c_{2,n}). \end{array} \right. \tag{5.9}$$

Alors pour quelque $\overline{x} \in \mathrm{zer}(L^*BL)$ et quelque $\overline{v} \in (\mathrm{ran}\, L)^\perp \cap B(L\overline{x})$ nous avons ce qui suit.

(i) $x_n \rightharpoonup \overline{x}$ *et* $v_n \rightharpoonup \overline{v}$.

(ii) *Supposons que B^{-1} soit uniformément monotone en $\overline{v}$. Alors $v_n \to \overline{v}$.*

Le Corollaire 5.5 fournit une alternative à la méthode de Douglas-Rachford pour résoudre $0 \in A_1x + A_2x$. Le nouveau algorithme utilise aussi des évaluations des résolvantes des deux opérateurs. Par ailleurs, le Corollaire 5.6 apporte une méthode qui surmonte certains désavantages de l'algorithme proposé dans [30] pour résoudre $0 \in L^*(B(Lx))$, notamment le calcul de la inverse généralisée à chaque itération et l'hypothèse que $\operatorname{ran} L$ soit fermé.

Dans le théorème suivant, nous proposons une méthode pour résoudre le cas où m opérateurs monotones composites interviennent.

Théorème 5.7 *Soit $z \in \mathcal{H}$ et soient $(\omega_i)_{1 \le i \le m}$ des nombres réels dans $]0,1]$ tels que $\sum_{i=1}^m \omega_i = 1$. Pour tout $i \in \{1,\ldots,m\}$, soit $(\mathcal{G}_i, \|\cdot\|_{\mathcal{G}_i})$ un espace hilbertien réel, soit $r_i \in \mathcal{G}_i$, soit $B_i\colon \mathcal{G}_i \to 2^{\mathcal{G}_i}$ un opérateur maximalement monotone et supposons que $0 \neq L_i \in \mathcal{B}(\mathcal{H}, \mathcal{G}_i)$. De plus, assumons que*

$$z \in \operatorname{ran} \sum_{i=1}^m \omega_i L_i^* \circ B_i \circ (L_i \cdot - r_i). \tag{5.10}$$

Considérons le problème

$$\text{trouver} \quad x \in \mathcal{H} \ \text{ tel que } \quad z \in \sum_{i=1}^m \omega_i L_i^* B_i(L_i x - r_i), \tag{5.11}$$

et le problème

$$\text{trouver } v_1 \in \mathcal{G}_1, \ldots, v_m \in \mathcal{G}_m \ \text{ tels que}$$
$$\sum_{i=1}^m \omega_i L_i^* v_i = z \text{ et } \ (\exists\, x \in \mathcal{H}) \begin{cases} v_1 \in B_1(L_1 x - r_1) \\ \quad \vdots \\ v_m \in B_m(L_m x - r_m). \end{cases} \tag{5.12}$$

Pour tout $i \in \{1,\ldots,m\}$, soient $(a_{1,i,n})_{n\in\mathbb{N}}$ et $(c_{1,i,n})_{n\in\mathbb{N}}$ des suites absolument sommables dans $\mathcal{H}$, soient $(a_{2,i,n})_{n\in\mathbb{N}}$, $(b_{i,n})_{n\in\mathbb{N}}$ et $(c_{2,i,n})_{n\in\mathbb{N}}$ des suites absolument sommables dans $\mathcal{G}_i$, soit $x_{i,0} \in \mathcal{H}$ et soit $v_{i,0} \in \mathcal{G}_i$. De plus, posons $\beta = \max_{1 \le i \le m} \|L_i\|$, soit $\varepsilon \in \left]0, 1/(\beta+1)\right[$ et soit $(\gamma_n)_{n\in\mathbb{N}}$ une suite dans $[\varepsilon, (1-\varepsilon)/\beta]$. Des suites $(v_{1,n})_{n\in\mathbb{N}}, \ldots, (v_{m,n})_{n\in\mathbb{N}}$ et $(x_n)_{n\in\mathbb{N}}$ sont générées comme suit.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} x_n = \sum_{i=1}^m \omega_i x_{i,n} \\ \text{Pour tout } i = 1,\ldots,m \\ \left\lfloor \begin{array}{l} y_{1,i,n} = x_{i,n} - \gamma_n (L_i^* v_{i,n} + a_{1,i,n}) \\ y_{2,i,n} = v_{i,n} + \gamma_n (L_i x_{i,n} + a_{2,i,n}) \end{array} \right. \\ p_{1,n} = \sum_{i=1}^m \omega_i y_{1,i,n} + \gamma_n z \\ \text{Pour tout } i = 1,\ldots,m \\ \left\lfloor \begin{array}{l} p_{2,i,n} = J_{\gamma_n B_i^{-1}}(y_{2,i,n} - \gamma_n r_i) + b_{i,n} \\ q_{1,i,n} = p_{1,n} - \gamma_n (L_i^* p_{2,i,n} + c_{1,i,n}) \\ q_{2,i,n} = p_{2,i,n} + \gamma_n (L_i p_{1,n} + c_{2,i,n}) \\ x_{i,n+1} = x_{i,n} - y_{1,i,n} + q_{1,i,n} \\ v_{i,n+1} = v_{i,n} - y_{2,i,n} + q_{2,i,n}. \end{array} \right. \end{array} \right. \tag{5.13}$$

Alors pour quelque solution $\overline{x}$ de (5.11) et quelque solution $(\overline{v}_i)_{1\leq i\leq m}$ de (5.80) telles que, pour tout $i \in \{1,\ldots,m\}$, $\overline{v}_i \in B_i(L_i\overline{x} - r_i)$, nous avons ce qui suit.

(i) $x_n \rightharpoonup \overline{x}$ *et, pour tout* $i \in \{1,\ldots,m\}$, $v_{i,n} \rightharpoonup \overline{v}_i$.

(ii) *Supposons que, pour tout $i \in \{1,\ldots,m\}$, B_i^{-1} soit fortement monotone en $\overline{v}_i$. Alors, pour tout $i \in \{1,\ldots,m\}$, $v_{i,n} \to \overline{v}_i$.*

Finalement, nous concluons l'article en étudiant le cas de problèmes de minimisation. Les résultats suivants sont des conséquences des résultats développés ci-dessus lorsque les opérateurs monotones sont les sous-différentiels de fonctions convexes.

Proposition 5.8 *Soit $f \in \Gamma_0(\mathcal{H})$, soit $g \in \Gamma_0(\mathcal{G})$, soit $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, soit $z \in \mathcal{H}$ et soit $r \in \mathcal{G}$. Supposons que $L \neq 0$ et que*

$$z \in \operatorname{ran}\big(\partial f + L^* \circ (\partial g) \circ (L \cdot - r)\big). \tag{5.14}$$

Considérons le problème primal

$$\underset{x\in\mathcal{H}}{\text{minimiser}} \quad f(x) + g(Lx - r) - \langle x \mid z \rangle\,, \tag{5.15}$$

et le problème dual

$$\underset{v\in\mathcal{G}}{\text{minimiser}} \quad f^*(z - L^*v) + g^*(v) + \langle v \mid r \rangle\,. \tag{5.16}$$

Soient $(a_{1,n})_{n\in\mathbb{N}}$, $(b_{1,n})_{n\in\mathbb{N}}$ et $(c_{1,n})_{n\in\mathbb{N}}$ des suites absolument sommables dans $\mathcal{H}$, et soient $(a_{2,n})_{n\in\mathbb{N}}$, $(b_{2,n})_{n\in\mathbb{N}}$ et $(c_{2,n})_{n\in\mathbb{N}}$ des suites absolument sommables dans $\mathcal{G}$. De plus, soit $x_0 \in \mathcal{H}$, soit $v_0 \in \mathcal{G}$, soit $\varepsilon \in \left]0, 1/(\|L\|+1)\right[$ et soit $(\gamma_n)_{n\in\mathbb{N}}$ une suite dans $[\varepsilon, (1-\varepsilon)/\|L\|\,]$. On génère des suites $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, $(p_n)_{n\in\mathbb{N}}$ et $(q_n)_{n\in\mathbb{N}}$ comme suit.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_{1,n} = x_n - \gamma_n(L^*v_n + a_{1,n}) \\ y_{2,n} = v_n + \gamma_n(Lx_n + a_{2,n}) \\ p_{1,n} = \operatorname{prox}_{\gamma_n f}(y_{1,n} + \gamma_n z) + b_{1,n} \\ p_{2,n} = \operatorname{prox}_{\gamma_n g^*}(y_{2,n} - \gamma_n r) + b_{2,n} \\ q_{1,n} = p_{1,n} - \gamma_n(L^*p_{2,n} + c_{1,n}) \\ q_{2,n} = p_{2,n} + \gamma_n(Lp_{1,n} + c_{2,n}) \\ x_{n+1} = x_n - y_{1,n} + q_{1,n} \\ v_{n+1} = v_n - y_{2,n} + q_{2,n}. \end{array} \right. \tag{5.17}$$

Alors pour quelque solution $\overline{x}$ de (5.15) et quelque solution $\overline{v}$ de (5.16) telles que $z - L^*\overline{v} \in \partial f(\overline{x})$ et $\overline{v} \in \partial g(L\overline{x} - r)$, nous avons ce qui suit.

(i) $x_n - p_{1,n} \to 0$ *et* $v_n - p_{2,n} \to 0$.

(ii) $x_n \rightharpoonup \overline{x}$, $p_{1,n} \rightharpoonup \overline{x}$, $v_n \rightharpoonup \overline{v}$ *et* $p_{2,n} \rightharpoonup \overline{v}$.

(iii) *Supposons que f soit uniformément convexe en $\overline{x}$. Alors $x_n \to \overline{x}$ et $p_{1,n} \to \overline{x}$.*

(iv) *Supposons que* g^* *soit uniformément convexe en* $\overline{v}$. *Alors* $v_n \to \overline{v}$ et $p_{2,n} \to \overline{v}$.

Proposition 5.9 *Soit* $z \in \mathcal{H}$ *et soient* $(\omega_i)_{1\leq i\leq m}$ *des nombres réels dans* $]0,1]$ *tels que* $\sum_{i=1}^m \omega_i = 1$. *Pour tout* $i \in \{1,\ldots,m\}$, *soit* $(\mathcal{G}_i, \|\cdot\|_{\mathcal{G}_i})$ *un espace hilbertien réel, soit* $r_i \in \mathcal{G}_i$, *soit* $g_i \in \Gamma_0(\mathcal{G}_i)$ *et supposons que* $0 \neq L_i \in \mathcal{B}(\mathcal{H},\mathcal{G}_i)$. *De plus, assumons que*

$$z \in \operatorname{ran} \sum_{i=1}^m \omega_i L_i^* \circ (\partial g_i) \circ (L_i \cdot - r_i). \tag{5.18}$$

Considérons le problème

$$\underset{x\in\mathcal{H}}{\text{minimiser}} \quad \sum_{i=1}^m \omega_i\, g_i(L_i x - r_i) - \langle x \mid z\rangle\,, \tag{5.19}$$

et le problème

$$\underset{\substack{v_1\in\mathcal{G}_1,\ldots,v_m\in\mathcal{G}_m\\ \sum_{i=1}^m \omega_i L_i^* v_i = z}}{\text{minimiser}} \quad \sum_{i=1}^m \omega_i \left(g_i^*(v_i) + \langle v_i \mid r_i\rangle\right). \tag{5.20}$$

Pour tout $i \in \{1,\ldots,m\}$, *soient* $(a_{1,i,n})_{n\in\mathbb{N}}$ *et* $(c_{1,i,n})_{n\in\mathbb{N}}$ *des suites absolument sommables dans* $\mathcal{H}$, *soient* $(a_{2,i,n})_{n\in\mathbb{N}}$, $(b_{i,n})_{n\in\mathbb{N}}$ *et* $(c_{2,i,n})_{n\in\mathbb{N}}$ *des suites absolument sommables dans* $\mathcal{G}_i$, *soit* $x_{i,0} \in \mathcal{H}$ *et soit* $v_{i,0} \in \mathcal{G}_i$. *De plus, posons* $\beta = \max_{1\leq i\leq m} \|L_i\|$, *soit* $\varepsilon \in \left]0, 1/(\beta+1)\right[$ *et soit* $(\gamma_n)_{n\in\mathbb{N}}$ *une suite dans* $[\varepsilon, (1-\varepsilon)/\beta]$. *Des suites* $(v_{1,n})_{n\in\mathbb{N}}, \ldots, (v_{m,n})_{n\in\mathbb{N}}$ *et* $(x_n)_{n\in\mathbb{N}}$ *sont générées comme suit.*

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} x_n = \sum_{i=1}^m \omega_i x_{i,n} \\ \text{Pour tout } i = 1,\ldots,m \\ \left\lfloor \begin{array}{l} y_{1,i,n} = x_{i,n} - \gamma_n(L_i^* v_{i,n} + a_{1,i,n}) \\ y_{2,i,n} = v_{i,n} + \gamma_n(L_i x_{i,n} + a_{2,i,n}) \end{array}\right. \\ p_{1,n} = \sum_{i=1}^m \omega_i y_{1,i,n} + \gamma_n z \\ \text{Pour tout } i = 1,\ldots,m \\ \left\lfloor \begin{array}{l} p_{2,i,n} = \operatorname{prox}_{\gamma_n g_i^*}(y_{2,i,n} - \gamma_n r_i) + b_{i,n} \\ q_{1,i,n} = p_{1,n} - \gamma_n(L_i^* p_{2,i,n} + c_{1,i,n}) \\ q_{2,i,n} = p_{2,i,n} + \gamma_n(L_i p_{1,n} + c_{2,i,n}) \\ x_{i,n+1} = x_{i,n} - y_{1,i,n} + q_{1,i,n} \\ v_{i,n+1} = v_{i,n} - y_{2,i,n} + q_{2,i,n}. \end{array}\right. \end{array}\right. \tag{5.21}$$

Alors pour quelque solution $\overline{x}$ *de* (5.19) *et quelque solution* $(\overline{v}_i)_{1\leq i\leq m}$ *de* (5.20) *telles que, pour tout* $i \in \{1,\ldots,m\}$, $\overline{v}_i \in \partial g_i(L_i\overline{x} - r_i)$, *nous avons ce qui suit.*

(i) $x_n \rightharpoonup \overline{x}$ *et, pour tout* $i \in \{1,\ldots,m\}$, $v_{i,n} \rightharpoonup \overline{v}_i$.

(ii) *Supposons que, pour tout* $i \in \{1,\ldots,m\}$, g_i^* *soit fortement convexe en* $\overline{v}_i$. *Alors, pour tout* $i \in \{1,\ldots,m\}$, $v_{i,n} \to \overline{v}_i$.

5.2 Article en anglais

A MONOTONE+SKEW SPLITTING MODEL FOR COMPOSITE MONOTONE INCLUSIONS IN DUALITY[1]

Abstract : The principle underlying this paper is the basic observation that the problem of simultaneously solving a large class of composite monotone inclusions and their duals can be reduced to that of finding a zero of the sum of a maximally monotone operator and a linear skew-adjoint operator. An algorithmic framework is developed for solving this generic problem in a Hilbert space setting. New primal-dual splitting algorithms are derived from this framework for inclusions involving composite monotone operators, and convergence results are established. These algorithms draw their simplicity and efficacy from the fact that they operate in a fully decomposed fashion in the sense that the monotone operators and the linear transformations involved are activated separately at each iteration. Comparisons with existing methods are made and applications to composite variational problems are demonstrated.

5.2.1 Introduction

A wide range of problems in areas such as optimization, variational inequalities, partial differential equations, mechanics, economics, signal and image processing, or traffic theory can be reduced to solving inclusions involving monotone set-valued operators in a Hilbert space $\mathcal{H}$, say

$$\text{find} \quad x \in \mathcal{H} \quad \text{such that} \quad z \in Mx, \tag{5.22}$$

where $M: \mathcal{H} \to 2^{\mathcal{H}}$ is monotone and $z \in \mathcal{H}$, e.g., [13, 14, 19, 20, 23, 26, 34, 38, 39, 42]. In many formulations of this type, the operator M can be expressed as the sum of two monotone operators, one of which is the composition of a monotone operator with a linear transformation and its adjoint. In such situations, it is often desirable to also solve an associated dual inclusion [1, 3, 4, 16, 22, 27, 28, 29, 31, 32, 33]. The present paper is concerned with the numerical solution of such composite inclusion problems in duality. More formally, the basic problem we consider is the following.

Problem 5.10 Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces, let $A: \mathcal{H} \to 2^{\mathcal{H}}$ and $B: \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone, let $L: \mathcal{H} \to \mathcal{G}$ be linear and bounded, let $z \in \mathcal{H}$, and let $r \in \mathcal{G}$. The problem is to solve the primal inclusion

$$\text{find} \quad x \in \mathcal{H} \quad \text{such that} \quad z \in Ax + L^*B(Lx - r) \tag{5.23}$$

1. L. M. Briceño-Arias and P. L. Combettes, A monotone+skew splitting model for composite monotone inclusions in duality, *SIAM Journal on Optimization*, à paraître.

together with the dual inclusion

$$\text{find} \quad v \in \mathcal{G} \quad \text{such that} \quad -r \in -LA^{-1}(z - L^*v) + B^{-1}v. \tag{5.24}$$

The set of solutions to (5.23) is denoted by $\mathcal{P}$ and the set of solutions to (5.24) by $\mathcal{D}$.

A classical instance of the duality scheme described in Problem 5.10 is the Fenchel-Rockafellar framework [33] which, under a suitable constraint qualification, corresponds to letting A and B be subdifferentials of proper lower semicontinuous convex functions $f\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ and $g\colon \mathcal{G} \to \left]-\infty, +\infty\right]$, respectively. In this scenario, the problems in duality are

$$\underset{x \in \mathcal{H}}{\text{minimize}} \;\; f(x) + g(Lx - r) - \langle x \mid z \rangle \tag{5.25}$$

and

$$\underset{v \in \mathcal{G}}{\text{minimize}} \;\; f^*(z - L^*v) + g^*(v) + \langle v \mid r \rangle\,. \tag{5.26}$$

Extensions of the Fenchel-Rockafellar framework to variational inequalities were considered in [1, 18, 22, 28], while extensions to saddle function problems were proposed in [25]. On the other hand, general monotone operators were investigated in [3, 4, 7, 27] in the case when $\mathcal{G} = \mathcal{H}$ and $L = \mathrm{Id}$. The general duality setting described in Problem 5.10 appears in [16, 29, 31].

Our objective is to devise an algorithm which solves (5.23) and (5.24) simultaneously, and which uses the operators A, B, and L separately. In the literature, several splitting algorithms are available for solving the primal problem (5.23), but they are restricted by stringent hypotheses. Let us set

$$A_1\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto -z + Ax \quad \text{and} \quad A_2\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto L^*B(Lx - r), \tag{5.27}$$

and observe that solving (5.23) is equivalent to finding a zero of $A_1 + A_2$. If B is single-valued and cocoercive (its inverse is strongly monotone), then so is A_2, and (5.23) can be solved by the forward-backward algorithm [10, 26, 39]. If B is merely Lipschitzian, or even just continuous, so is A_2, and (5.23) can then be solved via the algorithm proposed in [40]. These algorithms employ the resolvent of A_1, which is easily derived from that of A, and explicit applications of A_2, i.e., of B and L. They are however limited in scope by the fact that B must be single-valued and smooth. The main splitting algorithm to find a zero of $A_1 + A_2$ when both operators are set-valued is the Douglas-Rachford algorithm [11, 15, 24, 37]. This algorithm requires that both operators be maximally monotone and that their resolvents be computable to within some quantifiable error. Unfortunately, these conditions are seldom met in the present setting since A_2 may not be maximally monotone [29, 31] and, more importantly, since there is no convenient rule to compute the resolvent of A_2 in terms of L and the resolvent of B unless stringent conditions are imposed on L (see [6, Proposition 23.23] and [20]).

Our approach is motivated by the classical Kuhn-Tucker theory [36], which asserts that points $\overline{x} \in \mathcal{H}$ and $\overline{v} \in \mathcal{G}$ satisfying the conditions

$$(0,0) \in \big(-z + \partial f(\overline{x}) + L^*\overline{v},\, r + \partial g^*(\overline{v}) - L\overline{x} \big) \tag{5.28}$$

are solutions to (5.25) and (5.26), respectively. By analogy, it is natural to consider the following problem in conjunction with Problem 5.10.

Problem 5.11 In the setting of Problem 5.10, let $\boldsymbol{\mathcal{K}} = \mathcal{H} \oplus \mathcal{G}$ and set

$$\boldsymbol{M} \colon \boldsymbol{\mathcal{K}} \to 2^{\boldsymbol{\mathcal{K}}} \colon (x,v) \mapsto (-z+Ax)\times(r+B^{-1}v) \quad \text{and} \quad \boldsymbol{S} \colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \colon (x,v) \mapsto (L^*v, -Lx). \tag{5.29}$$

The problem is to

$$\text{find } \boldsymbol{x} \in \boldsymbol{\mathcal{K}} \quad \text{such that} \quad \boldsymbol{0} \in \boldsymbol{M}\boldsymbol{x} + \boldsymbol{S}\boldsymbol{x}. \tag{5.30}$$

The investigation of this companion problem may have various purposes [1, 16, 29, 31]. Ours is to exploit its simple structure to derive a new splitting algorithm to solve efficiently Problem 5.10. The crux of our approach is the simple observation that (5.30) reduces the original primal- dual problem (5.23)–(5.24) to that of finding a zero of the sum of a maximally monotone operator $\boldsymbol{M}$ and a bounded linear skew-adjoint transformation $\boldsymbol{S}$. In Section 5.2.2 we establish the convergence of an inexact splitting algorithm proposed in its original form in [40]. Each iteration of this forward-backward-forward scheme performs successively an explicit step on $\boldsymbol{S}$, an implicit step on $\boldsymbol{M}$, and another explicit step on $\boldsymbol{S}$. We then review the tight connections existing between Problem 5.10 and Problem 5.11 and, in particular, the fact that solving the latter provides a solution to the former. In Section 5.2.3, we apply the forward-backward-forward algorithm to the monotone+skew Problem 5.11 and obtain a new type of splitting algorithm for solving (5.23) and (5.24) simultaneously. The main feature of this scheme, that distinguishes it from existing techniques, is that at each iteration it employs the operators A, B, and L separately without requiring any additional assumption to those stated above except, naturally, existence of solutions. Using a product space technique, we then obtain a parallel splitting method for solving the m-term inclusion

$$\text{find } x \in \mathcal{H} \quad \text{such that} \quad z \in \sum_{i=1}^{m} L_i^* B_i(L_i x - r_i), \tag{5.31}$$

where each maximally monotone operator B_i acts on a Hilbert space $\mathcal{G}_i$, $r_i \in \mathcal{G}_i$, and $L_i \colon \mathcal{H} \to \mathcal{G}_i$ is linear and bounded. Applications to variational problems are discussed in Section 5.2.4, where we provide a proximal splitting scheme for solving the primal dual problem (5.25)–(5.26), as well as one for minimizing the sum of m composite functions.

Notation. We denote the scalar products of $\mathcal{H}$ and $\mathcal{G}$ by $\langle \cdot \mid \cdot \rangle$ and the associated norms by $\|\cdot\|$. $\mathcal{B}(\mathcal{H},\mathcal{G})$ is the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$,

$\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H},\mathcal{H})$, and the symbols $\rightharpoonup$ and $\to$ denote respectively weak and strong convergence. Moreover, $\mathcal{H} \oplus \mathcal{G}$ denotes the Hilbert direct sum of $\mathcal{H}$ and $\mathcal{G}$. The projector onto a nonempty closed convex set $C \subset \mathcal{H}$ is denoted by P_C, and its normal cone operator by N_C, i.e.,

$$N_C \colon \mathcal{H} \to 2^{\mathcal{H}} \colon x \mapsto \begin{cases} \big\{u \in \mathcal{H} \mid (\forall y \in C) \ \ \langle y - x \mid u \rangle \leq 0\big\}, & \text{if } x \in C; \\ \varnothing, & \text{otherwise.} \end{cases} \tag{5.32}$$

Let $M \colon \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. We denote by $\operatorname{ran} M = \big\{u \in \mathcal{H} \mid (\exists x \in \mathcal{H})\ u \in Mx\big\}$ the range of M, by $\operatorname{dom} M = \big\{x \in \mathcal{H} \mid Mx \neq \varnothing\big\}$ its domain, by $\operatorname{zer} M = \big\{x \in \mathcal{H} \mid 0 \in Mx\big\}$ its set of zeros, by $\operatorname{gr} M = \big\{(x,u) \in \mathcal{H} \times \mathcal{H} \mid u \in Mx\big\}$ its graph, and by M^{-1} its inverse, i.e., the operator with graph $\big\{(u,x) \in \mathcal{H} \times \mathcal{H} \mid u \in Mx\big\}$. The resolvent of M is $J_M = (\operatorname{Id} + M)^{-1}$. Moreover, M is monotone if

$$(\forall (x,y) \in \mathcal{H} \times \mathcal{H})(\forall (u,v) \in Mx \times My) \quad \langle x - y \mid u - v \rangle \geq 0, \tag{5.33}$$

and maximally so if there exists no monotone operator $\widetilde{M} \colon \mathcal{H} \to 2^{\mathcal{H}}$ such that $\operatorname{gr} M \subset \operatorname{gr} \widetilde{M} \neq \operatorname{gr} M$. In this case, J_M is a nonexpansive operator defined everywhere in $\mathcal{H}$. For background on convex analysis and monotone operator theory, the reader is referred to [6, 41].

5.2.2 Preliminary results

5.2.2.1 Technical facts

The following lemmas will be needed subsequently (see for instance [9, Lemma 3.1] and [9, Theorem 3.8], respectively).

Lemma 5.12 *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $[0,+\infty[$, let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence in $[0,+\infty[$, and let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a summable sequence in $[0,+\infty[$ such that $(\forall n \in \mathbb{N})\ \alpha_{n+1} \leq \alpha_n - \beta_n + \varepsilon_n$. Then $(\alpha_n)_{n\in\mathbb{N}}$ converges and $(\beta_n)_{n\in\mathbb{N}}$ is summable.*

Lemma 5.13 *Let C be a nonempty subset of $\mathcal{H}$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$. Suppose that, for every $x \in C$, there exists a summable sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $[0,+\infty[$ such that*

$$(\forall n \in \mathbb{N}) \ \ \|x_{n+1} - x\|^2 \leq \|x_n - x\|^2 + \varepsilon_n, \tag{5.34}$$

and that every sequential weak cluster point of $(x_n)_{n\in\mathbb{N}}$ is in C. Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in C.

We shall also require the following definition.

Définition 5.14 [2, Definition 2.3] An operator $M \colon \mathcal{H} \to 2^{\mathcal{H}}$ is *demiregular* at $x \in \operatorname{dom} M$ if, for every sequence $((x_n, u_n))_{n\in\mathbb{N}}$ in $\operatorname{gr} M$ and every $u \in Mx$ such that $x_n \rightharpoonup x$ and $u_n \to u$, we have $x_n \to x$.

Lemma 5.15 [2, Proposition 2.4] *Let $M\colon \mathcal{H} \to 2^{\mathcal{H}}$ and let $x \in \operatorname{dom} M$. Then M is demiregular at x in each of the following cases.*

(i) *M is uniformly monotone at x, i.e., there exists an increasing function $\phi\colon\ [0,+\infty[\to [0,+\infty]$ that vanishes only at 0 such that*

$$(\forall u \in Mx)(\forall (y,v) \in \operatorname{gr} M) \quad \langle x-y \mid u-v\rangle \geq \phi(\|x-y\|).$$

(ii) *M is α-strongly monotone, i.e., $M - \alpha\operatorname{Id}$ is monotone for some $\alpha \in \,]0,+\infty[$.*

(iii) *J_M is compact, i.e., for every bounded set $C \subset \mathcal{H}$, the closure of $J_M(C)$ is compact. In particular, $\operatorname{dom} M$ is boundedly relatively compact, i.e., the intersection of its closure with every closed ball is compact.*

(iv) *$M\colon \mathcal{H} \to \mathcal{H}$ is single-valued with a single-valued continuous inverse.*

(v) *M is single-valued on $\operatorname{dom} M$ and $\operatorname{Id} - M$ is demicompact, i.e., for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $\operatorname{dom} M$ such that $(Mx_n)_{n\in\mathbb{N}}$ converges strongly, $(x_n)_{n\in\mathbb{N}}$ admits a strong cluster point.*

5.2.2.2 An inexact forward-backward-forward algorithm

Our algorithmic framework will hinge on the following splitting algorithm, which was proposed in the error-free case in [40]. We provide an analysis of the asymptotic behavior of an inexact version of this method which is of interest in its own right.

Theorem 5.16 *Let $\boldsymbol{\mathcal{H}}$ be a real Hilbert space, let $\boldsymbol{A}\colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}$ be maximally monotone, and let $\boldsymbol{B}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}$ be monotone. Suppose that $\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B}) \neq \varnothing$ and that $\boldsymbol{B}$ is β-Lipschitzian for some $\beta \in \,]0,+\infty[$. Let $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$, and $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ be sequences in $\boldsymbol{\mathcal{H}}$ such that*

$$\sum_{n\in\mathbb{N}} \|\boldsymbol{a}_n\| < +\infty, \quad \sum_{n\in\mathbb{N}} \|\boldsymbol{b}_n\| < +\infty, \quad \text{and} \quad \sum_{n\in\mathbb{N}} \|\boldsymbol{c}_n\| < +\infty, \tag{5.35}$$

let $\boldsymbol{x}_0 \in \boldsymbol{\mathcal{H}}$, let $\varepsilon \in \,]0, 1/(\beta+1)[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\beta]$, and set

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \boldsymbol{y}_n = \boldsymbol{x}_n - \gamma_n(\boldsymbol{B}\boldsymbol{x}_n + \boldsymbol{a}_n) \\ \boldsymbol{p}_n = J_{\gamma_n \boldsymbol{A}}\, \boldsymbol{y}_n + \boldsymbol{b}_n \\ \boldsymbol{q}_n = \boldsymbol{p}_n - \gamma_n(\boldsymbol{B}\boldsymbol{p}_n + \boldsymbol{c}_n) \\ \boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n. \end{array} \right. \tag{5.36}$$

Then the following hold for some $\overline{\boldsymbol{x}} \in \operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$.

(i) *$\sum_{n\in\mathbb{N}} \|\boldsymbol{x}_n - \boldsymbol{p}_n\|^2 < +\infty$ and $\sum_{n\in\mathbb{N}} \|\boldsymbol{y}_n - \boldsymbol{q}_n\|^2 < +\infty$.*

(ii) *$\boldsymbol{x}_n \rightharpoonup \overline{\boldsymbol{x}}$ and $\boldsymbol{p}_n \rightharpoonup \overline{\boldsymbol{x}}$.*

(iii) *Suppose that one of the following is satisfied.*

(a) *$\boldsymbol{A}+\boldsymbol{B}$ is demiregular at $\overline{\boldsymbol{x}}$.*

(b) $\boldsymbol{A}$ *or* $\boldsymbol{B}$ *is uniformly monotone at* $\overline{\boldsymbol{x}}$.

(c) $\operatorname{int}\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B}) \neq \varnothing$.

Then $\boldsymbol{x}_n \to \overline{\boldsymbol{x}}$ *and* $\boldsymbol{p}_n \to \overline{\boldsymbol{x}}$.

Proof. Let us set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \widetilde{\boldsymbol{y}}_n = \boldsymbol{x}_n - \gamma_n \boldsymbol{B}\boldsymbol{x}_n \\ \widetilde{\boldsymbol{p}}_n = J_{\gamma_n \boldsymbol{A}}\, \widetilde{\boldsymbol{y}}_n \\ \widetilde{\boldsymbol{q}}_n = \widetilde{\boldsymbol{p}}_n - \gamma_n \boldsymbol{B}\widetilde{\boldsymbol{p}}_n \end{cases} \tag{5.37}$$

and

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{e}_n = \boldsymbol{y}_n - \boldsymbol{q}_n - \widetilde{\boldsymbol{y}}_n + \widetilde{\boldsymbol{q}}_n. \tag{5.38}$$

Then

$$(\forall n \in \mathbb{N}) \quad \gamma_n^{-1}(\widetilde{\boldsymbol{y}}_n - \widetilde{\boldsymbol{p}}_n) \in \boldsymbol{A}\widetilde{\boldsymbol{p}}_n. \tag{5.39}$$

Now let $\boldsymbol{x} \in \operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$ and let $n \in \mathbb{N}$. We first note that $(\boldsymbol{x}, -\gamma_n\boldsymbol{B}\boldsymbol{x}) \in \operatorname{gr}\gamma_n\boldsymbol{A}$. On the other hand, (5.39) yields $(\widetilde{\boldsymbol{p}}_n, \widetilde{\boldsymbol{y}}_n - \widetilde{\boldsymbol{p}}_n) \in \operatorname{gr}\gamma_n\boldsymbol{A}$. Hence, by monotonicity of $\gamma_n\boldsymbol{A}$, $\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \widetilde{\boldsymbol{p}}_n - \widetilde{\boldsymbol{y}}_n - \gamma_n\boldsymbol{B}\boldsymbol{x}\rangle \leq 0$. However, by monotonicity of $\boldsymbol{B}$, $\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \gamma_n\boldsymbol{B}\boldsymbol{x} - \gamma_n\boldsymbol{B}\widetilde{\boldsymbol{p}}_n\rangle \leq 0$. Upon adding these two inequalities, we obtain $\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \widetilde{\boldsymbol{p}}_n - \widetilde{\boldsymbol{y}}_n - \gamma_n\boldsymbol{B}\widetilde{\boldsymbol{p}}_n\rangle \leq 0$. In turn, we derive from (5.37) that

$$\begin{aligned} 2\gamma_n \langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \boldsymbol{B}\boldsymbol{x}_n - \boldsymbol{B}\widetilde{\boldsymbol{p}}_n\rangle &= 2\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \widetilde{\boldsymbol{p}}_n - \widetilde{\boldsymbol{y}}_n - \gamma_n\boldsymbol{B}\widetilde{\boldsymbol{p}}_n\rangle \\ &\quad + 2\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \gamma_n\boldsymbol{B}\boldsymbol{x}_n + \widetilde{\boldsymbol{y}}_n - \widetilde{\boldsymbol{p}}_n\rangle \\ &\leq 2\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \gamma_n\boldsymbol{B}\boldsymbol{x}_n + \widetilde{\boldsymbol{y}}_n - \widetilde{\boldsymbol{p}}_n\rangle \\ &= 2\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n\rangle \\ &= \|\boldsymbol{x}_n - \boldsymbol{x}\|^2 - \|\widetilde{\boldsymbol{p}}_n - \boldsymbol{x}\|^2 - \|\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n\|^2 \end{aligned} \tag{5.40}$$

and, therefore, using the Lipschitz continuity of $\boldsymbol{B}$, that

$$\begin{aligned} \|\boldsymbol{x}_n - \widetilde{\boldsymbol{y}}_n + \widetilde{\boldsymbol{q}}_n - \boldsymbol{x}\|^2 &= \|(\widetilde{\boldsymbol{p}}_n - \boldsymbol{x}) + \gamma_n(\boldsymbol{B}\boldsymbol{x}_n - \boldsymbol{B}\widetilde{\boldsymbol{p}}_n)\|^2 \\ &= \|\widetilde{\boldsymbol{p}}_n - \boldsymbol{x}\|^2 + 2\gamma_n\langle \widetilde{\boldsymbol{p}}_n - \boldsymbol{x} \mid \boldsymbol{B}\boldsymbol{x}_n - \boldsymbol{B}\widetilde{\boldsymbol{p}}_n\rangle + \gamma_n^2\|\boldsymbol{B}\boldsymbol{x}_n - \boldsymbol{B}\widetilde{\boldsymbol{p}}_n\|^2 \\ &\leq \|\boldsymbol{x}_n - \boldsymbol{x}\|^2 - \|\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n\|^2 + \gamma_n^2\|\boldsymbol{B}\boldsymbol{x}_n - \boldsymbol{B}\widetilde{\boldsymbol{p}}_n\|^2 \\ &\leq \|\boldsymbol{x}_n - \boldsymbol{x}\|^2 - (1 - \gamma_n^2\beta^2)\|\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n\|^2 \\ &\leq \|\boldsymbol{x}_n - \boldsymbol{x}\|^2 - \varepsilon^2\|\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n\|^2. \end{aligned} \tag{5.41}$$

We also derive from (5.36) and (5.37) the following inequalities. First,

$$\|\widetilde{\boldsymbol{y}}_n - \boldsymbol{y}_n\| = \gamma_n\|\boldsymbol{a}_n\| \leq \|\boldsymbol{a}_n\|/\beta. \tag{5.42}$$

Hence, since $J_{\gamma_n \boldsymbol{A}}$ is nonexpansive,

$$\begin{aligned}
\|\widetilde{\boldsymbol{p}}_n - \boldsymbol{p}_n\| &= \|J_{\gamma_n \boldsymbol{A}}\, \widetilde{\boldsymbol{y}}_n - J_{\gamma_n \boldsymbol{A}} \boldsymbol{y}_n - \boldsymbol{b}_n\| \\
&\leq \|J_{\gamma_n \boldsymbol{A}}\, \widetilde{\boldsymbol{y}}_n - J_{\gamma_n \boldsymbol{A}} \boldsymbol{y}_n\| + \|\boldsymbol{b}_n\| \\
&\leq \|\widetilde{\boldsymbol{y}}_n - \boldsymbol{y}_n\| + \|\boldsymbol{b}_n\| \\
&\leq \|\boldsymbol{a}_n\|/\beta + \|\boldsymbol{b}_n\|. \qquad (5.43)
\end{aligned}$$

In turn, we get

$$\begin{aligned}
\|\widetilde{\boldsymbol{q}}_n - \boldsymbol{q}_n\| &= \|\widetilde{\boldsymbol{p}}_n - \gamma_n \boldsymbol{B}\widetilde{\boldsymbol{p}}_n - \boldsymbol{p}_n + \gamma_n(\boldsymbol{B}\boldsymbol{p}_n + \boldsymbol{c}_n)\| \\
&\leq \|\widetilde{\boldsymbol{p}}_n - \boldsymbol{p}_n\| + \gamma_n\|\boldsymbol{B}\widetilde{\boldsymbol{p}}_n - \boldsymbol{B}\boldsymbol{p}_n\| + \gamma_n\|\boldsymbol{c}_n\| \\
&\leq (1 + \gamma_n\beta)\|\widetilde{\boldsymbol{p}}_n - \boldsymbol{p}_n\| + \gamma_n\|\boldsymbol{c}_n\| \\
&\leq 2(\|\boldsymbol{a}_n\|/\beta + \|\boldsymbol{b}_n\|) + \|\boldsymbol{c}_n\|/\beta. \qquad (5.44)
\end{aligned}$$

Combining (5.38), (5.42), and (5.44) yields $\|\boldsymbol{e}_n\| \leq \|\widetilde{\boldsymbol{y}}_n - \boldsymbol{y}_n\| + \|\widetilde{\boldsymbol{q}}_n - \boldsymbol{q}_n\| \leq 3\|\boldsymbol{a}_n\|/\beta + 2\|\boldsymbol{b}_n\| + \|\boldsymbol{c}_n\|/\beta$ and, in view of (5.35), it follows that

$$\sum_{k\in\mathbb{N}} \|\boldsymbol{e}_k\| < +\infty. \qquad (5.45)$$

Furthermore, (5.36), (5.38), and (5.41) imply that

$$\|\boldsymbol{x}_{n+1} - \boldsymbol{x}\| = \|\boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n - \boldsymbol{x}\| \leq \|\boldsymbol{x}_n - \widetilde{\boldsymbol{y}}_n + \widetilde{\boldsymbol{q}}_n - \boldsymbol{x}\| + \|\boldsymbol{e}_n\| \leq \|\boldsymbol{x}_n - \boldsymbol{x}\| + \|\boldsymbol{e}_n\|. \quad (5.46)$$

Thus, it follows from (5.45) and Lemma 5.12 that $(\boldsymbol{x}_k)_{k\in\mathbb{N}}$ is bounded, and we deduce from (5.37) that, since the operators $\boldsymbol{B}$ and $(J_{\gamma_k \boldsymbol{A}})_{k\in\mathbb{N}}$ are Lipschitzian, $(\widetilde{\boldsymbol{y}}_k)_{k\in\mathbb{N}}$, $(\widetilde{\boldsymbol{p}}_k)_{k\in\mathbb{N}}$, and $(\widetilde{\boldsymbol{q}}_k)_{k\in\mathbb{N}}$ are bounded. Consequently, $\mu = \sup_{k\in\mathbb{N}} \|\boldsymbol{x}_k - \widetilde{\boldsymbol{y}}_k + \widetilde{\boldsymbol{q}}_k - \boldsymbol{x}\| < +\infty$ and, using (5.36), (5.38), and (5.41), we obtain

$$\begin{aligned}
\|\boldsymbol{x}_{n+1} - \boldsymbol{x}\|^2 &= \|\boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n - \boldsymbol{x}\|^2 \\
&= \|\boldsymbol{x}_n - \widetilde{\boldsymbol{y}}_n + \widetilde{\boldsymbol{q}}_n - \boldsymbol{x} + \boldsymbol{e}_n\|^2 \\
&= \|\boldsymbol{x}_n - \widetilde{\boldsymbol{y}}_n + \widetilde{\boldsymbol{q}}_n - \boldsymbol{x}\|^2 + 2\left\langle \boldsymbol{x}_n - \widetilde{\boldsymbol{y}}_n + \widetilde{\boldsymbol{q}}_n - \boldsymbol{x} \mid \boldsymbol{e}_n \right\rangle + \|\boldsymbol{e}_n\|^2 \\
&\leq \|\boldsymbol{x}_n - \boldsymbol{x}\|^2 - \varepsilon^2\|\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n\|^2 + \varepsilon_n, \quad \text{where} \quad \varepsilon_n = 2\mu\|\boldsymbol{e}_n\| + \|\boldsymbol{e}_n\|^2. \qquad (5.47)
\end{aligned}$$

(i) : It follows from (5.45), (5.47), and Lemma 5.12 that

$$\sum_{n\in\mathbb{N}} \|\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n\|^2 < +\infty. \qquad (5.48)$$

On the other hand, since (5.35) and (5.43) imply that

$$\sum_{n\in\mathbb{N}} \|\widetilde{\boldsymbol{p}}_n - \boldsymbol{p}_n\| < +\infty, \qquad (5.49)$$

we have $\sum_{n\in\mathbb{N}} \|\widetilde{\boldsymbol{p}}_n - \boldsymbol{p}_n\|^2 < +\infty$. We therefore infer that $\sum_{n\in\mathbb{N}} \|\boldsymbol{x}_n - \boldsymbol{p}_n\|^2 < +\infty$. Furthermore, since (5.38) and (5.37) yield

$$
\begin{aligned}
(\forall n \in \mathbb{N}) \quad \|\boldsymbol{y}_n - \boldsymbol{q}_n\|^2 &= \|\widetilde{\boldsymbol{y}}_n - \widetilde{\boldsymbol{q}}_n + \boldsymbol{e}_n\|^2 \\
&= \|\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n - \gamma_n(\boldsymbol{B}\boldsymbol{x}_n - \boldsymbol{B}\widetilde{\boldsymbol{p}}_n) + \boldsymbol{e}_n\|^2 \\
&\leq 3\big(\|\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n\|^2 + \gamma_n^2\beta^2 \|\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n\|^2 + \|\boldsymbol{e}_n\|^2\big) \\
&\leq 6\|\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n\|^2 + 3\|\boldsymbol{e}_n\|^2,
\end{aligned}
\tag{5.50}
$$

we derive from (5.48) and (5.45) that $\sum_{n\in\mathbb{N}} \|\boldsymbol{y}_n - \boldsymbol{q}_n\|^2 < +\infty$.

(ii) : Set

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{u}_n = \gamma_n^{-1}(\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n) + \boldsymbol{B}\widetilde{\boldsymbol{p}}_n - \boldsymbol{B}\boldsymbol{x}_n. \tag{5.51}$$

Using (5.37) and (5.39), we get

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{u}_n = \gamma_n^{-1}(\widetilde{\boldsymbol{y}}_n - \widetilde{\boldsymbol{p}}_n) + \boldsymbol{B}\widetilde{\boldsymbol{p}}_n \in \boldsymbol{A}\widetilde{\boldsymbol{p}}_n + \boldsymbol{B}\widetilde{\boldsymbol{p}}_n. \tag{5.52}$$

On the other hand, using (5.48), the Lipschitz continuity of $\boldsymbol{B}$, and (5.51) we obtain

$$\boldsymbol{B}\widetilde{\boldsymbol{p}}_n - \boldsymbol{B}\boldsymbol{x}_n \to 0 \quad \text{and} \quad \boldsymbol{u}_n \to 0. \tag{5.53}$$

Now, let $\boldsymbol{w}$ be a weak sequential cluster point of $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$, say $\boldsymbol{x}_{k_n} \rightharpoonup \boldsymbol{w}$. It follows from (5.52) that $(\widetilde{\boldsymbol{p}}_{k_n}, \boldsymbol{u}_{k_n})_{n\in\mathbb{N}}$ lies in $\operatorname{gr}(\boldsymbol{A}+\boldsymbol{B})$, and from (5.48) and (5.53) that

$$\widetilde{\boldsymbol{p}}_{k_n} \rightharpoonup \boldsymbol{w} \quad \text{and} \quad \boldsymbol{u}_{k_n} \to \mathbf{0}. \tag{5.54}$$

Since $\boldsymbol{B}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}$ is monotone and continuous, it is maximally monotone [6, Corollary 20.25]. Furthermore, since $\operatorname{dom} \boldsymbol{B} = \boldsymbol{\mathcal{H}}$, $\boldsymbol{A}+\boldsymbol{B}$ is maximally monotone [6, Corollary 24.4(i)] and its graph is therefore sequentially closed in $\boldsymbol{\mathcal{H}}^{\text{weak}} \times \boldsymbol{\mathcal{H}}^{\text{strong}}$ [6, Proposition 20.33(ii)]. Therefore, $(\boldsymbol{w}, \mathbf{0}) \in \operatorname{gr}(\boldsymbol{A}+\boldsymbol{B})$. Using (5.47), (5.45), and Lemma 5.13, we conclude that there exists $\overline{\boldsymbol{x}} \in \operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$ such that $\boldsymbol{x}_n \rightharpoonup \overline{\boldsymbol{x}}$. Finally, in view of (i), $\boldsymbol{p}_n \rightharpoonup \overline{\boldsymbol{x}}$.

(iii)(a) : As shown in (ii), $\boldsymbol{p}_n \rightharpoonup \overline{\boldsymbol{x}}$. Hence (5.49) yields $\widetilde{\boldsymbol{p}}_n \rightharpoonup \overline{\boldsymbol{x}}$. Moreover, (5.53) yields $\boldsymbol{u}_n \to \mathbf{0}$ and (5.52) yields $(\forall n \in \mathbb{N})$ $(\widetilde{\boldsymbol{p}}_n, \boldsymbol{u}_n) \in \operatorname{gr}(\boldsymbol{A}+\boldsymbol{B})$. Altogether, Definition 5.14 implies that $\widetilde{\boldsymbol{p}}_n \to \overline{\boldsymbol{x}}$ and, therefore, using (5.49), that $\boldsymbol{p}_n \to \overline{\boldsymbol{x}}$. Finally, it results from (i) that $\boldsymbol{x}_n \to \overline{\boldsymbol{x}}$.

(iii)(b)⇒(iii)(a) : The assumptions imply that $\boldsymbol{A}+\boldsymbol{B}$ is uniformly monotone at $\overline{\boldsymbol{x}}$. Hence, the result follows from Lemma 5.15(i).

(iii)(c) : It follows from (5.47), (5.45), (ii), and [9, Proposition 3.10] that $\boldsymbol{x}_n \to \overline{\boldsymbol{x}}$. In turn, (i) yields $\boldsymbol{p}_n \to \overline{\boldsymbol{x}}$. □

Remark 5.17 The sequence $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$, and $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ in (5.36) model errors in the evaluation of the operators. In the error-free setting, the weak convergence of $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ to a zero of $\boldsymbol{A}+\boldsymbol{B}$ in Theorem 5.16(ii) follows from [40, Theorem 3.4(b)].

5.2.2.3 The monotone+skew model

Let us start with some elementary facts about the operators $\boldsymbol{M}$ and $\boldsymbol{S}$ appearing in Problem 5.11.

Proposition 5.18 *Consider the setting of Problem 5.10 and Problem 5.11. Then the following hold.*

(i) *$\boldsymbol{M}$ is maximally monotone.*

(ii) *$\boldsymbol{S} \in \mathcal{B}(\boldsymbol{\mathcal{K}})$, $\boldsymbol{S}^* = -\boldsymbol{S}$, and $\|\boldsymbol{S}\| = \|L\|$.*

(iii) *$\boldsymbol{M} + \boldsymbol{S}$ is maximally monotone.*

(iv) $(\forall \gamma \in \,]0,+\infty[)(\forall x \in \mathcal{H})(\forall v \in \mathcal{G})$ $J_{\gamma \boldsymbol{M}}(x,v) = \big(J_{\gamma A}(x+\gamma z)\,,\, J_{\gamma B^{-1}}(v-\gamma r)\big)$.

(v) $(\forall \gamma \in \,]0,+\infty[)(\forall x \in \mathcal{H})(\forall v \in \mathcal{G})$

$$J_{\gamma \boldsymbol{S}}(x,v) = \big((\mathrm{Id} + \gamma^2 L^*L)^{-1}(x-\gamma L^*v)\,,\,(\mathrm{Id}+\gamma^2 LL^*)^{-1}(v+\gamma Lx)\big).$$

Proof. (i) : Since A and B are maximally monotone, it follows from [6, Propositions 20.22 and 20.23] that $\boldsymbol{M}$ is likewise.

(ii) : The first two assertions are clear. Now let $(x,v) \in \boldsymbol{\mathcal{K}}$. Then $\|\boldsymbol{S}(x,v)\|^2 = \|(L^*v,-Lx)\|^2 = \|L^*v\|^2 + \|Lx\|^2 \leq \|L\|^2(\|v\|^2+\|x\|^2) = \|L\|^2\|(x,v)\|^2$. Thus, $\|\boldsymbol{S}\| \leq \|L\|$. Conversely, $\|x\| \leq 1 \Rightarrow \|(x,0)\| \leq 1 \Rightarrow \|Lx\| = \|\boldsymbol{S}(x,0)\| \leq \|\boldsymbol{S}\|$. Hence $\|L\| \leq \|\boldsymbol{S}\|$.

(iii) : By (i), $\boldsymbol{M}$ is maximally monotone. On the other hand, it follows from (ii) that $\boldsymbol{S}$ is monotone and continuous, hence maximally monotone [6, Example 20.29]. Altogether, since $\mathrm{dom}\,\boldsymbol{S} = \boldsymbol{\mathcal{K}}$, it follows from [6, Corollary 24.4(i)] that $\boldsymbol{M}+\boldsymbol{S}$ is maximally monotone.

(iv) : This follows from [6, Propositions 23.15(ii) and 23.16].

(v) : Let $(x,v) \in \boldsymbol{\mathcal{K}}$ and set $(p,q) = J_{\gamma \boldsymbol{S}}(x,v)$. Then $(x,v) = (p,q) + \gamma \boldsymbol{S}(p,q)$ and hence $x = p + \gamma L^* q$ and $v = q - \gamma Lp$. Hence, $Lx = Lp + \gamma LL^*q$ and $L^*v = L^*q - \gamma L^*Lp$. Thus, $x = p + \gamma L^*v + \gamma^2 L^*Lp$ and therefore $p = (\mathrm{Id}+\gamma^2L^*L)^{-1}(x-\gamma L^*v)$. Likewise, $v = q - \gamma Lx + \gamma^2 LL^*q$, and therefore $q = (\mathrm{Id}+\gamma^2LL^*)^{-1}(v+\gamma Lx)$. □

The next proposition makes the tight interplay between Problem 5.10 and Problem 5.11 explicit. An alternate proof of the equivalence (iii)⇔(iv)⇔(v) can be found in [29] (see also [3, 16, 27, 31] for partial results) ; we provide a direct argument for completeness.

Proposition 5.19 *Consider the setting of Problem 5.10 and Problem 5.11. Then*

(i) *$\mathrm{zer}(\boldsymbol{M}+\boldsymbol{S})$ is a closed convex subset of $\mathcal{P} \times \mathcal{D}$.*

Furthermore, the following are equivalent.

(ii) $z \in \mathrm{ran}(A + L^* \circ B \circ (L \cdot -r))$.

(iii) $\mathcal{P} \neq \varnothing$.

(iv) $\operatorname{zer}(\boldsymbol{M}+\boldsymbol{S}) \neq \varnothing$.

(v) $\mathcal{D} \neq \varnothing$.

(vi) $-r \in \operatorname{ran}(-L \circ A^{-1} \circ (z - L^*\cdot) + B^{-1})$.

Proof. The equivalences (ii)⇔(iii) and (v)⇔(vi) are clear. Now let $(x,v) \in \boldsymbol{\mathcal{K}}$.

(i) : We derive from from (5.29) that $(x,v) \in \operatorname{zer}(\boldsymbol{M}+\boldsymbol{S}) \Leftrightarrow (0 \in -z + Ax + L^*v$ and $0 \in r + B^{-1}v - Lx) \Leftrightarrow (z - L^*v \in Ax$ and $Lx - r \in B^{-1}v) \Leftrightarrow (z - L^*v \in Ax$ and $v \in B(Lx-r)) \Rightarrow (z - L^*v \in Ax$ and $L^*v \in L^*(B(Lx-r))) \Rightarrow z \in Ax + L^*(B(Lx-r)) \Leftrightarrow x \in \mathcal{P}$. Similarly, $(z - L^*v \in Ax$ and $Lx - r \in B^{-1}v) \Leftrightarrow (x \in A^{-1}(z - L^*v)$ and $r - Lx \in -B^{-1}v) \Rightarrow (Lx \in L(A^{-1}(z - L^*v))$ and $r - Lx \in -B^{-1}v) \Rightarrow r \in L(A^{-1}(z-L^*v)) - B^{-1}v \Leftrightarrow v \in \mathcal{D}$. Finally, since $\boldsymbol{M}+\boldsymbol{S}$ is maximally monotone by Proposition 5.18(iii), $\operatorname{zer}(\boldsymbol{M}+\boldsymbol{S})$ is closed and convex [6, Proposition 23.39].

(iii)⇒(iv) : In view of (5.29), $x \in \mathcal{P} \Leftrightarrow z \in Ax + L^*(B(Lx-r)) \Leftrightarrow (\exists\, w \in \mathcal{G})\ \big(z - L^*w \in Ax$ and $w \in B(Lx - r)\big) \Leftrightarrow \big((\exists\, w \in \mathcal{G})\ z \in Ax + L^*w$ and $-r \in B^{-1}w - Lx\big) \Leftrightarrow (\exists\, w \in \mathcal{G})\ (x,w) \in \operatorname{zer}(\boldsymbol{M}+\boldsymbol{S})$.

(iv)⇒(iii) and (iv)⇒(v) : These follow from (i).

(v)⇒(iv) : $v \in \mathcal{D} \Leftrightarrow r \in LA^{-1}(z - L^*v) - B^{-1}v \Leftrightarrow (\exists\, y \in \mathcal{H})\ (y \in A^{-1}(z - L^*v)$ and $r \in Ly - B^{-1}v) \Leftrightarrow (\exists\, y \in \mathcal{H})\ (0 \in -z + Ay + L^*v$ and $0 \in r + B^{-1}v - Ly) \Leftrightarrow (\exists\, y \in \mathcal{H})\ (y,v) \in \operatorname{zer}(\boldsymbol{M}+\boldsymbol{S})$. □

Remark 5.20 Suppose that $z \in \operatorname{ran}(A + L^*B(L\cdot - r))$. Then Proposition 5.19 asserts that solutions to (5.23) and (5.24) can be found as zeros of $\boldsymbol{M}+\boldsymbol{S}$. In principle, this can be achieved via the Douglas-Rachford algorithm applied to (5.30) : let $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$ be sequences in $\boldsymbol{\mathcal{K}}$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $]0,2[$ such that $\boldsymbol{b}_n \rightharpoonup \boldsymbol{0}$, $\sum_{n\in\mathbb{N}} \lambda_n(\|\boldsymbol{a}_n\| + \|\boldsymbol{b}_n\|) < +\infty$, and $\sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty$, let $\boldsymbol{y}_0 \in \boldsymbol{\mathcal{K}}$, let $\gamma \in]0,+\infty[$, and set

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \boldsymbol{x}_n = J_{\gamma \boldsymbol{S}}\, \boldsymbol{y}_n + \boldsymbol{b}_n \\ \boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \lambda_n\big(J_{\gamma \boldsymbol{M}}(2\boldsymbol{x}_n - \boldsymbol{y}_n) + \boldsymbol{a}_n - \boldsymbol{x}_n\big). \end{array} \right. \tag{5.55}$$

Then it follows from Proposition 5.18(i)–(iii) and [11, Theorem 2.1(i)(c)] that $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ converges weakly to a point in $\operatorname{zer}(\boldsymbol{M}+\boldsymbol{S})$. Now set $(\forall n \in \mathbb{N})$ $\boldsymbol{x}_n = (x_n, v_n)$, $\boldsymbol{y}_n = (y_{1,n}, y_{2,n})$, $\boldsymbol{a}_n = (a_{1,n}, a_{2,n})$, and $\boldsymbol{b}_n = (b_{1,n}, b_{2,n})$. Then, using Proposition 5.18(iv)&(v), (5.55) becomes

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} x_n = (\mathrm{Id} + \gamma^2 L^*L)^{-1}(y_{1,n} - \gamma L^* y_{2,n}) + b_{1,n} \\ v_n = (\mathrm{Id} + \gamma^2 LL^*)^{-1}(y_{2,n} + \gamma L y_{1,n}) + b_{2,n} \\ y_{1,n+1} = y_{1,n} + \lambda_n\big(J_{\gamma A}(2x_n - y_{1,n} + \gamma z) + a_{1,n} - x_n\big) \\ y_{2,n+1} = y_{2,n} + \lambda_n\big(J_{\gamma B^{-1}}(2v_n - y_{2,n} - \gamma r) + a_{2,n} - v_n\big). \end{array} \right. \tag{5.56}$$

Moreover, $(x_n)_{n\in\mathbb{N}}$ converges weakly towards a solution $\overline{x}$ to (5.23) and $(v_n)_{n\in\mathbb{N}}$ towards a solution $\overline{v}$ to (5.24) such that $z - L^*\overline{v} \in A\overline{x}$ and $\overline{v} \in B(L\overline{x} - r)$. However, a practical limitation of (5.56) is that it necessitates the inversion of two operators at each iteration, which may be quite demanding numerically.

Remark 5.21 It follows from (5.46) that the error-free version of the forward-backward-forward algorithm (5.36) is Fejér-monotone with respect to $\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$, i.e., for every $n \in \mathbb{N}$ and every $\boldsymbol{x} \in \operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$, $\|\boldsymbol{x}_{n+1}-\boldsymbol{x}\| \leq \|\boldsymbol{x}_n-\boldsymbol{x}\|$. Now let $n \in \mathbb{N}$. Then it follows from [5, Section 2] that there exist $\lambda_n \in [0,2]$ and a closed affine halfspace $\boldsymbol{H}_n \subset \boldsymbol{\mathcal{H}}$ containing $\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})$ such that

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \lambda_n(P_{\boldsymbol{H}_n}\boldsymbol{x}_n - \boldsymbol{x}_n). \tag{5.57}$$

In the setting of Problem 5.11, $\boldsymbol{H}_n$ and λ_n can be determined easily. To see this, consider Theorem 5.16 with $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{K}}$, $\boldsymbol{A} = \boldsymbol{M}$, and $\boldsymbol{B} = \boldsymbol{S}$. Let $\overline{\boldsymbol{x}} \in \operatorname{zer}(\boldsymbol{M}+\boldsymbol{S})$ and suppose that $\boldsymbol{q}_n \neq \boldsymbol{y}_n$ (otherwise, we trivially have $\boldsymbol{H}_n = \boldsymbol{\mathcal{K}}$). In view of (5.36), $\boldsymbol{y}_n - \boldsymbol{p}_n \in \gamma_n\boldsymbol{M}\boldsymbol{p}_n$ and $-\gamma_n\boldsymbol{S}\overline{\boldsymbol{x}} \in \gamma_n\boldsymbol{M}\overline{\boldsymbol{x}}$. Hence, using the monotonicity of $\gamma_n\boldsymbol{M}$ and Proposition 5.18(ii), we get $0 \leq \langle \boldsymbol{p}_n - \overline{\boldsymbol{x}} \mid \boldsymbol{y}_n - \boldsymbol{p}_n + \gamma_n\boldsymbol{S}\overline{\boldsymbol{x}}\rangle = \langle \boldsymbol{p}_n \mid \boldsymbol{y}_n - \boldsymbol{p}_n\rangle - \langle \boldsymbol{x} \mid \boldsymbol{y}_n - \boldsymbol{p}_n\rangle + \gamma_n\langle \boldsymbol{S}^*\boldsymbol{p}_n \mid \overline{\boldsymbol{x}}\rangle = \langle \boldsymbol{p}_n \mid \boldsymbol{y}_n - \boldsymbol{p}_n\rangle - \langle \overline{\boldsymbol{x}} \mid \boldsymbol{y}_n - \boldsymbol{p}_n + \gamma_n\boldsymbol{S}\boldsymbol{p}_n\rangle$. Therefore, we deduce from (5.36) that $\langle \overline{\boldsymbol{x}} \mid \boldsymbol{y}_n - \boldsymbol{q}_n\rangle \leq \langle \boldsymbol{p}_n \mid \boldsymbol{y}_n - \boldsymbol{p}_n\rangle = \langle \boldsymbol{p}_n \mid \boldsymbol{y}_n - \boldsymbol{q}_n\rangle$. Now set

$$\boldsymbol{H}_n = \big\{\boldsymbol{x} \in \boldsymbol{\mathcal{K}} \mid \langle \boldsymbol{x} \mid \boldsymbol{y}_n - \boldsymbol{q}_n\rangle \leq \langle \boldsymbol{p}_n \mid \boldsymbol{y}_n - \boldsymbol{q}_n\rangle\big\} \quad \text{and} \quad \lambda_n = 1 + \gamma_n^2\frac{\|\boldsymbol{S}(\boldsymbol{p}_n - \boldsymbol{x}_n)\|^2}{\|\boldsymbol{p}_n - \boldsymbol{x}_n\|^2}. \tag{5.58}$$

Then $\operatorname{zer}(\boldsymbol{M}+\boldsymbol{S}) \subset \boldsymbol{H}_n$ and $\lambda_n \leq 1 + \gamma_n^2\|\boldsymbol{S}\|^2 < 2$. Altogether, it follows from (5.36) and the skew-adjointness of $\boldsymbol{S}$ that

$$\begin{aligned}
\boldsymbol{x}_n + \lambda_n(\boldsymbol{P}_{\boldsymbol{H}_n}\boldsymbol{x}_n - \boldsymbol{x}_n) &= \boldsymbol{x}_n + \lambda_n\left(\frac{\langle \boldsymbol{p}_n - \boldsymbol{x}_n \mid \boldsymbol{y}_n - \boldsymbol{q}_n\rangle}{\|\boldsymbol{y}_n - \boldsymbol{q}_n\|^2}\right)(\boldsymbol{y}_n - \boldsymbol{q}_n)\\
&= \boldsymbol{x}_n + \lambda_n\left(\frac{\langle \boldsymbol{p}_n - \boldsymbol{x}_n \mid \boldsymbol{x}_n - \boldsymbol{p}_n + \gamma_n\boldsymbol{S}(\boldsymbol{p}_n - \boldsymbol{x}_n)\rangle}{\|\boldsymbol{x}_n - \boldsymbol{p}_n + \gamma_n\boldsymbol{S}(\boldsymbol{p}_n - \boldsymbol{x}_n)\|^2}\right)(\boldsymbol{y}_n - \boldsymbol{q}_n)\\
&= \boldsymbol{x}_n + \lambda_n\left(\frac{\|\boldsymbol{x}_n - \boldsymbol{p}_n\|^2}{\|\boldsymbol{x}_n - \boldsymbol{p}_n\|^2 + \gamma_n^2\|\boldsymbol{S}(\boldsymbol{p}_n - \boldsymbol{x}_n)\|^2}\right)(\boldsymbol{q}_n - \boldsymbol{y}_n)\\
&= \boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n = \boldsymbol{x}_{n+1}.
\end{aligned} \tag{5.59}$$

Thus, the updating rule of algorithm of Theorem 5.16 applied to $\boldsymbol{M}$ and $\boldsymbol{S}$ is given by (5.57)–(5.58). In turn, using results from [5], this iteration process can easily be modified to become strongly convergent.

5.2.3 Main results

The main result of the paper can now be presented. It consists of an application of Theorem 5.16 to find solutions to Problem 5.11, and thus obtain solutions to Problem 5.10. The resulting algorithm employs the operators A, B, and L separately. Moreover, the operators A and B can be activated in parallel and all the steps involving L are explicit.

Theorem 5.22 *In Problem 5.10, suppose that $L \neq 0$ and that $z \in \operatorname{ran}\big(A + L^* \circ B \circ (L \cdot -r)\big)$. Let $(a_{1,n})_{n\in\mathbb{N}}$, $(b_{1,n})_{n\in\mathbb{N}}$, and $(c_{1,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}$, and let $(a_{2,n})_{n\in\mathbb{N}}$, $(b_{2,n})_{n\in\mathbb{N}}$, and $(c_{2,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{G}$. Furthermore, let $x_0 \in \mathcal{H}$, let $v_0 \in \mathcal{G}$, let $\varepsilon \in \,]0, 1/(\|L\|+1)[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\|L\|\,]$, and set*

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{aligned} y_{1,n} &= x_n - \gamma_n(L^* v_n + a_{1,n}) \\ y_{2,n} &= v_n + \gamma_n(Lx_n + a_{2,n}) \\ p_{1,n} &= J_{\gamma_n A}(y_{1,n} + \gamma_n z) + b_{1,n} \\ p_{2,n} &= J_{\gamma_n B^{-1}}(y_{2,n} - \gamma_n r) + b_{2,n} \\ q_{1,n} &= p_{1,n} - \gamma_n(L^* p_{2,n} + c_{1,n}) \\ q_{2,n} &= p_{2,n} + \gamma_n(Lp_{1,n} + c_{2,n}) \\ x_{n+1} &= x_n - y_{1,n} + q_{1,n} \\ v_{n+1} &= v_n - y_{2,n} + q_{2,n}. \end{aligned} \right. \tag{5.60}$$

Then the following hold for some solution $\overline{x}$ to (5.23) and some solution $\overline{v}$ to (5.24) such that $z - L^*\overline{v} \in A\overline{x}$ and $\overline{v} \in B(L\overline{x} - r)$.

(i) $x_n - p_{1,n} \to 0$ *and* $v_n - p_{2,n} \to 0$.

(ii) $x_n \rightharpoonup \overline{x}$, $p_{1,n} \rightharpoonup \overline{x}$, $v_n \rightharpoonup \overline{v}$, *and* $p_{2,n} \rightharpoonup \overline{v}$.

(iii) *Suppose that A is uniformly monotone at $\overline{x}$. Then $x_n \to \overline{x}$ and $p_{1,n} \to \overline{x}$.*

(iv) *Suppose that B^{-1} is uniformly monotone at $\overline{v}$. Then $v_n \to \overline{v}$ and $p_{2,n} \to \overline{v}$.*

Proof. Consider the setting of Problem 5.11. As seen in Proposition 5.18, $\boldsymbol{M}$ is maximally monotone, and $\boldsymbol{S} \in \mathcal{B}(\boldsymbol{\mathcal{K}})$ is monotone and Lipschitzian with constant $\|L\|$. Moreover, Proposition 5.19 yields

$$\varnothing \neq \operatorname{zer}(\boldsymbol{M} + \boldsymbol{S}) \subset \mathcal{P} \times \mathcal{D}. \tag{5.61}$$

Now set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \boldsymbol{x}_n = (x_n, v_n) \\ \boldsymbol{y}_n = (y_{1,n}, y_{2,n}) \\ \boldsymbol{p}_n = (p_{1,n}, p_{2,n}) \\ \boldsymbol{q}_n = (q_{1,n}, q_{2,n}) \end{cases} \quad \text{and} \quad \begin{cases} \boldsymbol{a}_n = (a_{1,n}, a_{2,n}) \\ \boldsymbol{b}_n = (b_{1,n}, b_{2,n}) \\ \boldsymbol{c}_n = (c_{1,n}, c_{2,n}). \end{cases} \tag{5.62}$$

Then, using (5.29) and Proposition 5.18(iv), (5.60) can be written in $\boldsymbol{\mathcal{K}}$ as

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{aligned} \boldsymbol{y}_n &= \boldsymbol{x}_n - \gamma_n(\boldsymbol{S}\boldsymbol{x}_n + \boldsymbol{a}_n) \\ \boldsymbol{p}_n &= J_{\gamma_n \boldsymbol{M}}\, \boldsymbol{y}_n + \boldsymbol{b}_n \\ \boldsymbol{q}_n &= \boldsymbol{p}_n - \gamma_n(\boldsymbol{S}\boldsymbol{p}_n + \boldsymbol{c}_n) \\ \boldsymbol{x}_{n+1} &= \boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n, \end{aligned} \right. \tag{5.63}$$

which is precisely the form of (5.36). Moreover, our assumptions imply that (5.35) is satisfied. These observations allow us to establish the following.

(i)&(ii) : These follow from Theorem 5.16(i)&(ii) applied to $\boldsymbol{M}$ and $\boldsymbol{S}$ in $\boldsymbol{\mathcal{K}}$.

(iii) : As in (5.37), define

$$(\forall n \in \mathbb{N}) \quad (\widetilde{p}_{1,n}, \widetilde{p}_{2,n}) = \widetilde{\boldsymbol{p}}_n = J_{\gamma_n \boldsymbol{M}}\big(\boldsymbol{x}_n - \gamma_n \boldsymbol{S}\boldsymbol{x}_n\big). \tag{5.64}$$

Then, arguing as in (5.49), we obtain

$$p_{1,n} - \widetilde{p}_{1,n} \to 0 \quad \text{and} \quad p_{2,n} - \widetilde{p}_{2,n} \to 0. \tag{5.65}$$

On the other hand, (5.64) yields

$$(\forall n \in \mathbb{N}) \quad \gamma_n^{-1}(\boldsymbol{x}_n - \widetilde{\boldsymbol{p}}_n) - \boldsymbol{S}\boldsymbol{x}_n \in \boldsymbol{M}\widetilde{\boldsymbol{p}}_n, \tag{5.66}$$

i.e., via (5.29),

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \gamma_n^{-1}(x_n - \widetilde{p}_{1,n}) - L^*v_n \in A\widetilde{p}_{1,n} - z \\ \gamma_n^{-1}(v_n - \widetilde{p}_{2,n}) + Lx_n \in B^{-1}\widetilde{p}_{2,n} + r. \end{cases} \tag{5.67}$$

Since $\overline{x}$ solves (5.23), there exist $u \in \mathcal{H}$ and $v \in \mathcal{G}$ such that

$$u \in A\overline{x}, \quad v \in B(L\overline{x} - r), \quad \text{and} \quad z = u + L^*v. \tag{5.68}$$

Now let $n \in \mathbb{N}$. We derive from (5.67) that

$$\gamma_n^{-1}(x_n - \widetilde{p}_{1,n}) - L^*v_n + z \in A\widetilde{p}_{1,n} \quad \text{and} \quad \widetilde{p}_{2,n} \in B\big(\gamma_n^{-1}(v_n - \widetilde{p}_{2,n}) + Lx_n - r\big). \tag{5.69}$$

Now set

$$\alpha_n = \|x_n - \widetilde{p}_{1,n}\|\big(\varepsilon^{-1}\|\widetilde{p}_{1,n} - \overline{x}\| + \|L\|\,\|v_n - v\|\big) \quad \text{and} \quad \beta_n = \varepsilon^{-1}\|v_n - \widetilde{p}_{2,n}\|\,\|\widetilde{p}_{2,n} - v\|. \tag{5.70}$$

It follows from (5.68), (5.69), and the uniform monotonicity of A that there exists an increasing function $\phi\colon\ [0,+\infty[\ \to [0,+\infty]$ that vanishes only at 0 such that

$$\begin{aligned} &\alpha_n + \langle x_n - \overline{x} \mid L^*v - L^*v_n\rangle \\ &\quad \geq \varepsilon^{-1}\|\widetilde{p}_{1,n} - \overline{x}\|\,\|x_n - \widetilde{p}_{1,n}\| + \langle \widetilde{p}_{1,n} - x_n \mid L^*v - L^*v_n\rangle + \langle x_n - \overline{x} \mid L^*v - L^*v_n\rangle \\ &\quad = \varepsilon^{-1}\|\widetilde{p}_{1,n} - \overline{x}\|\,\|x_n - \widetilde{p}_{1,n}\| + \langle \widetilde{p}_{1,n} - \overline{x} \mid L^*v - L^*v_n\rangle \\ &\quad \geq \big\langle \widetilde{p}_{1,n} - \overline{x} \mid \gamma_n^{-1}(x_n - \widetilde{p}_{1,n}) - L^*v_n + L^*v\big\rangle \\ &\quad = \big\langle \widetilde{p}_{1,n} - \overline{x} \mid \gamma_n^{-1}(x_n - \widetilde{p}_{1,n}) - L^*v_n + z - u\big\rangle \\ &\quad \geq \phi(\|\widetilde{p}_{1,n} - \overline{x}\|). \end{aligned} \tag{5.71}$$

On the other hand, since B is monotone, (5.70), (5.68), and (5.69) yield

$$\begin{aligned} \beta_n + \langle x_n - \overline{x} \mid L^*\widetilde{p}_{2,n} - L^*v\rangle &\geq \big\langle \gamma_n^{-1}(v_n - \widetilde{p}_{2,n}) + L(x_n - \overline{x}) \mid \widetilde{p}_{2,n} - v\big\rangle \\ &= \big\langle (\gamma_n^{-1}(v_n - \widetilde{p}_{2,n}) + Lx_n - r) - (L\overline{x} - r) \mid \widetilde{p}_{2,n} - v\big\rangle \\ &\geq 0. \end{aligned} \tag{5.72}$$

Upon adding these two inequalities, we obtain

$$\alpha_n+\beta_n+\|x_n-\overline{x}\|\,\|L\|\,\|\widetilde{p}_{2,n}-v_n\|\geq\alpha_n+\beta_n+\langle x_n-\overline{x}\mid L^*(\widetilde{p}_{2,n}-v_n)\rangle\geq\phi(\|\widetilde{p}_{1,n}-\overline{x}\|). \tag{5.73}$$

Hence, since (ii), (i), and (5.65) imply that the sequences $(x_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, $(\widetilde{p}_{1,n})_{n\in\mathbb{N}}$, and $(\widetilde{p}_{2,n})_{n\in\mathbb{N}}$ are bounded, it follows from (5.70), (5.65), and (i) that $\phi(\|\widetilde{p}_{1,n}-\overline{x}\|)\to 0$, from which we infer that $\widetilde{p}_{1,n}\to\overline{x}$ and, by (5.65), that $p_{1,n}\to\overline{x}$. In turn, (i) yields $x_n\to\overline{x}$.

(iv) : Proceed as in (iii), using the dual objects. □

Remark 5.23 Using a well-known resolvent identity, the computation of $p_{2,n}$ in (5.60) can be performed in terms of the resolvent of B via the identity $J_{\gamma_n B^{-1}}y = y - \gamma_n J_{\gamma_n^{-1}B}(\gamma_n^{-1}y)$.

Remark 5.24 Set $\boldsymbol{\mathcal{Z}} = \big\{(x,v)\in\mathcal{P}\times\mathcal{D} \mid z-L^*v\in Ax \text{ and } v\in B(Lx-r)\big\}$. Since Theorem 5.22 is an application of Theorem 5.16 in $\boldsymbol{\mathcal{K}}$, we deduce from Remark 5.21 that, in the error-free case, the updating process for (x_n,v_n) in (5.60) results from a relaxed projection onto a closed affine halfspace $\boldsymbol{H}_n$ containing $\boldsymbol{\mathcal{Z}}$, namely

$$(x_{n+1},v_{n+1})=(x_n,v_n)+\lambda_n\big(P_{\boldsymbol{H}_n}(x_n,v_n)-(x_n,v_n)\big), \tag{5.74}$$

where

$$\boldsymbol{H}_n=\big\{(x,v)\in\boldsymbol{\mathcal{K}}\mid\langle x\mid y_{1,n}-q_{1,n}\rangle+\langle v\mid y_{2,n}-q_{2,n}\rangle\leq\langle p_{1,n}\mid y_{1,n}-q_{1,n}\rangle+\langle p_{2,n}\mid y_{2,n}-q_{2,n}\rangle\big\}$$
$$\text{and}\quad \lambda_n=1+\gamma_n^2\frac{\|L(p_{1,n}-x_n)\|^2+\|L^*(p_{2,n}-v_n)\|^2}{\|p_{1,n}-x_n\|^2+\|p_{2,n}-v_n\|^2}. \tag{5.75}$$

In the special case when $\mathcal{G}=\mathcal{H}$ and $L=\mathrm{Id}$, an analysis of such outer projection methods is provided in [17].

Corollary 5.25 *Let $A_1\colon\mathcal{H}\to 2^{\mathcal{H}}$ and $A_2\colon\mathcal{H}\to 2^{\mathcal{H}}$ be maximally monotone operators such that $\mathrm{zer}(A_1+A_2)\neq\varnothing$. Let $(b_{1,n})_{n\in\mathbb{N}}$ and $(b_{2,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}$, let x_0 and v_0 be in $\mathcal{H}$, let $\varepsilon\in\,]0,1/2[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon,1-\varepsilon]$, and set*

$$(\forall n\in\mathbb{N})\quad\left\lfloor\begin{array}{l} p_{1,n}=J_{\gamma_n A_1}(x_n-\gamma_n v_n)+b_{1,n}\\ p_{2,n}=J_{\gamma_n A_2^{-1}}(v_n+\gamma_n x_n)+b_{2,n}\\ x_{n+1}=p_{1,n}+\gamma_n(v_n-p_{2,n})\\ v_{n+1}=p_{2,n}+\gamma_n(p_{1,n}-x_n).\end{array}\right. \tag{5.76}$$

Then the following hold for some $\overline{x}\in\mathrm{zer}(A_1+A_2)$ and some $\overline{v}\in\mathrm{zer}(-A_1^{-1}\circ(-\mathrm{Id})+A_2^{-1})$ such that $-\overline{v}\in A_1\overline{x}$ and $\overline{v}\in A_2\overline{x}$.

(i) *$x_n\rightharpoonup\overline{x}$ and $v_n\rightharpoonup\overline{v}$.*

(ii) *Suppose that A_1 is uniformly monotone at $\overline{x}$. Then $x_n\to\overline{x}$.*

(iii) *Suppose that* A_2^{-1} *is uniformly monotone at* $\overline{v}$. *Then* $v_n \to \overline{v}$.

Proof. Apply Theorem 5.22 with $\mathcal{G} = \mathcal{H}$, $L = \mathrm{Id}$, $A = A_1$, $B = A_2$, $r = 0$, and $z = 0$. ☐

Remark 5.26 The most popular algorithm to find a zero of the sum of two maximally monotone operators is the Douglas-Rachford algorithm [11, 15, 24, 37] (see (5.55)). Corollary 5.25 provides an alternative scheme which is also based on evaluations of the resolvents of the two operators.

Corollary 5.27 *In Problem 5.10, suppose that* $L \neq 0$ *and that* $\mathrm{zer}(L^*BL) \neq \varnothing$. *Let* $(a_{1,n})_{n\in\mathbb{N}}$ *and* $(c_{1,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{H}$, *and let* $(a_{2,n})_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, *and* $(c_{2,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{G}$. *Let* $x_0 \in \mathcal{H}$, *let* $v_0 \in \mathcal{G}$, *let* $\varepsilon \in \left]0, 1/(\|L\|+1)\right[$, *let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon, (1-\varepsilon)/\|L\|\,]$, *and set*

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} s_n = \gamma_n(L^*v_n + a_{1,n}) \\ y_n = v_n + \gamma_n(Lx_n + a_{2,n}) \\ p_n = J_{\gamma_n B^{-1}} y_n + b_n \\ x_{n+1} = x_n - \gamma_n(L^*p_n + c_{1,n}) \\ v_{n+1} = p_n - \gamma_n(Ls_n + c_{2,n}). \end{array} \right. \tag{5.77}$$

Then the following hold for some $\overline{x} \in \mathrm{zer}(L^*BL)$ *and some* $\overline{v} \in (\mathrm{ran}\, L)^{\perp} \cap B(L\overline{x})$.

(i) $x_n \rightharpoonup \overline{x}$ *and* $v_n \rightharpoonup \overline{v}$.

(ii) *Suppose that* B^{-1} *is uniformly monotone at* $\overline{v}$. *Then* $v_n \to \overline{v}$.

Proof. Apply Theorem 5.22 with $A = 0$, $r = 0$, and $z = 0$. ☐

Remark 5.28 In connection with Corollary 5.27, a weakly convergent splitting method was proposed in [30] for finding a zero of L^*BL. This method requires the additional assumption that $\mathrm{ran}\, L$ be closed. In addition, unlike the algorithm described in (5.77), it requires the exact implementation of the generalized inverse of L at each iteration, which is a challenging task.

Next, we extend (5.23) to the problem of solving an inclusion involving the sum of m composite monotone operators. We obtain an algorithm in which the operators $(B_i)_{1\leq i\leq m}$ can be activated in parallel, and independently from the transformations $(L_i)_{1\leq i\leq m}$.

Theorem 5.29 *Let* $z \in \mathcal{H}$ *and let* $(\omega_i)_{1\leq i\leq m}$ *be real numbers in* $]0,1]$ *such that* $\sum_{i=1}^m \omega_i = 1$. *For every* $i \in \{1, \ldots, m\}$, *let* $(\mathcal{G}_i, \|\cdot\|_{\mathcal{G}_i})$ *be a real Hilbert space, let* $r_i \in \mathcal{G}_i$, *let* $B_i \colon \mathcal{G}_i \to 2^{\mathcal{G}_i}$ *be maximally monotone, and suppose that* $0 \neq L_i \in \mathcal{B}(\mathcal{H}, \mathcal{G}_i)$. *Moreover, assume that*

$$z \in \mathrm{ran} \sum_{i=1}^m \omega_i L_i^* \circ B_i \circ (L_i \cdot - r_i). \tag{5.78}$$

Consider the problem

$$\text{find} \quad x \in \mathcal{H} \ \text{ such that } \quad z \in \sum_{i=1}^{m} \omega_i L_i^* B_i(L_i x - r_i), \tag{5.79}$$

and the problem

$$\text{find} \quad v_1 \in \mathcal{G}_1, \ldots, v_m \in \mathcal{G}_m \quad \text{such that}$$

$$\sum_{i=1}^{m} \omega_i L_i^* v_i = z \quad \text{and} \quad (\exists\, x \in \mathcal{H}) \quad \begin{cases} v_1 \in B_1(L_1 x - r_1) \\ \quad \vdots \\ v_m \in B_m(L_m x - r_m). \end{cases} \tag{5.80}$$

Now, for every $i \in \{1, \ldots, m\}$, let $(a_{1,i,n})_{n\in\mathbb{N}}$ and $(c_{1,i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}$, let $(a_{2,i,n})_{n\in\mathbb{N}}$, $(b_{i,n})_{n\in\mathbb{N}}$, and $(c_{2,i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{G}_i$, let $x_{i,0} \in \mathcal{H}$, and let $v_{i,0} \in \mathcal{G}_i$. Furthermore, set $\beta = \max_{1\leq i\leq m} \|L_i\|$, let $\varepsilon \in \left]0, 1/(\beta+1)\right[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\beta]$, and set

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} x_n = \sum_{i=1}^m \omega_i x_{i,n} \\ \text{For } i = 1, \ldots, m \\ \left\lfloor \begin{array}{l} y_{1,i,n} = x_{i,n} - \gamma_n (L_i^* v_{i,n} + a_{1,i,n}) \\ y_{2,i,n} = v_{i,n} + \gamma_n (L_i x_{i,n} + a_{2,i,n}) \end{array} \right. \\ p_{1,n} = \sum_{i=1}^m \omega_i y_{1,i,n} + \gamma_n z \\ \text{For } i = 1, \ldots, m \\ \left\lfloor \begin{array}{l} p_{2,i,n} = J_{\gamma_n B_i^{-1}}(y_{2,i,n} - \gamma_n r_i) + b_{i,n} \\ q_{1,i,n} = p_{1,n} - \gamma_n (L_i^* p_{2,i,n} + c_{1,i,n}) \\ q_{2,i,n} = p_{2,i,n} + \gamma_n (L_i p_{1,n} + c_{2,i,n}) \\ x_{i,n+1} = x_{i,n} - y_{1,i,n} + q_{1,i,n} \\ v_{i,n+1} = v_{i,n} - y_{2,i,n} + q_{2,i,n}. \end{array} \right. \end{array} \right. \tag{5.81}$$

Then the following hold for some solution $\overline{x}$ to (5.79) and some solution $(\overline{v}_i)_{1\leq i\leq m}$ to (5.80) such that, for every $i \in \{1, \ldots, m\}$, $\overline{v}_i \in B_i(L_i \overline{x} - r_i)$.

(i) $x_n \rightharpoonup \overline{x}$ *and, for every $i \in \{1, \ldots, m\}$, $v_{i,n} \rightharpoonup \overline{v}_i$.*

(ii) *Suppose that, for every $i \in \{1, \ldots, m\}$, B_i^{-1} is strongly monotone at $\overline{v}_i$. Then, for every $i \in \{1, \ldots, m\}$, $v_{i,n} \to \overline{v}_i$.*

Proof. Let $\boldsymbol{\mathcal{H}}$ be the real Hilbert space obtained by endowing the Cartesian product $\mathcal{H}^m$ with the scalar product $\langle \cdot \mid \cdot \rangle_{\boldsymbol{\mathcal{H}}} \colon (\boldsymbol{x}, \boldsymbol{y}) \mapsto \sum_{i=1}^m \omega_i \langle x_i \mid y_i \rangle$, where $\boldsymbol{x} = (x_i)_{1\leq i\leq m}$ and $\boldsymbol{y} = (y_i)_{1\leq i\leq m}$ denote generic elements in $\boldsymbol{\mathcal{H}}$. The associated norm is $\|\cdot\|_{\boldsymbol{\mathcal{H}}} \colon \boldsymbol{x} \mapsto \sqrt{\sum_{i=1}^m \omega_i \|x_i\|^2}$. Likewise, let $\boldsymbol{\mathcal{G}}$ denote the real Hilbert space obtained by endowing $\mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ with the scalar product and the associated norm respectively defined by

$$\langle \cdot \mid \cdot \rangle_{\boldsymbol{\mathcal{G}}} \colon (\boldsymbol{y}, \boldsymbol{z}) \mapsto \sum_{i=1}^{m} \omega_i \langle y_i \mid z_i \rangle_{\mathcal{G}_i} \quad \text{and} \quad \|\cdot\|_{\boldsymbol{\mathcal{G}}} \colon \boldsymbol{y} \mapsto \sqrt{\sum_{i=1}^{m} \omega_i \|y_i\|_{\mathcal{G}_i}^2}. \tag{5.82}$$

Define

$$\boldsymbol{V} = \big\{(x,\ldots,x) \in \boldsymbol{\mathcal{H}} \mid x \in \mathcal{H}\big\} \quad \text{and} \quad \boldsymbol{j}\colon \mathcal{H} \to \boldsymbol{V}\colon x \mapsto (x,\ldots,x). \tag{5.83}$$

In view of (5.32), the normal cone operator of $\boldsymbol{V}$ is

$$N_{\boldsymbol{V}}\colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}\colon \boldsymbol{x} \mapsto \begin{cases} \boldsymbol{V}^{\perp} = \big\{\boldsymbol{u} \in \boldsymbol{\mathcal{H}} \mid \sum_{i=1}^{m} \omega_i u_i = 0\big\}, & \text{if } \boldsymbol{x} \in \boldsymbol{V};\\ \varnothing, & \text{otherwise.} \end{cases} \tag{5.84}$$

Now set

$$\boldsymbol{A} = N_{\boldsymbol{V}},\ \boldsymbol{B}\colon \boldsymbol{\mathcal{G}} \to 2^{\boldsymbol{\mathcal{G}}}\colon \boldsymbol{y} \mapsto \underset{i=1}{\overset{m}{\times}} B_i y_i,\ \boldsymbol{L}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{G}}\colon \boldsymbol{x} \mapsto (L_i x_i)_{1\leq i\leq m},\ \text{and } \boldsymbol{r} = (r_i)_{1\leq i\leq m}. \tag{5.85}$$

It is easily checked that $\boldsymbol{A}$ and $\boldsymbol{B}$ are maximally monotone with resolvents

$$(\forall \gamma \in \left]0,+\infty\right[) \quad J_{\gamma\boldsymbol{A}}\colon \boldsymbol{x} \mapsto P_{\boldsymbol{V}}\boldsymbol{x} = \boldsymbol{j}\left(\sum_{i=1}^{m} \omega_i x_i\right) \quad \text{and} \quad J_{\gamma\boldsymbol{B}^{-1}}\colon \boldsymbol{y} \mapsto \big(J_{\gamma B_i^{-1}} y_i\big)_{1\leq i\leq m}. \tag{5.86}$$

Moreover, $\boldsymbol{L} \in \mathcal{B}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{G}})$ and

$$\boldsymbol{L}^*\colon \boldsymbol{\mathcal{G}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{v} \mapsto \big(L_i^* v_i\big)_{1\leq i\leq m}. \tag{5.87}$$

Now, set

$$\begin{cases} \boldsymbol{\mathcal{P}} = \big\{\boldsymbol{x} \in \boldsymbol{\mathcal{H}} \mid \boldsymbol{j}(z) \in \boldsymbol{A}\boldsymbol{x} + \boldsymbol{L}^*\boldsymbol{B}(\boldsymbol{L}\boldsymbol{x} - \boldsymbol{r})\big\}\\ \boldsymbol{\mathcal{D}} = \big\{\boldsymbol{v} \in \boldsymbol{\mathcal{G}} \mid -\boldsymbol{r} \in -\boldsymbol{L}\boldsymbol{A}^{-1}(\boldsymbol{j}(z) - \boldsymbol{L}^*\boldsymbol{v}) + \boldsymbol{B}^{-1}\boldsymbol{v}\big\}. \end{cases} \tag{5.88}$$

Then, for every $x \in \mathcal{H}$,

$$\begin{aligned} x \text{ solves (5.79)} \quad &\Leftrightarrow \quad z \in \sum_{i=1}^{m} \omega_i L_i^*\big(B_i(L_i x - r_i)\big)\\ &\Leftrightarrow \quad \left(\exists\, (v_i)_{1\leq i\leq m} \in \underset{i=1}{\overset{m}{\times}} B_i(L_i x - r_i)\right) \quad z = \sum_{i=1}^{m} \omega_i L_i^* v_i\\ &\Leftrightarrow \quad \left(\exists\, (v_i)_{1\leq i\leq m} \in \underset{i=1}{\overset{m}{\times}} B_i(L_i x - r_i)\right) \quad \sum_{i=1}^{m} \omega_i (z - L_i^* v_i) = 0\\ &\Leftrightarrow \quad \big(\exists\, \boldsymbol{v} \in \boldsymbol{B}(\boldsymbol{L}\boldsymbol{j}(x) - \boldsymbol{r})\big) \quad \boldsymbol{j}(z) - \boldsymbol{L}^*\boldsymbol{v} \in \boldsymbol{V}^{\perp} = \boldsymbol{A}\boldsymbol{j}(x)\\ &\Leftrightarrow \quad \boldsymbol{j}(z) \in \boldsymbol{A}\boldsymbol{j}(x) + \boldsymbol{L}^*\boldsymbol{B}(\boldsymbol{L}\boldsymbol{j}(x) - \boldsymbol{r})\\ &\Leftrightarrow \quad \boldsymbol{j}(x) \in \boldsymbol{\mathcal{P}} \subset \boldsymbol{V}. \end{aligned} \tag{5.89}$$

Moreover, for every $\boldsymbol{v} \in \boldsymbol{\mathcal{G}}$,

$$\begin{aligned}
\boldsymbol{v} \text{ solves (5.80)} \quad &\Leftrightarrow \quad \sum_{i=1}^{m} \omega_i(z - L_i^* v_i) = 0 \quad \text{and} \quad (\exists x \in \mathcal{H}) \quad (v_i)_{1 \le i \le m} \in \bigtimes_{i=1}^{m} B_i(L_i x - r_i) \\
&\Leftrightarrow \quad (\exists x \in \mathcal{H}) \quad \boldsymbol{j}(z) - \boldsymbol{L}^* \boldsymbol{v} \in \boldsymbol{V}^\perp = \boldsymbol{A}\boldsymbol{j}(x) \quad \text{and} \quad \boldsymbol{v} \in \boldsymbol{B}(\boldsymbol{L}\boldsymbol{j}(x) - \boldsymbol{r}) \\
&\Leftrightarrow \quad (\exists x \in \mathcal{H}) \quad \boldsymbol{j}(x) \in \boldsymbol{A}^{-1}\big(\boldsymbol{j}(z) - \boldsymbol{L}^* \boldsymbol{v}\big) \quad \text{and} \quad \boldsymbol{L}\boldsymbol{j}(x) - \boldsymbol{r} \in \boldsymbol{B}^{-1}\boldsymbol{v} \\
&\Leftrightarrow \quad (\exists \boldsymbol{x} \in \boldsymbol{V} = \operatorname{dom} \boldsymbol{A}) \quad \boldsymbol{x} \in \boldsymbol{A}^{-1}\big(\boldsymbol{j}(z) - \boldsymbol{L}^* \boldsymbol{v}\big) \quad \text{and} \quad \boldsymbol{L}\boldsymbol{x} - \boldsymbol{r} \in \boldsymbol{B}^{-1}\boldsymbol{v} \\
&\Leftrightarrow \quad -\boldsymbol{r} \in -\boldsymbol{L}\boldsymbol{A}^{-1}\big(\boldsymbol{j}(z) - \boldsymbol{L}^* \boldsymbol{v}\big) + \boldsymbol{B}^{-1}\boldsymbol{v} \\
&\Leftrightarrow \quad \boldsymbol{v} \in \boldsymbol{\mathcal{D}}.
\end{aligned} \tag{5.90}$$

Altogether, solving the inclusion (5.79) in $\mathcal{H}$ is equivalent to solving the inclusion $\boldsymbol{j}(z) \in \boldsymbol{A}\boldsymbol{x} + \boldsymbol{L}^*\boldsymbol{B}(\boldsymbol{L}\boldsymbol{x} - \boldsymbol{r})$ in $\boldsymbol{\mathcal{H}}$ and solving (5.80) in $\boldsymbol{\mathcal{G}}$ is equivalent to solving $-\boldsymbol{r} \in \boldsymbol{B}^{-1}\boldsymbol{v} - \boldsymbol{L}\boldsymbol{A}^{-1}\big(\boldsymbol{j}(z) - \boldsymbol{L}^*\boldsymbol{v}\big)$ in $\boldsymbol{\mathcal{G}}$. Next, let us show that the algorithm described in (5.81) is a particular case of the algorithm described in (5.60) in Theorem 5.22. To this end define, for every $n \in \mathbb{N}$, $\boldsymbol{x}_n = (x_{i,n})_{1 \le i \le m}$, $\boldsymbol{v}_n = (v_{i,n})_{1 \le i \le m}$, $\boldsymbol{y}_{1,n} = (y_{1,i,n})_{1 \le i \le m}$, $\boldsymbol{y}_{2,n} = (y_{2,i,n})_{1 \le i \le m}$, $\boldsymbol{p}_{1,n} = \boldsymbol{j}(p_{1,n})$, $\boldsymbol{p}_{2,n} = (p_{2,i,n})_{1 \le i \le m}$, $\boldsymbol{q}_{1,n} = (q_{1,i,n})_{1 \le i \le m}$, $\boldsymbol{q}_{2,n} = (q_{2,i,n})_{1 \le i \le m}$, $\boldsymbol{a}_{1,n} = (a_{1,i,n})_{1 \le i \le m}$, $\boldsymbol{a}_{2,n} = (a_{2,i,n})_{1 \le i \le m}$, $\boldsymbol{b}_{2,n} = (b_{i,n})_{1 \le i \le m}$, $\boldsymbol{c}_{1,n} = (c_{1,i,n})_{1 \le i \le m}$, and $\boldsymbol{c}_{2,n} = (c_{2,i,n})_{1 \le i \le m}$. Then we deduce from (5.85), (5.86), and (5.87) that, in terms of these new variables, (5.81) can be rewritten as

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l}
\boldsymbol{y}_{1,n} = \boldsymbol{x}_n - \gamma_n(\boldsymbol{L}^*\boldsymbol{v}_n + \boldsymbol{a}_{1,n}) \\
\boldsymbol{y}_{2,n} = \boldsymbol{v}_n + \gamma_n(\boldsymbol{L}\boldsymbol{x}_n + \boldsymbol{a}_{2,n}) \\
\boldsymbol{p}_{1,n} = J_{\gamma_n \boldsymbol{A}}(\boldsymbol{y}_{1,n} + \gamma_n z) \\
\boldsymbol{p}_{2,n} = J_{\gamma_n \boldsymbol{B}^{-1}}(\boldsymbol{y}_{2,n} - \gamma_n r) + \boldsymbol{b}_{2,n} \\
\boldsymbol{q}_{1,n} = \boldsymbol{p}_{1,n} - \gamma_n(\boldsymbol{L}^*\boldsymbol{p}_{2,n} + \boldsymbol{c}_{1,n}) \\
\boldsymbol{q}_{2,n} = \boldsymbol{p}_{2,n} + \gamma_n(\boldsymbol{L}\boldsymbol{p}_{1,n} + \boldsymbol{c}_{2,n}) \\
\boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \boldsymbol{y}_{1,n} + \boldsymbol{q}_{1,n} \\
\boldsymbol{v}_{n+1} = \boldsymbol{v}_n - \boldsymbol{y}_{2,n} + \boldsymbol{q}_{2,n}.
\end{array} \right. \tag{5.91}$$

Moreover, $\|\boldsymbol{L}\| \le \max_{1 \le i \le m} \|L_i\| = \beta$, and our assumptions imply that the sequences $(\boldsymbol{a}_{1,n})_{n\in\mathbb{N}}$, $(\boldsymbol{c}_{1,n})_{n\in\mathbb{N}}$, $(\boldsymbol{a}_{2,n})_{n\in\mathbb{N}}$, $(\boldsymbol{b}_{2,n})_{n\in\mathbb{N}}$, and $(\boldsymbol{c}_{2,n})_{n\in\mathbb{N}}$ are absolutely summable. Furthermore, (5.78) and (5.89) assert that $\boldsymbol{j}(z) \in \operatorname{ran}(\boldsymbol{A} + \boldsymbol{L}^* \circ \boldsymbol{B} \circ (\boldsymbol{L} \cdot - \boldsymbol{r}))$.

(i) : It follows from Theorem 5.22(ii) that there exists $\overline{\boldsymbol{x}} \in \boldsymbol{\mathcal{P}}$ and $(\overline{v}_i)_{1 \le i \le m} = \overline{\boldsymbol{v}} \in \boldsymbol{\mathcal{D}}$ such that $\boldsymbol{j}(z) - \boldsymbol{L}^*\overline{\boldsymbol{v}} \in \boldsymbol{A}\overline{\boldsymbol{x}}$, $\overline{\boldsymbol{v}} \in \boldsymbol{B}(\boldsymbol{L}\overline{\boldsymbol{x}} - \boldsymbol{r})$, $\boldsymbol{x}_n \rightharpoonup \overline{\boldsymbol{x}}$, and $\boldsymbol{v}_n \rightharpoonup \overline{\boldsymbol{v}}$. Hence $\boldsymbol{j}(x_n) = P_{\boldsymbol{V}}\boldsymbol{x}_n \rightharpoonup P_{\boldsymbol{V}}\overline{\boldsymbol{x}} = \overline{\boldsymbol{x}}$. Since (5.89) asserts that there exists a solution $\overline{x}$ to (5.79) such that $\overline{\boldsymbol{x}} = \boldsymbol{j}(\overline{x})$, we obtain that $x_n = \boldsymbol{j}^{-1}(P_{\boldsymbol{V}}\boldsymbol{x}_n) \rightharpoonup \boldsymbol{j}^{-1}(\overline{\boldsymbol{x}}) = \overline{x}$. Altogether, by (5.90), for every $i \in \{1, \ldots, m\}$, $v_{i,n} \rightharpoonup \overline{v}_i$, where $(\overline{v}_i)_{1 \le i \le m}$ solves (5.80).

(ii) : Let $(\boldsymbol{w}_1, \boldsymbol{y}_1)$ and $(\boldsymbol{w}_2, \boldsymbol{y}_2)$ in $\operatorname{gr} \boldsymbol{B}^{-1}$. We derive from (5.85) that $(\forall i \in \{1, \ldots, m\})$ $y_{1,i} \in B_i^{-1} w_{1,i}$ and $y_{2,i} \in B_i^{-1} w_{2,i}$. Hence, since the operators $(B_i^{-1})_{1 \le i \le m}$ are strongly monotone, there exist constants $(\rho_i)_{1 \le i \le m}$ in $]0, +\infty[$ such that $\langle \boldsymbol{y}_1 - \boldsymbol{y}_2 \mid \boldsymbol{w}_1 - \boldsymbol{w}_2 \rangle_{\boldsymbol{\mathcal{G}}} = \sum_{i=1}^m \omega_i \langle y_{1,i} - y_{2,i} \mid w_{1,i} - w_{2,i} \rangle_{\mathcal{G}_i} \ge \sum_{i=1}^m \omega_i \rho_i \|w_{1,i} - w_{2,i}\|_{\mathcal{G}_i}^2 \ge$

$\rho\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|^2_{\boldsymbol{\mathcal{G}}}$, where $\rho = \min_{1\leq i\leq m}\rho_i \in \,]0,+\infty[$. Therefore, $\boldsymbol{B}^{-1}$ is strongly monotone and hence uniformly monotone. Thus, the result follows from Theorem 5.22(iv). ☐

5.2.4 Variational problems

We apply the results of the previous sections to minimization problems. Let us first recall some standard notation and results [6, 41]. We denote by $\Gamma_0(\mathcal{H})$ the class of lower semicontinuous convex functions $f\colon \mathcal{H} \to \,]-\infty,+\infty]$ such that $\operatorname{dom} f = \big\{x \in \mathcal{H} \mid f(x) < +\infty\big\} \neq \varnothing$. Now let $f \in \Gamma_0(\mathcal{H})$. The conjugate of f is the function $f^* \in \Gamma_0(\mathcal{H})$ defined by $f^*\colon u \mapsto \sup_{x\in\mathcal{H}}(\langle x \mid u\rangle - f(x))$. Moreover, for every $x \in \mathcal{H}$, $f + \|x - \cdot\|^2/2$ possesses a unique minimizer, which is denoted by $\operatorname{prox}_f x$. Alternatively,

$$\operatorname{prox}_f = (\operatorname{Id} + \partial f)^{-1} = J_{\partial f}, \tag{5.92}$$

where $\partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{u \in \mathcal{H} \mid (\forall y \in \mathcal{H}) \;\; \langle y - x \mid u\rangle + f(x) \leq f(y)\big\}$ is the subdifferential of f, which is a maximally monotone operator. Finally, let C be a convex subset of $\mathcal{H}$. The indicator function of C is denoted by ι_C, its support function by σ_C, and its strong relative interior (the set of points in $x \in C$ such that the cone generated by $-x+C$ is a closed vector subspace of $\mathcal{H}$) by $\operatorname{sri} C$. The following facts will also be required.

Proposition 5.30 *Let $f \in \Gamma_0(\mathcal{H})$, let $g \in \Gamma_0(\mathcal{G})$, let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, let $z \in \mathcal{H}$, and let $r \in \mathcal{G}$. Then the following hold.*

(i) $\operatorname{zer}(-z + \partial f + L^*\circ(\partial g)\circ(L\cdot - r)) \subset \operatorname{Argmin}(f - \langle\cdot \mid z\rangle + g\circ(L\cdot - r))$.

(ii) $\operatorname{zer}(r - (L\circ(\partial f^*)\circ(z - L^*\cdot)) + \partial g^*) \subset \operatorname{Argmin}(f^*(z - L^*\cdot) + g^* + \langle r \mid \cdot\rangle)$.

(iii) *Suppose that one of the following is satisfied.*

  (a) $\operatorname{Argmin}(f + g\circ(L\cdot - r) - \langle\cdot \mid z\rangle) \neq \varnothing$ *and* $r \in \operatorname{sri}(L(\operatorname{dom} f) - \operatorname{dom} g)$.

  (b) $\operatorname{Argmin}(f + g\circ(L\cdot - r) - \langle\cdot \mid z\rangle) \subset \operatorname{Argmin}(f - \langle\cdot \mid z\rangle) \cap \operatorname{Argmin} g\circ(L\cdot - r) \neq \varnothing$ *and* $r \in \operatorname{sri}(\operatorname{ran} L - \operatorname{dom} g)$.

  (c) $f = \iota_C$ *and* $g = \iota_D$*,* $z = 0$*, where C and D are closed convex subset of $\mathcal{H}$ and $\mathcal{G}$, respectively, such that* $C \cap L^{-1}(r + D) \neq \varnothing$ *and* $r \in \operatorname{sri}(\operatorname{ran} L - D)$.

  Then $z \in \operatorname{ran}(\partial f + L^*\circ(\partial g)\circ(L\cdot - r))$.

Proof. (i)&(ii) : By [6, Proposition 16.5(ii) and Theorem 16.2], $\operatorname{zer}(-z + \partial f + L^*\circ(\partial g)\circ(L\cdot - r)) \subset \operatorname{zer}(\partial(f - \langle\cdot \mid z\rangle + g\circ(L\cdot - r))) = \operatorname{Argmin}(f - \langle\cdot \mid z\rangle + g\circ(L\cdot - r))$. We obtain (ii) similarly.

(iii)(a) : By [6, Theorem 16.2 and Theorem 16.37(i)], we have

$$\begin{aligned}\varnothing \neq \operatorname{Argmin}(f + g\circ(L\cdot - r) - \langle\cdot \mid z\rangle) &= \operatorname{zer}\partial(f + g\circ(L\cdot - r) - \langle\cdot \mid z\rangle)\\ &= \operatorname{zer}(-z + \partial f + L^*\circ(\partial g)\circ(L\cdot - r)).\end{aligned} \tag{5.93}$$

(iii)(b) : Since $r \in \operatorname{sri}(\operatorname{ran} L - \operatorname{dom} g)$, using (i) and standard convex analysis, we obtain

$$\begin{aligned}\operatorname{Argmin}(f - \langle \cdot \mid z\rangle) \cap \operatorname{Argmin}(g \circ (L \cdot - r)) &= \operatorname{zer}(-z + \partial f) \cap \operatorname{zer} \partial(g \circ (L \cdot - r)) \\ &= \operatorname{zer}(-z + \partial f) \cap \operatorname{zer}(L^* \circ (\partial g) \circ (L \cdot - r)) \\ &\subset \operatorname{zer}(-z + \partial f + L^* \circ (\partial g) \circ (L \cdot - r)) \\ &\subset \operatorname{Argmin}(f + g \circ (L \cdot - r) - \langle \cdot \mid z\rangle).\end{aligned} \tag{5.94}$$

Therefore, the hypotheses yield $\operatorname{zer}(-z + \partial f + L^* \circ (\partial g) \circ (L \cdot - r)) = \operatorname{Argmin}(f - \langle \cdot \mid z\rangle) \cap \operatorname{Argmin}(g \circ (L \cdot - r)) \neq \varnothing$.

(iii)(c) : Since $\operatorname{dom}(\iota_C + \iota_D(L \cdot - r)) = C \cap L^{-1}(r + D)$,

$$\begin{aligned}\operatorname{Argmin}(\iota_C + \iota_D \circ (L \cdot - r)) &= \operatorname{Argmin} \iota_{C \cap L^{-1}(r+D)} \\ &= C \cap L^{-1}(r + D) \\ &= \operatorname{Argmin} \iota_C \cap \operatorname{Argmin}(\iota_D \circ (L \cdot - r)) \neq \varnothing.\end{aligned} \tag{5.95}$$

In view of (ii) applied to $f = \iota_C$, $g = \iota_D$, and $z = 0$, the proof is complete. ☐

Our first result is a new splitting method for the Fenchel-Rockafellar duality framework (5.25)–(5.26).

Proposition 5.31 *Let $f \in \Gamma_0(\mathcal{H})$, let $g \in \Gamma_0(\mathcal{G})$, let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, let $z \in \mathcal{H}$, and let $r \in \mathcal{G}$. Suppose that $L \neq 0$ and that*

$$z \in \operatorname{ran} \big(\partial f + L^* \circ (\partial g) \circ (L \cdot - r)\big). \tag{5.96}$$

Consider the primal problem

$$\underset{x \in \mathcal{H}}{\text{minimize}} \;\; f(x) + g(Lx - r) - \langle x \mid z\rangle, \tag{5.97}$$

and the dual problem

$$\underset{v \in \mathcal{G}}{\text{minimize}} \;\; f^*(z - L^*v) + g^*(v) + \langle v \mid r\rangle. \tag{5.98}$$

Let $(a_{1,n})_{n\in\mathbb{N}}$, $(b_{1,n})_{n\in\mathbb{N}}$, and $(c_{1,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}$, and let $(a_{2,n})_{n\in\mathbb{N}}$, $(b_{2,n})_{n\in\mathbb{N}}$, and $(c_{2,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{G}$. Furthermore, let $x_0 \in \mathcal{H}$, let $v_0 \in \mathcal{G}$, let $\varepsilon \in \left]0, 1/(\|L\|+1)\right[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\|L\|\,]$, and set

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{aligned} y_{1,n} &= x_n - \gamma_n(L^* v_n + a_{1,n}) \\ y_{2,n} &= v_n + \gamma_n(L x_n + a_{2,n}) \\ p_{1,n} &= \operatorname{prox}_{\gamma_n f}(y_{1,n} + \gamma_n z) + b_{1,n} \\ p_{2,n} &= \operatorname{prox}_{\gamma_n g^*}(y_{2,n} - \gamma_n r) + b_{2,n} \\ q_{1,n} &= p_{1,n} - \gamma_n(L^* p_{2,n} + c_{1,n}) \\ q_{2,n} &= p_{2,n} + \gamma_n(L p_{1,n} + c_{2,n}) \\ x_{n+1} &= x_n - y_{1,n} + q_{1,n} \\ v_{n+1} &= v_n - y_{2,n} + q_{2,n}. \end{aligned} \right. \tag{5.99}$$

Then the following hold for some solution $\overline{x}$ *to* (5.97) *and some solution* $\overline{v}$ *to* (5.98) *such that* $z - L^*\overline{v} \in \partial f(\overline{x})$ *and* $\overline{v} \in \partial g(L\overline{x} - r)$.

(i) $x_n - p_{1,n} \to 0$ *and* $v_n - p_{2,n} \to 0$.

(ii) $x_n \rightharpoonup \overline{x}$, $p_{1,n} \rightharpoonup \overline{x}$, $v_n \rightharpoonup \overline{v}$, *and* $p_{2,n} \rightharpoonup \overline{v}$.

(iii) *Suppose that* f *is uniformly convex at* $\overline{x}$. *Then* $x_n \to \overline{x}$ *and* $p_{1,n} \to \overline{x}$.

(iv) *Suppose that* g^* *is uniformly convex at* $\overline{v}$. *Then* $v_n \to \overline{v}$ *and* $p_{2,n} \to \overline{v}$.

Proof. Suppose that $A = \partial f$ and $B = \partial g$ in Problem 5.10. Then, since $A^{-1} = \partial f^*$ and $B^{-1} = \partial g^*$, we derive from Proposition 5.30(i)&(ii) that the solutions to (5.23) and (5.24) are solutions to (5.97) and (5.98), respectively. Moreover, (5.92) implies that (5.99) is a special case of (5.60). Finally, the uniform convexity of a function $\varphi \in \Gamma_0(\mathcal{H})$ at a point of the domain of $\partial\varphi$ implies the uniform monotonicity of $\partial\varphi$ at that point [41, Section 3.4]. Altogether, the results follow from Theorem 5.22. ☐

Remark 5.32 Here are some comments on Proposition 5.31.

(i) Sufficient conditions for (5.96) to hold are provided in Proposition 5.30.

(ii) As in Remark 5.23, if the proximity operator of g is simpler to implement than that of g^*, $p_{2,n}$ in (5.99) can be computed via the identity $\operatorname{prox}_{\gamma_n g^*} y = y - \gamma_n \operatorname{prox}_{\gamma_n^{-1} g}(\gamma_n^{-1} y)$.

(iii) In the special case when $\mathcal{H}$ and $\mathcal{G}$ are Euclidean spaces, an alternative primal-dual algorithm is proposed in [8], which also uses the proximity operators of f and g, and the operator L in separate steps. This method is derived there in the spirit of the proximal [35] and alternating direction (see [21] and the references therein) methods of multipliers.

(iv) Condition (iii) is satisfied when f is strongly convex, i.e., when $h = f - \alpha\|\cdot\|^2/2$ is convex for some $\alpha \in \left]0, +\infty\right[$. In this case, after rescaling, problem (5.96) can be written as

$$\underset{x \in \mathcal{H}}{\text{minimize}} \;\; h(x) + g(Lx - r) + \frac{1}{2}\|x - z\|^2. \tag{5.100}$$

An alternative primal-dual splitting method for solving this strongly convex problem is proposed in [12].

We now turn our attention to problems involving the sum of m composite functions.

Proposition 5.33 *Let* $z \in \mathcal{H}$ *and let* $(\omega_i)_{1 \le i \le m}$ *be reals in* $\left]0, 1\right]$ *such that* $\sum_{i=1}^m \omega_i = 1$. *For every* $i \in \{1, \ldots, m\}$, *let* $(\mathcal{G}_i, \|\cdot\|_{\mathcal{G}_i})$ *be a real Hilbert space, let* $r_i \in \mathcal{G}_i$, *let* $g_i \in \Gamma_0(\mathcal{G}_i)$, *and suppose that* $0 \neq L_i \in \mathcal{B}(\mathcal{H}, \mathcal{G}_i)$. *Moreover, assume that*

$$z \in \operatorname{ran} \sum_{i=1}^m \omega_i L_i^* \circ (\partial g_i) \circ (L_i \cdot - r_i). \tag{5.101}$$

Consider the problem

$$\underset{x\in\mathcal{H}}{\text{minimize}}\ \sum_{i=1}^{m}\omega_i\, g_i(L_ix-r_i)-\langle x\mid z\rangle\,, \tag{5.102}$$

and the problem

$$\underset{\substack{v_1\in\mathcal{G}_1,\ldots,v_m\in\mathcal{G}_m\\ \sum_{i=1}^m\omega_iL_i^*v_i=z}}{\text{minimize}}\ \sum_{i=1}^{m}\omega_i\left(g_i^*(v_i)+\langle v_i\mid r_i\rangle\right). \tag{5.103}$$

For every $i\in\{1,\ldots,m\}$, let $(a_{1,i,n})_{n\in\mathbb{N}}$ and $(c_{1,i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}$, let $(a_{2,i,n})_{n\in\mathbb{N}}$, $(b_{i,n})_{n\in\mathbb{N}}$, and $(c_{2,i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{G}_i$, let $x_{i,0}\in\mathcal{H}$, and let $v_{i,0}\in\mathcal{G}_i$. Furthermore, set $\beta=\max_{1\leq i\leq m}\|L_i\|$, let $\varepsilon\in\left]0,1/(\beta+1)\right[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon,(1-\varepsilon)/\beta]$, and set

$$(\forall n\in\mathbb{N})\quad \left\lfloor\begin{array}{l} x_n=\sum_{i=1}^m\omega_ix_{i,n}\\ \text{For } i=1,\ldots,m\\ \left\lfloor\begin{array}{l} y_{1,i,n}=x_{i,n}-\gamma_n(L_i^*v_{i,n}+a_{1,i,n})\\ y_{2,i,n}=v_{i,n}+\gamma_n(L_ix_{i,n}+a_{2,i,n})\end{array}\right.\\ p_{1,n}=\sum_{i=1}^m\omega_iy_{1,i,n}+\gamma_nz\\ \text{For } i=1,\ldots,m\\ \left\lfloor\begin{array}{l} p_{2,i,n}=\text{prox}_{\gamma_ng_i^*}(y_{2,i,n}-\gamma_nr_i)+b_{i,n}\\ q_{1,i,n}=p_{1,n}-\gamma_n(L_i^*p_{2,i,n}+c_{1,i,n})\\ q_{2,i,n}=p_{2,i,n}+\gamma_n(L_ip_{1,n}+c_{2,i,n})\\ x_{i,n+1}=x_{i,n}-y_{1,i,n}+q_{1,i,n}\\ v_{i,n+1}=v_{i,n}-y_{2,i,n}+q_{2,i,n}.\end{array}\right.\end{array}\right. \tag{5.104}$$

Then the following hold for some solution $\overline{x}$ to (5.102) and some solution $(\overline{v}_i)_{1\leq i\leq m}$ to (5.103) such that, for every $i\in\{1,\ldots,m\}$, $\overline{v}_i\in\partial g_i(L_i\overline{x}-r_i)$.

(i) $x_n\rightharpoonup\overline{x}$ *and, for every $i\in\{1,\ldots,m\}$, $v_{i,n}\rightharpoonup\overline{v}_i$.*

(ii) *Suppose that, for every $i\in\{1,\ldots,m\}$, g_i^* is strongly convex at $\overline{v}_i$. Then, for every $i\in\{1,\ldots,m\}$, $v_{i,n}\to\overline{v}_i$.*

Proof. Define $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{G}}$, $\boldsymbol{L}$, $\boldsymbol{V}$, $\boldsymbol{r}$, and $\boldsymbol{j}\colon\mathcal{H}\to\boldsymbol{V}$ as in the proof of Theorem 5.29. Moreover, set $\boldsymbol{f}=\iota_{\boldsymbol{V}}$ and $\boldsymbol{g}\colon\boldsymbol{\mathcal{G}}\to\left]-\infty,+\infty\right]:\boldsymbol{y}\mapsto\sum_{i=1}^m\omega_i\,g_i(y_i)$. Then, $\boldsymbol{f}\in\Gamma_0(\boldsymbol{\mathcal{H}})$, $\boldsymbol{g}\in\Gamma_0(\boldsymbol{\mathcal{G}})$, $\boldsymbol{f}^*=\iota_{\boldsymbol{V}^\perp}$, and $\boldsymbol{g}^*\colon\boldsymbol{v}\mapsto\sum_{i=1}^m\omega_i\,g_i^*(v_i)$. Therefore, (5.101) is equivalent to

$$\boldsymbol{j}(z)\in\text{ran}\left(\partial\boldsymbol{f}+\boldsymbol{L}^*\circ(\partial\boldsymbol{g})\circ(\boldsymbol{L}\cdot-\boldsymbol{r})\right). \tag{5.105}$$

Furthermore, (5.102) and (5.103) are equivalent to

$$\underset{\boldsymbol{x}\in\boldsymbol{\mathcal{H}}}{\text{minimize}}\ \boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{L}\boldsymbol{x}-\boldsymbol{r})-\langle\boldsymbol{x}\mid\boldsymbol{j}(z)\rangle_{\boldsymbol{\mathcal{H}}} \tag{5.106}$$

and

$$\underset{\boldsymbol{v}\in\boldsymbol{\mathcal{G}}}{\text{minimize}} \;\; \boldsymbol{f}^*(\boldsymbol{j}(z)-\boldsymbol{L}^*\boldsymbol{v})+\boldsymbol{g}^*(\boldsymbol{v})+\langle \boldsymbol{r}\mid \boldsymbol{v}\rangle_{\boldsymbol{\mathcal{G}}}\,, \tag{5.107}$$

respectively. On the other hand since, for every $\gamma\in\,]0,+\infty[$, $\operatorname{prox}_{\gamma\boldsymbol{f}}\colon \boldsymbol{x}\mapsto \boldsymbol{j}(\sum_{i=1}^m \omega_i x_i)$ and $\operatorname{prox}_{\gamma\boldsymbol{g}^*}=(\operatorname{prox}_{\gamma g_i^*})_{1\leq i\leq m}$, (5.104) is a particular case of (5.99). Finally, in (ii), $\boldsymbol{g}^*$ is strongly, hence uniformly, convex at $\overline{\boldsymbol{v}}$. Altogether, the results follow from Proposition 5.31. □

Remark 5.34 Suppose that (5.102) has a solution and that

$$(r_1,\ldots,r_m)\in \operatorname{sri}\big\{(L_1x-y_1,\ldots,L_mx-y_m)\;\big|\; x\in\mathcal{H},\, y_1\in\operatorname{dom}g_1,\ldots,\, y_m\in\operatorname{dom}g_m\big\}. \tag{5.108}$$

Then, with the notation of the proof of Proposition 5.33, (5.108) is equivalent to $\boldsymbol{r}\in \operatorname{sri}(\boldsymbol{L}(\boldsymbol{V})-\operatorname{dom}\boldsymbol{g})=\operatorname{sri}(\boldsymbol{L}(\operatorname{dom}\boldsymbol{f})-\operatorname{dom}\boldsymbol{g})$. Thus, Proposition 5.30(iii)(a) asserts that (5.101) holds.

5.3 Bibliographie

[1] G. Alduncin, Composition duality principles for mixed variational inequalities, *Math. Comput. Modelling,* vol. 41, pp. 639–654, 2005.

[2] H. Attouch, L. M. Briceño-Arias, and P. L. Combettes, A parallel splitting method for coupled monotone inclusions, *SIAM J. Control Optim.*, vol. 48, pp. 3246–3270, 2010.

[3] H. Attouch and M. Théra, A general duality principle for the sum of two operators, *J. Convex Anal.,* vol. 3, pp. 1–24, 1996.

[4] H. Attouch and M. Théra, A duality proof of the Hille-Yosida theorem, in : *Progress in Partial Differential Equations : the Metz Surveys,* vol. 4, pp. 18–35, 1996.

[5] H. H. Bauschke and P. L. Combettes, A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces, *Math. Oper. Res.,* vol. 26, pp. 248–264, 2001.

[6] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces.* Springer, New York, 2011.

[7] H. H. Bauschke, P. L. Combettes, and S. Reich, The asymptotic behavior of the composition of two resolvents, *Nonlinear Anal.*, vol. 60, pp. 283–301, 2005.

[8] G. Chen and M. Teboulle, A proximal-based decomposition method for convex minimization problems, *Math. Programming,* vol. 64, pp. 81–101, 1994.

[9] P. L. Combettes, Quasi-Fejérian analysis of some optimization algorithms, in : D. Butnariu, Y. Censor, S. Reich (Eds.), *Inherently Parallel Algorithms for Feasibility and Optimization*, pp. 115–152, Elsevier, Amsterdam, 2001.

[10] P. L. Combettes, Solving monotone inclusions via compositions of nonexpansive averaged operators, *Optimization,* vol. 53, pp. 475–504, 2004.

[11] P. L. Combettes, Iterative construction of the resolvent of a sum of maximal monotone operators, *J. Convex Anal.,* vol. 16, pp. 727–748, 2009.

[12] P. L. Combettes, Dinh Dũng, and B. C. Vũ, Dualization of signal recovery problems, *Set-Valued Var. Anal.,* vol. 18, pp. 373–404, 2010.

[13] P. L. Combettes and J.-C. Pesquet, Proximal splitting methods in signal processing, in : H. H. Bauschke, R. Burachik, P. L. Combettes, V. Elser, D. R. Luke, H. Wolkowicz (Eds.), *Fixed-Point Algorithms for Inverse Problems in Science and Engineering,* pp. 185–212, Springer-Verlag, New York, 2010.

[14] P. L. Combettes and V. R. Wajs, Signal recovery by proximal forward-backward splitting, *Multiscale Model. Simul.*, vol. 4, pp. 1168–1200, 2005.

[15] J. Eckstein and D. P. Bertsekas, On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators, *Math. Programming,* vol. 55, pp. 293–318, 1992.

[16] J. Eckstein and M. C. Ferris, Smooth methods of multipliers for complementarity problems, *Math. Programming,* vol. 86, pp. 65–90, 1999.

[17] J. Eckstein and B. F. Svaiter, A family of projective splitting methods for the sum of two maximal monotone operators, *Math. Programming,* vol. 111, pp. 173–199, 2008.

[18] I. Ekeland and R. Temam, *Analyse Convexe et Problèmes Variationnels,* Dunod, Paris, 1974 ; *Convex Analysis and Variational Problems,* SIAM, Philadelphia, PA, 1999.

[19] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer-Verlag, New York, 2003.

[20] M. Fukushima, The primal Douglas-Rachford splitting algorithm for a class of monotone mappings with applications to the traffic equilibrium problem, *Math. Programming,* vol. 72, pp. 1–15, 1996.

[21] M. Fortin and R. Glowinski (eds.), *Augmented Lagrangian Methods : Applications to the Numerical Solution of Boundary Value Problems*, North-Holland, Amsterdam, 1983.

[22] D. Gabay, Applications of the method of multipliers to variational inequalities, in : M. Fortin and R. Glowinski (Eds.), *Augmented Lagrangian Methods : Applications to the Numerical Solution of Boundary Value Problems,* pp. 299–331, North-Holland, Amsterdam, 1983.

[23] R. Glowinski and P. Le Tallec (eds.), *Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics*, SIAM, Philadelphia, 1989.

[24] P.-L. Lions and B. Mercier, Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Numer. Anal.*, vol. 16, pp. 964–979, 1979.

[25] L. McLinden, An extension of Fenchel's duality theorem to saddle functions and dual minimax problems, *Pacific J. Math.,* vol. 50, pp. 135–158, 1974.

[26] B. Mercier, *Topics in Finite Element Solution of Elliptic Problems* (Lectures on Mathematics, no. 63), Tata Institute of Fundamental Research, Bombay, 1979.

[27] B. Mercier, *Inéquations Variationnelles de la Mécanique* (Publications Mathématiques d'Orsay, no. 80.01), Université de Paris-XI, Orsay, France, 1980.

[28] U. Mosco, Dual variational inequalities, *J. Math. Anal. Appl.,* vol. 40, pp. 202–206, 1972.

[29] T. Pennanen, Dualization of generalized equations of maximal monotone type, *SIAM J. Optim.,* vol. 10, pp. 809–835, 2000.

[30] T. Pennanen, A splitting method for composite mappings, *Numer. Funct. Anal. Optim.*, vol. 23, pp. 875–890, 2002.

[31] S. M. Robinson, Composition duality and maximal monotonicity, *Math. Programming,* vol. 85, pp. 1–13, 1999.

[32] S. M. Robinson, Generalized duality in variational analysis, in : N. Hadjisavvas and P. M. Pardalos (Eds.), *Advances in Convex Analysis and Global Optimization,* pp. 205–219, Dordrecht, The Netherlands, Kluwer, 2001.

[33] R. T. Rockafellar, Duality and stability in extremum problems involving convex functions, *Pacific J. Math.,* vol. 21, pp. 167–187, 1967.

[34] R. T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM J. Control Optim.,* vol. 14, pp. 877–898, 1976.

[35] R. T. Rockafellar, Augmented Lagrangians and applications of the proximal point algorithm in convex programming, *Math. Oper. Res.,* vol. 1, 97–116, 1976.

[36] R. T. Rockafellar, *Conjugate Duality and Optimization*, SIAM, Philadelphia, PA, 1974.

[37] B. F. Svaiter, On weak convergence of the Douglas-Rachford method, *SIAM J. Control Optim.,* vol. 49, pp. 280–287, 2011.

[38] P. Tseng, Further applications of a splitting algorithm to decomposition in variational inequalities and convex programming, *Math. Programming,* vol. 48, no. 2, pp. 249–263, 1990.

[39] P. Tseng, Applications of a splitting algorithm to decomposition in convex programming and variational inequalities, *SIAM J. Control Optim.,* vol. 29, pp. 119–138, 1991.

[40] P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, *SIAM J. Control Optim.*, vol. 38, pp. 431–446, 2000.

[41] C. Zălinescu, *Convex Analysis in General Vector Spaces,* World Scientific, River Edge, NJ, 2002.

[42] E. Zeidler, *Nonlinear Functional Analysis and Its Applications,* vols. I–V, Springer-Verlag, New York, 1985–1993.

Chapitre 6

Décomposition de domaine dans les équations aux dérivées partielles

6.1 Description et résultats principaux

Ce chapitre est consacré aux problèmes de décomposition de domaine dans les équations aux dérivées partielles. Un de ces objectifs principaux est la résolution d'équations aux dérivées partielles et des problèmes de frontière associés par la décomposition du domaine original en des sous-domaines simples (voir [11, 17, 23, 25]).

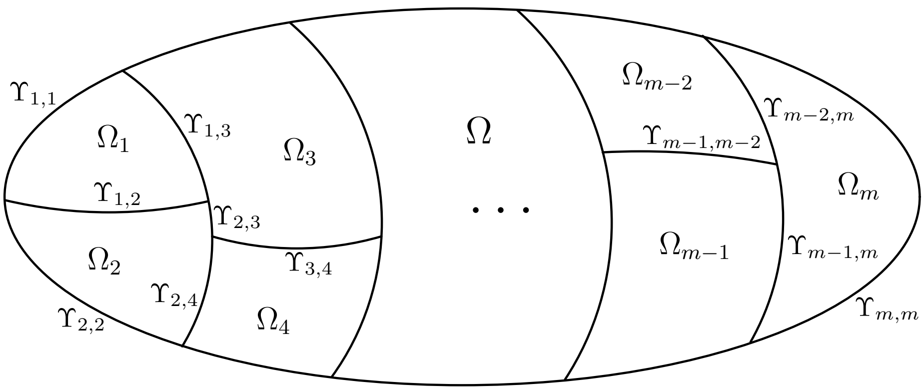


FIGURE 6.1 – Décomposition de domaine Ω.

Notre méthode repose sur un problème de minimisation. Le domaine original Ω est divisé en m sous-domaines $(\Omega_i)_{1\le i\le m}$ (voir Figure 6.1) et nous cherchons une solution

$(u_i)_{1\leq i\leq m}$ de

$$\underset{u_1\in\mathcal{H}_1,\ldots,u_m\in\mathcal{H}_m}{\text{minimiser}} \quad \sum_{i=1}^{m}\varphi_i(u_i)+\sum_{i=1}^{m-1}\sum_{j\in J(i+)}\psi_{ij}(\mathsf{T}_{ij}\,u_i-\mathsf{T}_{ji}\,u_j). \tag{6.1}$$

Dans cette formulation, chaque fonction $u_i\colon\Omega_i\to\mathbb{R}$ appartient à un espace de Sobolev hilbertien $\mathcal{H}_i$ approprié. De plus, $\varphi_i\colon\mathcal{H}_i\to\,]-\infty,+\infty]$ est une fonction semi-continue inférieurement et convexe, $J(i+)$ est l'ensemble d'indices $j>i$ des interfaces actives Υ_{ij} de Ω_i, $\mathsf{T}_{ij}\colon\mathcal{H}_i\to L^2(\Upsilon_{ij})$ est l'opérateur de trace relatif à l'interface Υ_{ij} et $\psi_{ij}\colon L^2(\Upsilon_{ij})\to\,]-\infty,+\infty]$ est une fonction semi-continue inférieurement et convexe. La composante découplée $(u_i)_{1\leq i\leq m}\mapsto\sum_{i=1}^m\varphi_i(u_i)$ dans (6.1) peut être non lisse et peut prendre la valeur $+\infty$. Dans certaines applications que nous examinerons ultérieurement, les fonctions $(\varphi_i)_{1\leq i\leq m}$ sont des potentiels d'énergie associées à des équations aux dérivées partielles définies sur les sous-domaines. Notre formulation peut incorporer des potentiels non quadratiques et, en particulier, elle peut modéliser des problèmes avec l'opérateur p-Laplacien. Par surcroît, vu que les potentiels $(\varphi_i)_{1\leq i\leq m}$ ne doivent pas être toujours finis, c'est possible d'inclure des contraintes sur chaque u_i. Par ailleurs, le potentiel de couplage $(u_i)_{1\leq i\leq m}\mapsto\sum_{i=1}^{m-1}\sum_{j\in J(i+)}\psi_{ij}(\mathsf{T}_{ij}\,u_i-\mathsf{T}_{ji}\,u_j)$ modélise des conditions de transmission à travers des interfaces. En particulier, en prenant $((\psi_{ij})_{j\in J(i+)})_{1\leq i\leq m}$ comme les fonctions indicatrices de $\{0\}$, le terme de couplage prend la valeur zéro si les sauts à travers des interfaces des fonctions $(u_i)_{1\leq i\leq m}$ sont nuls et il prend la valeur $+\infty$ ailleurs, d'où découle la propriété de continuité à travers les interfaces. Une grande avantage de cette approche est sa flexibilité : on peut traiter d'une manière unifiée des conditions de transmission non linéaires et/ou unilatérales, et divers potentiels d'énergie dans les sous-domaines.

Nous appliquons l'algorithme d'éclatement décrit dans la Proposition 5.8 à la résolution de (6.1), en l'adaptant à un cadre multicomposante sur des espaces de Sobolev hilbertiens. L'algorithme obtenu résout (6.1) d'une manière complètement décomposée : chaque pas élémentaire met en scène les constituants du problème (principalement u_i, φ_i, ψ_{ij}, et T_{ij}) séparément. De plus, la méthode résout aussi le problème dual associé à (6.1), c'est-à-dire

$$\underset{\substack{i\in I,j\in J(i+)\\ g_{ij}\in L^2(\Upsilon_{ij})}}{\text{minimiser}} \quad \sum_{i=1}^{m}\varphi_i^*\Bigg(\sum_{j\in J(i-)}\mathsf{T}_{ij}^*g_{ji}-\sum_{j\in J(i+)}\mathsf{T}_{ij}^*g_{ij}\Bigg)+\sum_{i=1}^{m-1}\sum_{j\in J(i+)}\psi_{ij}^*(g_{ij}), \tag{6.2}$$

où T_{ij}^* est l'opérateur adjoint de l'opérateur de trace T_{ij}, et φ_i^* et ψ_{ij}^* sont les conjugués de Legendre-Fenchel de φ_i et ψ_{ij}, respectivement. Dans ce problème dual la variable g_{ij} modélise la tension à travers l'interface Υ_{ij}.

Avant présenter en détail notre problème et l'algorithme pour le résoudre, nous présentons ci-dessous la notation utilisée dans ce chapitre et les hypothèses générales du problème.

Notation. Soit $\mathbb{R}^N$ l'espace euclidien de dimension N et notons par $|\cdot|$ sa norme, où $N \geq 2$. Soit Ω un sous-ensemble de $\mathbb{R}^N$ non vide, borné et ouvert avec frontière lipschitzienne $\operatorname{bdry}\Omega$. L'espace $H^1(\Omega) = \{u \in L^2(\Omega) \mid Du \in (L^2(\Omega))^N\}$, où D représente le gradient faible, est un espace hilbertien muni du produit scalaire $\langle \cdot \mid \cdot \rangle_{H^1(\Omega)} \colon (u, v) \mapsto \int_\Omega uv + \int_\Omega (Du)^\top Dv$. On note par S la mesure de surface définie sur $\operatorname{bdry}\Omega$ [22, Section 1.1.3]. De plus, soit Υ un sous-ensemble non vide et ouvert de $\operatorname{bdry}\Omega$ et soit $L^2(\Upsilon)$ l'espace de fonctions définies sur Υ dont le carré est S-intégrable. Muni du produit scalaire

$$\langle \cdot \mid \cdot \rangle_{L^2(\Upsilon)} \colon (g, h) \mapsto \int_\Upsilon gh\, dS, \tag{6.3}$$

$L^2(\Upsilon)$ est un espace hilbertien. L'opérateur de trace associé à Ω est l'unique opérateur $\mathsf{T} \in \mathcal{B}\left(H^1(\Omega), L^2(\operatorname{bdry}\Omega)\right)$ tel que $(\forall u \in \mathcal{C}^1(\overline{\Omega}))\ \mathsf{T}u = u|_{\operatorname{bdry}\Omega}$. Muni du produit scalaire

$$\langle \cdot \mid \cdot \rangle \colon (u, v) \mapsto \int_\Omega (Du)^\top Dv, \tag{6.4}$$

l'espace $H^1_{0,\Upsilon}(\Omega) = \{u \in H^1(\Omega) \mid \mathsf{T}\,u = 0 \text{ sur } \Upsilon\}$ est un espace hilbertien [27, Section 25.10]. Finalement, pour S-presque partout $\omega \in \operatorname{bdry}\Omega$, il existe un vecteur normal unitaire orienté vers l'extérieur $\nu(\omega)$. Pour détails et compléments, voir [1, 13, 16, 22, 27].

Hypothèse 6.1 Soit $m \geq 2$ un entier et soit $I = \{1, \dots, m\}$.

(A1) Ω est un sous-ensemble ouvert borné de $\mathbb{R}^N$ avec frontière lipschitzienne $\operatorname{bdry}\Omega$.

(A2) $(\Omega_i)_{i\in I}$ sont des sous-ensembles disjoints ouverts de Ω (voir Fig. 6.1) tels que les frontières $(\operatorname{bdry}\Omega_i)_{i\in I}$ sont lipschitziennes, $\overline{\Omega} = \bigcup_{i=1}^m \overline{\Omega_i}$ et

$$(\forall i \in I) \quad \Upsilon_{ii} = \operatorname{int}_{\operatorname{bdry}\Omega}(\operatorname{bdry}\Omega_i \cap \operatorname{bdry}\Omega) \neq \varnothing, \tag{6.5}$$

où $\operatorname{int}_{\operatorname{bdry}\Omega}$ est l'intérieur relatif à $\operatorname{bdry}\Omega$.

(A3) Pour tout $i \in I$, soit

$$(\forall j \in \{i+1, \dots, m\}) \quad \Upsilon_{ij} = \Upsilon_{ji} = \operatorname{int}_{\operatorname{bdry}\Omega_i}(\operatorname{bdry}\Omega_i \cap \operatorname{bdry}\Omega_j). \tag{6.6}$$

L'ensemble d'indices d'interfaces actives de Ω_i

$$J(i) = \{j \in I \smallsetminus \{i\} \mid \Upsilon_{ij} \neq \varnothing\} \tag{6.7}$$

est non vide.

(A4) Pour tout $i \in I$, $J(i-) = J(i) \cap \{1, \dots, i-1\}$ et $J(i+) = J(i) \cap \{i+1, \dots, m\}$, en considérant la convention $J(1-) = J(m+) = \varnothing$.

(A5) Pour tout $i \in I$, $\mathsf{T}_i \colon H^1(\Omega_i) \to L^2(\operatorname{bdry}\Omega_i)$ est l'opérateur de trace,

$$\mathcal{H}_i = H^1_{0,\Upsilon_{ii}}(\Omega_i) = \{u \in H^1(\Omega_i) \mid \mathsf{T}_i\, u = 0 \ \text{ sur } \Upsilon_{ii}\}, \tag{6.8}$$

et, pour tout $j \in J(i)$, $\mathsf{T}_{ij} \colon \mathcal{H}_i \to L^2(\Upsilon_{ij}) \colon u \mapsto (\mathsf{T}_i u)|_{\Upsilon_{ij}}$.

(A6) Pour tout $i \in I$,

$$\mathcal{G}_i = \bigoplus_{j \in J(i)} L^2(\Upsilon_{ij}), \tag{6.9}$$

$\nu_i(\omega)$ est le vecteur normale unitaire orienté vers l'extérieur sur $\omega \in \operatorname{bdry} \Omega_i$, et Q_i est l'opérateur qui associe, pour tout $(f, (g_j)_{j \in J(i)})$ dans $L^2(\Omega_i) \times \mathcal{G}_i$ l'unique solution faible dans $\mathcal{H}_i$ du problème Dirichlet-Neumann

$$\begin{cases} -\Delta u = f & \text{dans } \Omega_i, \\ u = 0 & \text{sur } \Upsilon_{ii}, \\ \nu_i^\top Du = g_j & \text{sur } \Upsilon_{ij}, \text{ pour tout } \; j \in J(i+), \\ \nu_i^\top Du = -g_j & \text{sur } \Upsilon_{ij}, \text{ pour tout } \; j \in J(i-). \end{cases} \tag{6.10}$$

Notons que, puisque $\operatorname{bdry} \Omega_i = \overline{\Upsilon_{ii}} \cup \overline{\bigcup_{j \in J(i+)} \Upsilon_{ij}} \cup \overline{\bigcup_{j \in J(i-)} \Upsilon_{ij}}$, l'existence et unicité de la solution de (6.10) sont assurées par [27, Theorem 25.I], d'où nous déduisons que Q_i est linéaire.

Problème 6.2 Supposons que l'Hypothèse 6.1 soit satisfaite, pour tout $i \in I$, soit $\varphi_i \in \Gamma_0(\mathcal{H}_i)$ et, pour tout $j \in J(i+)$, soit $\psi_{ij} \in \Gamma_0(L^2(\Upsilon_{ij}))$. Assumons que, pour tout $i \in I$, il existe $u_i \in \mathcal{H}_i$ et, pour tout $j \in J(i+)$, il existe $g_{ij} \in L^2(\Upsilon_{ij})$ telles que

$$(\forall i \in I) \quad \begin{cases} -Q_i\big(0, (g_{ij})_{j \in J(i+)}, (g_{ji})_{j \in J(i-)}\big) \in \partial \varphi_i(u_i) \\ (\forall j \in J(i+)) \quad g_{ij} \in \partial \psi_{ij}(\mathsf{T}_{ij} u_i - \mathsf{T}_{ji} u_j). \end{cases} \tag{6.11}$$

Le problème primal est

$$\underset{u_1 \in \mathcal{H}_1, \ldots, u_m \in \mathcal{H}_m}{\text{minimiser}} \quad \sum_{i=1}^m \varphi_i(u_i) + \sum_{i=1}^{m-1} \sum_{j \in J(i+)} \psi_{ij}(\mathsf{T}_{ij} u_i - \mathsf{T}_{ji} u_j), \tag{6.12}$$

et le problème dual est

$$\underset{\substack{i \in I, j \in J(i+) \\ g_{ij} \in L^2(\Upsilon_{ij})}}{\text{minimiser}} \quad \sum_{i=1}^m \varphi_i^*\Big(-Q_i\big(0, (g_{ij})_{j \in J(i+)}, (g_{ji})_{j \in J(i-)}\big)\Big) + \sum_{i=1}^{m-1} \sum_{j \in J(i+)} \psi_{ij}^*(g_{ij}). \tag{6.13}$$

Algorithme 6.3 On suppose que l'Hypothèse 6.1 soit satisfaite et, pour tout $i \in I$, soit $u_{i,0} \in \mathcal{H}_i$, soient $(a_{1i,n})_{n \in \mathbb{N}}$, $(b_{1i,n})_{n \in \mathbb{N}}$ et $(c_{1i,n})_{n \in \mathbb{N}}$ des suites absolument sommables dans $\mathcal{H}_i$, pour tout $j \in J(i+)$, soit $g_{ij,0} \in L^2(\Upsilon_{ij})$, soient $(a_{2ij,n})_{n \in \mathbb{N}}$, $(b_{2ij,n})_{n \in \mathbb{N}}$ et $(c_{2ij,n})_{n \in \mathbb{N}}$ des suites absolument sommables dans $L^2(\Upsilon_{ij})$, et soit $\beta_{ij} \in \left]0, \|\mathsf{T}_{ij}\|\right[$. Posons $\beta = (\sum_{i=1}^{m-1} \sum_{j \in J(i+)} \beta_{ij}^2)^{1/2}$, soit $\varepsilon \in \left]0, 1/(\beta+1)\right[$ et soit $(\gamma_n)_{n \in \mathbb{N}}$ une suite dans $[\varepsilon, (1-\varepsilon)/\beta]$. On génère des suites $(u_{1,n})_{n \in \mathbb{N}}, \ldots, (u_{m,n})_{n \in \mathbb{N}}$ et $(((g_{ij,n})_{n \in \mathbb{N}})_{j \in J(i+)})_{i \in I}$

comme suit.

$$
\begin{array}{l}
\text{For } n = 1, 2, \ldots \\
\left\lfloor
\begin{array}{l}
\text{Pour tout } \; i \in I \\
\left\lfloor
\begin{array}{l}
v_{i,n} = u_{i,n} - \gamma_n \Big(Q_i \big(0, (g_{ij,n})_{j \in J(i+)}, (g_{ji,n})_{j \in J(i-)} \big) + a_{1i,n} \Big) \\
p_{i,n} = \left(\underset{w \in \mathcal{H}_i}{\operatorname{argmin}} \; \gamma_n \varphi_i(w) + \dfrac{1}{2} \displaystyle\int_{\Omega_i} \big| Dw - Dv_{i,n} \big|^2 \right) + b_{1i,n} \\
\text{Pour tout } \; j \in J(i+) \\
\left\lfloor
\begin{array}{l}
z_{ij,n} = g_{ij,n} + \gamma_n (\mathsf{T}_{ij} u_{i,n} - \mathsf{T}_{ji} u_{j,n} + a_{2ij,n}) \\
q_{ij,n} = z_{ij,n} - \gamma_n \left(\underset{w \in L^2(\Upsilon_{ij})}{\operatorname{argmin}} \; \dfrac{1}{\gamma_n} \psi_{ij}(w) + \dfrac{1}{2} \displaystyle\int_{\Upsilon_{ij}} \left| w - \dfrac{z_{ij,n}}{\gamma_n} \right|^2 dS \right) + b_{2ij,n}
\end{array}
\right.
\end{array}
\right. \\
\text{Pour tout } \; i \in I \\
\left\lfloor
\begin{array}{l}
r_{i,n} = p_{i,n} - \gamma_n \Big(Q_i \big(0, (q_{ij,n})_{j \in J(i+)}, (q_{ji,n})_{j \in J(i-)} \big) + c_{1i,n} \Big) \\
u_{i,n+1} = u_{i,n} - v_{i,n} + r_{i,n} \\
\text{Pour tout } \; j \in J(i+) \\
\left\lfloor
\begin{array}{l}
s_{ij,n} = q_{ij,n} + \gamma_n \big(\mathsf{T}_{ij} p_{i,n} - \mathsf{T}_{ji} p_{j,n} + c_{2ij,n} \big) \\
g_{ij,n+1} = g_{ij,n} - z_{ij,n} + s_{ij,n}.
\end{array}
\right.
\end{array}
\right.
\end{array}
\right.
\end{array}
\tag{6.14}
$$

Dans la suite nous présentons le comportement asymptotique de l'Algorithme 6.3, ce qui est le résultat principal de ce chapitre. Nous présentons d'abord la definition suivante qui sera utile par la suite.

Définition 6.4 Une fonction $\Phi \colon \mathcal{H} \to]-\infty, +\infty]$ satisfait la *propriété Kadec'–Klee* si, pour toute suite $(u_n)_{n\in\mathbb{N}}$ dans $\operatorname{dom}\Phi$ et pour tout $u \in \mathcal{H}$, nous avons

$$
\begin{cases} u_n \rightharpoonup u \\ \varlimsup \Phi(u_n) \leq \Phi(u) \end{cases} \quad \Rightarrow \quad u_n \to u. \tag{6.15}
$$

Théorème 6.5 *Soient $(u_{1,n})_{n\in\mathbb{N}}, \ldots, (u_{m,n})_{n\in\mathbb{N}}$ et $(((g_{ij,n})_{n\in\mathbb{N}})_{j\in J(i+)})_{i\in I}$ les suites générées par l'Algorithme 6.15. Alors nous avons les résultats suivants pour quelque solution $(\overline{u}_1, \ldots, \overline{u}_m)$ de (6.65) et quelque solution $((\overline{g}_{ij})_{j\in J(i+)})_{i\in I}$ de (6.66) telles que (6.64) est satisfait.*

(i) *Pour tout $i \in I$ et $j \in J(i+)$, $u_{i,n} - p_{i,n} \to 0$ et $g_{ij,n} - q_{ij,n} \to 0$.*

(ii) *Pour tout $i \in I$ et $j \in J(i+)$, $u_{i,n} \rightharpoonup \overline{u}_i$, $p_{i,n} \rightharpoonup \overline{u}_i$, $g_{ij,n} \rightharpoonup \overline{g}_{ij}$, et $q_{ij,n} \rightharpoonup \overline{g}_{ij}$.*

(iii) *Supposons que la fonction*

$$
\boldsymbol{\mathcal{H}} \to]-\infty, +\infty] : (u_i)_{i\in I} \mapsto \sum_{i\in I} \varphi_i(u_i) \tag{6.16}
$$

satisfasse la propriété Kadec'–Klee sur $\boldsymbol{\mathcal{H}}$. Alors, pour tout $i \in I$, $u_{i,n} \to \overline{u}_i$ et $p_{i,n} \to \overline{u}_i$.

(iv) *Supposons que la fonction*

$$\boldsymbol{\mathcal{G}} \to \,]-\infty,+\infty] : ((g_{ij})_{j\in J(i+)})_{i\in I} \mapsto \sum_{i\in I}\sum_{j\in J(i+)} \psi_{ij}^*(g_{ij}) \tag{6.17}$$

satisfasse la propriété Kadec'–Klee sur $\boldsymbol{\mathcal{H}}$. Alors, pour tout $i \in I$ et $j \in J(i+)$, $g_{ij,n} \to \overline{g}_{ij}$ et $q_{ij,n} \to \overline{g}_{ij}$.

Enfin, nous appliquons l'Algorithme 6.3 aux problèmes de décomposition de domaine avec des conditions de continuité aux interfaces pour deux équations aux dérivées partielles classiques. La première application est consacrée au problème de Poisson avec des conditions de Dirichlet à la frontière

$$\begin{cases} -\Delta u = f, & \text{dans } \Omega; \\ u = 0, & \text{sur } \operatorname{bdry}\Omega, \end{cases} \tag{6.18}$$

dont la solution faible est la solution unique du problème de minimisation (voir [14, Chapter IV.2.1] et [27, Chapter 25.9])

$$\underset{u\in H_0^1(\Omega)}{\text{minimiser}} \;\; \frac{1}{2}\int_\Omega |Du|^2 - \int_\Omega fu. \tag{6.19}$$

Nous résolvons le problème suivant, qui est obtenu lorsque le domaine Ω est décomposé en m sous-domaines $\Omega_1, \ldots, \Omega_m$.

Problème 6.6 Soit $f \in L^2(\Omega)$. Supposons que l'Hypothèse 6.1 soit satisfaite et que, pour tout $i \in I$ et $j \in J(i+)$, Υ_{ij} et $\operatorname{bdry}\Omega$ soient de classe C^2. Le problème est de

$$\underset{\substack{u_1\in\mathcal{H}_1,\ldots,u_m\in\mathcal{H}_m \\ (\forall i\in I)(\forall j\in J(i+))\ \mathsf{T}_{ij}u_i=\mathsf{T}_{ji}u_j}}{\text{minimiser}} \;\; \sum_{i=1}^m \frac{1}{2}\int_{\Omega_i} |Du_i|^2 - \int_{\Omega_i} fu_i. \tag{6.20}$$

Nous montrons que le Problème 6.6 a une solution unique qui permet de construire la solution du problème (6.19), et nous appliquons l'Algorithme 6.3 au Problème 6.6 pour trouver cette solution. La convergence forte des itérés générés par l'Algorithme 6.3 est garantie.

La seconde application est consacrée au problème de p-Laplacien avec des conditions de Dirichlet à la frontière

$$\begin{cases} -\operatorname{div}(|Du|^{p-2}Du) = f, & \text{dans } \Omega; \\ u = 0, & \text{sur } \operatorname{bdry}\Omega, \end{cases} \tag{6.21}$$

dont la solution faible est la solution unique du problème de minimisation (voir [14, Section IV.2.2])

$$\underset{u\in W_0^{1,p}(\Omega)}{\text{minimiser}} \;\; \frac{1}{p}\int_\Omega |Du|^p - \int_\Omega fu. \tag{6.22}$$

Nous résolvons le problème suivant, qui est obtenu lorsque le domaine Ω est décomposé en m sous-domaines $\Omega_1, \ldots, \Omega_m$.

Problème 6.7 Soit $p \in \,]1, +\infty[$ et soit $f \in L^\infty(\Omega)$. Supposons que l'Hypothèse 6.1 soit satisfaite et que $\operatorname{bdry} \Omega$ et $(\Upsilon_{ij})_{i,j \in I}$ soient de classe C^2. De plus, posons

$$(\forall i \in I) \quad E_i^p = \big\{ u \in W^{1,p}(\Omega_i) \mid \mathsf{T}_i u = 0 \text{ sur } \Upsilon_{ii} \big\}. \tag{6.23}$$

Le problème est de

$$\underset{\substack{u_1 \in E_1^p, \ldots, u_m \in E_m^p \\ (\forall i \in I)(\forall j \in J(i+)) \; \mathsf{T}_{ij} u_i = \mathsf{T}_{ji} u_j}}{\text{minimiser}} \quad \sum_{i=1}^m \frac{1}{p} \int_{\Omega_i} |Du_i|^p - \int_{\Omega_i} f u_i. \tag{6.24}$$

Nous montrons que le Problème 6.7 a une solution unique qui permet de construire la solution du problème (6.21), et nous appliquons l'Algorithme 6.3 au Problème 6.7 pour trouver cette solution. La convergence forte dans $W^{1,p}(\Omega_1) \times \cdots \times W^{1,p}(\Omega_m)$ des itérés générés par l'Algorithme 6.3 est garantie sur certaines conditions de régularité de la solution.

6.2 Article en anglais

DOMAIN DECOMPOSITION FOR PARTIAL DIFFERENTIAL EQUATIONS VIA MONOTONE OPERATOR SPLITTING METHODS [1]

Abstract : We propose a primal-dual parallel proximal splitting method for domain decomposition of linear and nonlinear partial differential equations. It is obtained as an adaptation of a general splitting framework recently proposed for structured monotone inclusion problems. A key feature of our approach is that the continuity of the solution across the interfaces of the subdomains is enforced by modeling it as a constraint in the variational formulation. Weak convergence is obtained under mild regularity conditions, and sufficient conditions are provided for strong convergence. Our method can handle a wide range of linear and nonlinear problems, including unilateral transmission conditions through the interfaces (semipermeable membrane, fissured material), and p-Laplacian operators.

6.2.1 Introduction

Domain decomposition is an active, interdisciplinary research area concerned with the analysis, and implementation of coupling and decoupling strategies in mathematical and computational models of natural and engineered systems. One of the primary

1. H. Attouch, L. M. Briceño-Arias, and P. L. Combettes, Domain decomposition for partial differential equations via monotone operator splitting methods, prépublication.

objectives of domain decomposition is to solve partial differential equations and the associated boundary value problems on complex geometries, by partitioning the original domain in smaller, simpler subdomains. Among the rich literature which has been devoted to the subject, let us mention [11, 17, 23, 25]. In this paper we consider the case of non-overlapping domain decomposition, in which subdomains intersect only on their interfaces. This scenario allows for the design of approximation methods built from different discretizations in the different subdomains.

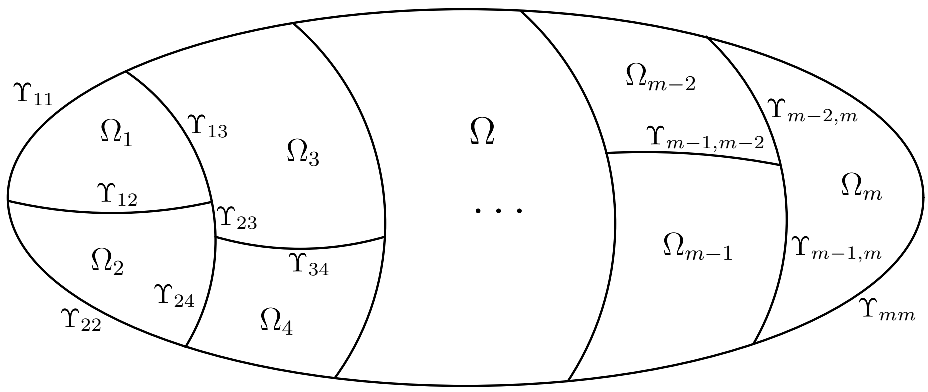


FIGURE 6.2 – Decomposition of the domain Ω.

Our method relies on a variational approach. The original domain Ω is partitioned into m subdomains $(\Omega_i)_{1\leq i\leq m}$ (see Figure 6.2), and we look for a solution $(\bar{u}_i)_{1\leq i\leq m}$ of the minimization problem

$$\underset{u_1\in\mathcal{H}_1,\dots,\,u_m\in\mathcal{H}_m}{\text{minimize}} \quad \sum_{i=1}^{m}\varphi_i(u_i)+\sum_{i=1}^{m-1}\sum_{j\in J(i+)}\psi_{ij}(\mathsf{T}_{ij}\,u_i-\mathsf{T}_{ji}\,u_j). \tag{6.25}$$

In this formulation, each function $\bar{u}_i\colon\Omega_i\to\mathbb{R}$ is the restriction to Ω_i of the solution to the original problem in Ω, and it lies in a suitable Hilbert Sobolev space $\mathcal{H}_i$. Moreover, $\varphi_i\colon\mathcal{H}_i\to\left]-\infty,+\infty\right]$ is a lower semicontinuous convex function, $J(i+)$ is the set of indices $j>i$ of the active interfaces Υ_{ij} of Ω_i, $\mathsf{T}_{ij}\colon\mathcal{H}_i\to L^2(\Upsilon_{ij})$ denotes the trace operator relative to the interface Υ_{ij}, and $\psi_{ij}\colon L^2(\Upsilon_{ij})\to\left]-\infty,+\infty\right]$ is lower semicontinuous and convex. The decoupled component $(u_i)_{1\leq i\leq m}\mapsto\sum_{i=1}^m\varphi_i(u_i)$ in (6.25) need be not smooth and may take on the value $+\infty$. In the applications presented in Section 6.2.5, $(\varphi_i)_{1\leq i\leq m}$ are energy functionals associated to some partial differential equations on the subdomains. Our formulation can deal with non quadratic functionals, and thereby captures for instance p-Laplacian problems or minimal surface problems. Furthermore, it is possible to impose hard constraints on $(u_i)_{1\leq i\leq m}$ since the potentials $(\varphi_i)_{1\leq i\leq m}$ need not be everywhere finite. On the other hand, the coupling function $(u_i)_{1\leq i\leq m}\mapsto\sum_{i=1}^{m-1}\sum_{j\in J(i+)}\psi_{ij}(\mathsf{T}_{ij}\,u_i-\mathsf{T}_{ji}\,u_j)$ models transmission conditions through the interfaces. In particular, by taking $((\psi_{ij})_{j\in J(i+)})_{1\leq i\leq m}$ to be indicator functions of $\{0\}$, the coupling term takes the value zero if the jumps of the functions $(u_i)_{1\leq i\leq m}$ across the interfaces are equal to zero and $+\infty$ elsewhere, whence the continuity property through the interfaces. A major advantage of this approch is its flexibility : one can treat in a unified fashion unilateral and/or nonlinear transmission conditions, and surface energy functionals.

To solve (6.25) we bring into play a proximal splitting method recently developed in [10] for solving convex minimization problems. This method will be adapted to solve the multicomponent variational problem (6.25) in a fully decomposed fashion, in that each elementary step of the algorithm involves the constituents of the problem (namely u_i, φ_i, ψ_{ij}, and T_{ij}) separately. In addition, it also solves the dual problem associated with (6.25), that is

$$\underset{\substack{i\in I, j\in J(i+)\\ g_{ij}\in L^2(\Upsilon_{ij})}}{\text{minimize}} \sum_{i=1}^{m}\varphi_i^*\left(\sum_{j\in J(i-)}\mathsf{T}_{ij}^*g_{ji}-\sum_{j\in J(i+)}\mathsf{T}_{ij}^*g_{ij}\right)+\sum_{i=1}^{m-1}\sum_{j\in J(i+)}\psi_{ij}^*(g_{ij}), \tag{6.26}$$

where T_{ij}^* denotes the adjoint of the trace operator T_{ij}, and φ_i^* and ψ_{ij}^* denote for the conjugates of φ_i and ψ_{ij}, respectively. In this dual problem the variables g_{ij} model tensions at the interfaces Υ_{ij}.

The paper is organized as follows. In Section 6.2.2 we present the notation and the algorithm which is the basis of our method. In addition, we examine some conditions for obtaining strong convergence. In Section 6.2.3 we formally state the domain decomposition problem and its assumptions. The algorithm for solving (6.25)–(6.26) is proposed and analyzed in Section 6.2.4, where we propose our algorithm and we study its asymptotic behavior. Finally, in Section 6.2.5, we present some applications to the domain decomposition in the Poisson problem, and the p–Laplacian problem.

6.2.2 Notation and preliminaries

Throughout the paper the following notation is adopted. We denote by $\mathbb{R}^N$ the usual N-dimensional Euclidean space and by $|\cdot|$ its norm, where $N\geq 2$. Weak and strong convergence are denoted by $\rightharpoonup$ and $\to$, respectively. Let Ω be a nonempty open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary $\operatorname{bdry}\Omega$. We denote by x a generic element of Ω, and by dx the restriction to Ω of the Lebesgue measure on $\mathbb{R}^N$. For every $p\in\,]1,+\infty[$, the space $W^{1,p}(\Omega)=\big\{v\in L^p(\Omega)\;\big|\;Dv\in(L^p(\Omega))^N\big\}$, where D denotes the weak gradient (derivatives in the distribution's sense), is a Banach space. In particular, we denote by $H^1(\Omega)=W^{1,2}(\Omega)$, which is a Hilbert space with scalar product $\langle\cdot\mid\cdot\rangle_{H^1(\Omega)}\colon(u,v)\mapsto\int_\Omega uv+\int_\Omega(Du)^\top Dv$. We denote by S the surface measure on $\operatorname{bdry}\Omega$ [22, Section 1.1.3]. Now let Υ be a nonempty open set in $\operatorname{bdry}\Omega$ and let $L^2(\Upsilon)$ be the space of square

S-integrable functions on Υ. Endowed with the scalar product

$$\langle \cdot \mid \cdot \rangle_{L^2(\Upsilon)} : (v, w) \mapsto \int_\Upsilon vw \, dS, \tag{6.27}$$

$L^2(\Upsilon)$ is a Hilbert space. The Sobolev trace operator $\mathsf{T}\colon W^{1,p}(\Omega) \to L^2(\operatorname{bdry}\Omega)$ is the unique linear bounded operator such that $(\forall v \in \mathcal{C}^1(\overline{\Omega}))\ \mathsf{T}v = v|_{\operatorname{bdry}\Omega}$. Endowed with the scalar product

$$\langle \cdot \mid \cdot \rangle : (u, v) \mapsto \int_\Omega (Du)^\top Dv, \tag{6.28}$$

the space $H^1_{0,\Upsilon}(\Omega) = \big\{u \in H^1(\Omega) \mid \mathsf{T}\,u = 0 \text{ on } \Upsilon\big\}$ is a Hilbert space [27, Section 25.10]. Finally, for S-almost every $\omega \in \operatorname{bdry}\Omega$, there exists a unit outward normal vector $\nu(\omega)$. For details and complements, see [1, 13, 16, 22, 26, 27].

Let $\mathcal{H}$ be a real Hilbert space and denote by $\Gamma_0(\mathcal{H})$ the set of lower semicontinuous convex functions $\varphi\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ which are proper in the sense that $\operatorname{dom}\varphi = \big\{u \in \mathcal{H} \mid \varphi(u) < +\infty\big\}$ is nonempty. Now let $\varphi \in \Gamma_0(\mathcal{H})$. The subdifferential of φ is the maximally monotone operator

$$\partial\varphi\colon \mathcal{H} \to 2^{\mathcal{H}}\colon u \mapsto \big\{u^* \in \mathcal{H} \mid (\forall v \in \mathcal{H})\ \ \varphi(u) + \langle v - u \mid u^*\rangle \leq \varphi(v)\big\} \tag{6.29}$$

and the conjugate of φ is the function in $\Gamma_0(\mathcal{H})$ defined by

$$\varphi^*\colon u^* \mapsto \sup_{u\in\mathcal{H}} \big(\langle u \mid u^*\rangle - \varphi(u)\big). \tag{6.30}$$

For background on convex analysis and monotone operators the reader is referred to [7].

A key tool in our approach is the proximity operator [7, 21].

Définition 6.8 Let $\varphi \in \Gamma_0(\mathcal{H})$. The proximity operator of φ is

$$\operatorname{prox}_\varphi\colon \mathcal{H} \to \mathcal{H}\colon u \mapsto \underset{v\in\mathcal{H}}{\operatorname{argmin}}\ \ \varphi(v) + \frac{1}{2}\|u - v\|^2. \tag{6.31}$$

We have

$$(\forall u \in \mathcal{H})(\forall p \in \mathcal{H}) \quad p = \operatorname{prox}_\varphi u \quad \Leftrightarrow \quad u - p \in \partial\varphi(p). \tag{6.32}$$

Définition 6.9 Let $\mathcal{H}$ be a real Hilbert space. A function $\Phi\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ satisfies the *Kadec'–Klee property* if, for every sequence $(u_n)_{n\in\mathbb{N}}$ in $\operatorname{dom}\Phi$ and every $u \in \mathcal{H}$, we have

$$\begin{cases} u_n \rightharpoonup u \\ \varlimsup \Phi(u_n) \leq \Phi(u) \end{cases} \quad \Rightarrow \quad u_n \to u. \tag{6.33}$$

Lemma 6.10 *Let $\Phi \in \Gamma_0(\mathcal{H})$ be uniformly convex, i.e., there exists an increasing function $\phi\colon\ [0,+\infty[\to [0,+\infty]$ vanishing only at 0 such that*

$$\begin{aligned}&(\forall(x,y) \in \operatorname{dom}\Phi \times \operatorname{dom}\Phi)(\forall \alpha \in]0,1[)\\ &\qquad \Phi(\alpha x + (1-\alpha)y) + \alpha(1-\alpha)\phi(\|x-y\|) \leq \alpha\Phi(x) + (1-\alpha)\Phi(y).\end{aligned} \tag{6.34}$$

Then Φ satisfies the Kadec'–Klee property.

Proof. Fix $\alpha \in]0,1[$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{dom}\Phi$ and let $u \in \mathcal{H}$ be such that $u_n \rightharpoonup u$ and $\overline{\lim}\,\Phi(u_n) \leq \Phi(u)$. Since Φ is lower semicontinuous and convex, it is weakly lower semicontinuous and therefore $\Phi(u) \leq \underline{\lim}\,\Phi(u_n)$. Hence, $\Phi(u_n) \to \Phi(u)$ and, since $\alpha u_n + (1-\alpha)u \rightharpoonup u$, we deduce from (6.34) that

$$\begin{aligned}\alpha(1-\alpha)\,\overline{\lim}\,\phi(\|u_n-u\|) &\leq \overline{\lim}\,\big(\alpha\Phi(u_n) + (1-\alpha)\Phi(u) - \Phi(\alpha u_n + (1-\alpha)u)\big)\\ &\leq \overline{\lim}\,\big(\alpha\Phi(u_n) + (1-\alpha)\Phi(u)\big) - \underline{\lim}\,\Phi(\alpha u_n + (1-\alpha)u)\\ &= \alpha\lim\Phi(u_n) + (1-\alpha)\Phi(u) - \underline{\lim}\,\Phi(\alpha u_n + (1-\alpha)u)\\ &\leq \Phi(u) - \Phi(u)\\ &= 0.\end{aligned} \tag{6.35}$$

Therefore, $\phi(\|u_n-u\|) \to 0$ and we conclude that $u_n \to u$. ☐

Our method hinges on the algorithm presented below, which derives from [10, Proposition 4.2].

Theorem 6.11 *Let $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{G}}$ be real Hilbert spaces, let $\boldsymbol{\Phi} \in \Gamma_0(\boldsymbol{\mathcal{H}})$, let $\boldsymbol{\Psi} \in \Gamma_0(\boldsymbol{\mathcal{G}})$, and let $\boldsymbol{\Lambda}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{G}}$ be a linear bounded operator. Suppose that $\boldsymbol{\Lambda} \neq 0$ and that*

$$\operatorname{zer}\big(\partial\boldsymbol{\Phi} + \boldsymbol{\Lambda}^* \circ (\partial\boldsymbol{\Psi}) \circ \boldsymbol{\Lambda}\big) \neq \varnothing. \tag{6.36}$$

Consider the primal problem

$$\underset{\boldsymbol{u}\in\boldsymbol{\mathcal{H}}}{\operatorname{minimize}}\ \ \boldsymbol{\Phi}(\boldsymbol{u}) + \boldsymbol{\Psi}(\boldsymbol{\Lambda}\boldsymbol{u}) \tag{6.37}$$

and the dual problem

$$\underset{\boldsymbol{g}\in\boldsymbol{\mathcal{G}}}{\operatorname{minimize}}\ \ \boldsymbol{\Phi}^*(-\boldsymbol{\Lambda}^*\boldsymbol{g}) + \boldsymbol{\Psi}^*(\boldsymbol{g}). \tag{6.38}$$

Let $(\boldsymbol{a}_{1,n})_{n\in\mathbb{N}}$, $(\boldsymbol{b}_{1,n})_{n\in\mathbb{N}}$, and $(\boldsymbol{c}_{1,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\boldsymbol{\mathcal{H}}$, and let $(\boldsymbol{a}_{2,n})_{n\in\mathbb{N}}$, $(\boldsymbol{b}_{2,n})_{n\in\mathbb{N}}$, and $(\boldsymbol{c}_{2,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\boldsymbol{\mathcal{G}}$. Furthermore, let $\boldsymbol{p}_0 \in \mathcal{H}$, let $\boldsymbol{q}_0 \in \mathcal{G}$, let $\varepsilon \in]0, 1/(\|\boldsymbol{\Lambda}\|+1)[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\|\boldsymbol{\Lambda}\|\,]$, *and set*

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \boldsymbol{v}_n = \boldsymbol{p}_n - \gamma_n(\boldsymbol{\Lambda}^*\boldsymbol{q}_n + \boldsymbol{a}_{1,n}) \\ \boldsymbol{u}_n = \operatorname{prox}_{\gamma_n \boldsymbol{\Phi}}(\boldsymbol{v}_n) + \boldsymbol{b}_{1,n} \\ \boldsymbol{z}_n = \boldsymbol{q}_n + \gamma_n(\boldsymbol{\Lambda}\boldsymbol{p}_n + \boldsymbol{a}_{2,n}) \\ \boldsymbol{g}_n = \operatorname{prox}_{\gamma_n \boldsymbol{\Psi}^*}(\boldsymbol{z}_n) + \boldsymbol{b}_{2,n} \\ \boldsymbol{r}_n = \boldsymbol{u}_n - \gamma_n(\boldsymbol{\Lambda}^*\boldsymbol{g}_n + \boldsymbol{c}_{1,n}) \\ \boldsymbol{p}_{n+1} = \boldsymbol{p}_n - \boldsymbol{v}_n + \boldsymbol{r}_n \\ \boldsymbol{s}_n = \boldsymbol{g}_n + \gamma_n(\boldsymbol{\Lambda}\boldsymbol{u}_n + \boldsymbol{c}_{2,n}) \\ \boldsymbol{q}_{n+1} = \boldsymbol{q}_n - \boldsymbol{z}_n + \boldsymbol{s}_n. \end{array} \right. \tag{6.39}$$

Then the following hold for some solution $\overline{\boldsymbol{u}}$ to (6.37) and some solution $\overline{\boldsymbol{g}}$ to (6.38) such that $-\boldsymbol{\Lambda}^*\overline{\boldsymbol{g}} \in \partial\boldsymbol{\Phi}(\overline{\boldsymbol{u}})$ and $\overline{\boldsymbol{g}} \in \partial\boldsymbol{\Psi}(\boldsymbol{\Lambda}\overline{\boldsymbol{u}})$.

(i) $\boldsymbol{u}_n - \boldsymbol{p}_n \to \mathbf{0}$ *and* $\boldsymbol{g}_n - \boldsymbol{q}_n \to \mathbf{0}$.

(ii) $\boldsymbol{u}_n \rightharpoonup \overline{\boldsymbol{u}}$, $\boldsymbol{p}_n \rightharpoonup \overline{\boldsymbol{u}}$, $\boldsymbol{g}_n \rightharpoonup \overline{\boldsymbol{g}}$, *and* $\boldsymbol{q}_n \rightharpoonup \overline{\boldsymbol{g}}$.

(iii) *Suppose that $\boldsymbol{\Phi}$ satisfies the Kadec'–Klee property. Then $\boldsymbol{u}_n \to \overline{\boldsymbol{u}}$ and $\boldsymbol{p}_n \to \overline{\boldsymbol{u}}$.*

(iv) *Suppose that $\boldsymbol{\Psi}^*$ satisfies the Kadec'–Klee property. Then $\boldsymbol{g}_n \to \overline{\boldsymbol{g}}$ and $\boldsymbol{q}_n \to \overline{\boldsymbol{g}}$.*

Proof. (i)–(ii) : The results follow from [10, Proposition 4.2].

(iii)–(iv) : Let $n \in \mathbb{N}$. Since $\overline{\boldsymbol{u}}$ and $\overline{\boldsymbol{g}}$ satisfy $-\boldsymbol{\Lambda}^*\overline{\boldsymbol{g}} \in \partial\boldsymbol{\Phi}(\overline{\boldsymbol{u}})$ and $\boldsymbol{\Lambda}\overline{\boldsymbol{u}} \in \partial\boldsymbol{\Psi}^*(\overline{\boldsymbol{g}})$, it follows from (6.29) that

$$\begin{cases} \boldsymbol{\Phi}(\overline{\boldsymbol{u}}) + \langle(\boldsymbol{u}_n - \boldsymbol{b}_{1,n}) - \overline{\boldsymbol{u}} \mid -\boldsymbol{\Lambda}^*\overline{\boldsymbol{g}}\rangle \leq \boldsymbol{\Phi}(\boldsymbol{u}_n - \boldsymbol{b}_{1,n}) \\ \boldsymbol{\Psi}^*(\overline{\boldsymbol{g}}) + \langle(\boldsymbol{g}_n - \boldsymbol{b}_{2,n}) - \overline{\boldsymbol{g}} \mid \boldsymbol{\Lambda}\overline{\boldsymbol{u}}\rangle \leq \boldsymbol{\Psi}^*(\overline{\boldsymbol{g}}_n - \boldsymbol{b}_{2,n}). \end{cases} \tag{6.40}$$

On the other hand, we deduce from (6.39) and (6.32) that

$$\begin{cases} (\boldsymbol{v}_n - \boldsymbol{u}_n + \boldsymbol{b}_{1,n})/\gamma_n \in \partial\boldsymbol{\Phi}(\boldsymbol{u}_n - \boldsymbol{b}_{1,n}) \\ (\boldsymbol{z}_n - \boldsymbol{g}_n + \boldsymbol{b}_{2,n})/\gamma_n \in \partial\boldsymbol{\Psi}^*(\boldsymbol{g}_n - \boldsymbol{b}_{2,n}), \end{cases} \tag{6.41}$$

and hence it follows from (6.29) that

$$\begin{cases} \boldsymbol{\Phi}(\boldsymbol{u}_n - \boldsymbol{b}_{1,n}) + \langle \overline{\boldsymbol{u}} - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \mid \boldsymbol{v}_n - \boldsymbol{u}_n + \boldsymbol{b}_{1,n}\rangle/\gamma_n \leq \boldsymbol{\Phi}(\overline{\boldsymbol{u}}) \\ \boldsymbol{\Psi}^*(\boldsymbol{g}_n - \boldsymbol{b}_{2,n}) + \langle \overline{\boldsymbol{g}} - \boldsymbol{g}_n + \boldsymbol{b}_{2,n} \mid \boldsymbol{z}_n - \boldsymbol{g}_n + \boldsymbol{b}_{2,n}\rangle/\gamma_n \leq \boldsymbol{\Psi}^*(\overline{\boldsymbol{g}}). \end{cases} \tag{6.42}$$

Combining (6.40) and (6.42) we obtain

$$\langle \overline{\boldsymbol{u}} - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \mid (\boldsymbol{v}_n - \boldsymbol{u}_n + \boldsymbol{b}_{1,n})\rangle/\gamma_n \leq \langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} - \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^*\overline{\boldsymbol{g}}\rangle \tag{6.43}$$

and

$$\langle \overline{\boldsymbol{g}} - \boldsymbol{g}_n + \boldsymbol{b}_{2,n} \mid (\boldsymbol{z}_n - \boldsymbol{g}_n + \boldsymbol{b}_{2,n})\rangle/\gamma_n \leq \langle \overline{\boldsymbol{g}} - \boldsymbol{g}_n + \boldsymbol{b}_{2,n} \mid \boldsymbol{\Lambda}\overline{\boldsymbol{u}}\rangle. \tag{6.44}$$

We derive from (6.43) and (6.39) that

$$\begin{aligned}
&\langle \overline{\boldsymbol{u}} - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \mid \boldsymbol{v}_n - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \rangle / \gamma_n \\
&\qquad = \big(\langle \overline{\boldsymbol{u}} - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \mid (\boldsymbol{v}_n - \boldsymbol{p}_n) \rangle + \langle \overline{\boldsymbol{u}} - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \mid \boldsymbol{p}_n - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \rangle \big) / \gamma_n \\
&\qquad = \langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} - \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^* \boldsymbol{q}_n + \boldsymbol{a}_{1,n} \rangle + \langle \overline{\boldsymbol{u}} - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \mid \boldsymbol{p}_n - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \rangle / \gamma_n \\
&\qquad = \langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} - \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^* \boldsymbol{g}_n \rangle + \langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} - \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^* (\boldsymbol{q}_n - \boldsymbol{g}_n) + \boldsymbol{a}_{1,n} \rangle \\
&\qquad\quad + \langle \overline{\boldsymbol{u}} - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \mid \boldsymbol{p}_n - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \rangle / \gamma_n \\
&\qquad = \langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} - \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^* (\boldsymbol{g}_n - \overline{\boldsymbol{g}}) \rangle + \langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} - \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^* \overline{\boldsymbol{g}} \rangle \\
&\qquad\quad + \langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} - \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^* (\boldsymbol{q}_n - \boldsymbol{g}_n) + \boldsymbol{a}_{1,n} \rangle \\
&\qquad\quad + \langle \overline{\boldsymbol{u}} - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \mid \boldsymbol{p}_n - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \rangle / \gamma_n \\
&\qquad = \langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} \mid \boldsymbol{\Lambda}^* (\boldsymbol{g}_n - \overline{\boldsymbol{g}}) \rangle - \langle \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^* (\boldsymbol{g}_n - \overline{\boldsymbol{g}}) \rangle \\
&\qquad\quad + \langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} - \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^* \overline{\boldsymbol{g}} \rangle + \langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} - \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^* (\boldsymbol{q}_n - \boldsymbol{g}_n) + \boldsymbol{a}_{1,n} \rangle \\
&\qquad\quad + \langle \overline{\boldsymbol{u}} - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \mid \boldsymbol{p}_n - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \rangle / \gamma_n.
\end{aligned} \tag{6.45}$$

Analogously, we derive from (6.44) that

$$\begin{aligned}
\langle \overline{\boldsymbol{g}} - \boldsymbol{g}_n + \boldsymbol{b}_{2,n} \mid \boldsymbol{z}_n - \boldsymbol{g}_n + \boldsymbol{b}_{2,n} \rangle / \gamma_n = &- \langle \boldsymbol{g}_n - \overline{\boldsymbol{g}} \mid \boldsymbol{\Lambda} (\boldsymbol{u}_n - \overline{\boldsymbol{u}}) \rangle + \langle \overline{\boldsymbol{g}} - \boldsymbol{g}_n + \boldsymbol{b}_{2,n} \mid \boldsymbol{\Lambda} \overline{\boldsymbol{u}} \rangle \\
&+ \langle \boldsymbol{b}_{2,n} \mid \boldsymbol{\Lambda} (\boldsymbol{u}_n - \overline{\boldsymbol{u}}) \rangle \\
&- \langle \boldsymbol{g}_n - \overline{\boldsymbol{g}} - \boldsymbol{b}_{2,n} \mid \boldsymbol{\Lambda} (\boldsymbol{p}_n - \boldsymbol{u}_n) + \boldsymbol{a}_{2,n} \rangle \\
&+ \langle \overline{\boldsymbol{g}} - \boldsymbol{g}_n + \boldsymbol{b}_{2,n} \mid \boldsymbol{q}_n - \boldsymbol{g}_n + \boldsymbol{b}_{2,n} \rangle / \gamma_n.
\end{aligned} \tag{6.46}$$

We deduce from (6.43) and (6.45) that

$$\begin{aligned}
0 &\geq \langle \overline{\boldsymbol{u}} - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \mid \boldsymbol{v}_n - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \rangle / \gamma_n - \langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} - \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^* \overline{\boldsymbol{g}} \rangle \\
&= \langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} \mid \boldsymbol{\Lambda}^* (\boldsymbol{g}_n - \overline{\boldsymbol{g}}) \rangle - \langle \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^* (\boldsymbol{g}_n - \overline{\boldsymbol{g}}) \rangle \\
&\quad + \langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} - \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^* (\boldsymbol{q}_n - \boldsymbol{g}_n) + \boldsymbol{a}_{1,n} \rangle + \langle \overline{\boldsymbol{u}} - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \mid \boldsymbol{p}_n - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \rangle / \gamma_n,
\end{aligned} \tag{6.47}$$

which yields

$$\begin{aligned}
\langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} \mid \boldsymbol{\Lambda}^* (\boldsymbol{g}_n - \overline{\boldsymbol{g}}) \rangle \leq \langle \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^* (\boldsymbol{g}_n - \overline{\boldsymbol{g}}) \rangle &- \langle \boldsymbol{u}_n - \overline{\boldsymbol{u}} - \boldsymbol{b}_{1,n} \mid \boldsymbol{\Lambda}^* (\boldsymbol{q}_n - \boldsymbol{g}_n) + \boldsymbol{a}_{1,n} \rangle \\
&- \langle \overline{\boldsymbol{u}} - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \mid \boldsymbol{p}_n - \boldsymbol{u}_n + \boldsymbol{b}_{1,n} \rangle / \gamma_n.
\end{aligned} \tag{6.48}$$

Analogously, we deduce from (6.46) and (6.44) that

$$\begin{aligned}
\langle \boldsymbol{g}_n - \overline{\boldsymbol{g}} \mid \boldsymbol{\Lambda} (\boldsymbol{u}_n - \overline{\boldsymbol{u}}) \rangle \geq \langle \boldsymbol{b}_{2,n} \mid \boldsymbol{\Lambda} (\boldsymbol{u}_n - \overline{\boldsymbol{u}}) \rangle &- \langle \boldsymbol{g}_n - \overline{\boldsymbol{g}} - \boldsymbol{b}_{2,n} \mid \boldsymbol{\Lambda} (\boldsymbol{p}_n - \boldsymbol{u}_n) + \boldsymbol{a}_{2,n} \rangle \\
&+ \langle \overline{\boldsymbol{g}} - \boldsymbol{g}_n + \boldsymbol{b}_{2,n} \mid \boldsymbol{q}_n - \boldsymbol{g}_n + \boldsymbol{b}_{2,n} \rangle / \gamma_n.
\end{aligned} \tag{6.49}$$

Note that, in view of (i) and (ii),

$$\boldsymbol{u}_k \rightharpoonup \overline{\boldsymbol{u}}, \; \boldsymbol{g}_k \rightharpoonup \overline{\boldsymbol{g}}, \; \boldsymbol{p}_k - \boldsymbol{u}_k \to \boldsymbol{0}, \; \text{and} \; \boldsymbol{q}_k - \boldsymbol{g}_k \to \boldsymbol{0}. \tag{6.50}$$

Hence, since $\boldsymbol{b}_{1,k} \to \mathbf{0}$ and $\boldsymbol{b}_{2,k} \to \mathbf{0}$, we have

$$\boldsymbol{u}_k - \overline{\boldsymbol{u}} - \boldsymbol{b}_{1,k} \rightharpoonup \mathbf{0} \quad \text{and} \quad \overline{\boldsymbol{g}} - \boldsymbol{g}_k + \boldsymbol{b}_{2,k} \rightharpoonup \mathbf{0}. \tag{6.51}$$

Moreover, since $\boldsymbol{a}_{1,k} \to 0$ and $\boldsymbol{a}_{2,k} \to \mathbf{0}$, we have $\boldsymbol{\Lambda}(\boldsymbol{p}_k - \boldsymbol{u}_k) + \boldsymbol{a}_{2,k} \to \mathbf{0}$, $\boldsymbol{\Lambda}^*(\boldsymbol{q}_k - \boldsymbol{g}_k) + \boldsymbol{a}_{1,k} \to \mathbf{0}$, $\boldsymbol{\Lambda}(\boldsymbol{u}_k - \overline{\boldsymbol{u}}) \rightharpoonup \mathbf{0}$, and $\boldsymbol{\Lambda}^*(\boldsymbol{g}_k - \overline{\boldsymbol{g}}) \rightharpoonup \mathbf{0}$. Altogether, since $\inf_{k\in\mathbb{N}} \gamma_k > 0$, taking the limsup in (6.48) and the liminf in (6.49) we obtain

$$\overline{\lim}\, \langle \boldsymbol{g}_k - \overline{\boldsymbol{g}} \mid \boldsymbol{\Lambda}(\boldsymbol{u}_k - \overline{\boldsymbol{u}})\rangle = \overline{\lim}\, \langle \boldsymbol{u}_k - \overline{\boldsymbol{u}} \mid \boldsymbol{\Lambda}^*(\boldsymbol{g}_k - \overline{\boldsymbol{g}})\rangle \leq 0 \leq \underline{\lim}\, \langle \boldsymbol{g}_k - \overline{\boldsymbol{g}} \mid \boldsymbol{\Lambda}(\boldsymbol{u}_k - \overline{\boldsymbol{u}})\rangle\,, \tag{6.52}$$

which yields

$$\langle \boldsymbol{g}_k - \overline{\boldsymbol{g}} \mid \boldsymbol{\Lambda}(\boldsymbol{u}_k - \overline{\boldsymbol{u}})\rangle = \langle \boldsymbol{\Lambda}^*(\boldsymbol{g}_k - \overline{\boldsymbol{g}}) \mid \boldsymbol{u}_k - \overline{\boldsymbol{u}}\rangle \to 0. \tag{6.53}$$

Hence, it follows from (6.45) and (6.46) that

$$\begin{cases} \langle \overline{\boldsymbol{u}} - \boldsymbol{u}_k + \boldsymbol{b}_{1,k} \mid \boldsymbol{v}_k - \boldsymbol{u}_k + \boldsymbol{b}_{1,k}\rangle / \gamma_k \to 0 \\ \langle \overline{\boldsymbol{g}} - \boldsymbol{g}_k + \boldsymbol{b}_{2,k} \mid \boldsymbol{z}_k - \boldsymbol{g}_k + \boldsymbol{b}_{2,k}\rangle / \gamma_k \to 0 \end{cases} \tag{6.54}$$

and therefore we conclude from (6.42) that

$$\overline{\lim}\, \boldsymbol{\Phi}(\boldsymbol{u}_k - \boldsymbol{b}_{1,k}) \leq \boldsymbol{\Phi}(\overline{\boldsymbol{u}}) \quad \text{and} \quad \overline{\lim}\, \boldsymbol{\Psi}^*(\boldsymbol{g}_k - \boldsymbol{b}_{2,k}) \leq \boldsymbol{\Psi}^*(\overline{\boldsymbol{g}}). \tag{6.55}$$

In turn,

$$\boldsymbol{u}_k - \boldsymbol{b}_{1,k} \rightharpoonup \overline{\boldsymbol{u}} \quad \text{and} \quad \boldsymbol{g}_k - \boldsymbol{b}_{2,k} \rightharpoonup \overline{\boldsymbol{g}}. \tag{6.56}$$

Therefore, it follows from (6.33), (6.56), and (6.55) that $\boldsymbol{u}_k - \boldsymbol{b}_{1,k} \to \overline{\boldsymbol{u}}$ and $\boldsymbol{g}_k - \boldsymbol{b}_{2,k} \to \overline{\boldsymbol{g}}$. Hence, $\boldsymbol{u}_k \to \overline{\boldsymbol{u}}$ and $\boldsymbol{g}_k \to \overline{\boldsymbol{g}}$. Finally, we deduce from (i) that $\boldsymbol{p}_k \to \overline{\boldsymbol{u}}$ and $\boldsymbol{q}_k \to \overline{\boldsymbol{g}}$. □

6.2.3 Problem statement

Our standing assumptions will be the following.

Assumption 6.12 Let $m \geq 2$ be an integer and set $I = \{1, \ldots, m\}$.

(A1) Ω is a nonempty open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary $\operatorname{bdry}\Omega$.

(A2) $(\Omega_i)_{i\in I}$ are disjoint open subsets of Ω (see Fig. 6.2) with Lipschitz boundaries $(\operatorname{bdry}\Omega_i)_{i\in I}$, $\overline{\Omega} = \bigcup_{i=1}^m \overline{\Omega_i}$, and

$$(\forall i \in I) \quad \Upsilon_{ii} = \operatorname{int}_{\operatorname{bdry}\Omega}(\operatorname{bdry}\Omega_i \cap \operatorname{bdry}\Omega) \neq \varnothing, \tag{6.57}$$

where $\operatorname{int}_{\operatorname{bdry}\Omega}$ denotes the interior relative to $\operatorname{bdry}\Omega$.

(A3) For every $i \in I$, set

$$(\forall j \in \{i+1,\dots,m\}) \quad \Upsilon_{ij} = \Upsilon_{ji} = \operatorname{int}_{\operatorname{bdry}\Omega_i}(\operatorname{bdry}\Omega_i \cap \operatorname{bdry}\Omega_j). \tag{6.58}$$

The set

$$J(i) = \big\{ j \in I \smallsetminus \{i\} \;\big|\; \Upsilon_{ij} \neq \varnothing \big\} \tag{6.59}$$

of indices of active interfaces of Ω_i is nonempty.

(A4) For every $i \in I$, $J(i-) = J(i) \cap \{1,\dots,i-1\}$ and $J(i+) = J(i) \cap \{i+1,\dots,m\}$, with the convention $J(1-) = J(m+) = \varnothing$.

(A5) For every $i \in I$, $\mathsf{T}_i \colon H^1(\Omega_i) \to L^2(\operatorname{bdry}\Omega_i)$ is the trace operator,

$$\mathcal{H}_i = H^1_{0,\Upsilon_{ii}}(\Omega_i) = \big\{ u \in H^1(\Omega_i) \;\big|\; \mathsf{T}_i\, u = 0 \;\text{ on } \Upsilon_{ii} \big\}, \tag{6.60}$$

and, for every $j \in J(i)$, $\mathsf{T}_{ij} \colon \mathcal{H}_i \to L^2(\Upsilon_{ij}) \colon u \mapsto (\mathsf{T}_i u)|_{\Upsilon_{ij}}$.

(A6) For every $i \in I$,

$$\mathcal{G}_i = \bigoplus_{j \in J(i)} L^2(\Upsilon_{ij}), \tag{6.61}$$

$\nu_i(\omega)$ is the unit outward normal vector at $\omega \in \operatorname{bdry}\Omega_i$, and

$$Q_i \colon L^2(\Omega_i) \times \mathcal{G}_i \to \mathcal{H}_i \tag{6.62}$$

is the operator that maps every $(f, (g_j)_{j \in J(i)})$ in $L^2(\Omega_i) \times \mathcal{G}_i$ into a weak solution in $\mathcal{H}_i$ of the Dirichlet-Neumann boundary problem

$$\begin{cases} -\Delta u = f & \text{on } \Omega_i,\\ u = 0 & \text{on } \Upsilon_{ii},\\ \nu_i^\top Du = g_j & \text{on } \Upsilon_{ij}, \;\text{ for every } \; j \in J(i+),\\ \nu_i^\top Du = -g_j & \text{on } \Upsilon_{ij}, \;\text{ for every } \; j \in J(i-). \end{cases} \tag{6.63}$$

Remark 6.13 Since $\operatorname{bdry}\Omega_i = \overline{\Upsilon_{ii}} \cup \overline{\bigcup_{j \in J(i+)} \Upsilon_{ij}} \cup \overline{\bigcup_{j \in J(i-)} \Upsilon_{ij}}$, the existence and uniqueness of the solution to (6.63) is guaranteed by [27, Theorem 25.I], from which we deduce that Q_i is linear.

Problem 6.14 Suppose that Assumption 6.12 holds and, for every $i \in I$, let $\varphi_i \in \Gamma_0(\mathcal{H}_i)$ and, for every $j \in J(i+)$, let $\psi_{ij} \in \Gamma_0(L^2(\Upsilon_{ij}))$. Assume that, for every $i \in I$ and $j \in J(i+)$, there exist $\overline{u}_i \in \mathcal{H}_i$ and $\overline{g}_{ij} \in L^2(\Upsilon_{ij})$ such that

$$(\forall i \in I) \quad \begin{cases} -Q_i\big(0, (\overline{g}_{ij})_{j \in J(i+)}, (\overline{g}_{ji})_{j \in J(i-)}\big) \in \partial\varphi_i(\overline{u}_i)\\ (\forall j \in J(i+)) \quad \overline{g}_{ij} \in \partial\psi_{ij}(\mathsf{T}_{ij}\,\overline{u}_i - \mathsf{T}_{ji}\,\overline{u}_j). \end{cases} \tag{6.64}$$

The primal problem is to

$$\underset{u_1\in\mathcal{H}_1,\dots,u_m\in\mathcal{H}_m}{\text{minimize}} \quad \sum_{i=1}^{m}\varphi_i(u_i)+\sum_{i=1}^{m-1}\sum_{j\in J(i+)}\psi_{ij}(\mathsf{T}_{ij}\,u_i-\mathsf{T}_{ji}\,u_j), \tag{6.65}$$

and the dual problem is to

$$\underset{\substack{i\in I, j\in J(i+)\\ g_{ij}\in L^2(\Upsilon_{ij})}}{\text{minimize}} \quad \sum_{i=1}^{m}\varphi_i^*\big(-Q_i\big(0,(g_{ij})_{j\in J(i+)},(g_{ji})_{j\in J(i-)}\big)\big)+\sum_{i=1}^{m-1}\sum_{j\in J(i+)}\psi_{ij}^*(g_{ij}). \tag{6.66}$$

As seen in the Introduction this is a flexible model in which we can incorporate constraints on the solutions as well as on the jumps across the interfaces. In addition, the associated dual problem represents the potential energy related to the dual variables g_{ij}, which are interpreted as tensions on the interfaces.

Assumption (6.64) will be seen to imply the existence of a solution $(\overline{u}_i)_{i\in I}$ to the primal Problem 6.65, and a solution $((\overline{g}_{ij})_{j\in J(i+)})_{i\in I}$ to the dual Problem 6.66. The assumption that each dual variable $\overline{g}_{ij}$ lie in $L^2(\Upsilon_{ij})$ is interpreted as a regularity property of the solution to the primal Problem 6.65. As will be seen in Section 6.2.5, several common problems satisfy these assumptions.

6.2.4 Algorithm and convergence

Our method for solving the primal-dual Problem 6.14 is an adaptation of Theorem 6.11. It results in a parallel decomposition method which uses separate implicit steps for minimizing the functions $(\varphi_i)_{i\in I}$ and $((\psi_{ij})_{j\in J(i+)})_{i\in I}$ and explicit steps in which the variables are coupled via linear mappings involving the traces operators and their adjoints.

Algorithm 6.15 Suppose that Assumption 6.12 holds. For every $i\in I$, let $p_{i,0}\in\mathcal{H}_i$, and let $(a_{1i,n})_{n\in\mathbb{N}}$, $(b_{1i,n})_{n\in\mathbb{N}}$, and $(c_{1i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}_i$, for every $j\in J(i+)$, let $q_{ij,0}\in L^2(\Upsilon_{ij})$, and let $(a_{2ij,n})_{n\in\mathbb{N}}$, $(b_{2ij,n})_{n\in\mathbb{N}}$, and $(c_{2ij,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $L^2(\Upsilon_{ij})$, and let $\beta_{ij}\in\,]0,\|\mathsf{T}_{ij}\|[$. Set $\beta=(\sum_{i=1}^{m-1}\sum_{j\in J(i+)}\beta_{ij}^2)^{1/2}$, let $\varepsilon \in \left]0, 1/(\beta+1)\right[$, and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\beta]$. Iterate

$$
\begin{array}{l}
\text{For } n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
\text{For every } \; i \in I \\
\left\lfloor
\begin{array}{l}
v_{i,n} = p_{i,n} - \gamma_n\Big(Q_i\big(0, (q_{ij,n})_{j\in J(i+)}, (q_{ji,n})_{j\in J(i-)}\big) + a_{1i,n}\Big) \\
u_{i,n} = \left(\underset{w\in\mathcal{H}_i}{\operatorname{argmin}} \; \gamma_n\varphi_i(w) + \dfrac{1}{2}\displaystyle\int_{\Omega_i} \big|Dw - Dv_{i,n}\big|^2\right) + b_{1i,n} \\
\text{For every } \; j \in J(i+) \\
\left\lfloor
\begin{array}{l}
z_{ij,n} = q_{ij,n} + \gamma_n\big(\mathsf{T}_{ij}p_{i,n} - \mathsf{T}_{ji}p_{j,n} + a_{2ij,n}\big) \\
g_{ij,n} = z_{ij,n} - \gamma_n\left(\underset{w\in L^2(\Upsilon_{ij})}{\operatorname{argmin}} \; \dfrac{1}{\gamma_n}\psi_{ij}(w) + \dfrac{1}{2}\displaystyle\int_{\Upsilon_{ij}} \left|w - \dfrac{z_{ij,n}}{\gamma_n}\right|^2 dS\right) + b_{2ij,n}
\end{array}
\right.
\end{array}
\right. \\
\text{For every } \; i \in I \\
\left\lfloor
\begin{array}{l}
r_{i,n} = u_{i,n} - \gamma_n\Big(Q_i\big(0, (g_{ij,n})_{j\in J(i+)}, (g_{ji,n})_{j\in J(i-)}\big) + c_{1i,n}\Big) \\
p_{i,n+1} = p_{i,n} - v_{i,n} + r_{i,n} \\
\text{For every } \; j \in J(i+) \\
\left\lfloor
\begin{array}{l}
s_{ij,n} = g_{ij,n} + \gamma_n\big(\mathsf{T}_{ij}u_{i,n} - \mathsf{T}_{ji}u_{j,n} + c_{2ij,n}\big) \\
q_{ij,n+1} = q_{ij,n} - z_{ij,n} + s_{ij,n}.
\end{array}
\right.
\end{array}
\right.
\end{array}
\right.
\end{array}
\tag{6.67}
$$

The following theorem is the central result of this paper. It establishes the convergence of the sequences generated by Algorithm 6.15 to primal and dual solutions to Problem 6.14. It will be convenient to introduce the Hilbert direct sums

$$
\boldsymbol{\mathcal{H}} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \quad \text{and} \quad \boldsymbol{\mathcal{G}} = \bigoplus_{i=1}^{m-1} \bigoplus_{j\in J(i+)} L^2(\Upsilon_{ij}). \tag{6.68}
$$

Recall that these Hilbert spaces are respectively equipped with the scalar products

$$
\begin{cases}
\langle\langle \cdot \mid \cdot \rangle\rangle \colon \boldsymbol{\mathcal{H}} \times \boldsymbol{\mathcal{H}} \to \mathbb{R} \colon (\boldsymbol{u}, \boldsymbol{v}) \mapsto \displaystyle\sum_{i=1}^{m} \langle u_i \mid v_i \rangle \\
\langle \cdot \mid \cdot \rangle_{\boldsymbol{\mathcal{G}}} \colon \boldsymbol{\mathcal{G}} \times \boldsymbol{\mathcal{G}} \to \mathbb{R} \colon (\boldsymbol{g}, \boldsymbol{h}) \mapsto \displaystyle\sum_{i=1}^{m-1} \sum_{j\in J(i+)} \langle g_{ij} \mid h_{ij} \rangle_{L^2(\Upsilon_{ij})},
\end{cases} \tag{6.69}
$$

where $\boldsymbol{u} = (u_i)_{i\in I}$ and $\boldsymbol{v} = (v_i)_{i\in I}$ are generic elements of $\boldsymbol{\mathcal{H}}$, and $\boldsymbol{g} = ((g_{ij})_{j\in J(i+)})_{i\in I}$ and $\boldsymbol{h} = ((h_{ij})_{j\in J(i+)})_{i\in I}$ are generic elements of $\boldsymbol{\mathcal{G}}$.

Theorem 6.16 *Let $(u_{1,n})_{n\in\mathbb{N}}, \ldots, (u_{m,n})_{n\in\mathbb{N}}$ and $((g_{ij,n})_{j\in J(i+)})_{i\in I}$ be the sequences generated by Algorithm 6.15. Then the following hold for some solution $(\overline{u}_1, \ldots, \overline{u}_m)$ to (6.65) and some solution $((\overline{g}_{ij})_{j\in J(i+)})_{i\in I}$ to (6.66) such that (6.64) holds.*

(i) *For every $i \in I$ and $j \in J(i+)$, $u_{i,n} - p_{i,n} \to 0$ and $g_{ij,n} - q_{ij,n} \to 0$.*

(ii) *For every* $i \in I$ *and* $j \in J(i+)$, $u_{i,n} \rightharpoonup \overline{u}_i$, $p_{i,n} \rightharpoonup \overline{u}_i$, $g_{ij,n} \rightharpoonup \overline{g}_{ij}$, *and* $q_{ij,n} \rightharpoonup \overline{g}_{ij}$.

(iii) *Suppose that the function*

$$\boldsymbol{\mathcal{H}} \to \left]-\infty,+\infty\right] : (u_i)_{i\in I} \mapsto \sum_{i\in I} \varphi_i(u_i) \tag{6.70}$$

satisfies the Kadec'–Klee property on $\boldsymbol{\mathcal{H}}$. *Then, for every* $i \in I$, $u_{i,n} \to \overline{u}_i$ *and* $p_{i,n} \to \overline{u}_i$.

(iv) *Suppose that the function*

$$\boldsymbol{\mathcal{G}} \to \left]-\infty,+\infty\right] : ((g_{ij})_{j\in J(i+)})_{i\in I} \mapsto \sum_{i\in I} \sum_{j\in J(i+)} \psi^*_{ij}(g_{ij}) \tag{6.71}$$

satisfies the Kadec'–Klee property. Then, for every $i \in I$ *and* $j \in J(i+)$, $g_{ij,n} \to \overline{g}_{ij}$ *and* $q_{ij,n} \to \overline{g}_{ij}$.

Proof. For every $i \in I$, the embedding $\mathcal{H}_i \hookrightarrow H^1(\Omega_i)$ is continuous [27, p. 1033] and therefore, for every $j \in J(i)$, $\mathsf{T}_i \colon H^1(\Omega_i) \to H^1(\Omega_i)$ and $\mathsf{T}_{ij} \colon \mathcal{H}_i \to L^2(\Upsilon_{ij})$ are linear bounded operators. Now define

$$\boldsymbol{\Lambda} \colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{G}} \colon (u_i)_{i\in I} \mapsto \big((\mathsf{T}_{ij}u_i - \mathsf{T}_{ji}u_j)_{j\in J(i+)}\big)_{i\in I}. \tag{6.72}$$

Since the operators $((\mathsf{T}_{ij})_{j\in J(i)})_{i\in I}$ are linear and bounded, $\boldsymbol{\Lambda}$ is linear and bounded and, for every $\boldsymbol{u} \in \boldsymbol{\mathcal{H}}$ and $\boldsymbol{g} \in \boldsymbol{\mathcal{G}}$,

$$\begin{aligned}
\langle\langle \boldsymbol{\Lambda u} \mid \boldsymbol{g} \rangle\rangle &= \sum_{i=1}^{m-1} \sum_{j\in J(i+)} \langle \mathsf{T}_{ij}u_i - \mathsf{T}_{ji}u_j \mid g_{ij} \rangle_{L^2(\Upsilon_{ij})} \\
&= \sum_{i=1}^{m-1} \sum_{j\in J(i+)} \big(\big\langle u_i \mid \mathsf{T}^*_{ij} g_{ij} \big\rangle - \big\langle u_j \mid \mathsf{T}^*_{ji} g_{ij} \big\rangle \big) \\
&= \sum_{i=1}^{m-1} \sum_{j\in J(i+)} \big\langle u_i \mid \mathsf{T}^*_{ij} g_{ij} \big\rangle - \sum_{j=2}^{m} \sum_{i\in J(j-)} \big\langle u_j \mid \mathsf{T}^*_{ji} g_{ij} \big\rangle \\
&= \sum_{i=1}^{m} \Bigg\langle u_i \,\Bigg|\, \sum_{j\in J(i+)} \mathsf{T}^*_{ij} g_{ij} - \sum_{j\in J(i-)} \mathsf{T}^*_{ij} g_{ji} \Bigg\rangle.
\end{aligned} \tag{6.73}$$

Thus,

$$\boldsymbol{\Lambda}^* \colon \big((g_{ij})_{j\in J(i+)}\big)_{i\in I} \mapsto \left(\sum_{j\in J(i+)} \mathsf{T}^*_{ij} g_{ij} - \sum_{j\in J(i-)} \mathsf{T}^*_{ij} g_{ji} \right)_{i\in I}. \tag{6.74}$$

In addition, for every $i \in I$, it follows from (6.28), (6.63), [27, Definition 25.31], and (6.27) that, for every $u \in \mathcal{H}_i$ and $\mathsf{g} = (g_j)_{j \in J(i)} \in \mathcal{G}_i$

$$\begin{aligned}
\langle u \mid Q_i(0,\mathsf{g})\rangle &= \int_{\Omega_i} (Du)^\top DQ_i(0,\mathsf{g}) \\
&= \int_{\operatorname{bdry}\Omega_i} (\mathsf{T}_i\, u)\big(\nu_i^\top DQ_i(0,\mathsf{g})\big)\, dS \\
&= \sum_{j \in J(i+)} \int_{\Upsilon_{ij}} (\mathsf{T}_{ij}\, u) g_j \, dS - \sum_{j \in J(i-)} \int_{\Upsilon_{ij}} (\mathsf{T}_{ij}\, u) g_j \, dS \\
&= \sum_{j \in J(i+)} \langle \mathsf{T}_{ij}\, u \mid g_j \rangle_{L^2(\Upsilon_{ij})} - \sum_{j \in J(i-)} \langle \mathsf{T}_{ij}\, u \mid g_j \rangle_{L^2(\Upsilon_{ij})} \\
&= \left\langle u \,\middle|\, \sum_{j \in J(i+)} \mathsf{T}_{ij}^* \, g_j - \sum_{j \in J(i-)} \mathsf{T}_{ij}^* \, g_j \right\rangle.
\end{aligned} \tag{6.75}$$

Hence, from (6.74) we have

$$(\forall \boldsymbol{g} \in \boldsymbol{\mathcal{G}}) \quad \Big(Q_i\big(0, (g_{ij})_{j \in J(i+)}, (g_{ji})_{j \in J(i-)}\big)\Big)_{i \in I} = \boldsymbol{\Lambda}^* \boldsymbol{g}. \tag{6.76}$$

Now set

$$\begin{cases}
\boldsymbol{\Phi} \colon \boldsymbol{\mathcal{H}} \to \left]-\infty, +\infty\right] : (u_i)_{i \in I} \mapsto \displaystyle\sum_{i=1}^{m} \varphi_i(u_i) \\
\boldsymbol{\Psi} \colon \boldsymbol{\mathcal{G}} \to \left]-\infty, +\infty\right] : \big((g_{ij})_{j \in J(i+)}\big)_{i \in I} \mapsto \displaystyle\sum_{i=1}^{m-1} \sum_{j \in J(i+)} \psi_{ij}(g_{ij}).
\end{cases} \tag{6.77}$$

Then, $\boldsymbol{\Phi} \in \Gamma_0(\boldsymbol{\mathcal{H}})$, $\boldsymbol{\Psi} \in \Gamma_0(\boldsymbol{\mathcal{G}})$, (6.64) yields (6.36), (6.65) is equivalent to (6.37), and, since $\boldsymbol{\Phi}^* \colon (u_i)_{i \in I} \mapsto \sum_{i=1}^{m} \varphi_i^*(u_i)$ and $\boldsymbol{\Psi}^* \colon ((g_{ij})_{j \in J(i+)})_{i \in I} \mapsto \sum_{i=1}^{m-1} \sum_{j \in J(i+)} \psi_{ij}^*(g_{ij})$, it follows from (6.76) that (6.66) is equivalent to (6.38). Now let $n \in \mathbb{N}$ and set

$$\begin{cases}
\boldsymbol{u}_n = (u_{i,n})_{i \in I} \\
\boldsymbol{v}_n = (v_{i,n})_{i \in I} \\
\boldsymbol{p}_n = (p_{i,n})_{i \in I} \\
\boldsymbol{r}_n = (r_{i,n})_{i \in I} \\
\boldsymbol{a}_{1,n} = (a_{1i,n})_{i \in I} \\
\boldsymbol{b}_{1,n} = (b_{1i,n})_{i \in I} \\
\boldsymbol{c}_{1,n} = (c_{1i,n})_{i \in I}
\end{cases}
\quad \text{and} \quad
\begin{cases}
\boldsymbol{g}_n = \big((g_{ij,n})_{j \in J(i+)}\big)_{i \in I \smallsetminus \{m\}} \\
\boldsymbol{z}_n = \big((z_{ij,n})_{j \in J(i+)}\big)_{i \in I \smallsetminus \{m\}} \\
\boldsymbol{q}_n = \big((q_{ij,n})_{j \in J(i+)}\big)_{i \in I \smallsetminus \{m\}} \\
\boldsymbol{s}_n = \big((s_{ij,n})_{j \in J(i+)}\big)_{i \in I \smallsetminus \{m\}} \\
\boldsymbol{a}_{2,n} = \big((a_{2ij,n})_{j \in J(i+)}\big)_{i \in I \smallsetminus \{m\}} \\
\boldsymbol{b}_{2,n} = \big((b_{2ij,n})_{j \in J(i+)}\big)_{i \in I \smallsetminus \{m\}} \\
\boldsymbol{c}_{2,n} = \big((c_{2ij,n})_{j \in J(i+)}\big)_{i \in I \smallsetminus \{m\}}.
\end{cases} \tag{6.78}$$

We derive from (6.67), (6.72), and (6.76) that

$$\begin{cases}
\boldsymbol{v}_n = \boldsymbol{p}_n - \gamma_n(\boldsymbol{\Lambda}^* \boldsymbol{q}_n + \boldsymbol{a}_{1,n}) \\
\boldsymbol{z}_n = \boldsymbol{q}_n + \gamma_n(\boldsymbol{\Lambda} \boldsymbol{p}_n + \boldsymbol{a}_{2,n}) \\
\boldsymbol{r}_n = \boldsymbol{u}_n - \gamma_n(\boldsymbol{\Lambda}^* \boldsymbol{g}_n + \boldsymbol{c}_{1,n}) \\
\boldsymbol{s}_n = \boldsymbol{g}_n + \gamma_n(\boldsymbol{\Lambda} \boldsymbol{u}_n + \boldsymbol{c}_{2,n}).
\end{cases} \tag{6.79}$$

In addition, we deduce from (6.67), (6.31), (6.27), and (6.28) that

$$(\forall i \in I)(\forall j \in J(i+)) \quad \begin{cases} u_{i,n} = \operatorname{prox}_{\gamma_n \varphi_i}(v_{i,n}) + b_{1i,n} \\ g_{ij,n} = z_{ij,n} - \gamma_n \operatorname{prox}_{\psi_{ij}/\gamma_n}(z_{ij,n}/\gamma_n) + b_{2ij,n}. \end{cases} \tag{6.80}$$

Hence it follows from [12, Lemma 2.9] and [7, Theorem 14.3] that

$$\begin{cases} \boldsymbol{u}_n = \operatorname{prox}_{\gamma_n \boldsymbol{\Phi}}(\boldsymbol{v}_n) + \boldsymbol{b}_{1,n} \\ \boldsymbol{g}_n = \operatorname{prox}_{\gamma_n \boldsymbol{\Psi}^*}(\boldsymbol{z}_n) + \boldsymbol{b}_{2,n}. \end{cases} \tag{6.81}$$

Therefore, (6.67) and (6.78) yield

$$\begin{cases} \boldsymbol{p}_{n+1} = \boldsymbol{p}_n - \boldsymbol{v}_n + \boldsymbol{r}_n \\ \boldsymbol{q}_{n+1} = \boldsymbol{q}_n - \boldsymbol{z}_n + \boldsymbol{s}_n \end{cases} \tag{6.82}$$

and it follows from (6.81), (6.79), and (6.78), that (6.67) is equivalent to (6.39).

(i)–(ii) : These follow from Theorem 6.11(i)&(ii).

(iii)–(iv) : These follow from (6.77) and Theorem 6.11(iii)&(iv). □

6.2.5 Special cases

In this section we study applications of Algorithm 6.15 to domain decomposition in some particular partial differential equations as the Poisson problem with Dirichlet conditions, p–Laplacian, obstacle problem.

For ease of presentation, we state our results without incorporating errors in the computations of the traces and of the subproblems (6.63). However, as seen in Theorem 6.16, the algorithms tolerate errors.

The following result will be useful.

Proposition 6.17 *Suppose that Assumption 6.12 holds, let $\gamma \in \,]0,+\infty[$, let $f \in L^2(\Omega)$, let $i \in I$, let $C_i \subset \mathcal{H}_i$ be a nonempty closed convex set, and let*

$$\varphi_i \colon \mathcal{H}_i \to \,]-\infty,+\infty] : u \mapsto \iota_{C_i}(u) + \frac{1}{2}\int_{\Omega_i} |Du|^2 - \int_{\Omega_i} fu. \tag{6.83}$$

Then the following hold.

(i) $\partial\varphi_i = N_{C_i} + \operatorname{Id} - Q_i(f,0,\ldots,0)$ *and*

$$\operatorname{prox}_{\gamma\varphi_i} = P_{C_i}\left(\frac{1}{1+\gamma}\operatorname{Id} + \frac{\gamma}{1+\gamma}Q_i(f,0,\ldots,0)\right). \tag{6.84}$$

(ii) *Suppose that* $C_i = \mathcal{H}_i$. *Then* φ_i *is Gâteaux–differentiable on* $\mathcal{H}_i$, $\nabla\varphi_i = \operatorname{Id} - Q_i(f, 0, \ldots, 0)$, *and*

$$\operatorname{prox}_{\gamma\varphi} = \frac{1}{1+\gamma}\operatorname{Id} + \frac{\gamma}{1+\gamma}Q_i(f, 0, \ldots, 0). \tag{6.85}$$

Proof. First note that the function

$$\phi_i \colon \mathcal{H}_i \to \mathbb{R} \colon u \mapsto \int_{\Omega_i} fu \tag{6.86}$$

is linear. Moreover, it follows from Cauchy-Schwarz, Poincaré's inequality [27, Appendix (53c)], and (6.28), that

$$(\exists\, \delta \in \,]0, +\infty[)(\forall u \in \mathcal{H}_i) \quad |\phi_i(u)| \leq \|f\|_{L^2(\Omega_i)}\|u\|_{L^2(\Omega_i)} \leq \delta\|f\|_{L^2(\Omega_i)}\|u\|. \tag{6.87}$$

Hence, the Riesz-Fréchet representation theorem asserts that there exists a unique $v_i \in \mathcal{H}_i$ such that

$$(\forall u \in \mathcal{H}_i) \quad \phi_i(u) = \int_{\Omega_i} fu = \int_{\Omega_i} Dv_i Du = \langle v_i \mid u\rangle\,. \tag{6.88}$$

Thus it follows from [27, Proposition 25.28] and (6.63) that $v_i = Q_i(f, 0, \ldots, 0)$. Using (6.28), we can therefore write (6.83) as

$$\varphi_i \colon u \mapsto \frac{1}{2}\|u\|^2 - \langle Q_i(f, 0, \ldots, 0) \mid u\rangle + \iota_{C_i}(u). \tag{6.89}$$

(i) : We deduce from standard subdifferential calculus [7, Section 16.4] that

$$\partial\varphi_i = \operatorname{Id} - Q_i(f, 0, \ldots, 0) + N_{C_i}, \tag{6.90}$$

which yields, for every u and p in $\mathcal{H}_i$,

$$\begin{aligned} p = \operatorname{prox}_{\gamma\varphi_i} u \quad &\Leftrightarrow \quad u - p \in \gamma\partial\varphi_i(p) \\ &\Leftrightarrow \quad u \in (1+\gamma)p - \gamma Q_i(f, 0, \ldots, 0) + N_{C_i}p \\ &\Leftrightarrow \quad \frac{1}{1+\gamma}u + \frac{\gamma}{1+\gamma}Q_i(f, 0, \ldots, 0) \in p + N_{C_i}p \\ &\Leftrightarrow \quad p = P_{C_i}\left(\frac{1}{1+\gamma}u + \frac{\gamma}{1+\gamma}Q_i(f, 0, \ldots, 0)\right). \end{aligned} \tag{6.91}$$

(ii) : Since $N_{C_i} \equiv \{0\}$ and $P_{C_i} = \operatorname{Id}$, the result follows from (i). □

6.2.5.1 Poisson problem

Let $f \in L^2(\Omega)$ and consider the Poisson problem with Dirichlet conditions

$$\begin{cases} -\Delta u = f, & \text{on } \Omega; \\ u = 0, & \text{on } \operatorname{bdry}\Omega. \end{cases} \tag{6.92}$$

This problem admits a unique weak solution $\overline{u} \in H_0^1(\Omega)$, which can be obtained by solving the strongly convex minimization problem (see [14, Chapter IV.2.1] or [27, Chapter 25.9])

$$\underset{u \in H_0^1(\Omega)}{\text{minimize}} \ \frac{1}{2}\int_\Omega |Du|^2 - \int_\Omega fu. \tag{6.93}$$

We are interested in solving this problem by decomposing the domain Ω in subdomains satisfying Assumption 6.12 and imposing continuity conditions on the interfaces.

Problem 6.18 Let $f \in L^2(\Omega)$. Suppose that Assumption 6.12 holds and that, for every $i \in I$ and $j \in J(i+)$, Υ_{ij} and $\operatorname{bdry}\Omega$ are of class $\mathcal{C}^2$. The problem is to

$$\underset{\substack{u_1 \in \mathcal{H}_1, \ldots, u_m \in \mathcal{H}_m \\ (\forall i \in I)(\forall j \in J(i+))\ \mathsf{T}_{ij}u_i = \mathsf{T}_{ji}u_j}}{\text{minimize}} \ \sum_{i=1}^m \frac{1}{2}\int_{\Omega_i} |Du_i|^2 - \int_{\Omega_i} fu_i. \tag{6.94}$$

First we show the equivalence between Problem 6.18 and (6.93).

Proposition 6.19 *Problem 6.18 has a unique solution $(\overline{u}_i)_{i\in I}$. Moreover,*

$$\overline{u}\colon \Omega \to \mathbb{R}\colon x \mapsto \begin{cases} \overline{u}_i(x), & \textit{if } (\exists\, i \in I) \quad x \in \Omega_i; \\ (\mathsf{T}_i\overline{u}_i)(x), & \textit{if } (\exists\, i \in I) \quad x \in \operatorname{bdry}\Omega_i \end{cases} \tag{6.95}$$

is the unique solution to (6.93).

Proof. First note that it follows from Assumption 6.12(A2) that $\overline{u}$ defined in (6.95) is well defined. Now since the objective function in (6.94) is strongly convex in $\boldsymbol{\mathcal{H}}$, there exists a unique solution $(\overline{u}_i)_{i\in I}$ to Problem 6.18. Now, for every $u \in L^2(\Omega)$, we deduce from [2, Lemma 6.4.1] that

$$u \in H^1(\Omega) \quad \Leftrightarrow \quad (\forall i \in I)(\forall j \in J(i+)) \quad u|_{\Omega_i} \in H^1(\Omega_i) \text{and} \mathsf{T}_{ij}(u|_{\Omega_i}) = \mathsf{T}_{ji}(u|_{\Omega_j}). \tag{6.96}$$

The characterization (6.96) expresses the fact that the jumps of every $u \in H^1(\Omega)$ across the interfaces $((\Upsilon_{ij})_{j\in J(i+)})_{i\in I}$ are zero. Correspondingly, by taking into account the Dirichlet boundary condition [14, Section 2.1], we deduce from (6.96) that

$$u \in H_0^1(\Omega) \quad \Leftrightarrow \quad (\forall i \in I)(\forall j \in J(i+)) \quad u|_{\Omega_i} \in \mathcal{H}_i \text{ and } \mathsf{T}_{ij}(u|_{\Omega_i}) = \mathsf{T}_{ji}(u|_{\Omega_j}). \tag{6.97}$$

Now from (6.95) we obtain, for every $i \in I$, $\overline{u}|_{\Omega_i} = \overline{u}_i \in \mathcal{H}_i$, and, for every $j \in J(i+)$, $\mathsf{T}_{ij}(\overline{u}|_{\Omega_i}) = \mathsf{T}_{ij}\overline{u}_i = \mathsf{T}_{ji}\overline{u}_j = \mathsf{T}_{ji}(\overline{u}|_{\Omega_j})$. Hence, (6.97) yields $\overline{u} \in H_0^1(\Omega)$ and, for every $u \in H_0^1(\Omega)$, the additive property of the integral yields (the sets $(\Omega_i)_{i\in I}$ are disjoint, and the Lebesgue measure of the interfaces is equal to zero)

$$\begin{aligned} \frac{1}{2}\int_\Omega |D\overline{u}|^2 - \int_\Omega f\overline{u} &= \sum_{i=1}^m \frac{1}{2}\int_{\Omega_i} |D\overline{u}_i|^2 - \int_{\Omega_i} f\overline{u}_i \\ &\leq \sum_{i=1}^m \frac{1}{2}\int_{\Omega_i} |D(u|_{\Omega_i})|^2 - \int_{\Omega_i} f(u|_{\Omega_i}) \\ &= \frac{1}{2}\int_\Omega |Du|^2 - \int_\Omega fu, \end{aligned} \tag{6.98}$$

which finishes the result. □

Now we propose our algorithm to solve Problem 6.18, which is a particular instance of Algorithm 6.15 without considering errors.

Algorithm 6.20 For every $i \in I$, let $p_{i,0} \in \mathcal{H}_i$, for every $j \in J(i+)$, let $q_{ij,0} \in L^2(\Upsilon_{ij})$, and let $\beta_{ij} \in \left]0, \|\mathsf{T}_{ij}\|\right[$. Set $\beta = (\sum_{i=1}^{m-1}\sum_{j\in J(i+)} \beta_{ij}^2)^{1/2}$, let $\varepsilon \in \left]0, 1/(\beta+1)\right[$, and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\beta]$. Iterate

$$\begin{array}{l} \text{For } n = 0, 1, \ldots \\ \left\lfloor \begin{array}{l} \text{For every } i \in I \\ \left\lfloor \begin{array}{l} v_{i,n} = p_{i,n} - \gamma_n\Big(Q_i\big(0, (q_{ij,n})_{j\in J(i+)}, (q_{ji,n})_{j\in J(i-)}\big)\Big) \\ u_{i,n} = \big(v_{i,n} + \gamma_n Q_i(f, 0, \ldots, 0)\big)/(1+\gamma_n) \\ \text{For every } j \in J(i+) \\ \lfloor\, g_{ij,n} = q_{ij,n} + \gamma_n(\mathsf{T}_{ij}p_{i,n} - \mathsf{T}_{ji}p_{j,n}) \end{array} \right. \\ \text{For every } i \in I \\ \left\lfloor \begin{array}{l} r_{i,n} = u_{i,n} - \gamma_n\Big(Q_i\big(0, (g_{ij,n})_{j\in J(i+)}, (g_{ji,n})_{j\in J(i-)}\big)\Big) \\ p_{i,n+1} = p_{i,n} - v_{i,n} + r_{i,n} \\ \text{For every } j \in J(i+) \\ \left\lfloor \begin{array}{l} s_{ij,n} = g_{ij,n} + \gamma_n\big(\mathsf{T}_{ij}u_{i,n} - \mathsf{T}_{ji}u_{j,n}\big) \\ q_{ij,n+1} = q_{ij,n} - g_{ij,n} + s_{ij,n}. \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \tag{6.99}$$

Proposition 6.21 *Let $(u_{1,n})_{n\in\mathbb{N}}, \ldots, (u_{m,n})_{n\in\mathbb{N}}$ be the sequences generated by Algorithm 6.20. Then, for every $i \in I$, $u_{i,n} \to u_i \in \mathcal{H}_i$ and $(u_i)_{i\in I}$ is the solution of Problem 6.18.*

Proof. Upon setting

$$(\forall i \in I) \quad \begin{cases} \varphi_i\colon u_i \mapsto \dfrac{1}{2}\displaystyle\int_{\Omega_i} |Du_i|^2 - \int_{\Omega_i} fu_i \\ (\forall j \in J(i+)) \quad \psi_{ij} = \iota_{\{0\}}, \end{cases} \tag{6.100}$$

Problem 6.18 appears as a particular case of Problem 6.14. Indeed, for every $i \in I$, $\varphi_i \colon \mathcal{H}_i \to \mathbb{R}$ is a continuous convex function and hence $\varphi_i \in \Gamma_0(\mathcal{H}_i)$ while, for every $j \in J(i+)$, $\psi_{ij} = \iota_{\{0\}} \in \Gamma_0(L^2(\Upsilon_{ij}))$.

Let us verify that condition (6.64) holds. Let $(\overline{u}_i)_{i\in I} \in \boldsymbol{\mathcal{H}}$ be the solution to Problem 6.18 guaranteed by Proposition 6.19 and let $\overline{u} \in H_0^1(\Omega)$ be as in (6.95). Since $\psi_{ij} = \iota_{\{0\}}$, we have $\partial\psi_{ij}(0) = L^2(\Upsilon_{ij})$, and hence the second condition in (6.64) is automatically verified. Since $\operatorname{bdry}\Omega$ and $((\Upsilon_{ij})_{j\in J(i+)})_{i\in I}$ are of class $\mathcal{C}^2$ we have from [16, Theorem 2.2.2.3] that $\overline{u} \in H^2(\Omega)$ and, for every $i \in I$ and $j \in J(i)$ we deduce from [16, Theorem 1.5.1.2] that the normal derivatives $\nu_i^\top D\overline{u}_i$ and $\nu_j^\top D\overline{u}_j$ belong to $L^2(\Upsilon_{ij})$. Thus the Euler equation associated with (6.94) yields [2, Theorem 6.4.1]

$$(\forall i \in I) \quad \begin{cases} -\Delta\overline{u}_i = f, & \text{on } \Omega_i; \\ \overline{u}_i = 0, & \text{on } \Upsilon_{ii}; \\ \mathsf{T}_{ij}\,\overline{u}_i = \mathsf{T}_{ji}\overline{u}_j, & \text{on } \Upsilon_{ij}, \text{ for every } j \in J(i); \\ \nu_i^\top D\overline{u}_i = -\nu_j^\top D\overline{u}_j, & \text{on } \Upsilon_{ij}, \text{ for every } j \in J(i), \end{cases} \tag{6.101}$$

which yields from (6.63) that, for every $i \in I$,

$$\overline{u}_i = Q_i(f, (-\nu_j^\top D\overline{u}_j)_{j\in J(i+)}, (\nu_j^\top D\overline{u}_j)_{j\in J(i-)}). \tag{6.102}$$

Let us stress that, because of the regularity $\overline{u} \in H^2(\Omega)$, the transmission conditions satisfied by $\overline{u}$ can be expressed as equalities in the spaces $L^2(\Upsilon_{ij})$, which fits in our abstract framework (indeed for all $j \in J(i)$ the normal derivative $\nu_i^\top D\overline{u}_i$ belongs to the fractional space $H^{1/2}(\Upsilon_{ij})$). Now let us show that the first condition in (6.64) holds with

$$(\forall i \in I)(\forall j \in J(i+)) \quad \overline{g}_{ij} = \nu_j^\top D\overline{u}_j \in L^2(\Upsilon_{ij}). \tag{6.103}$$

Since, for every $i \in I$ and $j \in J(i+)$, $\nu_i^\top D\overline{u}_i = -\nu_j^\top D\overline{u}_j$, we have

$$\overline{u}_i = Q_i\big(f, (-\nu_j^\top D\overline{u}_j)_{j\in J(i+)}, (\nu_j^\top D\overline{u}_j)_{j\in J(i-)}\big) = Q_i\big(f, (-\overline{g}_{ij})_{j\in J(i+)}, (-\overline{g}_{ji})_{j\in J(i-)}\big) \tag{6.104}$$

and hence, from Lemma 6.17(ii) and the linearity of Q_i, we obtain

$$\begin{aligned} \nabla\varphi_i(\overline{u}_i) &= \overline{u}_i - Q_i(f, 0, \ldots, 0) \\ &= Q_i\big(f, (-\overline{g}_{ij})_{j\in J(i+)}, (-\overline{g}_{ji})_{j\in J(i-)}\big) - Q_i(f, 0, \ldots, 0) \\ &= Q_i\big(0, (-\overline{g}_{ij})_{j\in J(i+)}, (-\overline{g}_{ji})_{j\in J(i-)}\big) \\ &= -Q_i\big(0, (\overline{g}_{ij})_{j\in J(i+)}, (\overline{g}_{ji})_{j\in J(i-)}\big), \end{aligned} \tag{6.105}$$

which is the first condition in (6.64).

On the other hand, it follows from (6.67) and (6.100) that, for every $i \in I$ and $j \in J(i+)$, $q_{ij,n} = z_{ij,n}$ in Algorithm 6.15. Hence, we deduce from Lemma 6.17(ii) that Algorithm 6.20 is a particular case of Algorithm 6.15 when $(\varphi_i)_{i\in I}$ and $((\psi_{ij})_{j\in J(i+)})_{i\in I}$ are defined by (6.100), and errors are zero. On the other hand, since the functions $(\varphi_i)_{i\in I}$ defined in (6.100) are strongly convex, so is $(x_i)_{i\in I} \mapsto \sum_{i\in I}\varphi_i(x_i)$. Hence, it is uniformly convex and it follows from Lemma 6.10 that it satisfies the Kadec'–Klee property. Altogether, the result follows from Theorem 6.16(iii). □

Remark 6.22

(i) The requirement that the dual variables $\overline{g}_{ij} = \nu_j^\top D\overline{u}_j$ belong to $L^2(\Upsilon_{ij})$ can be interpreted as a regularity property for the solution $\overline{u}$ of the primal Problem 6.65. This property is satisfied if $\overline{u} \in H^2(\Omega)$, which, by classical regularity results for elliptic equations, is verified if $\operatorname{bdry}\Omega$ and Υ_{ij} are $\mathcal{C}^2$ regular.

(ii) The preceding analysis can be conducted similarly in the case of the linear elasticity system. One needs to use Korn's inequality (instead of Poincaré's inequality). A key ingredient (and possible limitation) of our approach is the H^2 regularity property of the solution of the problem in the case of the linear elasticity system (see [5] and references therein).

(iii) Note that the ADMM is more complicated than our algorithm. Indeed in ADMM it is necessary to solve a linear system which involves several solutions of EDP's.

6.2.5.2 p-Laplacian

Since long it has been noticed that using convex real-extended lower semicontinuous functionals on Hilbert spaces, and the corresponding subdifferential calculus, allows us to study semi-linear or quasi-linear monotone problems [3, 9, 26]. We will follow a similar approach in applying our variational decomposition method to the p-Laplacian operator Δ_p.

Let $p \in \,]1, +\infty[$, let $f \in L^\infty(\Omega)$, and consider the partial differential equation governed by the p-Laplacian operator with Dirichlet boundary conditions

$$\begin{cases} -\mathrm{div}\,(|Du|^{p-2}Du) = f, & \text{on } \Omega; \\ u = 0, & \text{on } \operatorname{bdry}\Omega. \end{cases} \tag{6.106}$$

Note that, when $p = 2$ (6.106) reduces to (6.92).

This problem admits a unique weak solution $\overline{u} \in W_0^{1,p}(\Omega)$, which can be obtained by solving the strictly convex minimization problem [14, Section IV.2.2]

$$\underset{u \in W_0^{1,p}(\Omega)}{\text{minimize}} \ \frac{1}{p}\int_\Omega |Du|^p - \int_\Omega fu. \tag{6.107}$$

We are interested to solve this problem by decomposing the domain Ω in subdomains satisfying Assumption 6.12 and considering continuity conditions on the interfaces. More precisely, we are interested in the following problem.

Problem 6.23 Let $p \in \,]1, +\infty[$ and let $f \in L^\infty(\Omega)$. Suppose that Assumption 6.12 holds and that $\operatorname{bdry}\Omega$ and $(\Upsilon_{ij})_{i,j\in I}$ are of class $\mathcal{C}^2$. Moreover, denote by

$$(\forall i \in I) \quad E_i^p = \big\{u \in W^{1,p}(\Omega_i) \ \big|\ \mathsf{T}_i u = 0 \ \text{on } \Upsilon_{ii}\big\}. \tag{6.108}$$

The problem is to

$$\underset{\substack{u_1\in E_1^p,\ldots,u_m\in E_m^p\\(\forall i\in I)(\forall j\in J(i+))\ \mathsf{T}_{ij}u_i=\mathsf{T}_{ji}u_j}}{\text{minimize}} \quad \sum_{i=1}^{m}\frac{1}{p}\int_{\Omega_i}|Du_i|^p-\int_{\Omega_i}fu_i. \tag{6.109}$$

Before the presentation of our algorithm, we first show the equivalence between Problem 6.23 and (6.107).

Proposition 6.24 *Problem 6.23 has a unique solution $(\overline{u}_i)_{i\in I}$. Moreover, the function $\overline{u}$ defined in (6.95) is the unique solution to (6.107).*

Proof. Since the objective function in (6.109) is strictly convex and coercive in $E_1^p\times\cdots\times E_m^p$, there exists a unique solution $(\overline{u}_i)_{i\in I}$ to Problem 6.23. Now arguing similarly to [2, Lemma 6.4.1], we deduce that, for every $u\in L^2(\Omega)$,

$$u\in W^{1,p}(\Omega)\quad\Leftrightarrow\quad(\forall i\in I)(\forall j\in J(i+))\quad u|_{\Omega_i}\in W^{1,p}(\Omega_i)\text{ and }\mathsf{T}_{ij}(u|_{\Omega_i})=\mathsf{T}_{ji}(u|_{\Omega_j}), \tag{6.110}$$

and by taking into account the Dirichlet boundary condition [14, Section 2.1] we deduce

$$u\in W_0^{1,p}(\Omega)\quad\Leftrightarrow\quad(\forall i\in I)(\forall j\in J(i+))\quad u|_{\Omega_i}\in E_i^p\text{ and }\mathsf{T}_{ij}(u|_{\Omega_i})=\mathsf{T}_{ji}(u|_{\Omega_j}). \tag{6.111}$$

The proof is hence analogous to the proof of Proposition 6.19. □

Now we propose our algorithm to solve Problem 6.106, which is a particular instance of Algorithm 6.15 without considering errors.

Algorithm 6.25 For every $i\in I$, let $p_{i,0}\in\mathcal{H}_i$, for every $j\in J(i+)$, let $q_{ij,0}\in L^2(\Upsilon_{ij})$, and let $\beta_{ij}\in\,]0,\|\mathsf{T}_{ij}\|[$. Set $\beta=(\sum_{i=1}^{m-1}\sum_{j\in J(i+)}\beta_{ij}^2)^{1/2}$, let $\varepsilon\in\,]0,1/(\beta+1)[$, and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon,(1-\varepsilon)/\beta]$. Iterate

$$\begin{array}{l}
\text{For } n=0,1,\ldots\\
\left\lfloor\begin{array}{l}
\text{For every } i\in I\\
\left\lfloor\begin{array}{l}
v_{i,n}=p_{i,n}-\gamma_n\Big(Q_i\big(0,(q_{ij,n})_{j\in J(i+)},(q_{ji,n})_{j\in J(i-)}\big)\Big)\\
u_{i,n}=\text{argmin}\, w\in\mathcal{H}_i\cap E_i^p\gamma_n\left(\dfrac{1}{p}\displaystyle\int_{\Omega_i}|Dw|^p-\int_{\Omega_i}fw\right)+\dfrac{1}{2}\displaystyle\int_{\Omega_i}\big|Dw-Dv_{i,n}\big|^2\\
\text{For every } j\in J(i+)\\
\lfloor\, g_{ij,n}=q_{ij,n}+\gamma_n(\mathsf{T}_{ij}p_{i,n}-\mathsf{T}_{ji}p_{j,n})
\end{array}\right.\\
\text{For every } i\in I\\
\left\lfloor\begin{array}{l}
r_{i,n}=u_{i,n}-\gamma_n\Big(Q_i\big(0,(g_{ij,n})_{j\in J(i+)},(g_{ji,n})_{j\in J(i-)}\big)\Big)\\
p_{i,n+1}=p_{i,n}-v_{i,n}+r_{i,n}\\
\text{For every } j\in J(i+)\\
\left\lfloor\begin{array}{l}
s_{ij,n}=g_{ij,n}+\gamma_n\big(\mathsf{T}_{ij}u_{i,n}-\mathsf{T}_{ji}u_{j,n}\big)\\
q_{ij,n+1}=q_{ij,n}-g_{ij,n}+s_{ij,n}.
\end{array}\right.
\end{array}\right.
\end{array}\right.
\end{array} \tag{6.112}$$

Proposition 6.26 *Let $(u_{1,n})_{n\in\mathbb{N}},\ldots,(u_{m,n})_{n\in\mathbb{N}}$ be the sequences generated by Algorithm 6.25. Then, for every $i\in I$, $u_{i,n}\to\overline{u}_i$ in $W^{1,p}(\Omega_i)$ and $(\overline{u}_i)_{i\in I}$ is the solution to Problem 6.23.*

Proof. We consider two cases.

(a) $p\geq 2$: Since Ω is bounded, we have $W^{1,p}(\Omega)\subset H^1(\Omega)$ and hence it follows from (6.108) that $E_i^p\subset\mathcal{H}_i$. Thus, let us note that Problem 6.23 is a particular case of Problem 6.14 when

$$(\forall i\in I)\quad\begin{cases}\varphi_i\colon\mathcal{H}_i\to\,]-\infty,+\infty]:u_i\mapsto\begin{cases}\dfrac{1}{p}\displaystyle\int_{\Omega_i}|Du_i|^p-\int_{\Omega_i}fu_i, & \text{if } u_i\in E_i^p;\\ +\infty, & \text{otherwise}\end{cases}\\ (\forall j\in J(i+))\quad\psi_{ij}=\iota_{\{0\}}.\end{cases}\tag{6.113}$$

It is clear that the functions $((\psi_{ij})_{j\in J(i+)})_{i\in I}$ are proper, lower semicontinuous, and convex. Since the convexity of functions $(\varphi_i)_{i\in I}$ is clear, let us show that they are lower semicontinuous. Indeed, fix $i\in I$, take $\lambda\in\mathbb{R}$, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}_i$ such that, for every $n\in\mathbb{N}$, $\varphi_i(u_n)\leq\lambda$ and such that $u_n\to u\in\mathcal{H}_i$, where the convergence is in $\mathcal{H}_i$. From [2, Theorem 5.4.3] we deduce that the norm in $W^{1,p}(\Omega_i)$ and

$$u\mapsto\left(\int_{\Omega_i}|Du|^p\right)^{1/p}=\|Du\|_{L^p(\Omega_i)}\tag{6.114}$$

are equivalent in E_i^p, which yields the coercivity of φ_i in E_i^p. Therefore, $(u_n)_{n\in\mathbb{N}}$ is bounded in E_i^p and hence weakly converges to u in E_i^p. Moreover, the functional φ_i is convex and continuous on E_i^p, and hence weakly lower semicontinuous, which yields

$$\varphi_i(u)\leq\underline{\lim}\,\varphi_i(u_n)\leq\lambda.\tag{6.115}$$

Let us show that condition (6.64) holds. Let $(\overline{u}_i)_{i\in I}\in\boldsymbol{\mathcal{H}}$ be the solution to Problem 6.23, and let $\overline{u}\in H_0^1(\Omega)$ be as in (6.95). Since $\psi_{ij}=\iota_{\{0\}}$, we have $\partial\psi_{ij}(0)=L^2(\Upsilon_{ij})$, and hence the second condition in (6.64) is automatically verified. We now invoke the regularity properties for the solution to the p-Laplacian equation (see [8, 18, 24] for a recent account). Note that, by contrast with the case $p=2$, the degeneracy of the elliptic operator $-\Delta_p$ for $p\geq 2$ makes the regularity study more involved. In [8] the global $H^2(\Omega)$ regularity is obtained for the regularized operator $-\epsilon\Delta-\Delta_p$ $(\epsilon>0)$. As a general property, for smooth data, the local regularity $C^{1,\alpha}_{loc}(\Omega)$ holds $(\alpha>0)$. In order to avoid entering into too technical developments, we assume that the global regularity $\overline{u}\in C^{1,\alpha}(\overline{\Omega})$ holds, so that we can consider the normal derivative $\nu_i^\top|D\overline{u}_i|^{p-2}D\overline{u}_i$ belonging to $L^2(\Upsilon_{ij})$ for all $i,j\in I$.

Thus the Euler equation associated with Problem 6.23 yields

$$(\forall i \in I) \quad \begin{cases} -\text{div}\left(|D\overline{u}_i|^{p-2}D\overline{u}_i\right) = f, & \text{on } \Omega_i; \\ \overline{u}_i = 0, & \text{on } \Upsilon_{ii}; \\ \mathsf{T}_{ij}\,\overline{u}_i = \mathsf{T}_{ji}\overline{u}_j, & \text{on } \Upsilon_{ij}, \ \text{for every } j \in J(i); \\ |D\overline{u}_i|^{p-2}\nu_i^\top D\overline{u}_i = -|D\overline{u}_j|^{p-2}\nu_j^\top D\overline{u}_j, & \text{on } \Upsilon_{ij}, \ \text{for every } j \in J(i). \end{cases} \tag{6.116}$$

Now, for every $i \in I$, let us compute an element $v_i \in \partial\varphi_i(\overline{u}_i)$. By a classical directional derivation argument (see [2, Theorem 6.6.1] for a detailed demonstration) we obtain

$$(\forall u \in \mathcal{H}_i) \quad \int_{\Omega_i} (|D\overline{u}_i|^{p-2}D\overline{u}_i - Dv_i)^\top Du = \int_{\Omega_i} fu, \tag{6.117}$$

from which we deduce that v_i satisfies the boundary value problem

$$\begin{cases} -\Delta v_i = -f - \text{div}\left(|D\overline{u}_i|^{p-2}D\overline{u}_i\right), & \text{on } \Omega_i; \\ v_i = 0, & \text{on } \Upsilon_{ii}; \\ \nu_i^\top Dv_i = \nu_i^\top|D\overline{u}_i|^{p-2}D\overline{u}_i, & \text{on } \Upsilon_{ij}, \ \text{for every } j \in J(i). \end{cases} \tag{6.118}$$

Owing to (6.116), we thus have

$$\begin{cases} \Delta v_i = 0, & \text{on } \Omega_i; \\ v_i = 0, & \text{on } \Upsilon_{ii}; \\ \nu_i^\top Dv_i = \nu_i^\top|D\overline{u}_i|^{p-2}D\overline{u}_i, & \text{on } \Upsilon_{ij}, \ \text{for every } j \in J(i). \end{cases} \tag{6.119}$$

Let us now use the regularity property of $\nu_i^\top|D\overline{u}_i|^{p-2}D\overline{u}_i \in L^2(\Upsilon_{ij})$, and take precisely $v = (v_i)_{1\leq i\leq m}$ defined by (6.119). We have $v_i \in \partial\varphi_i(\overline{u}_i)$ and

$$-Q_i\big(0, (g_{ij})_{j\in J(i+)}, (g_{ji})_{j\in J(i-)}\big) \in \partial\varphi_i(\overline{u}_i), \tag{6.120}$$

where $g_{ij} = \nu_i^\top|D\overline{u}_i|^{p-2}D\overline{u}_i$ on Υ_{ij}. Hence condition (6.64) holds.

On the other hand, it follows from (6.67) and (6.113) that, for every $i \in I$ and $j \in J(i+)$, $q_{ij,n} = z_{ij,n}$ in Algorithm 6.15. Hence, since $\mathcal{H}_i \cap E_i^p = E_i^p$, we deduce from Lemma 6.17(ii) that Algorithm 6.25 is a particular case of Algorithm 6.15 when $(\varphi_i)_{i\in I}$ and $((\psi_{ij})_{j\in J(i+)})_{i\in I}$ are defined by (6.113), and errors are zero. Therefore, it follows from Theorem 6.16(i) that, for some solution $(\overline{u}_i)_{i\in I}$ to Problem 6.23 and for every $i \in I$, $u_{i,n} \rightharpoonup \overline{u}_i$ in $\mathcal{H}_i$. Hence, since, for every $i \in I$, the operator $u \mapsto \int_{\Omega_i} fu$ is linear and bounded, we obtain

$$(\forall i \in I) \quad \int_{\Omega_i} fu_{i,n} \to \int_{\Omega_i} f\overline{u}_i. \tag{6.121}$$

Since, for every $i \in I$, the norm of $W^{1,p}(\Omega_i)$ is equivalent to the norm defined in (6.114) in E_i^p, we deduce from (6.55), (6.113), and (6.121) that

$$\sum_{i=1}^{m} \|u_{i,n}\|_{W^{1,p}(\Omega_i)}^p \to \sum_{i=1}^{m} \|\overline{u}_i\|_{W^{1,p}(\Omega_i)}^p. \tag{6.122}$$

Thus, since, for every $i \in I$, the sequence $(u_{i,n})_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega_i)$ and $u_{i,n} \rightharpoonup \overline{u}_i$ in $\mathcal{H}_i$, we have that a subsequence of $(u_{i,n})_{n\in\mathbb{N}}$ converges weakly to $\overline{u}_i$ in $W^{1,p}(\Omega_i)$, and hence, by proceeding similarly, we obtain $u_{i,n} \rightharpoonup \overline{u}_i$ in $W^{1,p}(\Omega_i)$. Therefore, we deduce from (6.122) and [2, Proposition 2.4.10] that, for every $i \in I$, $u_{i,n} \to \overline{u}_i$ in $W^{1,p}(\Omega_i)$.

(b) $1 < p < 2$: In this case, for every $i \in I$, $\mathcal{H}_i \subset W^{1,p}(\Omega_i)$, with continuous embedding. Let us assume that the solution $\overline{u}$ of problem (6.107) belongs to $H_0^1(\Omega)$ (indeed we shall further state regularity properties of $\overline{u}$ which make this property satisfied). Combining this property with the density of $H_0^1(\Omega)$ in $W_0^{1,p}(\Omega)$ (for the norm topology of $W_0^{1,p}(\Omega)$), the variational problem (6.107) equivalently writes

$$\underset{u\in H_0^1(\Omega)}{\text{minimize}} \ \frac{1}{p}\int_\Omega |Du|^p - \int_\Omega fu. \tag{6.123}$$

By the same argument used in Proposition 6.19, this is equivalent to solve

$$\underset{\substack{u_1\in\mathcal{H}_1,\ldots,u_m\in\mathcal{H}_m\\(\forall i\in I)(\forall j\in J(i+))\ \mathsf{T}_{ij}u_i=\mathsf{T}_{ji}u_j}}{\text{minimize}} \quad \sum_{i=1}^{m} \frac{1}{p}\int_{\Omega_i} |Du_i|^p - \int_{\Omega_i} fu_i. \tag{6.124}$$

Thus we are led to set

$$(\forall i \in I) \quad \varphi_i \colon \mathcal{H}_i \to \mathbb{R} \colon u_i \mapsto \frac{1}{p}\int_{\Omega_i} |Du_i|^p - \int_{\Omega_i} fu_i, \tag{6.125}$$

which is continuous on $\mathcal{H}_i$.

The proof is identical to the case $p \geq 2$. Just notice that when $p < 2$, the p-Laplacian becomes a singular elliptic operator. The global regularity of the solution $\overline{u}$ to problem (6.107), with a globally continuous gradient, is a well established property [8, 19]. □

6.3 Bibliographie

[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces,* second edition, Academic Press, Amsterdam, 2003.

[2] H. Attouch, G. Buttazzo, and G. Michaille, *Variational Analysis in Sobolev and BV Spaces,* SIAM, Philadelphia, PA, 2006.

[3] H. Attouch and A. Damlamian, Application des méthodes de convexité et monotonie à l'étude de certaines équations quasi-linéaires, *Proc. Roy. Soc. Edinburgh Sect. A,* vol. 79, pp. 107–129, 1977.

[4] H. Attouch and C. Picard, Variational inequalities with varying obstacles : the general form of the limit problem, *J. Funct. Anal.*, vol. 50, pp. 329–386, 1983.

[5] C. Bacuta and J. H. Bramble, Regularity estimates for solutions of the equations of linear elasticity in convex plane polygonal domains, *Z. Angew. Math. Phys.*, vol. 54 , pp. 874–878, 2003.

[6] L. Badea, Convergence rate of a Schwarz multilevel method for the constrained minimization of nonquadratic functionals, *SIAM J. Numer. Anal.*, vol. 44, pp. 449–477, 2006.

[7] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces,* Springer, New York, 2011.

[8] H. Beirão da Veiga and F. Crispo, On the global regularity for nonlinear systems of the p-Laplacian type, http ://arxiv.org/abs/1008.3262.

[9] H. Brézis, Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations, in : E. Zarantonello (ed.), *Contributions to Nonlinear Functional Analysis*, Academic Press, New York, 1971.

[10] L. M. Briceño-Arias and P. L. Combettes, A monotone+skew splitting model for composite monotone inclusions in duality, http ://arxiv.org/abs/1011.5517

[11] T. F. Chan and T. P. Mathew, Domain decomposition algorithms, *Acta Numer.*, vol. 3, pp. 61–143, 1994.

[12] P. L. Combettes and V. R. Wajs, Signal recovery by proximal forward-backward splitting, *Multiscale Model. Simul.*, vol. 4, pp. 1168–1200, 2005.

[13] P. Drábek and J. Milota, *Methods of Nonlinear Analysis – Applications to Differential Equations,* Birkhäuser, Basel, MA, 2007.

[14] I. Ekeland and R. Temam, *Analyse Convexe et Problèmes Variationnels,* Paris : Dunod, 1974. English translation : *Convex Analysis and Variational Problems,* SIAM, Philadelphia, PA, 1999.

[15] J. Frehse, On the regularity of the solution of a second order variational inequality, *Boll. Un. Mat. Ital.*, vol. 4, pp. 312–315, 1972.

[16] P. Grisvard, *Elliptic Problems in Nonsmooth Domains,* Pitman, Boston, MA, 1985.

[17] P. Le Tallec, Domain decomposition methods in computational mechanics, *Comput. Mech. Adv.*, vol. 1, pp. 121–220, 1994.

[18] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.*, vol. 12, pp. 1203–1219, 1988.

[19] W. B. Liu and J. W. Barrett, A remark on the regularity of the solutions of the p-Laplacian and its application to their finite element approximation, *J. Math. Anal. Appl.*, vol. 178, pp. 470–487, 1993.

[20] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications,* Academic Press, New York, 1980.

[21] J.-J. Moreau, Proximité et dualité dans un espace hilbertien, *Bull. Soc. Math. France,* vol. 93, pp. 273–299, 1965.

[22] J. Nečas, *Les Méthodes Directes en Théorie des Équations Elliptiques,* Masson, Paris, 1967.

[23] A. Quarteroni and A. Valli, *Domain Decomposition Methods for Partial Differential Equations,* Oxford University Press, New York, 1999.

[24] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Differential Equations*, vol. 51, pp. 126–150, 1984.

[25] A. Toselli and O. Widlund, *Domain Decomposition Methods–Algorithms and Theory,* Springer-Verlag, Berlin, Germany, 2005.

[26] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/A–Linear Monotone Operators*, Springer-Verlag, New York, 1990.

[27] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/B–Nonlinear Monotone Operators*, Springer-Verlag, New York, 1990.

Chapitre 7

Construction d'équilibres de Nash de jeux sans potentiel

7.1 Description et résultats principaux

Nous considérons un jeu à $m \geq 2$ joueurs indexés par $i \in \{1, \ldots, m\}$. La stratégie x_i du joueur i appartient à l'espace hilbertien $\mathcal{H}_i$ et le problème est de trouver $x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m$ tels que, pour tout $i \in \{1, \ldots, m\}$,

$$x_i \in \underset{x \in \mathcal{H}_i}{\text{Argmin}}\, \boldsymbol{f}(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m) + \boldsymbol{g}_i(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m), \quad (7.1)$$

où, pour tout $i \in \{1, \ldots, m\}$, $\boldsymbol{g}_i$ représente la pénalité individuelle du joueur i, qui dépend des stratégies de tous les joueurs, et $\boldsymbol{f}$ est une pénalité convexe qui est commune à tous les joueurs et modèle le malaise collectif du groupe. À ce niveau de généralité, auncune méthode fiable existe pour résoudre (7.1) et alors certaines hypothèses sont nécessaires. Dans ce chapitre nous abordons le problème suivant.

Problème 7.1 Soit $m \geq 2$ un entier et soit $\boldsymbol{f} \colon \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \to \left]-\infty, +\infty\right]$ une fonction propre, semi-continue inférieurement et convexe. Pour tout $i \in \{1, \ldots, m\}$, soit $\boldsymbol{g}_i \colon \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \to \left]-\infty, +\infty\right]$ telle que, pour tout $(x_1, \ldots, x_m) \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$, la fonction $x \mapsto \boldsymbol{g}_i(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m)$ est convexe et différentiable sur $\mathcal{H}_i$, et notons par $\nabla_i\, \boldsymbol{g}_i(x_1, \ldots, x_m)$ sa dérivée en x_i. De plus, supposons que

$$\big(\forall (x_1, \ldots, x_m) \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m\big)\big(\forall (y_1, \ldots, y_m) \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m\big)$$
$$\sum_{i=1}^{m} \langle \nabla_i\, \boldsymbol{g}_i(x_1, \ldots, x_m) - \nabla_i\, \boldsymbol{g}_i(y_1, \ldots, y_m) \mid x_i - y_i \rangle \geq 0. \quad (7.2)$$

Le problème est de trouver $x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m$ tels que

$$\begin{cases} x_1 \in \underset{x \in \mathcal{H}_1}{\operatorname{Argmin}} \;\; \boldsymbol{f}(x, x_2, \ldots, x_m) + \boldsymbol{g}_1(x, x_2, \ldots, x_m) \\ \quad \vdots \\ x_m \in \underset{x \in \mathcal{H}_m}{\operatorname{Argmin}} \;\; \boldsymbol{f}(x_1, \ldots, x_{m-1}, x) + \boldsymbol{g}_m(x_1, \ldots, x_{m-1}, x). \end{cases} \tag{7.3}$$

Dans le cas particulier où, pour tout $i \in \{1, \ldots, m\}$, $\boldsymbol{g}_i = \boldsymbol{g}$, où $\boldsymbol{g}$ est convexe, (7.3) se réduit à trouver des équilibres de Nash d'un jeu de potentiel, i.e., un jeu où la pénalité de chaque joueur peut être représentée par une fonction de potentiel commune $\boldsymbol{f} + \boldsymbol{g}$ [13]. Alors, un équilibre de Nash peut être trouvé par la résolution du problème

$$\underset{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m}{\operatorname{minimiser}} \;\; \boldsymbol{f}(x_1, \ldots, x_m) + \boldsymbol{g}(x_1, \ldots, x_m). \tag{7.4}$$

Par conséquence, le problème est réduit à la minimisation de la somme de deux fonctions convexes sur l'espace hilbertien $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$ et plusieurs méthodes sont disponibles pour l'aborder sous certaines hypothèses (voir [5, Chapter 27]). Dans ce chapitre nous nous consacrons au problème sans potentiel, où les fonctions $(\boldsymbol{g}_i)_{1 \le i \le m}$ ne sont pas identiques ni convexes, mais elles doivent satisfaire la condition (7.2). Notons que, pour tout $i \in \{1, \ldots, m\}$, (7.2) implique la convexité de $\boldsymbol{g}_i$ par rapport à la i-ème variable.

Notre approche consiste à résoudre l'inclusion monotone

$$\text{trouver} \quad x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m \quad \text{tels que} \quad (0, \ldots, 0) \in \boldsymbol{A}(x_1, \ldots, x_m) + \boldsymbol{B}(x_1, \ldots, x_m), \tag{7.5}$$

où

$$\boldsymbol{A} = \partial \boldsymbol{f} \quad \text{and} \quad \boldsymbol{B} \colon (x_i)_{1 \le i \le m} \mapsto (\nabla_i \, \boldsymbol{g}_i(x_1, \ldots, x_m))_{1 \le i \le m}. \tag{7.6}$$

Notons que, la monotonie de $\boldsymbol{A}$ est une conséquence de $\boldsymbol{f} \in \Gamma_0(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m)$ et la monotonie de $\boldsymbol{B}$ est obtenue de (7.2). Nous montrons que toute solution de (7.5) est une solution du Problème 7.1, et ensuite nous appliquons deux méthodes pour résoudre (7.5) sous diverses hypothèses sur $\boldsymbol{B}$. Les résultats de convergence sont les suivants.

Théorème 7.2 *Dans le Problème 7.1, supposons qu'il existe* $(z_1, \ldots, z_m) \in \boldsymbol{\mathcal{H}}$ *tel que*

$$-\big(\nabla_1 \, \boldsymbol{g}_1(z_1, \ldots, z_m), \ldots, \nabla_m \, \boldsymbol{g}_m(z_1, \ldots, z_m)\big) \in \partial \boldsymbol{f}(z_1, \ldots, z_m) \tag{7.7}$$

et $\chi \in \left]0, +\infty\right[$ *tel que*

$$(\forall (x_1, \ldots, x_m) \in \boldsymbol{\mathcal{H}})(\forall (y_1, \ldots, y_m) \in \boldsymbol{\mathcal{H}}) \quad \sum_{i=1}^{m} \|\nabla_i \, \boldsymbol{g}_i(x_1, \ldots, x_m) - \nabla_i \, \boldsymbol{g}_i(y_1, \ldots, y_m)\|^2 \le \chi^2 \sum_{i=1}^{m} \|x_i - y_i\|^2. \tag{7.8}$$

De plus, soit $\varepsilon \in \,]0, 1/(\chi+1)[$*, soit* $(\gamma_n)_{n\in\mathbb{N}}$ *une suite dans* $[\varepsilon, (1-\varepsilon)/\chi]$*, pour tout* $i \in \{1,\ldots,m\}$*, soit* $x_{i,0} \in \mathcal{H}_i$ *et soit* $(a_{i,n})_{n\in\mathbb{N}}$*,* $(b_{i,n})_{n\in\mathbb{N}}$ *et* $(c_{i,n})_{n\in\mathbb{N}}$ *des suites absolument sommables dans* $\mathcal{H}_i$*. Des suites* $(x_{1,n})_{n\in\mathbb{N}},\ldots,(x_{m,n})_{n\in\mathbb{N}}$ *et* $(p_{1,n})_{n\in\mathbb{N}},\ldots,(p_{m,n})_{n\in\mathbb{N}}$ *sont générées comme suit.*

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \text{Pour } i = 1,\ldots,m \\ \lfloor\; y_{i,n} = x_{i,n} - \gamma_n(\nabla_i \boldsymbol{g}_i(x_{1,n},\ldots,x_{m,n}) + a_{i,n}) \\ (p_{1,n},\ldots,p_{m,n}) = \operatorname{prox}_{\gamma_n \boldsymbol{f}}(y_{1,n},\ldots,y_{m,n}) + (b_{1,n},\ldots,b_{m,n}) \\ \text{Pour } i = 1,\ldots,m \\ \left\lfloor \begin{array}{l} q_{i,n} = p_{i,n} - \gamma_n(\nabla_i \boldsymbol{g}_i(p_{1,n},\ldots,p_{m,n}) + c_{i,n}) \\ x_{i,n+1} = x_{i,n} - y_{i,n} + q_{i,n}. \end{array} \right. \end{array} \right. \tag{7.9}$$

Alors nous avons les résultats suivants pour quelque solution $(\overline{x}_1,\ldots,\overline{x}_m)$ *du Problème 7.1.*

(i) *Pour tout* $i \in \{1,\ldots,m\}$, $\sum_{n\in\mathbb{N}} \|x_{i,n} - p_{i,n}\|^2 < +\infty$ *et* $\sum_{n\in\mathbb{N}} \|y_{i,n} - q_{i,n}\|^2 < +\infty$.

(ii) *Pour tout* $i \in \{1,\ldots,m\}$, $x_{i,n} \rightharpoonup \overline{x}_i$ *et* $p_{i,n} \rightharpoonup \overline{x}_i$.

Théorème 7.3 *Dans le Problème 7.1, supposons qu'il existe* $(z_1,\ldots,z_m) \in \boldsymbol{\mathcal{H}}$ *tel que*

$$-\big(\nabla_1 \boldsymbol{g}_1(z_1,\ldots,z_m),\ldots,\nabla_m \boldsymbol{g}_m(z_1,\ldots,z_m)\big) \in \partial \boldsymbol{f}(z_1,\ldots,z_m) \tag{7.10}$$

et $\chi \in \,]0,+\infty[$ *tel que*

$$\begin{aligned} &(\forall (x_1,\ldots,x_m) \in \boldsymbol{\mathcal{H}})(\forall (y_1,\ldots,y_m) \in \boldsymbol{\mathcal{H}}) \\ &\sum_{i=1}^m \langle \nabla_i \boldsymbol{g}_i(x_1,\ldots,x_m) - \nabla_i \boldsymbol{g}_i(y_1,\ldots,y_m) \mid x_i - y_i \rangle \\ &\qquad\qquad \geq \chi^{-1} \sum_{i=1}^m \|\nabla_i \boldsymbol{g}_i(x_1,\ldots,x_m) - \nabla_i \boldsymbol{g}_i(y_1,\ldots,y_m)\|^2. \end{aligned} \tag{7.11}$$

De plus, soit $\varepsilon \in \,]0, 2/(\chi+1)[$*, soit* $(\gamma_n)_{n\in\mathbb{N}}$ *une suite dans* $[\varepsilon, (2-\varepsilon)/\chi]$*, pour tout* $i \in \{1,\ldots,m\}$*, soit* $x_{i,0} \in \mathcal{H}_i$*, soit* $(a_{i,n})_{n\in\mathbb{N}}$ *et* $(b_{i,n})_{n\in\mathbb{N}}$ *des suites absolument sommables dans* $\mathcal{H}_i$*, et soit* $(x_n)_{n\in\mathbb{N}}$ *la suite générée comme suit.*

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \text{Pour } i = 1,\ldots,m \\ \lfloor\; y_{i,n} = x_{i,n} - \gamma_n(\nabla_i \boldsymbol{g}_i(x_{1,n},\ldots,x_{m,n}) + a_{i,n}) \\ (x_{1,n+1},\ldots,x_{m,n+1}) = \operatorname{prox}_{\gamma_n \boldsymbol{f}}(y_{1,n},\ldots,y_{m,n}) + (b_{1,n},\ldots,b_{m,n}) \end{array} \right. \tag{7.12}$$

Alors nous avons les résultats suivants pour quelque solution $(\overline{x}_1,\ldots,\overline{x}_m)$ *du Problème 7.1.*

(i) *Pour tout* $i \in \{1,\ldots,m\}$, $x_{i,n} \rightharpoonup \overline{x}_i$.

(ii) *Pour tout* $i \in \{1,\ldots,m\}$, $\nabla_i \boldsymbol{g}_i(x_{1,n},\ldots,x_{m,n}) \to \nabla_i \boldsymbol{g}_i(\overline{x}_1,\ldots,\overline{x}_m)$.

Notons que les conditions (7.8) et (7.11) sont équivalentes au caractère χ-lipschizian et la χ^{-1}–cocercivité de $\boldsymbol{B}$, respectivement. Vu que tout opérateur cocoercif est monotone et lipschitzien, le Théorème 7.2 permet de résoudre une classe plus grande des problèmes que le Théorème 7.3. Par contre, à chaque itération la méthode (7.9) calcule une étape explicite supplémentaire par rapport à (7.12) et la suite $(\gamma_n)_{n\in\mathbb{N}}$ est restreinte à l'intervalle $]0,1/\chi[\,\subset\,]0,2/\chi[$.

Ensuite, nous appliquons les algorithmes précédents aux fonctions selles et jeux à somme nulle, aux équilibres de Nash généralisés et aux problèmes de proximation cyclique. Nous présentons ci-dessous les formulations des problèmes, les algorithmes et les résultats de convergence principaux.

Example 7.4 Fixons $\chi \in \,]0,+\infty[$. Soit $\boldsymbol{f} \in \Gamma_0(\mathcal{H}_1 \oplus \mathcal{H}_2)$ et soit $\boldsymbol{\mathcal{L}}\colon \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathbb{R}$ une fonction différentiable avec un gradient χ–lipschitzian telle que, pour tout $x_1 \in \mathcal{H}_1$, $\boldsymbol{\mathcal{L}}(x_1,\cdot)$ est concave et, pour tout $x_2 \in \mathcal{H}_2$, $\boldsymbol{\mathcal{L}}(\cdot,x_2)$ est convexe. Le problème est de trouver $x_1 \in \mathcal{H}_1$ et $x_2 \in \mathcal{H}_2$ tels que

$$\begin{cases} x_1 \in \underset{x\in\mathcal{H}_1}{\operatorname{Argmin}} \;\; \boldsymbol{f}(x,x_2) + \boldsymbol{\mathcal{L}}(x,x_2) \\ x_2 \in \underset{x\in\mathcal{H}_2}{\operatorname{Argmin}} \;\; \boldsymbol{f}(x_1,x) - \boldsymbol{\mathcal{L}}(x_1,x). \end{cases} \tag{7.13}$$

Proposition 7.5 *Dans l'Example 7.4, supposons qu'il existe $(z_1,z_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2$ tel que*

$$\big(-\nabla_1 \boldsymbol{\mathcal{L}}(z_1,z_2), \nabla_2 \boldsymbol{\mathcal{L}}(z_1,z_2)\big) \in \partial \boldsymbol{f}(z_1,z_2). \tag{7.14}$$

De plus, soit $\varepsilon \in \,]0,1/(\chi+1)[$, soit $(x_{1,0},x_{2,0}) \in \mathcal{H}_1 \oplus \mathcal{H}_2$, soient $(a_{1,n})_{n\in\mathbb{N}}$, $(b_{1,n})_{n\in\mathbb{N}}$ et $(c_{1,n})_{n\in\mathbb{N}}$ des suites absolument sommables dans $\mathcal{H}_1$, soient $(a_{2,n})_{n\in\mathbb{N}}$, $(b_{2,n})_{n\in\mathbb{N}}$ et $(c_{2,n})_{n\in\mathbb{N}}$ des suites absolument sommables dans $\mathcal{H}_2$, soit $(\gamma_n)_{n\in\mathbb{N}}$ une suite dans $[\varepsilon,(1-\varepsilon)/\chi]$, et soient $(x_{1,n})_{n\in\mathbb{N}}$, $(x_{2,n})_{n\in\mathbb{N}}$, $(p_{1,n})_{n\in\mathbb{N}}$ et $(p_{2,n})_{n\in\mathbb{N}}$ des suites générées comme suit.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_{1,n} = x_{1,n} - \gamma_n(\nabla_1 \boldsymbol{\mathcal{L}}(x_{1,n},x_{2,n}) + a_{1,n}) \\ y_{2,n} = x_{2,n} + \gamma_n(\nabla_2 \boldsymbol{\mathcal{L}}(x_{1,n},x_{2,n}) + a_{2,n}) \\ (p_{1,n},p_{2,n}) = \operatorname{prox}_{\gamma_n \boldsymbol{f}}(y_{1,n},y_{2,n}) + (b_{1,n},b_{2,n}) \\ q_{1,n} = p_{1,n} - \gamma_n(\nabla_1 \boldsymbol{\mathcal{L}}(p_{1,n},p_{2,n}) + c_{1,n}) \\ q_{2,n} = p_{2,n} + \gamma_n(\nabla_2 \boldsymbol{\mathcal{L}}(p_{1,n},p_{2,n}) + c_{2,n}) \\ x_{1,n+1} = x_{1,n} - y_{1,n} + q_{1,n} \\ x_{2,n+1} = x_{2,n} - y_{2,n} + q_{2,n}. \end{array} \right. \tag{7.15}$$

Alors nous avons les résultats suivants pour une solution $(\overline{x}_1,\overline{x}_2)$ de l'Example 7.4.

(i) $\sum_{n\in\mathbb{N}} \|x_{1,n} - p_{1,n}\|^2 < +\infty$, $\sum_{n\in\mathbb{N}} \|x_{2,n} - p_{2,n}\|^2 < +\infty$, $\sum_{n\in\mathbb{N}} \|y_{1,n} - q_{1,n}\|^2 < +\infty$ *et* $\sum_{n\in\mathbb{N}} \|y_{2,n} - q_{2,n}\|^2 < +\infty$.

(ii) $x_{1,n} \rightharpoonup \overline{x}_1$, $x_{2,n} \rightharpoonup \overline{x}_2$, $p_{1,n} \rightharpoonup \overline{x}_1$, *et* $p_{2,n} \rightharpoonup \overline{x}_2$.

Example 7.6 Soit $\boldsymbol{C} \subset \boldsymbol{\mathcal{H}}$ un ensemble non vide, fermé et convexe et, pour tout $i \in \{1, \ldots, m\}$, soit $\boldsymbol{g}_i : \boldsymbol{\mathcal{H}} \to]-\infty, +\infty]$ une fonction différentiable par rapport à l'i-ème variable. Supposons que

$$\big(\forall (x_1, \ldots, x_m) \in \boldsymbol{\mathcal{H}}\big)\big(\forall (y_1, \ldots, y_m) \in \boldsymbol{\mathcal{H}}\big)$$
$$\sum_{i=1}^{m} \langle \nabla_i\, \boldsymbol{g}_i(x_1, \ldots, x_m) - \nabla_i\, \boldsymbol{g}_i(y_1, \ldots, y_m) \mid x_i - y_i \rangle \geq 0 \quad (7.16)$$

et posons

$$(\forall (x_i)_{1 \leq i \leq m} \in \boldsymbol{\mathcal{H}}) \quad \begin{cases} \boldsymbol{Q}_1(x_2, \ldots, x_m) = \big\{x \in \mathcal{H}_1 \mid (x, x_2, \ldots, x_m) \in \boldsymbol{C}\big\} \\ \quad\vdots \\ \boldsymbol{Q}_m(x_1, \ldots, x_{m-1}) = \big\{x \in \mathcal{H}_m \mid (x_1, \ldots, x_{m-1}, x) \in \boldsymbol{C}\big\}. \end{cases} \quad (7.17)$$

Le problème est de trouver $x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m$ tels que

$$\begin{cases} x_1 \in \underset{x \in \boldsymbol{Q}_1(x_2, \ldots, x_m)}{\operatorname{Argmin}} \boldsymbol{g}_1(x, x_2, \ldots, x_m) \\ \quad\vdots \\ x_m \in \underset{x \in \boldsymbol{Q}_m(x_1, \ldots, x_{m-1})}{\operatorname{Argmin}} \boldsymbol{g}_m(x_1, \ldots, x_{m-1}, x). \end{cases} \quad (7.18)$$

Proposition 7.7 *Dans l'Example 7.6, supposons qu'il existe* $(z_1, \ldots, z_m) \in \boldsymbol{\mathcal{H}}$ *tel que*

$$-\big(\nabla_1\, \boldsymbol{g}_1(z_1, \ldots, z_m), \ldots, \nabla_m\, \boldsymbol{g}_m(z_1, \ldots, z_m)\big) \in N_{\boldsymbol{C}}(z_1, \ldots, z_m) \quad (7.19)$$

et $\chi \in]0, +\infty[$ *tel que*

$$(\forall (x_1, \ldots, x_m) \in \boldsymbol{\mathcal{H}})(\forall (y_1, \ldots, y_m) \in \boldsymbol{\mathcal{H}})$$
$$\sum_{i=1}^{m} \|\nabla_i\, \boldsymbol{g}_i(x_1, \ldots, x_m) - \nabla_i\, \boldsymbol{g}_i(y_1, \ldots, y_m)\|^2 \leq \chi^2 \sum_{i=1}^{m} \|x_i - y_i\|^2. \quad (7.20)$$

De plus, soit $\varepsilon \in]0, 1/(\chi+1)[$*, pour tout* $i \in \{1, \ldots, m\}$*, soit* $x_{i,0} \in \mathcal{H}_i$*, soient* $(a_{i,n})_{n\in\mathbb{N}}$*,* $(b_{i,n})_{n\in\mathbb{N}}$ *et* $(c_{i,n})_{n\in\mathbb{N}}$ *des suites absolument sommables dans* $\mathcal{H}_i$*, soit* $(\gamma_n)_{n\in\mathbb{N}}$ *une suite dans* $[\varepsilon, (1-\varepsilon)/\chi]$ *et soient* $(x_{1,n})_{n\in\mathbb{N}}, \ldots, (x_{m,n})_{n\in\mathbb{N}}$ *et* $(p_{1,n})_{n\in\mathbb{N}}, \ldots, (p_{m,n})_{n\in\mathbb{N}}$ *des suites générées comme suit.*

$$(\forall n \in \mathbb{N}) \quad \left| \begin{array}{l} \text{Pour } i = 1, \ldots, m \\ \lfloor\; y_{i,n} = x_{i,n} - \gamma_n(\nabla_i\, \boldsymbol{g}_i(x_{1,n}, \ldots, x_{m,n}) + a_{i,n}) \\ (p_{1,n}, \ldots, p_{m,n}) = P_{\boldsymbol{C}}(y_{1,n}, \ldots, y_{m,n}) + (b_{1,n}, \ldots, b_{m,n}) \\ \text{Pour } i = 1, \ldots, m \\ \left\lfloor \begin{array}{l} q_{i,n} = p_{i,n} - \gamma_n(\nabla_i\, \boldsymbol{g}_i(p_{1,n}, \ldots, p_{m,n}) + c_{i,n}) \\ x_{i,n+1} = x_{i,n} - y_{i,n} + q_{i,n}. \end{array} \right. \end{array} \right. \quad (7.21)$$

Alors nous avons les résultats suivants pour quelque solution $(\overline{x}_1, \ldots, \overline{x}_m)$ *de l'Example 7.6.*

(i) *Pour tout* $i \in \{1,\dots,m\}$, $\sum_{n\in\mathbb{N}} \|x_{i,n} - p_{i,n}\|^2 < +\infty$ *et* $\sum_{n\in\mathbb{N}} \|y_{i,n} - q_{i,n}\|^2 < +\infty$.

(ii) *Pour tout* $i \in \{1,\dots,m\}$, $x_{i,n} \rightharpoonup \overline{x}_i$ *et* $p_{i,n} \rightharpoonup \overline{x}_i$.

Example 7.8 Soit $\mathcal{G}$ un espace hilbertien réel, soit $\boldsymbol{f} \in \Gamma_0(\boldsymbol{\mathcal{H}})$ et, pour tout $i \in \{1,\dots,m\}$, soit $L_i\colon \mathcal{H}_i \to \mathcal{G}$ un opérateur linéaire et borné. Le problème est de trouver $x_1 \in \mathcal{H}_1,\dots,x_m \in \mathcal{H}_m$ tels que

$$\begin{cases} x_1 \in \underset{x\in\mathcal{H}_1}{\operatorname{Argmin}} \ \boldsymbol{f}(x,x_2,\dots,x_m) + \dfrac{1}{2}\|L_1x - L_2x_2\|^2 \\ x_2 \in \underset{x\in\mathcal{H}_2}{\operatorname{Argmin}} \ \boldsymbol{f}(x_1,x,\dots,x_m) + \dfrac{1}{2}\|L_2x - L_3x_3\|^2 \\ \quad\vdots \\ x_m \in \underset{x\in\mathcal{H}_m}{\operatorname{Argmin}} \ \boldsymbol{f}(x_1,\dots,x_{m-1},x) + \dfrac{1}{2}\|L_mx - L_1x_1\|^2. \end{cases} \tag{7.22}$$

Proposition 7.9 *Dans l'Example 7.8, supposons qu'il existe* $(z_1,\dots,z_m) \in \boldsymbol{\mathcal{H}}$ *tel que*

$$\big(L_1^*(L_2z_2 - L_1z_1),\dots,L_m^*(L_1z_1 - L_mz_m)\big) \in \partial\boldsymbol{f}(z_1,\dots,z_m). \tag{7.23}$$

Posons $\chi = 2\max_{1\le i\le m}\|L_i\|^2$, *soit* $\varepsilon \in \,]0, 2/(\chi+1)[$, *soit* $(\gamma_n)_{n\in\mathbb{N}}$ *une suite dans* $[\varepsilon,(2-\varepsilon)/\chi]$, *pour tout* $i \in \{1,\dots,m\}$, *soit* $x_{i,0} \in \mathcal{H}_i$, *soient* $(a_{i,n})_{n\in\mathbb{N}}$ *et* $(b_{i,n})_{n\in\mathbb{N}}$ *des suites absolument sommables dans* $\mathcal{H}_i$, *et soit* $(x_n)_{n\in\mathbb{N}}$ *une suite générée comme suit.*

$$(\forall n\in\mathbb{N}) \quad \left\lfloor \begin{array}{l} \textit{Pour } \ i = 1,\dots,m \\ \lfloor\, y_{i,n} = x_{i,n} - \gamma_n(L_i^*(L_ix_{i,n} - L_{i+1}x_{i+1,n}) + a_{i,n}) \\ (x_{1,n+1},\dots,x_{m,n+1}) = \operatorname{prox}_{\gamma_n\boldsymbol{f}}(y_{1,n},\dots,y_{m,n}) + (b_{1,n},\dots,b_{m,n}) \end{array} \right. \tag{7.24}$$

Alors nous avons les résultats suivants pour quelque solution $(\overline{x}_1,\dots,\overline{x}_m)$ *de l'Example 7.8.*

(i) *Pour tout* $i \in \{1,\dots,m\}$, $x_{i,n} \rightharpoonup \overline{x}_i$.

(ii) *Pour tout* $i \in \{1,\dots,m\}$, $L_i^*\big(L_i(x_{i,n} - \overline{x}_i) - L_{i+1}(x_{i+1,n} - \overline{x}_{i+1})\big) \to 0$.

7.2 Article en anglais

MONOTONE OPERATOR METHODS FOR NASH EQUILIBRIA IN NON-POTENTIAL GAMES [1]

Abstract : We observe that a significant class of Nash equilibrium problems in non-potential games can be associated with monotone inclusion problems. We propose splitting techniques to solve such problems and establish their convergence. Applications to generalized Nash equilibria, zero-sum games, and cyclic proximation problems are demonstrated.

7.2.1 Problem statement

Consider a game with $m \geq 2$ players indexed by $i \in \{1, \ldots, m\}$. The strategy x_i of the ith player lies in a real Hilbert space $\mathcal{H}_i$ and the problem is to find $x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m$ such that

$$(\forall i \in \{1, \ldots, m\}) \quad x_i \in \underset{x \in \mathcal{H}_i}{\operatorname{Argmin}}\, \boldsymbol{f}(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m) + \boldsymbol{g}_i(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m), \tag{7.25}$$

where $(\boldsymbol{g}_i)_{1 \leq i \leq m}$ represents the individual penalty of player i depending on the strategies of all players and $\boldsymbol{f}$ is a convex penalty which is common to all players and models the collective discomfort of the group. At this level of generality, no reliable method exists for solving (7.25) and some hypotheses are required. In this paper we focus on the following setting.

Problem 7.10 Let $m \geq 2$ be an integer and let $\boldsymbol{f} \colon \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \to \left]-\infty, +\infty\right]$ be a proper lower semicontinuous convex function. For every $i \in \{1, \ldots, m\}$, let $\boldsymbol{g}_i \colon \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \to \left]-\infty, +\infty\right]$ be such that, for every $(x_1, \ldots, x_m) \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$, the function $x \mapsto \boldsymbol{g}_i(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m)$ is convex and differentiable on $\mathcal{H}_i$, and denote by $\nabla_i\, \boldsymbol{g}_i(x_1, \ldots, x_m)$ its derivative at x_i. Moreover,

$$\big(\forall (x_1, \ldots, x_m) \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m\big)\big(\forall (y_1, \ldots, y_m) \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m\big) \quad \sum_{i=1}^{m} \langle \nabla_i\, \boldsymbol{g}_i(x_1, \ldots, x_m) - \nabla_i\, \boldsymbol{g}_i(y_1, \ldots, y_m) \mid x_i - y_i \rangle \geq 0. \tag{7.26}$$

1. L. M. Briceño-Arias and P. L. Combettes, Monotone operator methods for Nash equilibria in non-potential games, *Computational and Analytical Mathematics,* soumis.

The problem is to find $x_1 \in \mathcal{H}_1, \dots, x_m \in \mathcal{H}_m$ such that

$$\begin{cases} x_1 \in \underset{x \in \mathcal{H}_1}{\operatorname{Argmin}} \ \boldsymbol{f}(x, x_2, \dots, x_m) + \boldsymbol{g}_1(x, x_2, \dots, x_m) \\ \quad \vdots \\ x_m \in \underset{x \in \mathcal{H}_m}{\operatorname{Argmin}} \ \boldsymbol{f}(x_1, \dots, x_{m-1}, x) + \boldsymbol{g}_m(x_1, \dots, x_{m-1}, x). \end{cases} \tag{7.27}$$

In the special case when, for every $i \in \{1, \dots, m\}$, $\boldsymbol{g}_i = \boldsymbol{g}$, Problem 7.10 amounts to finding a Nash equilibrium of a potential game, i.e., a game in which the penalty of player i can be represented by a common potential $\boldsymbol{f} + \boldsymbol{g}$ [13]. Hence, Nash equilibria can be found by solving

$$\underset{x_1 \in \mathcal{H}_1, \dots, x_m \in \mathcal{H}_m}{\text{minimize}} \ \boldsymbol{f}(x_1, \dots, x_m) + \boldsymbol{g}(x_1, \dots, x_m). \tag{7.28}$$

Thus, the problem reduces to the minimization of the sum of two convex functions on the Hilbert space $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$ and various methods are available to tackle it under suitable assumptions (see for instance [5, Chapter 27]). In this paper we address the more challenging non-potential setting, in which the functions $(\boldsymbol{g}_i)_{1 \le i \le m}$ need not be identical nor convex, but they must satisfy (7.26). Let us note that (7.26) actually implies the convexity of $\boldsymbol{g}_i$ with respect to its ith variable.

Our methodology consists in using monotone operator splitting techniques for solving an auxiliary monotone inclusion, the solutions of which are Nash equilibria of Problem 7.10. In Section 7.2.2 we review the notation and background material needed subsequently. In Section 7.2.3 we introduce the auxiliary monotone inclusion problem and provide conditions ensuring the existence of solutions to the auxiliary problem. We also propose two methods for solving Problem 7.10 and establish their convergence. Finally, in Section 7.2.4 the proposed methods are applied to the construction of generalized Nash equilibria, to zero-sum games, and to cyclic proximation problems.

7.2.2 Notation and background

Throughout this paper, $\mathcal{H}$, $\mathcal{G}$, and $(\mathcal{H}_i)_{1 \le i \le m}$ are real Hilbert spaces. For convenience, their scalar products are all denoted by $\langle \cdot \mid \cdot \rangle$ and the associated norms by $\| \cdot \|$. Let $A \colon \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. The domain of A is $\operatorname{dom} A = \big\{x \in \mathcal{H} \mid Ax \neq \varnothing\big\}$, the set of zeros of A is $\operatorname{zer} A = \big\{x \in \mathcal{H} \mid 0 \in Ax\big\}$, the graph of A is $\operatorname{gr} A = \big\{(x, u) \in \mathcal{H} \times \mathcal{H} \mid u \in Ax\big\}$, the range of A is $\operatorname{ran} A = \big\{u \in \mathcal{H} \mid (\exists\, x \in \mathcal{H})\ u \in Ax\big\}$, the inverse of A is the set-valued operator $A^{-1} \colon \mathcal{H} \to 2^{\mathcal{H}} \colon u \mapsto \big\{x \in \mathcal{H} \mid u \in Ax\big\}$, and the resolvent of A is $J_A = (\operatorname{Id} + A)^{-1}$. In addition, A is monotone if

$$(\forall (x, y) \in \mathcal{H} \times \mathcal{H})(\forall (u, v) \in Ax \times Ay) \quad \langle x - y \mid u - v \rangle \ge 0 \tag{7.29}$$

and it is maximally monotone if, furthermore, every monotone operator $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ such that $\operatorname{gr} A \subset \operatorname{gr} B$ coincides with A.

We denote by $\Gamma_0(\mathcal{H})$ the class of lower semicontinuous convex functions $\varphi\colon \mathcal{H} \to]-\infty, +\infty]$ which are proper in the sense that $\operatorname{dom}\varphi = \big\{x \in \mathcal{H} \mid \varphi(x) < +\infty\big\} \neq \varnothing$. Let $\varphi \in \Gamma_0(\mathcal{H})$. The proximity operator of φ is

$$\operatorname{prox}_{\varphi}\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto \underset{y\in\mathcal{H}}{\operatorname{argmin}}\ \varphi(y) + \frac{1}{2}\|x-y\|^2 \tag{7.30}$$

and the subdifferential of φ is the maximally monotone operator

$$\partial\varphi\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{u \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \ \langle y-x \mid u\rangle + \varphi(x) \leq \varphi(y)\big\}. \tag{7.31}$$

We have

$$\underset{x\in\mathcal{H}}{\operatorname{Argmin}}\ \varphi(x) = \operatorname{zer}\partial\varphi \quad \text{and} \quad \operatorname{prox}_{\varphi} = J_{\partial\varphi}. \tag{7.32}$$

Let $\beta \in]0, +\infty[$. An operator $T\colon \mathcal{H} \to \mathcal{H}$ is β-cocoercive (or βT is firmly nonexpansive) if

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x-y \mid Tx-Ty\rangle \geq \beta\|Tx-Ty\|^2, \tag{7.33}$$

which implies that it is monotone and β^{-1}–Lipschitzian. Let C be a nonempty convex subset of $\mathcal{H}$. The indicator function of C is

$$\iota_C\colon \mathcal{H} \to]-\infty, +\infty] : x \mapsto \begin{cases} 0, & \text{if } x \in C;\\ +\infty, & \text{if } x \notin C \end{cases} \tag{7.34}$$

and $\partial\iota_C = N_C$ is the normal cone operator of C, i.e.,

$$N_C\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \begin{cases} \big\{u \in \mathcal{H} \mid (\forall y \in C)\ \ \langle y-x \mid u\rangle \leq 0\big\}, & \text{if } x \in C;\\ \varnothing, & \text{otherwise.} \end{cases} \tag{7.35}$$

If C is closed, for every $x \in \mathcal{H}$, there exists a unique point $P_C x \in C$ such that $\|x - P_C x\| = \inf_{y\in C}\|x-y\|$; $P_C x$ is called the projection of x onto C and we have $P_C = \operatorname{prox}_{\iota_C}$. In addition, the symbols $\rightharpoonup$ and $\to$ denote respectively weak and strong convergence. For a detailed account of the tools described above, see [5].

7.2.3 Model, algorithms, and convergence

We investigate an auxiliary monotone inclusion problem the solutions of which are Nash equilibria of Problem 7.10 and propose two splitting methods to solve it. Both involve the proximity operator $\operatorname{prox}_{\boldsymbol{f}}$, which can be computed explicitly in several instances [5, 7]. We henceforth denote by $\boldsymbol{\mathcal{H}}$ the direct sum of the Hilbert spaces $(\mathcal{H}_i)_{1\leq i\leq m}$, i.e., the product space $\mathcal{H}_1 \times \cdots \times \mathcal{H}_m$ equipped with the scalar product

$$\langle\langle \cdot \mid \cdot \rangle\rangle \colon \big((x_i)_{1\leq i\leq m}, (y_i)_{1\leq i\leq m}\big) \mapsto \sum_{i=1}^{m} \langle x_i \mid y_i \rangle \,. \tag{7.36}$$

We denote the associated norm by $|||\cdot|||$, a generic element of $\boldsymbol{\mathcal{H}}$ by $\boldsymbol{x} = (x_i)_{1\leq i\leq m}$, and the identity operator on $\boldsymbol{\mathcal{H}}$ by $\mathbf{Id}$.

7.2.3.1 A monotone inclusion model

With the notation and hypotheses of Problem 7.10, let us set

$$\boldsymbol{A} = \partial \boldsymbol{f} \quad \text{and} \quad \boldsymbol{B} \colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}} \colon \boldsymbol{x} \mapsto \big(\nabla_1 \boldsymbol{g}_1(\boldsymbol{x}), \ldots, \nabla_m \boldsymbol{g}_m(\boldsymbol{x})\big). \tag{7.37}$$

We consider the inclusion problem

$$\text{find} \quad \boldsymbol{x} \in \operatorname{zer}(\boldsymbol{A} + \boldsymbol{B}). \tag{7.38}$$

Since $\boldsymbol{f} \in \Gamma_0(\boldsymbol{\mathcal{H}})$, $\boldsymbol{A}$ is maximally monotone. On the other hand, it follows from (7.26) that $\boldsymbol{B}$ is monotone. The following result establishes a connection between the monotone inclusion problem (7.38) and Problem 7.10.

Proposition 7.11 *Using the notation and hypotheses of Problem 7.10, let $\boldsymbol{A}$ and $\boldsymbol{B}$ be as in (7.37). Then every point in $\operatorname{zer}(\boldsymbol{A} + \boldsymbol{B})$ is a solution to Problem 7.10.*

Proof. Suppose that $\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B}) \neq \varnothing$ and let $(x_1, \ldots, x_m) \in \boldsymbol{\mathcal{H}}$. Then [5, Proposition 16.6] asserts that

$$\boldsymbol{A}(x_1, \ldots, x_m) \subset \partial\big(\boldsymbol{f}(\cdot, x_2, \ldots, x_m)\big)(x_1) \times \cdots \times \partial\big(\boldsymbol{f}(x_1, \ldots, x_{m-1}, \cdot)\big)(x_m). \tag{7.39}$$

Hence, since $\operatorname{dom} \boldsymbol{g}_1(\cdot, x_2, \ldots, x_m) = \mathcal{H}_1$, ..., $\operatorname{dom} \boldsymbol{g}_m(x_1, \ldots, x_{m-1}, \cdot) = \mathcal{H}_m$, we derive from (7.37), (7.32), and [5, Corollary 16.38(iii)] that

$$\begin{aligned}
(x_1, \ldots, x_m) \in \operatorname{zer}(\boldsymbol{A} + \boldsymbol{B}) \quad &\Leftrightarrow \quad -\boldsymbol{B}(x_1, \ldots, x_m) \in \boldsymbol{A}(x_1, \ldots, x_m) \\
&\Rightarrow \quad \begin{cases} -\nabla_1 \boldsymbol{g}_1(x_1, \ldots, x_m) \in \partial\big(\boldsymbol{f}(\cdot, x_2, \ldots, x_m)\big)(x_1) \\ \qquad \vdots \\ -\nabla_m \boldsymbol{g}_m(x_1, \ldots, x_m) \in \partial\big(\boldsymbol{f}(x_1, \ldots, x_{m-1}, \cdot)\big)(x_m) \end{cases} \\
&\Leftrightarrow \quad (x_1, \ldots, x_m) \ \text{ solves Problem 7.10,}
\end{aligned} \tag{7.40}$$

which yields the result. □

Proposition 7.11 asserts that we can solve Problem 7.10 by solving (7.38), provided the latter has solutions. The following result provides instances in which this property is satisfied. First, we need the following definitions (see [5, Chapters 21–24]).

Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be monotone. Then A is 3^* monotone if $\operatorname{dom} A \times \operatorname{ran} A \subset \operatorname{dom} F_A$, where

$$F_A\colon \mathcal{H} \times \mathcal{H} \to \left]-\infty, +\infty\right] : (x, u) \mapsto \langle x \mid u \rangle - \inf_{(y,v) \in \operatorname{gr} A} \langle x - y \mid u - v \rangle . \tag{7.41}$$

On the other hand, A is uniformly monotone if there exists an increasing function $\phi\colon \left[0, +\infty\right[\to [0, +\infty]$ vanishing only at 0 such that

$$\big(\forall (x, y) \in \mathcal{H} \times \mathcal{H}\big)\big(\forall (u, v) \in Ax \times Ay\big) \quad \langle x - y \mid u - v \rangle \geq \phi(\|x - y\|). \tag{7.42}$$

A function $\varphi \in \Gamma_0(\mathcal{H})$ is uniformly convex if there exists an increasing function $\phi\colon \left[0, +\infty\right[\to [0, +\infty]$ vanishing only at 0 such that

$$(\forall (x, y) \in \operatorname{dom} \varphi \times \operatorname{dom} \varphi)(\forall \alpha \in \left]0, 1\right[) \\ \varphi(\alpha x + (1 - \alpha) y) + \alpha(1 - \alpha)\phi(\|x - y\|) \leq \alpha\varphi(x) + (1 - \alpha)\varphi(y). \tag{7.43}$$

The function ϕ in (7.42) and (7.43) is called the modulus of uniform monotonicity and of uniform convexity, respectively, and it is said to be supercoercive if $\lim_{t \to +\infty} \phi(t)/t = +\infty$.

Proposition 7.12 *With the notation and hypotheses of Problem 7.10, let $\boldsymbol{B}$ be as in (7.37). Suppose that $\boldsymbol{B}$ is maximally monotone and that one of the following holds.*

(i) $\lim_{|||\boldsymbol{x}||| \to +\infty} \inf |||\partial \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{x}||| = +\infty$.

(ii) *$\partial \boldsymbol{f} + \boldsymbol{B}$ is uniformly monotone with a supercoercive modulus.*

(iii) *$(\operatorname{dom} \partial \boldsymbol{f}) \cap \operatorname{dom} \boldsymbol{B}$ is bounded.*

(iv) *$\boldsymbol{f} = \iota_{\boldsymbol{C}}$, where $\boldsymbol{C}$ is a nonempty closed convex bounded subset of $\boldsymbol{\mathcal{H}}$.*

(v) *$\boldsymbol{f}$ is uniformly convex with a supercoercive modulus.*

(vi) *$\boldsymbol{B}$ is 3^* monotone, and $\partial \boldsymbol{f}$ or $\boldsymbol{B}$ is surjective.*

(vii) *$\boldsymbol{B}$ is uniformly monotone with a supercoercive modulus.*

(viii) *$\boldsymbol{B}$ is linear and bounded, there exists $\beta \in \left]0, +\infty\right[$ such that $\boldsymbol{B}$ is β–cocoercive, and $\partial \boldsymbol{f}$ or $\boldsymbol{B}$ is surjective.*

Then $\operatorname{zer}(\partial \boldsymbol{f} + \boldsymbol{B}) \neq \varnothing$. In addition, if (ii), (v), or (vii) holds, $\operatorname{zer}(\partial \boldsymbol{f} + \boldsymbol{B})$ is a singleton.

Proof. First note that, for every $\boldsymbol{x} = (x_i)_{1 \leq i \leq m} \in \boldsymbol{\mathcal{H}}$, $\operatorname{dom} \nabla_1 \boldsymbol{g}_1(\cdot, x_2, \ldots, x_m) = \mathcal{H}_1, \ldots, \operatorname{dom} \nabla_m \boldsymbol{g}_m(x_1, \ldots, x_{m-1}, \cdot) = \mathcal{H}_m$. Hence, it follows from (7.37) that $\operatorname{dom} \boldsymbol{B} = \boldsymbol{\mathcal{H}}$ and, therefore, from [5, Corollary 24.4(i)] that $\partial \boldsymbol{f} + \boldsymbol{B}$ is maximally monotone. In addition, it follows from [5, Example 24.9] that $\partial \boldsymbol{f}$ is 3^* monotone.

(i) : This follows from [5, Corollary 21.20]. (ii) : This follows from [5, Corollary 23.37(i)]. (iii) : Since $\operatorname{dom}(\partial \boldsymbol{f} + \boldsymbol{B}) = (\operatorname{dom} \partial \boldsymbol{f}) \cap \operatorname{dom} \boldsymbol{B}$, the result follows from [5, Proposition 23.36(iii)]. (iv)⇒(iii) : $\boldsymbol{f} = \iota_{\boldsymbol{C}} \in \Gamma_0(\boldsymbol{\mathcal{H}})$ and $\operatorname{dom} \partial \boldsymbol{f} = \boldsymbol{C}$ is bounded. (v)⇒(ii) : It follows from (7.37) and [5, Example 22.3(iii)] that $\partial \boldsymbol{f}$ is uniformly monotone. Hence, $\partial \boldsymbol{f} + \boldsymbol{B}$ is uniformly monotone. (vi) : This follows from [5, Corollary 24.22(ii)]. (vii)⇒(ii) : Clear. (viii)⇒(vi) : This follows from [5, Proposition 24.12]. □

7.2.3.2 Forward-backward-forward algorithm

Our first method for solving Problem 7.10 derives from an algorithm proposed in [6], which is itself a variant of a method proposed in [15].

Theorem 7.13 *In Problem 7.10, suppose that there exist* $(z_1, \ldots, z_m) \in \boldsymbol{\mathcal{H}}$ *such that*

$$-\big(\nabla_1 \boldsymbol{g}_1(z_1, \ldots, z_m), \ldots, \nabla_m \boldsymbol{g}_m(z_1, \ldots, z_m)\big) \in \partial \boldsymbol{f}(z_1, \ldots, z_m) \tag{7.44}$$

and $\chi \in \left]0, +\infty\right[$ *such that*

$$(\forall (x_1, \ldots, x_m) \in \boldsymbol{\mathcal{H}})(\forall (y_1, \ldots, y_m) \in \boldsymbol{\mathcal{H}})$$
$$\sum_{i=1}^m \|\nabla_i \boldsymbol{g}_i(x_1, \ldots, x_m) - \nabla_i \boldsymbol{g}_i(y_1, \ldots, y_m)\|^2 \leq \chi^2 \sum_{i=1}^m \|x_i - y_i\|^2. \tag{7.45}$$

Let $\varepsilon \in \left]0, 1/(\chi+1)\right[$ *and let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon, (1-\varepsilon)/\chi]$. *Moreover, for every* $i \in \{1, \ldots, m\}$, *let* $x_{i,0} \in \mathcal{H}_i$, *and let* $(a_{i,n})_{n\in\mathbb{N}}$, $(b_{i,n})_{n\in\mathbb{N}}$, *and* $(c_{i,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{H}_i$. *Now consider the following routine.*

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \text{For } i = 1, \ldots, m \\ \lfloor\, y_{i,n} = x_{i,n} - \gamma_n(\nabla_i \boldsymbol{g}_i(x_{1,n}, \ldots, x_{m,n}) + a_{i,n}) \\ (p_{1,n}, \ldots, p_{m,n}) = \operatorname{prox}_{\gamma_n \boldsymbol{f}}(y_{1,n}, \ldots, y_{m,n}) + (b_{1,n}, \ldots, b_{m,n}) \\ \text{For } i = 1, \ldots, m \\ \left\lfloor \begin{array}{l} q_{i,n} = p_{i,n} - \gamma_n(\nabla_i \boldsymbol{g}_i(p_{1,n}, \ldots, p_{m,n}) + c_{i,n}) \\ x_{i,n+1} = x_{i,n} - y_{i,n} + q_{i,n}. \end{array} \right. \end{array} \right. \tag{7.46}$$

Then there exists a solution $(\overline{x}_1, \ldots, \overline{x}_m)$ *to Problem 7.10 such that, for every* $i \in \{1, \ldots, m\}$, $x_{i,n} \rightharpoonup \overline{x}_i$ *and* $p_{i,n} \rightharpoonup \overline{x}_i$.

Proof. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be defined as (7.37). Then (7.44) yields $\operatorname{zer}(\boldsymbol{A} + \boldsymbol{B}) \neq \varnothing$ and, for every $\gamma \in \left]0, +\infty\right[$, (7.32) yields $J_{\gamma \boldsymbol{A}} = \operatorname{prox}_{\gamma \boldsymbol{f}}$. In addition, we deduce from (7.26) and (7.45) that $\boldsymbol{B}$ is monotone and χ–Lipschitzian. Now set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \boldsymbol{x}_n = (x_{1,n}, \ldots, x_{m,n}) \\ \boldsymbol{y}_n = (y_{1,n}, \ldots, y_{m,n}) \\ \boldsymbol{p}_n = (p_{1,n}, \ldots, p_{m,n}) \\ \boldsymbol{q}_n = (q_{1,n}, \ldots, q_{m,n}) \end{cases} \quad \text{and} \quad \begin{cases} \boldsymbol{a}_n = (a_{1,n}, \ldots, a_{m,n}) \\ \boldsymbol{b}_n = (b_{1,n}, \ldots, b_{m,n}) \\ \boldsymbol{c}_n = (c_{1,n}, \ldots, c_{m,n}). \end{cases} \tag{7.47}$$

Then (7.46) is equivalent to

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \boldsymbol{y}_n = \boldsymbol{x}_n - \gamma_n(\boldsymbol{B}\boldsymbol{x}_n + \boldsymbol{a}_n) \\ \boldsymbol{p}_n = J_{\gamma_n \boldsymbol{A}}\boldsymbol{y}_n + \boldsymbol{b}_n \\ \boldsymbol{q}_n = \boldsymbol{p}_n - \gamma_n(\boldsymbol{B}\boldsymbol{p}_n + \boldsymbol{c}_n) \\ \boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n. \end{array} \right. \tag{7.48}$$

Therefore, the result follows from [6, Theorem 2.5(ii)] and Proposition 7.11. □

Note that two (forward) gradient steps involving the individual penalties $(\boldsymbol{g}_i)_{1\leq i\leq m}$ and one (backward) proximal step involving the common penalty $\boldsymbol{f}$ are required at each iteration of (7.46).

7.2.3.3 Forward-backward algorithm

Our second method for solving Problem 7.10 is somewhat simpler than (7.46) but requires stronger hypotheses on $(\boldsymbol{g}_i)_{1\leq i\leq m}$. This method is an application of the forward-backward splitting algorithm (see [3, 8] and the references therein for background).

Theorem 7.14 *In Problem 7.10, suppose that there exist $(z_1,\ldots,z_m) \in \boldsymbol{\mathcal{H}}$ such that*

$$-\big(\nabla_1 \boldsymbol{g}_1(z_1,\ldots,z_m),\ldots,\nabla_m \boldsymbol{g}_m(z_1,\ldots,z_m)\big) \in \partial\boldsymbol{f}(z_1,\ldots,z_m) \tag{7.49}$$

and $\chi \in \left]0,+\infty\right[$ such that

$$\begin{aligned} &(\forall (x_1,\ldots,x_m) \in \boldsymbol{\mathcal{H}})(\forall (y_1,\ldots,y_m) \in \boldsymbol{\mathcal{H}}) \\ &\sum_{i=1}^m \langle \nabla_i \boldsymbol{g}_i(x_1,\ldots,x_m) - \nabla_i \boldsymbol{g}_i(y_1,\ldots,y_m) \mid x_i - y_i\rangle \\ &\qquad\qquad \geq \frac{1}{\chi}\sum_{i=1}^m \|\nabla_i \boldsymbol{g}_i(x_1,\ldots,x_m) - \nabla_i \boldsymbol{g}_i(y_1,\ldots,y_m)\|^2. \end{aligned} \tag{7.50}$$

Let $\varepsilon \in \left]0, 2/(\chi+1)\right[$ and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon,(2-\varepsilon)/\chi]$. Moreover, for every $i \in \{1,\ldots,m\}$, let $x_{i,0} \in \mathcal{H}_i$, and let $(a_{i,n})_{n\in\mathbb{N}}$ and $(b_{i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}_i$. Now consider the following routine.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \text{For } i = 1,\ldots,m \\ \lfloor\, y_{i,n} = x_{i,n} - \gamma_n(\nabla_i \boldsymbol{g}_i(x_{1,n},\ldots,x_{m,n}) + a_{i,n}) \\ (x_{1,n+1},\ldots,x_{m,n+1}) = \operatorname{prox}_{\gamma_n \boldsymbol{f}}(y_{1,n},\ldots,y_{m,n}) + (b_{1,n},\ldots,b_{m,n}). \end{array} \right. \tag{7.51}$$

Then there exists a solution $(\overline{x}_1,\ldots,\overline{x}_m)$ to Problem 7.10 such that, for every $i \in \{1,\ldots,m\}$, $x_{i,n} \rightharpoonup \overline{x}_i$ and $\nabla_i \boldsymbol{g}_i(x_{1,n},\ldots,x_{m,n}) \to \nabla_i \boldsymbol{g}_i(\overline{x}_1,\ldots,\overline{x}_m)$.

Proof. If we define $\boldsymbol{A}$ and $\boldsymbol{B}$ as in (7.37), (7.49) is equivalent to $\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B}) \neq \varnothing$, and it follows from (7.50) that $\boldsymbol{B}$ is χ^{-1}–cocoercive. Moreover, (7.51) can be recast as

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \boldsymbol{y}_n = \boldsymbol{x}_n - \gamma_n(\boldsymbol{B}\boldsymbol{x}_n + \boldsymbol{a}_n) \\ \boldsymbol{x}_{n+1} = J_{\gamma_n \boldsymbol{A}} \boldsymbol{y}_n + \boldsymbol{b}_n. \end{array} \right. \tag{7.52}$$

The result hence follows from Proposition 7.11 and [3, Theorem 2.8(i)&(ii)]. □

Theorem 7.14 imposes more restrictions on $(\boldsymbol{g}_i)_{1\leq i\leq m}$. However, unlike the forward-backward-forward algorithm used in Section 7.2.3.2, it employs only one forward step at each iteration. In addition, this method allows for larger gradient steps since the sequence $(\gamma_n)_{n\in\mathbb{N}}$ lies in $]0, 2/\chi[$, as opposed to $]0, 1/\chi[$ in Theorem 7.13.

7.2.4 Applications

The previous results can be used to solve a wide variety of instances of Problem 7.10. We discuss three examples.

7.2.4.1 Saddle functions and zero-sum games

We consider an instance of Problem 7.10 with $m = 2$ players whose individual penalties $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are saddle functions.

Example 7.15 Let $\chi \in\ ]0, +\infty[$, let $\boldsymbol{f} \in \Gamma_0(\mathcal{H}_1 \oplus \mathcal{H}_2)$, and let $\boldsymbol{\mathcal{L}} \colon \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathbb{R}$ be a differentiable function with a χ–Lipschitzian gradient such that, for every $x_1 \in \mathcal{H}_1$, $\boldsymbol{\mathcal{L}}(x_1, \cdot)$ is concave and, for every $x_2 \in \mathcal{H}_2$, $\boldsymbol{\mathcal{L}}(\cdot, x_2)$ is convex. The problem is to find $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$ such that

$$\begin{cases} x_1 \in \underset{x\in\mathcal{H}_1}{\operatorname{Argmin}}\ \ \boldsymbol{f}(x, x_2) + \boldsymbol{\mathcal{L}}(x, x_2) \\ x_2 \in \underset{x\in\mathcal{H}_2}{\operatorname{Argmin}}\ \ \boldsymbol{f}(x_1, x) - \boldsymbol{\mathcal{L}}(x_1, x). \end{cases} \tag{7.53}$$

Proposition 7.16 *In Example 7.15, suppose that there exists $(z_1, z_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2$ such that*

$$\big(-\nabla_1 \boldsymbol{\mathcal{L}}(z_1, z_2), \nabla_2 \boldsymbol{\mathcal{L}}(z_1, z_2)\big) \in \partial \boldsymbol{f}(z_1, z_2). \tag{7.54}$$

Let $\varepsilon \in\ ]0, 1/(\chi+1)[$ and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\chi]$. Moreover, let $(x_{1,0}, x_{2,0}) \in \mathcal{H}_1 \oplus \mathcal{H}_2$, let $(a_{1,n})_{n\in\mathbb{N}}$, $(b_{1,n})_{n\in\mathbb{N}}$, and $(c_{1,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}_1$, and let $(a_{2,n})_{n\in\mathbb{N}}$, $(b_{2,n})_{n\in\mathbb{N}}$, and $(c_{2,n})_{n\in\mathbb{N}}$ be absolutely summable sequences *in $\mathcal{H}_2$. Now consider the following routine.*

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_{1,n} = x_{1,n} - \gamma_n(\nabla_1 \boldsymbol{\mathcal{L}}(x_{1,n}, x_{2,n}) + a_{1,n}) \\ y_{2,n} = x_{2,n} + \gamma_n(\nabla_2 \boldsymbol{\mathcal{L}}(x_{1,n}, x_{2,n}) + a_{2,n}) \\ (p_{1,n}, p_{2,n}) = \operatorname{prox}_{\gamma_n \boldsymbol{f}}(y_{1,n}, y_{2,n}) + (b_{1,n}, b_{2,n}) \\ q_{1,n} = p_{1,n} - \gamma_n(\nabla_1 \boldsymbol{\mathcal{L}}(p_{1,n}, p_{2,n}) + c_{1,n}) \\ q_{2,n} = p_{2,n} + \gamma_n(\nabla_2 \boldsymbol{\mathcal{L}}(p_{1,n}, p_{2,n}) + c_{2,n}) \\ x_{1,n+1} = x_{1,n} - y_{1,n} + q_{1,n} \\ x_{2,n+1} = x_{2,n} - y_{2,n} + q_{2,n}. \end{array} \right. \tag{7.55}$$

Then there exists a solution $(\overline{x}_1, \overline{x}_1)$ to Example 7.15 such that $x_{1,n} \rightharpoonup \overline{x}_1$, $p_{1,n} \rightharpoonup \overline{x}_1$, $x_{2,n} \rightharpoonup \overline{x}_2$, and $p_{2,n} \rightharpoonup \overline{x}_2$.

Proof. Example 7.15 corresponds to the particular instance of Problem 7.10 in which $m = 2$, $\boldsymbol{g}_1 = \boldsymbol{\mathcal{L}}$, and $\boldsymbol{g}_2 = -\boldsymbol{\mathcal{L}}$. Indeed, it follows from [14, Theorem 1] that the operator

$$(x_1, x_2) \mapsto \big(\nabla_1 \boldsymbol{\mathcal{L}}(x_1, x_2), -\nabla_2 \boldsymbol{\mathcal{L}}(x_1, x_2)\big) \tag{7.56}$$

is monotone in $\mathcal{H}_1 \oplus \mathcal{H}_2$ and hence (7.26) holds. In addition, (7.54) implies (7.44) and, since $\nabla \boldsymbol{\mathcal{L}}$ is χ–Lipschitzian, (7.45) holds. Altogether, since (7.46) reduces to (7.55), the result follows from Theorem 7.13. □

Next, we examine an application of Proposition 7.16 to 2-player finite zero-sum games.

Example 7.17 We consider a 2-player finite zero-sum game (for complements and background on finite games, see [16]). Let S_1 be the finite set of pure strategies of player 1, with cardinality N_1, and let

$$C_1 = \left\{ (\xi_j)_{1 \le j \le N_1} \in [0,1]^{N_1} \;\middle|\; \sum_{j=1}^{N_1} \xi_j = 1 \right\} \tag{7.57}$$

be his set of mixed strategies (S_2, N_2, and C_2 are defined likewise). Moreover, let L be an $N_1 \times N_2$ real cost matrix such that

$$(\exists\, z_1 \in C_1)(\exists\, z_2 \in C_2) \quad -Lz_2 \in N_{C_1} z_1 \;\text{ and }\; L^\top z_1 \in N_{C_2} z_2. \tag{7.58}$$

The problem is to

$$\text{find} \quad x_1 \in \mathbb{R}^{N_1} \;\text{ and }\; x_2 \in \mathbb{R}^{N_2} \quad \text{such that} \quad \begin{cases} x_1 \in \underset{x \in C_1}{\operatorname{Argmin}} \; x^\top L x_2 \\ x_2 \in \underset{x \in C_2}{\operatorname{Argmax}} \; x_1^\top L x. \end{cases} \tag{7.59}$$

Since the penalty function of player 1 is $(x_1, x_2) \mapsto x_1^\top L x_2$ and the penalty function of player 2 is $(x_1, x_2) \mapsto -x_1^\top L x_2$, (7.59) is a zero-sum game. It corresponds to the particular instance of Example 7.15 in which $\mathcal{H}_1 = \mathbb{R}^{N_1}$, $\mathcal{H}_2 = \mathbb{R}^{N_2}$, $\boldsymbol{f}\colon (x_1, x_2) \mapsto \iota_{C_1}(x_1) + \iota_{C_2}(x_2)$, and $\boldsymbol{\mathcal{L}}\colon (x_1, x_2) \mapsto x_1^\top L x_2$. Indeed, since C_1 and C_2 are nonempty closed convex sets, $\boldsymbol{f} \in \Gamma_0(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Moreover, $x_1 \mapsto \boldsymbol{\mathcal{L}}(x_1, x_2)$ and $x_2 \mapsto -\boldsymbol{\mathcal{L}}(x_1, x_2)$ are convex, and $\nabla\boldsymbol{\mathcal{L}}\colon (x_1, x_2) \mapsto (Lx_2, L^\top x_1)$ is linear and bounded, with $\|\nabla\boldsymbol{\mathcal{L}}\| = \|L\|$. In addition, for every $\gamma \in \left]0, +\infty\right[$, $\operatorname{prox}_{\gamma\boldsymbol{f}} = (P_{C_1}, P_{C_2})$ [5, Proposition 23.30]. Hence, (7.55) reduces to (we set the error terms to zero for simplicity)

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_{1,n} = x_{1,n} - \gamma_n L x_{2,n} \\ y_{2,n} = x_{2,n} + \gamma_n L^\top x_{1,n} \\ p_{1,n} = P_{C_1} y_{1,n} \\ p_{2,n} = P_{C_2} y_{2,n} \\ q_{1,n} = p_{1,n} - \gamma_n L p_{2,n} \\ q_{2,n} = p_{2,n} + \gamma_n L^\top p_{1,n} \\ x_{1,n+1} = x_{1,n} - y_{1,n} + q_{1,n} \\ x_{2,n+1} = x_{2,n} - y_{2,n} + q_{2,n}, \end{array} \right. \tag{7.60}$$

where $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence in $[\varepsilon, (1-\varepsilon)/\|L\|]$ for some arbitrary $\varepsilon \in \left]0, 1/(\|L\|+1)\right[$. Since $\partial\boldsymbol{f}\colon (x_1, x_2) \mapsto N_{C_1}x_1 \times N_{C_2}x_2$, (7.58) yields (7.54). Altogether, Proposition 7.16 asserts that the sequences $(x_{1,n})_{n\in\mathbb{N}}$ and $(x_{2,n})_{n\in\mathbb{N}}$ generated by (7.60) converge to $\overline{x}_1 \in \mathbb{R}^{N_1}$ and $\overline{x}_2 \in \mathbb{R}^{N_2}$, respectively, such that $(\overline{x}_1, \overline{x}_2)$ is a solution to (7.59).

7.2.4.2 Generalized Nash equilibria

We consider the particular case of Problem 7.10 in which $\boldsymbol{f}$ is the indicator function of a closed convex subset of $\boldsymbol{\mathcal{H}} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$.

Example 7.18 Let $\boldsymbol{C} \subset \boldsymbol{\mathcal{H}}$ be a nonempty closed convex set and, for every $i \in \{1, \ldots, m\}$, let $\boldsymbol{g}_i\colon \boldsymbol{\mathcal{H}} \to \left]-\infty, +\infty\right]$ be a function which is differentiable with respect to its ith variable. Suppose that

$$\big(\forall (x_1, \ldots, x_m) \in \boldsymbol{\mathcal{H}}\big)\big(\forall (y_1, \ldots, y_m) \in \boldsymbol{\mathcal{H}}\big) \\ \sum_{i=1}^{m} \langle \nabla_i \boldsymbol{g}_i(x_1, \ldots, x_m) - \nabla_i \boldsymbol{g}_i(y_1, \ldots, y_m) \mid x_i - y_i \rangle \geq 0 \tag{7.61}$$

and set

$$(\forall (x_1, \ldots, x_m) \in \boldsymbol{\mathcal{H}}) \quad \begin{cases} \boldsymbol{Q}_1(x_2, \ldots, x_m) = \big\{x \in \mathcal{H}_1 \mid (x, x_2, \ldots, x_m) \in \boldsymbol{C}\big\} \\ \qquad\vdots \\ \boldsymbol{Q}_m(x_1, \ldots, x_{m-1}) = \big\{x \in \mathcal{H}_m \mid (x_1, \ldots, x_{m-1}, x) \in \boldsymbol{C}\big\}. \end{cases} \tag{7.62}$$

The problem is to find $x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m$ such that

$$\begin{cases} x_1 \in \underset{x \in \boldsymbol{Q}_1(x_2,\ldots,x_m)}{\operatorname{Argmin}} \; \boldsymbol{g}_1(x, x_2, \ldots, x_m) \\ \quad\vdots \\ x_m \in \underset{x \in \boldsymbol{Q}_m(x_1,\ldots,x_{m-1})}{\operatorname{Argmin}} \; \boldsymbol{g}_m(x_1, \ldots, x_{m-1}, x). \end{cases} \tag{7.63}$$

The solutions to Example 7.18 are called generalized Nash equilibria [10], social equilibria [9], or equilibria of abstract economies [1], and their existence has been studied in [1, 9]. We deduce from Proposition 7.11 that we can find a solution to Example 7.18 by solving a variational inequality in $\boldsymbol{\mathcal{H}}$, provided the latter has solutions. This observation is also made in [10], which investigates a Euclidean setting in which additional smoothness properties are imposed on $(\boldsymbol{g}_i)_{1\le i\le m}$. An alternative approach for solving Example 7.18 in Euclidean spaces is also proposed in [12] with stronger differentiability properties on $(\boldsymbol{g}_i)_{1\le i\le m}$ and a monotonicity assumption of the form (7.61). However, the convergence of the method is not guaranteed. Below we derive from Section 7.2.3.2 a weakly convergent method for solving Example 7.18.

Proposition 7.19 *In Example 7.18, suppose that there exist $(z_1, \ldots, z_m) \in \boldsymbol{\mathcal{H}}$ such that*

$$-\big(\nabla_1 \boldsymbol{g}_1(z_1, \ldots, z_m), \ldots, \nabla_m \boldsymbol{g}_m(z_1, \ldots, z_m)\big) \in N_{\boldsymbol{C}}(z_1, \ldots, z_m) \tag{7.64}$$

and $\chi \in \left]0, +\infty\right[$ such that

$$(\forall (x_1, \ldots, x_m) \in \boldsymbol{\mathcal{H}})(\forall (y_1, \ldots, y_m) \in \boldsymbol{\mathcal{H}}) \quad \sum_{i=1}^{m} \|\nabla_i \boldsymbol{g}_i(x_1, \ldots, x_m) - \nabla_i \boldsymbol{g}_i(y_1, \ldots, y_m)\|^2 \le \chi^2 \sum_{i=1}^{m} \|x_i - y_i\|^2. \tag{7.65}$$

Let $\varepsilon \in \left]0, 1/(\chi+1)\right[$ and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\chi]$. Moreover, for every $i \in \{1, \ldots, m\}$, let $x_{i,0} \in \mathcal{H}_i$, and let $(a_{i,n})_{n\in\mathbb{N}}$, $(b_{i,n})_{n\in\mathbb{N}}$, and $(c_{i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}_i$. Now consider the following routine.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \text{For } i = 1, \ldots, m \\ \lfloor\, y_{i,n} = x_{i,n} - \gamma_n(\nabla_i \boldsymbol{g}_i(x_{1,n}, \ldots, x_{m,n}) + a_{i,n}) \\ (p_{1,n}, \ldots, p_{m,n}) = P_{\boldsymbol{C}}(y_{1,n}, \ldots, y_{m,n}) + (b_{1,n}, \ldots, b_{m,n}) \\ \text{For } i = 1, \ldots, m \\ \left\lfloor \begin{array}{l} q_{i,n} = p_{i,n} - \gamma_n(\nabla_i \boldsymbol{g}_i(p_{1,n}, \ldots, p_{m,n}) + c_{i,n}) \\ x_{i,n+1} = x_{i,n} - y_{i,n} + q_{i,n}. \end{array} \right. \end{array} \right. \tag{7.66}$$

Then there exists a solution $(\overline{x}_1, \ldots, \overline{x}_m)$ to Example 7.18 such that, for every $i \in \{1, \ldots, m\}$, $x_{i,n} \rightharpoonup \overline{x}_i$ and $p_{i,n} \rightharpoonup \overline{x}_i$.

Proof. Example 7.18 corresponds to the particular instance of Problem 7.10 in which $\boldsymbol{f} = \iota_{\boldsymbol{C}}$. Hence, since $P_{\boldsymbol{C}} = \operatorname{prox}_{\boldsymbol{f}}$, the result follows from Theorem 7.13. □

7.2.4.3 Cyclic proximation problem

We consider the following problem in $\boldsymbol{\mathcal{H}} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$.

Example 7.20 Let $\mathcal{G}$ be a real Hilbert space, let $\boldsymbol{f} \in \Gamma_0(\boldsymbol{\mathcal{H}})$, and, for every $i \in \{1, \ldots, m\}$, let $L_i \colon \mathcal{H}_i \to \mathcal{G}$ be a bounded linear operator. The problem is to find $x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m$ such that

$$\begin{cases} x_1 \in \underset{x \in \mathcal{H}_1}{\operatorname{Argmin}} \ \boldsymbol{f}(x, x_2, \ldots, x_m) + \frac{1}{2}\|L_1 x - L_2 x_2\|^2 \\ x_2 \in \underset{x \in \mathcal{H}_2}{\operatorname{Argmin}} \ \boldsymbol{f}(x_1, x, \ldots, x_m) + \frac{1}{2}\|L_2 x - L_3 x_3\|^2 \\ \quad \vdots \\ x_m \in \underset{x \in \mathcal{H}_m}{\operatorname{Argmin}} \ \boldsymbol{f}(x_1, \ldots, x_{m-1}, x) + \frac{1}{2}\|L_m x - L_1 x_1\|^2. \end{cases} \tag{7.67}$$

For every $i \in \{1, \ldots, m\}$, the individual penalty function of player i models his desire to keep some linear transformation L_i of his strategy close to some linear transformation of that of the next player $i+1$. In the particular case when $\boldsymbol{f} \colon (x_i)_{1 \leq i \leq m} \mapsto \sum_{i=1}^m f_i(x_i)$, a similar formulation is studied in [2, Section 3.1], where an algorithm is proposed for solving (7.67). However, each step of the algorithm involves the proximity operator of a sum of convex functions, which is extremely difficult to implement numerically. The method described below circumvents this difficulty.

Proposition 7.21 *In Example 7.20, suppose that there exists $(z_1, \ldots, z_m) \in \boldsymbol{\mathcal{H}}$ such that*

$$\big(L_1^*(L_2 z_2 - L_1 z_1), \ldots, L_m^*(L_1 z_1 - L_m z_m)\big) \in \partial \boldsymbol{f}(z_1, \ldots, z_m). \tag{7.68}$$

Set $\chi = 2\max_{1 \leq i \leq m} \|L_i\|^2$, let $\varepsilon \in \left]0, 2/(\chi+1)\right[$ and let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence in $[\varepsilon, (2-\varepsilon)/\chi]$. For every $i \in \{1, \ldots, m\}$, let $x_{i,0} \in \mathcal{H}_i$, and let $(a_{i,n})_{n \in \mathbb{N}}$ and $(b_{i,n})_{n \in \mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}_i$. Now set $L_{m+1} = L_1$, for every $n \in \mathbb{N}$, set $x_{m+1,n} = x_{1,n}$, and consider the following routine.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \text{For } i = 1, \ldots, m \\ \lfloor\, y_{i,n} = x_{i,n} - \gamma_n\big(L_i^*(L_i x_{i,n} - L_{i+1} x_{i+1,n}) + a_{i,n}\big) \\ (x_{1,n+1}, \ldots, x_{m,n+1}) = \operatorname{prox}_{\gamma_n \boldsymbol{f}}(y_{1,n}, \ldots, y_{m,n}) + (b_{1,n}, \ldots, b_{m,n}). \end{array} \right. \tag{7.69}$$

Then there exists a solution $(\overline{x}_1, \ldots, \overline{x}_m)$ to Example 7.20 such that, for every $i \in \{1, \ldots, m\}$, $x_{i,n} \rightharpoonup \overline{x}_i$ and $L_i^*\big(L_i(x_{i,n} - \overline{x}_i) - L_{i+1}(x_{i+1,n} - \overline{x}_{i+1})\big) \to 0$.

Proof. Note that Example 7.20 corresponds to the particular instance of Problem 7.10 in which, for every $i \in \{1, \ldots, m\}$, $\boldsymbol{g}_i \colon (x_i)_{1 \leq i \leq m} \mapsto \|L_i x_i - L_{i+1} x_{i+1}\|/2$, where we set

$x_{m+1} = x_1$. Indeed, since

$$(\forall (x_1,\ldots,x_m) \in \boldsymbol{\mathcal{H}}) \quad \begin{cases} \nabla_1 \boldsymbol{g}_1(x_1,\ldots,x_m) = L_1^*(L_1x_1 - L_2x_2) \\ \qquad\vdots \\ \nabla_m \boldsymbol{g}_m(x_1,\ldots,x_m) = L_m^*(L_mx_m - L_1x_1), \end{cases} \tag{7.70}$$

the operator $(x_i)_{1\leq i\leq m} \mapsto (\nabla_i \boldsymbol{g}_i(x_1,\ldots,x_m))_{1\leq i\leq m}$ is linear and bounded. Thus, for every $(x_1,\ldots,x_m) \in \boldsymbol{\mathcal{H}}$,

$$\begin{aligned} \sum_{i=1}^m \langle \nabla_i \boldsymbol{g}_i(x_1,\ldots,x_m) \mid x_i\rangle &= \sum_{i=1}^m \langle L_i^*(L_ix_i - L_{i+1}x_{i+1}) \mid x_i\rangle \\ &= \sum_{i=1}^m \langle L_ix_i - L_{i+1}x_{i+1} \mid L_ix_i\rangle \\ &= \sum_{i=1}^m \|L_ix_i\|^2 - \sum_{i=1}^m \langle L_{i+1}x_{i+1} \mid L_ix_i\rangle \\ &= \frac{1}{2}\sum_{i=1}^m \|L_ix_i\|^2 + \frac{1}{2}\sum_{i=1}^m \|L_{i+1}x_{i+1}\|^2 - \sum_{i=1}^m \langle L_{i+1}x_{i+1} \mid L_ix_i\rangle \\ &= \sum_{i=1}^m \frac{1}{2}\|L_ix_i - L_{i+1}x_{i+1}\|^2 \\ &= \sum_{i=1}^m \frac{1}{2\|L_i\|^2}\|L_i\|^2\|L_ix_i - L_{i+1}x_{i+1}\|^2 \\ &\geq \chi^{-1}\sum_{i=1}^m \|L_i^*(L_ix_i - L_{i+1}x_{i+1})\|^2 \\ &= \chi^{-1}\sum_{i=1}^m \|\nabla_i \boldsymbol{g}_i(x_1,\ldots,x_m)\|^2, \end{aligned} \tag{7.71}$$

and hence (7.50) and (7.26) hold. In addition, (7.68) yields (7.49). Altogether, since (7.51) reduces to (7.69), the result follows from Theorem 7.14. ☐

We present below an application of Proposition 7.21 to cyclic proximation problems and, in particular, to cyclic projection problems.

Example 7.22 We apply Example 7.20 to cyclic evaluations of proximity operators. For every $i \in \{1,\ldots,m\}$, let $\mathcal{H}_i = \mathcal{H}$, let $f_i \in \Gamma_0(\mathcal{H})$, let $L_i = \mathrm{Id}$, and set $\boldsymbol{f}\colon (x_i)_{1\leq i\leq m} \mapsto \sum_{i=1}^m f_i(x_i)$. In view of (7.30), Example 7.20 reduces to finding $x_1 \in \mathcal{H},\ldots,x_m \in \mathcal{H}$ such that

$$\begin{cases} x_1 = \operatorname{prox}_{f_1} x_2 \\ x_2 = \operatorname{prox}_{f_2} x_3 \\ \quad\vdots \\ x_m = \operatorname{prox}_{f_m} x_1. \end{cases} \tag{7.72}$$

It is assumed that (7.72) has at least one solution. Since $\operatorname{prox}_{\boldsymbol{f}}\colon (x_i)_{1\leq i\leq m} \mapsto (\operatorname{prox}_{f_i} x_i)_{1\leq i\leq m}$ [5, Proposition 23.30], (7.69) becomes (we set errors to zero for simplicity)

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \text{For } i = 1,\ldots,m \\ \lfloor\, x_{i,n+1} = \operatorname{prox}_{\gamma_n f_i}((1-\gamma_n)x_{i,n} + \gamma_n x_{i+1,n}), \end{array} \right. \tag{7.73}$$

where $(x_{i,0})_{1\leq i\leq m} \in \mathcal{H}^m$ and $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence in $[\varepsilon, 1-\varepsilon]$ for some arbitrary $\varepsilon \in\,]0, 1/2[$. Proposition 7.21 asserts that the sequences $(x_{1,n})_{n\in\mathbb{N}}, \ldots, (x_{m,n})_{n\in\mathbb{N}}$ generated by (7.73) converge weakly to points $\overline{x}_1 \in \mathcal{H}, \ldots, \overline{x}_m \in \mathcal{H}$, respectively, such that $(\overline{x}_1, \ldots, \overline{x}_m)$ is a solution to (7.72).

In the particular case when, for every $i \in \{1,\ldots,m\}$, $f_i = \iota_{C_i}$, a solution of (7.72) represents a cycle of points in $C_1, \ldots, C_m$. It can be interpreted as a Nash equilibrium of the game in which, for every $i \in \{1,\ldots,m\}$, the strategies of player i, belong to C_i and its penalty function is $(x_i)_{1\leq i\leq m} \mapsto \|x_i - x_{i+1}\|^2$, that is, player i wants to have strategies as close as possible to the strategies of player $i+1$. Such schemes go back at least to [11]. It has recently been proved [4] that, in this case, if $m > 2$, the cycles are not minimizers of any potential, from which we infer that this problem cannot be reduced to a potential game. Note that (7.73) becomes

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \text{For } i = 1,\ldots,m \\ \lfloor\, x_{i,n+1} = P_{C_i}((1-\gamma_n)x_{i,n} + \gamma_n x_{i+1,n}) \end{array} \right. \tag{7.74}$$

and the sequences thus generated $(x_{1,n})_{n\in\mathbb{N}}, \ldots, (x_{m,n})_{n\in\mathbb{N}}$ converge weakly to points $\overline{x}_1 \in \mathcal{H}, \ldots, \overline{x}_m \in \mathcal{H}$, respectively, such that $(\overline{x}_1, \ldots, \overline{x}_m)$ is a cycle. The existence of cycles has been proved in [11] when one of the sets $C_1, \ldots, C_m$ is bounded. Thus, (7.74) is an alternative parallel algorithm to the method of successive projections [11].

7.3 Bibliographie

[1] K. J. Arrow and G. Debreu, Existence of an equilibrium for a competitive economy, *Econometrica*, vol. 22, pp. 265–290, 1954.

[2] H. Attouch, J. Bolte, P. Redont, and A. Soubeyran, Alternating proximal algorithms for weakly coupled convex minimization problems – Applications to dynamical games and PDE's, *J. Convex Anal.*, vol. 15, pp. 485–506, 2008.

[3] H. Attouch, L. M. Briceño-Arias, and P. L. Combettes, A parallel splitting method for coupled monotone inclusions, *SIAM J. Control Optim.*, vol. 48, pp. 3246–3270, 2010.

[4] J.-B. Baillon, P. L. Combettes, and R. Cominetti, There is no variational characterization of the cycles in the method of periodic projections, http ://arxiv.org/pdf/1102.1378v1.

[5] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, 2011.

[6] L. M. Briceño-Arias and P. L. Combettes, A monotone+skew splitting model for composite monotone inclusions in duality, *SIAM J. Optim.,* to appear.

[7] L. M. Briceño-Arias, P. L. Combettes, J.-C. Pesquet, and N. Pustelnik, Proximal algorithms for multicomponent image processing, *J. Math. Imaging Vision*, 2011. DOI : 10.1007/s10851-010-0243-1.

[8] P. L. Combettes, Solving monotone inclusions via compositions of nonexpansive averaged operators, *Optimization,* vol. 53, pp. 475–504, 2004.

[9] G. Debreu, A social equilibrium existence theorem, *Proc. Nat. Acad. Sci. USA*, vol. 38, pp. 886–893, 1952.

[10] F. Facchinei and C. Kanzow, Generalized Nash equilibrium problems, *Ann. Oper. Res.*, vol. 175, pp. 177–211, 2010.

[11] L. G. Gubin, B. T. Polyak, and E. V. Raik, The method of projections for finding the common point of convex sets, *Comput. Math. Math. Phys.*, vol. 7, pp. 1–24, 1967.

[12] A. Von Heusinger and C. Kanzow, Relaxation methods for generalized Nash equilibrium problems with inexact line search, *J. Optim. Theory Appl.*, vol. 143, pp. 159–183, 2009.

[13] D. Monderer and L. S. Shapley, Potential games, *Games Econom. Behav.*, vol. 14, pp. 124–143, 1996.

[14] R. T. Rockafellar, Monotone operators associated with saddle-functions and minimax problems, in : *Nonlinear Functional Analysis, Part 1*, F. E. Browder (Ed.), Proc. Sympos. Pure Math., vol. 18, Amer. Math. Soc., Providence, RI, pp. 241–250, 1970.

[15] P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, *SIAM J. Control Optim.*, vol. 38, pp. 431–446, 2000.

[16] J. W. Weibull, *Evolutionary Game Theory,* MIT Press, Cambridge, MA, 1995.

Chapitre 8

Résolution de problèmes de point fixe composites

8.1 Description et résultats principaux

Dans ce chapitre nous nous intéressons au problème de point fixe suivant.

Problème 8.1 Fixons $\varepsilon \in \,]0,1[$ et soit $(\beta_n)_{n\in\mathbb{N}}$ une suite dans $]0,1-\varepsilon]$. Pour tout $n \in \mathbb{N}$, soit $T_n\colon \mathcal{H} \to \mathcal{H}$ une contraction ferme, soit $R_n\colon \operatorname{dom} R_n \subset \mathcal{H} \to \mathcal{H}$ une pseudo contraction telle que $(\operatorname{Id} - R_n)$ est un opérateur β_n–lipschitzien et soit $S \subset \mathcal{H}$ un ensemble non vide, fermé et convexe. Le problème est de

$$\text{trouver} \quad x \in Z = S \cap \bigcap_{n\in\mathbb{N}} \operatorname{Fix} T_n R_n. \tag{8.1}$$

Dans la littérature il existe des algorithmes pour résoudre le problème ci-dessus dans quelques cas particuliers. En effet, si $S = \mathcal{H}$, $R_n \equiv \operatorname{Id}$ et $Z \neq \varnothing$ des méthodes peuvent être trouvées dans [3, 1] et si $S = \mathcal{H}$, $T_n \equiv \operatorname{Id}$ et $R_n \equiv R$, où R est une pseudo contraction lipschitzienne d'un ensemble convexe $C \subset \mathcal{H}$ dans lui même, des méthodes sont dans [4, 5, 6, 7]. Cependant, vu que la composition d'une contraction ferme avec une pseudo contraction lipschitzienne n'est pas une pseudo contraction en général, le Problème 8.1 ne peut pas être résolu par ces méthodes.

Nous proposons l'algorithme suivant qui, dans chaque itération $n \in \mathbb{N}$, effectue séquentiellement des calculs explicites des opérateurs R_n, T_n et R_n suivis d'une approximation extérieure de la contrainte S. Cette approximation est faite à partir d'une projection sur un demi-espace affine et fermé qui contient S. La méthode tolère des erreurs dans l'évaluation de chaque opérateur impliqué.

Algorithme 8.2 Soient $(T_n)_{n\in\mathbb{N}}$, $(R_n)_{n\in\mathbb{N}}$ et S définis comme dans le Problème 8.1. Pour tout $n \in \mathbb{N}$, on note par $Q_n\colon \mathcal{H} \to \mathcal{H}$ l'opérateur de projection sur un demi-espace affine fermé contenant S, et soient $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ et $(c_n)_{n\in\mathbb{N}}$ des suites dans $\mathcal{H}$ telles que $\sum_{n\in\mathbb{N}}\|a_n\| < +\infty$, $\sum_{n\in\mathbb{N}}\|b_n\| < +\infty$ et $\sum_{n\in\mathbb{N}}\|c_n\| < +\infty$. De plus, soit $\varepsilon \in\,]0,1[$, soit $(\lambda_n)_{n\in\mathbb{N}}$ une suite dans $[\varepsilon, 1]$ et soit $x_0 \in \operatorname{dom} R_0$. On génère des suites $(x_n)_{n\in\mathbb{N}}$ et $(z_n)_{n\in\mathbb{N}}$ comme suit.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_n = R_n x_n + a_n \\ q_n = T_n y_n + b_n \\ \text{Si } q_n \notin \operatorname{dom} R_n \text{ s'arrêter.} \\ \text{Sinon} \\ \left\lfloor \begin{array}{l} r_n = R_n q_n + c_n \\ z_n = x_n - y_n + r_n \\ x_{n+1} = x_n + \lambda_n (Q_n z_n - x_n) \end{array} \right. \\ \text{Si } x_{n+1} \notin \operatorname{dom} R_{n+1} \text{ s'arrêter.} \\ \text{Sinon } n = n+1. \end{array} \right. \tag{8.2}$$

Le résultat principal est le suivant.

Théorème 8.3 *Supposons que $Z \neq \varnothing$ dans le Problème 8.1 et que l'Algorithme 8.13 engendre des orbites infinies $(x_n)_{n\in\mathbb{N}}$ et $(z_n)_{n\in\mathbb{N}}$ telles que*

$$(\forall x \in \mathcal{H}) \quad \begin{cases} x_{k_n} \rightharpoonup x \\ x_n - T_n R_n x_n \to 0 \\ z_n - x_n \to 0 \\ z_n - Q_n z_n \to 0 \end{cases} \quad \Rightarrow \quad x \in Z. \tag{8.3}$$

Alors $(x_n)_{n\in\mathbb{N}}$ converge faiblement vers une solution du Problème 8.1.

Nous étudions deux applications de l'Algorithme 8.2. La première est une application au problème d'inclusions monotones suivant.

Problème 8.4 Soient $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ et $B\colon \operatorname{dom} B \subset \mathcal{H} \to \mathcal{H}$ deux opérateurs maximalement monotones tels que $\operatorname{dom} A \subset \operatorname{dom} B$ et supposons que $A+B$ soit maximalement monotone. Pour tout $i \in \{1,\ldots,m\}$, soit $f_i\colon \mathcal{H} \to \mathbb{R}$ une fonction semi-continue inférieurement et convexe, notons par $S = \operatorname{lev}_{\leq 0} f_1 \cap \cdots \cap \operatorname{lev}_{\leq 0} f_m \neq \varnothing$ et supposons que $S \subset \operatorname{dom} B$ et que B soit χ-lipschitzien sur $S \cup \operatorname{dom} A$ avec $\beta \in\,]0, +\infty[$. Le problème est de

$$\text{trouver } x \in \mathcal{H} \text{ tel que } \begin{cases} 0 \in Ax + Bx \\ f_1(x) \leq 0 \\ \quad\vdots \\ f_m(x) \leq 0. \end{cases} \tag{8.4}$$

Des conditions suffisantes pour que la somme de deux opérateurs maximalement monotones soit maximalement monotone peuvent être trouvées dans [9, Corollary 24.4].

Dans le cas particulier où $m = 1$, $f_1 = d_S$, un algorithme pour résoudre ce problème est proposé dans [8]. Cet algorithme ne tolère pas des erreurs de calcul et il ne peut être mis en œuvre que lorsque l'opérateur P_S est calculable. Cependant, vu que l'opérateur P_S n'est pas toujours calculable facilement, la méthode ne peut pas être utilisée pour résoudre le Problème 8.4. Nous proposons un nouveau algorithme qui résout ce problème et qui étend la méthode dans [8]. Cet algorithme procède en activant indépendamment les contraintes $f_1 \leq 0, \ldots, f_m \leq 0$ qui sont linéarisées. De plus, des erreurs dans le calcul des opérateurs impliqués sont tolérées. Pour la mise en œuvre de cette méthode nous utilisons le projecteur sous-différentiel par rapport à la fonction $f \in \Gamma_0(\mathcal{H})$, qui est défini par

$$G\colon x \mapsto \begin{cases} x - \dfrac{f(x)}{\|u\|^2}u, & \text{si } f(x) > 0; \\ x, & \text{sinon}, \end{cases} \tag{8.5}$$

où $u \in \partial f(x)$, et la fonction $\mathrm{i}\colon \mathbb{N} \to \{1, \ldots, m\}\colon n \mapsto 1 + \mathrm{rem}(n-1, m)$, où $\mathrm{rem}(\cdot, m)$ est le reste de la division par m.

Algorithme 8.5 Pour tout $i \in \{1, \ldots, m\}$, on note par $G_i\colon \mathcal{H} \to \mathcal{H}$ le projecteur sous-différentiel par rapport à f_i. Soient $(e_{1,n})_{n\in\mathbb{N}}$, $(e_{2,n})_{n\in\mathbb{N}}$ et $(e_{3,n})_{n\in\mathbb{N}}$ des suites dans $\mathcal{H}$ telles que $\sum_{n\in\mathbb{N}} \|e_{1,n}\| < +\infty$, $\sum_{n\in\mathbb{N}} \|e_{2,n}\| < +\infty$ et $\sum_{n\in\mathbb{N}} \|e_{3,n}\| < +\infty$. Soit $\varepsilon \in \left]0, 1/(\chi+1)\right[$, soit $(\gamma_n)_{n\in\mathbb{N}}$ une suite dans $\left]\varepsilon, (1-\varepsilon)/\chi\right[$ et soit $x_0 \in \mathrm{dom}\, B$. On génère une suite $(x_n)_{n\in\mathbb{N}}$ comme suit.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_n = x_n - \gamma_n(Bx_n + e_{1,n}) \\ q_n = J_{\gamma_n A}(y_n + e_{2,n}) \\ r_n = q_n - \gamma_n(Bq_n + e_{3,n}) \\ z_n = x_n - y_n + r_n \\ x_{n+1} = G_{\mathrm{i}(n)}\, z_n. \end{array} \right. \tag{8.6}$$

Dans l'Algorithme 8.5, les suites $(e_{1,n})_{n\in\mathbb{N}}$ et $(e_{3,n})_{n\in\mathbb{N}}$ représentent des erreurs dans le calcul de l'opérateur B. De plus, nous supposons que les résolvantes $(J_{\gamma_n A})_{n\in\mathbb{N}}$ peuvent être calculées approximativement, pour tout $n \in \mathbb{N}$, en résolvant l'inclusion perturbée

$$\text{trouver} \quad q \in \mathcal{H} \quad \text{tel que} \quad y_n - q + e_{2,n} \in \gamma_n A q. \tag{8.7}$$

Nous obtenons le résultat de convergence suivant.

Proposition 8.6 *Supposons que*

$$\bigcup_{i=1}^{m} \mathrm{ran}\, G_i \subset \mathrm{dom}\, B \quad \textit{et que} \quad S \cap \mathrm{zer}(A+B) \neq \varnothing. \tag{8.8}$$

Alors l'Algorithme 8.5 engendre une orbite infinie $(x_n)_{n\in\mathbb{N}}$ qui converge faiblement vers une solution du Problème 8.4.

La deuxième application est consacrée aux problèmes d'équilibre du type

$$\text{trouver} \quad x \in C \quad \text{tel que} \quad (\forall y \in C) \quad F(x,y) \geq 0, \tag{8.9}$$

où F et C satisfont l'hypothèse suivante.

Hypothèse 8.7 C est un sous-ensemble fermé convexe et non vide de $\mathcal{H}$ et $F\colon C^2 \to \mathbb{R}$ satisfait les conditions suivantes.

(i) $(\forall x \in C) \quad F(x,x) = 0$.

(ii) $(\forall (x,y) \in C^2) \quad F(x,y) + F(y,x) \leq 0$.

(iii) Pour tout x dans C, $F(x,\cdot)\colon C \to \mathbb{R}$ est semi-continue inférieurement et convexe.

(iv) $(\forall (x,y,z) \in C^3) \quad \overline{\lim}_{\varepsilon \to 0^+} F((1-\varepsilon)x + \varepsilon z, y) \leq F(x,y)$.

Le problème que nous étudions est plus général que (8.9) et il sera présenté après les préliminaires suivantes.

La résolvante de $F\colon C^2 \to \mathbb{R}$ est l'opérateur multivoque

$$J_F\colon \mathcal{H} \to 2^C\colon x \mapsto \big\{z \in C \mid (\forall y \in C) \quad F(z,y) + \langle z - x \mid y - z\rangle \geq 0\big\} \tag{8.10}$$

et, pour tout $\delta \in \left]0,+\infty\right[$, la δ–résolvante de $F\colon C^2 \to \mathbb{R}$ est l'opérateur multivoque

$$J_F^\delta\colon \mathcal{H} \to 2^C\colon x \mapsto \big\{z \in C \mid (\forall y \in C) \quad F(z,y) + \langle z - x \mid y - z\rangle \geq -\delta\big\}. \tag{8.11}$$

Nous aussi démontrons le lemme suivant.

Lemme 8.8 *Supposons que $F\colon C^2 \to \mathbb{R}$ satisfasse l'Hypothèse 8.7. Alors les propriétés suivantes sont satisfaites.*

(i) $\operatorname{dom} J_F = \mathcal{H}$.

(ii) *J_F est une contraction ferme.*

(iii) $(\forall x \in \mathcal{H})(\forall \delta \in \left]0,+\infty\right[) \quad J_F x \in J_F^\delta x$.

(iv) $(\forall x \in \mathcal{H})(\forall \delta \in \left]0,+\infty\right[) \quad J_F^\delta x \subset B(J_F x; \sqrt{\delta})$.

Problème 8.9 Supposons que F et C satisfassent l'Hypothèse 8.7. Soit $(S_i)_{i\in I}$ une famille dénombrable (finie or infinie dénombrable) de sous-ensembles convexes, fermés et non vides de $\mathcal{H}$ tels que $S = \cap_{i\in I} S_i \neq \varnothing$. Soit $B\colon \operatorname{dom} B \subset \mathcal{H} \to \mathcal{H}$ un opérateur monotone et χ–lipschitzien avec $\chi \in \left]0,+\infty\right[$ tel que $C \subset \operatorname{dom} B$ et supposons que

$$\bigcup_{i\in I} S_i \subset \operatorname{int} \operatorname{dom} B. \tag{8.12}$$

Le problème est de

$$\text{trouver} \quad x \in S \quad \text{tel que} \quad (\forall y \in C) \quad F(x,y) + \langle y - x \mid Bx\rangle \geq 0. \tag{8.13}$$

Le Problème 8.9 modélise un large éventail de problèmes, incluant des problèmes de complémentarité, d'optimisation, d'admissibilité, de point fixe, de la théorie des jeux, des inéquations variationnelles, entre autres [10, 2, 22, 23, 24, 25].

Dans la littérature il existe des algorithmes pour résoudre le problème d'équilibre

$$\text{trouver} \quad x \in C \quad \text{tel que} \quad (\forall y \in C) \quad F_1(x,y) + F_2(x,y) \geq 0, \tag{8.14}$$

où F_1 et F_2 satisfont l'Hypothèse 8.7. Ces méthodes exploitent des propriétés inhérentes de F_1 et F_2 séparément. Par exemple, des méthodes qui utilisent les résolvantes J_{F_1} et J_{F_2} sont proposées dans [26] avec seulement un résultat de convergence ergodique. Cependant, si $F_1 = F$ et $F_2\colon (x,y) \mapsto \langle y - x \mid Bx\rangle$ comme dans (8.13), nous avons $J_{F_2} = (\mathrm{Id} + B)^{-1}$ [2, Lemma 2.15(i)], ce qui est souvent difficile à calculer, même dans le cas où B est linéaire. De plus, les méthodes ergodiques présentées dans [26] incluent des paramètres évanescents qui génèrent des instabilités numériques, ce qui fait que ces méthodes ne soient pas très utiles en pratique. Dans [2, 27] une approche non ergodique est développée pour surmonter ces problèmes dans le cas où B est cocoercif. Dans ces méthodes l'opérateur B est calculé explicitement et la convergence vers une solution de (8.13) est établie dans le cas où $S = C$. Nous proposons un approche non ergodique qui considère un opérateur B monotone et lipschitzien et des contraintes dans la formulation du problème. L'algorithme est le suivant.

Algorithme 8.10 Soit $(I_n)_{n\in\mathbb{N}}$ une suite de sous-ensembles finis de I, soient $(e_{1,n})_{n\in\mathbb{N}}$ et $(e_{2,n})_{n\in\mathbb{N}}$ des suites dans $\mathcal{H}$ telles que $\sum_{n\in\mathbb{N}} \|e_{1,n}\| < +\infty$ et $\sum_{n\in\mathbb{N}} \|e_{2,n}\| < +\infty$, et soit $(\delta_n)_{n\in\mathbb{N}}$ une suite dans $]0,+\infty[$ telle que $\sum_{n\in\mathbb{N}} \sqrt{\delta_n} < +\infty$. De plus, soit $\varepsilon \in\,]0, 1/(\chi+1)[$, soit $(\gamma_n)_{n\in\mathbb{N}}$ une suite dans $[\varepsilon, (1-\varepsilon)/\chi]$, soit $\cup_{n\in\mathbb{N}}\{\omega_{i,n}\}_{i\in I_n} \subset [\varepsilon, 1]$ tel que, pour tout $n \in \mathbb{N}$, $\sum_{i\in I_n} \omega_{i,n} = 1$, et soit $x_0 \in \operatorname{dom} B$. On génère une suit $(x_n)_{n\in\mathbb{N}}$ comme suit.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_n = x_n - \gamma_n(Bx_n + e_{1,n}) \\ q_n \in J^{\delta_n}_{\gamma_n F} y_n \\ r_n = q_n - \gamma_n(Bq_n + e_{2,n}) \\ z_n = x_n - y_n + r_n \\ x_{n+1} = \sum_{i\in I_n} \omega_{i,n} P_{S_i} z_n. \end{array} \right. \tag{8.15}$$

Dans l'Algorithme 8.10, les suites $(e_{1,n})_{n\in\mathbb{N}}$ et $(e_{2,n})_{n\in\mathbb{N}}$ représentent des erreurs dans le calcul de l'opérateur B. De plus, nous déduisons de (8.15) et (8.11) que, pour tout $n \in \mathbb{N}$, q_n est une solution de

$$\text{trouver} \quad q \in C \quad \text{tel que} \quad (\forall y \in C) \quad F(q,y) + \langle y - y_n \mid y - q\rangle \geq -\delta_n. \tag{8.16}$$

Alors nous tirons de (8.10) que q_n peut être interprété comme un calcul approché de la résolvante $J_{\gamma_n F} y_n$.

Proposition 8.11 *Supposons qu'il existe des entiers strictement positifs* $(M_i)_{i\in I}$ *et* N *tels que*

$$(\forall (i,n) \in I \times \mathbb{N}) \quad i \in \bigcup_{k=n}^{n+M_i-1} I_k \quad \textit{et} \quad 1 \leq \operatorname{card} I_n \leq N \tag{8.17}$$

et que le Problème 8.9 admette au moins une solution. Alors l'Algorithme 8.10 génère une orbite infinie $(x_n)_{n\in\mathbb{N}}$ *qui converge faiblement vers une solution du Problème 8.9.*

8.2 Article en anglais

OUTER APPROXIMATION METHOD FOR CONSTRAINED COMPOSITE FIXED POINT PROBLEMS INVOLVING LIPSCHITZ PSEUDO CONTRACTIVE OPERATORS [1]

Abstract : We propose a method for solving constrained fixed point problems involving compositions of Lipschitz pseudo contractive and firmly nonexpansive operators in Hilbert spaces. Each iteration of the method uses separate evaluations of these operators and an outer approximation given by the projection onto a closed half-space containing the constraint set. Its convergence is established and applications to monotone inclusion splitting and constrained equilibrium problems are demonstrated.

8.2.1 Introduction

The problem under consideration in this paper is the following.

Problem 8.12 Let $\mathcal{H}$ be a real Hilbert space, fix $\varepsilon \in]0,1[$, and let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence in $]0,1-\varepsilon]$. For every $n \in \mathbb{N}$, let $T_n \colon \mathcal{H} \to \mathcal{H}$ be a firmly nonexpansive operator, let $R_n \colon \operatorname{dom} R_n \subset \mathcal{H} \to \mathcal{H}$ be a pseudo contraction such that $(\mathrm{Id} - R_n)$ is β_n–Lipschitzian, and let S be a closed convex subset of $\mathcal{H}$. The problem is to

$$\text{find} \quad x \in S \quad \text{such that} \quad (\forall n \in \mathbb{N}) \quad T_n R_n x = x. \tag{8.18}$$

The set of solutions to (8.18) is denoted by Z.

As will be seen subsequently, this formulation models a broad range of problems in numerical analysis, including monotone inclusions, variational inequalities, and equilibrium problems (see [1, 2] and the references therein). Methods can be found in the literature to solve Problem 8.12 in special cases. Thus, when $S = \mathcal{H}$, $R_n \equiv \mathrm{Id}$, and $Z \neq \varnothing$,

1. L. M. Briceño-Arias, Outer approximation method for constrained composite fixed point problems involving Lipschitz pseudo contractive operators, *Numerical Functional Analysis and Optimization*, à paraître, 2011.

algorithms can be found in [1, 3], and when $S = \mathcal{H}$, $T_n \equiv \mathrm{Id}$, and $R_n \equiv R$, where R is a Lipschitzian pseudo contraction from a convex set C into itself, methods can be found in [4, 5, 6, 7]. Since the composition between a firmly nonexpansive operator and a Lipschitzian pseudo contraction is not a pseudo contraction in general, Problem 8.12 can not be solved by the methods mentioned above. The purpose of the present paper is to provide an algorithm for solving Problem 8.12. It involves four elementary steps at each iteration n : the first three steps are successive computations of operators R_n, T_n, and R_n, and the last step is an outer approximation of the constraint. The latter is given by the projection onto a half-space containing S. In Section 8.2.2 we propose our algorithm and we prove its weak convergence to a solution to Problem 8.12. In Section 8.2.3 we study an application to monotone inclusions under convex constraints, and obtain an extension of a result of [8]. Finally, in Section 8.2.4, we study an application to equilibrium problems with convex constraints.

Notation. Throughout this paper $\mathcal{H}$ denotes a real Hilbert space, $\langle \cdot \mid \cdot \rangle$ denotes its scalar product, and $\| \cdot \|$ denotes the associated norm. For a single-valued operator $R\colon \operatorname{dom} R \subset \mathcal{H} \to \mathcal{H}$, the set of fixed points is $\operatorname{Fix} R = \big\{x \in \mathcal{H} \mid x = Rx\big\}$, R is χ–Lipschitzian for some $\chi \in \left]0, +\infty\right[$, if it satisfies

$$(\forall x \in \operatorname{dom} R)(\forall y \in \operatorname{dom} R) \quad \|Rx - Ry\| \leq \chi \|x - y\|, \tag{8.19}$$

R is pseudo contractive if it satisfies

$$(\forall x \in \operatorname{dom} R)(\forall y \in \operatorname{dom} R) \quad \|Rx - Ry\|^2 \leq \|x - y\|^2 + \|(\mathrm{Id} - R)x - (\mathrm{Id} - R)y\|^2, \tag{8.20}$$

R is firmly nonexpansive if it satisfies

$$(\forall x \in \operatorname{dom} R)(\forall y \in \operatorname{dom} R) \quad \|Rx - Ry\|^2 \leq \|x - y\|^2 - \|(\mathrm{Id} - R)x - (\mathrm{Id} - R)y\|^2, \tag{8.21}$$

or equivalently,

$$(\forall x \in \operatorname{dom} R)(\forall y \in \operatorname{dom} R) \quad \langle x - y \mid Rx - Ry \rangle \geq \|Rx - Ry\|^2, \tag{8.22}$$

and R is χ–cocoercive (or χ-inverse-strongly-monotone) if χR is firmly nonexpansive.

8.2.2 Algorithm and convergence

At each iteration $n \in \mathbb{N}$, our method for solving Problem 8.12 involves an outer approximation to S and separate computations of the operators T_n and R_n. Each approximation is computed by the projection onto a closed affine half-space containing S, and errors on the computation of the operators are modeled by the sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, and $(c_n)_{n\in\mathbb{N}}$.

Algorithm 8.13 Let $(T_n)_{n\in\mathbb{N}}$, $(R_n)_{n\in\mathbb{N}}$, and S be as in Problem 8.12. For every $n \in \mathbb{N}$, let $Q_n\colon \mathcal{H} \to \mathcal{H}$ be the projector operator onto a closed affine half-space containing

S, let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, and $(c_n)_{n\in\mathbb{N}}$ be sequences in $\mathcal{H}$ such that $\sum_{n\in\mathbb{N}}\|a_n\| < +\infty$, $\sum_{n\in\mathbb{N}}\|b_n\| < +\infty$, and $\sum_{n\in\mathbb{N}}\|c_n\| < +\infty$. Moreover, let $\varepsilon \in\,]0,1[$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon,1]$, let $x_0 \in \operatorname{dom} R_0$, and consider the following routine.

$$(\forall n\in\mathbb{N})\quad \left\lfloor \begin{array}{l} y_n = R_n x_n + a_n\\ q_n = T_n y_n + b_n\\ \text{If } q_n \notin \operatorname{dom} R_n \text{ stop.}\\ \text{Else}\\ \left\lfloor \begin{array}{l} r_n = R_n q_n + c_n\\ z_n = x_n - y_n + r_n\\ x_{n+1} = x_n + \lambda_n\big(Q_n z_n - x_n\big)\end{array}\right.\\ \text{If } x_{n+1}\notin \operatorname{dom} R_{n+1} \text{ stop.}\\ \text{Else } n = n+1.\end{array}\right. \tag{8.23}$$

Our main result is the following.

Theorem 8.14 *Suppose that $Z \neq \varnothing$ in Problem 8.12 and that Algorithm 8.13 generates infinite orbits $(x_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ such that*

$$(\forall x\in\mathcal{H})\quad \begin{cases} x_{k_n} \rightharpoonup x\\ x_n - T_n R_n x_n \to 0\\ z_n - x_n \to 0\\ z_n - Q_n z_n \to 0\end{cases} \quad\Rightarrow\quad x\in Z. \tag{8.24}$$

Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a solution to Problem 8.12.

Proof. Set

$$(\forall n\in\mathbb{N})\quad \widetilde{y}_n = R_n x_n,\quad \widetilde{q}_n = T_n\widetilde{y}_n,\quad\text{and}\quad \widetilde{r}_n = R_n\widetilde{q}_n, \tag{8.25}$$

fix $z\in Z$, and let $n\in\mathbb{N}$. Note that, since $z\in S$, we have

$$z = P_S z = Q_n z = T_n R_n z = R_n z + (\operatorname{Id} - R_n)T_n R_n z. \tag{8.26}$$

In addition, it follows from [9, Theorem 1] that $(\operatorname{Id} - R_n)$ is monotone, which yields $\langle(\operatorname{Id} - R_n)\widetilde{q}_n - (\operatorname{Id} - R_n)z \mid \widetilde{q}_n - z\rangle \geq 0$. Therefore, we deduce from (8.26), (8.25), and the firm nonexpansivity of T_n that

$$\begin{aligned} 2\,\langle\widetilde{q}_n - z \mid (\operatorname{Id} - R_n)x_n - (\operatorname{Id} - R_n)\widetilde{q}_n\rangle &= -2\,\langle\widetilde{q}_n - z \mid (\operatorname{Id} - R_n)\widetilde{q}_n - (\operatorname{Id} - R_n)z)\rangle\\ &\quad + 2\,\langle\widetilde{q}_n - z \mid x_n - z\rangle - 2\,\langle\widetilde{q}_n - z \mid R_n x_n - R_n z\rangle\\ &\leq 2\,\langle\widetilde{q}_n - z \mid x_n - z\rangle - 2\,\langle T_n\widetilde{y}_n - T_n R_n z \mid \widetilde{y}_n - R_n z\rangle\\ &\leq 2\,\langle\widetilde{q}_n - z \mid x_n - z\rangle - 2\|T_n\widetilde{y}_n - T_n R_n z\|^2\\ &= \big(2\,\langle\widetilde{q}_n - z \mid x_n - z\rangle - \|\widetilde{q}_n - z\|^2\big) - \|\widetilde{q}_n - z\|^2\\ &\leq \|x_n - z\|^2 - \|\widetilde{q}_n - x_n\|^2 - \|\widetilde{q}_n - z\|^2. \end{aligned} \tag{8.27}$$

Hence, since $\sup_{k\in\mathbb{N}} \beta_k^2 \leq (1-\varepsilon)^2 \leq 1-\varepsilon$, it follows from (8.25) and the β_n–Lipschitz property of $(\mathrm{Id} - R_n)$ that

$$\begin{aligned}\|x_n - \widetilde{y}_n + \widetilde{r}_n - z\|^2 &= \|\widetilde{q}_n - z + (x_n - \widetilde{y}_n) - (\widetilde{q}_n - \widetilde{r}_n)\|^2 \\ &= \|\widetilde{q}_n - z + (\mathrm{Id} - R_n)x_n - (\mathrm{Id} - R_n)\widetilde{q}_n\|^2 \\ &= \|\widetilde{q}_n - z\|^2 + \|(\mathrm{Id} - R_n)x_n - (\mathrm{Id} - R_n)\widetilde{q}_n\|^2 \\ &\qquad + 2\left\langle \widetilde{q}_n - z \mid (\mathrm{Id} - R_n)x_n - (\mathrm{Id} - R_n)\widetilde{q}_n\right\rangle \\ &\leq \|\widetilde{q}_n - z\|^2 + \beta_n^2\|\widetilde{q}_n - x_n\|^2 \\ &\qquad + 2\left\langle \widetilde{q}_n - z \mid (\mathrm{Id} - R_n)x_n - (\mathrm{Id} - R_n)\widetilde{q}_n\right\rangle \\ &\leq \|x_n - z\|^2 - (1-\beta_n^2)\|\widetilde{q}_n - x_n\|^2 \\ &\leq \|x_n - z\|^2 - \varepsilon\|\widetilde{q}_n - x_n\|^2, \end{aligned} \tag{8.28}$$

which yields

$$\|x_n - \widetilde{y}_n + \widetilde{r}_n - z\| \leq \|x_n - z\|. \tag{8.29}$$

We also derive from (8.23) and (8.25) the following inequalities. First, $\|y_n - \widetilde{y}_n\| = \|a_n\|$, and since T_n is nonexpansive, we obtain

$$\|q_n - \widetilde{q}_n\| = \|T_n y_n + b_n - T_n\widetilde{y}_n\| \leq \|\widetilde{y}_n - y_n\| + \|b_n\| = \|a_n\| + \|b_n\|. \tag{8.30}$$

In turn, it follows from the β_n–Lipschitz property of $(\mathrm{Id} - R_n)$ that

$$\begin{aligned}\|r_n - \widetilde{r}_n\| &= \|R_n q_n + c_n - R_n\widetilde{q}_n\| \\ &\leq \|(\mathrm{Id} - R_n)\widetilde{q}_n - (\mathrm{Id} - R_n)q_n\| + \|q_n - \widetilde{q}_n\| + \|c_n\| \\ &\leq (1+\beta_n)\|q_n - \widetilde{q}_n\| + \|c_n\| \\ &\leq 2(\|a_n\| + \|b_n\|) + \|c_n\|. \end{aligned} \tag{8.31}$$

Altogether, if we set

$$e_n = \widetilde{y}_n - y_n + r_n - \widetilde{r}_n, \tag{8.32}$$

we have

$$\|e_n\| = \|\widetilde{y}_n - y_n + r_n - \widetilde{r}_n\| \leq \|y_n - \widetilde{y}_n\| + \|r_n - \widetilde{r}_n\| \leq 3\|a_n\| + 2\|b_n\| + \|c_n\|, \tag{8.33}$$

and therefore $\sum_{k\in\mathbb{N}} \|e_k\| < +\infty$. Hence, from (8.23), (8.26), the nonexpansivity of Q_n, and (8.29) we get

$$\begin{aligned}\|x_{n+1} - z\| &= \|(1-\lambda_n)(x_n - z) + \lambda_n(Q_n z_n - Q_n z)\| \\ &\leq (1-\lambda_n)\|x_n - z\| + \lambda_n\|Q_n z_n - Q_n z\| \\ &\leq (1-\lambda_n)\|x_n - z\| + \lambda_n\|z_n - z\| \\ &\leq (1-\lambda_n)\|x_n - z\| + \lambda_n\big(\|x_n - \widetilde{y}_n + \widetilde{r}_n - z\| + \|e_n\|\big) \\ &\leq \|x_n - z\| + \|e_n\|, \end{aligned} \tag{8.34}$$

and we conclude from [10, Lemma 3.1] that

$$\xi = \sup_{k\in\mathbb{N}} \|x_k - z\| < +\infty. \tag{8.35}$$

Thus, from the convexity of $\|\cdot\|^2$, the firm nonexpansivity of Q_n, (8.26), (8.23), (8.32), (8.28), and (8.29) we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1-\lambda_n)\|x_n - z\|^2 + \lambda_n\|Q_n z_n - Q_n z\|^2 \\
&\leq (1-\lambda_n)\|x_n - z\|^2 + \lambda_n\big(\|z_n - z\|^2 - \|z_n - Q_n z_n\|^2\big) \\
&\leq (1-\lambda_n)\|x_n - z\|^2 + \lambda_n\big(\|x_n - \widetilde{y}_n + \widetilde{r}_n - z\|^2 + \|e_n\|^2 \\
&\qquad\qquad + 2\,\|x_n - \widetilde{y}_n + \widetilde{r}_n - z\|\,\|e_n\| - \|z_n - Q_n z_n\|^2\big) \\
&\leq (1-\lambda_n)\|x_n - z\|^2 + \lambda_n\big(\|x_n - z\|^2 - \varepsilon\|\widetilde{q}_n - x_n\|^2 \\
&\qquad\qquad + \|e_n\|^2 + 2\,\|x_n - z\|\,\|e_n\| - \|z_n - Q_n z_n\|^2\big) \\
&\leq \|x_n - z\|^2 - \varepsilon^2\|\widetilde{q}_n - x_n\|^2 - \varepsilon\,\|z_n - Q_n z_n\|^2 + \eta_n,
\end{aligned} \tag{8.36}$$

where $\eta_n = \|e_n\|^2 + 2\xi\|e_n\|$ satisfies $\sum_{k\in\mathbb{N}} \eta_k < +\infty$. Hence, from [10, Lemma 3.1] we deduce that

$$\sum_{k\in\mathbb{N}} \|T_k R_k x_k - x_k\|^2 = \sum_{k\in\mathbb{N}} \|\widetilde{q}_k - x_k\|^2 < +\infty \quad \text{and} \quad \sum_{k\in\mathbb{N}} \|z_k - Q_k z_k\|^2 < +\infty, \tag{8.37}$$

and therefore $T_n R_n x_n - x_n = \widetilde{q}_n - x_n \to 0$ and $z_n - Q_n z_n \to 0$. Thus, it follows from (8.23) and the nonexpansivity of T_n that

$$\begin{aligned}
\|z_n - x_n\| &= \|r_n - y_n\| \\
&= \|\widetilde{r}_n - \widetilde{y}_n + e_n\| \\
&\leq \|T_n\widetilde{q}_n - T_n x_n\| + \|e_n\| \\
&\leq \|\widetilde{q}_n - x_n\| + \|e_n\| \\
&\to 0.
\end{aligned} \tag{8.38}$$

Altogether, since (8.24) asserts that all the weak limits of the sequence $(x_k)_{k\in\mathbb{N}}$ are in Z, the result follows from [10, Theorem 3.8]. □

8.2.3 Monotone inclusions with convex constraints

We consider the problem

$$\text{find} \quad x \in S \quad \text{such that} \quad 0 \in Ax + Bx, \tag{8.39}$$

where $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ and $B\colon \operatorname{dom} B \subset \mathcal{H} \to \mathcal{H}$ are maximally monotone, and $S \subset \mathcal{H}$ is nonempty, closed, and convex. When B is cocoercive, $\operatorname{dom} B = \mathcal{H}$, and $S = \mathcal{H}$, (8.39) models wide variety of problems in nonlinear analysis, and it can be solved by the forward-backward splitting method [11, 12, 13, 14, 15, 16]. However, in several applications these assumptions are very restrictive. If the cocoercivity of B is relaxed to a Lipschitz property, (8.39) can be solved by the modified forward-backward splitting in [8]. We propose an extension of this method for solving (8.39) with a finite number of convex constraints. In addition, our method allows for errors in the computations of the operators involved.

Notation. For a set-valued operator $A\colon \mathcal{H} \to 2^{\mathcal{H}}$, $\operatorname{dom} A = \big\{x \in \mathcal{H} \mid Ax \neq \varnothing\big\}$ is the domain of A, $\operatorname{zer} A = \big\{x \in \mathcal{H} \mid 0 \in Ax\big\}$ is its set of zeros, and $\operatorname{gr} A = \big\{(x,u) \in \mathcal{H} \times \mathcal{H} \mid u \in Ax\big\}$ is its graph. The operator A is monotone if it satisfies, for every (x,u) and (y,v) in $\operatorname{gr} A$, $\langle x - y \mid u - v\rangle \geq 0$, and it is maximally monotone if its graph is not properly contained in the graph of any other monotone operator acting on $\mathcal{H}$. In this case, the resolvent of A, $J_A = (\operatorname{Id} + A)^{-1}$, is well defined, single-valued, $\operatorname{dom} J_A = \mathcal{H}$, and it is firmly nonexpansive. For every $\alpha \in \mathbb{R}$, the lower level set at height α of a function $f\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ is the closed convex set $\operatorname{lev}_\alpha f = \big\{x \in \mathcal{H} \mid f(x) \leq \alpha\big\}$ and the subdifferential of f is the operator

$$\partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{u \in \mathcal{H} \mid (\forall y \in \mathcal{H}) \quad \langle y - x \mid u\rangle + f(x) \leq f(y)\big\}. \tag{8.40}$$

Now let C be a nonempty subset of $\mathcal{H}$. Then $\operatorname{int} C$ is the interior of C and if C is nonempty, convex, and closed, then P_C denotes the projector operator onto C, which, for every $x \in \mathcal{H}$ satisfies $\|x - P_C x\| = \min_{y \in C} \|x - y\| = d_C(x)$, where d_C denotes the distance function of C. For further background in monotone operator theory and convex analysis see [17].

Problem 8.15 Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ and $B\colon \operatorname{dom} B \subset \mathcal{H} \to \mathcal{H}$ be two maximally monotone operators such that $\operatorname{dom} A \subset \operatorname{dom} B$ and suppose that $A + B$ is maximally monotone (see [17, Corollary 24.4] for some sufficient conditions). For every $i \in \{1, \ldots, m\}$, let $f_i\colon \mathcal{H} \to \mathbb{R}$ be lower semicontinuous and convex, denote by $S = \operatorname{lev}_{\leq 0} f_1 \cap \cdots \cap \operatorname{lev}_{\leq 0} f_m \neq \varnothing$, and assume that $S \subset \operatorname{dom} B$ and that B is χ-Lipschitzian on $S \cup \operatorname{dom} A$, for some $\chi \in \left]0, +\infty\right[$. The problem is to

$$\text{find} \quad x \in \mathcal{H} \quad \text{such that} \quad \begin{cases} x \in \operatorname{zer}(A + B) \\ f_1(x) \leq 0 \\ \quad\vdots \\ f_m(x) \leq 0. \end{cases} \tag{8.41}$$

Problem 8.15 models various applications to economics, traffic theory, Nash equilibrium problems, and network equilibrium problems among others (see [18, 19, 20] and the references therein).

In the particular case when $m = 1$, $f_1 = d_C$, and $C \subset \mathcal{H}$ is a nonempty closed convex set, an algorithm for solving Problem 8.15 is proposed in [8], without considering errors in the computations and assuming that P_C is easily computable (see also [21] for an approach using enlargements of maximally monotone operators). However, since P_S is not computable in general, Problem 8.15 can not be solved by this method. We propose an algorithm for solving Problem 8.15 in which the constraints $f_1 \leq 0,\ldots, f_m \leq 0$ are activated independently and linearized, and where errors in the computation of the operators involved are permitted. For the implementation of this method we use the subgradient projector with respect to $f \in \Gamma_0(\mathcal{H})$, which is defined by

$$G\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto \begin{cases} x - \dfrac{f(x)}{\|u\|^2}u, & \text{if } f(x) > 0;\\ x, & \text{otherwise,}\end{cases} \tag{8.42}$$

where $u \in \partial f(x)$, and the function $\mathrm{i}\colon \mathbb{N} \to \{1,\ldots,m\}\colon n \mapsto 1 + \mathrm{rem}(n-1, m)$, where $\mathrm{rem}(\cdot, m)$ is the remainder function of division by m.

Algorithm 8.16 For every $i \in \{1,\ldots,m\}$, denote by $G_i\colon \mathcal{H} \to \mathcal{H}$ the subgradient projector with respect to f_i. Let $(e_{1,n})_{n\in\mathbb{N}}$, $(e_{2,n})_{n\in\mathbb{N}}$, and $(e_{3,n})_{n\in\mathbb{N}}$ be sequences in $\mathcal{H}$ such that $\sum_{n\in\mathbb{N}} \|e_{1,n}\| < +\infty$, $\sum_{n\in\mathbb{N}} \|e_{2,n}\| < +\infty$, and $\sum_{n\in\mathbb{N}} \|e_{3,n}\| < +\infty$. Let $\varepsilon \in \left]0, 1/(\chi+1)\right[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\chi]$, let $x_0 \in \operatorname{dom} B$, and let $(x_n)_{n\in\mathbb{N}}$ be the sequence generated by the following routine.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_n = x_n - \gamma_n(Bx_n + e_{1,n})\\ q_n = J_{\gamma_n A}(y_n + e_{2,n})\\ r_n = q_n - \gamma_n(Bq_n + e_{3,n})\\ z_n = x_n - y_n + r_n\\ x_{n+1} = G_{\mathrm{i}(n)}\, z_n. \end{array}\right. \tag{8.43}$$

Remark 8.17 In Algorithm 8.16, the sequences $(e_{1,n})_{n\in\mathbb{N}}$ and $(e_{3,n})_{n\in\mathbb{N}}$ represent errors in the computation of the operator B. In addition, we suppose that the resolvents $(J_{\gamma_n A})_{n\in\mathbb{N}}$ can be computed approximatively by solving, for every $n \in \mathbb{N}$, the perturbed inclusion

$$\text{find} \quad q \in \mathcal{H} \quad \text{such that} \quad y_n - q + e_{2,n} \in \gamma_n A q. \tag{8.44}$$

Proposition 8.18 *Suppose that*

$$\bigcup_{i=1}^m \operatorname{ran} G_i \subset \operatorname{dom} B \quad \textit{and} \quad S \cap \operatorname{zer}(A+B) \neq \varnothing. \tag{8.45}$$

Then Algorithm 8.16 generates an infinite orbit $(x_n)_{n\in\mathbb{N}}$ which converges weakly to a solution to Problem 8.15.

Proof. Set

$$(\forall n \in \mathbb{N}) \quad \beta_n = \gamma_n\chi, \quad T_n = J_{\gamma_n A}, \quad \text{and} \quad R_n = \operatorname{Id} - \gamma_n B. \tag{8.46}$$

Note that $(\beta_n)_{n\in\mathbb{N}}$ is a sequence in $]0,1-\varepsilon]$ and, for every $n \in \mathbb{N}$, T_n is firmly nonexpansive and $\mathrm{Id} - R_n = \gamma_n B$ is β_n–Lipschitzian and monotone. Hence, it follows from [9, Theorem 1] that the operators $(R_n)_{n\in\mathbb{N}}$ are pseudo contractive. In addition, note that $x \in \mathrm{zer}(A+B) \Leftrightarrow (\forall n \in \mathbb{N})\ \ x - \gamma_n Bx \in x + \gamma_n Ax \Leftrightarrow (\forall n \in \mathbb{N})\ \ x \in \mathrm{Fix}\, T_n R_n$. Altogether, we deduce that Problem 8.15 is a particular case of Problem 8.12 and

$$Z = S \cap \bigcap_{n\in\mathbb{N}} \mathrm{Fix}\, T_n R_n = S \cap \mathrm{zer}(A+B) \neq \varnothing. \tag{8.47}$$

Now let us prove that Algorithm 8.16 is a particular case of Algorithm 8.13. Set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} a_n = -\gamma_n e_{1,n} \\ b_n = J_{\gamma_n A}(y_n + e_{2,n}) - J_{\gamma_n A} y_n \\ c_n = -\gamma_n e_{3,n} \end{cases} \quad \text{and} \quad \begin{cases} \lambda_n = 1 \\ Q_n = G_{\mathrm{i}(n)}. \end{cases} \tag{8.48}$$

Then, since $\sup_{n\in\mathbb{N}} \gamma_n < \chi^{-1}$, we have $\sum_{n\in\mathbb{N}} \|a_n\| < +\infty$ and $\sum_{n\in\mathbb{N}} \|c_n\| < +\infty$. Moreover, from the nonexpansivity of $(J_{\gamma_n A})_{n\in\mathbb{N}}$, we deduce that $\sum_{n\in\mathbb{N}} \|b_n\| < +\infty$, and, for every $x \in \mathcal{H}$ and $n \in \mathbb{N}$, $Q_n x$ is the projection onto the closed affine half-space $\big\{y \in \mathcal{H} \mid \langle x - y \mid u\rangle \geq f_{\mathrm{i}(n)}(x)\big\}$, for some $u \in \partial f_{\mathrm{i}(n)}(x)$, which contains $\mathrm{lev}_{\leq 0} f_{\mathrm{i}(n)} \supset S$. On the other hand, $x_0 \in \mathrm{dom}\, B$ and since, for every $i \in \{1,\ldots,m\}$, $\mathrm{ran}\, G_i \subset \mathrm{dom}\, B$, it follows from (8.43) that, for every $n \in \mathbb{N} \setminus \{0\}$, $x_n \in \mathrm{dom}\, B$. In addition, $q_n = J_{\gamma_n A}(y_n + e_{2,n}) \in \mathrm{dom}\, A \subset \mathrm{dom}\, B$. Altogether, from (8.46) and (8.48), we deduce that Algorithm 8.16 is a particular case of Algorithm 8.13 and that it generates an infinite orbit $(x_n)_{n\in\mathbb{N}}$.

Let us prove that condition (8.24) holds. Suppose that $x_{k_n} \rightharpoonup x$, $x_n - T_n R_n x_n \to 0$, $z_n - x_n \to 0$, $z_n - Q_n z_n \to 0$, and, for every $n \in \mathbb{N}$, denote by $p_n = T_n R_n x_n$. Hence, $p_{k_n} \rightharpoonup x$ and from (8.46) we obtain, for every $n \in \mathbb{N}$,

$$\begin{aligned} p_n = T_n R_n x_n \quad &\Leftrightarrow \quad x_n - \gamma_n B x_n \in p_n + \gamma_n A p_n \\ &\Leftrightarrow \quad \frac{1}{\gamma_n}(x_n - p_n) - B x_n \in A p_n \\ &\Leftrightarrow \quad \frac{1}{\gamma_n}(x_n - p_n) + B p_n - B x_n \in (A+B) p_n. \end{aligned} \tag{8.49}$$

Now, since $A+B$ is maximally monotone, from [17, Proposition 20.33], its graph is sequentially weak-strong closed. Therefore, since $x_{k_n} - p_{k_n} \to 0$, $\|B p_{k_n} - B x_{k_n}\| \leq \chi \|x_{k_n} - p_{k_n}\| \to 0$, $\gamma_{k_n} \geq \varepsilon > 0$, $p_{k_n} \rightharpoonup x$, we conclude from (8.49) that $x \in \mathrm{zer}(A+B)$. Now let us prove that, for every $i \in \{1,\ldots,m\}$, $f_i(x) \leq 0$. Fix $i \in \{1,\ldots,m\}$ and, for every $n \in \mathbb{N}$, let $j_n \in \mathbb{N}$ such that $k_n \leq j_n \leq k_n + m$ and $\mathrm{i}(j_n) = i$. We deduce from $z_n - x_n \to 0$ and $z_n - Q_n z_n \to 0$ that, for every $n \in \mathbb{N}$, $\|x_{n+1} - x_n\| = \|Q_n z_n - x_n\| \leq \|Q_n z_n - z_n\| + \|z_n - x_n\| \to 0$. Therefore,

$$(\forall n \in \mathbb{N}) \quad \|x_{j_n} - x_{k_n}\| \leq \sum_{\ell=k_n}^{j_n - 1} \|x_{\ell+1} - x_\ell\| \leq m \max_{k_n \leq \ell \leq k_n + m} \|x_{\ell+1} - x_\ell\| \to 0 \tag{8.50}$$

and hence it follows from $z_{j_n} - x_{j_n} \to 0$ and $x_{k_n} \rightharpoonup x$ that $z_{j_n} \rightharpoonup x$. Note that, from (8.48) and (8.42) we have, for some $u_{j_n} \in \partial f_i(z_{j_n})$,

$$(\forall n \in \mathbb{N}) \quad Q_{j_n} z_{j_n} - z_{j_n} = \begin{cases} -\dfrac{f_i(z_{j_n})}{\|u_{j_n}\|^2} u_{j_n}, & \text{if } f_i(z_{j_n}) > 0; \\ 0, & \text{otherwise}, \end{cases} \tag{8.51}$$

and, since $\|Q_{j_n} z_{j_n} - z_{j_n}\| \to 0$, we deduce that $\max\{0, f_i(z_{j_n})\} \to 0$. Thus, it follows from $z_{j_n} \rightharpoonup x$ that $f_i(x) \le \underline{\lim}\, f_i(z_{j_n}) \le \underline{\lim}\, \max\{0, f_i(z_{j_n})\} = 0$, and hence $x \in \operatorname{lev}_0 f_i$. We conclude that $x \in Z$ and the result follows from Theorem 8.14. □

Remark 8.19 Let us consider the particular case of Theorem 8.18 obtained when $e_{1,n} \equiv e_{2,n} \equiv e_{3,n} \equiv 0$, $m = 1$, and $f_1 = d_C$, where $C \subset \mathcal{H}$ is a nonempty closed convex set. Then, since $G_1 = P_C$, Algorithm 8.16 reduces to the method proposed in [8]. Moreover, since $S = C$, note that the assumption $\operatorname{ran} G_1 \subset \operatorname{dom} B$ is equivalent to $S \subset \operatorname{dom} B$, which was already assumed in Problem 8.15.

8.2.4 Equilibrium problems with convex constraints

We consider the problem

$$\text{find} \quad x \in C \quad \text{such that} \quad (\forall y \in C) \quad F(x,y) \ge 0, \tag{8.52}$$

where C and F satisfy the following assumption.

Assumption 8.20 C is a nonempty closed convex subset of $\mathcal{H}$ and $F\colon C^2 \to \mathbb{R}$ satisfies the following.

(i) $(\forall x \in C) \quad F(x,x) = 0$.

(ii) $(\forall (x,y) \in C^2) \quad F(x,y) + F(y,x) \le 0$.

(iii) For every x in C, $F(x,\cdot)\colon C \to \mathbb{R}$ is lower semicontinuous and convex.

(iv) $(\forall (x,y,z) \in C^3) \quad \overline{\lim}_{\varepsilon \to 0^+} F((1-\varepsilon)x + \varepsilon z, y) \le F(x,y)$.

We are interested in solving a more general problem than (8.52), which involves a finite or a countable infinite number of convex constraints. It will be presented after the following preliminaries.

Notation. The resolvent of $F\colon C^2 \to \mathbb{R}$ is the set valued operator

$$J_F\colon \mathcal{H} \to 2^C\colon x \mapsto \big\{ z \in C \mid (\forall y \in C) \quad F(z,y) + \langle y - z \mid z - x\rangle \ge 0 \big\} \tag{8.53}$$

and, for every $\delta \in \left]0,+\infty\right[$, the δ–resolvent of $F\colon C^2 \to \mathbb{R}$ is the set valued operator

$$J_F^\delta\colon \mathcal{H} \to 2^C\colon x \mapsto \big\{ z \in C \mid (\forall y \in C) \quad F(z,y) + \langle y - z \mid z - x\rangle \ge -\delta \big\}. \tag{8.54}$$

Lemma 8.21 *Suppose that $F\colon C^2 \to \mathbb{R}$ satisfies Assumption 8.20. Then the following hold.*

(i) $\operatorname{dom} J_F = \mathcal{H}$.

(ii) *J_F is single-valued and firmly nonexpansive.*

(iii) $(\forall x \in \mathcal{H})(\forall \delta \in \left]0,+\infty\right[) \quad J_F x \in J_F^\delta x$.

(iv) $(\forall x \in \mathcal{H})(\forall \delta \in \left]0,+\infty\right[) \quad J_F^\delta x \subset B(J_F x; \sqrt{\delta})$.

Proof. (i)&(ii) : [2, Lemma 2.12]. (iii) : This follows from (ii), (8.53), and (8.54). (iv) : Fix $x \in \mathcal{H}$ and $\delta \in \left]0,+\infty\right[$, and let $w \in J_F^\delta x$. We deduce from (8.53) and (8.54) that $F(J_F x, w) + \langle w - J_F x \mid J_F x - x\rangle \geq 0$ and $F(w, J_F x) + \langle J_F x - w \mid w - x\rangle \geq -\delta$, respectively. Adding both inequalities we obtain $F(w, J_F x) + F(J_F x, w) - \|J_F x - w\|^2 \geq -\delta$. Hence, it follows from Assumption (ii) that $\|J_F x - w\|^2 \leq \delta$, which yields the result. ☐

Problem 8.22 Let F and C be such that Assumption 8.20 holds. Let $(S_i)_{i\in I}$ be a countable (finite or countable infinite) family of closed convex subsets of $\mathcal{H}$ such that $S = \cap_{i\in I} S_i \neq \varnothing$. Let $B\colon \operatorname{dom} B \subset \mathcal{H} \to \mathcal{H}$ be a monotone and χ–Lipschitzian operator for some $\chi \in \left]0,+\infty\right[$ such that $C \subset \operatorname{dom} B$, and suppose that

$$\bigcup_{i\in I} S_i \subset \operatorname{int}\operatorname{dom} B. \tag{8.55}$$

The problem is to

$$\text{find} \quad x \in S \quad \text{such that} \quad (\forall y \in C) \quad F(x,y) + \langle y - x \mid Bx\rangle \geq 0. \tag{8.56}$$

Problem 8.22 models a wide variety of problems including complementarity problems, optimization problems, feasibility problems, Nash equilibrium problems, variational inequalities, and fixed point problems [10, 2, 22, 23, 24, 25].

In the literature, there exist some splitting algorithms for solving the equilibrium problem

$$\text{find} \quad x \in C \quad \text{such that} \quad (\forall y \in C) \quad F_1(x,y) + F_2(x,y) \geq 0, \tag{8.57}$$

where F_1 and F_2 satisfy Assumption 8.20. These methods take advantage of the properties of F_1 and F_2 separately. For instance, sequential and parallel splitting algorithms are proposed in [26], where the resolvents J_{F_1} and J_{F_2} are used. The ergodic convergence to a solution to (8.57) is established without additional assumptions. However, when $F_1 = F$ and $F_2\colon (x,y) \mapsto \langle y - x \mid Bx\rangle$ we have $J_{F_2} = J_B = (\mathrm{Id} + B)^{-1}$ [2, Lemma 2.15(i)], which is often difficult to compute, even in the linear case. Moreover, the ergodic method proposed in [26] involves vanishing parameters that leads to numerical instabilities, which make it of limited use in applications. In [2, 27] a different approach is developed to overcome this disadvantage when B is cocoercive. In their methods, the operator B is computed explicitly and the weakly convergence to a solution to (8.56) when $S = C$ is demonstrated.

In this section we propose the following non-ergodic algorithm for solving the general case considered in Problem 8.22. This approach can deal with errors in the computations of the operators involved. The convergence of the proposed method is a consequence of Theorem 8.14.

Algorithm 8.23 Let $(I_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of I, let $(e_{1,n})_{n\in\mathbb{N}}$ and $(e_{2,n})_{n\in\mathbb{N}}$ be sequences in $\mathcal{H}$ such that $\sum_{n\in\mathbb{N}} \|e_{1,n}\| < +\infty$ and $\sum_{n\in\mathbb{N}} \|e_{2,n}\| < +\infty$, and let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence in $]0,+\infty[$ such that $\sum_{n\in\mathbb{N}} \sqrt{\delta_n} < +\infty$. Let $\varepsilon \in \left]0, 1/(\chi+1)\right[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\chi]$, let $\cup_{n\in\mathbb{N}}\{\omega_{i,n}\}_{i\in I_n} \subset [\varepsilon, 1]$ be such that, for every $n \in \mathbb{N}$, $\sum_{i\in I_n} \omega_{i,n} = 1$, let $x_0 \in \operatorname{dom} B$, and let $(x_n)_{n\in\mathbb{N}}$ be the sequence generated by the following routine.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_n = x_n - \gamma_n(Bx_n + e_{1,n}) \\ q_n \in J^{\delta_n}_{\gamma_n F} y_n \\ r_n = q_n - \gamma_n(Bq_n + e_{2,n}) \\ z_n = x_n - y_n + r_n \\ x_{n+1} = \sum_{i\in I_n} \omega_{i,n} P_{S_i} z_n. \end{array} \right. \tag{8.58}$$

Remark 8.24 In Algorithm 8.23, the sequences $(e_{1,n})_{n\in\mathbb{N}}$ and $(e_{2,n})_{n\in\mathbb{N}}$ represent errors in the computation of the operator B. On the other hand, it follows from (8.58) and (8.54) that, for every $n \in \mathbb{N}$, q_n is a solution to

$$\text{find} \quad q \in C \quad \text{such that} \quad (\forall y \in C) \quad F(q,y) + \langle y - y_n \mid y - q\rangle \geq -\delta_n. \tag{8.59}$$

Thus, we obtain from (8.53) that q_n can be interpreted as an approximate computation of the resolvent $J_{\gamma_n F} y_n$.

Proposition 8.25 *Suppose that there exist strictly positive integers $(M_i)_{i\in I}$ and N such that*

$$(\forall (i,n) \in I \times \mathbb{N}) \quad i \in \bigcup_{k=n}^{n+M_i-1} I_k \quad \textit{and} \quad 1 \leq \operatorname{card} I_n \leq N, \tag{8.60}$$

and that Problem 8.22 admits at least one solution. Then Algorithm 8.23 generates an infinite orbit $(x_n)_{n\in\mathbb{N}}$ which converges weakly to a solution to Problem 8.22.

Proof. First, let us prove that Problem 8.22 is a particular case of Problem 8.12. Set

$$(\forall n \in \mathbb{N}) \quad \beta_n = \gamma_n\chi, \;\; T_n = J_{\gamma_n F}, \;\; \text{and} \;\; R_n = \operatorname{Id} - \gamma_n B. \tag{8.61}$$

Note that $(\beta_n)_{n\in\mathbb{N}}$ is a sequence in $]0, 1-\varepsilon]$ and, for every $n \in \mathbb{N}$, T_n is firmly nonexpansive [2, Lemma 2.12] and $\operatorname{Id} - R_n = \gamma_n B$ is β_n–Lipschitzian and monotone. Hence, it follows from [9, Theorem 1] that the operators $(R_n)_{n\in\mathbb{N}}$ are pseudo contractive. In addition, we deduce from (8.53) and (8.61) that $(\forall n \in \mathbb{N})\quad x \in \operatorname{Fix} T_n R_n \;\Leftrightarrow\; (\forall n \in \mathbb{N})(\forall y \in C)\ \gamma_n F(x,y) + \langle y - x \mid x - R_n x\rangle \geq 0 \;\Leftrightarrow\; (\forall y \in C)\ F(x,y) + \langle y - x \mid Bx\rangle \geq 0$. Altogether, we deduce that Problem 8.22 is a particular case of Problem 8.12 and

$$Z = S \cap \bigcap_{n\in\mathbb{N}} \operatorname{Fix} T_n R_n = S \cap \big\{x \in C \mid (\forall y \in C)\ F(x,y) + \langle y - x \mid Bx\rangle \geq 0\big\} \neq \varnothing. \quad (8.62)$$

Now let us show that Algorithm 8.23 is deduced from Algorithm 8.13. Set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} a_n = -\gamma_n e_{1,n} \\ b_n = q_n - J_{\gamma_n F} y_n \\ c_n = -\gamma_n e_{2,n} \end{cases} \quad \text{and} \quad \begin{cases} \lambda_n = 1 \\ Q_n = \sum_{i\in I_n} \omega_{i,n} P_{S_i}. \end{cases} \quad (8.63)$$

Then, since $\sup_{n\in\mathbb{N}} \gamma_n < \chi^{-1}$, we have $\sum_{n\in\mathbb{N}} \|a_n\| < +\infty$ and $\sum_{n\in\mathbb{N}} \|c_n\| < +\infty$. Moreover, it follows from (8.58) and Lemma (iv) that $\sum_{n\in\mathbb{N}} \|b_n\| < +\infty$, and, for every $x \in \mathcal{H}$ and $n \in \mathbb{N}$, $Q_n x$ is the projection onto the closed affine half-space $H_n(x) = \big\{z \in \mathcal{H} \mid \langle z - Q_n x \mid x - Q_n x\rangle \leq 0\big\}$, which satisfies $S \subset \cap_{i\in I_n} S_i = \operatorname{Fix} Q_n \subset H_n(x)$ [10, Proposition 2.4]. On the other hand, we have $x_0 \in \operatorname{dom} B$ and it follows from (8.55) and the convexity of $\operatorname{int}\operatorname{dom} B$ [28, Theorem 27.1] that

$$(\forall n \in \mathbb{N}) \quad \operatorname{ran}\left(\sum_{i\in I_n} \omega_{i,n} P_{S_i}\right) \subset \operatorname{conv}\left(\bigcup_{i\in I_n} S_i\right) \subset \operatorname{conv}\left(\bigcup_{i\in I} S_i\right) \subset \operatorname{int}\operatorname{dom} B. \quad (8.64)$$

Hence, we conclude from (8.58) that, for every $n \in \mathbb{N} \setminus \{0\}$, $x_n \in \operatorname{int}\operatorname{dom} B$. Moreover, for every $n \in \mathbb{N}$, $q_n \in C \subset \operatorname{dom} B$. Altogether, from (8.61) and (8.63), we deduce that Algorithm 8.23 is a particular case of Algorithm 8.13, which generates an infinite orbit $(x_n)_{n\in\mathbb{N}}$.

Finally, let us show that (8.24) holds. Suppose that $x_{k_n} \rightharpoonup x$, $x_n - T_n R_n x_n \to 0$, $z_n - x_n \to 0$, $z_n - Q_n z_n \to 0$, and, for every $n \in \mathbb{N}$, denote by $p_n = T_n R_n x_n$. Hence, $p_{k_n} \rightharpoonup x$ and it follows from (8.61) and (8.53) that, for every $n \in \mathbb{N}$,

$$\begin{aligned} p_n = T_n R_n x_n \;&\Leftrightarrow\; (\forall z \in C)\ \ F(p_n, z) + \frac{1}{\gamma_n}\langle z - p_n \mid p_n - x_n\rangle + \langle z - p_n \mid Bx_n\rangle \geq 0 \\ &\Leftrightarrow\; (\forall z \in C)\ \ F(p_n, z) + \frac{1}{\gamma_n}\langle z - p_n \mid p_n - x_n\rangle \\ &\qquad\qquad + \langle z - p_n \mid Bx_n - Bp_n\rangle + \langle z - p_n \mid Bp_n\rangle \geq 0 \\ &\Leftrightarrow\; (\forall z \in C)\ \ G(p_n, z) + \frac{1}{\gamma_n}\langle z - p_n \mid p_n - x_n\rangle \\ &\qquad\qquad + \langle z - p_n \mid Bx_n - Bp_n\rangle \geq 0, \end{aligned} \quad (8.65)$$

where

$$G\colon C^2 \to \mathbb{R}\colon (x,y) \mapsto F(x,y) + \langle y - x \mid Bx\rangle \quad (8.66)$$

satisfies the Assumption 8.20 [2, Lemma 2.15(i)]. Moreover, since $\inf_{n\in\mathbb{N}} \gamma_{k_n} > 0$, $x_{k_n} - p_{k_n} \to 0$, and $(p_{k_n})_{n\in\mathbb{N}}$ is bounded, we have $(\forall z \in C)\ \langle z - p_{k_n} \mid p_{k_n} - x_{k_n}\rangle / \gamma_{k_n} \to 0$, and from the Lipschitzian property of B we obtain $(\forall z \in C)\ \langle z - p_{k_n} \mid Bx_{k_n} - Bp_{k_n}\rangle \to 0$. Hence, we deduce from $p_{k_n} \rightharpoonup x$, Assumption 8.20(iii), Assumption 8.20(ii), and (8.65) that

$$\begin{aligned}(\forall z \in C)\quad G(z,x) &\le \underline{\lim}\, G(z,p_{k_n})\\ &\le \underline{\lim}\, -G(p_{k_n},z)\\ &\le \underline{\lim}\, \frac{1}{\gamma_{k_n}} \langle z - p_{k_n} \mid p_{k_n} - x_{k_n}\rangle + \langle z - p_{k_n} \mid Bx_{k_n} - Bp_{k_n}\rangle\\ &= 0. \end{aligned} \tag{8.67}$$

Now let $\varepsilon \in \,]0,1]$ and $y \in C$. By convexity of C we have $x_\varepsilon = (1-\varepsilon)x + \varepsilon y \in C$. Thus, Assumption 8.20(i), Assumption 8.20(iii), and (8.67) with $z = x_\varepsilon$ yield

$$0 = G(x_\varepsilon, x_\varepsilon) \le (1-\varepsilon)G(x_\varepsilon, x) + \varepsilon G(x_\varepsilon, y) \le \varepsilon G(x_\varepsilon, y), \tag{8.68}$$

whence $G(x_\varepsilon, y) \ge 0$. In view of Assumption 8.20(iv), we conclude that $G(x,y) \ge \overline{\lim}_{\varepsilon\to 0^+} G(x_\varepsilon, y) \ge 0$, which yields

$$(\forall y \in C)\quad G(x,y) = F(x,y) + \langle y - x \mid Bx\rangle \ge 0. \tag{8.69}$$

Now, let us prove that $x \in S$. Since $z_n - x_n \to 0$ and $z_n - Q_n z_n \to 0$, (8.63) yields

$$(\forall n \in \mathbb{N})\quad \|x_{n+1} - x_n\| = \|Q_n z_n - x_n\| \le \|Q_n z_n - z_n\| + \|z_n - x_n\| \to 0. \tag{8.70}$$

Now, fix $i \in I$. In view of (8.60), there exists a sequence $(j_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that, for every $n \in \mathbb{N}$, $k_n \le j_n \le k_n + M_i - 1$ and $i \in I_{j_n}$. For every $n \in \mathbb{N}$, it follows from (8.70) that

$$\|x_{j_n} - x_{k_n}\| \le \sum_{\ell=k_n}^{k_n+M_i-2} \|x_{\ell+1} - x_\ell\| \le (M_i - 1) \max_{k_n\le\ell\le k_n+M_i-2} \|x_{\ell+1} - x_\ell\| \to 0. \tag{8.71}$$

Thus, we deduce from $x_{k_n} \rightharpoonup x$ and $z_{j_n} - x_{j_n} \to 0$ that $z_{j_n} \rightharpoonup x$. On the other hand, let $z \in S$ and $n \in \mathbb{N}$. Since, for every $\ell \in I_{j_n}$, $P_{S_\ell} z = z$, and $\mathrm{Id} - P_{S_\ell}$ is firmly nonexpansive, from (8.58) and (8.63) we have

$$\begin{aligned}\|P_{S_i} z_{j_n} - z_{j_n}\|^2 &\le \max_{\ell\in I_{j_n}} \|P_{S_\ell} z_{j_n} - z_{j_n}\|^2\\ &\le \frac{1}{\varepsilon} \sum_{\ell\in I_{j_n}} \omega_{\ell,j_n} \|P_{S_\ell} z_{j_n} - z_{j_n}\|^2\\ &\le \frac{1}{\varepsilon} \sum_{\ell\in I_{j_n}} \omega_{\ell,j_n} \langle z - z_{j_n} \mid (\mathrm{Id} - P_{S_\ell})z - (\mathrm{Id} - P_{S_\ell})z_{j_n}\rangle\\ &= \frac{1}{\varepsilon}\left\langle z - z_{j_n} \,\middle|\, \sum_{\ell\in I_{j_n}} \omega_{\ell,j_n} P_{S_\ell} z_{j_n} - z_{j_n}\right\rangle\\ &\le \frac{1}{\varepsilon}\|z - z_{j_n}\|\, \|Q_{j_n} z_{j_n} - z_{j_n}\|. \end{aligned} \tag{8.72}$$

Hence, since $(z_{j_n})_{n\in\mathbb{N}}$ is a bounded sequence and $Q_{j_n}z_{j_n} - z_{j_n} \to 0$, we deduce that $P_{S_i}z_{j_n} - z_{j_n} \to 0$. The maximally monotonicity of $\operatorname{Id} - P_{S_i}$ yields that its graph is sequentially weakly-strongly closed, and since $z_{j_n} \rightharpoonup x$, we conclude that $x = P_{S_i}x \in S_i$. Altogether, from (8.69) and (8.62) we deduce that $x \in Z$, and the result follows from Theorem 8.14. □

Acknowledgement

I thank Professor Patrick L. Combettes for bringing this problem to my attention and for helpful discussions.

8.3 Problèmes de point fixe avec des pseudo contractions lipschitziennes

Dans les Sections 8.1 et 8.2 nous avons proposé l'Algorithme 8.2 pour résoudre le Problème 8.1. Dans le cas particulier où $T_n \equiv \operatorname{Id}$, (8.1) se réduit à

$$\text{trouver} \quad x \in S \cap \bigcap_{n\in\mathbb{N}} \operatorname{Fix} R_n, \tag{8.73}$$

où, pour tout $n \in \mathbb{N}$, R_n est une pseudo contraction et $\operatorname{Id} - R_n$ est une contraction stricte, ce qui implique que les opérateurs $(R_n)_{n\in\mathbb{N}}$ sont 2–lipschitziens. Cette section est consacrée au cas où les opérateurs impliqués sont des pseudo contractions lipschitziens avec des constantes de Lipschitz strictement positives quelconques. Plus précisément dans cette section on s'intéresse au problème suivant.

Problème 8.26 Soit S un sous-ensemble fermé, convexe et non vide de $\mathcal{H}$, pour tout $n \in \mathbb{N}$, soit $(\beta_n)_{n\in\mathbb{N}}$ une suite dans $\left]0, +\infty\right[$ telle que $\eta = \sup_{n\in\mathbb{N}} \beta_n < +\infty$, soit $U_n : \mathcal{H} \to \mathcal{H}$ une pseudo contraction β_n–lipschitzienne. Le problème est de

$$\text{trouver} \quad x \in Z = S \cap \bigcap_{n\in\mathbb{N}} \operatorname{Fix} U_n. \tag{8.74}$$

Dans le cas où $S = \mathcal{H}$ et $U_n \equiv U$, où U est une pseudo contraction lipschitzienne de un sous-ensemble fermé et convexe de $\mathcal{H}$ dans lui même, des méthodes pour résoudre le Problème 8.26 sont dans [4, 5, 6, 7]. Dans la suite nous présentons un algorithme qui résout le cas général. Le lemme suivant sera utile dans la suite.

Lemme 8.27 *Soit $\beta \in \left]0, +\infty\right[$, soit $U : \mathcal{H} \to \mathcal{H}$ une pseudo contraction β–lipschitzienne, soit $\alpha \in \left]0, 1/(1+\beta)\right[$ et posons $R = (1-\alpha)\operatorname{Id} + \alpha U$. Alors les conditions suivantes sont satisfaites.*

(i) *R est une pseudo contraction.*

(ii) $\operatorname{Id} - R$ *est une contraction stricte de constante* $\alpha(1+\beta) \in]0,1[$.

(iii) $\operatorname{Fix} R = \operatorname{Fix} U$.

Proof. (i) Vu que $\operatorname{Id} - R = \alpha(\operatorname{Id} - U)$, la monotonie de $\operatorname{Id} - U$ est équivalente à la monotonie de $\operatorname{Id} - R$. Le résultat est donc une conséquence de [9, Theorem 1]. (ii) Soient x et y dans $\mathcal{H}$. Vu que $\operatorname{Id} - R = \alpha(\operatorname{Id} - U)$, le caractère lipschitzien de U implique

$$\begin{aligned}\|(\operatorname{Id} - R)x - (\operatorname{Id} - R)y\| &= \alpha\|(\operatorname{Id} - U)x - (\operatorname{Id} - U)y\| \\ &\leq \alpha(\|x-y\| + \|Ux - Uy\|) \\ &\leq \alpha(1+\beta)\|x-y\|. \end{aligned} \tag{8.75}$$

(iii) Soit $x \in \mathcal{H}$. Nous avons

$$x \in \operatorname{Fix} U \quad \Leftrightarrow \quad 0 = Ux - x \quad \Leftrightarrow \quad x = x + \alpha(Ux - x) \quad \Leftrightarrow \quad x \in \operatorname{Fix} R, \tag{8.76}$$

ce qui conclut la démonstration. □

Algorithme 8.28 Soient $(e_{1,n})_{n\in\mathbb{N}}$ et $(e_{2,n})_{n\in\mathbb{N}}$ des suites dans $\mathcal{H}$ telles que $\sum_{n\in\mathbb{N}} \|e_{1,n}\| < +\infty$ et $\sum_{n\in\mathbb{N}} \|e_{2,n}\| < +\infty$. De plus, soit $\varepsilon \in]0,(2+\eta)^{-1}[$, soit $(\alpha_n)_{n\in\mathbb{N}}$ une suite dans $[\varepsilon, (1-\varepsilon)/(1+\eta)]$, soit $(\lambda_n)_{n\in\mathbb{N}}$ une suite dans $[\varepsilon, 1]$ et soit $x_0 \in \mathcal{H}$. On génère une suite $(x_n)_{n\in\mathbb{N}}$ comme suit.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_n = (1-\alpha_n)x_n + \alpha_n(U_n x_n + e_{1,n}) \\ r_n = (1-\alpha_n)y_n + \alpha_n(U_n y_n + e_{2,n}) \\ z_n = r_n + x_n - y_n \\ x_{n+1} = x_n + \lambda_n(P_S z_n - x_n). \end{array} \right. \tag{8.77}$$

Proposition 8.29 *On suppose que $Z \neq \varnothing$ et que*

$$(\forall x \in \mathcal{H}) \quad \begin{cases} x_{k_n} \rightharpoonup x \\ x_n - U_n x_n \to 0 \end{cases} \quad \Rightarrow \quad x \in \bigcap_{n\in\mathbb{N}} \operatorname{Fix} U_n \tag{8.78}$$

Alors la suite $(x_n)_{n\in\mathbb{N}}$ engendrée par l'Algorithme 8.28 converge faiblement vers un point dans Z.

Proof. Soient

$$(\forall n \in \mathbb{N}) \quad \beta_n = (1+\eta)\alpha_n, \quad T_n = \operatorname{Id} \quad \text{et} \quad R_n = (1-\alpha_n)\operatorname{Id} + \alpha_n U_n. \tag{8.79}$$

Notons que $(\beta_k)_{k\in\mathbb{N}}$ est une suite dans $]0, 1-\varepsilon]$ et que, pour tout $n \in \mathbb{N}$, T_n est une contraction ferme et que R_n est une pseudo contraction telle que $\operatorname{Id} - R_n$ est β_n–lipschitzienne (Lemme 8.27 (i)&(ii)). De plus, le Lemme 8.27 (iii) garantit que, pour tout $n \in \mathbb{N}$, $\operatorname{Fix} U_n = \operatorname{Fix} R_n$, d'où $\operatorname{Fix} U_n = \operatorname{Fix} T_n R_n$. Par conséquence, nous avons que le Problème 8.26 est un cas particulier du Problème 8.1, où

$$Z = S \cap \bigcap_{n \in \mathbb{N}} \operatorname{Fix} U_n = S \cap \bigcap_{n \in \mathbb{N}} \operatorname{Fix} T_n R_n \neq \varnothing. \tag{8.80}$$

Ensuite nous montrons que l'Algorithme 8.28 est un cas particulier de l'Algorithme 8.2. Posons

$$(\forall n \in \mathbb{N}) \quad a_n = \alpha_n e_{1,n}, \quad b_n = 0, \quad c_n = \alpha_n e_{2,n} \quad \text{et} \quad Q_n = P_S. \tag{8.81}$$

Nous avons $\sum_{k \in \mathbb{N}} \|a_k\| < +\infty$, $\sum_{k \in \mathbb{N}} \|b_k\| < +\infty$, $\sum_{k \in \mathbb{N}} \|c_k\| < +\infty$, et, pour tout $n \in \mathbb{N}$ et $x \in \mathcal{H}$, $Q_n x$ est la projection sur le demi-espace affine fermé $\big\{y \in \mathcal{H} \mid \langle y - P_S x \mid x - P_S x \rangle \leq 0\big\}$, qui contient S. Alors de (8.79) et (8.81) nous déduisons que l'Algorithme 8.28 est un cas particulier de l'Algorithme 8.2.

Enfin, nous montrons que la condition (8.3) est satisfaite. Supposons que, pour quelque $x \in \mathcal{H}$, $x_{k_n} \rightharpoonup x$, $x_n - T_n R_n x_n \to 0$, $z_n - x_n \to 0$ et $z_n - Q_n z_n \to 0$. On déduit de (8.79) et (8.81) que $x_n - T_n R_n x_n = \alpha_n (x_n - U_n x_n) \to 0$ et $z_n - P_S z_n \to 0$. Vu que $\inf_{n \in \mathbb{N}} \alpha_n \geq \varepsilon > 0$ nous avons $x_n - U_n x_n \to 0$. Alors la condition (8.78) implique que $x \in \bigcap_{n \in \mathbb{N}} \operatorname{Fix} U_n$. D'autre part, vu que $x_n - z_n \to 0$ nous avons que $z_{k_n} \rightharpoonup x$. Puisque P_S est une contraction, $\operatorname{Id} - P_S$ est maximalement monotone, et alors, de [17, Proposition 20.33] nous avons que son graphe est séquentiellement faible-forte fermé, ce qui implique $x \in \operatorname{Fix} P_S = S$. Par conséquence, $y \in Z$ et alors le résultat découle du Théorème 8.3. ☐

Remarque 8.30 En particulier, la condition (8.78) est satisfaite si $U_n \equiv U$, où U est une pseudo contraction lipschitzienne. En effet, le Lemme (i) implique que $\operatorname{Id} - U$ est continu et monotone, et donc maximalement monotone. Par suite, puisque $\bigcap_{n \in \mathbb{N}} \operatorname{Fix} U_n = \operatorname{Fix} U$, la condition (8.78) est une conséquence de [17, Proposition 20.33].

8.4 Bibliographie

[1] P. L. Combettes, Solving monotone inclusions via compositions of nonexpansive averaged operators, *Optimization*, vol. 53, pp. 475–504, 2004.

[2] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.*, vol. 6, pp. 117–136, 2005.

[3] F. E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces, *Math. Z.*, vol. 100, pp. 201–225, 1967.

[4] R. E. Bruck Jr., A strongly convergent iterative solution of $0 \in U(x)$ for a maximal monotone operator U in Hilbert space, *J. Math. Anal. Appl.*, vol. 48, pp. 114–126, 1974.

[5] S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, vol. 44, pp. 147–150, 1974.

[6] J. Schu, Approximating fixed points of Lipschitzian pseudocontractive mappings, *Houston J. Math.*, vol. 19, pp. 107–115, 1993.

[7] H. Zhou, Convergence theorems of fixed points for Lipschitz pseudo-contractions in Hilbert spaces, *J. Math. Anal. Appl.*, vol. 343, pp. 546–556, 2008.

[8] P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, *SIAM J. Control Optim.*, vol. 38, pp. 431–446, 2000.

[9] F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.*, vol. 20, pp. 197–228, 1967.

[10] P. L. Combettes, Quasi-Fejérian analysis of some optimization algorithms, in : D. Butnariu, Y. Censor, and S. Reich (Eds.), *Inherently Parallel Algorithms for Feasibility and Optimization*, pp. 115–152, Elsevier, New York, 2001.

[11] H. Attouch, L. M. Briceño-Arias, and P. L. Combettes, A parallel splitting method for coupled monotone inclusions, *SIAM J. Control Optim.*, vol. 48, pp. 3246–3270, 2010.

[12] P. L. Combettes and V. R. Wajs, Signal recovery by proximal forward-backward splitting, *Multiscale Model. Simul.*, vol. 4, pp. 1168–1200, 2005.

[13] F. Liu and M. Z. Nashed, Regularization of nonlinear ill-posed variational inequalities and convergence rates, *Set-Valued Anal.*, vol. 6, pp. 313–344, 1998.

[14] B. Mercier, *Inéquations Variationnelles de la Mécanique*, Publications Mathématiques d'Orsay, no. 80.01, Université de Paris-Sud, Orsay, 1980.

[15] P. Tseng, Further applications of a splitting algorithm to decomposition in variational inequalities and convex programming, *Math. Programming*, vol. 48, pp. 249–263, 1990.

[16] P. Tseng, Applications of a splitting algorithm to decomposition in convex programming and variational inequalities, *SIAM J. Control Optim.*, vol. 29, pp. 119–138, 1991.

[17] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, 2011.

[18] D. P. Bertsekas and E. M. Gafni, Projection methods for variational inequalities with application to the traffic assignment problem, *Math. Programming Stud.*, vol. 17, pp. 139–159, 1982.

[19] S. C. Dafermos and S. C. McKelvey, Partitionable variational inequalities with applications to network and economic equilibria, *J. Optim. Theory Appl.*, vol. 73, pp. 243–268, 1992.

[20] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer-Verlag, New York, 2003.

[21] M. V. Solodov and B. F. Svaiter, A hybrid approximate extragradient-proximal point algorithm using the enlargement of a maximal monotone operator, *Set-Valued Anal.*, vol. 7, pp. 323–345, 1999.

[22] H. H. Bauschke and J. M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Rev.*, vol. 38, pp. 367–426, 1996.

[23] M. Bianchi and S. Schaible, Generalized monotone bifunctions and equilibrium problems, *J. Optim. Theory Appl.*, vol. 90, pp. 31–43, 1996.

[24] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student.*, vol. 63, pp. 123–145, 1994.

[25] W. Oettli, A remark on vector-valued equilibria and generalized monotonicity, *Acta Math. Vietnamica*, vol. 22, pp. 215–221, 1997.

[26] A. Moudafi, On the convergence of splitting proximal methods for equilibrium problems in Hilbert spaces, *J. Math. Anal. Appl.*, vol. 359, pp. 508–513, 2009.

[27] A. Moudafi, Mixed equilibrium problems : sensitivity analysis and algorithmic aspect, *Comput. Math. Appl.*, vol. 44, pp. 1099–1108, 2002.

[28] S. Simons, *From Hahn-Banach to Monotonicity*, Lecture Notes in Math. 1693, Springer-Verlag, New York, 2008.

Chapitre 9

Bilan et perspectives

9.1 Bilan

Cette thèse a été consacrée à la conception et l'analyse de nouvelles méthodes d'éclatement pour résoudre des problèmes d'analyse non linéaire multivoque dans lesquels plusieurs variables interagissent. Le problème générique a été modélisé par une inclusion vis-à-vis d'une somme de $m \geq 2$ opérateurs monotones sur un espace hilbertien produit. Plusieurs méthodes ont été proposés dans les Chapitres 2–4 pour résoudre ce problème sous divers jeux d'hypothèses sur les opérateurs monotones impliqués. Elles sont appliquées au traitement du signal et de l'image, à la théorie des jeux, à la théorie du trafic, aux équations d'évolution, aux problèmes de meilleure approximation et à la décomposition de domaine dans les équations aux dérivées partielles.

Le Chapitre 5 a présenté un nouveau formalisme de décomposition monotone+anti-adjoint pour résoudre des inclusions monotones composites en dualité. La formulation présentée permet de traiter une ample variété de problèmes et fournit un outil puissant de modélisation en l'analyse non linéaire.

Le Chapitre 6 a proposé une approche générale primale-duale pour résoudre des problèmes de décomposition de domaine dans les équations aux dérivées partielles en considérant $m \geq 2$ sous-domaines disjoints et des contraintes dures aux interfaces. En particulier, la méthode permet de considérer conditions de continuité et/ou de transmission sur les interfaces aussi bien que divers problèmes dans chaque sous-domaine. De plus, la méthode proposée résout simultanément le problème dual associé, dont les solutions représentent les tensions aux interfaces.

Le Chapitre 7 a présenté la première utilisation systématique de méthodes d'éclatement d'opérateurs pour la construction d'équilibres de Nash. Certains cas particuliers de jeux sans potentiel à m joueurs ont été étudiés et les méthodes obtenues ont été appliquées aux jeux à somme nulle, aux problèmes d'équilibres de Nash généralisés et aux problèmes de proximations cycliques.

Le Chapitre 8 a traité les problèmes de point fixe avec contraintes où deux opérateurs sont impliqués : une contraction ferme et une pseudo contraction lipschitzienne. Une méthode pour résoudre ce problème a été proposée et appliquée aux inclusions monotones et aux problèmes d'équilibre.

9.2 Perspectives

Cette thèse ouvre des perspectives nouvelles sur diverses questions : inclusions d'évolution couplées, problèmes de la théorie du trafic, décomposition de domaine dans les équations aux dérivées partielles, inclusions monotones multicomposantes, problèmes d'équilibre et d'autres problèmes de l'analyse non linéaire.

Au Chapitre 2 nous avons proposé des méthodes pour résoudre des inclusions d'évolution couplées et des problèmes de théorie du trafic. Ces problèmes ont été réduits à une inclusion vis-à-vis d'une somme de deux opérateurs, dont l'un est séparable et l'autre est cocoercif. Il serait intéressant de relaxer ces hypothèses et d'appliquer certaines méthodes proposées dans les Chapitres 4 et 5 à ces problèmes. D'une part, la méthode proposée dans la Section 4.2.2 permettrait de remplacer l'hypothèse de cocoercivité par la monotonie plus le caractère lipschitzien et, d'autre part, l'hypothèse de monotonie suffirait pour utiliser la méthode décrite dans la Section 4.2.3. Par ailleurs, l'application de la méthode primale-duale proposée dans le Chapitre 5 aux inclusions d'évolution et à la théorie du trafic résoudrait de plus le problème dual associé, ce qui a des interprétations importantes dans chaque domaine. Ainsi, dans le cas de la théorie du trafic, cette application fournirait une manière de trouver les multiplicateurs de Lagrange associés aux contraintes de capacité du réseau, qui sont interprétés comme les retards à l'équilibre [3].

En ce qui concerne le problème de décomposition de domaine dans les équations aux dérivées partielles au Chapitre 6, nous avons étudié le cas où les sous-domaines ne se chevauchent pas en considérant une condition Dirichlet sur la frontière du domaine. Il serait intéressant d'étudier le cas du chevauchement et de permettre d'autres conditions sur la frontière. Une approche possible dans les problèmes avec chevauchement est de les réduire au cas sans chevauchement en considérant chaque intersection comme un nouveau sous-domaine. Dans le cas de $m = 2$ sous-domaines, un approche similaire a été étudiée dans [4]. D'autres conditions sur la frontière pourraient être modélisées en considérant des espaces de Sobolev différents.

En ce qui concerne les inclusions monotones multicomposantes, il serait intéressant d'appliquer la méthode développée au Chapitre 8 à la somme hilbertienne directe. Ceci permettrait de résoudre des inclusions monotones multicomposantes où plusieurs contraintes dans l'espace produit interviennent. Contrairement à la méthode proposée dans la Section 4.2.2, la méthode obtenue tolérerait des erreurs dans le calcul des opérateurs concernés et approcherait les contraintes par des projections sur certains demi-espaces définis sur l'espace produit.

D'autre part, nous avons étudié dans le Chapitre 8 une application aux problèmes d'équilibre. Contrairement aux approches utilisées dans [5, 7], la méthode proposée permet de résoudre le Problème 8.9 lorsque l'opérateur B est monotone et lipschitzien. Il serait intéressant de concevoir une nouvelle méthode d'éclatement pour résoudre un problème d'équilibre sous des hypothèses plus générales (voir (8.14)).

Une autre direction de recherche à poursuivre est la conception des méthodes pour la résolution de problèmes d'inclusions lorsque les opérateurs impliqués ne sont plus monotones. En particulier, il serait intéressant de concevoir des méthodes d'éclatement d'opérateurs co-hypomonotones, ceux qui apparaissent souvent dans plusieurs applications [6, 8] (voir aussi [1]).

Une autre problème intéressant à étudier est de trouver, s'ils existent, les avantages de la méthode proposée dans [9] par rapport à l'algorithme explicite-implicite décrit dans la Section 1.2 pour les problèmes de minimisation. Dans ce cas, l'opérateur univoque impliqué est le gradient d'une fonction convexe qui est supposé lipschitzien. Si le domaine du gradient est tout l'espace, il est bien connu que ces hypothèses sont équivalentes à la cocoercivité du gradient [2]. Cependant, on ne sait pas si une fonction convexe différentiable peut avoir un gradient lipschitzien mais pas cocoercif lorsque son domaine n'est pas tout l'espace.

Enfin, vu que dans tous les algorithmes proposés il est nécessaire de calculer des résolvantes d'opérateurs monotones, il serait intéressant de développer le calcul explicite de résolvantes dans la somme hilbertienne directe. Certains progrès dans cette direction ont été faits dans la Section 4.3 mais il reste beaucoup à faire.

9.3 Bibliographie

[1] Y. Alber, D. Butnariu, and I. Ryazantseva, Regularization and resolution of monotone variational inequalities with operators given by hypomonotone approximations, *J. Nonlinear Convex Anal.*, vol. 6, pp. 23–53, 2005.

[2] H. H. Bauschke and P. L. Combettes, The Baillon-Haddad theorem revisited, *J. Convex Anal.*, vol. 17, pp. 781–787, 2010.

[3] M. G. H. Bell, Stochastic user equilibrium assignment in networks with queues, *Transportation Res. Part B*, vol. 29, pp. 125–137, 1995.

[4] H. Ben Dhia and G. Rateau, Analyse mathématique de la méthode Arlequin mixte, *C. R. Acad. Sci. Paris Sér. I Math.*, vol. 332, pp. 649–654, 2001.

[5] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.*, vol. 6, pp. 117–136, 2005.

[6] P. L. Combettes and T. Pennanen, Proximal methods for cohypomonotone operators, *SIAM J. Control Optim.*, vol. 43, pp. 731–742, 2004.

[7] A. Moudafi, Mixed equilibrium problems : sensitivity analysis and algorithmic aspect, *Comput. Math. Appl.*, vol. 44, pp. 1099–1108, 2002.

[8] T. Pennanen, Local convergence of the proximal point algorithm and multiplier methods without monotonicity, *Math. Oper. Res.*, vol. 27, pp. 170–191, 2002.

[9] P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, *SIAM J. Control Optim.*, vol. 38, pp. 431–446, 2000.