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Abstract.

In a previous article about the homogenization of the classical problem of diffusion in a bounded domain with sufficiently smooth boundary we proved that the error is of order $\varepsilon^{1/2}$. Now, for an open set Ω with sufficiently smooth boundary ($\mathcal{C}^{1,1}$) and homogeneous Dirichlet or Neuman limits conditions we show that in any open set strongly included in Ω the error is of order ε. If the open set $\Omega \subset \mathbb{R}^n$ is of polygonal (n=2) or polyhedral (n=3) boundary we also give the global and interrior error estimates.

Résumé. Nous avons démontré dans un précédent article sur l'homogénéisation du problème type de la diffusion dans un domaine borné de frontière régulière que l'erreur est d'ordre $\varepsilon^{1/2}$. On montre maintenant pour un ouvert Ω de frontière régulière ($\mathcal{C}^{1,1}$) avec les conditions aux limites homogènes de Dirichlet ou de Neumann que dans tout ouvert fortement inclus dans Ω l'erreur est de l'ordre de ε. Si l'ouvert $\Omega \subset \mathbb{R}^n$ est de frontière polygonale (n=2) ou polyédrale (n=3) on donne également les estimations globale et intérieure de l'erreur.

Keywords : periodic homogenization, error estimate, unfolding method.

1. Introduction

This paper follows two previous studies [4,5] of the error estimates in the classical periodic homogenization problem. The first error estimates in periodic homogenization problem have been given by Bensoussan, Lions and Papanicolaou [1], by Oleinik, Shamaev and Yosifian [7], and by Cioranescu and Donato [3]. In all these works, the result is proved under the assumption that the correctors belong to $W^{1,\infty}(Y)$, $Y =]0,1[^n$ being the reference cell. The estimate is of order $\varepsilon^{1/2}$. The additional regularity of the correctors holds true when the coefficients of the operator are very regular which is not necessarily the situation in homogenization. In [4] we obtained an error estimate without any regularity hypothesis on the correctors but we supposed that the solution of the homogenized problem belonged to $W^{2,p}(\Omega)$ ($p > n$). The exponent of ε in the error estimate is inferior to $1/2$ and depends on n and p. In [5] we obtained an error estimate without any regularity hypothesis on the correctors but we supposed that the solution of the homogenized problem belonged to $H^2(\Omega)$. This holds true with a smooth boundary and homogeneous Dirichlet or Neuman limits conditions. The exponent of ε in the error estimate is equal to $1/2$.

The aim of this work is to give the interior error estimate and new error estimate with minimal hypothesis on the boundary of Ω.

The paper is organized as follows. Section 2 is dedicated to some projection theorems. Among them Theorems 2.3 and 2.6 are essential tools to obtain new estimates. These theorems are related to the periodic unfolding method (see [2] and [5]). We show that for any ϕ in $H^1(\Omega)$, where Ω is a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary, there exists a function $\widehat{\phi}_\varepsilon$ in $L^2(\Omega; H^1_{per}(Y))$, such that the distance between the unfolded $\mathcal{T}_\varepsilon(\nabla_x \phi)$ and $\nabla_x \phi + \nabla_y \widehat{\phi}_\varepsilon$ is of order ε in the space $[L^2(Y; (H^1(\Omega))')]^n$ (Theorem 2.3) and is of order ε^s in the space $[L^2(Y; (H^s(\Omega))')]^n$, $0 < s < 1$, (Theorem 2.6), provided that the norm of gradient ϕ in a neighbourhood (of thickness $4\varepsilon\sqrt{n}$) of the boundary of Ω is less than $\varepsilon^{1/2}$ in the first case and less than $\varepsilon^{s/2}$ in the second case.

In Theorem 3.2 in Section 3.1, we suppose that Ω has a smooth boundary, that the right handside of the homogenization problem belongs to $L^2(\Omega)$ and we consider the homogeneous Dirichlet or Neumann limits conditions. By transposition and thanks to Theorem 2.3 we show that the L^2 error estimate is of order ε and then we obtain the interior error estimate of the same order. The required condition in Theorem 2.3 is obtained thanks to the estimates of Theorems 4.1 and 4.2 of [5].

In Theorem 3.3 in Section 3.2, we suppose that the domain Ω is of polygonal ($n = 2$) or polyhedral ($n = 3$) boundary and the right handside of the homogenization problem in $L^2(\Omega)$. We show that the H^1 error estimate is at the most of order $\varepsilon^{1/4}$ and that the L^2 and the interior error estimates are at the most of order $\varepsilon^{1/2}$.

We use the notation of [2] and [5] throughout this study. In this article, the constants appearing in the estimates are independent from ε.

2. Preliminary results

Let Ω be a bounded domain in $\mathbb{R}^n$ with lipchitzian boundary. We put

$$\widehat{\Omega}_{\varepsilon,k} = \Big\{x \in \mathbb{R}^n \mid dist(x, \partial\Omega) < k\sqrt{n}\varepsilon\Big\}, \qquad \widetilde{\Omega}_{\varepsilon,k} = \Big\{x \in \mathbb{R}^n \mid dist(x, \Omega) < k\sqrt{n}\varepsilon\Big\}, \qquad k \in \{1,\, 2,\, 3,\, 4\},$$
$$\Omega_\varepsilon = \text{interior}\Big(\bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi + \overline{Y}\Big), \qquad \Xi_\varepsilon = \big\{\xi \in \mathbb{Z}^n \mid \varepsilon(\xi + Y) \cap \Omega \neq \emptyset\big\}, \qquad Y =]0, 1[^n,$$

where the open set $Y =]0, 1[^n$ is the reference cell and where ε is a strictly positive real. We have

$$\Omega \subset \Omega_\varepsilon \in \widetilde{\Omega}_{\varepsilon,1}$$

For almost any $x \in \mathbb{R}^n$, there exists a unique element in $\mathbb{Z}^n$ denoted $[x]$ such that

$$x = [x] + \{x\}, \qquad \{x\} \in Y.$$

The running point of Ω is denoted x, and the running point of Y is denoted y.

2.1 Projection theorems in $L^2(Y; (H^1(\Omega))')$.

Lemma 2.1 : *There exists a linear and continuous extension operator $\mathcal{P}_\varepsilon$ from $H^1(\Omega)$ into $H^1(\widetilde{\Omega}_{\varepsilon,3})$ such that*

$$\text{(2.1)} \qquad \begin{cases} ||\nabla\mathcal{P}_\varepsilon(\phi)||_{[L^2(\widetilde{\Omega}_{\varepsilon,3})]^n} \leq C||\nabla\phi||_{[L^2(\Omega)]^n} \qquad ||\nabla\mathcal{P}_\varepsilon(\phi)||_{[L^2(\widetilde{\Omega}_{\varepsilon,3}\setminus\Omega)]^n} \leq C||\nabla\phi||_{[L^2(\Omega\setminus\widehat{\Omega}_{\varepsilon,3})]^n} \\ ||\mathcal{P}_\varepsilon(\phi)||_{L^2(\widetilde{\Omega}_{\varepsilon,3})} \leq C\big\{||\phi||_{L^2(\Omega)} + \varepsilon||\nabla\phi||_{[L^2(\Omega\setminus\widehat{\Omega}_{\varepsilon,3})]^n}\big\} \end{cases}$$

The constants depend only on n and $\partial\Omega$.

Proof : There exists a finite open covering $(\Omega_j)_j$ of the boundary $\partial\Omega$ such that for each j there exists a Lipschitz diffeomorphism θ_j which maps Ω_j to the open set $\mathcal{O} =]-1, 1[^{n-1}\times]-1, 1[$ of $\mathbb{R}^n$ and $\Omega_j \cap \Omega$ to the open set $\mathcal{O}_+ =]-1, 1[^{n-1}\times]0, 1[$. To the covering of $\partial\Omega$ we associate a partition of the unity

$$\phi_j \in \mathcal{C}_0^1(\Omega_j), \qquad \sum_j \phi_j = 1 \qquad \text{in a neighbourhood of} \quad \partial\Omega.$$

Let ψ be in $H^1(\Omega)$. The function $(\phi_j\psi)\circ\theta_j^{-1}$ belongs to $H^1(\mathcal{O}_+)$. We use a reflexion argument to extend this function to an element $\widetilde{\psi}_j$ belonging to $H^1(\mathcal{O})$. In the neighbourhood of the boundary of Ω the extension is equal to $\sum_j \widetilde{\psi}_j \circ \theta_j$. This immediately gives the estimates of Lemma 2.1. □

From now on any function belonging to $H^1(\Omega)$ will be extended to a function belonging to $H^1(\widetilde{\Omega}_{\varepsilon,3})$. To make the notation simpler the extention of function ϕ will still be denoted ϕ.

In the sequel, we will make use of definitions and results from [2] and [5] concerning the periodic unfolding method. Let us recall the definition of the unfolding operator $\mathcal{T}_\varepsilon$ which asociates a function $\mathcal{T}_\varepsilon(\phi) \in L^1(\Omega\times Y)$ to each function $\phi \in L^1(\Omega_\varepsilon)$,

$$\mathcal{T}_\varepsilon(\phi)(x,y) = \phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) \qquad \text{for } x \in \Omega \text{ and } y \in Y.$$

We also recall the approximate integration formula

$$\Big|\int_\Omega v - \frac{1}{|Y|}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(v)\Big| \le ||v||_{L^1(\widehat{\Omega}_{\varepsilon,1})} \qquad \forall v \in L^1(\Omega_\varepsilon) \tag{2.2}$$

For the other properties of $\mathcal{T}_\varepsilon$, we refer the reader to [2] and [5].
Let $\phi \in H^1(\Omega)$ extended to $\widetilde{\Omega}_{\varepsilon,2}$. We have defined the scale-splitting operators $\mathcal{Q}_\varepsilon$ and $\mathcal{R}_\varepsilon$ (see [2]). The function $\mathcal{Q}_\varepsilon(\phi)$ is the restriction to Ω of Q_1-interpolate of the discrete function $M^\varepsilon_Y(\phi)$

$$M^\varepsilon_Y(\phi)(x) = \frac{1}{|Y|}\int_Y \phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon z\Big)dz \qquad x \in \Omega$$

and $\mathcal{R}_\varepsilon(\phi) = \phi - \mathcal{Q}_\varepsilon(\phi)$. The operator $\mathcal{Q}_\varepsilon$ is linear and continuous from $H^1(\Omega)$ to $H^1(\Omega)$ and we have the estimates

$$||\mathcal{Q}_\varepsilon(\phi)||_{H^1(\Omega)} \le C||\phi||_{H^1(\Omega)} \qquad ||\phi - \mathcal{Q}_\varepsilon(\phi)||_{L^2(\Omega)} \le C\varepsilon||\nabla\phi||_{[L^2(\Omega)]^n} \qquad \forall \phi \in H^1(\Omega).$$

The constants depend on n and $\partial\Omega$.

Theorem 2.2 : *Let ϕ be in $H^1(\Omega)$. There exists $\widehat{\psi}_\varepsilon$ belonging to $H^1_{per}(Y; L^2(\Omega))$ such that*

$$\begin{cases} ||\widehat{\psi}_\varepsilon||_{H^1(Y;L^2(\Omega))} \le C\big\{||\phi||_{L^2(\Omega)} + \varepsilon||\nabla\phi||_{[L^2(\Omega)]^n}\big\} \\ ||\mathcal{T}_\varepsilon(\phi) - \widehat{\psi}_\varepsilon||_{H^1(Y;(H^1(\Omega))')} \le C\varepsilon\big\{||\phi||_{L^2(\Omega)} + \varepsilon||\nabla\phi||_{[L^2(\Omega)]^n}\big\} \\ \qquad\qquad + C\sqrt{\varepsilon}\big\{||\phi||_{L^2(\widehat{\Omega}_{\varepsilon,2})} + \varepsilon||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,2})]^n}\big\} \end{cases} \tag{2.3}$$

The constants depend only on n and $\partial\Omega$.

Proof : In this proof we use the same notation and the same ideas as in Proposition 3.3 of [5].
Theorem 2.2 is proved in two steps. We reintroduce the unfolding operators $\mathcal{T}_{\varepsilon,i}$, defined in [5], which for any $\phi \in H^1(\Omega)$, allow us to estimate the difference between the restrictions to two neighbouring cells of the unfolded of ϕ in $L^2(Y;(H^1(\Omega))')$. Then we evaluate the periodic defect of the functions $y \longrightarrow \mathcal{T}_\varepsilon(\phi)(.,y)$ thanks to Theorem 2.2 of [5].
Let $K_i = Y \cup (\vec{e}_i + Y)$, $i \in \{1,\dots,n\}$. For any x in Ω, the cell $\varepsilon\Big(\Big[\frac{x}{\varepsilon}\Big]_Y + K_i\Big)$ is included in $\widetilde{\Omega}_{\varepsilon,2}$.
We recall that the unfolding operator $\mathcal{T}_{\varepsilon,i}$ from $L^2(\widetilde{\Omega}_{\varepsilon,2})$ into $L^2(\Omega \times K_i)$ is defined by

$$\forall \psi \in L^2(\widetilde{\Omega}_{\varepsilon,2}), \qquad \mathcal{T}_{\varepsilon,i}(\psi)(x,y) = \psi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) \qquad \text{for } x \in \Omega \text{ and a. e. } y \in K_i.$$

The restriction of $\mathcal{T}_{\varepsilon,i}(\psi)$ to $\Omega \times Y$ is equal to the unfolded $\mathcal{T}_\varepsilon(\psi)$ and we have the following equalities in $L^2(\Omega \times Y)$:

$$\mathcal{T}_{\varepsilon,i}(\psi)(.,..+\vec{e}_i) = \mathcal{T}_\varepsilon(\psi)(.+\varepsilon\vec{e}_i,..), \qquad i \in \{1,\dots,n\}$$

Step one. Let us take $\psi \in L^2(\widetilde{\Omega}_{\varepsilon,2})$. We evaluate the difference $\mathcal{T}_{\varepsilon,i}(\psi)(.,..+\vec{e}_i)-\mathcal{T}_{\varepsilon,i}(\psi)$ in $L^2(Y;(H^1(\Omega))')$. For any $\Psi \in H^1(\Omega)$, extended on $\widetilde{\Omega}_{\varepsilon,1}$, a linear change of variables and the relations above give

$$\text{for a. e. } y \in Y, \qquad \int_\Omega \mathcal{T}_{\varepsilon,i}(\psi)(x,y+\vec{e}_i)\Psi(x)dx = \int_\Omega \mathcal{T}_{\varepsilon,i}(\psi)(x+\varepsilon\vec{e}_i,y)\Psi(x)dx$$
$$= \int_{\Omega+\varepsilon\vec{e}_i} \mathcal{T}_{\varepsilon,i}(\psi)(x,y)\Psi(x-\varepsilon\vec{e}_i)dx$$

We deduce

$$\Big|\int_\Omega \big\{\mathcal{T}_{\varepsilon,i}(\psi)(.,y+\vec{e}_i)-\mathcal{T}_{\varepsilon,i}(\psi)(.,y)\big\}\Psi - \int_\Omega \mathcal{T}_{\varepsilon,i}(\psi)(.,y)\big\{\Psi(.-\varepsilon\vec{e}_i)-\Psi\big\}\Big|$$
$$\leq C||\mathcal{T}_{\varepsilon,i}(\psi)(.,y)||_{L^2(\widehat{\Omega}_{\varepsilon,1})}||\Psi||_{L^2(\widehat{\Omega}_{\varepsilon,1})} \qquad \text{for a. e. } y \in Y.$$

Since Ω is a bounded domain with lipschitzian boundary and since Ψ belongs to $H^1(\widetilde{\Omega}_{\varepsilon,1})$ we have

$$\begin{cases} ||\Psi||_{L^2(\widehat{\Omega}_{\varepsilon,1})} \leq C\sqrt{\varepsilon}\big\{||\Psi||_{L^2(\Omega)}+||\nabla\Psi||_{[L^2(\Omega)]^n}\big\}, \\ ||\Psi(.-\varepsilon\vec{e}_i)-\Psi||_{L^2(\Omega)} \leq C\varepsilon\Big\|\dfrac{\partial\Psi}{\partial x_i}\Big\|_{L^2(\Omega)}, \qquad i\in\{1,\dots,n\}, \end{cases} \tag{2.4}$$

hence

$$< \mathcal{T}_{\varepsilon,i}(\psi)(.,y+\vec{e}_i)-\mathcal{T}_{\varepsilon,i}(\psi)(.,y)\,,\,\Psi >_{(H^1(\Omega))',H^1(\Omega)}$$
$$= \int_\Omega \big\{\mathcal{T}_{\varepsilon,i}(\psi)(.,y+\vec{e}_i)-\mathcal{T}_{\varepsilon,i}(\psi)(.,y)\big\}\Psi$$
$$\leq C\varepsilon||\nabla\Psi||_{[L^2(\Omega)]^n}||\mathcal{T}_{\varepsilon,i}(\psi)(.,y)||_{L^2(\Omega)} + C\sqrt{\varepsilon}||\Psi||_{H^1(\Omega)}||\mathcal{T}_{\varepsilon,i}(\psi)(.,y)||_{L^2(\widehat{\Omega}_{\varepsilon,1})}.$$

We deduce that

$$||\mathcal{T}_{\varepsilon,i}(\psi)(.,y+\vec{e}_i)-\mathcal{T}_{\varepsilon,i}(\psi)(.,y)||_{(H^1(\Omega))'} \leq C\varepsilon||\mathcal{T}_{\varepsilon,i}(\psi)(.,y)||_{L^2(\Omega)} + C\sqrt{\varepsilon}||\mathcal{T}_{\varepsilon,i}(\psi)(.,y)||_{L^2(\widehat{\Omega}_{\varepsilon,1})}.$$

Which leads to the following estimate of the difference between $\mathcal{T}_{\varepsilon,i}(\psi)_{|\Omega\times Y}$ and one of its translated :

$$||\mathcal{T}_{\varepsilon,i}(\psi)(.,..+\vec{e}_i)-\mathcal{T}_{\varepsilon,i}(\psi)||_{L^2(Y;(H^1(\Omega))')} \leq C\varepsilon||\psi||_{L^2(\widetilde{\Omega}_{\varepsilon,3})} + C\sqrt{\varepsilon}||\psi||_{L^2(\widehat{\Omega}_{\varepsilon,2})}. \tag{2.5}$$

The constant depends only on n and on the boundary of Ω.

Step two. Let $\phi \in H^1(\Omega)$. The estimate (2.5) applied to ϕ and its partial derivatives give us

$$||\mathcal{T}_{\varepsilon,i}(\phi)(.,..+\vec{e}_i)-\mathcal{T}_{\varepsilon,i}(\phi)||_{L^2(Y;(H^1(\Omega))')} \leq C\varepsilon\big\{||\phi||_{L^2(\Omega)}+\varepsilon||\nabla\phi||_{[L^2(\Omega)]^n}\big\} + C\sqrt{\varepsilon}||\phi||_{L^2(\widehat{\Omega}_{\varepsilon,2})}$$
$$||\mathcal{T}_{\varepsilon,i}(\nabla\phi)(.,..+\vec{e}_i)-\mathcal{T}_{\varepsilon,i}(\nabla\phi)||_{[L^2(Y;(H^1(\Omega))'^n)} \leq C\big\{\varepsilon||\nabla\phi||_{[L^2(\Omega)]^n}+\sqrt{\varepsilon}||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,2})]^n}\big\}$$

We recall that $\nabla_y\big(\mathcal{T}_{\varepsilon,i}(\phi)\big) = \varepsilon\mathcal{T}_{\varepsilon,i}(\nabla\phi)$ (see [3]). The above estimates can also be written as follows :

$$||\mathcal{T}_{\varepsilon,i}(\phi)(.,..+\vec{e}_i)-\mathcal{T}_{\varepsilon,i}(\phi)||_{H^1(Y;(H^1(\Omega))')} \leq C\varepsilon\big\{||\phi||_{L^2(\Omega)}+\varepsilon||\nabla\phi||_{[L^2(\Omega)]^n}+\sqrt{\varepsilon}||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,2})]^n}\big\}$$
$$+ C\sqrt{\varepsilon}||\phi||_{L^2(\widehat{\Omega}_{\varepsilon,2})}$$

From these inequalities, for any $i \in \{1,\dots,n\}$, we deduce the estimate of the difference of the traces of $y \longrightarrow \mathcal{T}_\varepsilon(\phi)(.,y)$ on the faces Y_i and $\vec{e}_i+Y_i$

$$\begin{cases} ||\mathcal{T}_\varepsilon(\phi)(.,..+\vec{e}_i)-\mathcal{T}_\varepsilon(\phi)||_{H^{1/2}(Y_i;(H^1(\Omega))')} \leq C\varepsilon\big\{||\phi||_{L^2(\Omega)}+\varepsilon||\nabla\phi||_{[L^2(\Omega)]^n}\big\} \\ \qquad\qquad + C\sqrt{\varepsilon}\big\{||\phi||_{L^2(\widehat{\Omega}_{\varepsilon,2})}+\varepsilon||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,2})]^n}\big\} \end{cases}$$

It measures the periodic defect of $y \longrightarrow \mathcal{T}_\varepsilon(\phi)(.,y)$. We decompose $\mathcal{T}_\varepsilon(\phi)$ into the sum of an element belonging to $H^1_{per}(Y;L^2(\Omega))$ and an element belonging to $\big(H^1(Y;L^2(\Omega))\big)^\perp$ (the orthogonal of $H^1_{per}(Y;L^2(\Omega))$ in $H^1(Y;L^2(\Omega))$, see [5])

$$\mathcal{T}_\varepsilon(\phi) = \widehat{\psi}_\varepsilon + \overline{\phi}_\varepsilon, \qquad \widehat{\psi}_\varepsilon \in H^1_{per}(Y;L^2(\Omega)), \qquad \overline{\phi}_\varepsilon \in \big(H^1(Y;L^2(\Omega))\big)^\perp \tag{2.6}$$

From the Riesz Theorem the dual space $\big(H^1(\Omega)\big)'$ is a Hilbert space isomorphic to $H^1(\Omega)$. The function $y \longrightarrow \mathcal{T}_\varepsilon(\phi)(.,y)$ takes its values in a finite dimensionnal space,

$$\overline{\phi}_\varepsilon(.,..) = \sum_{\xi\in\Xi_\varepsilon} \overline{\phi}_{\varepsilon,\xi}(..)\chi_\xi(.)$$

where $\chi_\xi(.)$ is the characteristic function of the cell $\varepsilon(\xi+Y)$ and where $\overline{\phi}_{\varepsilon,\xi}(..) \in \big(H^1(Y)\big)^\perp$ (the orthogonal of $H^1_{per}(Y)$ in $H^1(Y)$, see [5]). Hence the decomposing (2.6) is the same in $H^1(Y;\big(H^1(\Omega)\big)')$. As the decomposing is orthogonal, we have

$$||\widehat{\psi}_\varepsilon||^2_{H^1(Y;L^2(\Omega))} + ||\overline{\phi}_\varepsilon||^2_{H^1(Y;L^2(\Omega))} = ||\mathcal{T}_\varepsilon(\phi)||^2_{H^1(Y;L^2(\Omega))} \leq C\big\{||\phi||_{L^2(\Omega)} + \varepsilon||\nabla\phi||_{[L^2(\Omega)]^n}\big\}^2$$

Hence we have the first inequality (2.3) and an estimate of $\overline{\phi}_\varepsilon$ in $H^1(Y;L^2(\Omega))$. From Theorem 2.2 of [5] and (2.5) we obtain a finer estimate of $\overline{\phi}_\varepsilon$ in $H^1(Y;\big(H^1(\Omega)\big)')$

$$||\overline{\phi}_\varepsilon||_{H^1(Y;(H^1(\Omega))')} \leq C\varepsilon\big\{||\phi||_{L^2(\Omega)} + \varepsilon||\nabla\phi||_{[L^2(\Omega)]^n} + \sqrt{\varepsilon}||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,2})]^n}\big\} + C\sqrt{\varepsilon}||\phi||_{L^2(\widehat{\Omega}_{\varepsilon,2})}$$

It is the second inequality in (2.3). □

Theorem 2.3 : *For any $\phi \in H^1(\Omega)$, there exists $\widehat{\phi}_\varepsilon \in H^1_{per}(Y;L^2(\Omega))$ such that*

$$\begin{cases} ||\widehat{\phi}_\varepsilon||_{H^1(Y;L^2(\Omega))} \leq C||\nabla\phi||_{[L^2(\Omega)]^n}, \\ ||\mathcal{T}_\varepsilon(\nabla_x\phi) - \nabla_x\phi - \nabla_y\widehat{\phi}_\varepsilon||_{[L^2(Y;(H^1(\Omega))')]^n} \leq C\varepsilon||\nabla\phi||_{[L^2(\Omega)]^n} + C\sqrt{\varepsilon}||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}. \end{cases} \tag{2.7}$$

The constants depend only on n and $\partial\Omega$.

Proof : Let $\phi \in H^1(\Omega)$. The function ϕ is decomposed

$$\phi = \Phi + \varepsilon\underline{\phi}, \quad \text{where } \Phi = \mathcal{Q}_\varepsilon(\phi) \quad \text{and} \quad \underline{\phi} = \frac{1}{\varepsilon}\mathcal{R}_\varepsilon(\phi).$$

with the following estimate :

$$||\nabla\Phi||_{[L^2(\Omega)]^n} + ||\underline{\phi}||_{L^2(\Omega)} + \varepsilon||\nabla\underline{\phi}||_{[L^2(\Omega)]^n} \leq C||\nabla\phi||_{[L^2(\Omega)]^n}. \tag{2.8}$$

We apply the Poincaré-Wirtinger inequality to the function ϕ in each cell of the form $\varepsilon(\xi+K_i)$ and of the form $\varepsilon(\xi+Y)$ included in $\widehat{\Omega}_{\varepsilon,3}$. We deduce that

$$\begin{aligned} & ||\nabla\mathcal{Q}_\varepsilon(\phi)||_{[L^2(\widehat{\Omega}_{\varepsilon,2})]^n} \leq C||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} \\ \Longrightarrow \quad & ||\nabla\underline{\phi}||_{[L^2(\widehat{\Omega}_{\varepsilon,2})]^n} \leq \frac{C}{\varepsilon}||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} \end{aligned}$$

We also have (see [3])

$$||\underline{\phi}||_{L^2(\widehat{\Omega}_{\varepsilon,2})} = \frac{1}{\varepsilon}||\phi - \mathcal{Q}_\varepsilon(\phi)||_{L^2(\widehat{\Omega}_{\varepsilon,2})} \leq C||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}$$

Theorem 3 applied to $\underline{\phi}$ gives us the existence of an element $\widehat{\phi}_\varepsilon$ in $H^1_{per}(Y; L^2(\Omega))$ such that

$$\text{(2.9)} \qquad \begin{cases} ||\widehat{\phi}_\varepsilon||_{H^1(Y;L^2(\Omega))} \leq C||\nabla\phi||_{[L^2(\Omega)]^n}, \\ ||\mathcal{T}_\varepsilon(\underline{\phi}) - \widehat{\phi}_\varepsilon||_{H^1(Y;(H^1(\Omega))')} \leq C\varepsilon||\nabla\phi||_{[L^2(\Omega)]^n} + C\sqrt{\varepsilon}||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}. \end{cases}$$

We evaluate $||\mathcal{T}_\varepsilon(\nabla\Phi) - \nabla\Phi||_{[L^2(Y;(H^1(\Omega))')]^n}$.

From Lemma 2.2 we have

$$\text{(2.10)} \qquad \Big\|\frac{\partial\Phi}{\partial x_i} - M^\varepsilon_Y\Big(\frac{\partial\Phi}{\partial x_i}\Big)\Big\|_{(H^1(\Omega))'} \leq C\varepsilon||\nabla\phi||_{[L^2(\Omega)]^n} + C\sqrt{\varepsilon}||\nabla\phi||_{L^2(\widehat{\Omega}_{\varepsilon,3})]^n}$$

From the definition of Φ it results that $y \longrightarrow \mathcal{T}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)(.,y)$ is linear with respect to each variable. For any $\psi \in H^1(\Omega)$, we have

$$\begin{aligned} < \mathcal{T}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(.,y) - M^\varepsilon_Y\Big(\frac{\partial\Phi}{\partial x_1}\Big), \psi >_{(H^1(\Omega))',H^1(\Omega)} &= \int_\Omega \Big\{\mathcal{T}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(.,y) - M^\varepsilon_Y\Big(\frac{\partial\Phi}{\partial x_1}\Big)\Big\}\psi \\ &= \int_{\Omega_\varepsilon} \Big\{\mathcal{T}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(.,y) - M^\varepsilon_Y\Big(\frac{\partial\Phi}{\partial x_1}\Big)\Big\}M^\varepsilon_Y(\psi) \\ &+ \int_{\Omega\setminus\Omega_\varepsilon} \Big\{\mathcal{T}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(.,y) - M^\varepsilon_Y\Big(\frac{\partial\Phi}{\partial x_1}\Big)\Big\}\psi \end{aligned}$$

We have

$$\int_{\Omega\setminus\Omega_\varepsilon} \Big\{\mathcal{T}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(.,y) - M^\varepsilon_Y\Big(\frac{\partial\Phi}{\partial x_1}\Big)\Big\}\psi \leq C\sqrt{\varepsilon}||\nabla\phi||_{L^2(\widehat{\Omega}_{\varepsilon,3})]^n}\big\{||\psi||_{L^2(\Omega)} + ||\nabla\psi||_{L^2(\Omega)]^n}\big\}$$

Besides, as in Theorem 3.4 of [5] we show that

$$\begin{aligned} \int_{\Omega_\varepsilon} \Big\{\mathcal{T}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(.,y) - M^\varepsilon_Y\Big(\frac{\partial\Phi}{\partial x_1}\Big)\Big\}M^\varepsilon_Y(\psi) &\leq C\varepsilon||\nabla\phi||_{[L^2(\Omega)]^n}||\nabla\psi||_{[L^2(\Omega)]^n} \\ &+ C\sqrt{\varepsilon}||\nabla\phi||_{L^2(\widehat{\Omega}_{\varepsilon,3})]^n}\big\{||\psi||_{L^2(\Omega)} + ||\nabla\psi||_{L^2(\Omega)]^n}\big\} \end{aligned}$$

and eventually

$$\forall y \in Y, \qquad \Big\|\mathcal{T}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(.,y) - M^\varepsilon_Y\Big(\frac{\partial\Phi}{\partial x_1}\Big)\Big\|_{(H^1(\Omega))'} \leq C\varepsilon||\nabla\phi||_{[L^2(\Omega)]^n} + C\sqrt{\varepsilon}||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}.$$

Considering (2.10) and all the partial derivatives, we obtain

$$||\mathcal{T}_\varepsilon(\nabla\Phi) - \nabla\Phi||_{[L^2(Y;(H^1(\Omega))')]^n} \leq C\varepsilon||\nabla\phi||_{[L^2(\Omega)]^n} + C\sqrt{\varepsilon}||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}$$

Moreover we have

$$\begin{aligned} &\int_\Omega \frac{\partial\underline{\phi}}{\partial x_i}\psi = \int_{\partial\Omega}\underline{\phi}n_i\psi - \int_\Omega \underline{\phi}\frac{\partial\psi}{\partial x_i} \leq C\big\{||\underline{\phi}||_{L^2(\partial\Omega)} + C||\underline{\phi}||_{L^2(\Omega)}\big\}||\psi||_{H^1(\Omega)} \\ &||\underline{\phi}||_{L^2(\partial\Omega)} \leq \frac{C}{\sqrt{\varepsilon}}||\underline{\phi}||_{L^2(\widehat{\Omega}_{\varepsilon,1})]} + C\sqrt{\varepsilon}||\nabla\underline{\phi}||_{[L^2(\widehat{\Omega}_{\varepsilon,1})]^n} \leq \frac{C}{\sqrt{\varepsilon}}||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} \end{aligned}$$

hence $||\varepsilon\nabla\underline{\phi}||_{[(H^1(\Omega))']^n} \leq C\varepsilon||\nabla\phi||_{[L^2(\Omega)]^n} + C\sqrt{\varepsilon}||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}$. Thanks to (2.9) and to the above inequalities the second estimate of (2.7) is proved. □

2.2 Projection theorems in $L^2(Y;(H^s(\Omega))')$, $0<s<1$.

The space $H^s(\Omega)$, $0<s<1$, is defined by

$$H^s(\Omega) = \Big\{\phi \in L^2(\Omega) \mid \int_{\Omega\times\Omega} \frac{|\phi(x)-\phi(x')|^2}{|x-x'|^{n+2s}} dxdx' < +\infty\Big\}.$$

Equipped with the inner product

$$< \phi, \psi >_s = \int_\Omega \phi\,\psi + \int_{\Omega\times\Omega} \frac{\big(\phi(x)-\phi(x')\big)\big(\psi(x)-\psi(x')\big)}{|x-x'|^{n+2s}} dxdx'$$

$H^s(\Omega)$ is a Hilbert separable space. We denote $||.||_{s,\Omega}$ the norm associated to this inner product.
As we have done in Lemma 2.1 we build a linear and continuous extension operator $\mathcal{P}$ from $H^s(\Omega)$, $0<s<1$, into $H^s(\widetilde{\Omega}_{\varepsilon,4})$ verifying

$$||\mathcal{P}(\phi)||_{s,\widetilde{\Omega}_{\varepsilon,4}} \leq C||\phi||_{s,\Omega}$$

The constant depends only on n, s and $\partial\Omega$.
From now on any function belonging to $H^s(\Omega)$ will be extended to a function belonging to $H^s(\widetilde{\Omega}_{\varepsilon,3})$, $0<s<1$. To make the notation simpler the extention of function ϕ will still be denoted ϕ.

Lemma 2.4 : *For any $\phi \in H^s(\Omega)$, $0<s<1$, we have*

$$\text{(2.11)} \qquad \begin{cases} ||\phi - M_Y^\varepsilon(\phi)||_{L^2(\Omega)} \leq C\varepsilon^s||\phi||_{s,\Omega} \qquad ||\phi - \mathcal{Q}_\varepsilon(\phi)||_{L^2(\Omega)} \leq C\varepsilon^s||\phi||_{s,\Omega} \\ ||\nabla\mathcal{Q}_\varepsilon(\phi)||_{[L^2(\widetilde{\Omega}_{\varepsilon,3})]^n} \leq C\varepsilon^{s-1}||\phi||_{s,\Omega}, \quad ||\phi||_{L^2(\widehat{\Omega}_{\varepsilon,3})} \leq C\varepsilon^{s/2}||\phi||_{s,\Omega} \\ ||\phi - \phi(.+\varepsilon\vec{e}_i)||_{L^2(\Omega)} \leq C\varepsilon^s||\phi||_{s,\Omega}, \qquad i\in\{1,\dots,n\} \\ ||\mathcal{Q}_\varepsilon(\phi)||_{L^2(\partial\Omega)} \leq C\varepsilon^{(s-1)/2}||\phi||_{s,\Omega} \end{cases}$$

The constants depend on n, s and $\partial\Omega$.
Proof : For any ψ belonging to $H^s(Y)$, $0<s<1$, we have the Poincaré-Wirtinger inequality

$$||\psi - M_Y(\psi)||_{L^2(Y)} \leq C||\psi||_{s,Y}$$

where $M_Y(\psi)$ is the mean of ψ in the cell Y. The constant depends only on n. We immediately deduce the upper bound $||\phi - M_Y^\varepsilon(\phi)||_{L^2(\widetilde{\Omega}_{\varepsilon,4})} \leq C\varepsilon^s||\phi||_{s,\Omega}$. We apply the Poincaré-Wirtinger inequality to the restriction of ϕ to two neighbouring cells included in $\widetilde{\Omega}_{\varepsilon,4}$ and we obtain the estimate of the gradient of $\mathcal{Q}_\varepsilon(\phi)$ in $\widetilde{\Omega}_{\varepsilon,3}$ ($||\nabla\mathcal{Q}_\varepsilon(\phi)||_{[L^2(\widetilde{\Omega}_{\varepsilon,3})]^n} \leq C\varepsilon^{s-1}||\phi||_{s,\Omega}$) and then the upper bound $||\phi - \mathcal{Q}_\varepsilon(\phi)||_{L^2(\widetilde{\Omega}_{\varepsilon,3})} \leq C\varepsilon^s||\phi||_{s,\Omega}$ thanks to the estimate of $||\phi - M_Y^\varepsilon(\phi)||_{L^2(\widetilde{\Omega}_{\varepsilon,3})}$. The function $\mathcal{Q}_\varepsilon(\phi)$ belongs to $H^1(\widetilde{\Omega}_{\varepsilon,3})$, hence considering a neighbourhood of $\partial\widetilde{\Omega}_{\varepsilon,3}$ (included in $\widetilde{\Omega}_{\varepsilon,3}$) of thickness ε^{1-s} we show that

$$||\mathcal{Q}_\varepsilon(\phi)||_{L^2(\widehat{\Omega}_{\varepsilon,3})} \leq C\varepsilon^{s/2}||\phi||_{s,\Omega} \qquad \Longrightarrow \qquad ||\phi||_{L^2(\widehat{\Omega}_{\varepsilon,3})} \leq C\varepsilon^{s/2}||\phi||_{s,\Omega}.$$

We have

$$\begin{aligned} ||\phi - \phi(.+\varepsilon\vec{e}_i)||_{L^2(\Omega)} \leq\ & ||\phi - \mathcal{Q}_\varepsilon(\phi)||_{L^2(\Omega)} + ||\mathcal{Q}_\varepsilon(\phi) - \mathcal{Q}_\varepsilon(\phi)(.+\varepsilon\vec{e}_i)||_{L^2(\Omega)} \\ & + ||\phi(.+\varepsilon\vec{e}_i) - \mathcal{Q}_\varepsilon(\phi)(.+\varepsilon\vec{e}_i)||_{L^2(\Omega)} \leq C\varepsilon^s||\phi||_{s,\Omega} \end{aligned}$$

thanks to the upper bounds of $||\phi-\mathcal{Q}_\varepsilon(\phi)||_{L^2(\widetilde{\Omega}_{\varepsilon,3})}$ and $||\nabla\mathcal{Q}_\varepsilon(\phi)||_{[L^2(\widetilde{\Omega}_{\varepsilon,3})]^n}$. The last inequality of the lemma is the consequence of the estimates $||\nabla\mathcal{Q}_\varepsilon(\phi)||_{[L^2(\widetilde{\Omega}_{\varepsilon,3})]^n} \leq C\varepsilon^s||\phi||_{s,\Omega}$ and $||\mathcal{Q}_\varepsilon(\phi)||_{L^2(\widetilde{\Omega}_{\varepsilon,3})} \leq C||\phi||_{s,\Omega}$. □

Corollary : *For any $s \in]0,1[$ and for any $\phi \in H^s(\Omega)$ we have*

$$\begin{cases} ||\mathcal{Q}_\varepsilon(\phi) - M_Y^\varepsilon(\phi)||_{L^2(\Omega)} \leq C\varepsilon^s||\phi||_{s,\Omega} \\ \quad ||\phi - \mathcal{T}_\varepsilon(\phi)||_{L^2(\Omega\times Y)} \leq C\varepsilon^s||\phi||_{s,\Omega} \end{cases} \tag{2.12}$$

The constants depend on n, s and $\partial\Omega$.

Proof : The inequalities (2.12) are the consequences of (2.11). □

Theorem 2.5 : *Let ϕ be in $H^1(\Omega)$. There exists $\widehat{\psi}_\varepsilon$ belonging to $H^1_{per}(Y;L^2(\Omega))$ such that for any $s \in]0,1[$*

$$\begin{cases} ||\widehat{\psi}_\varepsilon||_{H^1(Y;L^2(\Omega))} \leq C\big\{||\phi||_{L^2(\Omega)} + \varepsilon||\nabla\phi||_{[L^2(\Omega)]^n}\big\} \\ ||\mathcal{T}_\varepsilon(\phi) - \widehat{\psi}_\varepsilon||_{H^1(Y;(H^s(\Omega))')} \leq C\varepsilon^s\big\{||\phi||_{L^2(\Omega)} + \varepsilon||\nabla\phi||_{[L^2(\Omega)]^n}\big\} \\ \qquad\qquad + C\varepsilon^{s/2}\big\{||\phi||_{L^2(\widehat{\Omega}_{\varepsilon,2})} + \varepsilon||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,2})]^n}\big\} \end{cases} \tag{2.13}$$

The constants depend only on n, s and $\partial\Omega$.

Proof : With a few modifications we prove Theorem 2.5 as Theorem 2.2. Thanks to Lemma 2.4 we replace the inequalities (2.4) of step one in Theorem 2.2 by

$$\forall\Psi \in H^s(\Omega), \qquad \begin{cases} ||\Psi||_{L^2(\widehat{\Omega}_{\varepsilon,1})} \leq C\varepsilon^{s/2}||\Psi||_{s,\Omega}, \\ ||\Psi(.-\varepsilon\vec{e}_i) - \Psi||_{L^2(\Omega)} \leq C\varepsilon^s||\Psi||_{s,\Omega}, \qquad i \in \{1,\ldots,n\}. \end{cases}$$

□

Theorem 2.6 : *For any $\phi \in H^1(\Omega)$, there exists $\widehat{\phi}_\varepsilon \in H^1_{per}(Y;L^2(\Omega))$ such that*

$$\begin{cases} ||\widehat{\phi}_\varepsilon||_{H^1(Y;L^2(\Omega))} \leq C||\nabla\phi||_{[L^2(\Omega)]^n}, \\ ||\mathcal{T}_\varepsilon(\nabla_x\phi) - \nabla_x\phi - \nabla_y\widehat{\phi}_\varepsilon||_{[L^2(Y;(H^s(\Omega))')]^n} \leq C\varepsilon^s||\nabla\phi||_{[L^2(\Omega)]^n} + C\varepsilon^{s/2}||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}. \end{cases} \tag{2.14}$$

The constants depend only on n, s and $\partial\Omega$.

Proof : With a few modifications we prove Theorem 2.6 as Theorem 2.3. Proceeding as Theorem 3.4 in [5] and thanks to Lemma 2.4, we show that

$$||\mathcal{T}_\varepsilon(\nabla\Phi) - \nabla\Phi||_{[L^2(Y;(H^s(\Omega))')]^n} \leq C\varepsilon^s||\nabla\phi||_{[L^2(\Omega)]^n} + C\varepsilon^{s/2}||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}$$

where $\phi = \Phi + \varepsilon\underline{\phi}$, $\Phi = \mathcal{Q}_\varepsilon(\phi)$. Now let ψ be in $H^s(\Omega)$. We have

$$\begin{aligned} \int_\Omega \frac{\partial\underline{\phi}}{\partial x_i}\psi &= \int_\Omega \frac{\partial\underline{\phi}}{\partial x_i}\big(\psi - \mathcal{Q}_\varepsilon(\psi)\big) + \int_\Omega \frac{\partial\underline{\phi}}{\partial x_i}\mathcal{Q}_\varepsilon(\psi) = \int_\Omega \frac{\partial\underline{\phi}}{\partial x_i}\big(\psi - \mathcal{Q}_\varepsilon(\psi)\big) + \int_{\partial\Omega}\underline{\phi}n_i\mathcal{Q}_\varepsilon(\psi) - \int_\Omega \underline{\phi}\frac{\partial\mathcal{Q}_\varepsilon(\psi)}{\partial x_i} \\ &\leq ||\nabla\underline{\phi}||_{[L^2(\Omega)]^n}||\psi - \mathcal{Q}_\varepsilon(\psi)||_{L^2(\Omega)} + ||\underline{\phi}||_{L^2(\partial\Omega)}||\mathcal{Q}_\varepsilon(\psi)||_{L^2(\partial\Omega)} + ||\underline{\phi}||_{L^2(\Omega)}||\mathcal{Q}_\varepsilon(\psi)||_{[L^2(\Omega)]^n} \end{aligned}$$

hence $||\varepsilon\nabla\underline{\phi}||_{[(H^s(\Omega))']^n} \leq C\varepsilon^s||\nabla\phi||_{[L^2(\Omega)]^n} + C\varepsilon^{s/2}||\nabla\phi||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}$ thanks to the estimates of $\underline{\phi}$ (see Theorem 2.3) and the inequalities of Lemma 2.4. □

3. Error estimate in the classical homogenization problem

We consider the following homogenization problem :

$$\begin{cases} \phi^\varepsilon \in H^1_{\Gamma_0}(\Omega), \\ \displaystyle\int_\Omega A\big(\big\{\frac{\cdot}{\varepsilon}\big\}\big)\nabla\phi^\varepsilon.\nabla u = \int_\Omega fu, \\ \forall u \in H^1_{\Gamma_0}(\Omega), \end{cases} \tag{3.1}$$

where

- Ω is a bounded domain in $\mathbb{R}^n$ with lipschitzian boundary,
- Γ_0 is a measurable set of $\partial\Omega$ with measure nonnull or $\Gamma_0 = \emptyset$,
- $H^1_{\Gamma_0}(\Omega) = \big\{\phi \in H^1(\Omega) \mid \phi = 0 \text{ on } \Gamma_0\big\}$,
- $f \in L^2(\Omega)$,
- A is a square matrix of elements belonging to $L^\infty(Y)$, verifying the condition of uniform ellipticity $c|\xi|^2 \leq A(y)\xi.\xi \leq C|\xi|^2$ a.e. $y \in Y$, with c and C strictly positive constants.

If $\Gamma_0 = \emptyset$, we suppose that $\displaystyle\int_\Omega f = \int_\Omega \phi^\varepsilon = 0$

We have shown, see [2], that $\nabla\phi^\varepsilon - \nabla\Phi - \mathcal{U}_\varepsilon\big(\nabla_y\widehat{\phi}\big)$ strongly converges towards 0 in $[L^2(\Omega)]^n$, where $\mathcal{U}_\varepsilon$ is the averaging operator defined by

$$V \in L^2(\Omega \times Y) \qquad \mathcal{U}_\varepsilon(V)(x) = \int_Y V\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon z, \Big\{\frac{x}{\varepsilon}\Big\}\Big)dz, \qquad \mathcal{U}_\varepsilon(V) \in L^2(\Omega),$$

and where

$$(\Phi, \widehat{\phi}) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega, H^1_{per}(Y)/\mathbb{R})$$

is the solution of the limit problem of unfolding homogenization

$$\begin{cases} \forall (U, \widehat{u}) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega; H^1_{per}(Y)/\mathbb{R}) \\ \displaystyle\int_\Omega\int_Y A\big\{\nabla_x\Phi + \nabla_y\widehat{\phi}\big\}.\big\{\nabla_x U + \nabla_y\widehat{u}\big\} = \int_\Omega fU. \end{cases} \tag{3.2}$$

If $\Gamma_0 = \emptyset$, we take $\displaystyle\int_\Omega \Phi = 0$.
We recall that the correctors χ_i, $i \in \{1, \ldots, n\}$, are the solutions of the following variational problems :

$$\chi_i \in H^1_{per}(Y), \qquad \int_Y \chi_i = 0, \qquad \int_Y A(y)\nabla_y\big(\chi_i(y) + y_i\big)\nabla_y\psi(y)dy = 0, \qquad \forall\psi \in H^1_{per}(Y)$$

They allow us to express $\widehat{\phi}$ in terms of $\nabla\Phi$

$$\widehat{\phi} = \sum_{i=1}^n \frac{\partial\Phi}{\partial x_i}\chi_i,$$

and to give the homogenized problem verified by Φ

$$\int_\Omega \mathcal{A}\nabla\Phi\nabla U = \int_\Omega fU \qquad \forall U \in H^1_{\Gamma_0}(\Omega) \tag{3.3}$$

where (see [3])

$$\mathcal{A}_{ij} = \frac{1}{|Y|}\sum_{k,l=1}^{n}\int_Y a_{kl}\frac{\partial(y_i+\chi_i)}{\partial y_k}\frac{\partial(y_j+\chi_j)}{\partial y_l}.$$

3.1 First case : smooth boundary and homogeneous Dirichlet or Neumann limits conditions

In this paragraph we suppose that

- Ω *is a bounded domain in* $\mathbb{R}^n$ *with* $\mathcal{C}^{1,1}$ *boundary,*
- $\Gamma_0 = \partial\Omega$ *(homogeneous Dirichlet condition) or* $\Gamma_0 = \emptyset$ *(homogeneous Neumann condition).*

In Theorems 4.1 and 4.2 in [5] we gave the following error estimate for the solution of problem (3.1) :

$$||\phi^\varepsilon - \Phi||_{L^2(\Omega)} + ||\nabla\phi^\varepsilon - \nabla\Phi - \sum_{i=1}^{n}\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\nabla_y\chi_i\big(\{\frac{\cdot}{\varepsilon}\}\big)||_{[L^2(\Omega)]^n} \leq C\varepsilon^{1/2}||f||_{L^2(\Omega)}, \tag{3.4}$$

the constant depends on n, A and $\partial\Omega$. In Theorem 3.2 we are going to complete these estimates.

Lemma 3.1 : *We have*

$$||\nabla\phi^\varepsilon||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} \leq C\sqrt{\varepsilon}||f||_{L^2(\Omega)} \tag{3.5}$$

The constant depends on n, A and $\partial\Omega$.

Proof : The boundary of Ω being of class $\mathcal{C}^{1,1}$ we deduce that the solution Φ of the homogenized problem (3.3.i) belongs to $H^2(\Omega)$ and verifies $||\Phi||_{H^2(\Omega)} \leq C||f||_{L^2(\Omega)}$. The estimate of Lemma 3.1 is a consequence of (2.1), and of (3.4) and of the following inequality :

$$\begin{aligned}||\nabla\Phi - \sum_{i=1}^{n}\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\nabla_y\chi_i\big(\frac{\cdot}{\varepsilon}\big)||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} \leq& ||\nabla\Phi||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} + C||\nabla\mathcal{Q}_\varepsilon(\Phi)||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n}||\nabla_y\chi_i||_{[L^2(Y)]^n}\\ \leq& C||\nabla\Phi||_{[L^2(\widehat{\Omega}_{\varepsilon,4})]^n} \leq C\sqrt{\varepsilon}||\Phi||_{H^2(\Omega)} \leq C\sqrt{\varepsilon}||f||_{L^2(\Omega)}\end{aligned}$$

□

We denote by $\rho(x) = dist(x,\partial\Omega)$ the distance between $x \in \Omega$ and the boundary of Ω.

Theorem 3.2 : *The solution* ϕ^ε *of problem* (3.1) *verifies the following estimates :*

$$||\phi^\varepsilon - \Phi||_{L^2(\Omega)} \leq C\varepsilon||f||_{L^2(\Omega)}, \tag{3.6}$$

$$\Big\|\rho\Big(\nabla\phi^\varepsilon - \nabla\Phi - \sum_{i=1}^{n}\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\nabla_y\chi_i\big(\frac{\cdot}{\varepsilon}\big)\Big)\Big\|_{[L^2(\Omega)]^n} \leq C\varepsilon||f||_{L^2(\Omega)}. \tag{3.7}$$

The constants depend on n, A and $\partial\Omega$.

Proof : We put $\rho_\varepsilon(.) = \inf\Big\{\frac{\rho(.)}{\varepsilon}, 1\Big\}$.

Step one. Let $U \in H^1_{\Gamma_0}(\Omega)\cap H^2(\Omega)$. In problem (3.1) we take the test function U, then by unfolding we transform the equality we have obtained. Thanks to (2.2), (3.4) and thanks to the corollary of Proposition 3.1 of [5], we have

$$\begin{aligned}\Big|\int_\Omega A\big(\{\frac{\cdot}{\varepsilon}\}\big)\nabla\phi^\varepsilon.\nabla U - \int_{\Omega\times Y}A\mathcal{T}_\varepsilon(\nabla\phi^\varepsilon)\nabla U\Big| \leq& \Big|\int_\Omega A\big(\{\frac{\cdot}{\varepsilon}\}\big)\nabla\phi^\varepsilon.\nabla U - \int_{\Omega\times Y}\mathcal{T}_\varepsilon\Big(A\big(\{\frac{\cdot}{\varepsilon}\}\big)\nabla\phi^\varepsilon.\nabla U\Big)\Big|\\ &+\Big|\int_{\Omega\times Y}A\mathcal{T}_\varepsilon(\nabla\phi^\varepsilon)\big\{\mathcal{T}_\varepsilon(\nabla U) - \nabla U\big\}\Big|\\ &\leq C\big\{\sqrt{\varepsilon}||\nabla\phi^\varepsilon||_{[L^2(\widehat{\Omega}_{\varepsilon,1})]^n} + \varepsilon||\nabla\phi^\varepsilon||_{[L^2(\Omega)]^n}\big\}||\nabla U||_{[H^1(\Omega)]^n}\\ &\leq C\varepsilon||f||_{L^2(\Omega)}||\nabla U||_{[H^1(\Omega)]^n}\end{aligned}$$

We apply now Theorem 2.3 to the function ϕ^ε. There exists $\widehat{\phi}^\varepsilon \in H^1_{per}(Y; L^2(\Omega))$ such that

$$||\mathcal{T}_\varepsilon(\nabla_x \phi^\varepsilon) - \nabla_x \phi - \nabla_y \widehat{\phi}_\varepsilon||_{[L^2(Y;(H^1(\Omega))')]^n} \leq C\varepsilon ||f||_{L^2(\Omega)} \tag{3.8}$$

since from Lemma 3.1 we have $||\nabla \phi^\varepsilon||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} \leq C\sqrt{\varepsilon}||f||_{L^2(\Omega)}$. From the above estimates and from (3.1) we obtain

$$\Big|\int_\Omega f\, U - \int_{\Omega\times Y} A\big(\nabla_x \phi^\varepsilon + \nabla_y \widehat{\phi}_\varepsilon\big)\nabla_x U\Big| \leq C\varepsilon ||f||_{L^2(\Omega)}||\nabla U||_{[H^1(\Omega)]^n} \tag{3.9}$$

Now let $\overline{\chi}_i \in H^1_{per}(Y)$, $i \in \{1, \ldots, n\}$, be the solution of the variationnal problem

$$\int_Y A\nabla_y \theta \nabla_y\big(\overline{\chi}_i + y_i\big) = 0 \qquad \forall \theta \in H^1_{per}(Y) \tag{3.10}$$

If matrix A is symetric $\overline{\chi}_i = \chi_i$, χ_i are the correctors.

In problem (3.1) let us take the test function $u_\varepsilon(x) = \varepsilon \rho_\varepsilon(x) \sum_{i=1}^n \mathcal{Q}_\varepsilon(\frac{\partial U}{\partial x_i})(x)\overline{\chi}_i\big(\frac{x}{\varepsilon}\big)$. We have multiplied by ρ_ε so that the test function u_ε belongs to $H^1_0(\Omega)$. We immediately verify the inequalities ($i \in \{1, \ldots, n\}$)

$$\begin{aligned}
\Big|\int_\Omega A\big(\{\tfrac{\cdot}{\varepsilon}\}\big)\nabla\phi^\varepsilon \nabla u_\varepsilon\Big| = \Big|\int_\Omega f u_\varepsilon\Big| &\leq C\varepsilon ||f||_{L^2(\Omega)}||\nabla U||_{[L^2(\Omega)]^n} \\
\Big|\int_\Omega \varepsilon A\big(\{\tfrac{\cdot}{\varepsilon}\}\big)\nabla\phi^\varepsilon \nabla\rho_\varepsilon\, \mathcal{Q}_\varepsilon(\frac{\partial U}{\partial x_i})\overline{\chi}_i\big(\tfrac{\cdot}{\varepsilon}\big)\Big| &\leq C\sqrt{\varepsilon}||\nabla\phi^\varepsilon||_{[L^2(\widehat{\Omega}_{\varepsilon,1})]^n}||\nabla U||_{[H^1(\Omega)]^n} \\
\Big|\int_\Omega \varepsilon\rho_\varepsilon A\big(\{\tfrac{\cdot}{\varepsilon}\}\big)\nabla\phi^\varepsilon\, \nabla\mathcal{Q}_\varepsilon(\frac{\partial U}{\partial x_i})\overline{\chi}_i\big(\tfrac{\cdot}{\varepsilon}\big)\Big| &\leq C\varepsilon||\nabla\phi^\varepsilon||_{[L^2(\Omega)]^n}||\nabla U||_{[H^1(\Omega)]^n} \\
\Big|\int_\Omega (1-\rho_\varepsilon) A\big(\{\tfrac{\cdot}{\varepsilon}\}\big)\nabla\phi^\varepsilon\, \mathcal{Q}_\varepsilon(\frac{\partial U}{\partial x_i})\nabla_y\overline{\chi}_i\big(\tfrac{\cdot}{\varepsilon}\big)\Big| &\leq C\sqrt{\varepsilon}||\nabla\phi^\varepsilon||_{[L^2(\widehat{\Omega}_{\varepsilon,1})]^n}||\nabla U||_{[H^1(\Omega)]^n}
\end{aligned}$$

From these estimates, from (3.5) and the corollary of Proposition 3.1 in [5] we obtain

$$\begin{aligned}
&\Big|\int_\Omega A\big(\{\tfrac{\cdot}{\varepsilon}\}\big)\nabla\phi^\varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon(\frac{\partial U}{\partial x_i})\nabla_y\overline{\chi}_i\big(\tfrac{\cdot}{\varepsilon}\big)\Big| \leq C\varepsilon||f||_{L^2(\Omega)}||\nabla U||_{[H^1(\Omega)]^n} \\
\Longrightarrow\quad &\Big|\int_\Omega A\big(\{\tfrac{\cdot}{\varepsilon}\}\big)\nabla\phi^\varepsilon \sum_{i=1}^n M^\varepsilon_Y(\frac{\partial U}{\partial x_i})\nabla_y\overline{\chi}_i\big(\tfrac{\cdot}{\varepsilon}\big)\Big| \leq C\varepsilon||f||_{L^2(\Omega)}||\nabla U||_{[H^1(\Omega)]^n}
\end{aligned}$$

By unfolding we transform the left handside integral of the above second inequality. From (2.2) and (3.5) we have

$$\begin{aligned}
&\Big|\int_\Omega A\big(\{\tfrac{\cdot}{\varepsilon}\}\big)\nabla\phi^\varepsilon \sum_{i=1}^n M^\varepsilon_Y(\frac{\partial U}{\partial x_i})\nabla_y\overline{\chi}_i\big(\tfrac{\cdot}{\varepsilon}\big) - \int_{\Omega\times Y}\mathcal{T}_\varepsilon\Big(A\big(\{\tfrac{\cdot}{\varepsilon}\}\big)\nabla\phi^\varepsilon \sum_{i=1}^n M^\varepsilon_Y(\frac{\partial U}{\partial x_i})\nabla_y\overline{\chi}_i\big(\tfrac{\cdot}{\varepsilon}\big)\Big)\Big| \\
\leq &C\sqrt{\varepsilon}||\nabla\phi^\varepsilon||_{[L^2(\widehat{\Omega}_{\varepsilon,1})]^n}||\nabla U||_{[H^1(\Omega)]^n} \leq C\varepsilon||f||_{L^2(\Omega)}||\nabla U||_{[H^1(\Omega)]^n}
\end{aligned}$$

We reintroduce the partial derivatives of U. As a result we have

$$\Big|\int_{\Omega\times Y} A\mathcal{T}_\varepsilon(\nabla_x\phi^\varepsilon)\sum_{i=1}^n \frac{\partial U}{\partial x_i}\nabla_y\overline{\chi}_i\Big| \leq C\varepsilon||f||_{L^2(\Omega)}||\nabla U||_{[H^1(\Omega)]^n}$$

We replace $\mathcal{T}_\varepsilon(\nabla_x\phi^\varepsilon)$ by $\nabla_x\phi+\nabla_y\widehat{\phi}_\varepsilon$ thanks to (3.8), which gives us

$$\Big|\int_{\Omega\times Y} A\big(\nabla_x\phi^\varepsilon+\nabla_y\widehat{\phi^\varepsilon}\big)\,\nabla_y\Big(\sum_{i=1}^n\frac{\partial U}{\partial x_i}\overline{\chi}_i\Big)\Big|\le C\varepsilon||f||_{L^2(\Omega)}||\nabla U||_{[H^1(\Omega)]^n}$$

From the definition of the correctors χ_i we obtain $\displaystyle\int_{\Omega\times Y} A\big(\nabla_x\phi^\varepsilon+\sum_{i=1}^n\frac{\partial\phi^\varepsilon}{\partial x_i}\nabla_y\chi_i\big)\,\nabla_y\Big(\sum_{j=1}^n\frac{\partial U}{\partial x_j}\overline{\chi}_j\Big)=0$, we substract it from the left handside of the above inequality and thanks to (3.10) we deduce

$$\Big|\int_{\Omega\times Y} A\nabla_y\big(\widehat{\phi^\varepsilon}-\sum_{i=1}^n\frac{\partial\phi^\varepsilon}{\partial x_i}\chi_i\big)\,\nabla_x U\Big|\le C\varepsilon||f||_{L^2(\Omega)}||\nabla U||_{[H^1(\Omega)]^n}$$

and then from (3.9) we obtain

$$\Big|\int_\Omega \mathcal{A}\big(\nabla\phi^\varepsilon-\nabla\Phi\big)\nabla U\Big|\le C\varepsilon||f||_{L^2(\Omega)}||\nabla U||_{[H^1(\Omega)]^n}\qquad\forall U\in H^1_{\Gamma_0}(\Omega)\cap H^2(\Omega)\tag{3.11}$$

where $\mathcal{A}$ is the matrix of the homogenized problem.

Let $U_\varepsilon\in H^1_{\Gamma_0}(\Omega)$ be the solution of the variationnal problem

$$\int_\Omega \mathcal{A}\nabla v\nabla U_\varepsilon=\int_\Omega(\phi^\varepsilon-\Phi)v\qquad\forall v\in H^1_{\Gamma_0}(\Omega)\tag{3.12}$$

The boundary of Ω is of class $\mathcal{C}^{1,1}$ and we have the homogeneous Dirichlet or homogeneous Neumann limits conditions. As a result we have U_ε belonging to $H^1_{\Gamma_0}(\Omega)\cap H^2(\Omega)$. Moreover it verifies the estimate

$$||U_\varepsilon||_{H^2(\Omega)}\le C||\phi^\varepsilon-\Phi||_{L^2(\Omega)}$$

In (3.12) we take $v=\phi^\varepsilon-\Phi$ to obtain the estimate of the L^2 norm of $\phi^\varepsilon-\Phi$ thanks to (3.11).

Step two. Now we prove the estimate (3.7) of the theorem.

Let U be in $H^1_{\Gamma_0}(\Omega)$. From Theorem 3.4 in [5] there exists $\widehat{u}^\varepsilon\in H^1_{per}(Y;L^2(\Omega))$ such that

$$||\mathcal{T}_\varepsilon(\nabla U)-\nabla U-\nabla_y\widehat{u}_\varepsilon||_{[L^2(Y;H^{-1}(\Omega))]^n}\le C\varepsilon||\nabla U||_{[L^2(\Omega)]^n}\tag{3.13}$$

In problem (3.1) we take the test function ρU and in problem (3.2) the couple of test functions $(\rho U,\rho\widehat{u}^\varepsilon)$. We obtain

$$\begin{cases}\displaystyle\int_\Omega f\,\rho U=\int_\Omega A\big(\{\tfrac{\cdot}{\varepsilon}\}\big)\,\rho\nabla\phi^\varepsilon.\nabla U+\int_\Omega UA\big(\{\tfrac{\cdot}{\varepsilon}\}\big)\nabla\phi^\varepsilon\nabla\rho\\ \displaystyle\int_\Omega f\,\rho U=\int_{\Omega\times Y}A\rho\Big(\nabla_x\Phi+\sum_{i=1}^n\frac{\partial\Phi}{\partial x_i}\nabla_y\chi_i\Big)\big(\nabla_x U+\nabla_y\widehat{u}^\varepsilon\big)\\ \qquad\qquad\displaystyle+\int_{\Omega\times Y}UA\Big(\nabla_x\Phi+\sum_{i=1}^n\frac{\partial\Phi}{\partial x_i}\nabla_y\chi_i\Big)\nabla_x\rho\end{cases}\tag{3.14}$$

The solution Φ of homogenized problem (3.3.i) belongs to $H^2(\Omega)$ and verifies $||\Phi||_{H^2(\Omega)}\le C||f||_{L^2(\Omega)}$. Hence the function $\rho\nabla\Phi$ belongs to $[H^1_0(\Omega)]^n$. From (3.13) we have

$$\Big|\int_{\Omega\times Y}A\rho\big(\nabla_x\Phi+\sum_{i=1}^n\frac{\partial\Phi}{\partial x_i}\nabla_y\chi_i\big)\big(\mathcal{T}_\varepsilon(\nabla_x U)-\nabla_x U-\nabla_y\widehat{u}_\varepsilon\big)\Big|\le C\varepsilon||f||_{L^2(\Omega)}||\nabla U||_{[L^2(\Omega)]^n}$$

Now we introduce the discrete functions $M_Y^\varepsilon(\nabla\Phi)$, $M_Y^\varepsilon(\frac{\partial\Phi}{\partial x_i})$, $M_Y^\varepsilon(U)$, $M_Y^\varepsilon(\rho)$, $M_Y^\varepsilon(\nabla\rho)$ to replace $\nabla\Phi$, $\frac{\partial\Phi}{\partial x_i}$, U, ρ, $\nabla\rho$ thanks to the estimate of Proposition 3.1 of [5]). We use (2.2) to transform the integrals over $\Omega\times Y$ in integrals over Ω by inverse unfolding. Then we replace the discrete functions by $\nabla\Phi$, $\mathcal{Q}_\varepsilon(\frac{\partial\Phi}{\partial x_i})$, U, ρ, $\nabla\rho$ and to conclude we add the partial derivatives missing in the gradient of $\Phi+\varepsilon\sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\big(\frac{\cdot}{\varepsilon}\big)$ (for more details see the proof of Proposition 4.3 in [5]). We obtain

$$\Big|\int_\Omega f\,\rho U-\int_\Omega A\big(\{\tfrac{\cdot}{\varepsilon}\}\big)\rho\nabla\Big(\Phi+\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\big(\tfrac{\cdot}{\varepsilon}\big)\Big)\nabla U$$
$$-\int_\Omega UA\big(\{\tfrac{\cdot}{\varepsilon}\}\big)\nabla\Big(\Phi+\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\big(\tfrac{\cdot}{\varepsilon}\big)\Big)\nabla\rho\Big|\le C\varepsilon||f||_{L^2(\Omega)}||U||_{H^1(\Omega)}$$

The first equality of (3.14) and the above inequality give us

$$\Big|\int_\Omega A\big(\{\tfrac{\cdot}{\varepsilon}\}\big)\rho\nabla\Big(\phi^\varepsilon-\Phi-\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\big(\tfrac{\cdot}{\varepsilon}\big)\Big)\nabla U$$
$$+\int_\Omega UA\big(\{\tfrac{\cdot}{\varepsilon}\}\big)\nabla\Big(\phi^\varepsilon-\Phi-\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\big(\tfrac{\cdot}{\varepsilon}\big)\Big)\nabla\rho\Big|\le C\varepsilon||f||_{L^2(\Omega)}||U||_{H^1(\Omega)}$$

Now we choose $U=\rho\Big(\phi^\varepsilon-\Phi-\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\big(\frac{\cdot}{\varepsilon}\big)\Big)$. From the coercivity of matrix A there follows that

$$||\rho\nabla\Big(\phi^\varepsilon-\Phi-\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\big(\tfrac{\cdot}{\varepsilon}\big)\Big)||^2_{[L^2(\Omega)]^n}$$
$$\le C||\rho\nabla\Big(\phi^\varepsilon-\Phi-\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\big(\tfrac{\cdot}{\varepsilon}\big)\Big)||_{[L^2(\Omega)]^n}||\phi^\varepsilon-\Phi-\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\big(\tfrac{\cdot}{\varepsilon}\big)||_{L^2(\Omega)}$$
$$+C\varepsilon||f||_{L^2(\Omega)}\Big\{||\rho\nabla\Big(\phi^\varepsilon-\Phi-\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\big(\tfrac{\cdot}{\varepsilon}\big)\Big)||_{[L^2(\Omega)]^n}+||\phi^\varepsilon-\Phi-\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\big(\tfrac{\cdot}{\varepsilon}\big)||_{L^2(\Omega)}\Big\}$$

Thanks to (3.6) we obtain an upper bound of $||\rho\nabla\big(\phi^\varepsilon-\Phi-\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\big(\frac{\partial\Phi}{\partial x_i}\big)\chi_i\big(\frac{\cdot}{\varepsilon}\big)\big)||_{[L^2(\Omega)]^n}$. The functions $\mathcal{Q}_\varepsilon\big(\frac{\partial\Phi}{\partial x_i}\big)$, $i\in\{1,\dots,n\}$, are bounded in $H^1(\Omega)$, the estimate (3.7) immediately follows. □

Corollary : Let Ω' an open set strongly included in Ω, we have

$$||\phi^\varepsilon-\Phi-\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\big(\tfrac{\cdot}{\varepsilon}\big)||_{H^1(\Omega')}\le C\varepsilon||f||_{L^2(\Omega)}$$

The constant depends on n, A, Ω' and $\partial\Omega$. □

3.2 Second case : Lipschitz boundary

In Theorem 4.5 of [5], Γ_0 is a union of connected components of $\partial\Omega$ and we have shown that there exists γ in the interval $\big]0,1/3\big]$ depending on A, n and $\partial\Omega$ such that the solution of problem (3.1) verifies the following error estimate :

$$||\phi^\varepsilon-\Phi||_{L^2(\Omega)}+||\nabla\phi^\varepsilon-\nabla\Phi-\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\nabla_y\chi_i\big(\tfrac{\cdot}{\varepsilon}\big)||_{[L^2(\Omega)]^n}\le C\varepsilon^\gamma||f||_{L^2(\Omega)}\tag{3.15}$$

The constant depends on n, A and $\partial\Omega$.

In the sequel of this paragraph we suppose that

- *the open set Ω is a bounded domain in $\mathbb{R}^2$ of polygonal ($n=2$) or polyhedral ($n=3$) boundary,*
- *Ω is on one side only of its boundary,*
- *Γ_0 is the union of some edges ($n=2$) or some faces ($n=3$) of $\partial\Omega$,*
- *if $\Gamma_0 \neq \partial\Omega$ the homogenized matrix $\mathcal{A}$ is symetric.*

We know (see [6]) that for any $g \in L^2(\Omega)$ the solution of the variationnal problem

$$U \in H^1_{\Gamma_0}(\Omega), \qquad \int_\Omega \nabla U \nabla \phi = \int_\Omega g\phi \qquad \forall \phi \in H^1_{\Gamma_0}(\Omega) \tag{3.16}$$

belongs to $H^{1+s}(\Omega)$ for an s belonging to $]1/2,1[$ ($s=1$ if the domain is convex) depending only on $\partial\Omega$ and on the chosen limits conditions and verifies the estimate

$$||\nabla U||_{s,\Omega} \leq C||g||_{L^2(\Omega)}$$

Under a non singular linear transformation the variationnal problem (3.3) becomes (3.16). It is posed in a domain which is of the same kind as Ω. Hence, the solution Φ of the homogenized problem (3.3) belongs to $H^{1+s}(\Omega)$ for an s belonging to $]1/2,1[$ ($s=1$ if the domain is convex) depending only on $\partial\Omega$, on $\mathcal{A}$ and on the chosen limits conditions and verifies the estimate

$$||\nabla \Phi||_{s,\Omega} \leq C||f||_{L^2(\Omega)}$$

Theorem 3.3 : *The solution ϕ^ε of problem (3.1) verifies*

$$\begin{cases} \Big\| \nabla\phi^\varepsilon - \nabla\Phi - \displaystyle\sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\nabla_y\chi_i\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{[L^2(\Omega)]^n} \leq C\varepsilon^{s/2}||f||_{L^2(\Omega)}, \\ ||\phi^\varepsilon - \Phi||_{L^2(\Omega)} + \Big\|\rho\Big(\nabla\phi^\varepsilon - \nabla\Phi - \displaystyle\sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\nabla_y\chi_i\Big(\frac{\cdot}{\varepsilon}\Big)\Big)\Big\|_{[L^2(\Omega)]^n} \leq C\varepsilon^s||f||_{L^2(\Omega)}. \end{cases} \tag{3.17}$$

The constants depend on n, A and $\partial\Omega$.

Proof :

Step one. As in Proposition 4.3 of [5], we show that if $(\Phi, \widehat{\phi})$ is the solution of problem (3.2), then $\Phi + \displaystyle\sum_{i=1}^n \varepsilon\rho_\varepsilon \mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\Big(\frac{\cdot}{\varepsilon}\Big)$ is an approximate solution of problem (3.1). The function Φ is the solution of the homogenized problem (3.3).

Let $\Psi \in H^1_{\Gamma_0}(\Omega)$. Thanks to Theorem 2.6, there exists $\widehat{\psi}^\varepsilon \in H^1_{per}(Y; L^2(\Omega))$ verifying the estimates (2.15). We take $(\Psi, \widehat{\psi}^\varepsilon)$ as test-function in the unfolded problem (3.2). Since $\nabla\Phi$ belongs to $[H^s(\Omega)]^n$ and $||\nabla\Phi||_{s,\Omega} \leq C||f||_{L^2(\Omega)}$, we obtain

$$\Big|\int_\Omega f\Psi - \frac{1}{|Y|}\int_{\Omega\times Y} A(y)\Big(\nabla_x\Phi(x) + \sum_{i=1}^n \frac{\partial\Phi}{\partial x_i}(x)\nabla_y\chi_i(y)\Big)\mathcal{T}_\varepsilon\big(\nabla_x\Psi\big)\Big| \leq C\varepsilon^{s/2}||f||_{L^2(\Omega)}||\Psi||_{H^1(\Omega)}$$

We replace $\dfrac{\partial\Phi}{\partial x_i}$ by $\mathcal{Q}_\varepsilon\big(\dfrac{\partial\Phi}{\partial x_i}\big)$ and then, the following part of the proof is exactly the same as the proof of Proposition 4.3 in [5] because, thanks to Lemma 2.4 we have

$$\begin{cases} ||\nabla\Phi - \mathcal{Q}_\varepsilon(\nabla\Phi)||_{[L^2(\Omega)]^n} \leq C\varepsilon^s||f||_{L^2(\Omega)}, \\ ||\mathcal{Q}_\varepsilon(\nabla\Phi)||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} \leq C\varepsilon^{s/2}||f||_{L^2(\Omega)}, \\ ||\mathcal{Q}_\varepsilon(\nabla\Phi)||_{[L^2(\Omega)]^n} \leq C||f||_{L^2(\Omega)} \qquad ||\mathcal{Q}_\varepsilon(\nabla\Phi)||_{[H^1(\Omega)]^n} \leq C\varepsilon^{s-1}||f||_{L^2(\Omega)}. \end{cases} \tag{3.18}$$

Hence we obtain the first inequality of (3.17).

Step two. We now use the first inequality of (3.17) and again the estimates of Lemma 2.4 and as in Lemma 3.1 we prove the following upper bound of the L^2 norm of gradient ϕ^ε in the neighbourhood of Ω :

$$||\nabla\phi^\varepsilon||_{[L^2(\widehat{\Omega}_{\varepsilon,3})]^n} \leq C\varepsilon^{s/2}||f||_{L^2(\Omega)} \tag{3.19}$$

The constant depends on n, A and $\partial\Omega$.

Step three. Let U be in $H^1_{\Gamma_0}(\Omega)\cap H^{1+s}(\Omega)$. In problem (3.1) we take the test function U, then by unfolding we transform the equality we have obtained. Thanks to (2.2), (3.19) and thanks to the corollary of Lemma 2.4, we have

$$\begin{aligned}\Big|\int_\Omega A(\{\frac{\cdot}{\varepsilon}\})\nabla\phi^\varepsilon.\nabla U - \int_{\Omega\times Y} A\mathcal{T}_\varepsilon(\nabla\phi^\varepsilon)\nabla U\Big| &\leq \Big|\int_\Omega A(\{\frac{\cdot}{\varepsilon}\})\nabla\phi^\varepsilon.\nabla U - \int_{\Omega\times Y}\mathcal{T}_\varepsilon\Big(A(\{\frac{\cdot}{\varepsilon}\})\nabla\phi^\varepsilon.\nabla U\Big)\Big| \\ &+ \Big|\int_{\Omega\times Y} A\mathcal{T}_\varepsilon(\nabla\phi^\varepsilon)\big\{\mathcal{T}_\varepsilon(\nabla U)-\nabla U\big\}\Big| \\ &\leq C\big\{\varepsilon^{s/2}||\nabla\phi^\varepsilon||_{[L^2(\widehat{\Omega}_{\varepsilon,1})]^n} + \varepsilon^s||\nabla\phi^\varepsilon||_{[L^2(\Omega)]^n}\big\}||\nabla U||_{s,\Omega} \\ &\leq C\varepsilon^s||f||_{L^2(\Omega)}||\nabla U||_{s,\Omega}\end{aligned}$$

We now apply Theorem 2.6 to the function ϕ^ε. There exists $\widehat{\phi}^\varepsilon \in H^1_{per}(Y;L^2(\Omega))$ such that

$$||\mathcal{T}_\varepsilon(\nabla_x\phi^\varepsilon) - \nabla_x\phi - \nabla_y\widehat{\phi}_\varepsilon||_{[L^2(Y;(H^s(\Omega))')]^n} \leq C\varepsilon^s||f||_{L^2(\Omega)} \tag{3.20}$$

We go on as in step 1 of Theorem 3.2 to obtain

$$\Big|\int_\Omega \mathcal{A}\big(\nabla\phi^\varepsilon - \nabla\Phi\big)\nabla U\Big| \leq C\varepsilon^s||f||_{L^2(\Omega)}||\nabla U||_{s,\Omega} \qquad \forall U\in H^1_{\Gamma_0}(\Omega)\cap H^{1+s}(\Omega) \tag{3.21}$$

Let U_ε be the solution of the variationnal problem

$$U_\varepsilon \in H^1_{\Gamma_0}(\Omega), \qquad \int_\Omega \mathcal{A}\nabla v\nabla U_\varepsilon = \int_\Omega(\phi^\varepsilon-\Phi)v \qquad \forall v\in H^1_{\Gamma_0}(\Omega). \tag{3.22}$$

The function U_ε belongs to $H^1_{\Gamma_0}(\Omega)\cap H^{1+s}(\Omega)$. Moreover we have

$$||\nabla U_\varepsilon||_{s,\Omega} \leq C||\phi^\varepsilon-\Phi||_{L^2(\Omega)}$$

We take $v=\phi^\varepsilon-\Phi$ in (3.22) and thanks to (3.21) we obtain the estimate of the L^2 norm of $\phi^\varepsilon-\Phi$.

Step four. We now prove the upper bound of $\rho\Big(\nabla\phi^\varepsilon - \nabla\Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\nabla_y\chi_i\big(\frac{\cdot}{\varepsilon}\big)\Big)$.

We take a test function in $U\in H^1_{\Gamma_0}(\Omega)$ and as in step 2 of Theorem 2.5 we decompose the unfolded of its gradient thanks to Theorem 3.4 of [5]. In (3.1) we take ρU as test function and in (3.2) we take $(\rho U, \rho\widehat{u}^\varepsilon)$ as couple of test functions. We obtain both equalities (3.14). In the first line of the second equality of (3.14) we replace $\nabla\Phi$ and $\frac{\partial\Phi}{\partial x_i}$ by $\mathcal{Q}_\varepsilon(\nabla\Phi)$ and $\mathcal{Q}_\varepsilon(\frac{\partial\Phi}{\partial x_i})$. Thanks to (3.18) we have

$$\left\{\begin{aligned}&\Big|\int_\Omega f\,\rho U - \int_{\Omega\times Y} A\rho\Big(\mathcal{Q}_\varepsilon(\nabla_x\Phi)+\sum_{i=1}^n\mathcal{Q}_\varepsilon(\frac{\partial\Phi}{\partial x_i})\nabla_y\chi_i\Big)\big(\nabla_x U+\nabla_y\widehat{u}^\varepsilon\big)\\ &+\int_{\Omega\times Y} UA\Big(\nabla_x\Phi+\sum_{i=1}^n\frac{\partial\Phi}{\partial x_i}\nabla_y\chi_i\Big)\nabla_x\rho\Big| \leq C\varepsilon^s||f||_{L^2(\Omega)}||\nabla U||_{[L^2(\Omega)]^n}\end{aligned}\right.$$

From the belonging of $\rho\mathcal{Q}_\varepsilon(\nabla\Phi)$ to $[H_0^1(\Omega)]^n$, and from (3.18) and from (3.13) we deduce

$$\Big|\int_{\Omega\times Y} A\rho\Big(\mathcal{Q}_\varepsilon(\nabla_x\Phi)+\sum_{i=1}^n \mathcal{Q}_\varepsilon(\frac{\partial\Phi}{\partial x_i})\nabla_y\chi_i\Big)\Big(\mathcal{T}_\varepsilon(\nabla_x U)-\nabla_x U-\nabla_y\widehat{u}_\varepsilon\Big)\Big| \le C\varepsilon^s||f||_{L^2(\Omega)}||\nabla U||_{[L^2(\Omega)]^n}$$

We go on as in step 2 of Theorem 3.2. To conclude we use the upper bound of the L^2 norm of the function $\phi^\varepsilon-\Phi$ we obtained above. □

Corollary : Let $\Omega^{'}$ be an open set strongly included in Ω, we have

$$||\phi^\varepsilon-\Phi-\varepsilon\sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\Big(\frac{\cdot}{\varepsilon}\Big)||_{H^1(\Omega')} \le C\varepsilon^s||f||_{L^2(\Omega)}$$

The constant depends on n, A, $\Omega^{'}$ and $\partial\Omega$. □

Remark : If Ω is a convex domain we obtain the same estimates as in Theorem 3.2. □

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