Critical dimension for quadratic functional quantization

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Abstract

In this paper we tackle the asymptotics of the critical dimension for quadratic functional quantization of Gaussian stochastic processes as the quantization level goes to infinity, *i.e.* the smallest dimensional truncation of an optimal quantization of the process which is "fully" quantized. We first establish a lower bound for this critical dimension based on the regular variation index of the eigenvalues of the Karhunen-Loève expansion of the process. This lower bound is consistent with the commonly shared sharp rate conjecture (and supported by extensive numerical experiments). Moreover, we show that, conversely, constructive optimized quadratic functional quantizations based on this critical dimension rate are always asymptotically optimal (strong admissibility result).

Keywords : quadratic functional quantization; Karhunen-Loève expansion; Gaussian process ; optimal quantizer ; asymptotically optimal quantizer

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1 Introduction

The aim of this paper is two-folded: first we aim at providing a *constructive proof* of the sharp rate of functional quantization in the quadratic case for (a wide class of) Gaussian processes or more generally of Gaussian random vectors X taking values in a separable Hilbert space $\big(H,(.|.)_H\big)$. Secondly, we provide several results about the *critical quantization dimension* in this framework, especially a "sharp" asymptotic lower bound for the *genuine critical dimension* (sharp with respect to a conjecture supported by extensive numerical experiments carried out on the Brownian motion and the Brownian bridge, see [11]) and the sharp asymptotics of the *"asymptotic" critical dimension.*

By constructive proof we mean that we exhibit sequences of quantizers of size n which induces an asymptotically sharp rate quadratic mean quantization error (this sharp rate has been first established in [10]) as n goes to infinity. These quantizers live in finite dimensional subspaces spanned by the (first component of the) Karhunen-Loève expansion of the Gaussian process of interest. The genuine critical dimension at level n is the lowest dimension of such a subspace which contains an optimal n-quantizer whereas the "asymptotic" critical dimension corresponds to asymptotically optimal n-quantizers. Like the mean quantization rate, the asymptotics of the critical dimension is ruled by the rate of decay of the K-L eigenvalues (listed in a decreasing order).

The L^r-mean quantization error of a random variable X defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$ and having a finite r^{th} moment taking value in a separable Hilbert space is defined by

$$e_{n,r}(X)=\inf\Big\{\big\|\min_{a\in\alpha}|X-a|_H\big\|_r,\alpha\subset H,\,|\alpha|\le n\Big\}.$$

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where $|\alpha|$ denotes the cardinality of the set α. It can also be characterized as

$$e_{n,r}(X) = \min\{\| \, |X - Y|_{_H} \|_r, \; Y : (\Omega, \mathcal{A}, \mathbb{P}) \to H, \, |Y(\Omega)| \leq n\}$$

where $|\,.\,|_{_H}$ denotes the norm on the Hilbert space H. Any random vector which achieves the above minimum (there is always at least one) is called an (L^r-)optimal n-quantization. One can show that such an optimal quantization is always of the form $Y^* = \pi_{\alpha^*}(X)$ where π is a Borel projection on $\alpha^* = Y^*(\Omega)$ following the nearest neighbour rule. The subset α is called an *optimal Voronoi n-quantizer* (or *optimal Voronoi quantizer* at level n). By extension any random vector of the form $\pi_\alpha(X)$ is called a *Voronoi quantization* whereas α is often called an n-quantizer (if $|\alpha| = n$). The term Voronoi refers to the nearest neighbour projection. A sequence $(\alpha_n)_{n\geq 1}$ of n-quantizers is called *asymptotically n-optimal* if

$$\lim_n \frac{\| \min_{a\in\alpha_n} |X - a|_H \|_r}{e_{n,r}(X)} = 1.$$

In the quadratic framework ($r = 2$), we will drop the subscript r for simplicity.

When H is an infinite dimensional separable Hilbert space (typically H is function space like $L^2([0,T], dt)$), one often speaks of *functional quantization*. The first problem of interest (beyond the existence of optimal n-quantizers for every $n \in \mathbb{N}$) is the rate of decay of the mean (quadratic) quantization error. This sharp rate problem in a Hilbert has been solved for a wide class of H-valued Gaussian random vectors/processes X in [10] (see Theorem 2.1 below). Namely when the eigenvalues of the Karhunen-Loève eigensystem of an H-valued random vector X ordered in a non-increasing order read $\lambda_n = \varphi(n)$ where φ is a non-increasing regularly varying function with index $-b$, $b \geq 1$. Thus we know that for the Brownian motion (or any Gaussian processwhose parameter b is equal to 2),

$$\lim_n \sqrt{\log n}\, e_n(W) = \frac{\sqrt{\pi}}{2} T.$$

and the conjecture on the genuine critical dimension reads as follows

$$\lim_n (\log n)^{-1} d_n^W = 2.$$

Rather unexpectedly, this proof in [10] is not constructive and provides no straightforward information on the critical dimension d_n^X as n grows. In this paper we fill this two gaps (only partially as concerns the second one). In particular the adopted approach relied on a block product quantization where we let the size of the blocks go to infinity. Other proofs based on self-similarity arguments have been proposed (in an L^r-framework, see [4]). Note however in [11] a first attempt of asymptotically optimal quantization grids has been carried out using varying block sizes.

The paper is organized as follows: in Section 2 we provide some more rigorous background on Karhunen-Loève expansions of Gaussian random vectors and functional quantization. Then, we state our main results by exhibiting a a sequence of asymptotically optimal quantization grids and provide an asymptotic lower bound for the genuine critical dimension. In Section 3 and 4 we establish some upper and lower bounds respectively for the mean quantization error. Section 5 is devoted to the proofs and constructive aspects. We conclude by few numerical illustrations which support the conjecture.

Our main tools, beyond discrete optimization techniques used for the upper bounds, are the Shannon's Shannon's source coding Theorem and the connection between mean quantization error and Shannon ε-entropy (or rate-distortion function, see [3]).

Notations. • $|\,.\,|$ canonical Euclidean norm on $\mathbb{R}^d$.
• $\mathbb{N}^* = \{1, 2, 3, \dots\}$ the set of positive integers.

• $L^2_T = L^2([0,T],dt)$ equipped with its Hilbert norm $|f|_{L^2_T} = \Big(\int_0^T f^2(t)dt\Big)^{\frac{1}{2}}$.
• Let $X:(\Omega,\mathcal{A})\to H$.
• Let (a_n) and (b_n) be two sequences of real numbers. $a_n \sim b_n$ if there exists a sequence (u_n) such that $a_n = u_n b_n$ and $\lim_n u_n = 1$.
• $o(1)$ denotes a sequence indexed by $n\in\mathbb{N}^*$ going to 0 as $n\to\infty$ (which may vary from line to line)

2 Background on optimal functional quantization and main result

2.1 Karhunen-Loève expansion and main running assumption

Let $X:(\Omega,\mathcal{A},\mathbb{P})\to H$ be a centered Gaussian random vector taking values in a separable Hilbert space $(H,|\,.\,|_H)$ satisfying

$$\dim K_X = +\infty, \qquad \text{where } K_X \text{ is the self-reproducing space of } X.$$

A typical example is the case of a Gaussian stochastic process $X=(X_t)_{t\in[0,T]}$ with continuous paths. Clearly, for such a process $a.s.$ $t\mapsto X_t(\omega)$ lies in L^2_T so that X can be see as a random vector taking values in $\big(L^2_T,|.|_{L^2_T}\big)$.

Let $(\lambda_k^X,e_k^X)_{k\geq 1}$ be the orthonormal eigensystem of the (positive trace) covariance operator of X, also known as the Karhunen-Loève (K-L) orthonormal system of X. Since the sequence $(\lambda_n^X)_{n\geq 1}$ has only one limiting value, 0, one may assume without loss of generality that K-L eigensystem is indexed so that the sequence of eigenvalues is non-increasing. To alleviate notations we will drop the dependency of the eigenvalues in X by simply noting λ_n instead of λ_n^X.

Throughout the paper, the main results are obtained under the following assumption about the K-L eigenvalues:

$$(R)\equiv\left\{\begin{array}{l}\text{There exists } b\in[1,+\infty) \text{ and a non-increasing function } \varphi:(0,+\infty)\to(0,+\infty) \text{ with regular}\\ \text{variations at infinity of index } -b \text{ (hence going to 0 at infinity) such that } \lambda_k=\varphi(k),\ k\geq 1.\end{array}\right.$$

Then, the Karhunen-Loève decomposition of X reads

$$X=\sum_{k\geq 1}\sqrt{\lambda_k}\xi_k e_k^X$$

where $\xi_k = \dfrac{\langle X,e_k^X\rangle}{\sqrt{\lambda_k}}$, $k\geq 1$, defines an i.i.d. sequence of $\mathcal{N}(0;1)$-distributed random variables defined on $(\Omega,\mathcal{A},\mathbb{P})$. The convergence holds $a.s.$ in H.

2.2 Optimal quadratic functional quantization

The optimal quantization problem for X in $L^2_H(\mathbb{P})$ is

$$e_n(X)=\inf\Big\{\big\|\min_{a\in\alpha}|X-a|_H\big\|_2,\ \alpha\subset H,\ |\alpha|\leq n\Big\}. \tag{2.1}$$

For every integer $d\geq 1$, we define the H-orthogonal projection $X^{(d)}$ of X on $\mathrm{span}\{e_1^X,\ldots,e_d^X\}$, namely

$$X^{(d)}=\sum_{k=1}^{d}\sqrt{\lambda_k}\,\xi_k\,e_k^X$$

and

$$
\begin{aligned}
e_n(X^{(d)}) &= \inf\Big\{\big\|\min_{a\in\alpha}|X^{(d)}-a|_H\big\|_2,\ \alpha\subset\oplus_{1\le k\le d}\mathbb{R}e_k^X,\ |\alpha|\le n\Big\} \\
&= \inf\Big\{\big\|\min_{a\in\alpha}|\big(\sqrt{\lambda_k}Z_k)_{1\le k\le d}-a|\big\|_2,\ \alpha\subset\mathbb{R}^d,\ |\alpha|\le n\Big\}
\end{aligned}
\tag{2.2}
$$

where $Z=(Z_1,\dots,Z_d)\stackrel{d}{=}\mathcal{N}(0;I_d)$.

Finally we set,

$$e_n^2(X,d)=e_n^2(X^{(d)})+\sum_{k\ge d+1}\lambda_k$$

and

$$C(d)=\sup_{k\ge1}k^2e_k^2\Big(\mathcal{N}\big(0;I_d\big)\Big).$$

We know from [9] (see Proposition 2.1) that, for every $n\in\mathbb{N}^*$, the infimum in (2.1) holds as a minimum: there exists at least one optimal quantizer $\alpha^{*,n}$ which turns out to have full size n. Furthermore $\alpha^{*,n}$ lies in a finite dimensional space spanned by finitely many elements of the K-L basis. We can define the *(genuine) critical dimension* as the smallest vector subspace of $\mathrm{span}\{e_n^X,\ n\ge1\}$ in which some optimal n-quantizer lies, namely:

$$d_n=\min\big\{d\in\mathbb{N}^*\ :\ \exists\,\alpha^{*,n}\text{ optimal }n\text{-quantizer s.t. }\alpha^{*,n}\subset\mathrm{span}\{e_k^X,\ 1\le k\le d\}\big\}\,.$$

The sequence $(d_n)_{n\ge1}$ makes up a sequence satisfying

$$e_n^2(X)=e_n^2(X,d_n).$$

It is clear that d_n goes to infinity, otherwise one could extract a subsequence $d_{n'}$ such that $d_{n'}\le\widetilde{d}<+\infty$. If so, we would have

$$e_n^2(X)\ge\sum_{k\ge\widetilde{d}+1}\lambda_k$$

which contradicts the obvious fact that $e_n^2(X)$ goes to zero as n goes to infinity (see *e.g.* [9]). This last claim is a consequence of the fact that, if $(z_n)_{n\ge1}$ is everywhere dense in H, then

$$e_n^2(X)\le\mathbb{E}\Big(\min_{1\le i\le n}|X-z_i|_H^2\Big)\to0\text{ as }n\to\infty.$$

Otherwise very little is known on the sequence $(d_n)_{n\ge1}$, in particular we do not know whether this sequence is monotone.

We will use the following easy lemma

Lemma 2.1. *Let $n\ge1$ be an integer. The sequence $d\mapsto e_n^2(X,d)$ is non-increasing (and so is constant for $d\ge d_n$).*

Proof. Let $d\le d'$. Let $\alpha^{*,n}(d)$ be an optimal quadratic quantizer for $X^{(d)}$ of size (at most) n. It is clear that for every $a\in\alpha^{*,n}(d)$,

$$|X^{(d')}-a|_H^2=|X^{(d)}-a|_H^2+\sum_{k=d+1}^{d'}\lambda_k$$

since $\alpha^{*,n}(d) \subset \mathrm{span}\{e_1^X, \ldots, e_d^X\}$. As a consequence $e_n^2(X^{(d')}) \leq e_n^2(X^{(d)}) + \sum_{k=d+1}^{d'} \lambda_k$ and one concludes by adding the tail $\sum_{k \geq d'+1} \lambda_k$. $\square$

It holds as a conjecture for a long time that, under Assumption (R),

$$\lim_n \frac{d_n}{\log n} = \frac{2}{b}$$

whereas the sharp rate of quadratic quantization has been elucidated for long in [10] (with several extension to more general Banach settings obtained ever since for $L^p([0,T], dt)$-norms, $1 \leq p \leq +\infty$, see [4], etc).

Extensive computations carried out in [11] provide strong evidence that in fact, as concerns the standard Brownian motion and the Brownian bridge (which corresponds to $b = 2$), we even have that

$$d_n \in \{\lfloor \log n \rfloor,\ \lceil \log n \rceil\}.$$

These conjecture are also supported by results obtained for optimal block quantization results either with constant size blocks or varying size blocks (see [10, 11]).

The aims of this paper can now be summed up as follows : firstly to provide a *constructive* proof of the sharp rate theorem recalled below, secondly to provide a partial answer to the above conjecture and finally to provide a complete answer to the "asymptotic" dimension problem.

Theorem 2.1. *(see [9]) Assume (R). Let $\psi : (0, +\infty) \to (0, +\infty)$ be defined by*

$$\psi(x) = \begin{cases} \dfrac{1}{x\varphi(x)} & \text{if } b > 1 \text{ so that } \psi \text{ is a regularly varying function with index } b-1 \\ \dfrac{1}{\int_x^\infty \varphi(y)dy} & \text{if } b = 1 \text{ so that } \psi \text{ is a slowly varying function.} \end{cases} \tag{2.3}$$

Then $$\lim_n \psi(\log n) e_n^2(X) = \begin{cases} \left(\dfrac{b}{2}\right)^{b-1} \dfrac{b}{b-1} & \text{if } b > 1 \\ 1 & \text{if } b = 1. \end{cases}$$

2.3 Main result: critical dimension

Now we pass to results on the critical dimension, with an emphasis on the constructive aspects. While the genuine critical dimension is of considerable theoretical interest, for practical purposes the "asymptotic" critical dimension is more interesting. It corresponds to the smallest admissible dimension according to the definition below.

Definition 2.1. *Assume (R). Let ψ be defined as in Theorem 2.1. Let $(\delta_n)_{n \geq 1}$ be a sequence of positive integers going to infinity.*

(a) A sequence $(\delta_n)_{n \geq 1}$ is admissible for X if

$$\lim_n \frac{e_n^2(X, \delta_n)}{e_n^2(X)} = 1.$$

(b) If $b > 1$, an admissible sequence $(\delta_n)_{n \geq 1}$ is strongly admissible for X if

$$\lim_n \psi(\delta_n) e_n^2(X^{(\delta_n)}) = 1.$$

Admissibility suggests that such a sequence of "pseudo-critical dimensions" can be used to produce asymptotic optimal quantizers in practice.

Strong admissibility models the fact that, asymptotically no dimension, is useless in $X^{(\delta_n)}$ so that the dimension δ_n is (asymptotically) minimal in view of practical computations of optimal quadratic quantizers of a Gaussian process. For more insight on these numerical aspects, we refer to [11] and [12].

The following theorem provides a complete solution of the dimension problem in the asymptotic sense, at least for $b > 1$.

Theorem 2.2. *Assume* (R)*. Let* ψ *be defined as in Theorem 2.1.*

(a) *If* $b > 1$,

$$(\delta_n)_{n\geq 1} \text{ is admissible} \quad\Longleftrightarrow\quad \liminf_n \frac{\delta_n}{\log n} \geq \frac{2}{b}$$

and

$$(\delta_n)_{n\geq 1} \text{ is strongly admissible} \quad\Longleftrightarrow\quad \lim_n \frac{\delta_n}{\log n} = \frac{2}{b}.$$

(b) *If* $b = 1$,

$$(\delta_n)_{n\geq 1} \text{ is admissible} \quad\Longrightarrow\quad \liminf_n \frac{\psi(\delta_n)}{\psi(\log n)} \geq 1$$

and

$$\liminf_n \frac{\delta_n}{\log n} > 0 \quad\Longrightarrow\quad (\delta_n)_{n\geq 1} \text{ is admissible.}$$

As for the genuine critical dimension, we thus obtain the following lower bounds.

Corollary 2.1. *Assume* (R)*.* (a) *If* $b > 1$,

$$\liminf_n \frac{d_n}{\log n} \geq \frac{2}{b}.$$

(b) *If* $b = 1$,

$$\liminf_n \frac{\psi(d_n)}{\psi(\log n)} \geq 1.$$

where this time $\psi(x) = \dfrac{1}{\int_x^\infty \varphi(y)dy}$, $x > 0$, *is a slowly varying function.*

Furthermore, it follows from Theorem 2.2 that the above conjecture is true if and only if

$$\lim_{n\to+\infty} \psi(d_n) e_n^2(X^{(d_n)}) = 1.$$

3 Upper bound

Since we are trying to provide a fair new *constructive* proof of the sharp rate for quadratic functional quantization, we are not yet in position at this stage to claim that $\lim_n \psi(\log n) e_n^2(X)$ does exist. This is the reason why the claims in the proposition below involve $\limsup_n \psi(\log n) e_n^2(X)$ which always exists (the same will be true with Proposition 4.1 in the next section).

Proposition 3.1. *Assume* (R). *Let* $(\delta_n)_{n\geq 1}$ *be a sequence of integers going to infinity.*
(a) *If* $b > 1$ *and* $\liminf_n \dfrac{\delta_n}{\log n} \geq \dfrac{2}{b}$, *then*

$$\limsup_n \psi(\log n)\, e_n^2(X) \leq \limsup_n \psi(\log n)\, e_n^2(X, \delta_n) \leq \left(\frac{b}{2}\right)^{b-1} \frac{b}{b-1}.$$

Furthermore, if $\lim_n \dfrac{\delta_n}{\log n} = \dfrac{2}{b}$, *then*

$$\limsup_n \psi(\log n)\, e_n^2(X^{(\delta_n)}) \leq \left(\frac{b}{2}\right)^{b-1}.$$

(b) *If* $b = 1$ *and* $\liminf_n \frac{\delta_n}{\log n} = \kappa \in (0, +\infty)$, *then*

$$\limsup_n \psi(\log n) e_n^2(X) \leq \limsup_n \psi(\log n)\, e_n^2(X, \delta_n) \leq 1.$$

First we need two lemmas devoted two block quantization and their critical dimension which are the key of the proof. For every integer $d \geq 1$, we define set $\lambda_k^{(d)} = \lambda_{(k-1)d+1}$.

Lemma 3.1 (Block quantization)**.** *Let* $d, d_0 \in \mathbb{N}^*$, $d > d_0$. *Then, for every* $k \in \mathbb{N}^*$, $k \leq \frac{d}{d_0}$, *we have*

$$e_n^2(X^{(d)}) \leq C(d_0) \min\left\{\sum_{\ell=1}^{k} \lambda_\ell^{(d_0)} n_\ell^{-\frac{2}{d_0}},\, n_1, \ldots, n_k \in \mathbb{N}^*,\, \prod_{\ell=1}^{k} n_\ell \leq n\right\} + \sum_{i=kd_0+1}^{d} \lambda_i$$

Proof. We introduce the (sub-optimal) d_0-block product quantizer defined as follows

$$\widetilde{X}^{(d,d_0,k)} = \sum_{\ell=1}^{k} \sum_{i=1}^{d_0} \sqrt{\lambda_{(\ell-1)d_0+i}} \left(\mathrm{Proj}_{\alpha^{(\ell)}}\left((\xi_j)_{(\ell-1)d_0+1 \leq j \leq \ell d_0}\right)\right)_i e^X_{(\ell-1)d_0+i}$$

where $\alpha^{(\ell)} \subset \mathbb{R}^{d_0}$ is an optimal quadratic quantizer of size (at level) n_ℓ of $\mathcal{N}\big(0; I_{d_0}\big)$ and $\mathrm{Proj}_{\alpha^{(\ell)}} : \mathbb{R}^{d_0} \to \alpha^{(\ell)}$ a Borel nearest neighbour projection on $\alpha^{(\ell)}$.

Elementary computations based on Pythagorus's theorem (see Lemma 4.2) in [10]) show that

$$\begin{aligned}
\left\|X - \widetilde{X}^{(d)}\right\|_2 &= \sum_{\ell=1}^{k} \sum_{i=1}^{d_0} \lambda_{(\ell-1)d_0+i} \mathbb{E}\left|\left(\mathrm{Proj}_{\alpha^{(\ell)}}\left((\xi_j)_{(\ell-1)d_0+1 \leq j \leq \ell d_0}\right)\right)_i - \xi_{(\ell-1)d_0+i}\right|^2 + \sum_{i=kd_0+1}^{d} \lambda_i, \\
&\leq \sum_{\ell=1}^{k} \lambda_{(\ell-1)d_0+1} \mathbb{E}\left|\left(\mathrm{Proj}_{\alpha^{(\ell)}}\left((\xi_j)_{(\ell-1)d_0+1 \leq j \leq \ell d_0}\right)\right)_i - \xi_{(\ell-1)d_0+i}\right|^2 + \sum_{i=kd_0+1}^{d} \lambda_i, \\
&\leq \sum_{\ell=1}^{k} \lambda_{(\ell-1)d_0+1} e_{n_\ell}^2\Big(\mathcal{N}\big(0; I_{d_0}\big)\Big) + \sum_{i=kd_0+1}^{d} \lambda_i.
\end{aligned}$$

The definition of $C(d_0)$ completes the proof. $\square$

This optimal integer bit allocation has a formal almost optimal solution given by

$$n_\ell = \lfloor x_\ell \rfloor \quad \text{with} \quad x_\ell = \left(\lambda_\ell^{(d_0)}\right)^{\frac{d_0}{2}} \left(\prod_{j=1}^{k} \lambda_j^{(d_0)}\right)^{-\frac{d_0}{2k}} n^{\frac{1}{k}},\ \ell = 1, \ldots, k,$$

as suggested by considering the problem on $(\mathbb{R}_+)^k$ instead of $(\mathbb{N}^*)^k$. This solution is admissible as soon as all the n_ℓs are nonzero or equivalently since they are non-increasing in ℓ as soon as $n_k \geq 1$.

Lemma 3.2 (Critical dimension for block quantization)**.** *Let d, d_0 and k be like in Lemma 3.1.*

$$A(n,d_0)=\left\{k\geq 1:\left(\lambda_k^{(d_0)}\right)^{\frac{d_0}{2}}\left(\prod_{j=1}^{k}\lambda_j^{(d_0)}\right)^{-\frac{d_0}{2k}}n^{\frac{1}{k}}\geq 1\right\}.$$

(a) $A(n,d_0)=\{1,\ldots,k_n(d_0)\}$.

(b) Assume (R) Then $k_n(d_0)\sim\frac{2}{bd_0}\log n$.

(c) For every integer $k\leq\min\left(k_n(d_0),\frac{d}{d_0}\right)$,

$$e_n^2(X^{(d)})\leq 4^{\frac{1}{d_0}}C(d_0)k\lambda_k^{(d_0)}+\sum_{i=kd_0+1}^{d}\lambda_i.$$

Proof. (a) This follows from the fact that the sequence

$$a_k=a_k^{(d_0)}:=\frac{d_0}{2}\left(\sum_{\ell=1}^{k}\log\lambda_\ell^{(d_0)}-k\log\lambda_k^{(d_0)}\right)\tag{3.4}$$

is non-decreasing since $A(n,d_0)=\{k\,:\,a_k\leq\log n\}$.
(b) First note that $\lambda_j^{(d_0)}=\varphi((j-1)d_0+1)$ and that $\varphi((.-1)d_0+1)$ is regularly varying still with index b if φ is. As a consequence standard arguments on regularly varying functions show that

$$\frac{2a_k}{d_0}\sim bk\ \text{ or equivalently }\ a_k\sim\frac{bd_0k}{2}\quad\text{ as }\ k\to\infty$$

which in turn implies that $k_n(d_0)\sim\frac{2}{bd_0}\log n$ as $n\to\infty$.
(c) It is straightforward that

$$\begin{aligned}\sum_{\ell=1}^{k}\lambda^{(d_0)}n_\ell^{-\frac{2}{d_0}} &\leq 2^{\frac{2}{d_0}}\sum_{\ell=1}^{k}\lambda_\ell^{(d_0)}(n_\ell+1)^{-\frac{2}{d_0}}\\ &\leq 4^{\frac{1}{d_0}}\sum_{\ell=1}^{k}\lambda_\ell^{(d_0)}x_\ell^{-\frac{2}{d_0}}\\ &\leq 4^{\frac{1}{d_0}}k\lambda_k^{(d_0)}x_k^{-\frac{2}{d_0}}\\ &\leq 4^{\frac{1}{d_0}}k\lambda_k^{(d_0)}\end{aligned}$$

since $\lambda_\ell^{(d_0)}x_\ell^{-\frac{2}{d_0}}=\left(\prod_{j=1}^{k}\lambda_j^{(d_0)}\right)^{-\frac{d_0}{2k}}n^{\frac{1}{k}}$ does not depend on ℓ and $x_k\geq 1$. $\square$

Proof of Proposition 3.1. (a) First assume that $\lim_n\frac{\delta_n}{\log n}=\frac{2}{b}$. Let $d_0\in\mathbb{N}^*$ be a (temporarily) fixed integer. Set $k_n=k_n(d_0)\wedge\left\lfloor\frac{\delta_n}{d_0}\right\rfloor$ for n large enough to have $\delta_n\geq d_0$. It follows from Lemma 3.2(b)

that $k_n \sim \dfrac{2}{b}\dfrac{\log n}{d_0}$ and $k_n \le k_n(d_0)$ and $k_n \le \frac{\delta_n}{d_0}$ so that by Lemma 3.2(c) we get as soon as $n \ge n_{d_0}$,

$$\begin{aligned} e_n^2(X^{(\delta_n)}) &\le 4^{\frac{1}{d_0}} C(d_0) k_n \lambda_{k_n}^{(d_0)} + \sum_{i=k_n d_0+1}^{\delta_n} \lambda_i \\ &\le 4^{\frac{1}{d_0}} C(d_0) k_n \lambda_{k_n}^{(d_0)} + (\delta_n - d_0 k_n)\lambda_{k_n+1}^{(d_0)}. \end{aligned} \tag{3.5}$$

Now, mimicking arguments in [10] involving regularly varying functions, namely φ, we get

$$d_0 k_n \lambda_{k_n}^{(d_0)} = d_0 k_n \varphi(k_n d_0 + 1) \sim \frac{2}{b}\log n \left(\frac{2}{b}\right)^{-b} \varphi(\log n) = \left(\frac{2}{b}\right)^{1-b} \frac{1}{\psi(\log n)} \quad \text{as } n \to \infty.$$

Moreover

$$(\delta_n - d_0 k_n)\lambda_{k_n+1}^{(d_0)} = \Big(\frac{\delta_n}{k_n d_0} - 1\Big) k_n d_0 \lambda_{k_n+1}^{(d_0)} = o\Big(\frac{1}{\psi(\log n)}\Big)$$

since $\delta_n \sim k_n d_0 \sim \frac{2}{b}\log n$.

Consequently, by letting d_0 go to infinity, we get

$$\limsup_n \psi(\log n) e_n^2(X^{(\delta_n)}) \le \left(\frac{2}{b}\right)^{1-b} \limsup_{d_0} \frac{C(d_0)}{d_0}.$$

One concludes by using (see [10]) that owing to the converse Shannon theorem

$$\lim_d \frac{C(d)}{d} = 1.$$

On the other hand

$$\sum_{i \ge k_n d_0 + 1} \lambda_i^{(d_0)} \sim \frac{k_n d_0 \varphi(k_n d_0)}{b-1} \sim \frac{1}{(b-1)\psi\big(\frac{2}{b}\log n\big)} \sim \left(\frac{2}{b}\right)^{1-b} \frac{1}{(b-1)\psi\big(\log n\big)}.$$

which yields the announced result by sub-additivity of $\limsup_n$.

If $\liminf_n \dfrac{\delta_n}{\log n} \ge \dfrac{2}{b}$, then set $\delta'_n = \delta_n \wedge \left\lfloor \frac{2\log n}{b} \right\rfloor$, $n \ge 1$. Then $\lim_n \dfrac{\delta'_n}{\log n} = \dfrac{2}{b}$ whereas by Lemma 3.1 $e_n^2(X, \delta_n) \le e_n^2(X, \delta'_n)$ which implies

$$\limsup_n \psi(\log n) e_n^2(X, \delta_n) \le \limsup_n \psi(\log n) e_n^2(X, \delta'_n) \le \left(\frac{2}{b}\right)^{1-b} \frac{b}{b-1}.$$

(b) Assume first that $\lim_n \frac{\delta_n}{\log n} = \kappa \in (0, +\infty)$. Owing to Lemma 3.1, we can assume as above that k_n defined like in (a) satisfies $k_n \sim \kappa' \log n$ where $\kappa' = \kappa \wedge \big(\frac{2}{d_0}\big)$. First we derive that

$$\sum_{i \ge \delta_n + 1} \lambda_i^{(d_0)} \sim \frac{1}{\psi(\delta_n)} \sim \frac{1}{\psi(\kappa \log n)} \sim \frac{1}{\psi(\log n)}$$

since ψ has slow variation defined in Theorem 2.1 satisfying $x\varphi(x) = o(1/\psi(x))$ (see [2], Proposition 1.5.9 b). On the other hand,

$$d_0 k_n \lambda_{k_n}^{(d_0)} = d_0 k_n \varphi(d_0 k_n + 1) = o\Big(\frac{1}{\psi(d_0 k_n + 1)}\Big) = o\Big(\frac{1}{\psi(\log n)}\Big)$$

since ψ is slowly varying, and

$$\sum_{i=d_0k_n+1}^{\delta_n} \lambda_i \le \Big(\frac{\delta_n}{d_0k_n} - 1\Big) d_0 k_n \lambda_{k_n}^{(d_0)} = o\Big(\frac{1}{\psi(\log n)}\Big)$$

since $\frac{\delta_n}{d_0k_n} - 1$ has a finite limit $\frac{\kappa}{d_0\kappa'} - 1$. As a consequence

$$\limsup_n \psi(\log n) e_n^2(X, \delta_n) \le 1.$$

The extension to the general case $\liminf_n \frac{\delta_n}{\log n} = \kappa \in (0, +\infty)$ is straightforward up to the extraction of a subsequence. $\square$

Remark. Note that when $b = 1$, we do not need to let d_0 go to infinity. Since this rate is optimal (in view of Theorem 2.1) *i.e.*, this means in particular that *scalar product* quantization (*i.e.* block quantization with blocks of size $d_0 = 1$) is asymptotically optimal.

4 Lower bound

We will rely on the famous notion in Information Theory, the Shannon ε-entropy (or rate-distortion function) of P (see [13]). Let P be a probability measure on H. For $\varepsilon > 0$, it is defined by

$$\begin{aligned} R_P(\varepsilon) \;\; = \;\; \inf \Big\{ & \mathcal{H}(Q|P \otimes Q_2) : Q \text{ probability measure on } \; H \times H \\ & \text{with first marginal } Q_1 = P \text{ and } \int_{H\times H} \|x-y\|^2 dQ(x,y) \le \varepsilon^2 \Big\}, \end{aligned}$$

where $\mathcal{H}(Q|P \otimes Q_2)$ classically denotes the relative entropy (mutual information)

$$\mathcal{H}(Q|P\otimes Q_2) = \begin{cases} \displaystyle\int_H \log\Big(\frac{dQ}{dP\otimes Q_2}\Big) dQ & \text{if } Q \text{ is absolutely continuous with respect to } P \otimes Q_2 \\ \\ +\infty & \text{otherwise.} \end{cases}$$

The simple converse part of the source coding theorem (see [1] Theorem 3.2.2; [5], p.163) says that the minimal number $N(\varepsilon)$ of codewords needed in a codebook α such that $\mathbb{E} \min_{a\in\alpha} \|X - a\|^2 \le \varepsilon^2$ satisfies $\log N(\varepsilon) \ge R(\varepsilon)$ so that, in particular

$$R(e_n(X)) \le \log n.$$

We rely here on the closed form for Shannon's entropy of Gaussian vectors known as Kolmogorov-Ihara's formula (see [8, 6]) that we will apply to $P = \mathcal{L}(X^{(d)})$ and $H = \bigoplus_{k=1}^{d} \mathbb{R} e_k^X$ (or equivalently to the d-dimensional normal distribution $P = \mathcal{N}\Big(0; \mathrm{Diag}(\lambda_1, \ldots, \lambda_d)\Big)$ on the canonical space $H = \mathbb{R}^d$). Of course, the eigenvalues are still supposed to be ordered in a non-increasing way.

Theorem 4.1 (Kolmogorov-Ihara, see [8, 6])**.** *Let $d \ge 1$ and let $P = \mathcal{N}\Big(0; \mathrm{Diag}(\lambda_1, \ldots, \lambda_d)\Big)$ where $\lambda_1 \ge \cdots \ge \lambda_d$. For every $\varepsilon > 0$ such that $\varepsilon^2 \in (0, \lambda_1 + \cdots + \lambda_d)$,*

$$R(\varepsilon) = \frac{1}{2} \sum_{k=1}^{r(\varepsilon)} \log\Big(\frac{\lambda_k}{\theta(\varepsilon)}\Big) = \log\left(\prod_{k=1}^{r(\varepsilon)} \frac{\lambda_k}{\theta(\varepsilon)}\right)^{\frac{1}{2}}$$

where $r(\varepsilon) = \max\{k \in \{1,\ldots,d\} : \bar{\lambda}_k^d > \varepsilon^2\}$, *with* $\bar{\lambda}_k^d = k\lambda_k + \lambda_{k+1} + \cdots + \lambda_d$, $k = 1,\ldots,d$ *and* $\bar{\lambda}_k^d = 0$, $k \geq d+1$, *and* $\theta(\varepsilon)$ *is the unique solution to the equation*

$$\varepsilon^2 = r(\varepsilon)\theta(\varepsilon) + \sum_{k=r(\varepsilon)+1}^{d} \lambda_k.$$

Note that the definition of $r(\varepsilon)$ is consistent since $(\bar{\lambda}_k^d)_{1\leq k\leq d}$ is non-increasing; furthermore by construction $\theta(\varepsilon) \in [\lambda_{r(\varepsilon)+1}, \lambda_{r(\varepsilon)})$.

By the definition of optimal quantization *at level* n, we have, as recalled above (see also [10]),

$$\forall\, n \geq 1, \qquad R\big(e_n(X^{(d)})\big) \leq \log n.$$

Lemma 4.1. *Let* d, $n \in \mathbb{N}^*$. *Then* $e_n^2(X^{(d)}) \geq \min\left(n^{-\frac{2}{d}}\, d \left(\prod_{k=1}^{d} \lambda_k\right)^{\frac{1}{d}}, d\lambda_d\right)$.

Proof. If $e_n^2(X^{(d)}) =: \varepsilon^2 < \bar{\lambda}_d^d$, then $r(\varepsilon) = d$ and $\theta(\varepsilon) = \frac{\varepsilon^2}{d}$ so that

$$R(\varepsilon) = \log\left(\prod_{k=1}^{d} \lambda_k\right)^{\frac{1}{2}} - \frac{d}{2}\log\left(\frac{\varepsilon^2}{d}\right) \leq \log n$$

iff

$$e_n^2(X^{(d)}) = \varepsilon^2 \geq n^{-\frac{2}{d}}\, d\left(\prod_{k=1}^{d} \lambda_k\right)^{\frac{1}{d}}.$$

Proposition 4.1 (Lower bound)**.** *Assume* (R). *Let* δ_n *be a sequence of dimensions going to infinity.*
(a) If $b > 1$ *and* $\kappa = \limsup_n \frac{\delta_n}{\log n} \in [0, +\infty]$ *then, with standard conventions,*

$$\liminf_n \psi(\log n) e_n^2(X, \delta_n) \geq \kappa^{1-b}\left(\frac{1}{b-1} + e^{-2(\frac{1}{\kappa} - \frac{b}{2})_+}\right).$$

Furthermore, if $\limsup_n \psi(\log n) e_n^2(X, \delta_n) = \limsup_n \psi(\log n) e_n^2(X)$, *then*

$$\liminf_n \frac{\delta_n}{\log n} \geq \frac{2}{b}.$$

(b) If $b = 1$, *then*

$$\liminf_n \psi(\log n) e_n^2(X) \geq \liminf_n \frac{\psi(\log n)}{\psi(\delta_n)}.$$

Furthermore, if if $\limsup_n \psi(\log n) e_n^2(X, \delta_n) = \limsup_n \psi(\log n) e_n^2(X)$, *then*

$$\liminf_n \frac{\psi(\delta_n)}{\psi(\log n)} \geq 1.$$

Proof. (a) Having in mind that $\psi(x) = 1/(x\varphi(x))$, it follows from Lemma 4.1 that

$$\begin{aligned}\psi(\log n)e_n^2(X^{(\delta_n)}) &\geq \psi(\log n)\min\left(n^{-\frac{2}{\delta_n}}\delta_n\left(\prod_{k=1}^{\delta_n}\lambda_k\right)^{\frac{1}{\delta_n}}, \frac{1}{\psi(\delta_n)}\right)\\ &= \frac{\psi(\log n)}{\psi(\delta_n)}\min\left(e^{-2\frac{\log n}{\delta_n}}\frac{1}{\varphi(\delta_n)}e^{\frac{1}{\delta_n}\sum_{1\leq k\leq\delta_n}\log\varphi(k)}, 1\right).\end{aligned}$$

The function φ being regularly varying with index $-b$, $b > 1$, one checks (see [2])

$$\frac{1}{m}\sum_{k=1}^{m}\log\varphi(k) = b + \log\varphi(m) + o(1) \qquad \text{as } m \to \infty$$

so that,

$$\psi(\log n)e_n^2(X^{(\delta_n)}) \geq \frac{\psi(\log n)}{\psi(\delta_n)}\min\left(e^{-2\frac{\log n}{\delta_n}+b+o(1)}, 1\right)$$

which in turn implies that

$$\psi(\log n)e_n^2(X, \delta_n) \geq \frac{\psi(\log n)}{\psi(\delta_n)}\left(\min\left(e^{-2\frac{\log n}{\delta_n}+b+o(1)}, 1\right) + \frac{1+o(1)}{b-1}\right).$$

At this stage we introduce the function g_b defined on $[0, +\infty]$ (with the usual conventions) by

$$g_b(u) := \left(\min\left(e^{-2(\frac{1}{u}-\frac{b}{2})}, 1\right) + \frac{1}{b-1}\right)u^{1-b} = u^{1-b}\left(\frac{1}{b-1} + e^{-2(\frac{1}{u}-\frac{b}{2})_+}\right).$$

The function g_b is decreasing on $[0, +\infty]$ with

$$g_b\Big(\frac{2}{b}\Big) = \Big(\frac{b}{2}\Big)^{b-1}\frac{b}{b-1}.$$

Let (n') be a subsequence such that $\frac{\delta_{n'}}{\log n'} \to u \in [0, +\infty]$. Using that ψ is regularly varying with index $b-1$ we derive that

$$\liminf_n \psi(\log n')e_n^2(X, \delta_{n'}) \geq g_b(u)$$

so that finally

$$\liminf_n \psi(\log n)e_n^2(X, \delta_n) \geq \sup_{u\leq\kappa} g_b(u) = g_b(\kappa) \text{ where } \kappa = \limsup_n \frac{\delta_n}{\log n}.$$

Assume now that $\limsup_n \psi(\log n)e_n^2(X, \delta_n) = \limsup_n \psi(\log n)e_n^2(X)$. Let $c =: \liminf_n \frac{\delta_n}{\log n} \in [0, +\infty]$ and let $(\delta_{n'})_{n\geq 1}$ be a subsequence such that $\frac{\delta_{n'}}{\log n'} \to c$. Let $(\widetilde{\delta}_n)_{n\geq 1}$ be a sequence going to infinity and satisfying $\widetilde{\delta}_{n'} = \delta_{n'}$ and $\limsup_n \frac{\widetilde{\delta}_n}{\log n} = c$. Then one gets

$$\limsup_n \psi(\log n)e_n^2(X, \delta_n) \geq \liminf_n \psi(\log n')e_{n'}^2(X, \delta_{n'}) \geq \liminf_n \psi(\log n)e_n^2(X, \widetilde{\delta}_n) \geq g_b(c).$$

If $c = 0$, $g_b(0) = +\infty$ and we would have that $\psi(\log n)e_n^2(X, \delta_n) \to +\infty$ which is in contradiction with claim (a) in Proposition 3.1.

If $c \in (0, +\infty]$, the upper bound obtained in Proposition 3.1 implies $g_b(c) \le g_b\Big(\frac{2}{b}\Big)$ which in turn implies $c \ge \frac{2}{b}$.

(b) Using again standard results from [2] about regularly varying functions with index -1, we get

$$\sum_{i \ge \delta_n+1} \lambda_i \sim \frac{1}{\psi(\delta_n)} \quad \text{still with} \quad \psi(x) = \frac{1}{\int_x^{+\infty} \varphi(y)dy}.$$

Hence

$$\psi(\log n) e_n^2(X, \delta_n) \ge \psi(\log n) \sum_{i=\delta_n+1} \lambda_i \sim \frac{\psi(\log n)}{\psi(\delta_n)}.$$

Using the same trick (based on the sequence $(\widetilde{\delta}_n)_{n\ge1}$) as in the former case, we derive similarly that, if $\limsup_n \psi(\log n) e_n^2(X, \delta_n) = \limsup_n \psi(\log n) e_n^2(X)$, then $\limsup_n \frac{\psi(\log n)}{\psi(\delta_n)} \le 1$ *i.e.* the announced result. $\square$

5 Synthesis

5.1 Proof of Theorem 2.1

First we provide a proof of Theorem 2.1 based on the upper and lower bounds established in former sections and the following lemma (already established in [10] but reproduced here for the reader's convenience). Furthermore it has to be noticed that it provides an easily tractable (and asymptotically optimal) lower bound for the quadratic quantization error, keeping in mind that the sequence $(k_n(1))_{n\ge1}$ is defined in Lemma 3.2.

Lemma 5.1. *For every* $n \in \mathbb{N}^*$,

$$e_n^2(X) \ge k_n(1)\lambda_{k_n(1)+1} + \sum_{k \ge k_n(1)+1} \lambda_k$$

Proof. It follows from Kolmogorov-Ihara's formula that for every $\varepsilon^2 \in (0, \lambda_1 + \cdots + \lambda_d)$, $R(\varepsilon) > a^{(1)}_{r(\varepsilon)}$ since $\theta(\varepsilon) < \lambda_{r(\varepsilon)}$ (see Equation (3.4) for a definition of $a_k^{(1)}$). As a consequence, $a^{(1)}_{r(e_n(X^{(d)}))} \le \log n$. Consequently, it follows from Lemma 3.2(a) that $r(e_n(X^{(d)})) \le k_n(1)$ which in turn implies that, for every $d \in \mathbb{N}^*$, $\bar{\lambda}^d_{k_n(1)+1} \le e_n^2(X^{(d)})$. Noting that $e_n^2(X) \ge e_n^2(X^{(d)})$ and letting d go to infinity, we get, for every $n \in \mathbb{N}^*$,

$$e_n^2(X) \ge (k_n(1)+1)\lambda_{k_n(1)+1} + \sum_{k \ge k_n(1)+2} \lambda_k. \qquad \square$$

Proof of Theorem 2.1 Case $b > 1$. We know from Proposition 3.1 that $\limsup_n \psi(\log n) e_n^2(X) \le \left(\frac{b}{2}\right)^{b-1} \frac{b}{b-1}$.

On the other hand, combining the fact that $k_n(1) \sim \frac{2}{b}\log n$ and arguments based on regularly varying functions already used in Proposition 3.1 yield that

$$k_n(1)\lambda_{k_n(1)+1} \sim \frac{1}{\psi(k_n(1))} \sim \left(\frac{b}{2}\right)^{b-1} \frac{1}{\psi(\log n)} \text{ and } \sum_{k \ge k_n(1)+1} \lambda_k \sim \frac{1}{(b-1)\psi(k_n(1))} \sim \left(\frac{b}{2}\right)^{b-1} \frac{1}{(b-1)\psi(\log n)}.$$

so that $\liminf_n \psi(\log n) e_n^2(X) \ge \left(\frac{b}{2}\right)^{b-1} \dfrac{b}{b-1}$ which completes the proof.

Case $b = 1$. One concludes likewise since $k_n(1)\lambda_{k_n(1)+1} \sim \frac{1}{\psi(\log n)}$ and $\sum_{k \ge k_n(1)+1} \lambda_k = o(1/\psi(\log n))$. $\square$

5.2 Proof of Theorem 2.2

Proof of Theorem 2.2. (a) When $b > 1$, the direct claim on admissibility is a consequence of Proposition 4.1 The converse claim follows from Proposition 3.1(a) and Theorem 2.1.

As for strong admissibility, the direct claim is as follows: from the definition of strong admissibility, we get $e_n(X, \delta_n)^2 \sim e_n(X)^2$ (by admissibility) and

$$e_n(X, \delta_n)^2 \sim \frac{1}{\psi(\delta_n)} + \frac{1}{b-1}\frac{1}{\psi(\delta_n)} = \frac{b}{b-1}\frac{1}{\psi(\delta_n)}$$

so that $\frac{b}{b-1}\frac{1}{\psi(\delta_n)} \sim e_n(X)^2$. Then comparing with the sharp rate from Theorem **??**, we get

$$\frac{1}{\psi(\delta_n)} \sim \left(\frac{b}{2}\right)^{b-1} \frac{1}{\psi(\log n)}$$

which finally implies, having in mind that ψ is regularly varying with index $b-1$, that

$$\delta_n \sim \frac{b}{2}\log n.$$

The converse claim is a consequence of Proposition 3.1(a) and Theorem 2.1. Claim (b) follows the same lines and details are left to the reader. ☐

▷ Back to the conjecture(s) ($b > 1$). As concerns the conjecture $\lim_n \frac{d_n}{\log n} = \frac{2}{b}$ on the sharp asymptotics of the critical dimension d_n, strictly speaking, we only made half the way by proving that

$$\liminf_n \frac{d_n}{\log n} \geq \frac{2}{b}.$$

However the strong admissibility result in Theorem 2.2(a) can be seen as an answer *in the asymptotic sense* since it shows that if $\lim_n \frac{\delta_n}{\log n} = \frac{2}{b}$, then the resulting quadratic quantization error is asymptotically optimal and (asymptotically almost) all dimensions are used (strong admissibility).

This result is helpful from a numerical point of view since it shows, *e.g.* for the Brownian motion (for which $b = 2$, see below), that considering a truncation at $\delta_n = \lfloor \log n \rfloor$ or $\delta_n = \lceil \log n \rceil$ is at least asymptotically optimal whatever the future of the sharper conjecture

$$d_n \in \{\lfloor \log n \rfloor, \lceil \log n \rceil\}.$$

could be. Of course, such a choice is also asymptotically optimal for all processes whose K-L eigensystem has an varying index equal to 2 like the Brownian bridge, Ornstein-Uhlenbeck process, etc.

For other examples of families of processes satisfying Assumptions (R) (including multi-parameters processes like the Brownian sheet, we refer to [10]).

▷ Numerical experiments on the Brownian motion.

We know that the K-L eigensystem of the standard Brownian motion $W = (W_t)_{t \in [0,T]}$ over $[0, T]$ is given by

$$\lambda_k^W = \left(\frac{T}{\pi(k - \frac{1}{2})}\right)^2, \; e_n^W(t) := \sqrt{\frac{2}{T}} \sin\left(\frac{t}{\sqrt{\lambda_k}}\right), \quad k \geq 1.$$

so that $b = 2$. Then, Theorem 2.1 yields

$$\lim_n \log(n) e_n^2(W) = \frac{2T^2}{\pi^2} \approx 0.2026 \times T^2$$

Figure 1 depicts the graph of the $n \mapsto \log(n) e_n^2(W)$ (with $T = 1$) . One can see that it looks as a piecewise affine function with breaks in the slope. Note that the exponential function e^x satisfies

$$e^3 \approx 20.09 \approx 20 \qquad e^4 \approx 54.59 \approx 55 \qquad e^5 \approx 148.41 \approx 148.$$

These values graphically fit with the monotony slope breaks.

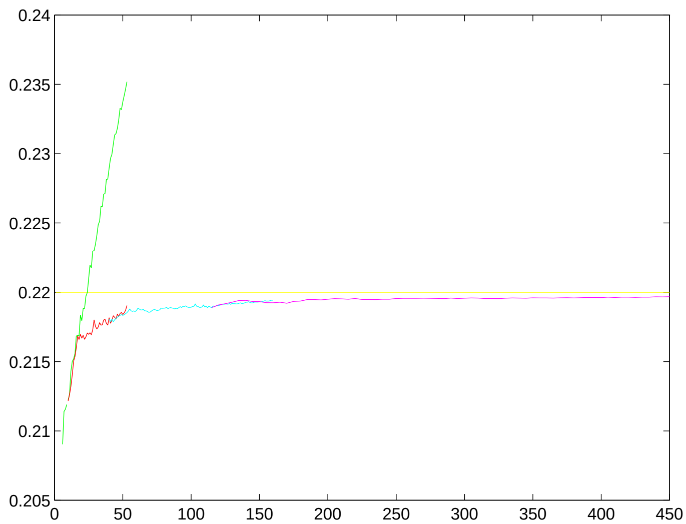

Figure 1: $n \mapsto \log(n) e_n^2(W)$; green: $d = 2$; red: $d = 3$; cyan : $d = 4$; magenta : $d = 5$.

The graph in Figure 1 suggests, at this (low) range of the computation, that the limiting value for $n \mapsto \log(n) e_n^2(W)$ is higher (≈ 0.22) than the theoretical one (≈ 0.2026). This impression is misleading since further computations not reproduced here show that the sequence $n \mapsto \log(n) e_n^2(W)$ starts to be slowly decreasing beyond $n \geq 1000$. The value 0.22 seems to be a local maximum. For further details on these (highly time consuming) computations we refer to [11].

The quantization grids computed at the occasion of these numerical experiments by stochastic optimization methods (randomized Lloyd's procedure, Competitive Learning Vector Quantization algorithm) for $N = 1$ up to 10^4 for the standard Brownian motion (when $T = 1$) can be downloaded from the website

www.quantize.maths-fi.com

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