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We present results of a Monte Carlo study for the ferromagnetic Ising model with long range interactions in two dimensions. This model has been simulated for a large range of interaction parameter σ and for large sizes. We observe that the results close to the change of regime from intermediate to short range do not agree with the renormalization group predictions.

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While the Ising model long range interactions (LRI) has been studied for a long time as a generalization of the Ising model with short range interactions, very few results have been obtained for the case of weak interactions decaying faster than the dimension d since the first works in the 70's [1]. We will recall some of these results after defining the model that we will consider in this letter. The Ising model with LRI is defined by the Hamiltonian:

$$\mathcal{H} = -\sum_{<ij>} \frac{J}{r_{ij}^{d+\sigma}} S_i S_j \ , \qquad (1)$$

with spins taking values $S_i = \pm 1$ on the sites i of a regular lattice and r_{ij} the distance between the spins on the sites i and j. The sum $< ij >$ is over all the pair of spins and we will consider only a ferromagnetic interaction $J > 0$.

In 1972, Fisher and al. [2] performed a renormalization group (RG) study for the $O(n)$ model with LRI (the Ising model corresponding to $n = 1$). They identified three regimes (for $d > 1$) : i) the classical regime with $\sigma \leq d/2$ which is believed to be with a mean-field behavior, ii) an intermediate regime for $d/2 < \sigma < 2$, iii) the short range regime for $\sigma \geq 2$. The value $\sigma = d/2$ marks the border of the mean field regime, meaning that the ordinary perturbation parameter $\epsilon = 4 - d$ is replaced by $2\sigma - d$. In [2] the relation $\eta = 2 - \sigma$ was also conjectured in the intermediate regime. This result was questioned since for $\sigma = 2$, the exponent η vanish while for $\sigma > 2$ its value has to be the one of the short range model η_{sr} which is positive for $d < 4$. Then it would imply a jump of the exponent η from 0 up to η_{sr} at $\sigma = 2$. This point was first considered by Sak [3], who, by taking in account higher order terms in the RG calculations, predicted that the change of behavior from the intermediate to the short range regime takes place at $\sigma = 2 - \eta_{sr}$. Many other studies have considered also this problem with various conclusions. In particular, van Enter [4] obtained that for $n \geq 2$, long range perturbations are relevant in the regime $2 - \eta_{sr} \leq \sigma \leq 2$ in contradiction with Sak results. Gusmão and Theumann [5], by considering a development in terms of $\epsilon' = 2\sigma - d$ in place of $\epsilon = 2 - \sigma$, obtained a similar result, namely the stability of the long range perturbation for $\sigma \leq 2$.

Note that all these studies are using a renormalisation group approach with an ϵ expansion of a Landau-Ginzburg effective Hamiltonian such that the propagator contains a p^σ term in addition to the ordinary p^2 term. While it is know that this approach gives accurate results for two and three dimensions for the short range model (with just the p^2 term) [6], it is only after comparing these predictions with other methods, numerical, high temperature expansions, (and the exact result in two dimensions), *etc.* that we can believe in these predictions. Similar comparisons need also to be done when considering the case with LRI and this is the main purpose of the study presented in this letter.

A first numerical study of the exponent η for $d = 2$ as a function of σ has already been done by Luijten and Blöte [7]. In particular, they obtained in the intermediate regime a result well described by the exponent $\eta = 2 - \sigma$ up to $2 - \sigma = \eta_{sr}$ and $\eta = \eta_{sr} = 1/4$ for larger σ. Thus their measured exponent is in agreement with $\eta = \max(2 - \sigma, 1/4)$ which corresponds to the RG predictions of [3].

In the present study, we improve Luijten and Blöte study. In particular, we repeat the measurement of η close to the region where its behavior is changing, *i.e.* for $\sigma \simeq 2 - \eta_{sr}$. We confirm their result that there is no discontinuity but we measure a clear deviation from the behavior predicted by Sak [3].

In order to be able to consider large lattices, we need to employ an efficient algorithm. Since the model is ferromagnetic, we can employ a cluster algorithm which will reduce the auto correlation time, that is the number of upgrades between two successive independent configurations. For the short range Ising model, Wolff algorithm, which builds a single cluster per update, is the most efficient one [8]. We will adapt this algorithm to the case with LRI. The algorithm can be summarized by the following steps. We start from a spin S_i at some randomly selected position i. Next, we add to this spin any other

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spin S_j with a probability

$$\delta_{S_i,S_j}\left(1-e^{-2\beta\frac{J}{r_{ij}^{d+\sigma}}}\right)=\delta_{S_i,S_j}(1-p(r_{ij}))\,, \qquad (2)$$

with β the inverse temperature. We repeat the same operation with all the added spins. In order to build a cluster containing M spins on a lattice of N sites, we have to compute the probability (2) for $\simeq N\times M$ bonds, then the number of operations is $\mathcal{O}(N\times M)$. This number of operations will be drastically reduced by our algorithm that we will describe now. We will present here only the main steps of this algorithm, a more complete version will be presented elsewhere [9].

The main idea is that since $p(r_{ij})$ is very small for large r_{ij}, then it is much faster to compute the probability that one (or more than one spin) among all the spins at this distance are connected to the original spin S_i. In practice, to build this cluster we proceed as follow. First, starting from some arbitrary position i, we order in some pile $\mathcal{P}_1(k=1,\cdots,N-1)$ of length $N-1$ all the other positions on the lattice, ordered in function of the distance. To be more precise, the pile will contain the difference of positions between the point i and j. The pile $\mathcal{P}_1$ will be the same for any i, so this operation is done only once. Next we consider all $n(r)$ spins at a distance r. This is a very fast operation if we build a second pile $\mathcal{P}_2(r)$ containing the position of the first spin in $\mathcal{P}_1$ at a distance r. Then $n(r)=\mathcal{P}_2(r')-\mathcal{P}_2(r)$, with r' the smallest distance on the lattice such that $r'>r$. For a given configuration of spins and a given starting point i, we need only to consider the spins with the same value as S_i and which are not already part of the cluster under construction. We denote by $n'(r)\leq n(r)$ the number of such spins. Then the decomposition

$$\begin{aligned}1 &= ((1-p(r))+p(r))^{n'(r)} \qquad (3)\\ &= p(r)^{n'(r)}+n'(r)p(r)^{n'(r)-1}(1-p(r))\\ &\quad +\cdots+(1-p(r))^{n'(r)}\\ &= \sum_{k=0}^{n'(r)}\frac{n'(r)!}{(n'(r)-k)!k!}p(r)^{n'(r)-k}(1-p(r))^k\end{aligned}$$

will contain the probability of having zero spins connected $p(r)^{n'(r)}$, of having one spin connected $n'(r)p(r)^{n'(r)-1}(1-p(r))$, two spins connected $n'(r)(n'(r)-1)p(r)^{n'(r)-2}(1-p(r))^2/2$, *etc.* Next we randomly generate a number ϵ in the interval $[0,1]$. If $\epsilon<p(r)^{n'(r)}$, no spin will be reversed at the distance r. We can then ignore all the $n(r)$ spins at distance r, at the cost of generating one random number. Otherwise, if $p(r)^{n'(r)}<\epsilon<p(r)^{n'(r)}+n'(r)p(r)^{n'(r)-1}(1-p(r))$, one spin has to be reverse among the possible $n'(r)$ spins. We then need a second random number to choose one spin among the $n'(r)$. Next if $p(r)^{n'(r)}+n'(r)p(r)^{n'(r)-1}(1-p(r))<\epsilon$, we continue the process until we obtain

$$\epsilon<\sum_{k=0}^{k_{max}}\frac{n'(r)!}{(n'(r)-k)!k!}p(r)^{n'(r)-k}(1-p(r))^k \qquad (4)$$

with k_{max} the number of spins that we have to reverse and thus the number of random numbers that we need to generate to choose these spins among the $n'(r)$.

For each distance r one needs also to compute $n'(r)$. This corresponds to the delta function in eq.(2). In principle, this means that we need to perform an additional number of operations $n(r)$ to determine $n'(r)$. In fact, in most cases, this part can be skipped. Indeed, it is much more convenient to first compare ϵ with $p(r)^{n(r)}<p(r)^{n'(r)}$. Thus we will compute $n'(r)$ only if $p(r)^{n(r)}<\epsilon$, which will be very rare for r large. Then in most cases one can consider all the $n(r)$ spins in the slice at distance r with the cost of generating a single random number.

A second improvement is to consider an ensemble of successive slices with $r_1\leq r\leq r_2$. The decomposition (3) is then replaced by

$$\begin{aligned}1 &= \prod_{r=r_1}^{r_2}((1-p(r))+p(r))^{n'(r)} \qquad (5)\\ &= \left(\prod_{r=r_1}^{r_2}p(r)^{n'(r)}\right)\left(1+\sum_{r=r_1}^{r_2}n'(r)\frac{(1-p(r))}{p(r)}+\cdots\right)\end{aligned}$$

The choices of r_1 and r_2 are guided by efficiency. Starting from a first slice r_1, we will add slices up to r_2 with the condition that $\prod_{r=r_1}^{r_2}p(r)^{n(r)}$ remains close enough to 1 to ensure that in most cases the randomly generated number ϵ will be such that $\epsilon<\prod_{r=r_1}^{r_2}p(r)^{n(r)}\leq\prod_{r=r_1}^{r_2}p(r)^{n'(r)}$. Again, we will first compare ϵ with $\prod_{r=r_1}^{r_2}p(r)^{n(r)}$. If ϵ is smaller than this product, we can then skip all the spins in the slices of distance $r_1\leq r\leq r_2$ for the cost of generating a single random number. Otherwise, we have to count the number of spins $n'(r)$ in each slice $r_1\leq r\leq r_2$ which can be connected to the cluster (*i.e.* spins with the good sign and not yet part of the cluster). This corresponds to $N_{r_1,r_2}=\sum_{r_1}^{r_2}n(r)$ operations. Now, if the condition $\prod_{r=r_1}^{r_2}p(r)^{n(r)}<\epsilon<\prod_{r=r_1}^{r_2}p(r)^{n'(r)}$ is satisfied, we can skip all the spins in the slices of distance r with $r_1\leq r\leq r_2$ and with a cost of generating a single random number + N_{r_1,r_2} operations. Otherwise, we have to reverse at least one spin and we have still some additional operations. It can be checked that the number of additional operations in that case is smaller than N_{r_1,r_2}.

The choice of r_1 and r_2 can now be obtained by requiring that the probability of reversing one spin times the number of operations is comparable to the number of operations for having to reverse no spins, *i.e.* when eq.(4) is satisfied with $k_{max}=0$. As said before, for

this case, the number of operations is $\mathcal{O}(1)$. As a first approximation, the probability of reversing one spin is $P_{r_1,r_2} \simeq \sum_{r=r_1}^{r_2} n(r)(1-p(r))$. Then the optimal choice r_1 and r_2 is such that $P_{r_1,r_2} \times N_{r_1,r_2} \simeq \mathcal{O}(1)$. A simple extension of this argument [9] leads to the result that the total number of operations for a system with N spins grows like $\mathcal{O}(N)$.

Our algorithm is similar in spirit with the one developed by Luijten and Blöte [7] but the implementation is rather different. In their algorithm, the second part of the weight (2) is obtained by building a cumulative bond probability. In order to perform this step efficiently, they need to approximate the interaction between two spins by a integral. While this approximation does not affect the universal critical properties, other nonuniversal quantities like the critical temperatures will be different from the one that we obtained. It was argued in [10] that Luijten and Blöte algorithm requires $\mathcal{O}(N \log N))$ operations.

Note also that while we present here results for the case of the Ising model in two dimensions, our algorithm is valid for any Potts model and in any dimensions. We will present elsewhere results for the case of the Ising model with long range interactions in three dimensions [9].

We will now report on simulation done on a triangular lattice of linear size up to $L = 5120$ with periodic boundary conditions. To implement these boundary conditions with long range interactions, we employed the minimum image convention. In order to be able to obtain a good precision, we had to accumulate a lot of statisics. For each value of $\sigma, K = \beta J$ and size L, we accumulated statistics over $10^6 \times \tau$ updates, with τ the autocorrelation time. The time for the update of one sample of size 5120×5120 is $\simeq 0.1$ second for $\sigma = 0.8$ and $\simeq 1$ second for $\sigma = 1.8$ on a recent PC. With the help of the Wolff cluster algorithm, the autocorrelation times are rather small even for the largest sizes considered. For $L = 5120$, we determined $\tau \simeq 200$ for $\sigma = 0.8$ and $\tau \simeq 66$ for $\sigma = 1.8$. The total computing time required to produce the data presented in this work corresponds to $\mathcal{O}(100)$ years of CPU time on a single core processor. In order to test the universality of our results and to have a more direct comparison with the results of Luijten and Blöte [7], we also performed a simulation for $\sigma = 1.75$ on a square lattice.

For each value of σ, we first had to determine the critical value K_c. This was done by considering a magnetic cumulant similar to the Binder cumulant [11] and which is an adimensional quantity. We will consider the magnetic cumulant defined by $B(L, K)$:

$$B(L, K) = \frac{\langle m^2 \rangle^2}{\langle m^4 \rangle} , \tag{6}$$

with $\langle m^i \rangle$ the thermal average of the magnetization to the power i. This cumulant will converge to the value

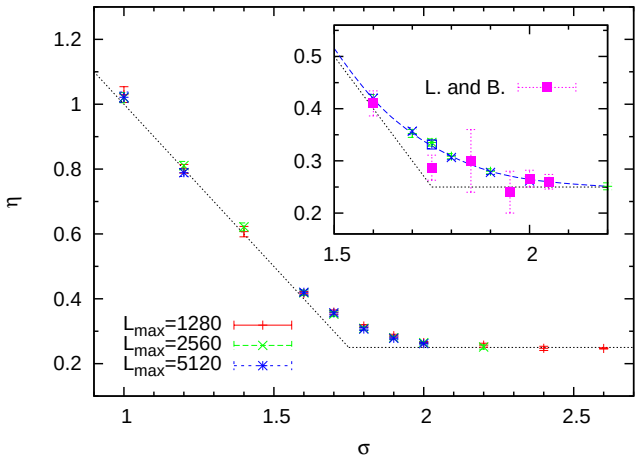

FIG. 1: (Color online) Exponent η vs. σ computed with data up to size $L = 1280, 2560$ and 5120. The inset contains a magnified part for $1.5 \leq \sigma \leq 2.2$ with in addition the data obtained by Luijten and Blöte [7]. The dotted lines correspond to the prediction from the RG analysis, $\eta = max(2-\sigma, 1/4)$. The dashed line connecting our measured points is a guide to the eye.

1 in the ferromagnetic phase and 1/3 for the paramagnetic phase. The curves describing the cumulant versus K for different sizes L will cross at a value $K_c(L)$ which will converge toward the real critical point in the large size limit. At the crossing point, the cumulant $B(L)$ will converge towards a finite value. For the critical Ising model with short range interaction, the limiting value was computed on the triangular lattice with a value of $\lim_{L\to\infty} B(L, K_c) = 0.85872528(3)$ [12]. Close to the critical point, the finite-size scaling behavior is expected to be of the form

$$B(L, K) \simeq f((K - K_c)L^{1/\nu}) , \tag{7}$$

with $f(x)$ a dimensionless function. Then the crossing of the curve $B(L, K)$ as a function of K for two different sizes L and L' takes place at $K = K_c$. In the following, we will always choose $L' = 2L$ and express the computed quantities as a function of L only. In practice, due to corrections to scaling, we need to take into account additional correction terms to $B(L, K)$. We will consider in the following the leading correction of the form :

$$B(L, K) \simeq f((K - K_c)L^{1/\nu}) + A_1 L^{-y_1} . \tag{8}$$

With such a term, the crossing of the curves will take place at a size dependent $K_c(L)$:

$$K_c(L) = K_c + \alpha_1 L^{-w_1} , \tag{9}$$

with α_1 a constant and $w_1 = y_1 + 1/\nu$. We will then determine the value of K_c by extrapolating the measured values of the crossing $K_c(L)$ while including one single

correction. We have checked that with such a procedure, the values of w_1 and K_c are stable if one keeps only large size data, $L \geq 100$. The same will be true for the computed values of η, see below. We have also tried to include further correction terms in (8). We have checked that the addition of subdominant corrections does not affect the results for $L_{max} \geq 1280$, *i.e.* the change in K_c is much smaller than the error bars on this value. We also performed a fit of $B(L, K)$ close to K_c with a development in powers of $(K - K_c)L^{1/\nu}$ and with terms L^{-y_i} corresponding to correction to scaling. The obtained exponents are always in good agreement with the ones from a direct fit of the form (8). In particular, we obtain that ν remains very close to 1. We quote $\nu = 0.96(2)$ for $\sigma = 1.6$. Finally, in [7], it was claimed that logarithmic corrections to scaling were needed in order to obtain a consistent analysis for $\sigma = 1.75$. This is not the case in the analysis of our data. In fact, even if we impose the existence of logarithmic corrections, we observe that most of our results are not affected. These corrections will mostly affect the exponents y_i without a measurable change on the exponent η for the largest sizes that we can simulate [9].

In the following, we will be interested in the region $\sigma \geq 1.4$ since this is where we obtain new results. Our main results are shown in Fig. 1. It contains our numerical results for the exponent $\eta = 2\beta/\nu$ versus σ obtained for three ranges of linear sizes, $L_{max} = 1280, 2560$ or 5120. In each case, L_{max} corresponds to the maximum sizes that we employed in order to determine the value of K_c as described in the previous section. Then we determined the value $\eta(K_c)$ by doing an extrapolation from the effective exponent η obtained from the data with $L_{max}/2$ and L_{max}.

The values of η that we obtained are reported in Table 1. We also report the value obtained for $\sigma = 1.75$ on the square lattice and this value extrapolates well with the ones on the triangular lattice. In this table, we can see that there is near no dependance in the size L_{max}. This confirms that our determination of K_c is not affected by further corrections.

From Fig. 1, we see that in the classical regime and in the intermediate regime up to $\sigma \simeq 1.5$ our results are in agreement with the prediction $\eta = 2 - \sigma$ (represented by a dotted line), confirming the results of Luijten and Blöte [7]. For $\sigma > 2$, η is in perfect agreement with the value for a short range model $\eta_{sr} = 1/4$ (represented also by a dotted line). In the remaining part for $1.6 \leq \sigma \leq 2$ and shown in the inset of Fig. 1, our results do not agree with the prediction of the RG analysis [2, 3]. On the contrary, we observe a set of values which interpolate smoothly between these two behaviors. The errors that we obtain on the measured values of η are much smaller than the deviation observed to the form $max(2 - \sigma, \eta_{sr})$. In particular, for $\sigma = 1.8$, the deviation corresponds to near 10 times the standard deviation, *i.e.* $\eta - \eta_{sr} \simeq 10d\eta$. In the inset of Fig. 1, we also show the results of Luijten and Blöte from [7]. We note that their results are compatible with ours but since our error bars are one order of magnitude smaller than theirs, we are able to observe a deviation from the RG predictions. In fact, we can observe that Luijten and Blöte results are compatible with both the RG predictions and our results [13].

σ	$\eta(L = 1280)$	$\eta(= 2560)$	$\eta(L = 5120)$	$max(2 - \sigma, \frac{1}{4})$
1.4	0.607 (11)	0.619 (12)	-	0.600
1.6	0.417 (6)	0.418 (9)	0.420 (8)	0.400
1.7	0.359 (7)	0.353 (8)	0.357 (7)	0.300
1.75	0.346 (7)	0.335 (7)	0.332 (8)	0.250
1.8	0.315 (5)	0.309 (5)	0.307 (5)	0.250
1.9	0.286 (5)	0.280 (5)	0.279 (5)	0.250
2.0	0.265 (3)	0.265 (4)	0.262 (4)	0.250
2.2	0.257 (6)	0.251 (7)	-	0.250

TABLE I: Exponent η for $\sigma = 1.4 - 2.2$. The second to fourth column contain the values obtained by a fit with data up to linear size $L = 1280, 2560$ and 5120. The last column contains the predictions from the RG analysis, $\eta = max(2 - \sigma, 1/4)$.

In this letter we obtained strong evidences that for the long range interaction Ising model in 2d, the exponent η does not behave as predicted by RG studies in [2, 3], in particular in the intermediate regime and close to the boundary with the short range regime as shown in Fig.1 and in Table 1. We believe that further studies are needed to recheck the RG analysis. In particular, it seems that the wave function normalization is not trivial for the long range interaction and this could then give further contributions which were not taken in account in the previous studies [14].

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