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Abstract

Let M be a triangulable compact manifold. We prove that, among closed subgroups of $\text{Homeo}_0(M)$ (the identity component of the group of homeomorphisms of M), the subgroup consisting of volume preserving elements is maximal.

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1 Introduction

The theory of of groups acting on the circle is very rich (see in particular the monographs [Ghy01, Nav07]). The theory is far less developed in higher dimension, where it seems difficult to discover more than some isolated islands in a sea of chaos. In this note, we are interested in the closed subgroups of the group $\text{Homeo}_0(M)$, the identity component of the group of homeomorphisms of some compact topological n-dimensional manifold M. We will show that, when $n \geq 2$, for any *good* (nonatomic and with total support) probability measure μ, the subgroup of elements that preserve μ is maximal among closed subgroups.

Let us recall some related results in the case when M is the circle. De La Harpe conjectured that $PSL(2, \mathbb{R})$ is a maximal closed subgroup ([Bes]). Ghys proposed a list of closed groups acting transitively, asking whether, up to conjugacy, the list was complete ([Ghy01]); the list consists in the whole group, $SO(2)$, $PSL(2, \mathbb{R})$, the group $\text{Homeo}_{k,0}(\mathbb{S}^1)$ of elements that commutes with some rotation of order k, and the group $PSL_k(2, \mathbb{R})$ which is defined analogously. The first conjecture was solved by Giblin and Markovic in [GM06]. These authors also answered Ghys's question affirmatively, under the additional hypothesis that the group contains some non trivial arcwise connected component. Thinking of the two-sphere with these results in mind, one is naturally led to the following questions.

Question 1. *Let G be a proper closed subgroup of* $\text{Homeo}_0(\mathbb{S}^2)$ *acting transitively. Assume that G is not a (finite dimensional) Lie group. Is G conjugate to one of the two subgroups: (1) the centralizer of the antipodal map $x \mapsto -x$, (2) the subgroup of area-preserving elements?*

Note that the centralizer of the antipodal map is the group of lifts of homeomorphisms of the projective plane; it is the spherical analog of the groups $\text{Homeo}_{k,0}(\mathbb{S}^1)$.

Question 2. *Is $PSL(2,\mathbb{C})$ maximal among closed subgroups of* $\mathrm{Homeo}_0(\mathbb{S}^2)$*?*

On the circle the group of measure-preserving elements coincides with $SO(2)$. It is not a maximal closed subgroup since it is included in $PSL(2,\mathbb{R})$. In contrast, we propose to prove that the closed subgroup of area-preserving homeomorphisms of the two-sphere is maximal. To put this into a general context, let M be a compact topological manifold whose dimension is greater or equal to 2. We assume that M is triangulable and (for simplicity) without boundary. Let us equip M with a probability measure μ which is assumed to be *good*: this means that every finite set has measure zero, and every non-empty open set has positive measure. We consider the group $\mathrm{Homeo}_0(M)$ of homeomorphisms of M that are isotopic to the identity, and the subgroup $\mathrm{Homeo}_0(M,\mu)$ of elements that preserve the measure μ. According to the famous Oxtoby-Ulam theorem ([OU41, GP75], see also [Fat80]), if μ' is another good probability measure on M then it is homeomorphic to μ, meaning that there exists an element $h \in \mathrm{Homeo}_0(M)$ such that $h_*\mu = \mu'$. In particular the subgroup $\mathrm{Homeo}_0(M,\mu')$ is isomorphic to $\mathrm{Homeo}_0(M,\mu)$. We equip these transformation groups with the topology of uniform convergence, which turns them into topological groups. The subgroup $\mathrm{Homeo}_0(M,\mu)$ is easily seen to be closed in $\mathrm{Homeo}_0(M)$. Note that according to Fathi's theorem (first theorem in [Fat80]), $\mathrm{Homeo}_0(M,\mu)$ coincides with the identity component in the group of measure preserving homeomorphisms. The aim of the present note is to prove the following.

Theorem. *The group* $\mathrm{Homeo}_0(M,\mu)$ *is maximal among closed subgroups of the group* $\mathrm{Homeo}_0(M)$*.*

In what follows we consider some element $f \in \mathrm{Homeo}_0(M)$ that does not preserves the measure μ, and we denote by G_f the subgroup of $\mathrm{Homeo}_0(M)$ generated by

$$\{f\} \cup \mathrm{Homeo}_0(M,\mu).$$

Our aim is to show that the group G_f is dense in $\mathrm{Homeo}_0(M)$.

2 Localization

In this section we show how to find some element in G_f that has small support and contracts the volume of some given ball.

Good balls A *ball* is any subset of M which is homeomorphic to a euclidean ball in $\mathbb{R}^n$, where n is the dimension of M. We will need to consider balls which are locally flat and whose boundary has measure zero. More precisely, let us denote by $B_r(0)$ the euclidean ball with radius r and center 0 in $\mathbb{R}^n$. A ball B will be called *good* if $\mu(\partial B) = 0$ and if there exists a topological embedding (continuous one-to-one map) $\gamma : B_2(0) \to M$ such that $\gamma(B_1(0)) = B$. Note that, due to countable additivity, if $\gamma : B_1(0) \to M$ is any topological embedding, then for almost every $r \in (0,1)$ the ball $\gamma(B_r(0))$ is good.

Oxtoby-Ulam theorem We will need the following consequence of the Oxtoby-Ulam theorem. Let B_1, B_2 be two good balls in the interior of some manifold M', with or without boundary (what we have in mind is either $M' = M$ or M' is a euclidean ball). Let μ' be a good probability measure on M' which assigns measure zero to the boundary $\partial M'$. Denote by $\mathrm{Homeo}_0(M', \mu')$ the identity component of the group of homeomorphisms of M' which are supported in the interior of M' and preserve μ'. Assume $\mu'(B_1) = \mu'(B_2)$. Then *there exists* $\phi \in \mathrm{Homeo}_0(M', \mu')$ *such that* $\phi(B_1) = B_2$. To construct ϕ, we first choose a good ball B in the interior of M' that contains B_1, B_2 in its interior. According to the annulus theorem ([Kir69, Qui82]), we may find a homeomorphism ϕ' supported in the ball B that sends B_1 onto B_2[1]. A first use of the Oxtoby-Ulam theorem provides a homeomorphism ϕ_1 supported in B_2 and sending the measure $(\phi'_* \mu')_{|B_2}$ to the measure $\mu'_{|B_2}$. A second use of the same theorem gives a homeomorphism ϕ_2 supported in $B \setminus B_2$ and sending the measure $(\phi'_* \mu')_{|B \setminus B_2}$ to the measure $\mu'_{|B \setminus B_2}$. Then ϕ is obtained as $\phi_2 \phi_1 \phi'$. Note that, since ϕ is supported in the ball B, Alexander's trick ([Ale23]) provides an isotopy from the identity to ϕ within the homeomorphisms of B that preserves the measure μ', which shows that ϕ belongs to $\mathrm{Homeo}_0(M', \mu')$.

Triangulations We will also need triangulations which have good properties with respect to the measure μ. We begin with any triangulation $\mathcal{T}$ of M. We would like the $(n-1)$-skeleton of $\mathcal{T}$ to have measure zero, but some $(n-1)$-dimensional simplices may have positive measure. We fix this as follows. Each n-dimensional simplex s of $\mathcal{T}$ is homeomorphic to the standard n-dimensional simplex ; let μ_s be a probability measure on s which is the homeomorphic image of the Lebesgue measure on the standard simplex. The measure

$$\mu' = \frac{1}{N} \sum \mu_s$$

(where N denotes the number of n-dimensional simplices of $\mathcal{T}$) is a good probability measure on M for which the $n-1$-dimensional simplices have measure zero. We apply the Oxtoby-Ulam theorem to get a homeomorphism h of M sending μ' to μ. Then we consider the image triangulation $\mathcal{T}_0 = h_*(\mathcal{T})$, whose $(n-1)$-skeleton has measure zero. In addition to this, all the simplices of $\mathcal{T}_0$ have the same mass. Using successive barycentric subdivisions we get a sequence $(\mathcal{T}_p)_{p \geq 0}$ of nested triangulations with both properties: the $(n-1)$-skeleton have no mass and all the simplices have the same mass. Denote by m_p the common mass of the simplices of $\mathcal{T}_p$, and by d_p the supremum of the diameters of the simplices of $\mathcal{T}_p$ (for some metric which is compatible with the topology on M). Then the sequences (m_p) and (d_p) tends to zero.

Here is a useful consequence. Let O be any open subset of M. We define inductively $\mathcal{O}_p$ as the set of all the n-dimensional open simplices of $\mathcal{T}_p$ that are included in O but not in some $s \in \mathcal{O}_{p-1}$. The elements of $\mathcal{O} := \cup \mathcal{O}_p$ are pairwise

[1] One may probably avoid the use of the annulus theorem here, since the ball B may be constructed explicitly by gluing the two good balls B_1 and B_2 to a piecewise linear tube connecting them.

disjoint and their closures cover O. Since the $(n-1)$-skeleton of our triangulations have no mass, we have the equality

$$\mu(O) = \sum_{U \in \mathcal{O}} \mu(U) \quad (1).$$

We call a (closed) simplex of some $\mathcal{T}_p$ *good* if it is a good ball in M. We notice that for every $p > 0$, all the n-dimensional simplices that are disjoint from the $(n-1)$-skeleton of $\mathcal{T}_0$ are good[2]. Thus equality (1) still holds if, in the definition of the $\mathcal{O}_p$'s, we replace the simplices by the simplices whose closure is good. As a consequence, if two probability measures μ, μ' give the same mass to all the good simplices of $\mathcal{T}_p$ for every p, then they are equal.

In the first Lemma we look for elements of the group G_f that do not preserve the measure and have small support.

Lemma 2.1. *For every positive ε there exists a good ball B of measure less than ε and an element $g \in G_f$ which is supported in B and does not preserve the measure μ.*

Proof. By hypothesis the probability measures μ and $f_*\mu$ are not equal. According to the discussion preceding the Lemma, there exists some $p > 0$ and some simplex of the triangulation $\mathcal{T}_p$ whose closure B_1 is a good ball, and such that $\mu(B_1) \neq \mu(f^{-1}(B_1))$. To fix ideas let us assume that

$$\mu(f^{-1}(B_1)) > \mu(B_1).$$

This implies the same inequality for at least one of the simplices of $\mathcal{T}_{p+1}$ that are included in B_1 ; thus, by induction, we see that we may choose p to be arbitrarily large. Note that we have $\mu(f^{-1}(M \setminus B_1)) < \mu(M \setminus B_1)$. Thus the same reasoning, applied to $M \setminus B_1$, provides a (closed) simplex B_2 of some $\mathcal{T}_{p'}$, disjoint from B_1, such that

$$\mu(f^{-1}(B_2)) < \mu(B_2).$$

Again, by induction, we may assume that $p' = p$ and this is an arbitrarily large integer. In particular B_1 and B_2 are good balls with the same mass. Let B' be a ball whose interior contains B_1 and B_2. Since B_1 and B_2 have the same measure, by the above mentioned version of the Oxtoby-Ulam theorem there exists $\phi \in \mathrm{Homeo}_0(M, \mu)$ supported in B' and sending B_1 onto B_2. Now we consider the element

$$g = f^{-1} \phi f$$

of the group G_f. It has support in the ball $B = f^{-1}(B')$. It sends the ball $f^{-1}(B_1)$ to the ball $f^{-1}(B_2)$, and we have

$$\mu(f^{-1}(B_1)) > \mu(B_1) = \mu(B_2) > \mu(f^{-1}(B_2))$$

so that g does not preserve the measure μ, as required by the Lemma.

It remains to see that in the above construction we may have chosen B to be a good ball of arbitrarily small measure. Since μ has no atom, for every $\varepsilon > 0$

[2] Note that there may be simplices in $\mathcal{T}_0$ that fail to be good balls if $\mathcal{T}_0$ is a triangulation but not a PL-triangulation.

there exists some $\eta > 0$ such that every subset of M of diameter less than η has measure less than ε. Thus by choosing $p = p'$ large enough we may require that

$$\mu(f^{-1}(B_1)) + \mu(f^{-1}(B_2)) < \varepsilon.$$

Then we choose B as a ball whose interior contains $f^{-1}(B_1)$ and $f^{-1}(B_2)$ and which still has measure less than ε. Finally we shrink B a little bit to turn it into a good ball. This completes the proof of the Lemma. □

We subdivide the euclidean unit ball $B_1(0)$ of $\mathbb{R}^n$ into the half-balls $B_1^- = B_1(0) \cap \{x \leq 0\}$ and $B_1^+ = B_1(0) \cap \{x \geq 0\}$. Let Σ be the disk $B_1^- \cap B_1^+$ that separates the half-balls. We consider a given ball B and some homeomorphism g supported in B. For every homeomorphism $\gamma : B_1(0) \to B$ we let $\gamma^\pm = \gamma(B_1^\pm)$; we say that γ is *thin* if $\gamma(\Sigma)$ has measure zero. We now consider the set $\mathcal{I}(\gamma, g)$ of all the numbers of the type

$$\mu(g(\gamma^+)) - \mu(\gamma^+)$$

where γ is thin.

Lemma 2.2. *If g does not preserve the measure μ then $\mathcal{I}(\gamma, g)$ contains an interval $[a^-, a^+]$ with $a^- < 0 < a^+$.*

Proof. First we want to prove that there exists some $\gamma : B_1(0) \to B$ which is thin and such that $\mu(g(\gamma^+)) \neq \mu(\gamma^+)$. Since g does not preserve the measure μ, we may find some good ball b in the interior of B such that $\mu(b) \neq \mu(f^{-1}(b))$. To fix ideas we assume that $\mu(b) < \mu(f^{-1}(b))$. Thanks to the Oxtoby-Ulam theorem we may identify B with a euclidean ball in $\mathbb{R}^n$, b with another euclidean ball inside B, and μ with the restriction of the Lebesgue measure on $\mathbb{R}^n$. All our balls are centered at the origin. Let b' be a ball slightly greater than b, and T be a thin tube in $B \setminus b'$ connecting the boundary of B and that of b'. There exists a homeomorphism $\gamma : B_1(0) \to B$ such that $\gamma^+ = T \cup b'$. The construction may be done so that the (Lebesgue) measure of γ^+ is arbitrarily close to that of b, and then we have $\mu(\gamma^+)) < \mu(g^{-1}(\gamma^+))$, as wanted.

We can find a continuous family $(R_t)_{t\in[0,1]}$ of rotations of $B_1(0)$ such that R_0 is the identity and R_1 is a rotation that exchanges B_1^- and B_1^+. Setting $\gamma_t := \gamma \circ R_t$, we have $\gamma_1^+ = \gamma_0^- = \gamma^-$. Note that it may happen that $\gamma_t(\Sigma)$ has positive measure for some t. To remedy for this we consider $\gamma' = \phi \circ \gamma$, where $\phi : B \to B$ is a homeomorphism that fixes $\gamma(\Sigma)$, such that the image under γ' of the Lebesgue measure on $B_1(0)$ is equivalent to the restriction of μ to the ball B, in the sense that both measures share the same measure zero sets; such a ϕ is provided by the Oxtoby-Ulam theorem. This ensures that $\gamma'_t := \gamma' \circ R_t$ is thin for every t. Note that ${\gamma'_0}^\pm = \gamma_0^\pm$ and ${\gamma'_1}^\pm = \gamma_1^\pm$. We have

$$\begin{aligned}
\mu(g({\gamma'_1}^+)) - \mu({\gamma'_1}^+) &= \mu(g({\gamma'_0}^-)) - \mu({\gamma'_0}^-) \\
&= (1 - \mu(g({\gamma'_0}^+))) - (1 - \mu({\gamma'_0}^+)) \\
&= -(\mu(g({\gamma'_0}^+)) - \mu({\gamma'_0}^+)) \neq 0.
\end{aligned}$$

Thus the set $\mathcal{I}(\gamma, g)$ contains the interval

$$\{\mu(g({\gamma'_t}^+)) - \mu({\gamma'_t}^+), t \in [0, 1]\}$$

which contains both a positive and a negative number, as required by the lemma. ☐

Corollary 2.3. *Let $\gamma_0 : B_1(0) \to M$ be a topological embedding in M with $\mu(\gamma_0(\Sigma)) = 0$, let $B_0 = \gamma_0(B_1(0))$, and let $\varepsilon > 0$ be less than the measure of γ_0^+. Then there exists some element $g \in G_f$, supported in B_0, such that*

$$\mu(g(\gamma_0^+)) = \mu(\gamma_0^+) - \varepsilon.$$

In the situation of the corollary we will say that g *transfers a mass ε from γ_0^+ to γ_0^-.*

Proof. Lemma 2.1 provides some element $g' \in G_f$ that does not preserve the measure μ, and which is supported on a good ball B whose measure is less than the minimum of $\mu(\gamma_0^+) - \varepsilon$ and $\mu(\gamma_0^-)$. Then Lemma 2.2 provides some homeomorphism $\gamma : B_1(0) \to B$ which is thin and such that g' transfers some mass a from γ^+ to γ^-:

$$\mu(g'(\gamma^+)) - \mu(\gamma^+) = a.$$

Since such a number a may be chosen freely in an open interval containing 0, we may assume that $a = \frac{\varepsilon}{N}$ for some positive integer N. Choose some homeomorphism $\Phi_1 \in \mathrm{Homeo}_0(M, \mu)$ that sends B inside B_0, γ^+ inside γ_0^+ and γ^- inside γ_0^-. Such a Φ_1 is provided by Oxtoby-Ulam theorem, thanks to the fact that we have chosen the measure of B to be small enough and that $\mu(\gamma(\Sigma)) = \mu(\gamma_0(\Sigma)) = 0$. Now the conjugate $g_1 = \Phi_1 g' \Phi_1^{-1}$ transfers a mass a from γ_0^+ to γ_0^-:

$$\mu(g_1(\gamma_0^+)) \quad = \quad \mu(\gamma_0^+) - a.$$

We repeat the process with $\gamma_1 = g_1 \circ \gamma_0$ instead of γ_0, getting an element $g_2 \in G_f$ that tranfers a mass a from γ_1^+ to γ_1^-:

$$\begin{aligned}\mu(g_2 g_1(\gamma_0^+)) &= \mu(g_2(\gamma_1^+)) \\ &= \mu(\gamma_1^+) - a \\ &= \mu(g_1(\gamma_0^+)) - a \\ &= \mu(\gamma_0^+) - 2a.\end{aligned}$$

We repeat the process N times, and get the final homeomorphism g as a composition of the N homeomorphisms $g_N, \dots, g_1$. ☐

3 Proof of the theorem

We consider as before some element $f \in \mathrm{Homeo}_0(M) \setminus \mathrm{Homeo}_0(M, \mu)$. Let g be some other element in $\mathrm{Homeo}_0(M)$. In order to prove the theorem we want to approximate g with some element in the group G_f generated by f and $\mathrm{Homeo}_0(M, \mu)$. We fix a triangulation $\mathcal{T}_0$ for which the $(n-1)$-skeleton has zero measure. The first step of the proof consists in finding an element $g' \in G_f$ satisfying the following property: *for every simplex s of $\mathcal{T}_0$, the measure of $g'(s)$ coincides with the measure of $g^{-1}(s)$.* To achieve this, the (very natural) idea is to use corollary 2.3 to progressively transfer some mass from the simplices s whose mass is larger than the mass of their image under g^{-1}, to those for which the opposite holds.

Here are some details. Given a triangulation $\mathcal{T}$ for which the $(n-1)$-skeleton has zero measure, we choose two n-dimensional simplices s, s' of $\mathcal{T}$, and some positive ε less than $\mu(s)$; let us explain how to transfer a mass ε from s to s'. First assume that s and s' are adjacent. Then we may choose an embedding $\gamma : B_1(0) \to s \cup s'$ with $\gamma(\Sigma) \subset s \cap s'$, $\gamma^+ \subset s$ and $\gamma^- \subset s'$, and we apply corollary 2.3. Thus we get an element $h \in G_f$, supported in $s \cup s'$, such that $\mu(h(s)) = \mu(s) - \varepsilon$, and consequently $\mu(h(s')) = \mu(s') + \varepsilon$. Now consider the general case, when s and s' are not adjacent. Since M is connected, there exists a sequence $s_0 = s, \ldots, s_\ell = s'$ of simplices of $\mathcal{T}$ in which two successive elements are adjacent. As described before we may transfer mass ε from s_0 to s_1, then from s_1 to s_2, and so on. Thus by successive adjacent transfers of mass we get some element in $h \in G_f$ that transfers mass ε from s to s'. Note that the masses of all the other elements do not change, that is, $\mu(h(\sigma)) = \mu(\sigma)$ for every simplex σ of $\mathcal{T}$ different from s and s'.

Now we go back to our triangulation $\mathcal{T}_0$, and we construct g' the following way. If each simplex s has the same measure as its inverse image $g^{-1}(s)$ then there is nothing to do. In the opposite case there exists some simplex s of $\mathcal{T}_0$ such that $\mu(s) > \mu(g^{-1}(s))$. We also select some other simplex s' such that $\mu(s') \neq \mu(g^{-1}(s'))$, and we use the previously described construction of a homeomorphism $g_1 \in G_f$ that transfers the mass $\mu(s) - \mu(g^{-1}(s))$ from the simplex s to the simplex s'. After doing so the number of simplices $g_1(s) \in g_{1*}\mathcal{T}_0$ whose mass differs from the mass of $g^{-1}(s)$ has decreased by at least one compared to $\mathcal{T}_0$. We proceed recursively until we get an element $g' \in G_f$ such that $\mu(g'(s)) = \mu(g^{-1}(s))$ for every simplex s in $\mathcal{T}_0$, as wanted for this first step.

For the second and last step we consider the triangulations $(g^{-1})_*(\mathcal{T}_0)$ and $g'_*(\mathcal{T}_0)$. The homeomorphism $g'g$ sends the first one to the second one, and each simplex $g^{-1}(s) \in (g^{-1})_*(\mathcal{T}_0)$ has the same measure as its image $g'(s) \in g'_*(\mathcal{T}_0)$. We apply Oxtoby-Ulam theorem independently on each $g'(s)$ to get a homeomorphism $\Phi_s : g'(s) \to g'(s)$, which is the identity on $\partial g'(s)$, and which sends the measure $(g'g)_*(\mu_{|g^{-1}(s)})$ to the measure $\mu_{|g'(s)}$. The homeomorphism

$$\Phi := \left(\prod_s \Phi_s \right) g'g$$

preserves the measure μ. Furthermore by Alexander's trick each Φ_s is isotopic to the identity, thus Φ is isotopic to the identity, and belongs to the group $\mathrm{Homeo}_0(M, \mu)$. Now the homeomorphism $g'' = g'^{-1}\Phi$ belongs to the group G_f and for each simplex s of the triangulation $\mathcal{T}_0$ we have $g''^{-1}(s) = g^{-1}(s)$. We may have chosen the triangulation $\mathcal{T}_0$ so that each simplex has diameter less than some given ε. Every point x in M belongs to some n-dimensional closed simplex $g^{-1}(s)$ of the triangulation $(g^{-1})_*\mathcal{T}_0$, and since both $g(x)$ and $g''(x)$ belong to s they are a distance less than ε apart. In other words the uniform distance from g to g'' is less than ε. This proves that g belongs to the closure of G_f, and completes the proof of the theorem.

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