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Abstract

We present integrable lattice equations on a two dimensional square lattice with coupled vertex and bond variables. In some of the models the vertex dynamics is independent of the evolution of the bond variables, and one can write the equations as non-autonomous "Yang-Baxter maps". We also present a model in which the vertex and bond variables are fully coupled. Integrability is tested with algebraic entropy as well as multidimensional consistency.

1 Introduction

Along with the classification results for integrable lattice maps on quad-graphs with one component fields on vertices [1], various more general integrable lattice equations have been proposed recently. This includes the "Yang-Baxter maps" (solutions to the functional Yang-Baxter equations [2]) with variables on bonds [3, 4, 5], and number of higher order as well as multi-component cases [6, 7, 8].

We describe here integrable lattice models which have both vertex and bond variables.[1] These models were found in the analysis of a coarse graining process of known integrable models on a square lattice [10]. We first describe the basic procedure we used, and the way to test integrability via algebraic entropy (section 2). We then list various models obtained in

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[1] The term "lattice models with bond and vertex variables" has already been used in the context of quantum spin models [9], but the setting is then quite different: the vertex and bond variables are independent variables and take on discrete values, the main data being the Boltzmann weight of the configuration, and one is the interested in the statistical mechanics of the system. Here, in contrast, the lattice equations define a 1+1 dimensional evolution of the variables.

this way, giving explicitly the evolution they define and their entropy. We also show that the evolution may be determined from a set of algebraic relations which have special rationality properties (section 3). We finally examine the three dimensional consistency of these models, and show that some of them can be interpreted as solutions of non-autonomous functional Yang-Baxter equations (section 4).

2 A road map

2.1 The setting

The starting point is a regular planar square lattice. Its vertices are labeled by two integers (n, m). We introduce two kinds of variables: the vertex variables, denoted $w_{n,m}$, and the bond variables denoted $X_{n,m}$ for the horizontal bonds, and $Y_{n,m}$ for the vertical bonds. The bond variables are indexed by their base vertex, which is to the left (respectively down) along the bond, as shown in Figure 1(a). As usual we use the shorthand notation where only shifts are indicated, e.g., $X_2 = X_{n,m+1}, w_{12} = w_{n+1,m+1}$, etc...

The models are given by three relations between the eight variables associated to each square cell, in such a way that they define an evolution: from the three defining relations, it is possible to calculate X_2, Y_1, w_{12} (open circles in Figure 1(a)) from w, X, Y, w_1, w_2 (black disks). To fully define an evolution, we have to give a suitable set of initial data. Such data may be given on a diagonal staircase, after which the vertex and bond variables may be evaluated on the entire lattice, using the local relations on each cell.

We will be interested in models where the evolution is rational, i.e., X_2, Y_1, w_{12} are rationally expressed in terms of w, X, Y, w_1, w_2. For the models we will consider, one may actually permute the role of the four corners, and in particular define a backward evolution calculating w, X, Y in terms of $Y_1, X_2, w_1, w_2, w_{12}$. This is a form of rational invertibility, similar to birationality for maps. For the models we describe, it is actually possible to define rational evolution in all four directions on the lattice.

2.2 A construction procedure

There is one very simple way to generate such integrable maps from known one-component lattice maps.

Suppose we are given an integrable quad map. As shown in Figure 1(b), consider $\{x, x_{11}, x_{22}, x_{1122}\}$ as vertex variables of a coarser lattice (renaming them $\{w, w_1, w_2, w_{12}\}$, the rule being $w_{n,m} = x_{2n,2m}$), and $\{x_2, x_{112}, x_1, x_{122}\}$ as bond variables (and renaming them $\{Y, Y_1, X, X_2\}$, i.e., $X_{n,m} = x_{2n+1,2m}$, $Y_{n,m} = x_{2n,2m+1}$). The quad map determines x_{12} in terms of x, x_1, x_2, and x_{122} in terms of x_2, x_{22}, x_{12}, and so on, after which one may forget the central x_{12}. One signature of this construction is that X_2 does not depend on w_1, and Y_1 does not depend on w_2.

The purpose of this paper is to present a number of models which cannot be obtained

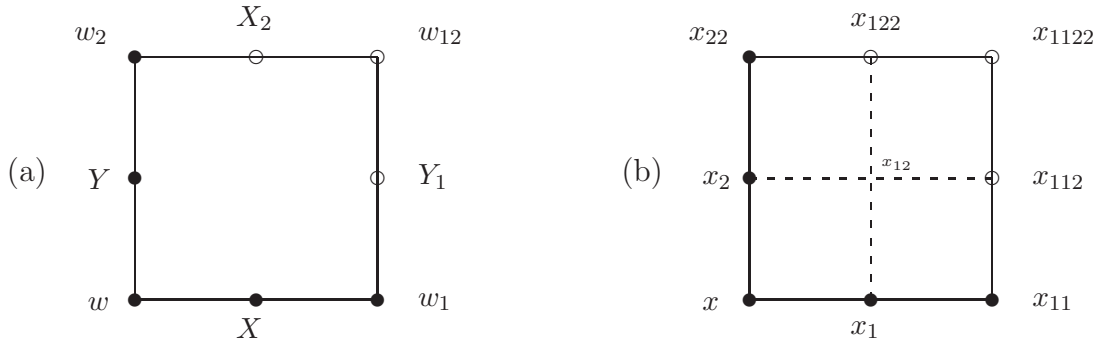

Figure 1: (a): the basic lattice square, and (b): its precursor, a 2×2 sublattice.

in the simple way described above, and to analyze their properties. They will in particular satisfy

$$\frac{\partial X_{n,m+1}}{\partial w_{n+1,m}} \cdot \frac{\partial Y_{n+1,m}}{\partial w_{n,m+1}} \neq 0, \, \forall n, m. \tag{1}$$

In order to construct such maps we start from quad maps having a 2×2 Lax pair [11, 12]. We use this Lax pair to write the zero curvature condition on the larger square of Figure 1(b). From the size of the Lax matrices, we see that this zero curvature condition can give at most three scalar conditions. We know that one rational solution can be obtained in the way described above, we call this solution "regular". *What is remarkable is that for certain quad maps, there exists another rational solution*, which we called "exotic" [10].

In other words, we use the fact that the zero curvature condition written on a coarser lattice is ambiguous. In most cases there is only one solution giving *rational* evolution, it is the regular one. There are usually other solutions that are not rational, but we discard them. Only in some exceptional cases do we find rational solutions satisfying (1).[2]

2.3 Testing integrability

When an exotic solution exists, there is no guarantee a priori that it leads to an integrable evolution, and we have to check its integrability.

The setting is appropriate to use the vanishing algebraic entropy criterion [14, 15]. Given initial data on a basic diagonal staircase, one evaluates the w's, X's and Y's away from the initial diagonal in terms of these data. The entropy can then be extracted from the sequences $\{d_n\}$ of the successive degrees respectively for the w's, X's and Y's. The entropy is the limit

$$\eta = \lim_{n \to \infty} \frac{1}{n} \log(d_n).$$

This limit always exists. *The characteristic of integrability is the vanishing of* η. It is

[2] In some respects this is similar to using conservation laws to derive dynamics, which sometimes allows multiple solutions[13].

equivalent to polynomial growth of the sequences $\{d_n\}$. Non-vanishing of the entropy, that is to say exponential growth of a sequence $\{d_n\}$, means non-integrability.

Remark : The simplest way to find the exact value of the entropy is by fitting, if possible, the generating function $\zeta(s) = \sum_n d_n s^n$ of the sequence of degrees by a rational fraction. The entropy is then determined by the position of the poles of that fraction [14, 15].

2.4 Simplified defining equations

We get the exotic solutions from a set of three algebraic conditions equivalent to a zero curvature condition, having more than one solution. It is possible, for all the models we describe below, to write three simpler equations, which have, *as a unique solution*, the exotic model.

These equations define a rational variety of dimension 5 in an 8 dimensional space. They have special (multi)-rationality properties: for any choice of a corner in Figure 1(a), the three corresponding variables (one vertex variable and the two adjacent bond variables) can be expressed rationally in terms of the five others. This special rationality property allows one to define rational evolutions in all four directions of the square lattice. This is a generalized form of the notion of quadrirationality introduced in [4].

Equivalently, choosing two adjacent bonds, the relations define $2 \mapsto 2$ birational maps between the variables attached to these bonds and the two remaining ones. These maps and their inverses actually happen to have the same form, but they are not involutions a priori.

3 The models

3.1 dpKdV (H1)

The simplest case is obtained from the lattice potential KdV (H1 in [1]). Computing the zero curvature condition on the coarser lattice we found two rational solutions [10]. We give here for reference the regular one, which does not fulfill condition (1):

$$X_2 = Y + \frac{(q-p)(Y-X)}{(Y-X)(w-w_2)-(q-p)}, \tag{2a}$$

$$Y_1 = X + \frac{(p-q)(X-Y)}{(X-Y)(w-w_1)-(p-q)}, \tag{2b}$$

$$w_{12} = w + (p-q)\frac{(p-q)(w_1+w_2-2w)+2(w-w_1)(w-w_2)(X-Y)}{(p-q)^2-(w-w_1)(w-w_2)(X-Y)^2}. \tag{2c}$$

The exotic solution is:

$$\begin{aligned} w_{12} &= w_1 + w_2 - w, && \text{(3a)} \\ X_2 &= Y + P, && \text{(3b)} \\ Y_1 &= X + P, && \text{(3c)} \\ P &= -\frac{(X-Y)[(p-r)(w-w_2)+(q-r)(w-w_1)]}{(p-r)(w-w_2)-(q-r)(w-w_1)-(w-w_1)(w-w_2)(X-Y)}. && \text{(3d)} \end{aligned}$$

The parameters p, q, r appearing in the solutions come from the construction of the models: they are present in the Lax pairs we used.

The sequence of degrees for the bond variables is

$$\{d_n\}_b = 1,\ 4,\ 13,\ 28,\ 49,\ 76,\ 109,\ 148,\ 193,\ 244,\ 301\ldots \tag{4}$$

The generating function of the sequence (4) is

$$\zeta(s) = \frac{1 + 4\,s^2 + s}{(1-s)^3}$$

The sequence has quadratic growth and the entropy vanishes.

Simplified form: By taking suitable linear combinations of equations (3) we can also write the equations in the form

$$\begin{aligned} w_{12} - w_1 - w_2 + w &= 0, && \text{(5a)} \\ X + X_2 - Y - Y_1 &= 0, && \text{(5b)} \\ X\,X_2 - Y\,Y_1 - (X - X_2)\frac{q-r}{w-w_2} + (Y - Y_1)\frac{p-r}{w-w_1} &= 0. && \text{(5c)} \end{aligned}$$

The equations (5) have the form typical for most of our results: an independent linear or linearizable w equation, a linear or linearizable equation for X, Y, X_2, Y_1, and a coupling equation. It is then easy to see that the evolution is rational in any direction: solve the w of any corner from the first equation, then its adjacent X, Y variables can be solved rationally from the remaining two.

Remark: The parameters appearing in the results (here and below) follow from the Lax set-up [10], p, q, r corresponding to the three coordinate directions of the consistency cube. In the end the regular solution typically depends only on $p - q$, while the exotic solution depends on $p - r$ and $q - r$. Clearly any finite r can be absorbed into p, q. The limit $r \to \infty$ is also possible, but in the present case it produces a model that is linear in X, Y.

3.2 H1ϵ

This quad map was introduced in [16]. There in fact are two kinds of integrable models of type H1ϵ, related by inversion.

$$(x - x_{12})(x_1 - x_2) - (p - q)(1 + \epsilon x x_{12}) = 0 \tag{6}$$

$$(x - x_{12})(x_1 - x_2) - (p - q)(1 + \epsilon x_1 x_2) = 0, \tag{7}$$

When extended to the whole lattice they must alternate thereby forming a black-white (checkerboard) lattice [18, 10].

We may consider two different configurations for the 2×2 basic cell

α :

(7)	(6)
(6)	(7)

β :

(6)	(7)
(7)	(6)

To each of these patterns corresponds a zero curvature condition. Each of these again have two rational solutions, the regular one and an exotic one. The exotic solutions are as follows:

3.2.1 H1$\epsilon\alpha$

For configuration α, we have:

Original form:

$$\begin{aligned} X_2 =& Y + P/Q, \quad Y_1 = X + P/Q, && \text{(8a)}\\ P =& (X-Y)[(p-r)(w-w_2)(1+\epsilon w w_1) + (q-r)(w-w_1)(1+\epsilon w w_2)] \\ &+ \epsilon(p+q-2r)(p-q)(w-w_1)(w-w_2) && \text{(8b)}\\ Q =& (w-w_1)(w-w_2)(X-Y) \\ &- (p-r)(w-w_2)(1+\epsilon w w_1) + (q-r)(w-w_1)(1+\epsilon w w_2) && \text{(8c)}\\ w_{12} + w =& \frac{(w_1+w_2)(1+\epsilon w^2)}{1+\epsilon(w w_1 + w w_2 - w_1 w_2)}. && \text{(8d)} \end{aligned}$$

Entropy: The sequence of degrees are respectively:

$$\{d_n\}_v = 1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 15,\ 17,\ 19,\ 21, \ldots \tag{9}$$

for the vertex variables, and

$$\{d_n\}_b = 1,\ 5,\ 17,\ 37,\ 65,\ 101,\ 145,\ 197,\ 257,\ 325, 401,\ 485,\ 577, \ldots \tag{10}$$

for the bond variables. The sequence (9) has linear growth, signaling the linearizability of the evolution of the vertex variables (see below). The sequence (10) has the generating function

$$\zeta(s) = \sum_n d_n s^n = \frac{1 + 5\,s^2 + 2\,s}{(1-s)^3} \tag{11}$$

which means quadratic growth of the degrees, i.e. integrability.

Simplified form: Transforming the vertex variables by the Moebius transformation $w \mapsto (w-1)/[\kappa(w+1)]$ with $\epsilon = -\kappa^2$ etc, takes (8d) into

$$w\, w_{12} = w_1 w_2. \tag{12}$$

Then we can write the equations in the form

$$\begin{aligned} w\, w_{12} - w_1 w_2 &= 0, && \text{(13a)}\\ X_2 + X - Y_1 - Y &= 0, && \text{(13b)}\\ X_2 X - Y_1 Y - (X - X_2)(q-r)\kappa\frac{w+w_2}{w-w_2} + (Y - Y_1)(p-r)\kappa\frac{w+w_1}{w-w_1} & \\ +\kappa^2[(p-r)^2 - (q-r)^2] &= 0. && \text{(13c)} \end{aligned}$$

In comparison with (5), which was linear in w, this is multiplicative.

3.2.2 H1$\epsilon\beta$

For the configuration β we have the exotic solution:

Original form:

$$\begin{aligned} X_2 = Y + P_2/Q_2, \quad Y_1 = X + P_1/Q_1 && \text{(14a)}\\ P_1 =& (X-Y)(1+\epsilon X^2)[(p-r)(w-w_2) + (q-r)(w-w_1)], && \text{(14b)}\\ Q_1 =& (X-Y)(w-w_1)(w-w_2) - (p-r)(w-w_2)(1+\epsilon X^2) & \\ &+ (q-r)(w-w_1)(1-\epsilon X(X-2Y)), && \text{(14c)}\\ P_2 =& (X-Y)(1+\epsilon Y^2)[(p-r)(w-w_2) + (q-r)(w-w_1)], && \text{(14d)}\\ Q_2 =& (X-Y)(w-w_1)(w-w_2) - (p-r)(w-w_2)(1-\epsilon Y(Y-2X)) & \\ &+ (q-r)(w-w_1)(1+\epsilon Y^2), && \text{(14e)}\\ w_{12} =& w_1 + w_2 - w. && \text{(14f)} \end{aligned}$$

Entropy: The entropy calculation leads to the same conclusion as for the configuration α: the vertex evolution is linear, and independent of the bonds. The degree sequence for the bonds is the same as (10).

Simplified form: Starting with (14) and using the Moebius transformation $X \mapsto (X-1)/[\kappa(X+1)]$, etc. with $\epsilon = -\kappa^2$, we get:

$$\begin{aligned} w_{12} - w_1 - w_2 + w &= 0, && \text{(15a)}\\ X + X_2 - Y - Y_1 + 2\kappa\frac{q-r}{w-w_2}(X - X_2) - 2\kappa\frac{p-r}{w-w_1}(Y - Y_1) &= 0, && \text{(15b)}\\ X X_2 - Y Y_1 &= 0. && \text{(15c)} \end{aligned}$$

Remarks: H1$\epsilon\alpha$ and H1$\epsilon\beta$ are deformations that in the limit $\epsilon \to 0$ reduce back to H1. As a consequence the original forms (8) and (14) reduce to (3), as can be readily verified. However, the simplified forms (13) and (15) are obtained with a transformation that is singular in ϵ. In these equations ϵ appears through κ and the $\kappa \to 0$ limit trivializes them.

The models again depend on $p-r$ and $q-r$, so r can be eliminated, except that $r \to \infty$ is a possible limit. In the H1$\epsilon\beta$ case the $r \to \infty$ limit leads to a nonlinear model, which may be interesting on its own.

3.3 dmKdV ($H3_{\delta=0}$)

If we start from the discrete modified KdV equation ($H3$ with $\delta = 0$ in [1]) we obtain the following:

Original form:

$$
\begin{aligned}
w_{12} w &= w_1 w_2, && \text{(16a)}\\
X_2 X &= Y_1 Y, && \text{(16b)}\\
\frac{X_2}{Y} = \frac{Y_1}{X} &= \frac{(q^2 w_2 - r^2 w)(w - w_1) pX - (p^2 w_1 - r^2 w)(w - w_2) qY}{(r^2 w_2 - q^2 w)(w_1 - w) pY - (r^2 w_1 - p^2 w)(w_2 - w) qX} && \text{(16c)}
\end{aligned}
$$

Entropy: The sequence of degrees we get are respectively

$$\{d_n\}_v = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11, \ldots \tag{17}$$

for the vertex variables, and

$$\{d_n\}_b = 1,\ 4,\ 13,\ 28,\ 49,\ 76,\ 109,\ 148,\ 193,\ 244,\ 301, \ldots \tag{18}$$

for the bond variables, actually the same as for $H1$. The vertex evolution is linearizable, and the bond evolution is integrable.

Simplified form: By taking suitable linear combinations of the original equations we can write the result also as

$$
\begin{aligned}
w\, w_{12} - w_1\, w_2 &= 0, && \text{(19a)}\\
(X + X_2) p (q^2 + r^2) - (X - X_2) p (q^2 - r^2) \frac{w + w_2}{w - w_2} & \\
-(Y + Y_1) q (p^2 + r^2) + (Y - Y_1) q (p^2 - r^2) \frac{w + w_1}{w - w_1} &= 0, && \text{(19b)}\\
X\, X_2 - Y\, Y_1 &= 0. && \text{(19c)}
\end{aligned}
$$

Remark: For H3 the parameter dependence is through p/r and q/r. Therefore we may put $r = 1$, but the special limits $r = 0$ and $r \to \infty$ also produce interesting sub-cases.

3.4 dSKdV ($Q1_{\delta=0}$)

This model was obtained from the discrete Schwarzian KdV equation [19] (Q_1 with $\delta = 0$ in [1]). For the elementary square we used the Lax matrices

$$L(x_1, x; p) = \begin{pmatrix} px_1 + r(x - x_1) & -pxx_1 \\ p & -px + r(x - x_1) \end{pmatrix}, \tag{20}$$

where $L(x_1, x; p)$ describes the parallel transport from x to x_1, and $L(x_2, x; q)$ from x to x_2. When constructing the Lax pair for the bigger 2×2 lattices, we get expressions depending on the corner variables $x, x_{11}, x_{22}, x_{1122}$ which will be relabeled as w, w_1, w_2, w_{12}, respectively, while $x_1, x_2, x_{112}, x_{122}$ will become bond variables X, Y, Y_1, X_2, respectively.

The results also depend on the three parameters (p, q, r) in an homogeneous way. The "true" parameters are thus p/r and q/r. If r is not vanishing we do get a rational model, but that model is not integrable, as we will see. (We will not give here the explicit solution for $r \neq 0$, as it may be extracted from the simplified form of the defining equations given at the end of this section.) In the $r \to 0$ limit, we have the following defining relations:

Original form:

$$\begin{aligned}
w_{12} &= \frac{A}{B} \qquad \text{with} \qquad (21)\\
A &= -(X-Y)(2w-X-Y)\,{w_1}^2{w_2}^2 + \left(-2YX^2 - 4Yw^2 + XY^2 + 4wYX + w^3\right){w_1}^2 w_2 \\
&\quad -2w(X-Y)\left(w^2 - YX\right) w_1 w_2 - \left(-2Y^2X - 4w^2X + 4wYX + wX^2 + w^3\right) w_1 {w_2}^2 \\
&\quad +Y^2(w-X)^2\,{w_1}^2 - X^2(w-Y)^2\,{w_2}^2 - wY^2(w-X)^2 w_1 + wX^2(w-Y)^2 w_2, \\
B &= -\left(2w^3X - Y^2X^2 + 4wXY^2 - 4w^2XY - w^2Y^2\right) w_1 - w(w-X)^2\,{w_2}^2 \\
&\quad +\left(-4w^2XY + 2Yw^3 - Y^2X^2 + 4wX^2Y - X^2w^2\right) w_2 + w(w-Y)^2\,{w_1}^2 \\
&\quad +2\,(X-Y)\left(w^2 - YX\right) w_1 w_2 + (w-X)^2 w_1 {w_2}^2 - (w-Y)^2\,{w_1}^2 w_2 \\
&\quad +w^2(X-Y)(wX + Yw - 2YX),
\end{aligned}$$

and

$$\begin{aligned}
X_2 &= \frac{C}{D} \qquad \text{with} \qquad (22)\\
C &= X^2(w-Y)^2 w_2 - Y^2(w-X)^2 w_1 + Y(w-X)^2\,{w_1}^2 \\
&\quad -\left(2Y^2X - YX^2 - Yw^2 - 2wYX + 2w^2X\right) w_1 w_2 + (X-Y)(-X + 2w - Y)\,{w_1}^2 w_2, \\
D &= -\left(2Y^2X - YX^2 - Yw^2 - 2wYX + 2w^2X\right) w_1 \\
&\quad +Y(w-X)^2 w_2 + (w-Y)^2\,{w_1}^2 - (w-X)^2 w_1 w_2 + w(X-Y)(-2YX + wX + wY).
\end{aligned}$$

By symmetry, Y_1 is obtained from X_2 by the exchange $(X \leftrightarrow Y, w_1 \leftrightarrow w_2)$

Entropy: For $r = 0$ the sequence of degrees for the vertex variables is

$$\{d_n\}_v = 1, 10, 28, 71, 139, 248, 398, 605, 869, 1206, 1616, 2115, 2703, \ldots \tag{23}$$

fitted by the generating function

$$\zeta_v(s) = \frac{1 + 7\,s + 9\,s^3 - s^4}{(1+s)\,(1-s)^4} \tag{24}$$

For the bond variables we get the sequence of degrees

$$\{d_n\}_b = 1, 8, 33, 92, 201, 376, 633, 988, 1457, 2056, 2801, 3708, 4793, \ldots \tag{25}$$

fitted by

$$\zeta_b(s) = \frac{(1+s)\,(1+3\,s+4\,s^2)}{(1-s)^4} \tag{26}$$

The growth of the degrees is *cubic*, showing integrability. It is interesting to notice that this example does not have the quadratic growth so commonly observed. The vertex and bond variables are moreover intertwined in a non-trivial way.

The behaviour when $r = 0$ has to be contrasted with the generic $r \neq 0$ behaviour, where the degree calculation yields for the vertex variables

$$\{d_n\}_v = 1,\ 10,\ 38,\ 149,\ 565,\ 2110,\ 7882,\ 29425,\ 109817,\ 409850,\ 1529590, \ldots$$

and for the bond variables

$$\{d_n\}_b = 1,\ 8,\ 45,\ 186,\ 711,\ 2672,\ 9991,\ 37304,\ 139239,\ 519666,\ 1939437, \ldots$$

fitted respectively by the generating functions

$$\zeta_v(s) = \frac{1 + 6\,s - s^2 + 6\,s^3 + s^4}{(1-s)\,(1+s+s^2)\,(1-4\,s+s^2)}, \quad \zeta_b(s) = \frac{1 + 14\,s^2 + 13\,s^3 + 8\,s^4 + 4\,s}{(1-s)\,(1+s+s^2)\,(1-4\,s+s^2)},$$

indicating non-integrability (the rate of growth of the degrees is given by the roots of $s^2 - 4s + 1$, i.e. $2 + \sqrt{3}$).

Simplified form: The transformation

$$X \mapsto \frac{Xw - w_1}{X - 1},\ Y \mapsto \frac{Yw - w_2}{Y - 1},\ X_2 \mapsto \frac{X_2 w_2 - w_{12}}{X_2 - 1},\ Y_1 \mapsto \frac{Y_1 w_1 - w_{12}}{Y_1 - 1},$$

simplifies the equations to

$$\begin{aligned} X_2 Y - Y_1 X &= 0, && (27a)\\ Y_1 Y (w - w_1) + w_{12} - w_2 &= 0, && (27b)\\ X_2 X (w - w_2) + w_{12} - w_1 &= 0, && (27c) \end{aligned}$$

Here is also, for reference, the (nonintegrable) form before taking the $r \to 0$ limit:

$$\begin{aligned} X_2 Y - Y_1 X &= r(AX_2 + BY_1),\\ Y_1 Y (w - w_1) + w_{12} - w_2 &= 0,\\ X_2 X (w - w_2) + w_{12} - w_1 &= 0, \end{aligned}$$

where

$$\begin{aligned} A &= \frac{(w-w_1)(w-w_2)\,q\,XY - ((w-w_2)X - w_1 + w_2)^2 p}{p\,q\,(w-w_1)(w_1-w_2)},\\ B &= \frac{(w-w_1)(w-w_2)\,p\,XY - ((w-w_1)Y - w_1 + w_2)^2 q}{p\,q\,(w-w_2)(w_1-w_2)}. \end{aligned}$$

4 Non-autonomous functional Yang-Baxter equations, three dimensional consistency

One interesting property of the four models proposed in Sec. 3.1-3.3 is that they provide non-autonomous solutions to the functional form of the Yang-Baxter equations [2], alias Yang-Baxter maps. This may be seen by examining the 3 dimensional consistency of the models on the configuration shown in Figure 2. The vertices introduce position dependence, and the maps induced on the bonds verify a modified functional Yang-Baxter equations if the model is 3D consistent.

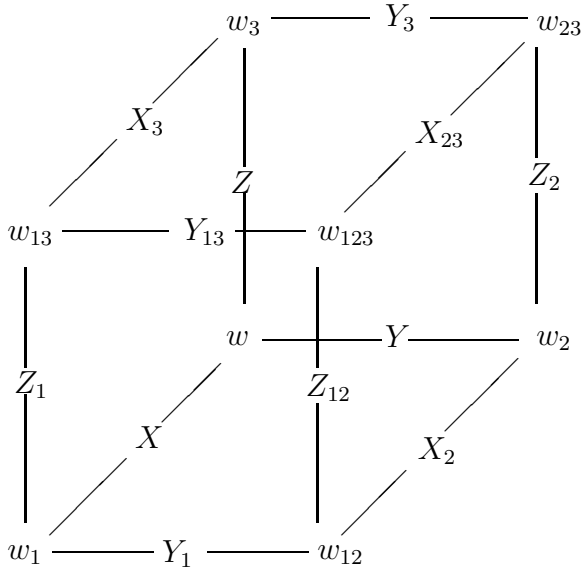

Figure 2: The consistency cube

Suppose that $w, w_1, w_2, w_3, X, Y, Z$ are given. Then the relation on each face allows one to evaluate $X_2, X_3, Y_1, Y_2, Z_1, Z_2, w_{12}, w_{13}, w_{23}$ in a unique way, X_{23}, Y_{13}, Z_{12} in two different ways, and w_{123} may be computed in three different ways. Consistency means they all have to give the same results. *We have checked that all the integrable models described here verify this 3 dimensional consistency.*

It should be noted that for the model presented in section 3.4, the consistency is not verified when $r \neq 0$, precisely in the case where it is not integrable, but is verified for $r = 0$ (that is to say the limit $p = q = s = \infty$), where it is integrable, as ascertained by the vanishing of the algebraic entropy.

The first four sets of simplified equations have similar structure. The vertex equation is independent of the bonds, and the values w might be considered as parameters. They introduce non-autonomy in the bond evolution, which turns out to be the functional equivalent of the quantum Yang-Baxter equation introduced in [20]. The vertex equations are either linear (additive case), or linearizable (multiplicative case), and can be solved explicitly. In the additive case $w + w_{12} - w_1 - w_2 = 0$ the general solution is $w_{n,m} = F(n) + G(m)$, and in the multiplicative case $w\, w_{12} - w_1\, w_2 = 0$, we have $w_{n,m} = F(n)\, G(m)$, with F and G arbitrary functions. The equations determining X and Y contain one linear part, and a

coupling equation which introduces non-linearity.

4.1 dpKdV and $H1\epsilon$, configuration α

We can solve dpKdV (5) and $H1\epsilon$, configuration α, (13) together. The vertex evolution is additive for (5a) and multiplicative for (13a) and is solved as described above. Equations (5c) and (13c) can then be cast in the common form:

$$X\,X_2 - Y Y_1 - g(m)\,(X - X_2) + f(n)\,(Y - Y_1) + \omega = 0, \tag{28}$$

where f and g are readily expressible in terms of the aforementioned $F(n)$ and $G(m)$ and ω is a constant.

We can solve (28) together with the constraint $X+X_2 = Y+Y_1$ for arbitrary $f(n), g(m), \omega$, obtaining

$$X_2 = Y + R, \qquad Y_1 = X + R, \qquad \text{with} \qquad R = \frac{f(n)^2 - g(m)^2 - \omega}{Y - X + f(n) - g(m)}.$$

Redefining the X and Y by the shifts

$$X = X' + f(n) + \sigma(n) + \rho(m), \qquad Y = Y' + g(m) + \sigma(n) + \rho(m),$$

where σ, ρ are determined from $\sigma(n+1) - \sigma(n) = 2\,f(n)$ and $\rho(m+1) - \rho(m) = 2\,g(m)$, we finally get

$$X_2' = Y' + P, \qquad Y_1' = X' + P, \qquad \text{with} \qquad P = \frac{f(n)^2 - g(m)^2 - \omega}{X' - Y'}. \tag{29}$$

This is a non-autonomous version of the Adler map [17] (F_V in the classification of [4]).

4.2 $H1\epsilon$, configuration β and dmKdV ($H3_{\delta=0}$)

Equations (15) and (19) can also be solved together. In both cases the coupling equations (15b) and (19b) can be written as

$$\mu\,(X + X_2) - \nu\,(Y + Y_1) - g(m)\,(X - X_2) + f(n)\,(Y - Y_1) = 0, \tag{30}$$

where f and g are, as above, some functions related to $F(n), G(m)$, and μ and ν are constants. Equation (30) together with the constraint $X\;X_2 - Y\;Y_1 = 0$ is solved in the generic case by

$$X_2 = Y\;Q, \qquad Y_1 = X\;Q, \qquad Q = \frac{(\mu - g(m))\,X - (\nu - f(n))\,Y}{(\nu + f(n))\,X - (\mu + g(m))\,Y}, \tag{31}$$

Scaling X and Y by

$$X = X'\sigma(n)\rho(m)/(f(n) + \nu), \qquad Y = Y'\sigma(n)\rho(m)/(g(m) + \mu), \tag{32}$$

where σ, ρ now solve

$$\sigma(n+1) = \sigma(n)\,\frac{\nu - f(n)}{\nu + f(n)} \qquad \rho(m+1) = \rho(m)\,\frac{\mu - g(m)}{\mu + g(m)}, \tag{33}$$

we get the equations in the form

$$X_2' = \frac{Y'}{\alpha(n)}P, \quad Y_1' = \frac{X'}{\beta(m)}P, \quad P = \frac{\alpha(n)X' - \beta(m)Y'}{X' - Y'}, \tag{34}$$

with

$$\alpha(n) = \frac{1}{\nu^2 - f(n)^2}, \qquad \beta(m) = \frac{1}{\mu^2 - g(m)^2}.$$

Equation (34) is a non-autonomous version of F_{III} of [4].

It should be noticed that the non-autonomous nature of the above equations parallels the results of [21].

4.3 dSKdV ($Q1_{\delta=0}$)

In the case of equation (27) the w evolution is not linearizable, as indicated by the algebraic entropy analysis. If we consider (27) as three equations for two variables X, Y then we can derive from their compatibility an equation for w, which can be written as

$$\frac{T(n,m+2)}{T(n,m)} = \frac{H(n+2,m)}{H(n,m)}, \tag{35a}$$

where T, H are 3-point Schwarzian-like derivatives of n, m, respectively:

$$T(n,m) := \frac{w(n+2,m) - w(n+1,m)}{w(n+1,m) - w(n,m)}, \quad H(n,m) := \frac{w(n,m+2) - w(n,m+1)}{w(n,m+1) - w(n,m)}. \tag{35b}$$

Equation (35) connects the points of a 3×3 sublattice, except the center point (in the above $w(n+1,m+1)$). The algebraic entropy analysis indicates the rare cubic growth. Equations defined on a 3×3 sublattice are typical for Boussinesq-type lattice equations [22], but they usually have quadratic growth.

The Schwarzian KdV equation is given by

$$S(n,m) := \frac{[w(n,m) - w(n+1,m)][w(n,m+1) - w(n+1,m+1)]}{[w(n,m) - w(n,m+1)][w(n+1,m) - w(n+1,m+1)]} = \frac{p}{q}. \tag{36}$$

From (35) one can now derive

$$S(n+1,m+1)S(n,m) = S(n+1,m)S(n,m+1), \tag{37}$$

which can be solved and we find that w solves the non-autonomous SKdV equation

$$\frac{[w(n,m) - w(n+1,m)][w(n,m+1) - w(n+1,m+1)]}{[w(n,m) - w(n,m+1)][w(n+1,m) - w(n+1,m+1)]} = \frac{f(n)}{g(m)}. \tag{38}$$

Thus when the model (27) is considered on larger cells made of four adjacent squares, we find an independent non-trivial evolution of the vertex variables (35), driving the evolution of the bond variables according to (27b,c).

5 Conclusion and perspectives

We have presented lattice models with values given at both the vertices and the bonds of a 2D square lattice. They were constructed using the ambiguity of the zero curvature condition on a coarse grained lattice. The integrability of these models is guaranteed by the vanishing of the algebraic entropy and by the Consistency-Around-the-Cube property.

Four out of five models provide non autonomous generalizations of known Yang-Baxter maps, the fifth case (27) (related to dSKdV) being seemingly different.

They all share some remarkable algebraic properties, in particular their multi-rationality, which might be the frame for further examples and an eventual classification.

Many more aspects will have to be examined (especially for (27)), like the existence of Bäcklund transforms, proper Lax pairs, symmetries, reductions (periodic reductions as well as similarity reductions) which should produce interesting integrable maps, including discrete Painlevé equations. Their continuous limits should also be considered. All these go beyond the scope of this paper, and require further studies.

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