Lifting in Sobolev spaces

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Introduction.

Let $\Omega \subset \mathbb{R}^n$ be a (smooth) bounded domain which is connected and simply connected. Given a function $u : \Omega \to S^1$ (i.e., $u : \Omega \to \mathbb{C}$ and $|u(x)| = 1$ a.e.) we may write pointwise

$$u(x) = e^{i\varphi(x)}$$

for some function $\varphi : \Omega \to \mathbb{R}$. The objective is to find a lifting φ "as regular as u permits." For example, if u is continuous one may choose φ to be continuous and if $u \in C^k$ one may also choose φ to be C^k. A more delicate result asserts that if $u \in$ VMO (= vanishing means oscillation), then one may choose φ to be also VMO (see R. Coifman and Y. Meyer [1] and H. Brezis and L. Nirenberg [1]). In this paper we study the question of lifting in the framework of the Sobolev spaces $W^{s,p}$ with $0 < s < \infty$ and $1 < p < \infty$. The motivation comes from problems of the Ginzburg-Landau type where one considers questions such as Min $\int |\nabla u|^2$ in the class of functions $u : \Omega \to S^1$ (see e.g. F. Bethuel, H. Brezis and F. Hélein [1]).

The first result in that direction is

Theorem (F. Bethuel and X. Zheng [1]). *Assume*

$$u \in W^{1,p}(\Omega; S^1) \quad \textit{with } p \geq 2,$$

then u may be written as $u = e^{i\varphi}$ for some $\varphi \in W^{1,p}(\Omega; \mathbb{R})$.

Surprisingly the restriction $p \geq 2$ is optimal in any dimension $n \geq 2$, i.e., given any $p < 2$ there is some $u \in W^{1,p}$ which cannot be lifted by a $\varphi \in W^{1,p}$ (such examples will be given later; see Section 4).

We address the same questions in all Sobolev spaces $W^{s,p}$. Here is a summary of our main results:

Theorem 1. *Assume $n = 1$, $0 < s < \infty$ and $1 < p < \infty$. Then the answer to the lifting question in $W^{s,p}$ is always positive.*

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Theorem 2. *Assume $n \geq 2$, $0 < s < 1$ and $1 < p < \infty$. The answer to the lifting question in $W^{s,p}$ is:*

a) positive if $sp < 1$,

b) negative if $1 \leq sp < n$,

c) positive if $sp \geq n$.

Theorem 3. *Assume $n \geq 2$, $1 \leq s < \infty$ and $1 < p < \infty$. The answer to the lifting question in $W^{s,p}$ is:*

a) negative if $sp < 2$,

b) positive if $sp \geq 2$.

In these statements "positive" means that every $u \in W^{s,p}(\Omega; S^1)$ may be written as $u = e^{i\varphi}$ for some $\varphi \in W^{s,p}(\Omega; \mathbb{R})$ and "negative" means that for some u's in $W^{s,p}(\Omega; S^1)$ there is no $\varphi \in W^{s,p}(\Omega; \mathbb{R})$ such that $u = e^{i\varphi}$.

As a simple consequence of the theorems when $p = 2$, i.e., for $H^s = W^{s,2}$, we have

Corollary 1. *When $n = 1$ the answer to the lifting problem in H^s is always positive.*

When $n \geq 2$ the answer to the lifting problem in H^s is:

a) positive if $0 < s < 1/2$,

b) negative if $1/2 \leq s < 1$,

c) positive if $s \geq 1$.

The proof of Theorems 1 and 2 when $sp < 1$ turns out to be quite involved (even for the H^s case, $s < 1/2$, and even when $n = 1$). It relies on a characterization, due to G. Bourdaud [1] (see also the earlier paper of R. Devore and V. A. Popov [1]), of the $W^{s,p}$ space when $sp < 1$; for the convenience of the reader, and also because we make use of sharp estimates, we have presented a proof in a separate section, Appendix A.

In view of the Corollary for $n \geq 2$, a function $u \in H^{1/2}(\Omega; S^1)$ need not have a lifting $\varphi \in H^{1/2}(\Omega; \mathbb{R})$; however, it has a lifting φ in H^s, $\forall s < 1/2$. We prove (see Appendix E)

Theorem 4. *Assume Q is a cube in $\mathbb{R}^n, n \geq 1$. For every $u \in H^s(Q; S^1)$ with $0 < s < 1/2$ one may find a φ in H^s such that $u = e^{i\varphi}$ and satisfying the (optimal) estimate*

$$\|\varphi\|_{H^s} \leq C(1 - 2s)^{-1/2}\|u\|_{H^s}$$

with C independent of u and independent of s (for s near $1/2$).

Such an estimate is useful in deriving bounds for the Ginzburg-Landau functional when the boundary condition belongs to $H^{1/2}$. For example, let Q be a cube of $\mathbb{R}^n, n \geq 1$, and let $\Omega = Q \times (0, 1)$. For any function $g \in H^{1/2}(Q; \mathbb{C})$, set

$$H^1_g(\Omega = \{u(x,t) : \Omega \to \mathbb{C} \ ; \int_\Omega |\nabla u|^2 dxdt < \infty \quad \text{and } u(x,0) = g(x) \text{ on } Q\},$$

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (|u|^2 - 1)^2,$$

where ∇ denotes the full gradient (in (x,t)).

Theorem 5. *For every $g \in H^{1/2}(Q;S^1)$ we have, for $\varepsilon > 0$,*

$$E_\varepsilon = \operatorname*{Min}_{u \in H^1_g(\Omega)} E_\varepsilon(u) \le C \log(1/\varepsilon) \|g\|^2_{H^{1/2}}$$

where C is independent of ε and of g.

For variants of Theorem 5, see Remark 8 in Section 5.

The plan of the paper is the following:

1. Proof of Theorems 1 and 2 when $sp < 1$
2. Proof of Theorem 1 when $sp \ge 1$ and of Theorem 2 when $sp \ge n$
3. Proof of Theorem 3 when $sp \ge 2$
4. Examples of obstruction in Theorems 2 and 3
5. Control of lifting in the H^s-norm for $s \overrightarrow{<} \frac{1}{2}$ and application to Ginzburg-Landau

Appendix A. A characterization of $W^{s,p}(\Omega;\mathbb{R})$ when $sp < 1$

Appendix B. Functions in $W^{s,p}(\Omega;\mathbb{Z})$ are constant when $sp \ge 1$

Appendix C. Composition in fractional Sobolev spaces

Appendix D. Gagliardo-Nirenberg inequalities and products in fractional Sobolev spaces

Appendix E. Behaviour of the H^s-norms of lifting for $s \overrightarrow{<} \frac{1}{2}$. Proof of Theorem 4

Appendix F. Martingale representation and lifting in $H^{s,p}$

1. Proof of Theorems 1 and 2 when sp $<$ 1.

Here, the assumption that Ω is simply connected is not needed since we may always extend the given function by a constant outside Ω; the resulting function still belongs to $W^{s,p}$ since $sp < 1$ (this is a well-known fact, see e.g. Lions-Magenes [1], Section 1.11 when $p = 2$ and the references therein; it is also a consequence of the characterization of $W^{s,p}$ in Appendix A). Thus, we may assume that $\Omega = (0,1)^n$ and we use the same notation as in Appendix A.

Let $u \in W^{s,p}(\Omega;S^1)$. For each $j = 0,1,\dots$, consider the function $U_j \in \mathcal{E}_j$ defined by

$$U_j(x) = \begin{cases} \frac{E_j(u)(x)}{|E_j(u)(x)|} & \text{if } E_j(u)(x) \ne 0 \\ 1 & \text{if } E_j(u)(x) = 0. \end{cases}$$

Clearly $U_j \to u$ a.e. on Ω (since $E_j(u) \to u$ a.e. and $|u| = 1$ a.e.) For each $j = 0,1,\dots$ we construct a real-valued function $\varphi_j \in \mathcal{E}_j$ such that

$$e^{i\varphi_j} = U_j \quad \text{on } \Omega, \tag{1.1}$$

$$|\varphi_j - \varphi_{j-1}| \le C|U_j - U_{j-1}| \quad \text{on } \Omega. \tag{1.2}$$

Note that (1.2) can be achieved by induction on j, for example with $C = \pi/2$.

On the other hand, observe that for every $\xi, \eta, \zeta \in \mathbb{C}$ with $|\zeta| = 1$, we have

$$\left| \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right| \le 4(|\zeta - \xi| + |\zeta - \eta|) \tag{1.3}$$

with the convention that $\frac{0}{0} = 1$ (consider separately the case where $|\xi|, |\eta| \ge 1/2$ and the case where either $|\xi| < 1/2$ or $|\eta| < 1/2$).

Applying (1.3) to $\xi = E_j(u)(x)$, $\eta = E_{j-1}(u)(x)$ and $\zeta = u(x)$ we obtain a.e. on Ω

$$|U_j - U_{j-1}| \le 4(|u - E_j(u)| + |u - E_{j-1}(u)|). \tag{1.4}$$

Combining this with (1.2) yields

$$|\varphi_j - \varphi_{j-1}| \le C(|u - E_j(u)| + |u - E_{j-1}(u)|) \tag{1.5}$$

and thus

$$\sum_{j \ge 1} 2^{spj} \|\varphi_j - \varphi_{j-1}\|_{L^p}^p \le C \sum_{j \ge 0} 2^{spj} \|u - E_j(u)\|_{L^p}^p. \tag{1.6}$$

Applying Theorem A.1 and Corollary A.1 in Appendix A, we conclude that $\varphi_j \to \varphi$ in L^p with $\varphi \in W^{s,p}$, $e^{i\varphi} = u$, and

$$\|\varphi\|_{W^{s,p}} \le C\|u\|_{W^{s,p}}. \tag{1.7}$$

We may always assume (by adding to φ an integer multiple of 2π) that

$$|\int_\Omega \varphi| \le 2\pi.$$

Thus, we have constructed a function $\varphi \in W^{s,p}$ such that $e^{i\varphi} = u$ and

$$\|\varphi\|_{L^p} + \|\varphi\|_{W^{s,p}} \le C(1 + \|u\|_{W^{s,p}}). \tag{1.8}$$

Remark 1. One should observe the linear dependence while in the continuous case there is **no** bound whatsoever for $\|\varphi\|_{L^\infty}$ in terms of $\|u\|_{L^\infty}$; see also Remark 3 where we show that there is no bound for φ in $H^{1/2}$ in terms for $\|u\|_{H^{1/2}}$ in one dimension despite the fact that every $u \in H^{1/2}$ has a (unique) lifting in $H^{1/2}$.

Remark 2. The function φ constructed above also belongs to every $L^q, q < \infty$. This may be easily seen by observing that $u \in W^{s,p} \cap L^\infty \subset W^{\sigma,q}$ for every $\sigma < s$ with $\sigma q = sp$ (by the Gagliardo-Nirenberg inequality, see Appendix D). Therefore φ belongs to every such $W^{\sigma,q}$. Choosing σ close to zero we obtain a q which is arbitrarily large.

2. Proof of Theorem 1 when $sp \geq 1$ and of Theorem 2 when $sp \geq n$.

When $sp > 1$ in Theorem 1 or $sp > n$ in Theorem 2, u is continuous by the Sobolev imbedding theorem and, locally, we may consider $\varphi = -i \log u$ which is well-defined and singlevalued. To conclude, we rely on a lemma about composition:

Lemma 1. *Assume $n \geq 1$, $0 < s < \infty$ and $1 < p < \infty$. Let $v \in W^{s,p}(\Omega) \cap L^\infty(\Omega)$ and let $\Phi \in C^\infty$. Then $\Phi \circ v \in W^{s,p}(\Omega)$.*

The proof is very simple when $0 < s < 1$ (using the definition of $W^{s,p}$ and the fact that Φ is Lipschitz on the range of v). This lemma is also well-known when s is an integer, with the help of the Gagliardo-Nirenberg inequality. When $s > 1$ is not an integer the argument is more delicate; we refer to Escobedo [1] and Lemma C.1 in Appendix C.

We now turn to the proof of Theorem 1 when $s = 1/p$; the proof of Theorem 2 when $s = n/p$ is identical and we omit it. Set $I = \Omega = (0, 1)$.

By standard trace theory there is some $\tilde{u} \in W^{s+1/p,p}(I^2; \mathbb{R}^2)$ such that

$$\tilde{u}(x, 0) = u(x).$$

Since u takes its values into S^1 one may expect that, near $I \times \{0\}$, $\tilde{u}$ takes its values "close" to S^1. This is not true for a general extension $\tilde{u}$. However, **special** extensions have that property. For example

$$\tilde{u}(x, y) = \frac{1}{2y} \int_{x-y}^{x+y} u(t) dt$$

(u is extended by symmetry to the interval $(-2, +2)$) has the property that $\tilde{u} \in W^{s+1/p,p}$, and moreover, $|\tilde{u}(x, y)| \to 1$ **uniformly** in x as $y \to 0$. This is a consequence of the fact that $W^{s,p} \subset \text{VMO}$ in the limiting case of the Sobolev imbedding (see e.g. Boutet de Monvel-Berthier, Georgescu and Purice [1],[2], Brezis and Nirenberg [1]). Similarly, any harmonic extension $\tilde{u}$ of u in I^2 has also the same property (see Brezis and Nirenberg [2], Appendix 3). If we consider $v = \tilde{u}/|\tilde{u}|$ in a neighborhood ω of $I \times \{0\}$ in I^2 we have an extension v of u such that

$$v \in W^{s+1/p,p}(\omega; S^1).$$

Here, we have used again Lemma 1.

Let us now explain how to complete the proof of the theorem when $p = 2$, i.e., $u \in H^{1/2}(I; S^1)$. From the above discussion we have some extension v of u, with

$$v \in H^1(\omega; S^1).$$

Applying the theorem of Bethuel and Zheng we may write

$$v = e^{i\psi}$$

for some $\psi \in H^1(\omega; \mathbb{R})$ and then $\varphi = \psi_{|_I}$ has the required properties.

We now turn to the general case. Here, we shall use the following lemma about products in fractional Sobolev spaces. Its proof is presented in Appendix D when $\Omega = \mathbb{R}^n$ (see Lemma D.2). The case of a smooth domain Ω follows by extending the functions to $\mathbb{R}^n$.

Lemma 2. *Assume $s \geq 1$ and $1 < p < \infty$. Let*

$$f, g \in W^{s,p}(\Omega; \mathbb{R}) \cap L^\infty(\Omega; \mathbb{R})$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$. Then

$$fDg \in W^{s-1,p}(\Omega).$$

Proof of Theorem 1 completed. We recall that there is a neighborhood Q of $I \times \{0\}$ in I^2 and an extension v of u such that

$$v \in W^{s+(1/p),p}(Q; S^1).$$

Applying once more the same construction we find some

$$w \in W^{s+(2/p),p}(U; S^1)$$

where U is a neighborhood of $Q \times \{0\}$ in $Q \times I$. (This construction is possible since $(s + 1/p)p = 2$, so that we are again in a limiting case for the Sobolev imbedding and thus $v \in \mathrm{VMO}$. Iterating this construction we find some

$$\zeta \in W^{s+(k/p),p}(G; S^1)$$

where G is a domain in $\mathbb{R}^{k+1}$. Consider the first integer $k \geq 1$ such that

$$s + (k/p) \geq 1.$$

This choice of k implies that

$$s + \frac{j}{p} < 1, \quad j = 0, 1, \dots , k - 1,$$

so that, at each step, standard trace theory applies (recall that a function in $W^{s,p}$ has an extension in $W^{s+1/p,p}$ provided s is not an integer).

¿From the Gagliardo-Nirenberg inequality (see Lemma D.1) we have

$$\zeta \in W^{1,k+1}(G; S^1).$$

Applying the theorem of Bethuel and Zheng, we may write

$$\zeta = e^{i\psi} \tag{2.1}$$

for some $\psi \in W^{1,k+1}(G; \mathbb{R})$. Differentiating (2.1) we find

$$D\psi = -i\bar{\zeta} D\zeta.$$

By Lemma 2 we have

$$D\psi \in W^{s+(k/p)-1,p}(G)$$

and hence

$$\psi \in W^{s+(k/p),p}(G).$$

Taking back traces we obtain

$$\varphi = \psi_{|_I} \in W^{s,p}(I)$$

and

$$u = e^{i\varphi}.$$

Remark 3. In one dimension we have established that every $u \in H^{1/2}(\Omega; S^1)$ admits a lifting $\varphi \in H^{1/2}(\Omega; S^1)$. Moreover, this lifting is unique modulo an additive constant (see Appendix B) and the map $u \mapsto \varphi$ is continuous from $H^{1/2}$ into $H^{1/2}$ (this can be established using the same argument as in Step 7 of the proof of Theorem 4 in Brezis-Nirenberg [1]). Surprisingly there is **no bound** whatsoever for $\|\varphi\|_{H^{1/2}}$ in terms of $\|u\|_{H^{1/2}}$. Here is an example of a sequence (φ_n) such that $\|\varphi_n\|_{H^{1/2}} \to +\infty$ while $\|e^{i\varphi_n}\|_{H^{1/2}} \leq C$. On $\Omega = (-1, +1)$ consider the sequence of functions φ_n defined by

$$\varphi_n(x) = \begin{cases} 0 & \text{for } -1 < x < 0 \\ 2\pi n x & \text{for } 0 < x < 1/n \\ 2\pi & \text{for } 1/n < x < 1. \end{cases}$$

Clearly $\|\varphi_n\|_{H^{1/2}} \to +\infty$ (since $\varphi_n \to \varphi = \mathbf{1}_{(0,1)}$ in L^2 and φ does not belong to $H^{1/2}$). In fact, a more precise computation left to the reader shows that $\|\varphi_n\|_{H^{1/2}} \geq c(\log n)^{1/2}$ with $c > 0$. On the other hand the reader will easily check (for example by scaling) that $\|e^{i\varphi_n} - 1\|_{H^{1/2}}$ remains bounded. The same conclusion holds when $H^{1/2}$ is replaced by $W^{1/p,p}$ with any p, $1 < p < \infty$.

Remark 4. As we have just pointed out there is no control of φ in $H^{1/2}$ in terms of $e^{i\varphi}$ in $H^{1/2}$. There is, however, (in dimension one), an estimate for $(\varphi - \fint \varphi)$ in the space $H^{1/2} + W^{1,1}$, equipped with its usual norm, in terms of $\|e^{i\varphi}\|_{H^{1/2}}$. Here is the argument, working for simplicity with periodic functions. We may also assume (by density) that φ is smooth. Observe that the dual of $H^{1/2} + W^{1,1}$ is $H^{-1/2} \cap W^{-1,\infty}$. Given any $T \in H^{-1/2} \cap W^{-1,\infty}$, write $T = \psi' + c$ for some $\psi \in H^{1/2} \cap L^\infty$ and some constant c. Then

$$\langle T, \varphi - \fint \varphi \rangle = \langle \psi', \varphi - \fint \varphi \rangle = -\langle \psi, \varphi' \rangle.$$

But if we set $u = e^{i\varphi}$, then $\varphi' = -i\bar{u}u'$ and thus

$$|\langle T, \varphi - \fint \varphi \rangle| = |\langle \psi, i\bar{u}u' \rangle| = |\langle u', i\psi\bar{u} \rangle| \leq \|u\|_{H^{1/2}} \|\psi u\|_{H^{1/2}}.$$

Recall that $H^{1/2} \cap L^\infty$ is an algebra (see e.g. Appendix D) and that

$$\begin{aligned}\|\psi u\|_{H^{1/2}} &\leq C(\|\psi\|_{H^{1/2}} + \|\psi\|_{L^\infty})(\|u\|_{H^{1/2}} + \|u\|_{L^\infty}) \\ &\leq C\|T\|_{H^{-1/2}\cap W^{-1,\infty}}(\|u\|_{H^{1/2}} + 1).\end{aligned}$$

We conclude that

$$\|\varphi - \fint \varphi\|_{H^{1/2}+W^{1,1}} \leq C\|u\|_{H^{1/2}}(\|u\|_{H^{1/2}} + 1).$$

The same estimate holds in higher dimensions if u belongs to the closure of $C^\infty(\bar{\Omega}; S^1)$ in $H^{1/2}(\Omega; S^1)$; however, the argument is much more delicate and will be presented in our forthcoming paper, Bourgain, Brezis and Mironescu [1].

3. Proof of Theorem 3 when sp $\geq$ 2.

The case $s = 1$ in Theorem 3 coincides with the theorem of Bethuel and Zheng. For the sake of completeness we present a proof which is simpler than the original one (see also Carbou [1] for a similar idea).

Proof of the Bethuel-Zheng theorem. The idea is to assume that φ is known and to derive some consequences. Writing $u = u_1 + iu_2$ with $u_1 = \cos\varphi$ and $u_2 = \sin\varphi$ we have

$$Du_1 = -(\sin\varphi)D\varphi = -u_2 D\varphi$$

and

$$Du_2 = (\cos\varphi)D\varphi = u_1 D\varphi.$$

Hence

$$D\varphi = u_1 Du_2 - u_2 Du_1. \tag{3.1}$$

The strategy is now to find φ by solving (3.1) with the help of a generalized form of Poincaré's lemma,

Lemma 3. *Let $1 \leq p < \infty$ and let $f \in L^p(\Omega; \mathbb{R}^n)$. The following properties are equivalent:*
a) there is some $\varphi \in W^{1,p}(\Omega; \mathbb{R})$ such that

$$f = D\varphi,$$

b) one has

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad \forall\, i, j, \ \ 1 \leq i, j \leq n \tag{3.2}$$

in the sense of distributions, i.e.,

$$\int f_i \frac{\partial \psi}{\partial x_j} = \int f_j \frac{\partial \psi}{\partial x_i} \qquad \forall\, \psi \in C_0^\infty(\Omega).$$

We emphasize that the assumption that Ω is simply connected is needed in this lemma.

Proof of Lemma 3. The implication $a) \Rightarrow b)$ is obvious. To prove the converse, let $\bar{f}$ be the extension of f by 0 outside Ω and let $\bar{f}_\varepsilon = \rho_\varepsilon \star \bar{f}$ where (ρ_ε) is a sequence of mollifiers. The $\bar{f}_\varepsilon$'s satisfy (3.2) on every compact subset of Ω (for ε sufficiently small). In particular, on every smooth simply connected domain $\omega \subset \Omega$ with compact closure in Ω, there is a function ψ_ε such that

$$D\psi_\varepsilon = \bar{f}_\varepsilon \quad \text{in } \omega.$$

(Here we have used the standard Poincaré lemma). Passing to the limit we obtain some $\psi \in W^{1,p}(\omega)$ such that $D\psi = f$ in ω. Finally, we write Ω as an increasing union of ω_n as above and obtain a corresponding sequence ψ_n. In the limit we find some $\varphi \in L^1_{\text{loc}}(\Omega)$ with $D\varphi = f$ in Ω. Using the regularity of Ω and a standard property of Sobolev spaces (see e.g. Maz'ja [1], Corollary in Section 1.1.11) we conclude that $\varphi \in W^{1,p}(\Omega)$.

Proof of the Bethuel-Zheng theorem completed. We will first verify condition $b)$ of the lemma for

$$f = u_1 D u_2 - u_2 D u_1 \tag{3.3}$$

i.e.,

$$f_i = u_1 \frac{\partial u_2}{\partial x_i} - u_2 \frac{\partial u_1}{\partial x_i}.$$

Formally, property (3.2) is clear. Indeed, if u_1 and u_2 are smooth, then

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 2\left(\frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i} - \frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j} \right).$$

On the other hand, if we differentiate the relation

$$|u|^2 = u_1^2 + u_2^2 = 1$$

we find

$$u_1 \frac{\partial u_1}{\partial x_i} + u_2 \frac{\partial u_2}{\partial x_i} = 0 \quad \forall\, i = 1, 2, \dots, n. \tag{3.4}$$

Thus, in $\mathbb{R}^2$, the vector $(\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i})$ is orthogonal to (u_1, u_2). It follows that the vectors $(\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i})$ and $(\frac{\partial u_1}{\partial x_j}, \frac{\partial u_2}{\partial x_j})$ are colinear and therefore

$$\det \begin{pmatrix} \frac{\partial u_1}{\partial x_i} & \frac{\partial u_2}{\partial x_i} \\ \frac{\partial u_1}{\partial x_j} & \frac{\partial u_2}{\partial x_j} \end{pmatrix} = \frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j} - \frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i} = 0. \tag{3.5}$$

Hence (3.2) holds. To make this argument rigorous we rely on the density of smooth functions in the Sobolev space $W^{1,p}(\Omega;\mathbb{R})$ (see e.g. Adams [1], Chap. III or Brezis [1], Chap. IX): there are sequences (u_{1n}) and (u_{2n}) in $C^\infty(\overline{\Omega};\mathbb{R})$ such that $u_{1n} \to u_1$ and $u_{2n} \to u_2$ in $W^{1,p}(\Omega;\mathbb{R})$ and $\|u_{1n}\|_{L^\infty} \le 1, \|u_{2n}\|_{L^\infty} \le 1$.

[**Warning:** We do not claim that $u_n = (u_{1n}, u_{2n})$ takes its values in S^1. The density of $C^\infty(\bar\Omega; N)$ in $W^{1,p}(\Omega; N)$, where N is a compact manifold without boundary, e.g. $N = S^1$, is a delicate matter which has been extensively studied by Bethuel [1]. As a matter of fact, the Bethuel-Zheng theorem can be used to prove the density of $C^\infty(\bar\Omega; S^1)$ in $W^{1,p}(\Omega; S^1)$ for $p \ge 2$.]

Set

$$f_n = u_{1n} Du_{2n} - u_{2n} Du_{1n},$$

so that

$$f_n \to f \quad \text{in } L^p$$

and

$$\frac{\partial f_{in}}{\partial x_j} - \frac{\partial f_{jn}}{\partial x_i} = 2(\frac{\partial u_{1n}}{\partial x_j}\frac{\partial u_{2n}}{\partial x_i} - \frac{\partial u_{1n}}{\partial x_i}\frac{\partial u_{2n}}{\partial x_j}) \tag{3.6}$$

converges in $L^{p/2}$ to $2\left(\frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i} - \frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j}\right)$. Multiplying (3.6) by $\psi \in C_0^\infty(\Omega)$, integrating by parts and passing to the limit (using the fact that $p \ge 2$) we obtain

$$-\int_\Omega (f_i \frac{\partial \psi}{\partial x_j} - f_j \frac{\partial \psi}{\partial x_i}) = 2\int_\Omega (\frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i} - \frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j})\psi.$$

On the other hand (3.4) and (3.5) hold a.e. (even for any $u \in W^{1,p}(\Omega; S^1)$, $1 \le p < \infty$) It follows that f satisfies $b)$ of Lemma 3, and therefore there is some $\varphi \in W^{1,p}(\Omega;\mathbb{R})$ such that

$$f = D\varphi.$$

We will now prove that this φ is essentially the one in the conclusion of the Bethuel-Zheng theorem.

Recall that if $g, h \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $1 \le p < \infty$, then $gh \in W^{1,p}$ and

$$\frac{\partial}{\partial x_i}(gh) = g\frac{\partial h}{\partial x_i} + h\frac{\partial g}{\partial x_i}.$$

Set

$$v = ue^{-i\varphi},$$

so that $v \in W^{1,p}$ and

$$\begin{aligned} Dv &= e^{-i\varphi}(Du - iD\varphi) = ue^{-i\varphi}(\bar u Du - iD\varphi) \\ &= ue^{-i\varphi}(\bar u Du - if) = ue^{-i\varphi}(u_1 Du_1 + u_2 Du_2) = 0 \quad \text{by (3.4).} \end{aligned}$$

We deduce that v is a constant and since $|v| = 1$ we may write $v = e^{iC}$ for some constant $C \in \mathbb{R}$. Hence $u = e^{i(\varphi+C)}$ and the function $\varphi + C$ has the desired properties.

We now turn to the proof of Theorem 3 when $sp \geq 2$. In fact, we have a more precise statement:

Lemma 4. *Assume $n \geq 1, s \geq 1, 1 < p < \infty$ and $sp \geq 2$. Then any $u \in W^{s,p}(\Omega; S^1)$ may be lifted as $u = e^{i\varphi}$ with $\varphi \in W^{s,p}(\Omega;\mathbb{R}) \cap W^{1,sp}(\Omega;\mathbb{R})$.*

Proof. Observe that

$$W^{s,p} \cap L^\infty \subset W^{1,sp}$$

by the Gagliardo-Nirenberg inequality (see Lemma D.1). Since $sp \geq 2$ we may apply the Bethuel-Zheng theorem and write $u = e^{i\varphi}$ for some $\varphi \in W^{1,sp}(\Omega;\mathbb{R})$. Using Lemma 2 we find that

$$D\varphi = -i\bar{u}Du \in W^{s-1,p},$$

so that $\varphi \in W^{s,p}$.

4. Examples of obstruction in Theorems 2 and 3.

We start with an example of obstruction in Theorem 2, i.e., when $0 < s < 1$ and $1 \leq sp < n$.

Lemma 5. *Assume $n \geq 2$. Given any s and any p with $0 < s < 1$, $1 < p < \infty$, and $1 \leq sp < n$, there is some $u \in W^{s,p}(\Omega; S^1)$ which cannot be lifted, i.e., for this u no $\varphi \in W^{s,p}(\Omega;\mathbb{R})$ exists such that $u = e^{i\varphi}$.*

Proof. Without loss of generality we may assume that Ω is the unit ball. Let

$$\psi(x) = \frac{1}{|x|^\alpha} \quad \text{with } \frac{n-sp}{p} \leq \alpha < \frac{n-sp}{sp}$$

and let

$$u = e^{i\psi}.$$

We claim that

$$u \in W^{s,p}(\Omega; S^1). \tag{4.1}$$

Indeed it is clear that

$$\psi \in W^{1,q} \quad \forall\, q \text{ with } 1 < q < \frac{n}{\alpha+1},$$

and thus

$$\psi \in W^{\sigma,q} \quad \forall\, \sigma \text{ with } 0 < \sigma < 1\,, \quad \forall\, q \text{ with } 1 < q < \frac{n}{\alpha+1}.$$

Since $u \in L^\infty$, we also know, by the Gagliardo-Nirenberg inequality (see Lemma D.1 in Appendix D), that

$$u \in W^{t,r} \ \ \forall\, t \in (0,1) \ \ \forall\, r \in (1,\infty) \quad \text{with } tr < \frac{n}{\alpha+1}.$$

In particular, we may choose $t = s$ and $r = p$ since $sp < n/(\alpha+1)$, i.e., (4.1) holds.

Next we claim that there is no $\varphi \in W^{s,p}(\Omega;\mathbb{R})$ such that $u = e^{i\varphi}$. Assume, by contradiction, that such φ exists. Set

$$\eta = \frac{1}{2\pi}(\varphi - \psi),$$

so that η takes its values in $\mathbb{Z}$ and

$$\eta \in W^{s,p}_{\text{loc}}(\Omega\backslash\{0\};\mathbb{Z})$$

(because ψ is smooth on $\Omega\backslash\{0\}$). Since $sp \geq 1$ and $\Omega\backslash\{0\}$ is connected we conclude, using Lemma B.1 in Appendix B, that η is a constant. Thus $\psi \in W^{s,p}(\Omega;\mathbb{R})$. Note that, by scaling,

$$A(r) = \int_{B_r}\int_{B_r} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{n+sp}}dxdy$$

satisfies $A(1) = r^\beta A(r)$ with $\beta = (\alpha+s)p - n \geq 0$ (by assumption on α). If $A(1) < \infty$, then $A(1) = 0$ (by letting $r \to 0$). But this is impossible. Thus $A(1) = \infty$, i.e., $\psi \notin W^{s,p}$. A contradiction.

A topological obstruction. There is an alternative example of obstruction to lifting, which is of topological nature.

Consider first the case $n = 2$. Set

$$u(x) = \frac{x}{|x|} \quad \text{on the unit ball } \Omega \text{ of } \mathbb{R}^2. \tag{4.2}$$

Since

$$|Du(x)| \leq C/|x|$$

we see that $u \in W^{1,q}(\Omega;S^1)$ for every $q < 2$ and therefore $u \in W^{s,p}(\Omega;S^1)$ for every $s \in (0,1)$ and every $p \in (1,\infty)$ with $sp < 2$ (by the Gagliardo-Nirenberg inequality; see Lemma D.1), If, in addition, we assume $sp \geq 1$ then there is no $\varphi \in W^{s,p}(\Omega;\mathbb{R})$ such that $u = e^{i\varphi}$. Indeed set

$$\Omega' = \Omega\backslash([0,1]\times\{0\})$$

and

$$\theta \in (0,2\pi) \qquad \text{with } e^{i\theta} = u.$$

Clearly $\theta \in C^\infty(\Omega')$ and θ has a jump of 2π along the segment $[0,1]\times\{0\}$. Assume, by contradiction, that u has a lifting $\varphi \in W^{s,p}(\Omega;\mathbb{R})$. Arguing as above we would conclude that $\theta \in W^{s,p}(\Omega;\mathbb{R})$ but this is impossible since θ has a jump of 2π along the segment $(0,1)\times\{0\}$ and such a function cannot belong to $W^{s,p}$ with $sp \geq 1$.

When $n \geq 3$, the same construction as above with

$$u(x) = \frac{(x_1,x_2)}{(x_1^2+x_2^2)^{1/2}} \quad x=(x_1,x_2,\ldots,x_n)$$

provides an example of a function $u \in W^{s,p}(\Omega;S^1)$ for every $s\in(0,1)$ and every $p\in(1,\infty)$ with $sp<2$ and which has no lifting in $W^{s,p}$ when $sp\geq 1$. However, this example does not reach the optimal condition $sp<n$ when $n\geq 3$.

Remark 5. The topological obstruction provides an example of loss of regularity in lifting. To explain the phenomenon consider the simple case where $p=2$. Recall (see Corollary 1) that if $u\in H^s(\Omega;S^1)$ with $1/2<s<1$, then, in general, u has no lifting in H^s. From the positive part in Corollary 1 one knows that u has a lifting in $H^{(1/2-\varepsilon)}$. Roughly speaking, we lose $(s-1/2)$ derivative in the lifting.

Open Problem: When $n\geq 3$ the precise loss of regularity in lifting is not fully understood. For simplicity consider the case $n=3$ and $p=4$. First a summary of the known results:

a) If $s<1/4$, any $u\in W^{s,4}$ has a lifting in $W^{s,4}$.

b) If $s\geq 3/4$, any $u\in W^{s,4}$ has a lifting in $W^{s,4}$.

c) If $1/4\leq s<3/4$ some u's in $W^{s,4}$ have no lifting in $W^{s,4}$.

d) The topological example provides an example of a function $u\in W^{s,4}\ \forall\, s<1/2$, and this u has no lifting even in $W^{1/4,4}$.

It would be interesting to find an example of a function $u\in W^{s,4}$ $\forall s<3/4$ which has no lifting even in $W^{1/4,4}$.

Finally, case *b*) in Theorem 3 relies on

Lemma 6. *Assume $n\geq 2$. Given any s and any p with $s\geq 1$ and $1<p<\infty$ with $sp<2$, there is some $u\in W^{s,p}(\Omega;S^1)$ which cannot be lifted by a function $\varphi\in W^{s,p}(\Omega;\mathbb{R})$.*

Proof. Use the topological example u above. It is easy to see that $u\in W^{s,p}$ $\forall s\in(0,\infty)$, $\forall p\in(1,\infty)$ with $sp<2$. This u has no lifting even in $W^{1/p,p}$.

5. Control of lifting in the H^s-norm for $s \overset{\rightarrow}{<} \frac{1}{2}$ and application to Ginzburg-Landau.

We return to the particular issue of lifting a function $u\in H^s(\Omega;S^1)$ when $s<1/2$ and $s\to 1/2$. Recall (see Corollary 1) that, for every $s<1/2$, u admits a lifting $\varphi\in H^s(\Omega;\mathbb{R})$, i.e.,

$$u=e^{i\varphi} \tag{5.1}$$

We also know (see (1.7)) that we may find a $\varphi \in H^s$ such that

$$\|\varphi\|_{H^s} \le C_s \|u\|_{H^s}.$$

Our aim is to find an optimal control for the constant C_s as $s \to 1/2$. Such a control will then be used in the study of the Ginzburg-Landau energy E_ε as $\varepsilon \to 0$.

If we follow the proof in Section 1 we obtain a φ as a limit of sequence φ_j such that

$$\sum_{j\ge 1} 4^{sj} \|\varphi_j - \varphi_{j-1}\|_{L^2}^2 \le C \sum_{j\ge 0} 4^{sj} \|u - E_j(u)\|_{L^2}^2 \tag{5.2}$$

where here, and in what follows, C without a subscript s denotes a constant which remains bounded as $s \to 1/2$. Following the proof of Corollary 1 we obtain

$$\sum_{j\ge 1} 4^{sj} \|E_j(\varphi) - E_{j-1}(\varphi)\|_{L^2}^2 \le C \sum_{j\ge 1} 4^{sj} \|\varphi_j - \varphi_{j-1}\|_{L^2}^2. \tag{5.3}$$

We also recall (see Step 3 in Appendix A) that

$$\sum_{j\ge 0} 4^{sj} \|u - E_j(u)\|_{L^2}^2 \le C \|u\|_{H^s}^2. \tag{5.4}$$

Combining (5.2), (5.3) and (5.4) yields

$$\sum_{j\ge 1} 4^{sj} \|E_j(\varphi) - E_{j-1}(\varphi)\|_{L^2}^2 \le C \|u\|_{H^s}^2. \tag{5.5}$$

Finally we know (see Corollary A.2 in Appendix A) that

$$\|\varphi\|_{H^s} \le C_s \left(\sum_{j\ge 1} 4^{sj} \|E_j(\varphi) - E_{j-1}(\varphi)\|_{L^2}^2 \right)^{1/2} \tag{5.6}$$

and the optimal constant C_s for the inequality (5.6) is of the order of $(1-2s)^{-1}$. Hence we deduce that the φ constructed by this technique satisfies

$$\|\varphi\|_{H^s} \le C(1-2s)^{-1} \|u\|_{H^s}. \tag{5.7}$$

In fact, there is a more refined construction of lifting which yields a better estimate. For simplicity we work in a cube Q of $\mathbb{R}^d$, $d \ge 1$; for more general domains see Remark E.2 in Appendix E.

Theorem 4. *For every $u \in H^s(Q;S^1)$ with $0 < s < 1/2$ one may construct a $\varphi \in H^s(Q;\mathbb{R})$ satisfying (5.1) and the (optimal) estimate*

$$\|\varphi\|_{H^s} \le C(1-2s)^{-1/2}\|u\|_{H^s}, \tag{5.8}$$

where C is independent of u and independent of s as $s \to 1/2$.

The reason why the previous construction does not yield the correct asymptotic as $s \to 1/2$ is due to "edge-singularities" at the nodes of our dyadic partitions P_j. To overcome this, we rely on an argument of translations which is explained in Appendix E where we present the proof of Theorem 4. That type of argument has been exploited earlier in slightly different contexts (for instance in comparing the usual and dyadic BMO-norms, see Garnett and Jones [1]).

The next result is an application to the Ginzburg-Landau functional. Let Q be a cube of $\mathbb{R}^d$, $d \ge 1$, and let $\Omega = Q \times (0,1)$. For any function $g \in H^{1/2}(Q;\mathbb{C})$ set

$$H^1_g(\Omega) = \left\{ u(x,t) : \Omega \to \mathbb{C};\int_\Omega |\nabla u|^2 dxdt < \infty \text{ and } u(x,0) = g(x) \text{ on } Q \right\},$$
$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (|u|^2-1)^2,$$

where ∇ denotes the full gradient (in (x,t)).

Theorem 5. *For every $g \in H^{1/2}(Q;S^1)$ we have, for $\varepsilon > 0$,*

$$E_\varepsilon = \operatorname*{Min}_{u \in H^1_g(\Omega)} E_\varepsilon(u) \le C \log(1/\varepsilon)\|g\|^2_{H^{1/2}} \tag{5.9}$$

where C is independent of ε and of g.

Proof. Let $s = s(\varepsilon) < 1/2$ to be specified. It follows from Theorem 4 (applied to g) that $g = e^{i\varphi}$ for some $\varphi \in H^s(Q;\mathbb{R})$ satisfying

$$\|\varphi\|_{H^s} \le C(1-2s)^{-1/2}\|g\|_{H^{1/2}}. \tag{5.10}$$

Denote φ_δ a δ-smoothing of φ (with δ to be chosen later). Thus, we have

$$\|\varphi - \varphi_\delta\|_{L^2(Q)} \le C\delta^s\|\varphi\|_{H^s(Q)} \le C\delta^s(1-2s)^{-1/2}\|g\|_{H^{1/2}(Q)} \tag{5.11}$$

also, by (5.10),

$$\|\varphi_\delta\|_{H^{1/2}(Q)} \le C\delta^{s-1/2}\|\varphi\|_{H^s(Q)} \le C(1-2s)^{-1/2}\delta^{s-1/2}\|g\|_{H^{1/2}(Q)}. \tag{5.12}$$

Taking

$$1-2s \sim (\log 1/\delta)^{-1} \tag{5.13}$$

we conclude that

$$\|\varphi_\delta\|_{H^{1/2}(Q)} \leq C(\log 1/\delta)^{1/2}\|g\|_{H^{1/2}(Q)}. \tag{5.14}$$

Let $\tilde{\varphi}_\delta$ denote some harmonic extension of φ_δ to Ω with

$$\|\tilde{\varphi}_\delta\|_{H^1(\Omega)} \leq C(\log 1/\delta)^{1/2}\|g\|_{H^{1/2}(Q)} \tag{5.15}$$

and set

$$G_\delta = e^{i\tilde{\varphi}_\delta} \tag{5.16}$$

so that

$$\|G_\delta\|_{H^1(\Omega)} \leq C(\log 1/\delta)^{1/2}\|g\|_{H^{1/2}(Q)}. \tag{5.17}$$

Let P denote some harmonic extension of $(g - e^{i\varphi_\delta})$ to Ω satisfying the following three estimates

$$\begin{aligned}\|P\|_{H^1(\Omega)} &\leq C\|g - e^{i\varphi_\delta}\|_{H^{1/2}(Q)}\\ &\leq C(\|g\|_{H^{1/2}(Q)} + \|\varphi_\delta\|_{H^{1/2}(Q)})\\ &\leq C(\log 1/\delta)^{1/2}\|g\|_{H^{1/2}(Q)} \text{ by (5.14),}\end{aligned} \tag{5.18}$$

$$\|P\|_{L^\infty(\Omega)} \leq C\|g - e^{i\varphi_\delta}\|_{L^\infty(Q)} \leq C, \tag{5.19}$$

and

$$\begin{aligned}\|P\|_{L^2(\Omega)} &\leq C\|g - e^{i\varphi_\delta}\|_{L^2(Q)}\\ &\leq C\|\varphi - \varphi_\delta\|_{L^2(Q)} \leq C\delta^{1/2}(\log 1/\delta)^{1/2}\|g\|_{H^{1/2}(Q)} \text{ by (5.11).}\end{aligned} \tag{5.20}$$

Define

$$u = G_\delta + P \tag{5.21}$$

so that by construction $u_{|t=0} = g$ on Q.

¿From (5.17) and (5.18) we have

$$\|u\|^2_{H^1(\Omega)} \leq C\log(1/\delta)\|g\|^2_{H^{1/2}(Q)}. \tag{5.22}$$

On the other hand, using (5.19) we find

$$\big||u|^2-1\big| \le C\big||u|-1\big|\big||u|+1\big| \le C\big||u|-1\big|$$

and since

$$\big||u|-1\big| = \big||u|-|G_\delta|\big| \le |u-G_\delta| = |P|$$

we are led to

$$\int_\Omega (|u|^2-1)^2 \le C\int_\Omega |P|^2 \le C\delta(\log 1/\delta)\|g\|_{H^{1/2}(Q)} \text{ by (5.20).} \tag{5.23}$$

Combining (5.22) and (5.23) we obtain

$$E_\varepsilon(u) \le C(1+\delta/\varepsilon^2)\log(1/\delta)\|g\|^2_{H^{1/2}(Q)}.$$

Choosing $\delta=\varepsilon^2$ yields the desired estimate (5.9).

Remark 6. In dimension $d=1$, E_ε remains bounded as $\varepsilon\to 0$ since we may write $g=e^{i\varphi}$ with some $\varphi\in H^{1/2}$ and then take $u=e^{i\tilde\varphi}$ where $\tilde\varphi$ is some harmonic extension of φ. However, the bound for E_ε depends on g, not just on $\|g\|_{H^{1/2}}$ (see also Remark 3).

Remark 7. In dimension $d\ge 2$, estimate (5.9) is optimal. This may be seen, for example in dimension $d=2$, by choosing for g the topological example described in Section 4,

$$g(x)=\frac{x}{|x|} \quad \text{on } Q.$$

We claim that $E_\varepsilon \ge \alpha\log(1/\varepsilon)$ for some constant $\alpha>0$. Indeed we may write for any $u\in H^1_g(\Omega)$,

$$E_\varepsilon(u) \ge \alpha\int_{1/2}^1 dr\int_{\Sigma_r}\left(\frac12|\nabla_\sigma u|^2+\frac{1}{4\varepsilon}(|u|^2-1)^2\right)d\sigma$$

where $\Sigma_r=\{(x,t)\in\Omega\,;|x|^2+t^2=r^2\}$ and ∇_σ denote the tangential gradient on Σ_r. We then invoke the lower bound

$$\frac12\int_{\Sigma_r}|\nabla_\sigma u|^2+\frac{1}{4\varepsilon^2}\int_{\Sigma_r}(|u|^2-1)^2 \ge c(\log 1/\varepsilon)$$

which is known for a 2-dimensional flat disk (see Bethuel, Brezis and Hélein [1], Theorem V.3) and can be transported to Σ_r by a smooth diffeomorphism.

The fact that (5.9) is optimal when $d\ge 2$ shows in turn that (5.8) is also optimal for $d\ge 2$. Indeed an estimate of the form $\|\varphi\|_{H^s}\le o((1-2s)^{-1/2})$ in place of (5.8), would yield $E_\varepsilon\le o(\log 1/\varepsilon)$, which is impossible. When $d=1$, estimate (5.8) is still optimal, but this requires a separate argument (see Remark E.1 in Appendix E).

Remark 8. Theorem 4 is still valid for a general smooth domain Q in $\mathbb{R}^d$ (without any topological assumption); see Remark E.2 in Appendix E. As a result, Theorem 5 is also true in that situation. In Theorem 5 we may also take for Ω any smooth bounded domain in $\mathbb{R}^{d+1}, d \geq 1$ and $Q = \partial\Omega$; this is a consequence of the fact that Theorem 4 is still valid when Q is a smooth d-dimensional manifold (see Remark E.2 in Appendix E). In that case a more elementary (and simple) proof of (5.9) was obtained recently by T. Rivière [3]. Estimate (5.9) plays a fundamental role in the asymptotic analysis (as $\varepsilon \to 0$) of Ginzburg-Landau minimizers (see Rivière [1], [2], Lin and Rivière [1], Sandier [1] and also the forthcoming paper Bourgain, Brezis and Mironescu [1]).

APPENDIX A

A characterization of $\mathbf{W}^{\mathbf{s},p}(\mathbf{\Omega};\mathbb{R})$ when $\mathbf{sp} < 1$

Let $\Omega = (0,1)^n$. For $j = 0, 1, \dots$ we denote by $\mathcal{P}_j$ the dyadic partition of Ω into 2^{jn} cubes of side 2^{-j} and by $\mathcal{E}_j$ the space of functions from Ω into $\mathbb{R}$ (or $\mathbb{C}$) which are constant on each cube of $\mathcal{P}_j$. Given a function $f \in L^p(\Omega)$ we consider the function $f_j = E_j(f) \in \mathcal{E}_j$ defined as follows: every $x \in \Omega$ belongs to one of the cubes, say $Q_j(x)$, of the partition $\mathcal{P}_j$ and we set

$$f_j(x) = E_j(f)(x) = \fint_{Q_j(x)} f.$$

Clearly we have

$$\|E_j(f)\|_{L^p} \leq \|f\|_{L^p} \quad \forall\, j, \tag{A.1}$$

$$E_j(f) \to f \quad \text{in } L^p \text{ and a.e. as } j \to \infty. \tag{A.2}$$

Theorem A.1. *Assume $sp < 1$. Then*

$$\begin{aligned} \|f\|^p_{W^{s,p}} &\sim \sum_{j\geq 1} 2^{spj} \|E_j(f) - E_{j-1}(f)\|^p_{L^p} \\ &\sim \sum_{j\geq 0} 2^{spj} \|f - E_j(f)\|^p_{L^p}. \end{aligned}$$

Remark A.1. Theorem A.1 is due to G. Bourdaud [1] (see his Théorème 5 with $m = 0$ and also the earlier paper of R. Devore and V. A. Popov [1]). It gives a positive answer to a conjecture of H. Triebel [1] (Conjecture 1) for the Haar system $\{h_j^{(-1,0)}\}$ in the spaces $B^s_{p,p} = W^{s,p}$. The parameter $\ell = -1+1-0 = 0$ and (for $s > 0$), the condition $s < \ell+(1/p)$ is indeed $sp < 1$. For the convenience of the reader, and also because we are interested in the behaviour of the sharp constants in the norm equivalence as $sp \to 1$, we present below a proof of Theorem A.1.

We have also made use of the

Corollary A.1. *Assume $sp < 1$ and let $(\varphi_j)_{j=0,1,\dots}$ be a sequence of functions on Ω such that*

$$\varphi_j \in \mathcal{E}_j \quad \forall j = 0, 1 \dots \tag{A.3}$$

and

$$\sum_{j \geq 1} 2^{spj} \|\varphi_j - \varphi_{j-1}\|_{L^p}^p < \infty. \tag{A.4}$$

Then $\varphi_j \to \varphi$ in L^p and $\varphi \in W^{s,p}$ with

$$\|\varphi\|_{W^{s,p}}^p \leq C \sum_{j \geq 1} 2^{spj} \|\varphi_j - \varphi_{j-1}\|_{L^p}^p. \tag{A.5}$$

Remark A.2. Here $\|f\|_{W^{s,p}}$ denotes the standard semi-norm,

$$\|f\|_{W^{s,p}}^p = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dxdy.$$

To work with a norm it suffices to add $|\int f|$.

Proof of Corollary A.1. From (A.4) we see that φ_j is a Cauchy sequence in L^p and thus $\varphi_j \to \varphi$ in L^p. In order to prove that $\varphi \in W^{s,p}$ it suffices, in view of Theorem A.1, to check that

$$\sum_{j \geq 1} 2^{spj} \|E_j(\varphi) - E_{j-1}(\varphi)\|_{L^p}^p < \infty. \tag{A.6}$$

Note that

$$E_j(\varphi) - E_{j-1}(\varphi) = E_j(\varphi - \varphi_j) - E_{j-1}(\varphi - \varphi_{j-1}) + \varphi_j - \varphi_{j-1} \tag{A.7}$$

and thus

$$\|E_j(\varphi) - E_{j-1}(\varphi)\|_{L^p} \leq \|\varphi - \varphi_j\|_{L^p} + \|\varphi - \varphi_{j-1}\|_{L^p} + \|\varphi_j - \varphi_{j-1}\|_{L^p} \tag{A.8}$$

On the other hand, if we write

$$\varphi_j - \varphi = (\varphi_j - \varphi_{j+1}) + (\varphi_{j+1} - \varphi_{j+2}) + \cdots,$$

we see that

$$\|\varphi_j - \varphi\|_{L^p} \leq \sum_{k \geq j} \|\varphi_k - \varphi_{k+1}\|_{L^p}$$

so that, by (A.8), we have

$$\|E_j(\varphi) - E_{j-1}(\varphi)\|_{L^p} \le 3 \sum_{k\ge j} \|\varphi_k - \varphi_{k-1}\|_{L^p}. \tag{A.9}$$

Thus, by Hölder,

$$\begin{aligned}\|E_j(\varphi) - E_{j-1}(\varphi)\|_{L^p} &\le 3 \sum_{k\ge j} (k-j+1)\|\varphi_k - \varphi_{k-1}\|_{L^p} \frac{1}{(k-j+1)} \\ &\le 3 \left(\sum_{k\ge j} (k-j+1)^p \|\varphi_k - \varphi_{k-1}\|_{L^p}^p\right)^{1/p} \left(\sum_{k\ge j} \frac{1}{(k-j+1)^{p'}}\right)^{1/p'}\end{aligned}$$

and therefore

$$\|E_j(\varphi) - E_{j-1}(\varphi)\|_{L^p}^p \le C \sum_{k\ge j} (k-j+1)^p \|\varphi_k - \varphi_{k-1}\|_{L^p}^p. \tag{A.10}$$

Consequently

$$\begin{aligned}\sum_{j\ge 1} 2^{spj} \|E_j(\varphi) - E_{j-1}(\varphi)\|_{L^p}^p &\le C \sum_{j\ge 1}\sum_{k\ge j} 2^{spj}(k-j+1)^p \|\varphi_k - \varphi_{k-1}\|_{L^p}^p \\ &= C \sum_{k\ge 1} 2^{spk} \|\varphi_k - \varphi_{k-1}\|_{L^p}^p a_k \end{aligned} \tag{A.11}$$

where

$$\begin{aligned} a_k &= \sum_{1\le j\le k} 2^{sp(j-k)}(k-j+1)^p \\ &= 2^{sp} \sum_{1\le \ell\le k} \frac{\ell^p}{2^{sp\ell}} \le a_\infty = 2^{sp} \sum_{\ell=1}^{\infty} \frac{\ell^p}{2^{sp\ell}}. \end{aligned}$$

We deduce from (A.11) and Theorem A.1 that $\varphi \in W^{s,p}$ and

$$\|\varphi\|_{W^{s,p}}^p \le C \sum_{j\ge 1} 2^{spj} \|\varphi_j - \varphi_{j-1}\|_{L^p}^p.$$

Proof of Theorem A.1. Set

$$\begin{aligned} X &= \|f\|_{W^{s,p}}^p \\ Y &= \sum_{j\ge 1} 2^{spj} \|E_j(f) - E_{j-1}(f)\|_{L^p}^p \\ Z &= \sum_{j\ge 0} 2^{spj} \|f - E_j(f)\|_{L^p}^p. \end{aligned}$$

We will prove that $Y \sim Z$ and $Z \leq CX$ without assuming $sp < 1$. That condition enters only to prove that $X \leq CY$.

Step 1: $Y \leq Z$

Proof. We have, since $E_{j-1}(f) \in \mathcal{E}_{j-1} \subset \mathcal{E}_j$,

$$E_j(E_{j-1}(f)) = E_{j-1}(f)$$

and thus

$$|E_j(f) - E_{j-1}(f)| = |E_j(f) - E_j(E_{j-1}(f))|.$$

Therefore

$$\|E_j(f) - E_{j-1}(f)\|_{L^p} \leq \|f - E_{j-1}(f)\|_{L^p}$$

and the estimate $Y \leq Z$ follows.

Step 2: $Z \leq CY$. Here the condition $sp < 1$ is not used; it suffices to have $s > 0$.

Proof. Set $\varphi_j = E_j(f)$; as in the proof of Corollary A.1 we obtain

$$\|f - \varphi_j\|_{L^p} \leq \sum_{k \geq j+1} \|\varphi_k - \varphi_{k-1}\|_{L^p}$$

and, by Hölder,

$$\|f - \varphi_j\|_{L^p} \leq \left(\sum_{k \geq j+1} (k-j)^p \|\varphi_k - \varphi_{k-1}\|^p_{L^p} \right)^{1/p} \left(\sum_{k \geq j+1} \frac{1}{(k-j)^{p'}} \right)^{1/p'}.$$

Thus

$$\|f - \varphi_j\|^p_{L^p} \leq C \sum_{k \geq j+1} (k-j)^p \|\varphi_k - \varphi_{k-1}\|^p_{L^p}$$

and consequently

$$\begin{aligned} \sum_{j \geq 0} 2^{spj} \|f - \varphi_j\|^p_{L^p} &\leq C \sum_{j \geq 0} \sum_{k \geq j+1} 2^{spj} (k-j)^p \|\varphi_k - \varphi_{k-1}\|^p_{L^p} \\ &= C \sum_{k \geq 1} 2^{spk} a_k \|\varphi_k - \varphi_{k-1}\|^p_{L^p} \end{aligned}$$

where

$$a_k = \sum_{0 \leq j \leq k-1} 2^{sp(j-k)} (k-j)^p \leq a_\infty = \sum_{\ell=1}^{\infty} \frac{\ell^p}{2^{sp\ell}} < \infty.$$

Hence

$$Z \leq C a_\infty Y.$$

Step 3: $Z \le CX$. Here, again, the condition $sp < 1$ is not used.

Proof. Recall that $Q_j(x)$ is the cube in the partition $\mathcal{P}_j$ containing the point x. Write

$$f(x) - E_j(f)(x) = f(x) - \fint_{Q_j(x)} f(y)dy = \fint_{Q_j(x)} (f(x) - f(y))dy$$
$$= 2^{nj} \int_{Q_j(x)} (f(x) - f(y))dy$$

and thus, by Hölder,

$$|f(x) - E_j(f)(x)|^p \le 2^{nj} \int_{Q_j(x)} |f(x) - f(y)|^p dy.$$

Therefore

$$\|f - E_j(f)\|_{L^p}^p \le 2^{nj} \int_\Omega dx \int_{Q_j(x)} |f(x) - f(y)|^p dy, \tag{A.12}$$

so that

$$Z = \sum_{j\ge 0} 2^{spj} \|f - E_j(f)\|_{L^p}^p \le \sum_{j\ge 0} 2^{(n+sp)j} \int_\Omega dx \int_{Q_j(x)} |f(x) - f(y)|^p dy$$
$$= \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} a(x,y) dxdy,$$

where

$$a(x,y) = |x-y|^{n+sp} \sum_{j\ge 0} 2^{(n+sp)j} \mathbf{1}_{Q_j(x)}(y)$$

and $\mathbf{1}$ denotes the characteristic function. Clearly

$$a(x,y) \le (4n)^{(n+sp)/2} \quad \forall x, y \in \Omega$$

and the conclusion follows.

Step 4: $X \le CY$ when $sp < 1$.

Proof. For $h \in \mathbb{R}^n$ set

$$(\delta_h f)(x) = f(x+h) - f(x), \ x \in \Omega_h = \Omega \cap (\Omega - h).$$

A quantity equivalent to X is

$$X' = \int_{|h|<1} \frac{dh}{|h|^{n+sp}} \int_{\Omega_h} |(\delta_h f)(x)|^p dx. \tag{A.13}$$

We will use the following two lemmas

Lemma A.1. *We have, with some constant C (depending only on p, α and β), for all $h \in \mathbb{R}^n$ and all $j \geq 1$*

$$\|\delta_h f\|^p_{L^p(\Omega_h)} \leq C \left(\sum_{k=1}^{j} 2^{\alpha(j-k)p} \|\delta_h(f_k - f_{k-1})\|^p_{L^p(\Omega_h)} + \sum_{k=j+1}^{\infty} 2^{\beta(k-j)p} \|f_k - f_{k-1}\|^p_{L^p(\Omega)} \right),$$

where $\alpha > 0$ and $\beta > 0$ will be chosen later.

Proof. As above, write

$$f = f_0 + \sum_{k\geq 1} (f_k - f_{k-1})$$

and thus

$$\delta_h f = \sum_{k \geq 1} \delta_h (f_k - f_{k-1}),$$

so that

$$\|\delta_h f\|_{L^p(\Omega_h)} \leq \sum_{k=1}^{j} \|\delta_h(f_k - f_{k-1})\|_{L^p(\Omega_h)} + 2 \sum_{k=j+1}^{\infty} \|f_k - f_{k-1}\|_{L^p(\Omega)},$$

and the conclusion follows from Hölder's inequality.

Lemma A.2. *We have, for all $h \in \mathbb{R}^n$ and all $\psi \in \mathcal{E}_k, k \geq 1$,*

$$\|\delta_h \psi\|^p_{L^p(\Omega_h)} \leq C|h|2^k \|\psi\|^p_{L^p(\Omega)} \tag{A.14}$$

where C depends only on p and n.

Proof. Write

$$\psi = \sum_{Q \in \mathcal{P}_k} a_Q \mathbf{1}_Q$$

and thus

$$\delta_h \psi = \sum_Q a_Q (\delta_h \mathbf{1}_Q).$$

Therefore, by Hölder

$$|\delta_h \psi|^p \leq \left(\sum_Q |a_Q|^p |\delta_h \mathbf{1}_Q| \right) \left(\sum_Q |\delta_h \mathbf{1}_Q| \right)^{p-1}.$$

But

$$\sum_Q |\delta_h \mathbf{1}_Q| \leq 2$$

and thus

$$\int_{\Omega_h} |\delta_h \psi|^p \le C \sum_Q |a_Q|^p \int_{\Omega_h} |\delta_h \mathbf{1}_Q|. \tag{A.15}$$

On the other hand

$$\int_{\Omega_h} |\delta_h \mathbf{1}_Q| \le |Q\backslash(Q-h)| + |(Q-h)\backslash Q| \le C\frac{|h|}{2^{(n-1)k}} \tag{A.16}$$

and

$$\|\psi\|^p_{L^p(\Omega)} = \frac{1}{2^{nk}} \sum_Q |a_Q|^p. \tag{A.17}$$

Combining (A.15), (A.16) and (A.17) yields (A.14).

Proof of Step 4 completed. In view of (A.13) we have

$$X \le C \sum_{j=1}^{\infty} \int_{\frac{1}{2^j}<|h|<\frac{1}{2^{j-1}}} \frac{dh}{|h|^{n+sp}} \int_{\Omega_h} |(\delta_h f)(x)|^p dx.$$

Combining this with Lemma A.1 we find

$$X \le C(I_1 + I_2)$$

where

$$I_1 = \sum_{j=1}^{\infty} \int_{\frac{1}{2^j}<|h|<\frac{1}{2^{j-1}}} 2^{(n+sp)j} \sum_{k=1}^{j} 2^{\alpha(j-k)p} \|\delta_h(f_k - f_{k-1})\|^p_{L^p(\Omega_h)} dh \tag{A.18}$$

and

$$I_2 = \sum_{j=1}^{\infty} \int_{\frac{1}{2^j}<|h|<\frac{1}{2^{j-1}}} 2^{(n+sp)j} \sum_{k=j+1}^{\infty} 2^{\beta(k-j)p} \|f_k - f_{k-1}\|^p_{L^p(\Omega)} dh. \tag{A.19}$$

The estimate for I_2 is very simple since

$$\begin{aligned} I_2 &\le C \sum_{j=1}^{\infty} \sum_{k=j+1}^{\infty} 2^{spj} 2^{\beta(k-j)p} \|f_k - f_{k-1}\|^p_{L^p} \\ &= C \sum_{k=2}^{\infty} 2^{spk} b_k \|f_k - f_{k-1}\|^p_{L^p} \end{aligned}$$

where

$$b_k = \sum_{j=1}^{k-1} 2^{sp(j-k)} 2^{\beta(k-j)p} \le b = \sum_{\ell=1}^{\infty} 2^{(\beta-s)\ell p} < \infty$$

provided we choose $0 < \beta < s$. Therefore $I_2 \le CY$.

To estimate I_1 we apply Lemma A.2 with $\psi = (f_k - f_{k-1})$. Inserting (A.14) in (A.18) we obtain

$$\begin{aligned} I_1 &\le C \sum_{j=1}^{\infty} 2^{spj} \sum_{k=1}^{j} 2^{(k-j)} 2^{\alpha(j-k)p} \|f_k - f_{k-1}\|_{L^p}^p \\ &= Cc \sum_{k=1}^{\infty} 2^{spk} \|f_k - f_{k-1}\|_{L^p}^p \end{aligned}$$

with

$$c = \sum_{\ell=0}^{\infty} 2^{(sp-1+\alpha p)\ell} < \infty,$$

provided we choose $0 < \alpha < (1-sp)/p$ (this is the only place where we use the assumption $sp < 1$). Thus we have proved that $I_1 \le CY$ and the proof of Step 4 is complete.

Returning to Theorem A.1 it is a natural question to ask how the norm-equivalence deteriorates when $sp \to 1$. It was already observed that the inequality

$$\sum_{j\ge 1} 2^{spj} \|\Delta_j f\|_{L^p}^p \le C \|f\|_{W^{s,p}}^p,$$

where $\Delta_j f = E_j(f) - E_{j-1}(f)$, is independent of the assumption $sp < 1$. Concerning the other direction, one has the following more precise result when sp is close to 1.

Proposition A.1. *Assume $sp < 1$. Then*

$$\|f\|_{W^{s,p}} \le \frac{C}{s(1-sp)} \left(\sum_{j\ge 1} 2^{spj} \|\Delta_j f\|_{L^p}^p \right)^{1/p} \tag{A.20}$$

where C is an absolute constant.

Proof. Following the proof of Step 4 with

$$\alpha = (1-sp)/2p \quad \text{and } \beta = s/2$$

and using the fact that

$$\sum_{\ell=1}^{\infty} 2^{-a\ell} \le \int_0^\infty \frac{dx}{2^{ax}} = C/a,$$

we obtain

$$X \le \left(1+\frac{C}{\alpha p'}+\frac{C}{\beta p'}\right)^{p-1}(I_1+I_2)$$

and then

$$I_2 \le C(1+\frac{1}{sp})Y$$

$$I_1 \le \frac{C}{1-sp}Y.$$

Combining these inequalities yields (A.20).

In particular, with $p=2$, we find

Corollary A.2. *For $1/4<s<1/2$ we have*

$$\|f\|_{H^s} \le C(1-2s)^{-1}\left(\sum_{j\ge 1} 4^{sj}\|\Delta_j f\|_{L^2}^2\right)^{1/2}$$

where C is an absolute constant.

The dependence in $(1-2s)^{-1}$ for $s\to 1/2$ in Corollary A.2 is optimal as can be seen from the following example.

Lemma A.3. *Let $0<s<\frac{1}{2}$. Let $\Omega=(-1,1)$ equipped with standard dyadic partition $\{\mathcal{P}_j\}$ and*

$$f=(\log\frac{1}{x})\chi_{[0<x<1]}.$$

Then

$$(i)\quad \|f\|_{H^s} \gtrsim (1-2s)^{-3/2}$$

$$(ii)\quad (\sum_{j\ge 1}4^{js}\|\Delta_j f\|_{L^2}^2)^{1/2} \sim (1-2s)^{-1/2}.$$

Proof.

(*i*)

$$\begin{aligned}\|f\|_{H^s}^2=\iint\frac{|f(x+h)-f(x)|^2}{|h|^{1+2s}}dxdh &\ge \iint_{x<0<x+h} h^{-(1+2s)}(\log\frac{1}{x+h})^2dxdh\\ &\ge \sum_j 4^{js}\int_{-2^{-j+1}}^{-2^{-j}}(\log\frac{1}{x})^2dx\\ &\sim \sum_j j^2 2^{-j(1-2s)}\\ &\sim (1-2s)^{-3}.\end{aligned}$$

(*ii*) We need to evaluate the increments $\Delta_j f$. Let $I \in \mathcal{P}_{j-1}$,

$$I = [a, a + 2^{-(j-1)}] \subset [0, 1].$$

Thus the value of $|\Delta_j f|$ on I is

$$2^j \left| \int_a^{a+2^{-j}} f - \int_{a+2^{-j}}^{a+2^{-j+1}} f \right| = 2^j \left| F(a + 2^{-j+1}) + F(a) - 2F(a + 2^{-j}) \right| \tag{A.21}$$

where

$$F(x) = x \log \frac{1}{x} + x.$$

For $a = 0$,

$$(A.21) = 2^j |F(2^{-j+1}) - 2F(2^{-j})| = 2^j |2^{-j+1}(j-1) - 2^{-j+1} j\big| = 2. \tag{A.22}$$

For $a = r2^{-(j-1)}, r \geq 1$

$$(A.21) \quad \lesssim 2^j 4^{-j} \|F''\|_{L^\infty(I)} = 2^{-j} \|\frac{1}{x}\|_{L^\infty(I)} \sim \frac{1}{r}. \tag{A.23}$$

It follows in particular from (A.22), (A.23) that

$$\|\Delta_j f\|_2^2 \leq C 2^{-j} \sum_{r \geq 1} r^{-2} = C 2^{-j}$$
$$\sum 4^{js} \|\Delta_j f\|_2^2 \leq C \sum 2^{-j(1-2s)} \sim (1 - 2s)^{-1}.$$

APPENDIX B

Functions in $\mathbf{W^{s,p}(\Omega; \mathbb{Z})}$ are constant when sp≥1.

A continuous function on a connected space with values into $\mathbb{Z}$ must be constant. Functions in the Sobolev space $W^{s,p}$ with $sp \geq 1$ have the same property although they need not be continuous. More precisely we have

Theorem B.1. *Assume Ω is a connected open set in $\mathbb{R}^n, n \geq 1$. Let $0 < s < \infty$ and $1 < p < \infty$ be such that*

$$sp \geq 1, \tag{B.1}$$

including $s = 1$ and $p = 1$. Then any function $f \in W^{s,p}(\Omega; \mathbb{Z})$ must be constant.

Remark B.1. Hardt, Kinderlehrer and Lin [1] have stated the same conclusion when $s = 1/2$ and $p = 2$ with a sketch of proof. Bethuel and Demengel [1] have also obtained the same result when $sp > 1$ and the proof we present follows their argument with an additional ingredient to cover the case $sp = 1$.

Proof. It is convenient to split the proof into two steps:

Step 1: the case $\mathbf{n = 1}$.

If $sp > 1$, the conclusion is obvious since f is continuous by the Sobolev imbedding theorem. If $sp = 1$, a borderline for the Sobolev imbedding, f need not be continuous, but f is VMO (see e.g. Brezis and Nirenberg [1], Section I.2). Therefore, the essential range of f is connected (see Brezis and Nirenberg [1], Section I.5) and thus f is constant. For the convenience of the reader we reproduce the argument. Set

$$f_\varepsilon(x) = \fint_{B_\varepsilon(x)} f(y)dy$$

and note that

$$\text{dist}(f_\varepsilon(x), \mathbb{Z}) \leq \fint_{B_\varepsilon(x)} |f(y) - f_\varepsilon(x)|dy \to 0$$

uniformly in x as $\varepsilon \to 0$ (since $f \in$ VMO). On the other hand $f_\varepsilon(\Omega)$ is connected and consequently there is some integer $k_\varepsilon \in \mathbb{Z}$ such that

$$\|f_\varepsilon - k_\varepsilon\|_{L^\infty} \to 0 \quad \text{as } \varepsilon \to 0.$$

It follows that $k_\varepsilon \to k$ as $\varepsilon \to 0$ with $k \in \mathbb{Z}$ and $f = k$ a.e. on Ω.

Step 2: the case $\boldsymbol{n} \geq \mathbf{2}$.

It suffices to prove that f is locally constant a.e. and thus we may assume, without loss of generality, that $\Omega = (0,1)^n$. For a.e. $x' = (x_1, \dots, x_{i-1}, x_{i+1}, \dots x_n)$ in $(0,1)^{n-1}$ the function

$$t \mapsto \psi(t) = f(x_1, \dots x_{i-1}, t, x_{i+1}, \dots x_n) \tag{B.2}$$

belongs to $W^{s,p}(0,1)$. This is a consequence of the fact that an equivalent norm for $W^{s,p}(\mathbb{R}^n)(0 < s < 1)$ is given by

$$|||f|||^p = \|f\|^p_{L^p} + \sum_{i=1}^n \int_0^1 \int_{\mathbb{R}^n} \frac{|f(x + te_i) - f(x)|^p}{t^{1+sp}} dxdt$$

where (e_i) denotes the canonical basis of $\mathbb{R}^n$ (see e.g. Adams [1], p.208-214). Applying Step 1 we know that for a.e. $x' \in (0,1)^{n-1}$ the function ψ is constant. To complete Step 2 we rely on the following simple measure theoretical lemma (see e.g. Lemma 2 in Brezis, Li, Mironescu and Nirenberg [1])

Lemma B.1. *Let $\Omega = (0,1)^n$ and let f be a measurable function on Ω such that for each $1 \le i \le n$ and for a.e. $x' = (x_1, \dots, x_{i-1}, x_{i+1}, \dots x_n)$ in $(0,1)^{n-1}$ the function ψ defined in (B.2) is constant a.e. on $(0,1)$. Then f is constant a.e. on Ω.*

Remark B.2. Assumption (B.1) cannot be weakened. Indeed, the characteristic function of any smooth domain ω compactly contained in Ω belongs to $W^{s,p}$ for any s, p with $sp < 1$.

Remark B.3. The conclusion of Theorem B.1 is still valid if $f : \Omega \to \mathbb{Z}$ is a sum of functions in different Sobolev space, i.e., $f = \sum_{i=1}^{k} f_i$ with $f_i \in W^{s_i,p_i}(\Omega;\mathbb{R})$ and $s_i p_i \ge 1$ for all i. The proof is identical to the one we have presented above. In particular the conclusion holds if $f \in H^{1/2} + W^{1,1}$; this fact will be used in our forthcoming paper Bourgain, Brezis and Mironescu [1].

APPENDIX C
Composition in fractional Sobolev spaces

We investigate here the question whether $\Phi \circ v$ belongs to $W^{s,p}(\Omega)$ when v belongs to $W^{s,p}(\Omega)$ and Φ is smooth. For simplicity we consider only the case $\Omega = \mathbb{R}^n$. Of course, here, we also assume that $\Phi(0) = 0$. The case of a domain can be treated by extending the functions to $\mathbb{R}^n$.

Lemma C.1. *Let $0 < s < \infty$ and $1 < p < \infty$. Assume*

$$v \in W^{s,p}(\Omega) \cap L^\infty(\Omega). \tag{C.1}$$

Then

$$\Phi \circ v \in W^{s,p}(\Omega). \tag{C.2}$$

Proof. When s is an integer the conclusion is easy via the Gagliardo-Nirenberg inequality. For example, when $s = 2$

$$D^2(\Phi \circ v) = \Phi'(v) D^2 v + \Phi''(v)(Dv)^2 \in L^p$$

since $W^{2,p} \cap L^\infty \subset W^{1,2p}$ by the Gagliardo-Nirenberg inequality. A similar argument holds for higher order derivatives.

We now turn to the case where s is fractional. The conclusion is obvious when $0 < s < 1$. Suppose now that $1 < s < 2$. One has to show that

$$D(\Phi \circ v) = \Phi'(v) Dv \in W^{s-1,p}.$$

This would require a lemma about products which eludes us.

Instead of this strategy one relies on a characterization of $W^{s,p}$ via finite differences. Set

$$(\delta_h u)(x) = u(x+h) - u(x),$$

so that

$$(\delta_h^2 u)(x) = u(x+2h) - 2u(x+h) + u(x).$$

Then

$$u \in W^{s,p} \Leftrightarrow \iint \frac{|\delta_h^2 u(x)|^p}{|h|^{n+sp}} dh dx < \infty, \tag{C.3}$$

(see Triebel [2], p.110).

The key observation is that $\delta_h^2(\Phi \circ v)$ can be estimated in terms of $\delta_h^2 v$ and $\delta_h v$. This is the purpose of our next computation.

Set

$$\begin{aligned} X &= v(x+2h) \\ Y &= v(x+h) \\ Z &= v(x). \end{aligned}$$

Then

$$\Phi(X) - \Phi(Y) = \Phi'(Y)(X-Y) + O(|X-Y|^2) \tag{C.4}$$

and

$$\Phi(Z) - \Phi(Y) = \Phi'(Y)(Z-Y) + O(|Z-Y|^2). \tag{C.5}$$

Since

$$\delta_h^2(\Phi \circ v)(x) = (\Phi(X) - \Phi(Y)) + (\Phi(Z) - \Phi(Y)),$$

one finds

$$|\delta_h^2(\Phi \circ v)(x)| \le C(|\delta_h^2 v(x)| + |\delta_h v(x+h)|^2 + |\delta_h v(x)|^2). \tag{C.6}$$

Consequently

$$\iint \frac{|\delta_h^2(\Phi \circ v)(x)|^p}{|h|^{n+sp}} \le C \iint \frac{|\delta_h^2 v(x)|^p}{|h|^{n+sp}} + C \iint \frac{|\delta_h v(x)|^{2p}}{|h|^{n+sp}}. \tag{C.7}$$

The first term on the righthand side of (C.7) is finite since $v \in W^{s,p}$ and for the second term we observe that

$$\iint \frac{|\delta_h v(x)|^{2p}}{|h|^{n+sp}} = \|v\|^{2p}_{W^{\frac{s}{2},2p}} \le C \|v\|^p_{L^\infty} \|v\|^p_{W^{s,p}}$$

by the Gagliardo-Nirenberg inequality (see Lemma D.1). Hence we have proved that $\Phi \circ v \in W^{s,p}$. The same argument extends to a general $s > 2$, s non integer (see e.g. Escobedo [1]).

APPENDIX D

Gagliardo-Nirenberg inequalities and products in fractional Sobolev spaces

We establish here some Gagliardo- Nirenberg type inequalities used in the paper. We also present a proof of Lemma 2 concerning products in fractional Sobolev spaces. These results are presumably known to the experts. For simplicity we work on $\mathbb{R}^n$; the case of a domain can be treated by extending the functions to $\mathbb{R}^n$.

Lemma D.1. *Let $0 < s < \infty, 1 < p < \infty$. Assume*

$$u \in W^{s,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

Then

$$u \in W^{r,q}, \ \ \forall\, r \in (0, s) \quad \text{with } q = \frac{sp}{r}, \tag{D.1}$$

and

$$|||u|||_{W^{r,q}} \le C \|u\|_{L^\infty}^{1-(r/s)} |||u|||_{W^{s,p}}^{r/s}, \tag{D.2}$$

provided that either (i) both r, s are non integers or (ii) r is an integer.

Here, we use the following semi-norm on $W^{s,p}$ (see e.g. Triebel [2]):

$$|||u|||_{W^{s,p}} = \begin{cases} \|D^s u\|_{L^p}, & \text{if } s \text{ is an integer} \\ (\iint \frac{|\delta_h^M u(x)|^p}{|h|^{n+sp}} dx dh)^{1/p} & \text{if } s \text{ is not an integer} \end{cases}$$

(as usual, $M > s$ is any integer).

Proof of Lemma D.1. It is convenient to observe that, for every $s \in (0, \infty)$ and every $p \in (1, \infty)$,

$$|||u|||^p_{W^{s,p}(\mathbb{R}^n)} \sim \int_{S^{n-1}} d\sigma \int_{y\cdot\sigma=0} |||u(t\sigma + y)|||^p_{W^{s,p}(\mathbb{R})} dy. \tag{D.3}$$

(When s is not an integer, (D.3) is clear. When s is an integer, (D.3) follows from the fact that the function

$$A \mapsto \left(\int_{S^{n-1}} |A(\sigma, \sigma, ..., \sigma)|^p d\sigma \right)^{\frac{1}{p}}$$

is a norm on the space of s-linear symmetric forms on $\mathbb{R}^n$.) Using (D.3) one sees that the proof of (D.2) reduces to the one-dimensional case.

Also, note that the desired inequality (D.2) is clear when both s and r are not integers. Indeed, in this case, we have, for $M > s$ (and hence $M > r$)

$$\begin{aligned} |||u|||^q_{W^{r,q}} &= \iint \frac{|\delta_h^M u(x)|^q}{|h|^{n+rq}} dxdh \le \|\delta_h^M u\|^{q-p}_{L^\infty} \iint \frac{|\delta_h^M u(x)|^p}{|h|^{n+rq}} dxdh \\ &\le C\|u\|^{q-p}_{L^\infty} |||u|||^p_{W^{s,p}}. \end{aligned}$$

Therefore, it suffices to establish (D.2) for $n = 1$ and $s \ge 1$. We follow the proof of Nirenberg [1]. By the Sobolev imbedding theorem, we have (since $sp > 1$),

$$W^{s,p}([0,1]) \subset W^{r,q}([0,1]).$$

Hence

$$|||u|||_{W^{r,q}([0,1])} \le C(\|u\|_{L^p([0,1])} + |||u|||_{W^{s,p}([0,1])}), u \in W^{s,p}([0,1]). \tag{D.4}$$

It then follows that

$$|||u|||_{W^{r,q}([0,1])} \le C(\|u\|_{L^\infty([0,1])} + |||u|||_{W^{s,p}([0,1])}), u \in W^{s,p}([0,1]). \tag{D.5}$$

By scaling, we find

$$\begin{aligned} |||u|||^q_{W^{r,q}([0,\ell])} &\le C(\ell^{1-sp}\|u\|^q_{L^\infty([0,\ell])} + \ell^{(\frac{s}{r}-1)(sp-1)}|||u|||^q_{W^{s,p}([0,\ell])}), \\ &= C(A(\ell) + B(\ell)), u \in W^{s,p}([0,\ell]). \end{aligned} \tag{D.6}$$

It clearly suffices to prove (D.2) in $[0,\infty)$ and we may assume that $\|u\|_{W^{s,p}} = 1$. Fix some $\varepsilon > 0$. We construct inductively a sequence of disjoint intervals $I_1, I_2, \dots$ such that

$$[0,+\infty) = I_1 \cup I_2 \cup \cdots$$

as follows:

We compare $A(\varepsilon)$ and $B(\varepsilon)$. If $B(\varepsilon) \ge A(\varepsilon)$, then we take $I_1 = [0,\varepsilon)$ and next construct I_2. Otherwise, note that $\lim_{\ell\to\infty} A(\ell) = 0$, $\lim_{\ell\to\infty} B(\ell) = \infty$ (unless $u \equiv 0$, which is not the case). Hence there is some $\varepsilon < \ell < \infty$ such that $A(\ell) = B(\ell)$. It then follows that

$$|||u|||^q_{W^{r,q}([0,\ell])} \le C\|u\|^{q-p}_{L^\infty([0,\ell])} |||u|||^p_{W^{s,p}([0,\ell])}.$$

In this case we take $I_1 = [0,\ell)$. We next start the above procedure from the endpoint of I_1. Since at each step we have $|I_j| \ge \varepsilon$, we clearly cover in this way $[0,\infty)$ with a sequence of intervals. Denote the first type of intervals by I_j and the second type by K_j. Using the assumption that r is an integer we have

$$\begin{aligned}|||u|||^q_{W^{r,q}([0,\infty))} &= \sum_{I_j} |||u|||^q_{W^{r,q}(I_j)} + \sum_{K_j} \cdots \\ &\le C\varepsilon^{(\frac{s}{r}-1)(sp-1)} \sum_{I_j} |||u|||^q_{W^{s,p}(I_j)} \\ &\quad + C\|u\|^{q-p}_{L^\infty(\mathbb{R})} \sum_{K_j} |||u|||^p_{W^{s,p}(K_j)}.\end{aligned}$$

Note that, since $q > p$, we have

$$\sum_{I_j} |||u|||^p_{W^{s,p}(I_j)} \le 1 \Rightarrow \sum_{I_j} |||u|||^q_{W^{s,p}(I_j)} \le 1.$$

Hence

$$|||u|||^q_{W^{r,q}([0,\infty])} \le C\varepsilon^{(\frac{s}{r}-1(sp-1)} + C\|u\|^{q-p}_{L^\infty(\mathbb{R})}|||u|||^p_{W^{s,p}(\mathbb{R})}. \tag{D.7}$$

We conclude by letting $\varepsilon \to 0$ in (D.7) (the constants C are independent of ε).

Remark D.1. The conclusion of Lemma D.1 fails when $s = 1$ and $p = 1$. For example $W^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is not contained in $W^{1/2,2}(\mathbb{R})$—because this would imply the inequality $\|u\|_{W^{1/2,2}} \le C\|u\|_{W^{1,1}}$ which is clearly wrong (use for example the sequence in Remark 3).

Remark D.2. In the general case (no restrictions on r and s), the conclusions of Lemma D.1 are still true (the remaining case, i.e., s integer and r non integer, is treated in T. Runst [1], Lemma 5.2.1).

We next prove a regularity result for products in Sobolev spaces.

Lemma D.2. *Let $n \ge 1$, $1 < s < \infty, 1 < p < \infty$. Let $u, v \in W^{s,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then*

$$uDv \in W^{s-1,p}(\mathbb{R}^n).$$

Proof of Lemma D.2. If s is an integer, the conclusion follows easily from the Gagliardo-Nirenberg inequality. We henceforth assume that s is not an integer.

We use a Littlewood- Paley decomposition technique (see e.g. Bony [1], Alinhac and Gérard [1] or Chemin [1]). Let $\psi_0 \in C_0^\infty(\mathbb{R}^n)$ be such that

$$\psi_0(\xi) = 1 \quad \text{if } |\xi| \le 1 \quad \text{and } \psi_o(\xi) = 0 \quad \text{if } |\xi| \ge 2.$$

Set

$$\psi_j(\xi) = \psi_0(2^{-j}\xi) - \psi_0(2^{-j+1}\xi),\ j \ge 1 \quad \text{and } \varphi_j = \mathcal{F}^{-1}(\psi_j),\ j \ge 0.$$

For $f \in \mathcal{S}'$, let $f_j = f * \varphi_j$, so that $f = \sum_{j\geq 0} f_j$ in $\mathcal{S}'$.

We have $uDv = \sum(r_j + s_j)$, where

$$r_j = u_j \sum_{k\leq j-1} Dv_k \quad \text{ and } s_j = Dv_j \sum_{k\leq j} u_k.$$

Since clearly

$$\Big\| \sum_{k\leq j} \varphi_k \Big\|_{L^1} \leq C, \quad \Big\| \sum_{k\leq j} D\varphi_k \Big\|_{L^1} \leq C2^j, \quad \forall j \geq 0,$$

we obtain

$$\Big\| \sum_{k\leq j} v_k \Big\|_{L^q} \leq C\|v\|_{L^q}, \quad \forall q, \tag{D.8}$$

$$\Big\| \sum_{k\leq j} Dv_k \Big\|_{L^q} \leq C2^j\|v\|_{L^q}, \quad \forall q, \tag{D.9}$$

and the same inequalities hold for u. Therefore,

$$\|r_j\|^p_{L^p} \leq C\|u_j\|^p_{L^p} \Big\| \sum_{k\leq j-1} Dv_k \Big\|^p_{L^\infty} \leq C2^{jp}\|u_j\|^p_{L^p}\|v\|^p_{L^\infty}. \tag{D.10}$$

On the other hand, $v_j = \sum_{k\leq j+2}(v_j)_k$, since, for $k \geq j+3$,

$$\mathcal{F}((v_j)_k) = \mathcal{F}(v)\psi_j\psi_k = 0.$$

Therefore,

$$\|Dv_j\|_{L^q} = \Big\| \sum_{k\leq j+2} D(v_j)_k \Big\|_{L^q} \leq C2^j\|v_j\|_{L^q}, \ \forall q,$$

by (D.9) applied to v_j. Consequently,

$$\|s_j\|^p_{L^p} \leq C\|u\|^p_{L^\infty}\|Dv_j\|^p_{L^p} \leq C2^{jp}\|v_j\|^p_{L^p}\|u\|^p_{L^\infty}. \tag{D.11}$$

We now recall two basic facts about $W^{\sigma,p}$, $\sigma > 0$, σ non integer, $1 < p < \infty$. Let $f \in W^{\sigma,p}$ and let $f_j = f * \varphi_j$ as above. Then

$$\|f\|^p_{W^{\sigma,p}} \sim \sum_{j\geq 0} 2^{\sigma jp}\|f_j\|^p_{L^p} \tag{D.12}$$

(see e.g. Triebel [2], p. 46).

Conversely, let g_j be a sequence in L^p such that $supp\mathcal{F}(g_j) \subset B_{2^j}$. Then

$$\Big\| \sum_{j\geq 0} g_j \Big\|^p_{W^{\sigma,p}} \leq C \sum_{j\geq 0} 2^{\sigma jp} \|g_j\|^p_{L^p} \tag{D.13}$$

(see e.g. Chemin [1], p. 27). Using (D.10), (D.11) and (D.12) (with $\sigma = s$), we find

$$\sum_{j\geq 0} 2^{(s-1)jp} \|r_j + s_j\|^p_{L^p} \leq C\big(\|u\|^p_{L^\infty} \|v\|^p_{W^{s,p}} + \|v\|^p_{L^\infty} \|u\|^p_{W^{s,p}}\big). \tag{D.14}$$

Since $supp\mathcal{F}(r_j + s_j) \subset B_{2^{j+3}}$, (D.13) (applied with $\sigma = s-1$ and $g_j = r_j + s_j$) combined with (D.14) yields that $uDv \in W^{s-1,p}$ and that

$$\|uDv\|_{W^{s-1,p}} \leq C(\|u\|_{L^\infty} \|v\|_{W^{s,p}} + \|v\|_{L^\infty} \|u\|_{W^{s,p}}). \tag{D.15}$$

Remark D.3. As a consequence of Lemma D.2, we derive the well-known fact that $W^{s,p} \cap L^\infty$ is an algebra.

APPENDIX E

Behaviour of the H^s-norm of lifting for $s \overset{\to}{<} \frac{1}{2}$. Proof of Theorem 4

We return to the particular issue of lifting of an unimodular function F in H^s, $s < \frac{1}{2}$. As we have pointed out in Section 5 the construction described in Appendix A of a lifting

$$F = e^{i\varphi}, \ \varphi \in H^s \tag{E.1}$$

does not lead to the optimal estimate of $\|\varphi\|_{H^s}$ when $s \to \frac{1}{2}$. Our aim is to prove

Theorem E.1. *Let Q be a cube of $\mathbb{R}^d, d \geq 1$. For every $F \in H^s(Q; S^1)$ with $0 < s < 1/2$ one may construct a $\varphi \in H^s(Q;\mathbb{R})$ satisfying (E.1) and the (optimal) estimate*

$$\|\varphi\|_{H^s} \leq C(1-2s)^{-1/2} \|F\|_{H^s} \tag{E.2}$$

where C is a constant independent of F and independent of s as $s \to 1/2$.

Proof. Given an unimodular H^s-function F on a cube, say $Q = [0, \frac{1}{2}]^d \subset \mathbb{R}^d$, we may extend F to a 1-periodic unimodular function in $H^s_{loc}(\mathbb{R}^d)$ by the usual procedure of reflections and periodic continuation. Hence, we may assume $F \in H^s(\mathbb{T}^d; S^1)$, where $\mathbb{T}^d = d$-dim torus. This setting is particularly convenient to perform our translation averaging. On $\Omega = \mathbb{T}^d$, we fix again a system $\{\mathcal{P}_j\}_{j=0,1,2,\dots}$ of refining dyadic partitions (thus the atoms of $\mathcal{P}_j$ are d-intervals of size $\sim 2^{-j}$) and denote E_j the corresponding expectation operators. Denote also τ_θ the shift operators on $\mathbb{T}^d$.

We perform the following construction. Given $F \in H^s(\Omega; S^1)$, denote $F_\theta = F \circ \tau_\theta$ and $\varphi[\theta]$ the lifting of F_θ gotten from the construction described in Section 1 (with fixed $\mathcal{P}_j$'s). Thus

$$F_\theta = e^{i\varphi[\theta]} \text{ and } F = e^{i(\varphi[\theta]\circ\tau_{-\theta})} \tag{E.3}$$

and $\varphi[\theta] \circ \tau_{-\theta} = \varphi$ is a lifting for F. Thus Theorem 4 will follow immediately from the next statement.

Lemma E.1. *We have*

$$\int_{\mathbb{T}^d} \|\varphi[\theta]\|_{H^s} d\theta \le C(1-2s)^{-1/2}\|F\|_{H^s}.$$

Proof. We show in fact that

$$\int \|\varphi[\theta]\|^2_{H^s} d\theta \le C(1-2s)^{-1}\|F\|^2_{H^s}. \tag{E.4}$$

The lefthand side of (E.4) equals

$$\begin{aligned}&\iiint \frac{|\varphi[\theta]-\tau_h\varphi[\theta]|^2(x)}{|h|^{2s+d}}\, dxdhd\theta\\ &\sim \sum_{j\ge 0} 2^{(2s+d)j}\iint_{|h|\sim 2^{-j}} \|\varphi[\theta]-\tau_h\varphi[\theta]\|_2^2 dhd\theta.\end{aligned} \tag{E.5}$$

Denote $\varphi[\theta]$ by φ for simplicity. Fix j.
Writing

$$\varphi = E_j\varphi + \sum_{j'>j}\Delta_{j'}\varphi \qquad (\Delta_{j'} = E_{j'} - E_{j'-1}) \tag{E.6}$$

estimate

$$\|\varphi-\tau_h\varphi\|_2^2 \lesssim \|E_j\varphi - \tau_h E_j\varphi\|_2^2 + \sum_{j'>j}(j'-j)^2\|\Delta_{j'}\varphi\|_2^2. \tag{E.7}$$

Recall inequality (1.5) in Section 1

$$|\varphi_j - \varphi_{j-1}| \le C(|F_\theta - E_j(F_\theta)| + |F_\theta - E_{j-1}(F_\theta)|). \tag{E.8}$$

Hence, since $\varphi_j = E_j(\varphi_j)$, we have

$$\begin{aligned}\|\Delta_j\varphi\|_2 &\le \|E_j(\varphi-\varphi_j)\|_2 + \|E_{j-1}(\varphi-\varphi_{j-1})\|_2 + \|\varphi_j-\varphi_{j-1}\|_2 && \text{(E.9)}\\ &\le C\sum_{j'\ge j}\|\varphi_{j'}-\varphi_{j'-1}\|_2\\ &\le C\sum_{j'\ge j-1}\|F_\theta - E_{j'}(F_\theta)\|_2\\ &\le C\sum_{j'\ge j-1}(j'-j+2)\|\Delta_{j'}F_\theta\|_2 && \text{(E.10)}\end{aligned}$$

and estimate in (E.7)

$$\|\Delta_{j'}\varphi\|_2^2 \le C \sum_{j''\ge j'-1} (j''-j'+2)^4 \|\Delta_{j''}F_\theta\|_2^2. \tag{E.11}$$

Thus the contribution of the second term in (E.7) is bounded by

$$\begin{aligned}
&\sum_{j\ge 0} 2^{(2s+d)j} \iint_{|h|\sim 2^{-j}} \Big\{ \sum_{j'>j} (j'-j)^2 \|\Delta_{j'}\varphi\|_2^2 \Big\} dh d\theta \\
&\le C \int d\theta \Big\{ \sum_{j\ge 0} 2^{2sj} \sum_{j''+2\ge j'>j} (j'-j)^2 (j''-j'+2)^4 \|\Delta_{j''}F_\theta\|_2^2 \Big\} \\
&\le C \int d\theta \Big\{ \sum_{j''>0} 2^{2sj''} \|\Delta_{j''}F_\theta\|_2^2 \Big\}.
\end{aligned} \tag{E.12}$$

Recalling the proof of Theorem A1 (in particular the inequality $Y \le CX$ independent of the assumption $2s<1$) we have

$$(E.12) \le C \int d\theta \|F_\theta\|_{H^s}^2 \le C\|F\|_{H^s}^2. \tag{E.13}$$

Thus the θ-integration is irrelevant here.

The main point is the contribution of the first term $\|E_j\varphi - \tau_h E_j\varphi\|_2^2$ in (E.5), thus

$$\sum_{j\ge 0} 2^{(2s+d)j} \iint_{|h|\sim 2^{-j}} \int |E_j\varphi - \tau_h E_j\varphi|^2 d\theta dh dx. \tag{E.14}$$

Estimate

$$|E_j\varphi - \tau_h E_j\varphi| \le \sum_{j'\le j} |\Delta_{j'}\varphi - \tau_h \Delta_{j'}\varphi|. \tag{E.15}$$

Write

$$\Delta_{j'}\varphi = \sum_{I\in\mathcal{P}_{j'}} a_I \chi_{_I}. \tag{E.16}$$

Then, for $|h|<2^{-j}$, one easily verifies that

$$|\Delta_{j'}\varphi - \tau_h\Delta_{j'}\varphi| \le \sum_{I\in\mathcal{P}_{j'}} |a_I|\,|\chi_{_I} - \tau_h\chi_{_I}| \le C(|\Delta_{j'}\varphi| * P_{2^{-j'}})\chi_{j',2^{-j}} \tag{E.17}$$

where $\chi_{j',2^{-j}}$ denotes the characteristic function of the set

$$\{x; \text{dist } (x, \partial I) \le 2^{-j} \text{ for some } I \in \mathcal{P}_{j'}\} \tag{E.18}$$

and P_ε denotes the usual Poisson-kernel for instance.
Thus

$$\int \chi_{j',2^{-j}} = \text{mes } (E.18) \le C 2^{j'd} 2^{-j'(d-1)} 2^{-j} \le C 2^{j'-j}. \tag{E.19}$$

Substituting (E.17) in (E.15) implies (since $\cup_{I\in\mathcal{P}_{j_1'}} \partial I \subset \cup_{I\in\mathcal{P}_{j_2'}} \partial I$ for $j_1' < j_2'$)

$$|E_j\varphi - \tau_h E_j \varphi|^2 \le \sum_{\substack{j_1'\le j, j_2'\le j\\ j_1'\le j_2'}} (|\Delta_{j_1'}\varphi| * P_{2^{-j_1'}})(|\Delta_{j_2'}\varphi| * P_{2^{-j_2'}})\chi_{j_1',2^{-j}}. \tag{E.20}$$

Next,

$$\Delta_{j'}\varphi = E_{j'}(\varphi - \varphi_{j'}) - E_{j'-1}(\varphi - \varphi_{j'-1}) + \varphi_{j'} - \varphi_{j'-1} \tag{E.21}$$

and again from inequality (E.8)

$$|\varphi - \varphi_{j'}| \le C \sum_{j''>j'} (j'' - j')|\Delta_{j''} F_\theta|. \tag{E.22}$$

We get

$$|\Delta_{j'}\varphi| * P_{2^{-j'}} \le C \sum_{j''\ge j'} (j'' - j' + 1)(|\Delta_{j''}F_\theta| * P_{2^{-j'}}). \tag{E.23}$$

Substituting (E.23) in (E.20) and then in (E.14) gives

$$\sum_{j\ge 0} 2^{2sj} \iint dx d\theta \sum_{\substack{j_1'\le j, j_2'\le j, j_1'\le j_2'\\ j_1''\ge j_1', j_2''\ge j_2'}} (j_1'' - j_1' + 1)(j_2'' - j_2' + 1)(|\Delta_{j_1''}F_\theta| * P_{2^{-j_1'}})(|\Delta_{j_2''}F_\theta| * P_{2^{-j_2'}})\chi_{j_1',2^{-j}}(x). \tag{E.24}$$

The role of the θ-translation is that we introduced an extra variable to estimate (E.24). Write F as a Fourier series in $\mathbb{T}^d$

$$F = \sum_{n\in\mathbb{Z}^d} \widehat{F}(n) e^{inx}.$$

Then

$$\Delta_j(F_\theta) = \sum \widehat{F}(n) e^{in\theta} \Delta_j(e^{in\cdot}) \tag{E.25}$$

$$|(|\Delta_j F_\theta| * P_\varepsilon)(x)|^2 \le \int \Big| \sum \widehat{F}(n) e^{in\theta} \Delta_j(e^{in\cdot})(x-y)|^2 P_\varepsilon(y) dy. \tag{E.26}$$

Integrating (E.26) in θ gives clearly

(E.27) $$\| |\Delta_j F_\theta| * P_\varepsilon\|^2_{L^2_\theta} \leq \sum |\widehat{F}(n)|^2 \, \|\Delta_j(e^{in\cdot})\|^2_\infty \lesssim \sum |\widehat{F}(n)|^2 (1 \wedge |n|2^{-j})^2.$$

To estimate (E.24), perform first the θ-integration using Cauchy-Schwarz and (E.27).
This gives, recalling (E.19)

(E.28)
$$\sum_{j\geq 0} 2^{2sj} \sum_{j'_\alpha \leq j, j'_\alpha \leq j''_\alpha, j'_1 \leq j'_2} 2^{j'_1 - j}(j''_1 - j'_1 + 1)(j''_2 - j'_2 + 1)\Big[\sum_n |\widehat{F}(n)|^2 (1 \wedge |n|2^{-j''_1})^2\Big]^{1/2} \Big[\sum_n |\widehat{F}(n)|^2 (1 \wedge |n|2^{-j''_2})^2\Big]^{1/2}.$$

To evaluate (E.28), denote

(E.29) $$\ell_\alpha = j''_\alpha - j'_\alpha \geq 0 \quad (\alpha = 1, 2)$$

(E.30) $$m = j'_2 - j'_1 \geq 0$$

so that

(E.31) $$(E.28) = \sum_{m, \ell_1 . \ell_2 \geq 0} (\ell_1 + 1)(\ell_2 + 1) \sum_{j'_1} 2^{j'_1} \Big(\sum_{j \geq j'_1} 2^{(2s-1)j}\Big). \Big[\sum_n |\hat{F}(n)|^2 (1 \wedge |n|2^{-j'_1 - \ell_1})^2\Big]^{1/2} \Big[\sum_n |\hat{F}(n)|^2 (1 \wedge |n|2^{-j'_1 - m - \ell_2})^2\Big]^{1/2}.$$

Applying Cauchy-Schwarz for the j'_1-summation

(E.32)
$$(E.31) \leq C \sum_{m, \ell_1, \ell_2} (\ell_1 + 1)(\ell_2 + 1)(1 - 2s)^{-1} \Big[\sum_{n, j'_1} |\hat{F}(n)|^2 2^{2sj'_1} (1 \wedge |n|2^{-j'_1 - \ell_1})^2\Big]^{1/2} \Big[\sum_{n, j'_1} |\hat{F}(n)^2 2^{2sj'_1} (1 \wedge |n|2^{-j'_1 - m - \ell_2})^2\Big]^{1/2}.$$

Writing

(E.33) $$\sum_j 2^{2sj} (1 \wedge |n|2^{-j-\ell})^2 \sim 2^{-2s\ell} (1 + |n|)^{2s}$$

it follows that

(E.34) $$\begin{aligned}(E.32) &\leq \frac{C}{1 - 2s} \sum_{m, \ell_1, \ell_2} (\ell_1 + 1)(\ell_2 + 1) 2^{-s(\ell_1 + \ell_2 + m)} \Big(\sum_n |\hat{F}(n)|^2 (1 + |n|)^{2s}\Big) \\ &\leq C(1 - 2s)^{-1} \|F\|^2_{H^s}.\end{aligned}$$

Since (E.5) is bounded by the sum of (E.13) and (E.34), this proves Lemma E.1.

Remark E.1. The optimality of the bound (E.2) when $d = 2$ was proved in Remark 7. The case $d \geq 3$ is similar by choosing

$$g(x) = \frac{(x_1, x_2)}{(x_1^2 + x_2^2)^{1/2}} \qquad x = (x_1, x_2, \ldots, x_d)$$

and proceeding as in the 2-dimensional case. The optimality of (E.2) when $d = 1$ is more delicate and will be established in the forthcoming paper Bourgain, Brezis and Mironescu [1].

Remark E.2. Theorem E.1 is still valid if the cube Q is replaced by a smooth domain Ω in $\mathbb{R}^d, d \geq 2$ (without any topological assumption on Ω). The proof can be modified as follows. Consider a neighborhood $\tilde{\Omega}$ of $\bar{\Omega}$ and a function still denoted F, $F \in H^s(\tilde{\Omega}; S^1)$ which extends the original F (this can be done by the standard procedure of local reflexion across the boundary). Next, construct a finite sequence of disjoint cubes (Q_α), having the same size, and such that $\Omega \subset \bigcup\limits_\alpha Q_\alpha \Subset \tilde{\Omega}$. The construction described in Section 1 is still valid on $\bigcup\limits_\alpha Q_\alpha$ and we obtain a lifting $\varphi \in H^s(\bigcup\limits_\alpha Q_\alpha; \mathbb{R})$. For $\theta \in \mathbb{R}^d$ with $|\theta| < \delta, \delta$ sufficiently small, $F_\theta = F \circ \tau_\theta$ is well defined on $\bigcup\limits_\alpha Q_\alpha$ has a lifting $\varphi[\theta]$. The proof of Lemma E.1 described above can be adapted and yields

$$\int_{|\theta\| < \delta} \|\varphi[\theta]\|_{H^s} d\theta \leq C(1 - 2s)^{-1/2} \|F\|_{H^s}.$$

Theorem E.1 is also valid if the cube Q is replaced by a smooth d-dimensional manifold M, $d \geq 1$, say without boundary. The dyadic partition of Q is replaced by some dyadic "triangulation" of M. The shift operators τ_θ are replaced by a finite family $\{S_i(t)\}$, $1 \leq i \leq N$ of 1-parameter group of transformations on M such that, at each $x \in M$, the generators $V_i(x) = \frac{d}{dt} S_i(t) x_{|_{t=0}}$ span the tangent space $T_x(M)$. Such a family can be easily constructed as integral curves for the differential equations $\dot{x}(t) = V_i(x(t))$ and the vector-fields $V_i(x)$ are obtained via local coordinates and a partition of unity. The shift operators τ_θ are replaced by the shifts along the S_i, i.e., $\sigma_\theta = \prod\limits_i S_i(t_i)$, where $\theta = (t_1, t_2, \ldots, t_N)$, and then $F_\theta = F \circ \sigma_\theta$. Adapting the proof of Lemma E.1 we find

$$\int_{\theta \in \mathbb{R}^N, |\theta| < 1} \|\varphi[\theta]\| d\theta \leq C(1 - 2s)^{-1/2} \|F\|_{H^s}.$$

APPENDIX F

Martingale representation and lifting in $\mathbf{H^{s,p}}$

The question of representation and lifting can be raised in other function spaces. For instance, in the $H^{s,p}$ space.

Recall the definition of the $H^{s,p}$-norm $(0 < s < 1)$

$$\|f\|_{H^{s,p}} = \left[\int \left(\int \frac{|f(x+h) - f(x)|^2}{|h|^{2s+d}} dh \right)^{p/2} dx \right]^{1/p}. \tag{F.1}$$

This space is a bit more delicate to deal with then $W^{s,p}$. The natural martingale counterpart of (F.1) is given by

$$\left\| \left(\sum 2^{2js} |\Delta_j f|^2 \right)^{1/2} \right\|_p \tag{F.2}$$

where $\Delta_j f = E_j(f) - E_{j-1}(f)$ and E_j is the conditional expectation operator with respect to $\mathcal{P}_j$ (as before). This situation is a bit different from $W^{s,p}$. We show the following

Proposition F.1. (i) *We have*

$$\big\| \big(\sum 4^{js} |\Delta_j f|^2 \big)^{1/2} \big\|_p \leq C |f\|_{H^{s,p}} \tag{F.3}$$

(ii) *If $sp < 1$ and $p \geq 2$, then the converse inequality holds*

$$\|f\|_{H^{s,p}} \leq C \left\| \left(\sum 4^{js} |\Delta_j f|^2 \right)^{1/2} \right\|_p \tag{F.4}$$

(iii) *Inequality (F.4) fails for $s > \frac{1}{2}$.*

Proposition F.1 leaves some cases unanswered and they will possibly be addressed elsewhere. Again, Proposition F.1 is relevant to the question of Triebel [1] concerning the representation of Besov and Sobolev spaces in the Haar-system. It implies that for the spaces $H^{s,p} = F^s_{p,2}$, the conjecture is valid if $ps < 1, p \geq 2$ but fails for $s > \frac{1}{2}$.

In the proof of Proposition F.1, we will make use of some standard martingale inequalities (which the reader may find in Garsia [1] for instance).

Proposition F.2. *We have*

$$\Big\| \sum E_j(g_j) \Big\|_p \le C_p \Big\| \sum |g_j| \Big\|_p \quad \textit{for} \ \ 1 \le p < \infty \tag{F.5}$$

and

$$\Big\| \Big(\sum |E_j(g_j)|^2 \Big)^{1/2} \Big\|_p \le C_p \Big\| \Big(\sum |g_j|^2 \Big)^{1/2} \Big\|_p \quad \textit{for} \ \ 1 < p < \infty. \tag{F.6}$$

In both statements, the sequence $\{g_j\}$ consists of arbitrary functions.

Remark F.1. In (F.5), (F.6), the expectation operators E_j may get replaced by convolution operator $P_{2^{-j}}$ for instance, where P_ε stands for the usual Poisson kernel (cf. Stein [l]).

Proof of Proposition F.1.

(i) By (F.6)

$$\Big\| \Big(\sum 4^{js} |\Delta_j f|^2 \Big)^{1/2} \Big\|_p \le C \Big\| \Big(\sum 4^{js} |f - E_{j-1}(f)|^2 \Big)^{1/2} \Big\|_p . \tag{F.7}$$

Again

$$|(f - E_{j-1}(f))(x)| \le 2^{jd} \int_{|h| < 2^{-j}} |f(x) - f(x+h)| dh$$

$$|f - E_{j-1}(f)|^2 \le 2^{jd} \int_{|h| < 2^{-j}} |f - \tau_h f|^2 dh. \tag{F.8}$$

where τ_h is the translation operator.

Substituting (F.8) in (F.7) implies

$$\begin{aligned} (F.7) &\le \Big\| \Big\{ \int dh \ |f - \tau_h f|^2 \Big[\sum_{|h| < 2^{-j}} 4^{js} 2^{jd} \Big] \Big\}^{1/2} \Big\|_p \\ &\sim \Big\| \Big\{ \int \ |f - \tau_h f|^2 \ |h|^{-(d+2s)} dh \Big\}^{1/2} \Big\|_p \\ &= \|f\|_{H^{s,p}} . \end{aligned} \tag{F.9}$$

(ii) Write

$$\int |f-\tau_h f|^2\,|h|^{-(d+2s)}dh \sim \sum_j 2^{j(d+2s)}\int_{|h|\sim 2^{-j}} |f-\tau_h f|^2 dh. \tag{F.10}$$

Fix j. Estimate

$$\begin{aligned} &|f-\tau_h f| \le |f_j-\tau_h f_j| + |f-f_j| + \tau_h|f-f_j| \\ &|f-\tau_h f|^2 \lesssim \sum_{j'<j}(j-j')^2|\Delta_{j'}f-\tau_h(\Delta_{j'}f)|^2 + |f-f_j|^2 + \tau_h|f-f_j|^2 \end{aligned} \tag{F.11}$$

and substituting (F.11) in (F.10), we get the following contributions

$$(F.10) \ \le C\sum_{j'<j} 2^{j(d+2s)}(j-j')^2\int_{|h|\sim 2^{-j}} |\Delta_{j'}f-\tau_h(\Delta_{j'}f)|^2 dh \tag{F.12}$$

$$+\sum_j 4^{js}|f-f_j|^2 \tag{F.13}$$

$$+\sum_j 4^{js}[P_{2^{-j}}*(|f-f_j|^2)]. \tag{F.14}$$

Contribution of (F.13)

Write

$$\begin{aligned} \|(F.13)^{1/2}\|_p &\le \left\|\left[\sum_j 4^{js}\sum_{j'\ge j}(j'-j)^2|\Delta_{j'}f|^2\right]^{1/2}\right\|_p \\ &\sim \left\|\left(\sum_j 4^{j's}\,|\Delta_{j'}f|^2\right)^{1/2}\right\|_p. \end{aligned} \tag{F.15}$$

Contribution of (F.14)

$$\|(F.14)^{1/2}\|_p = \left\{\int\left\{\sum_j 4^{js}[P_{2^{-j}}*(|f-f_j|^2)]\right\}^{p/2}\right\}^{\frac{1}{p}}. \tag{F.16}$$

Use the general inequality (see Remark F.1)

$$\left\|\sum_j P_{2^{-j}}g_j\right\|_q \le C_q\left\|\sum_j |g_j|\right\|_q \quad \text{for} \ \ 1\le q<\infty. \tag{F.17}$$

Thus, since $p \geq 2$, letting $q = p/2$ in (F.17), it follows

$$(F.16) \leq C\Big[\int\Big(\sum_j 4^{js}\,|f - f_j|^2\Big)^{p/2}\Big]^{1/p}$$

$$\leq C\Big\|\Big(\sum_j 4^{js}|\Delta_j f|^2\Big)^{1/2}\Big\|_p. \tag{F.18}$$

Contribution of (F.12)

Denoting $\ell = j - j' \geq 0$, write

(F.19)

$$\|(F.12)^{1/2}\|_p \leq \sum_{\ell\geq 0} \ell 2^{\ell s}\Big(\Big\|\Big[\sum_{j'} 4^{j's}\Big(2^{(j'+\ell)d}\int_{|h|\leq 2^{-(j'+\ell)}} |\Delta_{j'}f - \tau_h(\Delta_{j'}f)|^2 dh\Big)\Big]^{1/2}\Big\|_p.$$

To bound (F.19), fix ℓ and consider the map

$$T_\ell : L^p_{\ell^2} \to L^p_{L^2_h \ell^2} \tag{F.20}$$

defined by

$$T_\ell \bar{g} = T_\ell(\{g_j\}) = \{(E_j g_j - \tau_h E_j g_j)2^{(j+\ell)d/2}\ \chi_{[|h|<2^{-(j+\ell)}]}\} \tag{F.21}$$

Thus the components of $T_\ell \bar{g}$ are functions of x and h.

Denote $\|T_\ell\|_p$ the norm of (F.20). We estimate $\|T_\ell\|_p$, $2 \leq p$, by interpolation between 2 and some large q.

Fixing $2 < q < \infty$, we may bound

$$\|T_\ell \bar{g}\|_{L^q_{L^2_h\ell^2}} \leq \|E_j|g_j|.2^{(j+\ell)d/2}\chi_{[|h|<2^{-(j+\ell)}]}\|_{L^q_{L^2_h\ell^2}} + \|\tau_h(E_j|g_j|).2^{(j+\ell)d/2}\ \chi_{[|h|<2^{-(j+\ell)}]}\|_{L^q_{L^2_h\ell^2}}$$
$$= (F.22) + (F.23).$$

Thus, invoking (F.6)

$$(F.22) \sim \Big\|\Big[\sum (E_j|g_j|)^2\Big]^{1/2}\Big\|_q \leq C_q\|\bar{g}\|_{L^q_{\ell^2}}. \tag{F.24}$$

Also, since $q > 2$ and using inequalities (F.17), (F.6)

$$(F.23) \leq C \Big\| \Big[\sum_j (E_j|g_j|)^2 * P_{2^{-(j+\ell)}} \Big]^{1/2} \Big\|_q = \Big\| \sum_j (E_j|g_j|)^2 * P_{2^{-(j+\ell)}} \Big\|_{q/2}^{1/2}$$

$$\leq C \Big\| \sum_j (E_j|g_j|)^2 \Big\|_{q/2}^{1/2} \leq C\|\bar{g}\|_{L^q_{\ell^2}}. \tag{F.25}$$

Thus $\|T_\ell \bar{g}\|_{L^q_{L^2_h \ell^2}} \leq C_q \|\bar{g}\|_{L^q_{\ell^2}}$, i.e.

$$\|T_\ell\|_q \leq C_q \quad \text{for} \quad 2 \leq q < \infty. \tag{F.26}$$

Next, for $p = 2$, a direct calculation gives

(F.27)

$$\|T_\ell \bar{g}\|_{L^2_x L^2_h \ell^2} = \Big[\sum_j 2^{(j+\ell)d} \iint_{|h|<2^{-(j+\ell)}} |(E_j g_j)(x) - (E_j g_j)(x+h)|^2 dx dh \Big]^{1/2}$$

$$\leq C 2^{-\ell/2} \Big(\sum_j \|E_j g_j\|_2^2 \Big)^{1/2} \tag{F.28}$$

$$\leq C 2^{-\ell/2} \|\bar{g}\|_{L^2_{\ell^2}}. \tag{F.29}$$

The estimate (F.28) simply results from the fact that for $I \in \mathcal{P}_j$ and $|h| < 2^{-(j+\ell)}$

$$\|\chi_I(x) - \chi_I(x+h)\|_{L^2_x} \leq C 2^{(-d-1)j/2 - \frac{j+\ell}{2}} = C 2^{-\ell/2} 2^{-dj/2}. \tag{F.30}$$

¿From (F.29),

$$\|T_\ell\|_2 \leq C 2^{-\ell/2}. \tag{F.31}$$

Interpolating $2 < p < q$, it results from (F.26), (F.31) that

$$\|T_\ell\|_p < C_\varepsilon 2^{-\ell(\frac{1}{p} - \varepsilon)} \text{ for all } \varepsilon > 0. \tag{F.32}$$

Returning to (F.19), we define thus

$$g_{j'} = 2^{j's} \Delta_{j'} f \tag{F.33}$$

so that, by (F.32)

$$
\begin{aligned}
(F.19) &\leq \sum_{\ell \geq 0} \ell 2^{\ell s} \|T_\ell\{g_{j'}\}\|_{L^p_{L^2_h \ell^2}} \\
&\leq C_\varepsilon \sum_{\ell \geq 0} \ell 2^{\ell s} 2^{-\ell(\frac{1}{p}-\varepsilon)} \ \|\{g_{j'}\}\|_{L^p_{\ell^2}}.
\end{aligned}
\tag{F.34}
$$

Since $sp < 1$, we may take ε sufficiently small to ensure boundedness of the factor in (F.34), leading again to the bound $\big\|\big(\sum 4^{j's}|\Delta_j f|^2\big)^{1/2}\big\|_p$.

This establishes inequality (F.4).

(iii) Take $d = 1$ and define

$$
f_j = 2^{-js} \sum_{r=1}^{2^j} (-1)^r \chi_{I_r} \text{ where } \mathcal{P}_j = \{I_1, \dots, I_{2^j}\}. \tag{F.35}
$$

Fix a large integer R and let $\{j_r\}_{r=1,\dots,R}$ be a lacunary sequence.

Define

$$
f = \sum_{r=1}^{R} \varepsilon_r f_{j_r} \tag{F.36}
$$

where $\varepsilon_r = \pm 1$ are independent signs. Thus $\Delta_{j_r} f = \varepsilon_r f_{j_r}$ and trivially

$$
\left\| \left(\sum 4^{js} |\Delta_j f|^2 \right)^{1/2} \right\|_p = R^{1/2}. \tag{F.37}
$$

Next, take $\delta > 0$ a small number and write

$$
\int |f - \tau_h f|^2 |h|^{-(1+2s)} dh \geq \sum_{r=1}^{R} (\delta 2^{-j_r})^{-(1+2s)} \int_{|h|<\delta 2^{-j_r}} |f - \tau_h f|^2 dh. \tag{F.38}
$$

Averaging over the $\pm$ signs ε_r in (F.36) permits us clearly to ensure that

$$
(F.38) \geq \sum_r (\delta 2^{-j_r})^{-(1+2s)} \int_{|h|<\delta 2^{-j_r}} |f_{j_r} - \tau_h f_{j_r}|^2 dh. \tag{F.39}
$$

Recalling (F.35), one sees that

(F.40)
$$(F.39) \geq c \sum_r (\delta 2^{-j_r})^{-(1+2s)} (\delta 2^{-j_r}) 4^{-j_r s} \sum_{I \in \mathcal{P}_{j_r}} \chi_{[\text{dist } (x,\partial I) < \frac{1}{2}\delta 2^{-j_r}]}$$

(F.41)
$$= c\delta^{-2s} \sum_r \sum_{I \in \mathcal{P}_{j_r}} \chi_{[\text{dist } (x,\partial I) < \frac{1}{2}\delta |I|]}.$$

Therefore

(F.42)
$$\|f\|_{H^{s,p}} \geq c\delta^{-s} \left\| \left\{ \sum_{r=1}^{R} \sum_{I \in \mathcal{P}_{j_r}} \chi_{[\text{dist } (x,\partial I) < \frac{1}{2}\delta |I|]} \right\}^{1/2} \right\|_p .$$

Fixing $\delta > 0$ and letting $R > R(\delta)$ be sufficiently large, the reader will easily convince himself that

(F.43)
$$(F.42) \geq c\delta^{-s} (\delta R)^{1/2} = c\delta^{\frac{1}{2}-s}.(F.37).$$

Consequently, letting $\delta \to 0$, we see that inequality (F.4) cannot hold for $s > \frac{1}{2}$. This completes the proof of Proposition F.1.

There is the following application of Proposition F.1 to the lifting problem of unimodular functions.

Corollary F.1. *Let $s > 0, sp < 1, p \geq 2$ and $F \in H^{s,p}(\Omega; S^1)$, where Ω is a cube in $\mathbb{R}^d$. Then*

(F.44)
$$F = e^{i\varphi}$$

for some $\varphi \in H^{s,p}(\Omega)$.

Remark F.2. The other cases not covered by the corollary have not been investigated.

Proof. The function φ is constructed as in the $W^{s,p}$-case (see Section 1). ¿From Proposition F.1, $(i), (ii)$ and similar calculations as in the $W^{s,p}$-estimate, we obtain (with the notations from Section 1)

$$\|\varphi\|_{H^{s,p}} \leq C\Big\|\Big(\sum 4^{js}|\Delta_j\varphi|^2\Big)^{1/2}\Big\|_p$$

$$\leq C\Big\|\Big(\sum 4^{js}E_j(\varphi-\varphi_j)|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum 4^{js}|\varphi_j-\varphi_{j-1}|^2\Big)^{1/2}\|_p \tag{F.45}$$

$$\overset{\text{by } (F.6)}{\leq} C\Big\|\Big(\sum 4^{js}|\varphi-\varphi_j|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum 4^{js}|\varphi_j-\varphi_{j-1}|^2\Big)^{1/2}\Big\|_p$$

$$\leq C\Big\|\Big(\sum_{j'>j} 4^{js}(j'-j)^2|\varphi_{j'}-\varphi_{j'-1}|^2\Big)^{1/2}\Big\|_p \tag{F.46}$$

$$\overset{\text{by} y(1.5)}{\leq} C\Big\|\Big(\sum_{j'>j} 4^{js}(j'-j)^2\ |F-E_{j'-1}F|^2\Big)^{1/2}\Big\|_p \tag{F.47}$$

$$\leq C\Big\|\Big(\sum_{j''\geq j'>j} 4^{js}(j'-j)^2(j''-j'+1)^2\ |\Delta_{j''}F|^2\Big)^{1/2}\Big\|_p \tag{F.48}$$

$$\leq C\Big\|\Big(\sum_{j''} 4^{j''s}\ |\Delta_{j''}F|^2\Big)^{1/2}\Big\|_p$$

$$\leq C\|F\|_{H^{s,p}}. \tag{F.49}$$

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