Étude de quelques problèmes inverses pour le système de Stokes. Application aux poumons.

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Directrices de thèse : Céline Grandmont et Muriel Boulakia

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UPMC PARIS UNIVERSITAS

Inria informatiques mathématiques

Plan

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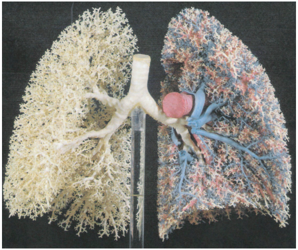
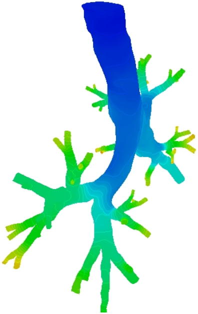
Motivations



Molding of human lung realized by E. R. Weibel.



Reconstructed bronchial tree,

[Baffico, Grandmont, Maury '10].

- airflow in the lungs,

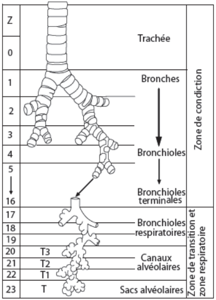
  [Baffico, Grandmont, Maury '10],

- blood flow in the cardiovascular system,

  [Quarteroni, Veneziani '03],
  [Vignon-Clementel, Figueroa , Jansen, Taylor '06].

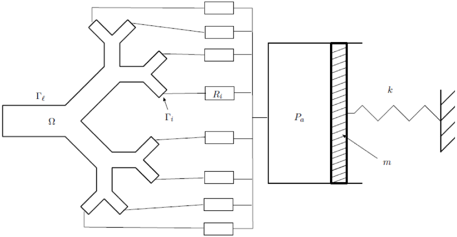
Modeling of the respiratory tract

[Baffico, Grandmont, Maury '10]



Decomposition of the respiratory tract into three stages:

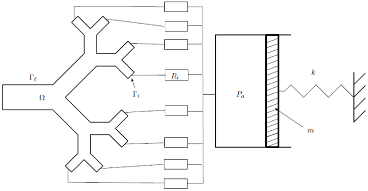
- the upper part (up to the sixth generation),
- the distal part (from the seventh to the 17^{th} generation),
- the acini.



Modeling of the respiratory tract

$$\left\{\begin{array}{rcll} \rho\partial_t u + \rho(u\cdot\nabla)u - \mu\Delta u + \nabla p & = & 0, & \text{in } (0,T)\times\Omega, \\ div\ u & = & 0, & \text{in } (0,T)\times\Omega, \\ u & = & 0, & \text{on } (0,T)\times\Gamma_l, \\ \mu\frac{\partial u}{\partial n} - pn & = & -P_0 n, & \text{on } (0,T)\times\Gamma_0, \\ \mu\frac{\partial u}{\partial n} - pn & = & -P_a n - R_i(\int_{\Gamma_i} u\cdot n)n, & \text{on } (0,T)\times\Gamma_i, \\ m\ddot{x} + kx & = & f_{ext} + SP_a, & \text{on } (0,T), \\ S\dot{x} & = & \sum_{i=1}^N \int_{\Gamma_i} u\cdot n = -\int_{\Gamma_0} u\cdot n, & \text{on } (0,T). \end{array}\right. \quad (1)$$

- u: fluid velocity,
- p: fluid pressure,
- x: position of the diaphragm,
- R_i: airflow resistance,
- k: stiffness of the diaphragm.



Modeling of the respiratory tract

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u: fluid velocity,

p: fluid pressure,

x: position of the diaphragm,

R_i: airflow resistance,

k: stiffness of the diaphragm.

Physiological interpretation:

- $\nearrow$ R_i in asthma,
- $\searrow$ k in emphysema.

⇝ To identify R_i et k from measurements available at mouth.

Spirometry

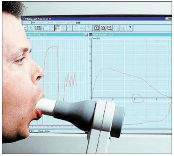


Figure: Evaluation of lung function with a spirometer

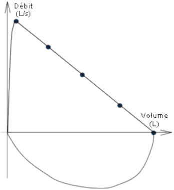


Figure: Normal profil

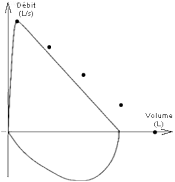


Figure: Restrictive lung disease

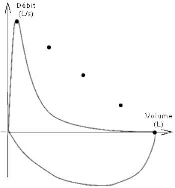
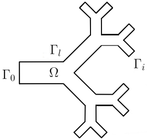


Figure: Obstructive lung disease

Direct problem: existence of solution



- $\partial\Omega = \Gamma_l \cup \left(\bigcup_{i=0}^{N} \Gamma_i\right)$,
- $\overline{\Gamma}_i \cap \overline{\Gamma}_j = \emptyset$,
- $\Gamma_i \perp \Gamma_l$.

Difficulty: to estimate the nonlinear term

$$\rho \int_\Omega (u \cdot \nabla)\, u \cdot u = \frac{\rho}{2} \sum_{i=0}^{N} \int_{\Gamma_i} |u|^2 u \cdot n.$$

⇝ Existence of a solution $u \in L^2(0, t^*; \mathcal{D}(A))$, with $\mathcal{D}(A) \subset H^{\frac{3}{2}+\epsilon}(\Omega)$

⇝ By replacing the boundary conditions

$$\frac{\partial u}{\partial n} - pn = -P_i n - R_i \left(\int_{\Gamma_i} u \cdot n\right) n, \text{ on } \Gamma_i,$$

by

$$\begin{cases} \left(\frac{\partial u}{\partial n} - pn\right) \cdot n = -P_i - R_i \left(\int_{\Gamma_i} u \cdot n\right), & \text{on } \Gamma_i, \\ u \cdot \tau_k = 0, & \text{on } \Gamma_i, \text{ for } k = 1, \dots, d-1, \end{cases}$$

we obtain more regularity in space: $u \in L^2(0, t^*; \mathcal{D}(A))$, with $\mathcal{D}(A) \subset H^2(\Omega)$

Inverse problem

- no coupling with the spring,
- no nonlinear convective term,
- the dissipative boundary conditions on Γ_i:

$$\frac{\partial u}{\partial n} - pn + R_i \left(\int_{\Gamma_i} u \cdot n \right) n = 0,$$

  are replaced by Robin boundary conditions

$$\frac{\partial u}{\partial n} - pn + qu = 0,$$

- simplified geometry.

Our problem

Let $T > 0$ and $\Omega \subset \mathbb{R}^d$ be a bounded connected open set.

$$\left\{\begin{array}{ccll} \partial_t u - \Delta u + \nabla p & = & 0, & \text{in } (0,T)\times\Omega, \\ div\ u & = & 0, & \text{in } (0,T)\times\Omega, \\ u & = & 0, & \text{on } (0,T)\times\Gamma_l, \\ \frac{\partial u}{\partial n} - pn & = & g, & \text{on } (0,T)\times\Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu & = & 0, & \text{on } (0,T)\times\Gamma_{out}, \\ u(0) & = & u_0, & \text{on } \Omega. \end{array}\right. \qquad (P_q)$$

Let $\Gamma \subseteq \Gamma_0$ and (u_j, p_j) be a solution of (P_{q_j}) for $j = 1, 2$.

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Let $\Gamma \subseteq \Gamma_0$ and (u_j, p_j) be a solution of (P_{q_j}) for $j = 1, 2$.

- **Identifiability:** Does $\mathcal{M}_{(0,T)\times\Gamma}(u_1, p_1) = \mathcal{M}_{(0,T)\times\Gamma}(u_2, p_2)$ imply $q_1 = q_2$?
- **Stability:** Is it possible to obtain stability estimate like

$$\|(q_1 - q_2)_{|(0,T)\times\Gamma_{out}}\| \leq f\left(\|(u_1 - u_2)_{|(0,T)\times\Gamma}\| + \|(p_1 - p_2)_{|(0,T)\times\Gamma}\|\right),$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that $\lim_{x\to 0} f(x) = 0$?

Our problem

Let $T > 0$ and $\Omega \subset \mathbb{R}^d$ be a bounded connected open set.

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Let $\Gamma \subseteq \Gamma_0$ and (u_j, p_j) be a solution of (P_{q_j}) for $j = 1, 2$.

Thanks to boundary conditions on Γ_{out}:

$$u_1(q_2 - q_1) = \left(\frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n}\right) - (p_1 - p_2)n + q_2(u_1 - u_2),$$

To do: estimate

$$\left\| \left(\frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n}\right)_{(0,T)\times\Gamma_{out}} \right\| + \|(p_1 - p_2)_{(0,T)\times\Gamma_{out}}\| + \|(u_1 - u_2)_{(0,T)\times\Gamma_{out}}\|$$

by boundary terms on $(0,T)\times\Gamma$.

Plan

State of the art

Unique continuation properties:

[Fabre, Lebeau '96],

[Regbaoui '99],

[Lin, Uhlmann,Wang '10], ...

Inverse problems:

[Imanuvilov, Yamamoto '00],

[Alvarez, Conca, Friz, Kavian, Ortega '05],

[Ballerini '10],

[Conca, Schwindt, Takahashi '12], ...

Related field:

[Fernández-Cara, Guerrero, Imanuvilov, Puel '04],

[Imanuvilov, Puel, Yamamoto '09], ...

State of the art for the Laplace equation

Stationary case:

- *using analytic functions theory*,

  [Chaabane, Jaoua '99],

  [Alessandrini, Del Piero, Rondi '03],

  [Chaabane, Fellah, Jaoua, Leblond '04],

  [Sincich '07], ...

- *using Carleman inequalities*,

  [Bellassoued, Cheng, Choulli '08],

  [Cheng, Choulli, Lin '08], ...

Nonstationary case: Up to our knowledge, largely open question.

Plan

General setting

Let $d \in \mathbb{N}^*$ and $\Omega \subset \mathbb{R}^d$ be a connected bounded open set. We consider:

$$\left\{\begin{array}{rcll} -\Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ div\ u & = & 0, & \text{in } \Omega, \\ u & = & 0, & \text{on } \Gamma_l, \\ \frac{\partial u}{\partial n} - pn & = & g, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu & = & 0, & \text{on } \Gamma_{out}. \end{array}\right. \qquad (P_q)$$

General setting

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Identifiability: Let $x_0 \in \Gamma_0$, $r > 0$.

Assume that g is non identically zero on Γ_0.

Let $(u_j, p_j) \in H^1(\Omega) \times L^2(\Omega)$ be the solution of (P_{q_j}) for $j = 1, 2$.

If $u_1 = u_2$ on $\Gamma_0 \cap \mathcal{B}(x_0, r)$, then $q_1 = q_2$ on Γ_{out}.

Remark

It is a corollary of Fabre-Lebeau unique continuation result.

General setting

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Stability estimates: Let $\Gamma \subseteq \Gamma_0$. We obtained two kind of results:

- two logarithmic stability estimates of type

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{C}{\left(\ln\left(\frac{C_1}{\|u_1-u_2\|_{L^2(\Gamma)}+\|p_1-p_2\|_{L^2(\Gamma)}+\left\|\frac{\partial p_1}{\partial n}-\frac{\partial p_2}{\partial n}\right\|_{L^2(\Gamma)}}\right)\right)^{\lambda}},$$

  with $0 < \lambda < 1$ and $K \subseteq \{x \in \Gamma_{out} / u_1(x) \neq 0\}$,
- a Lipschitz stability estimate of type

$$\|q_1 - q_2\|_{L^\infty(\Gamma_{out})} \leq C\left(\|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\|\frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n}\right\|_{L^2(\Gamma)}\right),$$

  under the *a priori* assumption that the Robin coefficient is piecewise constant on Γ_{out}.

A logarithmic stability estimate

Let $d \in \mathbb{N}^*$ and $\Omega \subset \mathbb{R}^d$ be a connected bounded open set. We consider:

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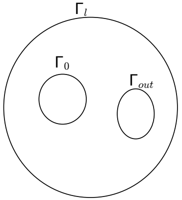

Figure: Example of such an open set Ω in dimension 2.

The boundary of the domain is composed of three parts

$$\Gamma_0 \cup \Gamma_{out} \cup \Gamma_l = \partial\Omega,$$

pairwise disjoint:

- $\overline{\Gamma}_0 \cap \overline{\Gamma}_{out} = \emptyset$,
- $\overline{\Gamma}_0 \cap \overline{\Gamma}_l = \emptyset$,
- $\overline{\Gamma}_l \cap \overline{\Gamma}_{out} = \emptyset$.

Logarithmic stability estimate

Theorem (M. Boulakia, A.-C. E., C. Grandmont)

Let $\alpha > 0$, $M_1 > 0$, $M_2 > 0$, $k \in \mathbb{N}^*$ *such that* $k + 2 > \frac{d}{2}$. *Assume that:*

- $\Gamma \subseteq \Gamma_0$ *and* Γ_{out} *are of class* $\mathcal{C}^\infty$,
- $g \in H^{\frac{1}{2}+k}(\Gamma_0)$ *is non identically zero and* $\|g\|_{H^{\frac{1}{2}+k}(\Gamma_0)} \leq M_1$,
- $q_j \in H^s(\Gamma_{out})$, *with* $s > \frac{d-1}{2}$ *and* $s \geq \frac{1}{2} + k$, *is such that* $q_j \geq \alpha$ *a. e. on* Γ_{out} *and* $\|q_j\|_{H^{\frac{1}{2}+k}(\Gamma_{out})} \leq M_2$.

Let K *be a compact subset of* $\{x \in \Gamma_{out} / u_1 \neq 0\}$ *and* $m > 0$ *be such that* $|u_1| \geq m$ *on* K.

Then, $\forall\ \beta \in (0,1)$, $\exists\ C(\alpha, M_1, M_2) > 0$ *and* $C_1(\alpha, M_1, M_2) > 0$ *such that*

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left(\ln\left(\frac{C_1(\alpha,M_1,M_2)}{\|u_1-u_2\|_{L^2(\Gamma)} + \|p_1-p_2\|_{L^2(\Gamma)} + \left\|\frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n}\right\|_{L^2(\Gamma)}}\right)\right)^{\frac{3\beta}{4}}}.$$

Note that [Bellassoued, Cheng, Choulli '08] has a similar result for the laplacian.

Sketch of the proof

Let us denote by $u = u_1 - u_2$ and $p = p_1 - p_2$.

$$(q_2 - q_1)u_1 = q_2 u + \frac{\partial u}{\partial n} - pn, \text{ on } \Gamma_{out}.$$

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} C(M_2) \left(\|u\|_{L^2(K)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(K)} + \|p\|_{L^2(K)} \right).$$

Using the **continuity of the trace mapping**:

$$\begin{array}{ccc} H^{\frac{3}{2}+\epsilon}(\omega) & \to & L^2(K) \times L^2(K) \\ v & \to & \left(v_{|K}, \frac{\partial v}{\partial n}_{|K} \right), \end{array}$$

and the following **interpolation inequality**:

$$\|v\|_{H^{\frac{3}{2}+\epsilon}(\omega)} \leq C \|v\|^{\theta}_{H^1(\omega)} \|v\|^{1-\theta}_{H^3(\omega)},$$

we obtain

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} C(\alpha, M_1, M_2) \left(\|u\|^{\theta}_{H^1(\omega)} + \|p\|^{\theta}_{L^2(\omega)} \right).$$

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To do

$\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)} \leq f\left(\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right)$, *where* $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ *is an increasing function.*

Sketch of the proof

Theorem

Assume that Ω *is of class* $\mathcal{C}^\infty$. *Let* $0 < \nu \leq \frac{1}{2}$ *and let* Γ *be a nonempty open subset of the boundary of* Ω. *There exists* $d_0 > 0$ *such that for all* $\gamma \in \left(0, \frac{1}{2} + \nu\right)$, *for all* $d > d_0$, *there exists* $c > 0$, *such that we have*

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq c \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\left(ln \left(d \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)}} \right) \right)^\gamma},$$

for all couples $(u,p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$ *solution of* $\left\{ \begin{array}{rcll} -\Delta u + \nabla p & = & 0, & in\ \Omega, \\ div\ u & = & 0, & in\ \Omega. \end{array} \right.$

Sketch of the proof

Theorem

Assume that Ω is of class $\mathcal{C}^\infty$. Let $0 < \nu \leq \frac{1}{2}$ and let Γ be a nonempty open subset of the boundary of Ω. There exists $d_0 > 0$ such that for all $\gamma \in (0, \frac{1}{2} + \nu)$, for all $d > d_0$, there exists $c > 0$, such that we have

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq c \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\left(ln \left(d \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)}} \right) \right)^\gamma},$$

for all couples $(u,p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$ solution of $\begin{cases} -\Delta u + \nabla p = 0, & in\ \Omega, \\ div\ u = 0, & in\ \Omega. \end{cases}$

⇝ **End of the proof of the logarithmic stability estimate:**

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} C(\alpha, M_1, M_2) \left(\|u\|^\theta_{H^1(\omega)} + \|p\|^\theta_{L^2(\omega)} \right).$$

We apply the previous theorem in ω with $\nu = \frac{1}{2}$ and with γ suitably chosen.

Remark

- $(u,p) \in H^3(\omega) \times H^2(\omega)$,

$$\exists\, C > 0,\ \exists\, \theta = \frac{3}{4}\left(1 - \frac{2\epsilon}{3}\right),\ \|u\|_{H^{\frac{3}{2}+\epsilon}(\omega)} \leq C \|u\|^{\theta}_{H^1(\omega)} \|u\|^{1-\theta}_{H^3(\omega)},$$

$$\Rightarrow \|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left(\ln\left(\frac{dC_1(\alpha, M_1, M_2)}{\|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\|\frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n}\right\|_{L^2(\Gamma)}}\right)\right)^{\frac{3\beta}{4}}}$$

- $(u,p) \in H^{k+2}(\omega) \times H^{k+1}(\omega)$, $k \in \mathbb{N}^*$ such that $k + 2 > \frac{d}{2}$,

$$\exists\, C > 0,\ \exists\, \tilde{\theta} = \frac{1/2 + k}{1 + k} - \frac{\epsilon}{1 + k},\ \|u\|_{H^{\frac{3}{2}+\epsilon}(\omega)} \leq C \|u\|^{\tilde{\theta}}_{H^1(\omega)} \|u\|^{1-\tilde{\theta}}_{H^{k+2}(\omega)},$$

$$\Rightarrow \|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left(\ln\left(\frac{dC_1(\alpha, M_1, M_2)}{\|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\|\frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n}\right\|_{L^2(\Gamma)}}\right)\right)^{\frac{1/2+k}{1+k}}}$$

A quantitative estimate of a unique continuation result

Assume that Ω is of class $\mathcal{C}^\infty$. Let $0 < \nu \leq \frac{1}{2}$ and let $\Gamma \subseteq \partial\Omega$ be a nonempty open subset.

Theorem

- *There exists $d_0 > 0$ such that for all $\gamma \in (0, \frac{1}{2} + \nu)$, for all $d > d_0$, there exists $c > 0$, such that we have*

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq c \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\left(ln\left(d\frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^2(\Gamma)} + \left\|\frac{\partial p}{\partial n}\right\|_{L^2(\Gamma)}}\right)\right)^\gamma},$$

- *for all $\beta \in (0, \frac{1}{2} + \nu)$, there exists $c > 0$, such that for all $\epsilon > 0$, we have*

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq e^{\frac{c}{\epsilon}}\left(\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^2(\Gamma)} + \left\|\frac{\partial p}{\partial n}\right\|_{L^2(\Gamma)}\right) + \epsilon^\beta(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}),$$

for all couples $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$ solution of $\left\{\begin{array}{rcll} -\Delta u + \nabla p & = & 0, & in\ \Omega, \\ div\ u & = & 0, & in\ \Omega. \end{array}\right.$

A quantitative estimate of a unique continuation result

We adapt to the Stokes system a quantitative estimate of unique continuation results for the Laplace equation:

- *Remarques sur l'observabilité pour l'équation de Laplace*, Kim-Dang Phung, 2003.

The proof is based on local Carleman estimates:

- inside the domain [Hörmander],
- near the boundary [Lebeau-Robbiano '95].

For all $(u,p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$ solution of

$$\left\{ \begin{array}{rcll} -\Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ div\ u & = & 0, & \text{in } \Omega, \end{array} \right. \qquad (2)$$

$$\|u\|_{H^1(\tilde{\omega}\cap\Omega)} + \|p\|_{H^1(\tilde{\omega}\cap\Omega)} \leq e^{\frac{c}{\epsilon}}(\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}) + \epsilon^\beta(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)})$$

$$\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)} \leq \frac{c}{\epsilon}\left(\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} + \left\|\frac{\partial u}{\partial n}\right\|_{L^2(\Gamma)} + \left\|\frac{\partial p}{\partial n}\right\|_{L^2(\Gamma)}\right) + \epsilon^s(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)})$$

and for all $(u,p) \in H^1(\Omega) \times L^2(\Omega)$ solution of (2)

$$\|u\|_{H^1(\tilde{\omega})} + \|p\|_{L^2(\tilde{\omega})} \leq \frac{c}{\epsilon}\left(\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)}\right) + \epsilon^s\left(\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)}\right)$$

Carleman estimate

Proposition (Lebeau-Robbiano)

- *Let* $K = \{x \in \mathbb{R}^n_+ / |x| \leq R_0\}$.
- *Let* P *be a second-order differential operator whose coefficients are* $\mathcal{C}^\infty$ *in a neighborhood of* K *defined by* $P(x, \partial_x) = -\partial^2_{x_n} + R(x, \frac{1}{i}\partial_{x'})$. *Let us denote by* $r(x, \xi')$ *the principal symbol of* R *and assume that* $r(x, \xi') \in \mathbb{R}$ *and that there exists a constant* $c > 0$ *such that* $(x, \xi') \in K \times \mathbb{R}^{n-1}$, *we have* $r(x, \xi') \geq c|\xi'|^2$.
- *Let* $\phi = \phi(x) \in \mathcal{C}^\infty$ *be a function defined in a neighborhood of* K. *We assume that the function* ϕ *satisfies the Hörmander hypoellipicity property on* K *and*

$$\partial_{x_n}\phi(x) \neq 0, \; \forall \, x \in K.$$

Then, there exists $c > 0$ *and* $h_1 > 0$ *such that for all* $h \in (0, h_1)$ *we have:*

$$\int_{\mathbb{R}^n_+} |y(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_{\mathbb{R}^n_+} |\nabla y(x)|^2 e^{2\phi(x)/h} dx \leq ch^3 \int_{\mathbb{R}^n_+} |P(x, \partial_x) y(x)|^2 e^{2\phi(x)/h} dx$$
$$+ c \int_{\mathbb{R}^{n-1}} (|y(x', 0)|^2 + |h\partial_{x'} y(x', 0)|^2 + |h\partial_{x_n} y(x', 0)|^2) e^{2\phi(x', 0)/h} dx',$$

for all function $y \in \mathcal{C}^\infty(\mathbb{R}^n_+)$ *with support in* K.

Estimates near the boundary

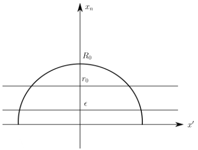
Key points of the proof:

- go back to the half plane by passing in geodesic normal coordinates,
- suitably choose the weight function ϕ,
- apply the Carleman inequality twice: one time to the velocity and another time to the pressure.

More precisely, for the first one:

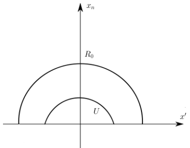
$$\phi(x) = e^{\lambda x_n}$$

+ Hardy inequality



For the second one:

$$\phi(x) = e^{-\lambda(x_n+|x|)}$$



Estimate inside the domain

- To prove the third one, we use the 'classical' local Carleman inequality for the Laplace equation inside the domain.
- Caccioppoli inequality allows to discard of the gradient of p in the right hand side.
- In particular, we prove a *three balls inequality* (see below).

Lemma (Three balls inequality)

Let $0 < r_1 < r_2 < r_3$ *and* $q \in \mathbb{R}^d$. *There exist* $C > 0$, $\alpha > 0$ *such that for all function* $(u, p) \in H^1(B(q, r_3)) \times H^1(B(q, r_3))$ *solution of*

$$\begin{cases} -\Delta u + \nabla p & = & 0, \\ div\ u & = & 0, \end{cases}$$

in $B(q, r_3)$ *there exists* $\alpha > 0$ *such that:*

$$\|u\|_{H^1(B(q,r_2))} + \|p\|_{L^2(B(q,r_2))} \leq C \left(\|u\|_{H^1(B(q,r_1))} + \|p\|_{L^2(B(q,r_1))} \right)^{\alpha} \left(\|u\|_{H^1(B(q,r_3))} + \|p\|_{L^2(B(q,r_3))} \right)^{1-\alpha}.$$

⇝ Useful to prove the Lipschitz stability estimate when the Robin coefficient is piecewise constant.

Extension

⇝ We also proved another logarithmic stability estimate valid in dimension $d = 2$.

Main differences:

Regularity on Ω	Regularity needed on (u, p)	Valid in dimension
$\mathcal{C}^{3,1}$	$(u, p) \in H^4(\Omega) \times H^3(\Omega)$	2
locally $\mathcal{C}^\infty$	$(u, p) \in H^{k+2}(\Omega) \times H^{k+1}(\Omega)$ for $k \in \mathbb{N}^*$ be such that $k + 2 > \frac{d}{2}$	in any dimension d

⇝ We can extend previous stability estimates to the nonstationary problem by using inequalities coming from analytic semigroup properties.

It leads to measurements in infinite time:

$$\|u_1 - u_2\|_{L^\infty(0,+\infty;L^2(\Gamma))} + \|p_1 - p_2\|_{L^\infty(0,+\infty;L^2(\Gamma))} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^\infty(0,+\infty;L^2(\Gamma))}.$$

[M. Boulakia, A.-C. E., C. Grandmont, Accepted for publication in *Mathematical control and related field*].

A Lipschitz stability estimate

Let $d = 2, 3$ and $\Omega \subset \mathbb{R}^d$ be a connected bounded open set. We consider:

$$\begin{cases} -\Delta u + \nabla p &= 0, & \text{in } \Omega, \\ div\ u &= 0, & \text{in } \Omega, \\ u &= 0, & \text{on } \Gamma_l, \\ \frac{\partial u}{\partial n} - pn &= g, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu &= 0, & \text{on } \Gamma_{out}. \end{cases}$$

We assume that q is piecewise constant on $\Gamma_{out} = \bigcup_{i=1}^{N} \Gamma_i$:

$$q_{|\Gamma_i} = q_i, \text{ with } q_i \in \mathbb{R}^+ \text{ for } i = 1, \ldots, N.$$

Remark

Due to the mixed boundary conditions, we do no have global regularity on the solution.

A Lipschitz stability estimate

Theorem (A.-C. E.)

Let $m > 0$, $M_1 > 0$, $R_M > 0$. Assume that:

- $\Gamma \subseteq \Gamma_0$ *is of class* $\mathcal{C}^\infty$ *and is such that* $(\overline{\Gamma} \cap \overline{\Gamma}_l) \bigcup (\overline{\Gamma} \cap \overline{\Gamma}_{out}) = \emptyset$,
- Γ_{out} *is of class* $\mathcal{C}^{2,1}$,
- $g \in H^{\frac{3}{2}}(\Gamma_0)$ *is non identically zero and* $\|g\|_{H^{\frac{3}{2}}(\Gamma_0)} \leq M_1$,
- $q_{j\,|\Gamma_i} = q_j^i$ *with* $q_j^i \in \mathbb{R}^+$ *and* $q_j^i \leq R_M$ *for* $i = 1, \ldots, N$ *and* $j = 1, 2$.

We assume that there exits $x_i \in \left\{ x \in \Gamma_i / d\left(x, \overline{\partial\Omega \backslash \Gamma_i}\right) > 0 \right\}$ *such that* $|u_2(x_i)| > m$ *for all* $i = 1, \ldots, N$.
Then, $\exists\, C(m, R_M, M_1, N) > 0$ *such that*

$$\|q_1 - q_2\|_{L^\infty(\Gamma_{out})} \leq C(m, R_M, M_1, N) \left(\|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma)} \right).$$

Note that [Sincich '07] has a similar result for the Laplace equation.

Sketch of the proof

We consider

$$(w,\pi) = \left(\frac{u_1 - u_2}{\sum_{j=1}^{N} |q_j^1 - q_j^2|}, \frac{p_1 - p_2}{\sum_{j=1}^{N} |q_j^1 - q_j^2|} \right).$$

Since for $j = 1, 2$, q_j is piecewise constant, (w,π) is solution of:

$$\begin{cases} -\Delta w + \nabla \pi &= 0, & \text{in } \Omega, \\ \text{div } w &= 0, & \text{in } \Omega, \\ w &= 0, & \text{on } \Gamma_l, \\ \dfrac{\partial w}{\partial n} - \pi n &= 0, & \text{on } \Gamma_0, \\ \dfrac{\partial w}{\partial n} - \pi n + q_1 w &= \dfrac{(q_2 - q_1)}{\sum_{j=1}^{N} |q_j^1 - q_j^2|} u_2, & \text{on } \Gamma_{out}. \end{cases}$$

We prove that there exists $C > 0$ such that:

$$\|w\|_{L^2(\Gamma)} + \|\pi\|_{L^2(\Gamma)} + \left\| \frac{\partial \pi}{\partial n} \right\|_{L^2(\Gamma)} \geq C.$$

Sketch of the proof

To do so, we use:

- previous unique continuation estimates for the Stokes system,
- a sequence of balls which approachs the boundary,
- Hölder regularity of the solution in a neighborhood of the boundary.

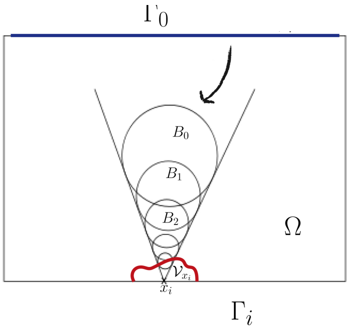


Figure: Scheme of how information is transmitted.

Plan

Back to the dissipative boundary conditions

Let (u_k, p_k) be solution of

$$\begin{cases} -\Delta u + \nabla p &= 0, & \text{in } \Omega, \\ div\ u &= 0, & \text{in } \Omega, \\ u &= 0, & \text{on } \Gamma_l, \\ \frac{\partial u}{\partial n} - pn &= -P_0 n, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + R_i \left(\int_{\Gamma_i} u \cdot n \right) n &= 0, & \text{on } \Gamma_i, \text{ for } i = 1, \dots, N, \end{cases}$$

with $R_i = R_i^k$ for $i = 1, \dots, N$ and $k = 1, 2$.

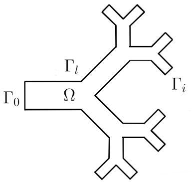
Back to the dissipative boundary conditions

Let (u_k, p_k) be solution of

$$\begin{cases} -\Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ div\ u & = & 0, & \text{in } \Omega, \\ u & = & 0, & \text{on } \Gamma_l, \\ \frac{\partial u}{\partial n} - pn & = & -P_0 n, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + R_i \left(\int_{\Gamma_i} u \cdot n \right) n & = & 0, & \text{on } \Gamma_i, \text{ for } i = 1, \dots, N, \end{cases}$$

with $R_i = R_i^k$ for $i = 1, \dots, N$ and $k = 1, 2$.

In a domain like



- $\partial\Omega = \Gamma_l \cup \left(\bigcup_{i=0}^{N} \Gamma_i \right)$,
- $\overline{\Gamma}_i \cap \overline{\Gamma}_j = \emptyset$,
- $\Gamma_i \perp \Gamma_l$.

Back to the dissipative boundary conditions

Let (u_k, p_k) be solution of

$$\begin{cases} -\Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ div\ u & = & 0, & \text{in } \Omega, \\ u & = & 0, & \text{on } \Gamma_l, \\ \frac{\partial u}{\partial n} - pn & = & -P_0 n, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + R_i \left(\int_{\Gamma_i} u \cdot n \right) n & = & 0, & \text{on } \Gamma_i, \text{ for } i = 1, \ldots, N, \end{cases}$$

with $R_i = R_i^k$ for $i = 1, \ldots, N$ and $k = 1, 2$.

We denote by $(v, \pi) = (u_1 - u_2, p_1 - p_2)$. Thanks to the boundary conditions on Γ_i, we have:

$$(R_i^2 - R_i^1) \left(\int_{\Gamma_i} u_1 \cdot n \right) = R_i^2 \left(\int_{\Gamma_i} v \cdot n \right) + \frac{\partial v}{\partial n} - \pi n.$$

$$|R_i^1 - R_i^2| \left| \int_{\Gamma_i} u_1 \cdot n \right| \leq |R_i^2| \left| \int_{\Gamma_i} v \cdot n \right| + \frac{1}{|\mathcal{K}|} \left\| \frac{\partial v}{\partial n} \right\|_{L^2(\mathcal{K})} + \frac{1}{|\mathcal{K}|} \|\pi\|_{L^2(\mathcal{K})},$$

where $\mathcal{K} \subseteq \Gamma_i$ is a non empty set.

Back to the dissipative boundary conditions

Let (u_k, p_k) be solution of

$$\begin{cases} -\Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ div\ u & = & 0, & \text{in } \Omega, \\ u & = & 0, & \text{on } \Gamma_l, \\ \frac{\partial u}{\partial n} - pn & = & -P_0 n, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + R_i \left(\int_{\Gamma_i} u \cdot n \right) n & = & 0, & \text{on } \Gamma_i, \text{ for } i = 1, \dots, N, \end{cases}$$

with $R_i = R_i^k$ for $i = 1, \dots, N$ and $k = 1, 2$.

We denote by $(v, \pi) = (u_1 - u_2, p_1 - p_2)$. Thanks to the boundary conditions on Γ_i, we have:

$$(R_i^2 - R_i^1) \left(\int_{\Gamma_i} u_1 \cdot n \right) = R_i^2 \left(\int_{\Gamma_i} v \cdot n \right) + \frac{\partial v}{\partial n} - \pi n.$$

$$|R_i^1 - R_i^2| \left| \int_{\Gamma_i} u_1 \cdot n \right| \leq |R_i^2| \left| \int_{\Gamma_i} v \cdot n \right| + \frac{1}{|\mathcal{K}|} \left\| \frac{\partial v}{\partial n} \right\|_{L^2(\mathcal{K})} + \frac{1}{|\mathcal{K}|} \|\pi\|_{L^2(\mathcal{K})},$$

where $\mathcal{K} \subseteq \Gamma_i$ is a non empty set.

$\rightsquigarrow$ Non local term !

A Lipschitz stability estimate in a particular case

We assume that $N = 1$ (only one outlet).

Assume that:

- $R_m \leq R_1^j \leq R_M$, for $j = 1, 2$
- P_0 be a nonzero constant.

Then, there exists $C(R_M, R_m, P_0) > 0$ such that

$$|R_1^1 - R_1^2| \leq C(R_M, R_m, P_0) \left(\|u_1 - u_2\|_{L^2(\Gamma_0)} + \|p_1 - p_2\|_{L^2(\Gamma_0)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma_0)} \right)$$

if $\|u_1 - u_2\|_{L^2(\Gamma_0)} + \|p_1 - p_2\|_{L^2(\Gamma_0)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma_0)}$ is small enough.

Key point: Since u is divergence-free,

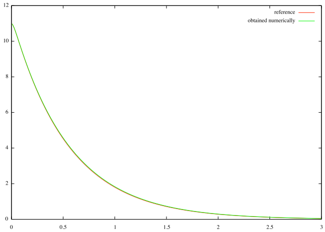
$$\int_{\Gamma_0} u \cdot n = -\int_{\Gamma_1} u \cdot n.$$

Remark

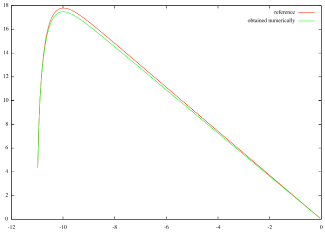
The constant involved in the stability estimate depends only on the data.

Back to the initial model: numerical point of view

$$\left\{\begin{array}{rcll}
\rho\partial_t u + \rho(u\cdot\nabla)u - \mu\Delta u + \nabla p & = & 0, & \text{in } (0,T)\times\Omega,\\
div\ u & = & 0, & \text{in } (0,T)\times\Omega,\\
u & = & 0, & \text{on } (0,T)\times\Gamma_l,\\
\mu\frac{\partial u}{\partial n} - pn & = & -P_0 n, & \text{on } (0,T)\times\Gamma_0,\\
\mu\frac{\partial u}{\partial n} - pn & = & -P_a n - R_i(\int_{\Gamma_i} u\cdot n)n, & \text{on } (0,T)\times\Gamma_i,\\
m\ddot{x} + kx & = & f_{ext} + SP_a, & \text{on } (0,T),\\
S\dot{x} & = & \sum_{i=1}^N \int_{\Gamma_i} u\cdot n = -\int_{\Gamma_0} u\cdot n, & \text{on } (0,T).
\end{array}\right.$$



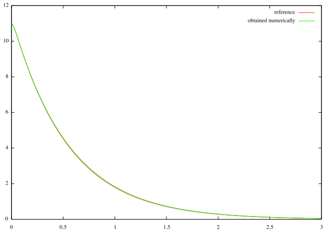
Volume curve



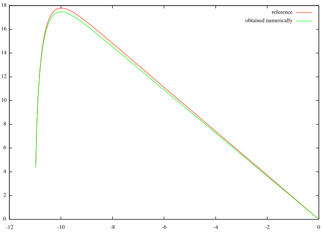
Flow-volume curve

Back to the initial model: numerical point of view

$$
\left\{
\begin{array}{rcll}
\rho \partial_t u + \rho (u \cdot \nabla) u - \mu \Delta u + \nabla p & = & 0, & \text{in } (0,T) \times \Omega, \\
div\ u & = & 0, & \text{in } (0,T) \times \Omega, \\
u & = & 0, & \text{on } (0,T) \times \Gamma_l, \\
\mu \frac{\partial u}{\partial n} - pn & = & -P_0 n, & \text{on } (0,T) \times \Gamma_0, \\
\mu \frac{\partial u}{\partial n} - pn & = & -P_a n - R_i (\int_{\Gamma_i} u \cdot n) n, & \text{on } (0,T) \times \Gamma_i, \\
m \ddot{x} + kx & = & f_{ext} + S P_a, & \text{on } (0,T), \\
S \dot{x} & = & \sum_{i=1}^{N} \int_{\Gamma_i} u \cdot n = - \int_{\Gamma_0} u \cdot n, & \text{on } (0,T).
\end{array}
\right.
$$



Volume curve



Flow-volume curve

- Resolution of the direct problem with FreeFem++ [Devys, Grandmont, Grec, Maury '10]
- Resolution of the inverse problem: we use Genetic algorithm [Dumas]

Physiological data:

- $m = 0,3\ kg$, total mass of the lung,
- $S = 0,011\ m^2$, surface of the moving box,
- $E = 3,32.10^5\ N \cdot m^{-5}$, the lung elastance,
- $k_0 = E \times S^2 = 40,172\ N \cdot m^{-1}$, the stiffness of the spring,
- $R_i = 1,33.10^5\ Pa \cdot s \cdot m^{-3}$, the resistance at the outlet Γ_i.

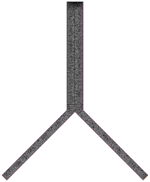


Figure: Domain Ω.

Presentation of results

- **Estimation of the stiffness constant**

	Volume curve	Flow-volume loop
parameter reference	40.172	40.172
obtained parameter	40.175608	40.140012

- **Estimation of two resistances**

	Volume curve	
reference parameters	133000	123000
parameters obtained	134846.11	121444.9

	Flow-volume loop	
reference parameters	133000	123000
parameters obtained	131654.7	124181.23

Plan

Conclusion

- Can we relax the regularity assumption needed on the boundary?

  [Bourgeois '10]
  [Bourgeois, Dardé'10]

Conclusion

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- Can we obtain logarithmic stability estimates on the whole Γ_{out} or on any compact $\kappa \subset \Gamma_{out}$?

Conclusion

- Can we relax the regularity assumption needed on the boundary?

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- Can we obtain logarithmic stability estimates on the whole Γ_{out} or on any compact $\kappa \subset \Gamma_{out}$?

- Can we obtain stability estimate for the unsteady problem in finite time?

Conclusion

- Can we relax the regularity assumption needed on the boundary?

  [Bourgeois '10]
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- Can we obtain logarithmic stability estimates on the whole Γ_{out} or on any compact $\kappa \subset \Gamma_{out}$?
- Can we obtain stability estimate for the unsteady problem in finite time?
- Can we obtain stability estimate with less measurement?

Merci pour votre attention !

ご清聴ありがとうございました！