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CARLEMAN ESTIMATES FOR ANISOTROPIC ELLIPTIC OPERATORS WITH JUMPS AT AN INTERFACE

JÉRÔME LE ROUSSEAU AND NICOLAS LERNER

Abstract. We consider a second-order selfadjoint elliptic operator with an anisotropic diffusion matrix having a jump across a smooth hypersurface. We prove the existence of a weight-function such that a Carleman estimate holds true. We moreover prove that the conditions imposed on the weight function are sharp.

Contents

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1. Introduction

1.1. **Carleman estimates.** Let $P(x, D_x)$ be a differential operator defined on some open subset of $\mathbb{R}^n$. A *Carleman estimate* for this operator is the following weighted a priori inequality

$$\|e^{\tau\varphi} P w\|_{L^2(\mathbb{R}^n)} \gtrsim \|e^{\tau\varphi} w\|_{L^2(\mathbb{R}^n)}, \tag{1.1}$$

where the weight function φ is real-valued with a non-vanishing gradient, τ is a large positive parameter and w is any smooth compactly supported function. This type of estimate was used for the first time in 1939 in T. Carleman's article [12] to handle uniqueness properties for the Cauchy problem for non-hyperbolic operators. To this day, it remains essentially the only method to prove unique continuation properties for ill-posed problems[1], in particular to handle uniqueness of the Cauchy problem for elliptic operators with non-analytic coefficients[2]. This tool has been refined, polished and generalized by manifold authors. The 1958 article by A.P. Calderón [11] gave a very important development of the Carleman method with a proof of an estimate of the form of (1.1) using a pseudo-differential factorization of the operator, giving a new start to singular-integral methods in local analysis. In the article [17] and in his first PDE book (Chapter VIII, [18]), L. Hörmander showed that local methods could provide the same estimates, with weaker assumptions on the regularity of the coefficients of the operator.

For instance, for second-order elliptic operators with real coefficients[3] in the principal part, Lipschitz continuity of the coefficients suffices for a Carleman estimate to hold and thus for unique continuation across a $\mathscr{C}^1$ hypersurface. Naturally, pseudo-differential methods require more derivatives, at least tangentially, i.e., essentially on each level surface of the weight function φ. Chapters 17 and 28 in the 1983-85 four-volume book [20] by L. Hörmander contain more references and results.

Furthermore, it was shown by A. Pliś [42] that Hölder continuity is not enough to get unique continuation: this author constructed a real homogeneous linear differential equation of second order and of elliptic type on $\mathbb{R}^3$ without the unique continuation property although the coefficients are Hölder-continuous with any exponent less

[1]The 1960 article by F. John [26] showed that, although Hadamard well-posedness property is a privilege of hyperbolic operators, some weaker type of continuous dependence, called in [26] *Hölder continuous well-behaviour*, could occur. Strong connections between the well-behavior property and Carleman estimates can be found in an article by H. Bahouri [3].

[2]For analytic operators, Holmgren's theorem provides uniqueness for the non-characteristic Cauchy problem, but that analytical result falls short of giving a control of the solution from the data.

[3]The paper [1] by S. Alinhac shows nonunique continuation property for second-order elliptic operators with non-conjugate roots; of course, if the coefficients of the principal part are real, this is excluded.

than one. The constructions by K. Miller in [41], and later by N. Mandache [39] and N. Filonov in [15], showed that Hölder continuity is not sufficient to obtain unique continuation for second-order elliptic operators, even in divergence form (see also [9] and [44] for the particular 2D case where boundedness is essentially enough to get unique continuation for elliptic equations in the case of $W^{1,2}$ solutions).

The results cited above are related to the regularity of the principal part of the second-order operator. For strong-unique-continuation properties for second-order operator with Lipschitz-continuous coefficients, many results are also available for differential inequalities with singular potentials, originating with the seminal work of D. Jerison and C. Kenig [24]. The reader is also referred to the work of C. Sogge [45] and some of the most recent and general results of H. Koch and D. Tataru [28, 29].

In more recent years, the field of applications of Carleman estimates has gone beyond the original domain. They are also used in the study of inverse problems (see e.g. [8, 23, 21, 27]) and control theory for PDEs. Through unique continuation properties, they are used for the exact controllability of hyperbolic equations [5]. They also yield the null controllability of linear parabolic equations [35] and the null controllability of classes of semi-linear parabolic equations [16, 4, 14].

1.2. **Jump discontinuities.** Although the situation seems to be almost completely clarified by the previous results, with a minimal and somewhat necessary condition on Lipschitz continuity, we are interested in the following second-order elliptic operator $\mathcal{L}$,

(1.2)
$$\mathcal{L}w = -\operatorname{div}(A(x)\nabla w),\ A(x) = (a_{jk}(x))_{1\le j,k\le n} = A^T(x),\ \inf_{\|\xi\|_{\mathbb{R}^n}=1}\langle A(x)\xi,\xi\rangle > 0,$$

in which the matrix A has a jump discontinuity across a smooth hypersurface. However we shall impose some stringent –yet natural– restrictions on the domain of functions w, which will be required to satisfy some homogeneous *transmission conditions*, detailed in the next sections. Roughly speaking, it means that w must belong to the domain of the operator, with continuity at the interface, so that ∇w remains bounded and continuity of the flux across the interface, so that $\operatorname{div}(A\nabla w)$ remains bounded, avoiding in particular the occurrence of a simple or multiple layer at the interface[4].

The article [13] by A. Doubova, A. Osses, and J.-P. Puel tackled that problem, in the isotropic case (the matrix A is scalar $c\,\mathrm{Id}$) with a monotonicity assumption: the observation takes place in the region where the diffusion coefficient c is the 'lowest'. (Note that the work of [13] concerns the case of a parabolic operator but an adaptation to an elliptic operator is straightforward.) In the one-dimensional case, the monotonicity assumption was relaxed for general piecewise $\mathscr{C}^1$ coefficients by A. Benabdallah, Y. Dermenjian and J. Le Rousseau [6], and for coefficients with bounded variations [30]. The case of an arbitrary dimension without any monotonicity condition in the elliptic case was solved by J. Le Rousseau and L. Robbiano in [32]: there the isotropic case is treated as well as a particular case of anisotropic

[4]In the sections below we shall also consider non-homogeneous boundary conditions.

medium. An extension of their approach to the case of parabolic operators can be found in [33]. A. Benabdallah, Y. Dermenjian and J. Le Rousseau also tackled the situation in which the interface meets the boundary, a case that is typical of stratified media [7]. They treat particular forms of anisotropic coefficients.

The purpose of the present article is to show that a Carleman estimate can be proven for any operator of type (1.2) without an isotropy assumption: $A(x)$ is a symmetric positive-definite matrix with a jump discontinuity across a smooth hypersurface. We also provide conditions on the Carleman weight function that are rather simple to handle and we prove that they are sharp.

The approach we follow differs from that of [32] where the authors base their analysis on the usual Carleman method for certain microlocal regions and on Calderón projectors for others. The regions they introduce are determined by the ellipticity or non-ellipticity of the conjugated operator. The method in [7] exploits a particular structure of the anisotropy that allows one to use Fourier series. The analysis is then close to that of [32, 33] in the sense that second-order operators are inverted in some frequency ranges. Here, our approach is somewhat closer to A. Calderón's original work on unique continuation [11]: the conjugated operator is factored out in first-order (pseudo-differential) operators for which estimates are derived. Naturally, the quality of these estimates depends on their elliptic or non-elliptic nature; we thus recover microlocal regions that correspond to that of [32]. Note that such a factorization is also used in [22] to address non-homogeneous boundary conditions.

1.3. **Notation and statement of the main result.** Let Ω be an open subset of $\mathbb{R}^n$ and Σ be a $\mathscr{C}^\infty$ oriented hypersurface of Ω: we have the partition

$$\Omega = \Omega_+ \cup \Sigma \cup \Omega_-, \quad \overline{\Omega_\pm} = \Omega_\pm \cup \Sigma, \quad \Omega_\pm \text{ open subsets of } \mathbb{R}^n, \tag{1.3}$$

and we introduce the following Heaviside-type functions

$$H_\pm = \mathbf{1}_{\Omega_\pm}. \tag{1.4}$$

We consider the elliptic second-order operator

$$\mathcal{L} = D \cdot AD = -\operatorname{div}(A(x)\nabla), \qquad (D = -i\nabla), \tag{1.5}$$

where $A(x)$ is a symmetric positive-definite $n \times n$ matrix, such that

$$A = H_- A_- + H_+ A_+, \quad A_\pm \in \mathscr{C}^\infty(\Omega). \tag{1.6}$$

We shall consider functions w of the following type:

$$w = H_- w_- + H_+ w_+, \quad w_\pm \in \mathscr{C}^\infty(\Omega). \tag{1.7}$$

We have $dw = H_- dw_- + H_+ dw_+ + (w_+ - w_-)\delta_\Sigma \nu$, where δ_Σ is the Euclidean hypersurface measure on Σ and ν is the unit conormal vector field to Σ pointing into Ω_+. To remove the singular term, we assume

$$w_+ = w_- \quad \text{at } \Sigma, \tag{1.8}$$

so that $Adw = H_- A_- dw_- + H_+ A_+ dw_+$ and

$$\operatorname{div}(Adw) = H_- \operatorname{div}(A_- dw_-) + H_+ \operatorname{div}(A_+ dw_+) + \langle A_+ dw_+ - A_- dw_-, \nu \rangle \delta_\Sigma.$$

Moreover, we shall assume that

$$\langle A_+ dw_+ - A_- dw_-, \nu\rangle = 0 \quad \text{at } \Sigma, \text{ i.e. } \langle dw_+, A_+\nu\rangle = \langle dw_-, A_-\nu\rangle, \tag{1.9}$$

so that

$$\operatorname{div}(A dw) = H_- \operatorname{div}(A_- dw_-) + H_+ \operatorname{div}(A_+ dw_+). \tag{1.10}$$

Conditions (1.8)-(1.9) will be called *transmission conditions* on the function w and we define the vector space

$$\mathcal{W} = \{H_- w_- + H_+ w_+\}_{w_\pm \in \mathscr{C}^\infty(\Omega) \text{ satisfying } (1.8)\text{-}(1.9)}. \tag{1.11}$$

Note that (1.8) is a continuity condition of w across Σ and (1.9) is concerned with the continuity of $\langle A dw, \nu\rangle$ across Σ, i.e. the continuity of the flux of the vector field $A dw$ across Σ. A weight function "suitable for observation from Ω_+" is defined as a Lipschitz continuous function φ on Ω such that

$$\varphi = H_-\varphi_- + H_+\varphi_+, \quad \varphi_\pm \in \mathscr{C}^\infty(\Omega), \quad \varphi_+ = \varphi_-, \quad \langle d\varphi_\pm, X\rangle > 0 \quad \text{at } \Sigma, \tag{1.12}$$

for any positively transverse vector field X to Σ (i.e. $\langle \nu, X\rangle > 0$).

Theorem 1.1. *Let $\Omega, \Sigma, \mathcal{L}, \mathcal{W}$ be as in (1.3), (1.5) and (1.11). Then for any compact subset K of Ω, there exist a weight function φ satisfying (1.12) and positive constants C, τ_1 such that for all $\tau \geq \tau_1$ and all $w \in \mathcal{W}$ with $\operatorname{supp} w \subset K$,*

$$\begin{aligned} C\|e^{\tau\varphi}\mathcal{L}w\|_{L^2(\mathbb{R}^n)} \geq \;& \\ &\tau^{3/2}\|e^{\tau\varphi}w\|_{L^2(\mathbb{R}^n)} + \tau^{1/2}\|H_+ e^{\tau\varphi}\nabla w_+\|_{L^2(\mathbb{R}^n)} + \tau^{1/2}\|H_- e^{\tau\varphi}\nabla w_-\|_{L^2(\mathbb{R}^n)} \\ &+ \tau^{3/2}|(e^{\tau\varphi}w)_{|\Sigma}|_{L^2(\Sigma)} + \tau^{1/2}|(e^{\tau\varphi}\nabla w_+)_{|\Sigma}|_{L^2(\Sigma)} + \tau^{1/2}|(e^{\tau\varphi}\nabla w_-)_{|\Sigma}|_{L^2(\Sigma)}. \end{aligned} \tag{1.13}$$

Remark 1.2. The proof of Theorem 1.1 provides an explicit construction of the weight function φ. The precise properties of φ are given in Section 2.4, viz., (2.22), (2.24) and (2.26). The weight function is at first constructed only depending on x_n. Dependency upon the other variables, i.e. convexification with respect to $\{x_n = 0\}$, is introduced in Section 4.5

Remark 1.3. It is important to notice that whenever a true discontinuity occurs for the vector field $A\nu$, then the space $\mathcal{W}$ does *not* contain $\mathscr{C}^\infty(\Omega)$: the inclusion $\mathscr{C}^\infty(\Omega) \subset \mathcal{W}$ implies from (1.9) that for all $w \in \mathscr{C}^\infty(\Omega)$, $\langle dw, A_+\nu - A_-\nu\rangle = 0$ at Σ so that $A_+\nu = A_-\nu$ at Σ, that is continuity for $A\nu$. The Carleman estimate which is proven in the present paper takes naturally into account these transmission conditions on the function w and it is important to keep in mind that the occurrence of a jump is excluding many smooth functions from the space $\mathcal{W}$. On the other hand, we have $\mathcal{W} \subset \operatorname{Lip}(\Omega)$.

Remark 1.4. We can also point out the geometric content of our assumptions, which do not depend on the choice of a coordinate system. For each $x \in \Omega$, the matrix $A(x)$ is a positive-definite symmetric mapping from $T_x(\Omega)^*$ onto $T_x(\Omega)$ so that $A(x)dw(x)$ belongs indeed to $T_x(\Omega)$ and $A dw$ is a vector field with a L^2 divergence (Inequality (1.13) yields the L^2 bound by density).

1.4. **Examples of applications.** Here we mention some applications of the Carleman estimate of Theorem 1.1, namely controllability for parabolic equations and stabilization for hyperbolic equations.

Following the work of [35, 37] (see also [32]) we first deduce the following interpolation inequality. With $\alpha \in (0, X_0/2)$, we set $X = (0, X_0) \times \Omega$, $Y = (\alpha, X_0 - \alpha) \times \Omega$.

Theorem 1.5. *There exist $C \geq 0$ and $\delta \in (0,1)$ such that for $u \in H^1(X)$ that satisfies $u_\pm = u|_{(0,X_0)\times\Omega_\pm} \in H^2((0, X_0) \times \Omega_\pm)$,*

$$u_+ = u_- \quad \text{and} \quad \langle du_+, A_+\nu\rangle = \langle du_-, A_-\nu\rangle \quad \text{at } (0, X_0) \times \Sigma,$$

and

$$u(x_0, x)|_{x\in\partial\Omega} = 0,\ x_0 \in (0, X_0), \quad \text{and} \quad u(0, x) = 0,\ x \in \Omega,$$

we have

$$\|u\|_{H^1(Y)} \leq C\|u\|^{\delta}_{H^1(X)} \left(\|(D^2_{x_0} + \mathcal{L})u\|_{L^2(X)} + \|\partial_{x_0} u(0, x)\|_{L^2(\omega)}\right)^{1-\delta}.$$

This interpolation inequality was first proven in [35, 37] for second-order elliptic operators with smooth coefficients and in [32] in the case of an isotropic diffusion coefficient with a jump at an interface. Here, a jump for the whole diffusion matrix is permitted.

Remark 1.6. In fact, the interpolation inequality of Theorem 1.5 rather follows from the non-homogeneous version of theorem 1.1 stated in Theorem 2.2 below.

From Theorem 1.5 we can prove an estimation of the loss of orthogonality for the eigenfunctions $\phi_j(x)$, $j \in \mathbb{N}$, of the operator $\mathcal{L}$, with Dirichlet boundary conditions, when these eigenfunctions are restricted to some subset ω of Ω (see [37, 25] and also [31]). We denote by μ_j, $j \in \mathbb{N}$, the associated eigenvalues, sorted in an increasing sequence.

Theorem 1.7. *There exists $C > 0$ such that for any $(a_j)_{j\in\mathbb{N}} \subset \mathbb{C}$ we have:*

$$\Big(\sum_{\mu_j\leq\mu} |a_j|^2\Big)^{\frac{1}{2}} = \Big\|\sum_{\mu_j\leq\mu} a_j\phi_j\Big\|_{L^2(\Omega)} \leq Ce^{C\sqrt{\mu}}\Big\|\sum_{\mu_j\leq\mu} a_j\phi_j\Big\|_{L^2(\omega)}, \quad \mu > 0. \tag{1.14}$$

In turn this yields the following null-controllability result for the associated anisotropic parabolic equation with jumps in the coefficients across Σ (see [35, 37, 32] and also [31]).

Theorem 1.8. *For an arbitrary time $T > 0$ and an arbitrary non-empty open subset $\omega \subset \Omega$ and an initial condition $y_0 \in L^2(\Omega)$, there exists $v \in L^2((0, T) \times \Omega)$ such that the solution y of*

$$\begin{cases} \partial_t y + \mathcal{L}y = 1_\omega u & \text{in } (0, T) \times \Omega, \\ y(t, x) = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, x) = y_0(x) & \text{in } \Omega, \end{cases} \tag{1.15}$$

satisfies $y(T) = 0$ a.e. in Ω.

The interpolation inequality of Theorem 1.5 also yields the stabilization of the following hyperbolic equation

$$\begin{cases} \partial_{tt} y + \mathcal{L} y + a(x)\partial_t y = 0 & \text{in } (0,T)\times\Omega, \\ y(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1.16}$$

where a is a nonvanishing nonnegative smooth function. From [34, 36], we can obtain a resolvent estimates which in turn yields the following energy decay estimate (see [10, Theorem 3]).

Theorem 1.9. *For all $k \in \mathbb{N}$ there exists $C > 0$ such that we have*

$$\|\partial_t y(t)\|_{L^2(\Omega)} + \|y(t)\|_{H^1(\Omega)} \leq \frac{C}{[\log(2+t)]^k} \Big(\|\partial_t y_{|t=0}\|_{D(\mathcal{L}^{\frac{k}{2}})} + \|y_{|t=0}\|_{D(\mathcal{L}^{\frac{k+1}{2}})} \Big), \quad t > 0,$$

for y solution to (1.16).

The same decay can also be obtained in the case of a boundary damping (see [36]).

Remark 1.10. Exponential decay cannot be achieved if the set $\mathcal{O} = \{a > 0\}$ does not satisfy the geometrical control condition of [43, 5]. Because of the jump in the matrix coefficient $A(x)$ here, some bicharacteristics of the hyperbolic operators $\partial_{tt} + \mathcal{L}$ can be trapped in Ω_+ or Ω_- and may remain away from the stabilization region $\mathcal{O}$.

1.5. **Sketch of the proof.** We provide in this subsection an outline of the main arguments used in our proof. To avoid technicalities, we somewhat simplify the geometric data and the weight function, keeping of course the anisotropy. We consider the operator

$$\mathcal{L}_0 = \sum_{1\leq j\leq n} D_j c_j D_j,\ c_j(x) = H_+ c_j^+ + H_- c_j^-,\ c_j^\pm > 0 \text{ constants},\ H_\pm = \mathbf{1}_{\{\pm x_n > 0\}}, \tag{1.17}$$

with $D_j = \frac{\partial}{i\partial x_j}$, and the vector space $\mathcal{W}_0$ of functions $H_+ w_+ + H_- w_-$, $w_\pm \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, such that

$$\text{at } x_n = 0,\ w_+ = w_-,\ c_n^+ \partial_n w_+ = c_n^- \partial_n w_- \ \text{(transmission conditions across } x_n = 0). \tag{1.18}$$

As a result, for $w \in \mathcal{W}_0$, we have $D_n w = H_+ D_n w_+ + H_- D_n w_-$ and

$$\mathcal{L}_0 w = \sum_j (H_+ c_j^+ D_j^2 w_+ + H_- c_j^- D_j^2 w_-). \tag{1.19}$$

We also consider a weight function[5]

$$\varphi = \underbrace{(\alpha_+ x_n + \beta x_n^2/2)}_{\varphi_+} H_+ + \underbrace{(\alpha_- x_n + \beta x_n^2/2)}_{\varphi_-} H_-, \quad \alpha_\pm > 0, \quad \beta > 0, \tag{1.20}$$

a positive parameter τ and the vector space $\mathcal{W}_\tau$ of functions $H_+ v_+ + H_- v_-$, $v_\pm \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, such that at $x_n = 0$,

$$v_+ = v_-, \tag{1.21}$$

$$c_n^+ (D_n v_+ + i\tau\alpha_+ v_+) = c_n^- (D_n v_- + i\tau\alpha_- v_-). \tag{1.22}$$

[5] In the main text, we shall introduce some minimal requirements on the weight function and suggest other possible choices.

Observe that $w \in \mathcal{W}_0$ is equivalent to $v = e^{\tau\varphi} w \in W_\tau$. We have

$$e^{\tau\varphi}\mathcal{L}_0 w = \underbrace{e^{\tau\varphi}\mathcal{L}_0 e^{-\tau\varphi}}_{\mathcal{L}_\tau}(e^{\tau\varphi}w)$$

so that proving a weighted a priori estimate $\|e^{\tau\varphi}\mathcal{L}_0 w\|_{L^2(\mathbb{R}^n)} \gtrsim \|e^{\tau\varphi}w\|_{L^2(\mathbb{R}^n)}$ for $w \in \mathcal{W}_0$ amounts to getting $\|\mathcal{L}_\tau v\|_{L^2(\mathbb{R}^n)} \gtrsim \|v\|_{L^2(\mathbb{R}^n)}$ for $v \in \mathcal{W}_\tau$.

Step 1: pseudo-differential factorization. Using Einstein convention on repeated indices $j \in \{1, \dots, n-1\}$, we have

$$\mathcal{L}_\tau = (D_n + i\tau\varphi')c_n(D_n + i\tau\varphi') + D_j c_j D_j$$

and for $v \in \mathcal{W}_\tau$, from (1.19), with $m_\pm = m_\pm(D') = (c_n^\pm)^{-1/2}(c_j^\pm D_j^2)^{1/2}$,

$$\mathcal{L}_\tau v = H_+ c_n^+\big((D_n + i\tau\varphi'_+)^2 + m_+^2\big)v_+ + H_- c_n^-\big((D_n + i\tau\varphi'_-)^2 + m_-^2\big)v_-$$

so that

$$\begin{aligned}(1.23)\quad \mathcal{L}_\tau v = H_+ c_n^+\big(D_n + i(\overbrace{\tau\varphi'_+ + m_+}^{e_+})\big)\big(D_n + i(\overbrace{\tau\varphi'_+ - m_+}^{f_+})\big)v_+ \\ + H_- c_n^-\big(D_n + i(\underbrace{\tau\varphi'_- - m_-}_{f_-})\big)\big(D_n + i(\underbrace{\tau\varphi'_- + m_-}_{e_-})\big)v_-.\end{aligned}$$

Note that $e_\pm$ are elliptic positive in the sense that $e_\pm = \tau\alpha_\pm + m_\pm \gtrsim \tau + |D'|$. We want at this point to use some natural estimates for first-order factors on the half-lines $\mathbb{R}_\pm$: let us for instance check on $t > 0$ for $\omega \in \mathscr{C}_c^\infty(\mathbb{R})$, λ, γ positive,

(1.24)

$$\begin{aligned}&\|D_t\omega + i(\lambda + \gamma t)\omega\|^2_{L^2(\mathbb{R}_+)} \\ &\quad = \|D_t\omega\|^2_{L^2(\mathbb{R}_+)} + \|(\lambda + \gamma t)\omega\|^2_{L^2(\mathbb{R}_+)} + 2\operatorname{Re}\langle D_t\omega, iH(t)(\lambda + \gamma t)\omega\rangle \\ &\quad \geq \int_0^{+\infty}\big((\lambda + \gamma t)^2 + \gamma\big)|\omega(t)|^2 dt + \lambda|\omega(0)|^2 \geq (\lambda^2 + \gamma)\|\omega\|^2_{L^2(\mathbb{R}_+)} + \lambda|\omega(0)|^2,\end{aligned}$$

which is somehow a perfect estimate of elliptic type, suggesting that the first-order factor containing e_+ should be easy to handle. Changing λ in $-\lambda$ gives

$$\begin{aligned}\|D_t\omega + i(-\lambda + \gamma t)\omega\|^2_{L^2(\mathbb{R}_+)} &\geq 2\operatorname{Re}\langle D_t\omega, iH(t)(-\lambda + \gamma t)\omega\rangle \\ &= \int_0^{+\infty}\gamma|\omega(t)|^2 dt - \lambda|\omega(0)|^2,\end{aligned}$$

so that $\|D_t\omega + i(-\lambda + \gamma t)\omega\|^2_{L^2(\mathbb{R}_+)} + \lambda|\omega(0)|^2 \geq \gamma\|\omega\|^2_{L^2(\mathbb{R}_+)}$, an estimate of lesser quality, because we need to secure a control of $\omega(0)$ to handle this type of factor.

Step 2: case $f_+ \geq 0$. Looking at formula (1.23), since the factor containing e_+ is elliptic in the sense given above, we have to discuss on the sign of f_+. Identifying the operator with its symbol, we have $f_+ = \tau(\alpha_+ + \beta x_n) - m_+(\xi')$, and thus $\tau\alpha_+ \geq m_+(\xi')$ yielding a non negative f_+. Iterating the method outlined above on the half-line $\mathbb{R}_+$, we get a nice estimate of the form of (1.24) on $\mathbb{R}_+$; in particular we obtain a control[6]

[6]In the case $f_+(0) = 0$, one needs to consider the estimation of

$$\|(D_n + ie_+)(D_n + if_+)v_+\|_{L^2(\mathbb{R}^+)} + \|(D_n + if_+)(D_n + ie_+)v_+\|_{L^2(\mathbb{R}^+)}$$

of $v_+(0)$ and $D_n v_+(0)$. From the transmission condition, we have $v_+(0) = v_-(0)$ and hence this amounts to also controlling $v_-(0)$. That control along with the natural estimates on $\mathbb{R}_-$ are enough to prove an inequality of the form of the sought Carleman estimate.

Step 3: case $f_+ < 0$. Here, we assume that $\tau\alpha_+ < m_+(\xi')$. We can still use on $\mathbb{R}_+$ the factor containing e_+, and by (1.23) and (1.24) control the following quantity

$$c_n^+(D_n + if_+)v_+(0) = \overbrace{c_n^+(D_n v_+ + i\tau\alpha_+)v_+(0)}^{=\mathcal{V}_+} - c_n^+ i m_+ v_+(0). \tag{1.25}$$

Our key assumption is

$$f_+(0) < 0 \Longrightarrow f_-(0) \leq 0. \tag{1.26}$$

Under that hypothesis, we can use the negative factor f_- on $\mathbb{R}_-$ (note that f_- is increasing with x_n, so that $f_-(0) \leq 0 \Longrightarrow f_-(x_n) < 0$ for $x_n < 0$). We then control

$$c_n^-(D_n + ie_-)v_-(0) = \underbrace{c_n^-(D_n v_- + i\tau\alpha_-)v_-(0)}_{=\mathcal{V}_-} + c_n^- i m_- v_-(0). \tag{1.27}$$

Nothing more can be achieved with inequalities on each side of the interface. At this point we however notice that the second transmission condition in (1.22) implies $\mathcal{V}_- = \mathcal{V}_+$, yielding the control of the difference of (1.27) and (1.25), i.e., of

$$c_n^- i m_- v_-(0) + c_n^+ i m_+ v_+(0) = i\big(c_n^- m_- + c_n^+ m_+\big)v(0).$$

Now, as $c_n^- m_- + c_n^+ m_+$ is elliptic positive, this gives a control of $v(0)$ in (tangential) H^1-norm, which is enough then to get an estimate on both sides that leads to the sought Carleman estimates.

Step 4: patching estimates together. The analysis we have sketched here relies on a separation into two zones in the (τ, ξ') space. Patching the estimates of the form of (1.13) in each zone together allows us to conclude the proof of the Carleman estimate.

1.6. **Explaining the key assumption.** In the first place, our key assumption, condition (1.26), can be reformulated as

$$\forall \xi' \in \mathbb{S}^{n-2}, \qquad \frac{\alpha_+}{\alpha_-} \geq \frac{m_+(\xi')}{m_-(\xi')}. \tag{1.28}$$

In fact [7], (1.26) means $\tau\alpha_+ < m_+(\xi') \Longrightarrow \tau\alpha_- \leq m_-(\xi')$ and since $\alpha_\pm, m_\pm$ are all positive, this is equivalent to having $m_+(\xi')/\alpha_+ \leq m_-(\xi')/\alpha_-$, which is (1.28).

from below to obtain a control of $v_+(0)$ and $D_n v_+(0)$ with the previous estimates used in cascade. Indeed the first term will give an estimate of $D_n v_+(0)$ and the second term one of $v_+(0)$.

[7] For the main theorem, we shall in fact require the stronger strict inequality

$$\frac{\alpha_+}{\alpha_-} > \frac{m_+(\xi')}{m_-(\xi')}. \tag{1.29}$$

This condition is then stable under perturbations, whereas (1.28) is not. This gives freedom to introduce microlocal cutoff in the analysis below.

However, we shall see in Section 5 that in the particular case presented here, where the matrix A is piecewise constant and the weight function φ solely depends on x_n the inequality (1.28) is actually a *necessary and sufficient* condition to obtain a Carleman estimate with weight φ.

An analogy with an estimate for a first-order factor may shed some light on this condition. With

$$f(t) = H(t)(\tau\alpha_+ + \beta t - m_+) + H(-t)(\tau\alpha_- + \beta t - m_-), \quad \tau, \alpha_\pm, \beta, m_\pm \text{ positive constants},$$

we want to prove an injectivity estimate of the type $\|D_t v + if(t)v\|_{L^2(\mathbb{R})} \gtrsim \|v\|_{L^2(\mathbb{R})}$, say for $v \in \mathscr{C}_c^\infty(\mathbb{R})$. It is a classical fact (see e.g. Lemma 3.1.1 in [38]) that such an estimate (for a smooth f) is equivalent to the condition that $t \mapsto f(t)$ does not change sign from $+$ to $-$ while t increases: it means that the adjoint operator $D_t - if(t)$ satisfies the so-called condition (Ψ). Looking at the function f, we see that it increases on each half-line $\mathbb{R}_\pm$, so that the only place to get a "forbidden" change of sign from $+$ to $-$ is at $t = 0$: to get an injectivity estimate, we have to avoid the situation where $f(0^+) < 0$ and $f(0^-) > 0$, that is, we have to make sure that $f(0^+) < 0 \Longrightarrow f(0^-) \leq 0$, which is indeed the condition (1.28). The function f is increasing affine on $\mathbb{R}_\pm$ with the same slope β on both sides, with a possible discontinuity at 0.

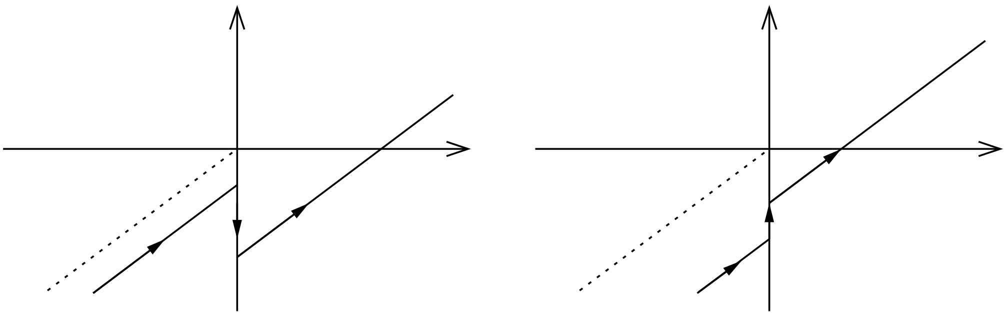

FIGURE 1. $f(0^-) \leq 0$; $f(0^+) < 0$.

When $f(0^+) < 0$ we should have $f(0^-) \leq 0$ and the line on the left cannot go above the dotted line, in such a way that the discontinuous zigzag curve with the arrows has only a change of sign from $-$ to $+$.

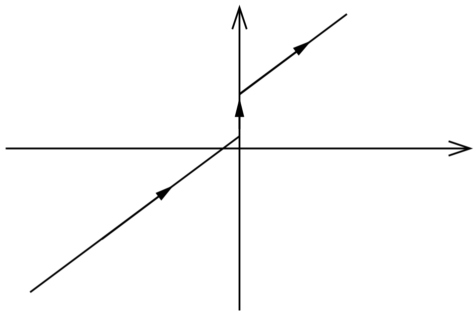

FIGURE 2. $f(0^-) \gtrless 0$; $f(0^+) \geq 0$.

When $f(0^+) \geq 0$, there is no other constraint on $f(0^-)$: even with a discontinuity, the change of sign can only occur from $-$ to $+$.

We prove below (Section 5) that condition (1.28) is relevant to our problem in the sense that it is indeed necessary to have a Carleman estimate with this weight:

if (1.28) is violated, we are able for this model to construct a quasi-mode for $\mathcal{L}_\tau$, i.e. a τ-family of functions v with L^2-norm 1 such that $\|\mathcal{L}_\tau v\|_{L^2} \ll \|v\|_{L^2}$, as τ goes to ∞, ruining any hope to prove a Carleman estimate. As usual for this type of construction, it uses some type of complex geometrical optics method, which is easy in this case to implement directly, due to the simplicity of the expression of the operator.

Remark 1.11. A very particular case of anisotropic medium was tackled in [32] for the purpose of proving a controllability result for linear parabolic equations. The condition imposed on the weight function in [32] (Assumption 2.1 therein) is much more demanding than what we impose here. In the isotropic case, $c_j^\pm = c_\pm$ for all $j \in \{1, \dots, n\}$, we have $m_+ = m_- = |\xi'|$ and our condition (1.29) reads $\alpha_+ > \alpha_-$. Note also that the isotropic case $c_- \geq c_+$ was already considered in [13].

In [32], the controllability result concerns an isotropic parabolic equation. The Carleman estimate we derive here extends this result to an anisotropic parabolic equation.

2. Framework

2.1. Presentation.

Let Ω, Σ be as in (1.3). With

$$\Xi = \{\text{positive-definite } n \times n \text{ matrices}\},$$

we consider $A_\pm \in \mathscr{C}^\infty(\Omega; \Xi)$ and let $\mathcal{L}, \varphi$ be as in (1.5) and (1.12). We set

$$\mathcal{L}_\pm = D \cdot A_\pm D = -\operatorname{div}(A_\pm \nabla).$$

Here, we generalize our analysis to non-homogeneous transmission conditions: for θ and Θ smooth functions of the interface Σ we set

$$w_+ - w_- = \theta, \quad \text{and} \quad \langle A_+ dw_+ - A_- dw_-, \nu\rangle = \Theta \quad \text{at } \Sigma, \tag{2.1}$$

(compare with (1.8)-(1.9)) and introduce

$$\mathcal{W}_0^{\theta,\Theta} = \left\{H_- w_- + H_+ w_+\right\}_{w_\pm \in \mathscr{C}_c^\infty(\Omega) \text{ satisfying (2.1)}}. \tag{2.2}$$

For $\tau \geq 0$ we define the affine space

$$\mathcal{W}_\tau^{\theta,\Theta} = \{e^{\tau\varphi} w\}_{w \in \mathcal{W}_0^{\theta,\Theta}}. \tag{2.3}$$

For $v \in \mathcal{W}_\tau^{\theta,\Theta}$, we have $v = e^{\tau\varphi} w$ with $w \in \mathcal{W}_0^{\theta,\Theta}$ so that, using the notation introduced in (1.4), (1.7), with $v_\pm = e^{\tau\varphi_\pm} w_\pm$, we have

$$v = H_- v_- + H_+ v_+, \tag{2.4}$$

and we see that the transmission conditions (2.1) on w read for v as

$$v_+ - v_- = \theta_\varphi, \quad \langle dv_+ - \tau v_+ d\varphi_+, A_+\nu\rangle - \langle dv_- - \tau v_- d\varphi_-, A_-\nu\rangle = \Theta_\varphi, \quad \text{at } \Sigma, \tag{2.5}$$

with

$$\theta_\varphi = e^{\tau\varphi_{|\Sigma}}\theta, \quad \Theta_\varphi = e^{\tau\varphi_{|\Sigma}}\Theta. \tag{2.6}$$

Observing that $e^{\tau\varphi_\pm} D e^{-\tau\varphi_\pm} = D + i\tau d\varphi_\pm$, for $w \in \mathcal{W}^{\theta,\Theta}$, we obtain

$$e^{\tau\varphi_\pm}\mathcal{L}_\pm w_\pm = e^{\tau\varphi_\pm} D \cdot A_\pm D e^{-\tau\varphi_\pm} v_\pm = (D + i\tau d\varphi_\pm) \cdot A_\pm (D + i\tau d\varphi_\pm) v_\pm$$

We define

$$\mathcal{P}_\pm = (D + i\tau d\varphi_\pm)\cdot A_\pm(D + i\tau d\varphi_\pm). \tag{2.7}$$

Proposition 2.1. *Let $\Omega, \Sigma, \mathcal{L}, \mathcal{W}_\tau^{\theta,\Theta}$ be as in (1.3), (1.5) and (2.3). Then for any compact subset K of Ω, there exist a weight function φ satisfying (1.12) and positive constants C, τ_1 such that for all $\tau \geq \tau_1$ and all $v \in \mathcal{W}_\tau$ with* $\operatorname{supp} v \subset K$

$$C\big(\|H_-\mathcal{P}_-v_-\|_{L^2(\mathbb{R}^n)} + \|H_+\mathcal{P}_+v_+\|_{L^2(\mathbb{R}^n)} + \mathcal{T}_{\theta,\Theta}\big) \geq \tau^{3/2}|v_\pm|_{L^2(\Sigma)} + \tau^{1/2}|(\nabla v_\pm)|_{L^2(\Sigma)} \\ + \tau^{3/2}\|v\|_{L^2(\mathbb{R}^n)} + \tau^{1/2}\|H_+\nabla v_+\|_{L^2(\mathbb{R}^n)} + \tau^{1/2}\|H_-\nabla v_-\|_{L^2(\mathbb{R}^n)},$$

where $\mathcal{T}_{\theta,\Theta} = \tau^{3/2}|\theta_\varphi|_{L^2(\Sigma)} + \tau^{1/2}|\nabla_\Sigma\theta_\varphi|_{L^2(\Sigma)} + \tau^{1/2}|\Theta_\varphi|_{L^2(\Sigma)}$.

Here, ∇_Σ denotes the tangential gradient to Σ. The proof of this proposition will occupy a large part of the remainder of the article (Sections 3 and 4) as it implies the result of the following theorem, a non-homogenous version of Theorem 1.1.

Theorem 2.2. *Let $\Omega, \Sigma, \mathcal{L}, \mathcal{W}_0^{\theta,\Theta}$ be as in (1.3), (1.5) and (2.2). Then for any compact subset K of Ω, there exist a weight function φ satisfying (1.12) and positive constants C, τ_1 such that for all $\tau \geq \tau_1$ and all $w \in \mathcal{W}$ with* $\operatorname{supp} w \subset K$,

$$\begin{aligned} &C\big(\|H_-e^{\tau\varphi_-}\mathcal{L}_-w_-\|_{L^2(\mathbb{R}^n)} + \|H_+e^{\tau\varphi_+}\mathcal{L}_+w_+\|_{L^2(\mathbb{R}^n)} + T_{\theta,\Theta}\big) \\ &\quad \geq \tau^{3/2}\|e^{\tau\varphi}w\|_{L^2(\mathbb{R}^n)} + \tau^{1/2}\big(\|H_+e^{\tau\varphi}\nabla w_+\|_{L^2(\mathbb{R}^n)} + \|H_-e^{\tau\varphi}\nabla w_-\|_{L^2(\mathbb{R}^n)}\big) \\ &\qquad + \tau^{3/2}|e^{\tau\varphi}w_\pm|_{L^2(\Sigma)} + \tau^{1/2}|e^{\tau\varphi}\nabla w_\pm|_{L^2(\Sigma)}. \end{aligned} \tag{2.8}$$

where $T_{\theta,\Theta} = \tau^{3/2}|e^{\tau\varphi_{|\Sigma}}\theta|_{L^2(\Sigma)} + \tau^{1/2}|e^{\tau\varphi_{|\Sigma}}\nabla_\Sigma\theta|_{L^2(\Sigma)} + \tau^{1/2}|e^{\tau\varphi_{|\Sigma}}\Theta|_{L^2(\Sigma)}$.

Theorem 1.1 corresponds to the case $\theta = \Theta = 0$ since by (1.10) we then have

$$\|e^{\tau\varphi}\mathcal{L}w\|_{L^2(\mathbb{R}^n)} = \|H_-e^{\tau\varphi_-}\mathcal{L}_-w_-\|_{L^2(\mathbb{R}^n)} + \|H_+e^{\tau\varphi_+}\mathcal{L}_+w_+\|_{L^2(\mathbb{R}^n)}$$

Remark 2.3. It is often useful to have such a Carleman estimate at hand for the case non-homogeneous transmission conditions, for examples when on tries to patch such local estimates together in the neighborhood of the interface.

Here, we derive local Carleman estimates. We can in fact consider similar geometrical situation on a Riemannian manifold (with or without boundary) with a metric exhibiting jump discontinuities across interfaces. For the associated Laplace-Beltrami operator the local estimates we derive can be patched together to yield a global estimate. We refer to [33, Section 5] for such questions.

Proof that Proposition 2.1 implies Theorem 2.2. Replacing v by $e^{\tau\varphi}w$, we get

$$\begin{aligned} &\|H_-e^{\tau\varphi_-}\mathcal{L}_-w_-\|_{L^2(\mathbb{R}^n)} + \|H_+e^{\tau\varphi_+}\mathcal{L}_+w_+\|_{L^2(\mathbb{R}^n)} + T_{\theta,\Theta} \\ &\quad \gtrsim \tau^{3/2}\|e^{\tau\varphi}w\|_{L^2(\mathbb{R}^n)} + \tau^{1/2}\big(\|H_+\nabla e^{\tau\varphi}w_+\|_{L^2(\mathbb{R}^n)} + \|H_-\nabla e^{\tau\varphi}w_-\|_{L^2(\mathbb{R}^n)}\big) \\ &\qquad + \tau^{3/2}|e^{\tau\varphi}w_\pm|_{L^2(\Sigma)} + \tau^{1/2}|\nabla e^{\tau\varphi}w_\pm|_{L^2(\Sigma)}. \end{aligned} \tag{2.9}$$

Commuting ∇ with $e^{\tau\varphi}$ produces

$$\begin{aligned} C\big(\|H_-e^{\tau\varphi_-}\mathcal{L}_-w_-\|_{L^2(\mathbb{R}^n)} + \|H_+e^{\tau\varphi_+}\mathcal{L}_+w_+\|_{L^2(\mathbb{R}^n)} + T_{\theta,\Theta}\big) \\ + C_1\tau^{3/2}\|e^{\tau\varphi}w\|_{L^2(\mathbb{R}^n)} + C_2\tau^{3/2}\big(|e^{\tau\varphi}w_{\pm|\Sigma}|_{L^2(\Sigma)}\big) \\ \geq \tau^{1/2}\|H_-e^{\tau\varphi}Dw_-\|_{L^2(\mathbb{R}^n)} + \tau^{1/2}\|H_+e^{\tau\varphi}Dw_+\|_{L^2(\mathbb{R}^n)} + \tau^{3/2}\|e^{\tau\varphi}w\|_{L^2(\mathbb{R}^n)} \\ + \tau^{1/2}|e^{\tau\varphi}Dw_\pm|_{L^2(\Sigma)} + \tau^{3/2}|e^{\tau\varphi}w_\pm|_{L^2(\Sigma)}, \end{aligned}$$

but from (2.9) we have

$$\begin{aligned} C_1\tau^{3/2}\|e^{\tau\varphi}w\| + C_2\tau^{3/2}|e^{\tau\varphi}w| \\ \leq C\max(C_1, C_2)\big(\|H_-e^{\tau\varphi_-}\mathcal{L}_-w_-\|_{L^2(\mathbb{R}^n)} + \|H_+e^{\tau\varphi_+}\mathcal{L}_+w_+\|_{L^2(\mathbb{R}^n)} + T_{\theta,\Theta}\big), \end{aligned}$$

proving the implication. ■

2.2. **Description in local coordinates.** Carleman estimates of types (1.13) and (2.8) can be handled locally as they can be patched together. Assuming as we may that the hypersurface Σ is given locally by the equation $\{x_n = 0\}$, we have, using the Einstein convention on repeated indices $j \in \{1, \dots, n-1\}$, and noting from the ellipticity condition that $a_{nn} > 0$ (the matrix $A(x) = (a_{jk}(x))_{1\leq j,k\leq n}$),

$$\begin{aligned} \mathcal{L} &= D_n a_{nn} D_n + D_n a_{nj} D_j + D_j a_{jn} D_n + D_j a_{jk} D_k, \\ &= D_n a_{nn}\big(D_n + a_{nn}^{-1}a_{nj}D_j\big) + D_j a_{jn} D_n + D_j a_{jk} D_k, \end{aligned}$$

With $T = a_{nn}^{-1}a_{nj}D_j$ we have

$$\mathcal{L} = \big(D_n + T^*)a_{nn}\big(D_n + T\big) - T^*a_{nn}D_n - T^*a_{nn}T + D_j a_{jn} D_n + D_j a_{jk} D_k.$$

and since $T^* = D_j a_{nn}^{-1}a_{nj}$, we have $T^*a_{nn}D_n = D_j a_{nj} D_n = D_j a_{jn} D_n$ and

$$\mathcal{L} = \big(D_n + T^*)a_{nn}\big(D_n + T\big) + D_j b_{jk} D_k, \tag{2.10}$$

where the $(n-1)\times(n-1)$ matrix (b_{jk}) is positive-definite since with $\xi' = (\xi_1, \dots, \xi_{n-1})$ and $\xi = (\xi', \xi_n)$,

$$\langle B\xi', \xi'\rangle = \sum_{1\leq j,k\leq n-1} b_{jk}\xi_j\xi_k = \langle A\xi, \xi\rangle,$$

where $a_{nn}\xi_n = -\sum_{1\leq j\leq n-1} a_{nj}\xi_j$. Note also that $b_{jk} = a_{jk} - (a_{nj}a_{nk}/a_{nn})$.

Remark 2.4. The positive-definite quadratic form B is the restriction of $\langle A\xi, \xi\rangle$ to the hyperplane $\mathcal{H}$ defined by $\{\langle A\xi, \xi\rangle, x_n\} = \partial_{\xi_n}\big(\langle A\xi, \xi\rangle\big) = 0$, where $\{\cdot, \cdot\}$ stands for the Poisson bracket. In fact the principal symbol of $\mathcal{L}$ is $\langle A(x)\xi, \xi\rangle$ and if Σ is defined by the equation $\psi(x) = 0$ with $d\psi \neq 0$ at Σ, we have

$$\frac{1}{2}\Big\{\langle A(x)\xi, \xi\rangle, \psi\Big\} = \langle A(x)\xi, d\psi(x)\rangle$$

so that $\mathcal{H}_x = \big(A(x)d\psi(x)\big)^\perp = \{\xi \in T^*_x(\Omega), \langle \xi, A(x)d\psi(x)\rangle_{T^*_x(\Omega),T_x(\Omega)} = 0\}$. When $x \in \Sigma$, that set does not depend on the choice of the defining function ψ of Σ and we have simply

$$\mathcal{H}_x = \big(A(x)\nu(x)\big)^\perp = \{\xi \in T^*_x(\Omega), \langle \xi, A(x)\nu(x)\rangle_{T^*_x(\Omega),T_x(\Omega)} = 0\}$$

where $\nu(x)$ is the conormal vector to Σ at x (recall that from Remark 1.4, $\nu(x)$ is a cotangent vector at x, $A(x)\nu(x)$ is a tangent vector at x). Now, for $x \in \Sigma$, we can restrict the quadratic form $A(x)$ to $\mathcal{H}_x$: this is the positive-definite quadratic form $B(x)$, providing a coordinate-free definition.

For $w \in \mathcal{W}_0^{\theta,\Theta}$, we have

$$\mathcal{L}_\pm w_\pm = (D_n + T_\pm^*) a_{nn}^\pm (D_n + T_\pm) w_\pm + D_j b_{jk}^\pm D_k w_\pm \tag{2.11}$$

and the non-homogeneous transmission conditions (2.1) read

$$w_+ - w_- = \theta, \quad a_{nn}^+ (D_n + T_+) w_+ - a_{nn}^- (D_n + T_-) w_- = \Theta, \quad \text{at } \Sigma. \tag{2.12}$$

2.3. **Pseudo-differential factorization on each side.** At first we consider the weight function $\varphi = H_+\varphi_+ + H_-\varphi_-$ with $\varphi_\pm$ that solely depend on x_n. Later on we shall allow for some dependency upon the tangential variables x' (see Section 4.5). We define for $m \in \mathbb{R}$ the class of tangential standard symbols $\mathcal{S}^m$ as the smooth functions on $\mathbb{R}^n \times \mathbb{R}^{n-1}$ such that, for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^{n-1}$,

$$\sup_{(x,\xi') \in \mathbb{R}^n \times \mathbb{R}^{n-1}} \langle \xi' \rangle^{-m+|\beta|} |(\partial_x^\alpha \partial_{\xi'}^\beta a)(x, \xi')| < \infty, \tag{2.13}$$

with $\langle \xi' \rangle = \big(1 + |\xi'|^2\big)^{\frac{1}{2}}$. Some basic properties of standard pseudo-differential operators are recalled in Appendix 6.1. Section 2.2 and formulæ (2.7), (2.11) give

$$\mathcal{P}_\pm = \big(D_n + i\tau\varphi'_\pm + T_\pm^*\big) a_{nn}^\pm \big(D_n + i\tau\varphi'_\pm + T_\pm\big) + D_j b_{jk}^\pm D_k. \tag{2.14}$$

We define $m_\pm \in \mathcal{S}^1$ such that

$$\text{for } |\xi'| \geq 1,\ m_\pm = \Big(\frac{b_{jk}^\pm}{a_{nn}^\pm} \xi_j \xi_k\Big)^{\frac{1}{2}}, \quad m_\pm \geq C\langle \xi' \rangle, \quad M_\pm = \operatorname{op}^w(m_\pm). \tag{2.15}$$

We have then $M_\pm^2 \equiv D_j b_{jk}^\pm D_k \mod \operatorname{op}(\mathcal{S}^1)$.

We define

$$\Psi^1 = \operatorname{op}(\mathcal{S}^1) + \tau \operatorname{op}(\mathcal{S}^0) + \operatorname{op}(\mathcal{S}^0) D_n. \tag{2.16}$$

Modulo the operator class Ψ^1 we may write

$$\mathcal{P}_+ \equiv \mathcal{P}_{E+} a_{nn}^+ \mathcal{P}_{F+}, \qquad \mathcal{P}_- \equiv \mathcal{P}_{F-} a_{nn}^- \mathcal{P}_{E-}, \tag{2.17}$$

where

$$\mathcal{P}_{E\pm} = D_n + S_\pm + i(\underbrace{\tau\varphi'_\pm + M_\pm}_{E_\pm}), \quad \mathcal{P}_{F\pm} = D_n + S_\pm + i(\underbrace{\tau\varphi'_\pm - M_\pm}_{F_\pm}), \tag{2.18}$$

with

(2.19)

$$S_\pm = s^w(x, D'), \quad s_\pm = \sum_{1 \leq j \leq n-1} \frac{a_{nj}^\pm}{a_{nn}^\pm} \xi_j, \quad \text{so that } S_\pm^* = S_\pm, \quad S_\pm = T_\pm + \frac{1}{2} \operatorname{div} T_\pm,$$

where

$$T_\pm \text{ is the vector field} \quad \sum_{1 \leq j \leq n-1} \frac{a_{nj}^\pm}{i a_{nn}^\pm} \partial_j. \tag{2.20}$$

We denote by $f_\pm$ and $e_\pm$ the homogeneous principal symbols of $F_\pm$ and $E_\pm$ respectively, determined modulo the symbol class $\mathcal{S}^1 + \tau\mathcal{S}^0$. The transmission conditions (2.12) with our choice of coordinates read, at $x_n = 0$,

$$\begin{cases} v_+ - v_- = \theta_\varphi = e^{\tau\varphi_{|x_n=0}}\theta, \\ a_{nn}^+(D_n + T_+ + i\tau\varphi_+')v_+ - a_{nn}^-(D_n + T_- + i\tau\varphi_-')v_- = \Theta_\varphi = e^{\tau\varphi_{|x_n=0}}\Theta. \end{cases} \tag{2.21}$$

Remark 2.5. Note that the Carleman estimate we shall prove is insensitive to terms in Ψ^1 in the conjugated operator $\mathcal{P}$. Formulæ (2.17) and (2.18) for $\mathcal{P}_+$ and $\mathcal{P}_-$ will thus be the base of our analysis.

Remark 2.6. In the articles [32, 33], the zero crossing of the roots of the symbol of $\mathcal{P}_\pm$, as seen as a polynomial in ξ_n, is analyzed. Here the factorization into first-order operators isolates each root. In fact, $f_\pm$ changes sign and we shall impose a condition on the weight function at the interface to obtain a certain scheme for this change of sign. See Section 4.

2.4. **Choice of weight-function.** The weight function can be taken of the form

$$\varphi_\pm(x_n) = \alpha_\pm x_n + \beta x_n^2/2, \quad \alpha_\pm > 0, \quad \beta > 0. \tag{2.22}$$

The choice of the parameters $\alpha_\pm$ and β will be done below and that choice will take into account the geometric data of our problem: $\alpha_\pm$ will be chosen to fulfill a geometric condition at the interface and $\beta > 0$ will be chosen large. Here, we shall require $\varphi' \geq 0$, that is, an "observation" region on the rhs of Σ. As we shall need β large, this amounts to working in a small neighborhood of the interface, i.e., $|x_n|$ small. Also, we shall see below (Section 4.5) that this weight can be perturbed by any smooth function with a small gradient.

Other choices for the weight functions are possible. In fact, two sufficient conditions can be put forward. We shall describe them now.

The operators $M_\pm$ have a principal symbol $m_\pm(x, \xi')$ in $\mathcal{S}^1$, which is positively-homogeneous[8] of degree 1 and elliptic, i.e. there exists $\lambda_0^\pm, \lambda_1^\pm$ positive such that for $|\xi'| \geq 1, x \in \mathbb{R}^n$,

$$\lambda_0^\pm |\xi'| \leq m_\pm(x, \xi') \leq \lambda_1^\pm |\xi'|. \tag{2.23}$$

We choose $\varphi'_{|x_n=0^\pm} = \alpha_\pm$ such that

$$\frac{\alpha_+}{\alpha_-} > \sup_{\substack{x', \xi' \\ |\xi'| \geq 1}} \frac{m_+(x', \xi')_{|x_n=0^+}}{m_-(x', \xi')_{|x_n=0^-}}. \tag{2.24}$$

The consequence of this condition will be made clear in Section 4. We shall also prove that this condition is sharp in Section 5: a strong violation of this condition, viz., $\alpha_+/\alpha_- < \sup(m_+/m_-)_{|x_n=0}$, ruins any possibility of deriving a Carleman estimate of the form of Theorem 1.1.

Condition (2.24) concerns the behavior of the weight function at the interface. Conditions away from the interface are also needed. These conditions are more

[8] The homogeneity property means as usual $m_\pm(x, \rho\xi') = \rho m_\pm(x, \xi')$ for $\rho \geq 1, |\xi'| \geq 1$.

classical. From (2.14), the symbols of $\mathcal{P}_\pm$, modulo the symbol class $\mathcal{S}^1+\tau\mathcal{S}^0+\mathcal{S}^0\xi_n$, are given by $p_\pm(x,\xi,\tau)=a^\pm_{nn}\big(q_2^\pm+2iq_1^\pm\big)$, with

$$q_2^\pm=(\xi_n+s_\pm)^2+\frac{b^\pm_{jk}}{a^\pm_{nn}}\xi_j\xi_k-\tau^2(\varphi'_\pm)^2,\qquad q_1^\pm=\tau\varphi'_\pm(\xi_n+s_\pm),$$

for φ solely depending on x_n, and from the construction of $m_\pm$, for $|\xi'|\geq 1$, we have

$$q_2^\pm=(\xi_n+s_\pm)^2+m_\pm^2-(\tau\varphi'_\pm)^2=(\xi_n+s_\pm)^2-f_\pm e_\pm. \tag{2.25}$$

We can then formulate the usual *sub-ellipticity* condition, with *loss of a half-derivative*:

$$q_2^\pm=0 \text{ and } q_1^\pm=0 \quad\Longrightarrow\quad \{q_2^\pm,q_1^\pm\}>0, \tag{2.26}$$

which can be achieved by choosing β sufficiently large. It is important to note that this property is coordinate free. For second-order elliptic operators with real smooth coefficients this property is necessary and sufficient for a Carleman estimate as that of Theorem 1.1 to hold (see [18] or e.g. [31]).

With the weight functions provided in (2.22) we choose $\alpha_\pm$ according to condition (2.24) and we choose $\beta>0$ large enough and we restrict ourselves to a small neighborhood of Σ, i.e., $|x_n|$ small to have $\varphi'>0$, and so that (2.26) is fulfilled.

Remark 2.7. Other "classical" forms for the weight function φ are also possible. For instance, one may use $\varphi(x_n)=e^{\beta\phi(x_n)}$ with the function ϕ depending solely on x_n of the form

$$\phi=H_-\phi_-+H_+\phi_+,\quad \phi_\pm\in\mathscr{C}_c^\infty(\mathbb{R}),$$

such that ϕ is *continuous* and $|\phi'_\pm|\geq C>0$. In this case, property (2.24) can be fulfilled by properly choosing $\phi'_{|x_n=0^\pm}$ and (2.26) by choosing β sufficiently large.

Property (2.26) concerns the conjugated second-order operator. We show now that this condition concerns in fact only one of the first-order terms in the pseudo-differential factorization that we put forward above, namely $\mathcal{P}_{F\pm}$.

Lemma 2.8. *There exist $C>0$, $\tau_1>1$ and $\delta>0$ such that for $\tau\geq\tau_1$*

$$|f_\pm|\leq\delta\lambda\quad\Longrightarrow\quad C^{-1}\tau\leq|\xi'|\leq C\tau\ \ \textit{and}\ \ \{\xi_n+s_\pm,f_\pm\}\geq C'\lambda,$$

with $\lambda^2=\tau^2+|\xi'|^2$.

See Appendix 6.2.1 for a proof. This is the form of the sub-ellipticty condition, with loss of half derivative, that we shall use. This will be further highlighted by the estimates we derive in Section 3 and by the proof of the main theorem.

3. Estimates for first-order factors

Unless otherwise specified, the notation $\|\cdot\|$ will stand for the $L^2(\mathbb{R}^n)$-norm and $|\cdot|$ for the $L^2(\mathbb{R}^{n-1})$-norm. The $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^{n-1})$ dot-products will be both denoted by $\langle\cdot,\cdot\rangle$.

In this section we shall use the following function space

$$\mathscr{S}_{\mathrm{c}}(\mathbb{R}^n)=\big\{u\in\mathscr{S}(\mathbb{R}^n);\ \exists L>0,\ \mathrm{supp}(u)\subset\mathbb{R}^{n-1}\times(-L,L)\big\}.$$

3.1. **Preliminary estimates.** Most of our pseudo-differential arguments concern a calculus with the large parameter $\tau \geq 1$: with

$$\lambda^2 = \tau^2 + |\xi'|^2, \tag{3.1}$$

we define for $m \in \mathbb{R}$ the class of tangential symbols $\mathcal{S}^m_\tau$ as the smooth functions on $\mathbb{R}^n \times \mathbb{R}^{n-1}$, depending on the parameter $\tau \geq 1$, such that, for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^{n-1}$,

$$\sup_{(x,\xi') \in \mathbb{R}^n \times \mathbb{R}^{n-1}} \lambda^{-m+|\beta|} |(\partial_x^\alpha \partial_{\xi'}^\beta a)(x, \xi', \tau)| < \infty. \tag{3.2}$$

Some basic properties of the calculus of the associated pseudo-differential operators are recalled in Appendix 6.1.2. We shall refer to this calculus as to the semi-classical calculus (with a large parameter). In particular we introduce the following Sobolev norms:

$$\|u\|_{\mathcal{H}^s} := \|\Lambda^s u\|_{L^2(\mathbb{R}^{n-1})}, \quad \text{with } \Lambda^s := \operatorname{op}(\lambda^s). \tag{3.3}$$

For $s \geq 0$ note that we have $\|u\|_{\mathcal{H}^s} \sim \tau^s \|u\|_{L^2(\mathbb{R}^{n-1})} + \|\langle D' \rangle^s u\|_{L^2(\mathbb{R}^{n-1})}$. Observe also that we have

$$\|u\|_{\mathcal{H}^s} \leq C \tau^{s-s'} \|u\|_{\mathcal{H}^{s'}}, \qquad s \leq s'.$$

In what follows we shall often refer implicitly to this inequality when invoking a large value for the paramter τ.

The operator $M_\pm$ is of pseudo-differential nature in the standard calculus. Observe however that in any region where $\tau \gtrsim |\xi'|$ the symbol $m_\pm$ does not satisfy the estimates of $\mathcal{S}^1_\tau$. We shall circumvent this technical point by introducing a cut-off procedure.

Let $C_0, C_1 > 0$ be such that $\varphi' \geq C_0$ and

$$(M_\pm u, H^+ u) \leq C_1 \|H^+ u\|^2_{L^2(\mathbb{R}; H^{\frac{1}{2}}(\mathbb{R}^{n-1}))}. \tag{3.4}$$

We choose $\psi \in \mathscr{C}^\infty(\mathbb{R}^+)$ nonnegative such that $\psi = 0$ in $[0, 1]$ and $\psi = 1$ in $[2, +\infty)$. We introduce the following Fourier multiplier

$$\psi_\epsilon(\tau, \xi') = \psi\Big(\frac{\epsilon \tau}{\langle \xi \rangle}\Big) \in \mathcal{S}^0_\tau, \qquad \text{with } 0 < \epsilon \leq \epsilon_0. \tag{3.5}$$

such that $\tau \gtrsim \langle \xi' \rangle / \epsilon$ in its support. We choose ϵ_0 sufficiently small so that $\operatorname{supp}(\psi_\epsilon)$ is disjoint from a conic neighborhood (for $|\xi'| \geq 1$) of the sets $\{f_\pm = 0\}$ (see Figure 3).

The following lemma states that we can obtain very natural estimates on both sides of the interface in the region $|\xi'| \ll \tau$, i.e. for ϵ small. We refer to Appendix 6.2.2 for a proof.

Lemma 3.1. *Let $\ell \in \mathbb{R}$. There exist $\tau_1 \geq 1$, $0 < \epsilon_1 \leq \epsilon_0$ and $C > 0$ such that*

$$C \|H_+ \mathcal{A}_+ \operatorname{op}(\psi_\epsilon) \omega\|_{L^2(\mathbb{R}; \mathcal{H}^\ell)} \geq |\operatorname{op}(\psi_\epsilon) \omega_{|x_n = 0^+}|_{\mathcal{H}^{\ell + \frac{1}{2}}} + \|H_+ \operatorname{op}(\psi_\epsilon) \omega\|_{L^2(\mathbb{R}; \mathcal{H}^{\ell+1})},$$

$$C \Big(\|H_- \mathcal{A}_- \operatorname{op}(\psi_\epsilon) \omega\|_{L^2(\mathbb{R}; \mathcal{H}^\ell)} + |\operatorname{op}(\psi_\epsilon) \omega_{|x_n = 0^-}|_{\mathcal{H}^{\ell + \frac{1}{2}}} \Big) \geq \|H_- \operatorname{op}(\psi_\epsilon) \omega\|_{L^2(\mathbb{R}; \mathcal{H}^{\ell+1})},$$

for $0 < \epsilon \leq \epsilon_1$, with $A_+ = \mathcal{P}_{E+}$ or $\mathcal{P}_{F+}$, $\mathcal{A}_- = \mathcal{P}_{E-}$ or $\mathcal{P}_{F-}$, for $\tau \geq \tau_1$ and $\omega \in \mathscr{S}_c(\mathbb{R}^n)$.

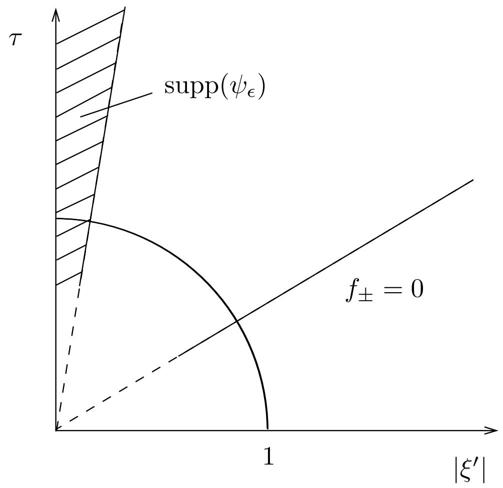

FIGURE 3. Relative positions of $\operatorname{supp}(\psi_\epsilon)$ and the sets $\{f_\pm = 0\}$.

3.2. **Positive imaginary part on a half-line.** We have the following estimates for the operators $\mathcal{P}_{E+}$ and $\mathcal{P}_{E-}$.

Lemma 3.2. *Let $\ell \in \mathbb{R}$. There exist $\tau_1 \geq 1$ and $C > 0$ such that*

$$C\|H_+\mathcal{P}_{E_+}\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \geq |\omega_{|x_n=0^+}|_{\mathcal{H}^{\ell+\frac{1}{2}}} + \|H_+\omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})} + \|H_+D_n\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)}, \tag{3.6}$$

and

$$C\Big(\|H_-\mathcal{P}_{E_-}\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + |\omega_{|x_n=0^-}|_{\mathcal{H}^{\ell+\frac{1}{2}}}\Big) \geq \|H_-\omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})} + \|H_+D_n\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)}, \tag{3.7}$$

for $\tau \geq \tau_1$ and $\omega \in \mathscr{S}_c(\mathbb{R}^n)$.

Note that the first estimate, in $\mathbb{R}_+$, is of very good quality as both the trace and the volume norms are dominated: we have a perfect elliptic estimate. In $\mathbb{R}_-$, we obtain an estimate of lesser quality. Observe also that no assumption on the weight function, apart from the positivity of φ', is used in the proof below.

Proof. Let ψ_ϵ be defined as in Section 3.1. We let $\tilde\psi \in \mathscr{C}^\infty(\mathbb{R}^+)$ be nonnegative and such that $\tilde\psi = 1$ in $[4, +\infty)$ and $\tilde\psi = 0$ in $[0, 3]$. We then define $\tilde\psi_\epsilon$ according to (3.5) and we have $\tau \lesssim \langle\xi'\rangle$ in $\operatorname{supp}(1 - \tilde\psi_\epsilon)$ and $\operatorname{supp}(1 - \psi_\epsilon) \cap \operatorname{supp}(\tilde\psi_\epsilon) = \emptyset$. We set $\tilde m_\pm = m_\pm(1 - \tilde\psi_\epsilon)$ and observe that $\tilde m_\pm \in \mathcal{S}^1_\tau$. We define

$$\tilde e_\pm = \tau\varphi' + \tilde m_\pm \in \mathcal{S}^1_\tau, \quad \tilde E_\pm = \operatorname{op}^w(\tilde e_\pm),$$

Observe that from the definition of $\tilde\psi_\epsilon$ we have

$$\tilde e_\pm \geq C\lambda. \tag{3.8}$$

Next, we note that

$$M_\pm\operatorname{op}(1 - \psi_\epsilon)\omega = \operatorname{op}^w(\tilde m_\pm)\operatorname{op}(1 - \psi_\epsilon)\omega + \operatorname{op}^w(m_\pm\tilde\psi_\epsilon)\operatorname{op}(1 - \psi_\epsilon)\omega,$$

and, since $m_\pm\tilde{\psi}_\epsilon \in \mathcal{S}^1$ and $1-\psi_\epsilon \in \mathcal{S}^0_\tau$, with the latter vanishing in a region $\langle \xi' \rangle \leq C\tau$, Lemma 6.4 yields

$$M_\pm \mathrm{op}(1-\psi_\epsilon)\omega = \mathrm{op}^w(\tilde{m}_\pm)\mathrm{op}(1-\psi_\epsilon)\omega + R_1\omega, \quad \text{with} \quad R_1 \in \mathrm{op}(\mathcal{S}_\tau^{-\infty}). \tag{3.9}$$

We set $u = \mathrm{op}(1-\psi_\epsilon)\omega$. For $s = 2\ell+1$, we compute,

$$\begin{aligned} 2\,\mathrm{Re}\langle \mathcal{P}_{E_+}u, iH_+\Lambda^s u\rangle &= \langle i[D_n, H_+]u, \Lambda^s u\rangle + \langle i[S_+, \Lambda^s]u, H_+u\rangle + 2\,\mathrm{Re}\langle E_+u, H_+\Lambda^s u\rangle \\ &\geq |u_{|x_n=0^+}|^2_{\mathcal{H}^{\ell+\frac12}} + 2\,\mathrm{Re}\langle E_+u, H_+\Lambda^s u\rangle - C\|H_+u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+\frac12})}. \end{aligned} \tag{3.10}$$

By (3.9) we have $E_+u = \tilde{E}_+u + R_1\omega$. This yields

$$\mathrm{Re}\langle E_+u, H_+\Lambda^s u\rangle + \|H_+\omega\|^2 \gtrsim \mathrm{Re}\langle \tilde{E}_+u, H_+\Lambda^s u\rangle \gtrsim \|H_+u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})},$$

for τ sufficiently large by (3.8) and Lemma 6.2. We thus obtain

$$\begin{aligned} \mathrm{Re}\langle \mathcal{P}_{E_+}u, iH_+\Lambda^s u\rangle + \|H_+u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+\frac12})} + \|H_+\omega\|^2 & \\ \gtrsim |u_{|x_n=0^+}|^2_{\mathcal{H}^{\ell+\frac12}} + \|H_+u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}&, \end{aligned}$$

With the Young inequality and taking τ sufficiently large we then find

$$\|H_+\mathcal{P}_{E_+}u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\| \gtrsim |u_{|x_n=0^+}|_{\mathcal{H}^{\ell+\frac12}} + \|H_+u\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

We now invoke the corresponding estimate provided by Lemma 3.1,

$$\|H_+\mathcal{P}_{E+}\mathrm{op}(\psi_\epsilon)\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \gtrsim |\mathrm{op}(\psi_\epsilon)\omega_{|x_n=0^+}|_{\mathcal{H}^{\ell+\frac12}} + \|H_+\mathrm{op}(\psi_\epsilon)\omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

Adding the two estimates, with the triangular inequality, we obtain

$$\begin{aligned} \|H_+\mathcal{P}_{E_+}\mathrm{op}(1-\psi_\epsilon)\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\mathcal{P}_{E+}\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\| & \\ \gtrsim |\omega_{|x_n=0^+}|_{\mathcal{H}^{\ell+\frac12}} + \|H_+\omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}&. \end{aligned}$$

Lemma 6.4 gives $\big[\mathcal{P}_{E_+}, \mathrm{op}(1-\psi_\epsilon)\big] \in \mathrm{op}(\mathcal{S}^0_\tau)$. We thus have

$$\begin{aligned} \|H_+\mathcal{P}_{E_+}\mathrm{op}(1-\psi_\epsilon)\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} &\lesssim \|H_+\mathrm{op}(1-\psi_\epsilon)\mathcal{P}_{E_+}\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \\ &\lesssim \|H_+\mathcal{P}_{E_+}\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)}. \end{aligned}$$

By taking τ sufficiently large, we thus obtain

$$\|H_+\mathcal{P}_{E_+}\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \gtrsim |\omega_{|x_n=0^+}|_{\mathcal{H}^{\ell+\frac12}} + \|H_+\omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}. \tag{3.11}$$

The term $\|H_+D_n\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)}$ can simply be introduced on the rhs of this estimates, to yield (3.6), thanks to the form of the first-order operator $\mathcal{P}_{E_+}$. To obtain estimate (3.7) we compute $2\,\mathrm{Re}\langle \mathcal{P}_{E\_}\omega, iH_-\omega\rangle$. The argument is similar whereas the trace term comes out with the opposite sign. ∎

For the operator $\mathcal{P}_{F+}$ we can also obtain a microlocal estimate. We place ourselves in a microlocal region where $f_+ = \tau\varphi^+ - m_+$ is positive. More precisely, let $\chi(x,\tau,\xi') \in \mathcal{S}^0_\tau$ be such that $|\xi'| \leq C\tau$ and $f_+ \geq C_1\lambda$ in $\mathrm{supp}(\chi)$, $C_1 > 0$, and $|\xi'| \geq C'\tau$ in $\mathrm{supp}(1-\chi)$.

Lemma 3.3. *Let $\ell \in \mathbb{R}$. There exist $\tau_1 \geq 1$ and $C > 0$ such that*

$$C\Big(\|H_+\mathcal{P}_{F_+}\mathrm{op}^w(\chi)\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\|\Big) \\ \geq |\mathrm{op}^w(\chi)\omega_{|x_n=0^+}|_{\mathcal{H}^{\ell+\frac{1}{2}}} + \|H_+\mathrm{op}^w(\chi)\omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})} + \|H_+D_n\mathrm{op}^w(\chi)\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)},$$

for $\tau \geq \tau_1$ and $\omega \in \mathscr{S}_c(\mathbb{R}^n)$.

As for (3.6) of Lemma 3.2, up to a harmless remainder term, we obtain an elliptic estimate in this microlocal region.

Proof. Let ψ_ϵ be as defined in Section 3.1 and let $\tilde{\psi}_\epsilon$ be as in the proof of Lemma 3.2. We set

$$\tilde{f}_\pm = \tau\varphi' - \tilde{m}_\pm \in \mathcal{S}^1_\tau, \quad \tilde{F}_\pm = \mathrm{op}^w(\tilde{f}_\pm). \tag{3.12}$$

Observe that we have

$$\tilde{f}_\pm = \tau\varphi' - \tilde{m}_\pm = \tau\varphi' - m_\pm(1-\tilde{\psi}_\epsilon) = f_\pm + \tilde{\psi}_\epsilon m_\pm \geq f_\pm.$$

This gives $\tilde{f}_+ \geq C\lambda$ in $\mathrm{supp}(\chi)$.

We set $u = \mathrm{op}(1-\psi_\epsilon)\mathrm{op}^w(\chi)\omega$. Following the proof of Lemma 3.2, for $s = 2\ell+1$, we obtain

$$\mathrm{Re}\langle \mathcal{P}_{F+}u, iH_+\Lambda^s u\rangle + \|H_+\omega\|^2 + \|H_+u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+\frac{1}{2}})} \\ \gtrsim |u_{|x_n=0^+}|^2_{\mathcal{H}^{\ell+\frac{1}{2}}} + \mathrm{Re}\langle \tilde{F}_+u, H_+\Lambda^s u\rangle$$

Let now $\tilde{\chi} \in \mathcal{S}^0_\tau$ satisfy the same properties as χ, with moreover $\tilde{\chi} = 1$ on a neighborhood of $\mathrm{supp}(\chi)$. We then write

$$\tilde{f}_+ = \check{f}_+ + r, \quad \text{with} \quad \check{f}_+ = \tilde{f}_+\tilde{\chi} + \lambda(1-\tilde{\chi}) \in \mathcal{S}^1_\tau, \quad r = (\tilde{f}_+ - \lambda)(1-\tilde{\chi}) \in \mathcal{S}^1_\tau.$$

As $\mathrm{supp}(1-\tilde{\chi}) \cap \mathrm{supp}(\chi) = \emptyset$, we find $r\sharp(1-\psi_\epsilon)\sharp\chi \in \mathcal{S}^{-\infty}_\tau$. Since $\check{f}_+ \geq C\lambda$ by construction, with Lemma 6.2 we obtain

$$\mathrm{Re}\langle \mathcal{P}_{F+}u, iH_+\Lambda^s u\rangle + \|H_+\omega\|^2 + \|H_+u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+\frac{1}{2}})} \\ \gtrsim |u_{|x_n=0^+}|^2_{\mathcal{H}^{\ell+\frac{1}{2}}} + \|H_+u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

With the Young inequality, taking τ sufficiently large, we obtain

$$\|H_+\mathcal{P}_{F+}u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\| \gtrsim |u_{|x_n=0^+}|_{\mathcal{H}^{\ell+\frac{1}{2}}} + \|H_+u\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

Invoking the corresponding estimate provided by Lemma 3.1 for $\mathrm{op}^w(\chi)\omega$,

$$\|H_+\mathcal{P}_{F+}\mathrm{op}(\psi_\epsilon)\mathrm{op}^w(\chi)\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \gtrsim |\mathrm{op}(\psi_\epsilon)\mathrm{op}^w(\chi)\omega_{|x_n=0^+}|_{\mathcal{H}^{\ell+\frac{1}{2}}} \\ + \|H_+\mathrm{op}(\psi_\epsilon)\mathrm{op}^w(\chi)\omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})},$$

and arguing as in the end of the proof of Lemma 3.2 we obtain the result. ∎

For the operator $\mathcal{P}_{F-}$ we can also obtain a microlocal estimate. We place ourselves in a microlocal region where $f_- = \tau\varphi^- - m_-$ is positive. More precisely, let $\chi(x,\tau,\xi') \in \mathcal{S}^0_\tau$ be such that $|\xi'| \leq C\tau$ and $f_- \geq C_1\lambda$ in $\mathrm{supp}(\chi)$, $C_1 > 0$, and $|\xi'| \geq C'\tau$ in $\mathrm{supp}(1-\chi)$.

Lemma 3.4. *Let $\ell \in \mathbb{R}$. There exist $\tau_1 \geq 1$ and $C > 0$ such that*

$$C\Big(\|H_-\mathcal{P}_{F_-}u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_-\omega\| + \|H_-D_n\omega\| + |u_{|x_n=0^-}|_{\mathcal{H}^{\ell+\frac12}}\Big) \geq \|H_-u\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}, \tag{3.13}$$

for $\tau \geq \tau_1$ and $u = a_{nn}^-\mathcal{P}_{E-}\mathrm{op}^w(\chi)\omega$ with $\omega \in \mathscr{S}_c(\mathbb{R}^n)$.

Proof. Let ψ_ϵ be defined as in Section 3.1. We define $\tilde f_-$ and $\tilde F_-$ as in (3.12). We have $\tilde f_- \geq f_- \geq C\lambda$ in $\mathrm{supp}(\chi)$. We set $z = \mathrm{op}(1-\psi_\epsilon)u$ and for $s = 2\ell+1$ we compute

$$\begin{aligned}
&2\,\mathrm{Re}\langle \mathcal{P}_{F_-}z, iH_-\Lambda^s z\rangle\\
&\quad= \langle i[D_n, H_-]z, \Lambda^s z\rangle + i\langle [S_-, \Lambda^s]z, H_-z\rangle + 2\,\mathrm{Re}\langle F_-z, H_-\Lambda^s z\rangle\\
&\quad\geq -|z_{|x_n=0^-}|^2_{\mathcal{H}^{\ell+\frac12}} + 2\,\mathrm{Re}\langle F_-z, H_-\Lambda^s z\rangle - C\|H_-z\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+\frac12})}.
\end{aligned}$$

Arguing as in the proof of Lemma 3.2 (see (3.9) and (3.10)) we obtain

$$2\,\mathrm{Re}\langle \mathcal{P}_{F_-}z, iH_-\Lambda^s z\rangle + C\|H_-u\|^2 + |z_{|x_n=0^-}|^2_{\mathcal{H}^{\ell+\frac12}} + C\|H_-z\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+\frac12})} \geq 2\,\mathrm{Re}\langle \tilde F_-z, H_-\Lambda^s z\rangle.$$

Let now $\tilde\chi \in \mathcal{S}^0_\tau$ satisfy the same properties as χ, with moreover $\tilde\chi = 1$ on a neighborhood of $\mathrm{supp}(\chi)$. We then write

$$\tilde f_- = \check f_- + r, \quad \text{with} \quad \check f_- = \tilde f_-\tilde\chi + \lambda(1-\tilde\chi) \in \mathcal{S}^1_\tau, \quad r = (\tilde f_- - \lambda)(1-\tilde\chi) \in \mathcal{S}^1_\tau.$$

As $\check f_- \geq C\lambda$ and $\mathrm{supp}(1-\tilde\chi)\cap\mathrm{supp}(\chi) = \emptyset$ with Lemma 6.2 we obtain, for τ large,

$$2\,\mathrm{Re}\langle \mathcal{P}_{F_-}z, iH_-\Lambda^s z\rangle + C\|H_-u\|^2 + |z_{|x_n=0^-}|^2_{\mathcal{H}^{\ell+\frac12}} + C\|H_-z\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+\frac12})} + \|H_-\omega\|^2 + \|H_-D_n\omega\|^2 \geq C'\|H_-z\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

With the Young inequality and taking τ sufficiently large we then find

$$\|H_-\mathcal{P}_{F_-}z\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_-u\| + |z_{|x_n=0^-}|_{\mathcal{H}^{\ell+\frac12}} + \|H_-\omega\| + \|H_-D_n\omega\| \gtrsim \|H_-z\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

Invoking the corresponding estimate provided by Lemma 3.1 for u yields

$$\|H_-\mathcal{P}_{F_-}\mathrm{op}(\psi_\epsilon)u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + |\mathrm{op}(\psi_\epsilon)u_{|x_n=0^-}|_{\mathcal{H}^{\ell+\frac12}} \gtrsim \|H_-\mathrm{op}(\psi_\epsilon)u\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

and arguing as in the end of Lemma 3.2 we obtain the result. ■

3.3. **Negative imaginary part on the negative half-line.** Here we place ourselves in a microlocal region where $f_- = \tau\varphi^- - m_-$ is negative. More precisely, let $\chi(x,\tau,\xi') \in \mathcal{S}^0_\tau$ be such that $|\xi'| \geq C\tau$ and $f_- \leq -C_1\lambda$ in $\mathrm{supp}(\chi)$, $C_1 > 0$. We have the following lemma whose form is adapted to our needs in the next section. Up to harmless remainder terms, this can also be considered as a good elliptic estimate.

Lemma 3.5. *There exist $\tau_1 \geq 1$ and $C > 0$ such that*

$$C\Big(\|H_-\mathcal{P}_{F_-}u\| + \|H_-\omega\| + \|H_-D_n\omega\|\Big) \geq |u_{|x_n=0^-}|_{\mathcal{H}^{\frac12}} + \|H_-u\|_{L^2(\mathbb{R};\mathcal{H}^1)}, \tag{3.14}$$

for $\tau \geq \tau_1$ and $u = a_{nn}^-\mathcal{P}_{E-}\mathrm{op}^w(\chi)\omega$ with $\omega \in \mathscr{S}_c(\mathbb{R}^n)$.

Proof. We compute

$$\begin{aligned}
&2\,\mathrm{Re}\langle \mathcal{P}_{F_-}u, -iH_-\Lambda^1 u\rangle \\
&\quad = \langle i[D_n, -H_-]u, \Lambda^1 u\rangle - i\langle [S_-,\Lambda^1]u, H_-u\rangle + 2\,\mathrm{Re}\langle -F_-u, H_-\Lambda^1 u\rangle \\
&\quad \geq |u_{|x_n=0^-}|^2_{\mathcal{H}^{\frac12}} + 2\,\mathrm{Re}\langle -F_-u, H_-\Lambda^1 u\rangle - C\|H_-u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\frac12})}.
\end{aligned}$$

Let now $\tilde\chi \in \mathcal{S}^0_\tau$ satisfy the same properties as χ, with moreover $\tilde\chi = 1$ on a neighborhood of $\mathrm{supp}(\chi)$. We then write

$$f_- = \check f_- + r, \quad \text{with} \ \ \check f_- = f_-\tilde\chi - \lambda(1-\tilde\chi), \quad r = (f_- + \lambda)(1-\tilde\chi).$$

Observe that $f_-\tilde\chi \in \mathcal{S}^1_\tau$ because of the support of $\tilde\chi$. Hence $\check f_- \in \mathcal{S}^1_\tau$. As $-\check f_- \geq C\lambda$ with Lemma 6.2 we obtain, for τ large, $\mathrm{Re}\langle -\mathrm{op}^w(\check f_-)u, H_-\Lambda^1 u\rangle \gtrsim \|H_-u\|^2_{L^2(\mathbb{R};\mathcal{H}^1)}$. Note that r does not satisfy the estimates of the semi-classical calculus because of the term $m_-(1-\tilde\chi)$. However, we have

$$\mathrm{op}^w(r)u = \mathrm{op}^w(r)a^-_{nn}\mathrm{op}^w(\chi)D_n\omega + \mathrm{op}^w(r)a^-_{nn}S_-\mathrm{op}^w(\chi)\omega + i\mathrm{op}^w(r)a^-_{nn}E_-\mathrm{op}^w(\chi)\omega.$$

Applying Lemma 6.4, using that $1-\tilde\chi \in \mathcal{S}^0_\tau \subset \mathcal{S}^0$, yields

$$\mathrm{op}^w(r)u = R\omega \qquad \text{with} \ \ R \in \mathrm{op}(\mathcal{S}^1_\tau)D_n + \mathrm{op}(\mathcal{S}^2_\tau).$$

As $\mathrm{supp}(1-\tilde\chi)\cap\mathrm{supp}(\chi) = \emptyset$, the composition formula (6.7) (which is valid in this case – see Lemma 6.4) yields moreover $R \in \mathrm{op}(\mathcal{S}^{-\infty}_\tau)D_n + \mathrm{op}(\mathcal{S}^{-\infty}_\tau)$. We thus find, for τ sufficiently large

$$\mathrm{Re}\langle \mathcal{P}_{F_-}u, -iH_-\Lambda^1 u\rangle + \|H_-\omega\|^2 + \|H_-D_n\omega\|^2 \gtrsim |u_{|x_n=0^-}|^2_{\mathcal{H}^{\frac12}} + \|H_-u\|^2_{L^2(\mathbb{R};\mathcal{H}^1)},$$

and we conclude with the Young inequality. ∎

3.4. **Increasing imaginary part on a half-line.** Here we allow the symbols $f_\pm$ to change sign. For the first-order factor $\mathcal{P}_{F_\pm}$ this will lead to an estimate that exhibits a loss of a half derivative as can be expected.

Let ψ_ϵ be as defined in Section 3.1 and let $\tilde\psi_\epsilon$ be as in the proof of Lemma 3.2. We define $\tilde f_\pm$ and $\tilde F_\pm$ as in (3.12) and set $\tilde{\mathcal{P}}_{F_\pm} = D_n + S_\pm + i\tilde F_\pm$.

As $\mathrm{supp}(\tilde\psi_\epsilon)$ remains away from the sets $\{f_\pm = 0\}$ the sub-ellipticy property of Lemma 2.8 is preserved for $\tilde f_\pm$ in place of $f_\pm$. We shall use the following inequality.

Lemma 3.6. *There exist $C>0$ such that for $\mu>0$ sufficiently large we have*

$$\rho_\pm = \mu\tilde f^2_\pm + \tau\left\{\xi_n + s_\pm, \tilde f_\pm\right\} \geq C\lambda^2,$$

with $\lambda^2 = \tau^2 + |\xi'|^2$.

Proof. If $|\tilde f_\pm| \leq \delta\lambda$, for δ small, then $\tilde f_\pm = f_\pm$ and $\tau\left\{\xi_n + s_\pm, \tilde f_\pm\right\} \geq C\lambda^2$ by Lemma 2.8.

If $|\tilde f_\pm| \geq \delta\lambda$, observing that $\tau\left\{\xi_n + s_\pm, \tilde f_\pm\right\} \in \tau\mathcal{S}^1_\tau \subset \mathcal{S}^2_\tau$, we obtain $\rho_\pm \geq C\lambda^2$ by choosing μ sufficiently large. ∎

We now prove the following estimate for $\mathcal{P}_{F_\pm}$.

Lemma 3.7. *Let $\ell \in \mathbb{R}$. There exist $\tau_1 \geq 1$ and $C > 0$ such that*

$$C\Big(\|H_\pm \mathcal{P}_{F\pm}\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + |\omega_{|x_n=0^\pm}|_{\mathcal{H}^{\ell+\frac{1}{2}}}\Big) \geq \tau^{-\frac{1}{2}}\Big(\|H_\pm\omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})} + \|H_\pm D_n\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)}\Big),$$

for $\tau \geq \tau_1$ and $\omega \in \mathscr{S}_c(\mathbb{R}^n)$.

Proof. we set $u = \mathrm{op}(1-\psi_\epsilon)\omega$. We start by invoking (3.9), and the fact that $[\tilde{\mathcal{P}}_{F+}, \Lambda^\ell] \in \mathrm{op}(\mathcal{S}^\ell_\tau)$, and write

$$\begin{aligned}\|H_+\tilde{\mathcal{P}}_{F+}\Lambda^\ell u\| &\lesssim \|H_+\Lambda^\ell\tilde{\mathcal{P}}_{F+}u\| + \|H_+[\tilde{\mathcal{P}}_{F+},\Lambda^\ell]u\| \\ &\lesssim \|H_+\tilde{\mathcal{P}}_{F+}u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \\ &\lesssim \|H_+\mathcal{P}_{F+}u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\| + \|H_+u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)}\end{aligned} \tag{3.15}$$

We set $u_\ell = \Lambda^\ell u$. We then have

$$\begin{aligned}\|H_+\tilde{\mathcal{P}}_{F+}u_\ell\|^2 &= \|H_+(D_n+S_+)u_\ell\|^2 + \|H_+\tilde{F}_+u_\ell\|^2 + 2\,\mathrm{Re}\langle (D_n+S_+)u_\ell, iH_+\tilde{F}_+u_\ell\rangle \\ &\geq \tau^{-1}\,\mathrm{Re}\langle \big(\mu\tilde{F}_+^2 + i\tau\big[D_n+S_+,\tilde{F}_+\big]\big)u_\ell, H_+u_\ell\rangle + \langle i[D_n,H_+]u_\ell, \tilde{F}_+u_\ell\rangle,\end{aligned}$$

if $\mu\tau^{-1} \leq 1$. As the principal symbol (in the semi-classical calculus) of $\mu\tilde{F}_+^2 + i\tau\big[D_n + S_+, \tilde{F}_+\big]$ is $\rho_+ = \mu\tilde{f}_+^2 + \tau\left\{\xi_n + s_+, \tilde{f}_+\right\}$, Lemmas 3.6 and 6.2 yield

$$\|H_+\tilde{\mathcal{P}}_{F+}u_\ell\|^2 + |u_\ell|^2_{\mathcal{H}^{\frac{1}{2}}} \gtrsim \tau^{-1}\|H_+u_\ell\|^2_{L^2(\mathbb{R};\mathcal{H}^1)},$$

for μ large, i.e., τ large. With (3.15) we obtain, for τ sufficiently large,

$$\|H_+\mathcal{P}_{F+}u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\| + |u|_{\mathcal{H}^{\ell+\frac{1}{2}}} \gtrsim \tau^{-\frac{1}{2}}\|H_+u\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

We now invoke the corresponding estimate provided by Lemma 3.1,

$$\|H_+\mathcal{P}_{F+}\mathrm{op}(\psi_\epsilon)\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \gtrsim |\mathrm{op}(\psi_\epsilon)\omega_{|x_n=0^+}|_{\mathcal{H}^{\ell+\frac{1}{2}}} + \|H_+\mathrm{op}(\psi_\epsilon)\omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}$$

and we proceed as in the end of the proof of Lemma 3.2 to obtain the result for $\mathcal{P}_{F+}$. The same computation and arguments, *mutatis mutandis*, give the result for $\mathcal{P}_{F-}$. ∎

4. Proof of the Carleman estimate

From the estimates for the first-order factors obtained in Section 3 we shall now prove Proposition 2.1 which gives the result of Theorem 1.1 and Theorem 2.2 (see the end of Section 2.1).

The Carleman estimates we prove are well known away from the interface $\{x_n = 0\}$. Since local Carleman estimates can be patched together, we may thus assume that the compact set K in the statements of Theorem 1.1 and Theorem 2.2 is such that $|x_n|$ is sufficiently small for the arguments below to be carried out. Hence we shall assume the functions $w_\pm$ in Theorem 2.2 (resp. $v_\pm$ in Proposition 2.1) have small supports near 0 in the x_n-direction.

4.1. **The geometric hypothesis.** In section 2.4 we chose a weight function φ that satisfies the following condition

$$\frac{\alpha_+}{\alpha_-} > \sup_{\substack{x',\xi' \\ |\xi'|\geq 1}} \frac{m_+(x',\xi')_{|x_n=0^+}}{m_-(x',\xi')_{|x_n=0^-}}, \qquad \alpha_\pm = \partial_{x_n}\varphi_{\pm|x_n=0^\pm}. \tag{4.1}$$

Let us explain the immediate consequences of that assumption: first of all, we can reformulate it by saying that

$$\exists \sigma > 1, \quad \frac{\alpha_+}{\alpha_-} = \sigma^2 \sup_{\substack{x',\xi' \\ |\xi'|\geq 1}} \frac{m_+(x',\xi')_{|x_n=0^+}}{m_-(x',\xi')_{|x_n=0^-}}. \tag{4.2}$$

Let $1 < \sigma_0 < \sigma$.

First, consider $(x',\xi',\tau) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^{+,*}$, $|\xi'| \geq 1$, such that

$$\tau\alpha_+ \geq \sigma_0 m_+(x',\xi')_{|x_n=0^+}. \tag{4.3}$$

Observe that we then have

(4.4)
$$\tau\alpha_+ - m_+(x',\xi')_{|x_n=0^+} \geq \tau\alpha_+(1-\sigma_0^{-1}) \geq \frac{\sigma_0 - 1}{2\sigma_0}\tau\alpha_+ + \frac{\sigma_0-1}{2} m_+(x',\xi')_{|x_n=0^+}$$
$$\geq C\lambda.$$

We choose τ sufficiently large, say $\tau \geq \tau_2 > 0$, so that this inequality remains true for $0 \leq |\xi'| \leq 2$. It also remains true for $x_n > 0$ small. As $f_+ = \tau(\varphi' - \alpha_+) + \tau\alpha_+ - m_+(x,\xi')$, for $|x_n|$ small, we obtain $f_+ \geq C\lambda$, which means that f_+ is elliptic positive in that region.

Second, if we now have $|\xi'| \geq 1$ and

$$\tau\alpha_+ \leq \sigma m_+(x',\xi')_{|x_n=0^+}, \tag{4.5}$$

we get that $\tau\alpha_- \leq \sigma^{-1} m_-(x',\xi')_{|x_n=0^-}$: otherwise we would have $\tau\alpha_- > \sigma^{-1}m_-(x',\xi')_{|x_n=0^-}$ and thus

$$\frac{m_-(x',\xi')_{|x_n=0^-}}{\sigma\alpha_-} < \tau \leq \frac{\sigma m_+(x',\xi')_{|x_n=0^+}}{\alpha_+},$$

implying

$$\frac{\alpha_+}{\alpha_-} < \sigma^2 \frac{m_+(x',\xi')_{|x_n=0^+}}{m_-(x',\xi')_{|x_n=0^-}} \leq \sigma^2 \sup_{\substack{x',\xi' \\ |\xi'|\geq 1}} \frac{m_+(x',\xi')_{|x_n=0^+}}{m_-(x',\xi')_{|x_n=0^-}} = \frac{\alpha_+}{\alpha_-} \quad \text{which is impossible.}$$

As a consequence we have

$$\tau\alpha_- - m_-(x',\xi')_{|x_n=0^-} \leq -m_-(x',\xi')_{|x_n=0^-}\frac{(\sigma-1)}{\sigma} \tag{4.6}$$
$$\leq -m_-(x',\xi')_{|x_n=0^-}\frac{(\sigma-1)}{2\sigma} - \frac{(\sigma-1)}{2}\tau\alpha_- \leq -C\lambda.$$

With $f_- = \tau(\varphi' - \alpha_-) + \tau\alpha_- - m_-(x,\xi')$, for $|x_n|$ sufficiently small, we obtain $f_- \leq -C\lambda$, which means that f_- is elliptic negative in that region.

We have thus proven the following result.

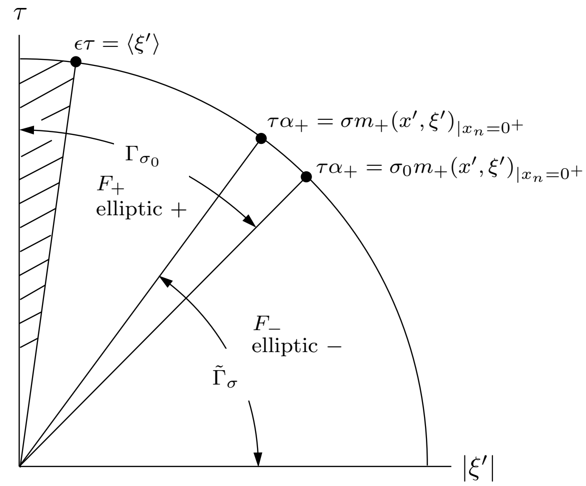

FIGURE 4. The overlapping microlocal regions Γ_{σ_0}, and $\widetilde{\Gamma}_\sigma$ in the $\tau, |\xi'|$ plane above a point x'. Dashed is the region used in Section 3.1 which is kept away from the overlap of Γ_{σ_0}, and $\widetilde{\Gamma}_\sigma$.

Lemma 4.1. *Let $\sigma > \sigma_0 > 1$, and $\alpha_\pm$, be positive numbers such that (4.2) holds. For $s > 0$, we define the following "cones" in $\mathbb{R}^{n-1}_{x'} \times \mathbb{R}^{n-1}_{\xi'} \times \mathbb{R}^*_+$ by*

$$\Gamma_s = \left\{(x',\tau,\xi');\ |\xi'| < 2 \text{ or } \tau\alpha_+ > s m_+(x',\xi')_{|x_n=0^+}\right\},$$
$$\widetilde{\Gamma}_s = \left\{(x',\tau,\xi');\ |\xi'| > 1 \text{ and } \tau\alpha_+ < s m_+(x',\xi')_{|x_n=0^+}\right\}.$$

For $|x_n|$ sufficiently small and τ sufficiently large, we have $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^*_+ = \Gamma_{\sigma_0} \cup \widetilde{\Gamma}_\sigma$ and

$$\Gamma_{\sigma_0} \subset \left\{(x',\xi',\tau) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^*_+;\ \ f_+(x,\xi') \geq C\lambda, \text{ if } 0 \leq x_n \text{ small}\right\},$$

$$\widetilde{\Gamma}_\sigma \subset \left\{(x',\xi',\tau) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^*_+;\ f_-(x,\xi') \leq -C\lambda,\ \text{ if } |x_n| \text{ small},\ x_n \leq 0\right\}.$$

N.B. The key result for the sequel is that property (4.1) is securing the fact that the overlapping open regions Γ_{σ_0} and $\widetilde{\Gamma}_\sigma$ are such that on Γ_{σ_0}, f_+ is elliptic positive and on $\widetilde{\Gamma}_\sigma$, f_- is elliptic negative. Using a partition of unity and symbolic calculus, we shall be able to assume that either F_+ is elliptic positive, or F_- is elliptic negative.

N.B. Note that we can keep the preliminary cut-off region of Section 3.1 away from the overlap of Γ_{σ_0} and $\widetilde{\Gamma}_\sigma$ by choosing ϵ sufficiently small (see (3.5) and Lemma 3.1). This is illustrated in Figure 4.

With the two overlapping "cones", for $\tau \geq \tau_2$, we introduce an homogeneous partition of unity

$$1 = \chi_0(x',\xi',\tau) + \chi_1(x',\xi',\tau), \quad \underbrace{\operatorname{supp}(\chi_0) \subset \Gamma_{\sigma_0}}_{|\xi'| \lesssim \tau,\ f_+ \text{ elliptic} > 0}, \quad \underbrace{\operatorname{supp}(\chi_1) \subset \widetilde{\Gamma}_\sigma}_{|\xi'| \gtrsim \tau,\ f_- \text{ elliptic} < 0}. \tag{4.7}$$

Note that χ'_j, $j = 0, 1$, are supported at the overlap of the regions Γ_{σ_0} and $\widetilde{\Gamma}_\sigma$, where $\tau \lesssim |\xi'|$. Hence, χ_0 and χ_1 satisfy the estimates of the semi-classical calculus and we have $\chi_0,\ \chi_1 \in \mathcal{S}^0_\tau$. With these symbols we associate the following operators.

$$\Xi_j = \mathrm{op}^w(\chi_j),\ \ j = 0, 1 \text{ and we have } \Xi_0 + \Xi_1 = \mathrm{Id}. \tag{4.8}$$

Remark 4.2. Here we have chosen to let χ_0 and χ_1 (resp. Ξ_0 and Ξ_1) be independent of x_n. As the functions $v_\pm$ have supports in which $|x_n|$ is small (see the introductory paragraph of this section), we can further introduce a cut-off in the x_n direction. The lemmata of Section 3 can then be applied directly.

From the transmission conditions (2.21) we find

$$\Xi_j v_{+|x_n=0^+} - \Xi_j v_{-|x_n=0^-} = \Xi_j \theta_\varphi, \tag{4.9}$$

and

$$\begin{aligned} a^+_{nn}(D_n + T_+ + i\tau\varphi'_+)\Xi_j v_{+|x_n=0^+} - a^-_{nn}(D_n + T_- + i\tau\varphi'_-)\Xi_j v_{-|x_n=0^-} \\ = \Xi_j \Theta_\varphi + \mathrm{op}^w(\kappa_0) v_{|x_n=0^+} + \mathrm{op}^w(\tilde\kappa_0)\theta_\varphi, \quad j = 0, 1, \end{aligned}$$

with $\kappa_0, \tilde\kappa_0 \in \mathcal{S}^0_\tau$ that originate from commutators and (4.9). Defining

$$\mathcal{V}_{j,\pm} = a^\pm_{nn}(D_n + S_\pm + i\tau\varphi'_\pm)\Xi_j v_{\pm|x_n=0^\pm} \tag{4.10}$$

and recalling (2.19) we find

$$\mathcal{V}_{j,+} - \mathcal{V}_{j,-} = \Xi_j\Theta_\varphi + \mathrm{op}^w(\kappa_1) v_{|x_n=0^+} + \mathrm{op}^w(\tilde\kappa_1)\theta_\varphi, \quad \kappa_1, \tilde\kappa_1 \in \mathcal{S}^0_\tau. \tag{4.11}$$

We shall now prove microlocal Carleman estimates in the two regions Γ_{σ_0} and $\widetilde\Gamma_\sigma$.

4.2. **Region Γ_{σ_0}: both roots are positive on the positive half-line.** On the one hand, from Lemma 3.2 we have

$$\|H_+\mathcal{P}_+\Xi_0 v_+\| \gtrsim |\mathcal{V}_{0,+} - i a^+_{nn} M_+ \Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}} + \|H_+\mathcal{P}_{F+}\Xi_0 v_+\|_{L^2(\mathbb{R};\mathcal{H}^1)}, \tag{4.12}$$

where the operator $\mathcal{P}_+$ is defined in (2.7) (see also (2.17)). The positive ellipticity of F_+ on the $\operatorname{supp}\chi_0 \cap \operatorname{supp}(v_+)$ allows us to reiterate the estimate by Lemma 3.3 to obtain

$$\begin{aligned} \|H_+\mathcal{P}_+\Xi_0 v_+\| + \|H_+ v_+\| \gtrsim |\mathcal{V}_{0,+} - i a^+_{nn} M_+ \Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}} + |\Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{3/2}} \\ + \|H_+\Xi_0 v_+\|_{L^2(\mathbb{R};\mathcal{H}^2)} + \|H_+ D_n \Xi_0 v_+\|_{L^2(\mathbb{R};\mathcal{H}^1)}. \end{aligned}$$

Since we have also

$$|\mathcal{V}_{0,+}|_{\mathcal{H}^{\frac12}} \lesssim |\mathcal{V}_{0,+} - i a^+_{nn} M_+ \Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}} + |\Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{3/2}}, \tag{4.13}$$

writing the $\mathcal{H}^{\frac12}$ norm as $|.|_{\mathcal{H}^{\frac12}} \sim \tau^{\frac12}|.|_{L^2} + |.|_{H^{\frac12}}$ and using the regularity of $M_+ \in \mathrm{op}(\mathcal{S}^1)$ in the standard calculus, we obtain

$$\begin{aligned} \|H_+\mathcal{P}_+\Xi_0 v_+\| + \|H_+ v_+\| \gtrsim |\mathcal{V}_{0,+}|_{\mathcal{H}^{\frac12}} + |\Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{3/2}} \\ + \|H_+\Xi_0 v_+\|_{L^2(\mathbb{R};\mathcal{H}^2)} + \|H_+\Xi_0 D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^1)}. \end{aligned} \tag{4.14}$$

On the other hand, with Lemma 3.7 we have, for $k = 0$ or $k = \frac12$,

$$\begin{aligned} \|H_-\mathcal{P}_-\Xi_0 v_-\|_{L^2(\mathbb{R};\mathcal{H}^{-k})} + |\mathcal{V}_{0,-} + i a^-_{nn} M_- \Xi_0 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12 - k}} \\ \gtrsim \tau^{-\frac12}\|H_-\mathcal{P}_{E-}\Xi_0 v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}. \end{aligned}$$

This gives

$$\|H_-\mathcal{P}_-\Xi_0 v_-\| + \tau^k |\mathcal{V}_{0,-} + i a^-_{nn} M_- \Xi_0 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12-k}} \gtrsim \tau^{k-\frac12} \|H_-\mathcal{P}_{E-}\Xi_0 v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})},$$

which with Lemma 3.2 yields

$$\begin{aligned}\|H_-\mathcal{P}_-\Xi_0 v_-\| + \tau^k |\mathcal{V}_{0,-} + i a^-_{nn} M_- \Xi_0 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12-k}} + \tau^{k-\frac12} |\Xi_0 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac32-k}} \\ \gtrsim \tau^{k-\frac12}\Big(\|H_-\Xi_0 v_-\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \|H_-\Xi_0 D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big).\end{aligned}$$

Arguing as for (4.13) we find

$$\begin{aligned}\|H_-\mathcal{P}_-\Xi_0 v_-\| + \tau^k |\mathcal{V}_{0,-}|_{\mathcal{H}^{\frac12-k}} + \tau^k |\Xi_0 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac32-k}} \\ \gtrsim \tau^{k-\frac12}\Big(\|H_-\Xi_0 v_-\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \|H_-\Xi_0 D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big).\end{aligned} \tag{4.15}$$

Now, from the transmission conditions (4.9)–(4.11), by adding $\varepsilon(4.15) + (4.14)$ we obtain

$$\begin{aligned}\|H_-\mathcal{P}_-\Xi_0 v_-\| + \|H_+\mathcal{P}_+\Xi_0 v_+\| + \tau^k\Big(|\theta_\varphi|_{\mathcal{H}^{\frac32-k}} + |\Theta_\varphi|_{\mathcal{H}^{\frac12-k}} + |v_{|x_n=0^+}|_{\mathcal{H}^{\frac12-k}}\Big) + \|H_+ v_+\| \\ \gtrsim \tau^k\Big(|\mathcal{V}_{0,-}|_{\mathcal{H}^{\frac12-k}} + |\mathcal{V}_{0,+}|_{\mathcal{H}^{\frac12-k}} + |\Xi_0 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac32-k}} + |\Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac32-k}}\Big) \\ + \tau^{k-\frac12}\Big(\|\Xi_0 v\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \|H_-\Xi_0 D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \|H_+\Xi_0 D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big).\end{aligned}$$

by choosing $\varepsilon > 0$ sufficiently small and τ sufficiently large. Finally, recalling the form of $\mathcal{V}_{0,\pm}$, arguing as for (4.13) we obtain

(4.16)

$$\begin{aligned}\|H_-\mathcal{P}_-\Xi_0 v_-\| + \|H_+\mathcal{P}_+\Xi_0 v_+\| + \tau^k\Big(|\theta_\varphi|_{\mathcal{H}^{\frac32-k}} + |\Theta_\varphi|_{\mathcal{H}^{\frac12-k}} + |v_{|x_n=0^+}|_{\mathcal{H}^{\frac12-k}}\Big) + \|H_+ v_+\| \\ \gtrsim \tau^k\Big(|\Xi_0 D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12-k}} + |\Xi_0 D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12-k}} + |\Xi_0 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac32-k}} + |\Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac32-k}}\Big) \\ + \tau^{k-\frac12}\Big(\|\Xi_0 v\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \|H_-\Xi_0 D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \|H_+\Xi_0 D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big),\end{aligned}$$

for $k = 0$ or $k = \frac12$.

Remark 4.3. Note that in the case $k = 0$, recalling the form of the second-order operators $\mathcal{P}_\pm$, we can estimate the additional terms $\tau^{-\frac12}\|H_\pm \Xi_0 D_n^2 v_\pm\|$.

4.3. **Region $\tilde\Gamma_\sigma$: only one root is positive on the positive half-line.** This case is more difficult a priori since we cannot expect to control $v_{|x_n=0^+}$ directly from the estimates of the first-order factors. Nevertheless when the positive ellipticity of F_+ is violated, then F_- is elliptic negative: this is the result of our main geometric assumption in Lemma 4.1.

As in (4.12) we have

$$\|H_+\mathcal{P}_+\Xi_1 v_+\| \gtrsim |\mathcal{V}_{1,+} - i a^+_{nn} M_+ \Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12}} + \|H_+\mathcal{P}_{F+}\Xi_1 v_+\|_{L^2(\mathbb{R};\mathcal{H}^1)}.$$

and using Lemma 3.5 for the negative half-line, we have

$$\begin{aligned}\|H_-\mathcal{P}_-\Xi_1 v_-\| + \|H_- v_-\| + \|H_- D_n v_-\| \\ \gtrsim |\mathcal{V}_{1,-} + i a^-_{nn} M_- \Xi_1 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12}} + \|H_-\mathcal{P}_{E-}\Xi_1 v_-\|_{L^2(\mathbb{R};\mathcal{H}^1)}.\end{aligned}$$

A quick glance at the above estimates show that none could be iterated in a favorable manner, since F_+ could be negative on the positive half-line and E_- is indeed positive on the negative half-line. We have to use the additional information given by the transmission conditions. From the above inequalities, we control

$$\tau^k\Big(|\mathcal{V}_{1,-} + ia^-_{nn}M_-\Xi_1 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12-k}} + |-\mathcal{V}_{1,+} + ia^+_{nn}M_+\Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12-k}}\Big),$$

for $k=0$ or $\frac12$, which, by the transmission conditions (4.9)–(4.11) implies the control of

$$\begin{aligned}&\tau^k|\mathcal{V}_{1,-} - \mathcal{V}_{1,+} + ia^-_{nn}M_-\Xi_1 v_{-|x_n=0^-} + ia^+_{nn}M_+\Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12-k}}\\ &\geq \tau^k|(a^-_{nn}M_- + a^+_{nn}M_+)\Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12-k}}\\ &\quad - C\tau^k\big(|\Theta_\varphi|_{\mathcal{H}^{\frac12-k}} + |\theta_\varphi|_{\mathcal{H}^{\frac32-k}} + |v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12-k}}\big).\end{aligned}$$

Let now $\tilde\chi_1 \in \mathcal{S}^0_\tau$ satisfying the same properties as χ_1, with moreover $\tilde\chi_1 = 1$ on a neighborhood of $\operatorname{supp}(\chi_1)$. We then write

$$m_\pm = \check m_\pm + r, \quad \text{with} \quad \check m_\pm = m_\pm\tilde\chi_1 + \lambda(1-\tilde\chi_1), \quad r = (m_\pm + \lambda)(1-\tilde\chi_1).$$

We have $\check m_\pm \geq C\lambda$ and $\check m_\pm \in \mathcal{S}^1_\tau$ because of the support of $\tilde\chi_1$. Because of the supports of $1-\tilde\chi_1$ and χ_1, in particular $\tau \lesssim |\xi'|$ in $\operatorname{supp}(\chi_1)$, Lemma 6.4 yields $r\sharp\chi_1 \in \mathcal{S}^{-\infty}_\tau$. With Lemma 6.2 and (4.9) we thus obtain

$$\begin{aligned}&|\mathcal{V}_{1,-} + ia^-_{nn}M_-\Xi_1 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12-k}} + |-\mathcal{V}_{1,+} + ia^+_{nn}M_+\Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12-k}}\\ &+ |\Theta_\varphi|_{\mathcal{H}^{\frac12-k}} + |\theta_\varphi|_{\mathcal{H}^{\frac32-k}} + |v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12-k}} \gtrsim |\Xi_1 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac32-k}} + |\Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac32-k}}.\end{aligned}$$

From the form of $\mathcal{V}_{1,+}$ we moreover obtain

$$\begin{aligned}&|\mathcal{V}_{1,-} + ia^-_{nn}M_-\Xi_1 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12-k}} + |-\mathcal{V}_{1,+} + ia^+_{nn}M_+\Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12-k}}\\ &+ |\Theta_\varphi|_{\mathcal{H}^{\frac12-k}} + |\theta_\varphi|_{\mathcal{H}^{\frac32-k}} + |v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12-k}} \gtrsim |\Xi_1 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac32-k}} + |\Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac32-k}}\\ &\qquad + |\Xi_1 D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12-k}} + |\Xi_1 D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12-k}}.\end{aligned}$$

We thus have

$$\begin{aligned}&\|H_-\mathcal{P}_-\Xi_1 v_-\| + \|H_+\mathcal{P}_+\Xi_1 v_+\| + \tau^k\Big(|\Theta_\varphi|_{\mathcal{H}^{\frac12-k}} + |\theta_\varphi|_{\mathcal{H}^{\frac32-k}} + |v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12-k}}\Big) + \|H_-v_-\|\\ &+ \|H_-D_n v_-\| \gtrsim \tau^k\Big(|\Xi_1 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac32-k}} + |\Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac32-k}} + |\Xi_1 D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac12-k}}\\ &\quad + |\Xi_1 D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac12-k}} + \|H_-\mathcal{P}_{E-}\Xi_1 v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \|H_+\mathcal{P}_{F+}\Xi_1 v_+\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big),\end{aligned}$$

for $k=0$ or $\frac12$. The remaining part of the discussion is very similar to the last part of the argument in the previous subsection. By Lemmas 3.2 and 3.7 we have

$$\begin{aligned}&\|H_-\mathcal{P}_{E-}\Xi_1 v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + |\Xi_1 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac32-k}}\\ &\qquad\gtrsim \|H_-\Xi_1 v_-\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \|H_-\Xi_1 D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\end{aligned}$$

and

$$\begin{aligned}&\|H_+\mathcal{P}_{F_+}\Xi_1 v_+\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + |\Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac32-k}}\\ &\qquad\gtrsim \tau^{-\frac12}\Big(\|H_+\Xi_1 v_+\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \|H_+\Xi_1 D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big).\end{aligned}$$

Since $|\Xi_1 v_{\pm|x_n=0^\pm}|_{\mathcal{H}^{\frac{3}{2}-k}}$ are already controlled, we control as well the rhs of the above inequalities and have

(4.17)
$$\|H_-\mathcal{P}_-\Xi_1 v_-\| + \|H_+\mathcal{P}_+\Xi_1 v_+\| + \tau^k\Big(|\Theta_\varphi|_{\mathcal{H}^{\frac{1}{2}-k}} + |\theta_\varphi|_{\mathcal{H}^{\frac{3}{2}-k}} + |v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{1}{2}-k}}\Big) + \|H_- v_-\| + \|H_- D_n v_-\| \gtrsim \tau^k\Big(|\Xi_1 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{3}{2}-k}} + |\Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{3}{2}-k}} + |\Xi_1 D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{1}{2}-k}} + |\Xi_1 D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{1}{2}-k}}\Big) + \tau^{k-\frac{1}{2}}\Big(\|\Xi_1 v\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \|H_-\Xi_1 D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \|H_+\Xi_1 D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big).$$

Remark 4.4. Note that in the case $k=0$, recalling the form of the second-order operators $\mathcal{P}_\pm$, we can estimate the additional terms $\tau^{-\frac{1}{2}}\|H_\pm \Xi_1 D_n^2 v_\pm\|$.

4.4. **Patching together microlocal estimates.** We now sum estimates (4.16) and (4.17) together. By the triangular inequality, this gives, for $k = 0$ or $\frac{1}{2}$,

$$\sum_{j=0,1}\Big(\|H_-\mathcal{P}_-\Xi_j v_-\| + \|H_+\mathcal{P}_+\Xi_j v_+\|\Big) + \tau^k\Big(|\Theta_\varphi|_{\mathcal{H}^{\frac{1}{2}-k}} + |\theta_\varphi|_{\mathcal{H}^{\frac{3}{2}-k}} + |v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{1}{2}-k}}\Big) + \|H_+ v_+\| + \|H_- v_-\| + \|H_- D_n v_-\|$$
$$\gtrsim \tau^k\Big(|v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{3}{2}-k}} + |v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{3}{2}-k}} + |D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{1}{2}-k}} + |D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{1}{2}-k}}\Big) + \tau^{k-\frac{1}{2}}\Big(\|v\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \|H_- D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \|H_+ D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big).$$

For τ sufficiently large we now obtain

$$\sum_{j=0,1}\Big(\|H_-\mathcal{P}_-\Xi_j v_-\| + \|H_+\mathcal{P}_+\Xi_j v_+\|\Big) + \tau^k\Big(|\Theta_\varphi|_{\mathcal{H}^{\frac{1}{2}-k}} + |\theta_\varphi|_{\mathcal{H}^{\frac{3}{2}-k}}\Big)$$
$$\gtrsim \tau^k\Big(|v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{3}{2}-k}} + |v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{3}{2}-k}} + |D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{1}{2}-k}} + |D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{1}{2}-k}}\Big) + \tau^{k-\frac{1}{2}}\Big(\|v\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \|H_- D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \|H_+ D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big).$$

Arguing with commutators, as in the end of Lemma 3.2, noting here that the second order operators $\mathcal{P}_\pm$ belong to the semi-classical calculus, i.e. $\mathcal{P}_\pm \in \mathcal{S}^2_\tau$, we otbain, for τ sufficiently large,

$$\|H_-\mathcal{P}_- v_-\| + \|H_+\mathcal{P}_+ v_+\| + \tau^k\Big(|\Theta_\varphi|_{\mathcal{H}^{\frac{1}{2}-k}} + |\theta_\varphi|_{\mathcal{H}^{\frac{3}{2}-k}}\Big)$$
$$\gtrsim \tau^k\Big(|v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{3}{2}-k}} + |v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{3}{2}-k}} + |D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{1}{2}-k}} + |D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{1}{2}-k}}\Big) + \tau^{k-\frac{1}{2}}\Big(\|v\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \|H_- D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \|H_+ D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big).$$

In particular this estimate allows us to absorb the perturbation in Ψ^1 as defined by (2.16) by taking τ large enough. For $k = \frac{1}{2}$ we obtain the result of Proposition 2.1, which concludes the proof of the Carleman estimate.

N.B. The case $k = 0$ provides higher Sobolev norm estimates of the trace terms $v_{\pm|x_n=0^\pm}$ and $D_n v_{\pm|x_n=0^\pm}$. It also allows one to estimate $\tau^{-\frac{1}{2}}\|H_\pm D_n^2 v_\pm\|$ as noted in

Remarks 4.3 and 4.4. These estimation are obtained at the price of higher requirements (one additional tangential half derivative) on the non-homogeneous transmission condition functions θ and Θ.

4.5. **Convexification.** We want now to modify slightly the weight function φ, for instance to allow some convexification. We started with $\varphi = H_+\varphi_+ + H_-\varphi_-$ where $\varphi_\pm$ were given by (2.22) and our proof relied heavily on a smooth factorization in first-order factors. We modify $\varphi_\pm$ into

$$\Phi_\pm(x', x_n) = \underbrace{\alpha_\pm x_n + \frac{1}{2}\beta x_n^2}_{\varphi_\pm(x_n)} + \kappa(x', x_n), \quad \kappa \in \mathscr{C}^\infty(\Omega; \mathbb{R}), \quad |d\kappa| \text{ bounded on } \Omega.$$

We shall prove below that the Carleman estimates of Theorem 1.1 and Theorem 2.2 also holds in this case if we choose $\|\kappa'\|_{L^\infty}$ sufficiently small.

We start by inspecting what survives in our factorization argument. We have from (2.7), $\mathcal{P}_\pm = (D + i\tau d\Phi_\pm) \cdot A_\pm(D + i\tau d\Phi_\pm)$, so that, modulo Ψ^1,

$$\mathcal{P}_\pm \equiv a^\pm_{nn}\Big(\big[D_n + S_\pm(x, D') + i\tau\big(\partial_n\Phi_\pm + S_\pm(x, \partial_{x'}\Phi_\pm)\big)\big]^2 + \frac{b^\pm_{jk}}{a^\pm_{nn}}(D_j + i\tau\partial_j\Phi_\pm)(D_k + i\tau\partial_k\Phi_\pm)\Big). \tag{4.18}$$

(See also (2.10).) The new difficulty comes from the fact that the roots in the variable D_n are not necessarily smooth: when Φ does not depend on x', the symbol of the term $b^\pm_{jk}(D_j + i\tau\partial_j\Phi_\pm)(D_k + i\tau\partial_k\Phi_\pm)$ equals $b^\pm_{jk}\xi_j\xi_k$ and thus is positive elliptic with a smooth positive square root. It is no longer the case when we have an actual dependence of Φ upon the variable x'; nevertheless, we have, as $\partial_{x'}\Phi_\pm = \partial_{x'}\kappa$,

$$\begin{aligned}\operatorname{Re}\Big(\frac{b^\pm_{jk}}{a^\pm_{nn}}(\xi_j + i\tau\partial_j\kappa)(\xi_k + i\tau\partial_k\kappa)\Big) &= \frac{b^\pm_{jk}}{a^\pm_{nn}}\xi_j\xi_k - \tau^2\frac{b^\pm_{jk}}{a^\pm_{nn}}\partial_j\kappa\partial_k\kappa \\ &\geq (\lambda_0^\pm)^2|\xi'|^2 - \tau^2(\lambda_1^\pm)^2|\partial_{x'}\kappa|^2 \geq \frac{3}{4}(\lambda_0^\pm)^2|\xi'|^2, \qquad \text{if } \tau\|\partial_{x'}\kappa\|_{L^\infty} \leq \frac{\lambda_0^\pm}{2\lambda_1^\pm}|\xi'|,\end{aligned}$$

where

$$\lambda_0^\pm = \inf_{\substack{x',\xi\\|\xi'|=1}}\Big(\frac{b^\pm_{jk}}{a^\pm_{nn}}\xi_j\xi_k\Big)^{\frac12}_{|x_n=0^\pm}, \qquad \lambda_1^\pm = \sup_{\substack{x',\xi\\|\xi'|=1}}\Big(\frac{b^\pm_{jk}}{a^\pm_{nn}}\xi_j\xi_k\Big)^{\frac12}_{|x_n=0^\pm},$$

As a result, the roots are smooth when $\tau\|\partial_{x'}\kappa\|_{L^\infty} \leq \frac{\lambda_0^\pm}{2\lambda_1^\pm}|\xi'|$.

In this case, we define $\mathfrak{m}_\pm \in \mathcal{S}^1$ such that

$$\text{for } |\xi'| \geq 1,\ \mathfrak{m}_\pm(x, \xi') = \Big(\frac{b^\pm_{jk}}{a^\pm_{nn}}(\xi_j + i\tau\partial_j\kappa)(\xi_k + i\tau\partial_k\kappa)\Big)^{\frac12}, \quad \mathfrak{m}_\pm(x, \xi') \geq C\langle\xi'\rangle.$$

Here we use the principal value of the square root function for complex numbers.

Introducing

$$\begin{aligned}\mathfrak{e}_\pm &= \tau\big(\partial_n\Phi_\pm + S_\pm(x, \partial_{x'}\kappa)\big) + \operatorname{Re}\mathfrak{m}_\pm(x, \xi'),\\ \mathfrak{f}_\pm &= \tau\big(\partial_n\Phi_\pm + S_\pm(x, \partial_{x'}\kappa)\big) - \operatorname{Re}\mathfrak{m}_\pm(x, \xi')\end{aligned}$$

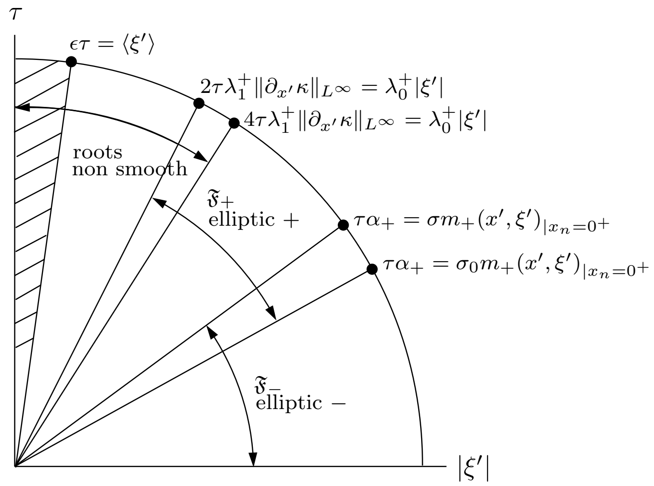

FIGURE 5. The overlapping microlocal regions in the case of a convex weight function.

we set $\mathfrak{E}_\pm = \text{op}(\mathfrak{e}_\pm)$ and $\mathfrak{F}_\pm = \text{op}(\mathfrak{f}_\pm)$ and

$$\mathcal{P}_{\mathfrak{E}\pm} = D_n + S_\pm(x, D') - \text{op}^w(\operatorname{Im} \mathfrak{m}_\pm) + i\mathfrak{E}_\pm,$$
$$\mathcal{P}_{\mathfrak{F}\pm} = D_n + S_\pm(x, D') + \text{op}^w(\operatorname{Im} \mathfrak{m}_\pm) + i\mathfrak{F}_\pm.$$

Modulo the operator class Ψ^1, as in Section 2.3, we may write

$$\mathcal{P}_+ \equiv \mathcal{P}_{\mathfrak{E}+} a_{nn}^+ \mathcal{P}_{\mathfrak{F}+}, \qquad \mathcal{P}_- \equiv \mathcal{P}_{\mathfrak{F}-} a_{nn}^- \mathcal{P}_{\mathfrak{E}-},$$

We keep the notation $m_\pm$ for the symbols that correspond to the previous sections, i.e., if κ vanishes:

$$m_\pm(x, \xi') = \Big(\frac{b_{jk}^\pm}{a_{nn}^\pm} \xi_j \xi_k \Big)^{\frac{1}{2}}, \quad |\xi'| \geq 1,$$

As above, see (4.1), we choose the weight function such that the following property is fulfilled

$$\frac{\alpha_+}{\alpha_-} > \sup_{\substack{x',\xi' \\ |\xi'|\geq 1}} \frac{m_+(x',\xi')_{|x_n=0^+}}{m_-(x',\xi')_{|x_n=0^-}}, \qquad \alpha_\pm = \partial_{x_n} \varphi_{\pm|x_n=0^\pm},$$

and we let $\sigma > 1$ be such that

$$\frac{\alpha_+}{\alpha_-} = \sigma^2 \sup_{\substack{x',\xi' \\ |\xi'|\geq 1}} \frac{m_+(x',\xi')_{|x_n=0^+}}{m_-(x',\xi')_{|x_n=0^-}}.$$

We also introduce $1 < \sigma_0 < \sigma$. As in Section 2.3 we set $f_\pm = \tau\varphi'_\pm - m_\pm$ (compare with $\mathfrak{f}_\pm$ above).

We can choose $\alpha_+/\|\partial_{x'}\kappa\|_{L^\infty}$ large enough so that

$$\frac{\sigma m^+_{|x_n=0^+}}{\alpha_+} < \frac{\lambda_0^+|\xi'|}{4\lambda_1^+\|\partial_{x'}\kappa\|_{L^\infty}}$$

and

$$f_\pm \geq C\lambda, \quad \text{if } \tau \geq |\xi'| \frac{\lambda_0^+}{4\lambda_1^+\|\partial_{x'}\kappa\|_{L^\infty}} \quad \text{for } |x_n| \text{ sufficiently small.} \tag{4.19}$$

We may then consider the following cases.

(1) When $\tau\alpha_+ \leq \sigma m^+(x',\xi')_{|x_n=0^+}$, then arguing as for (4.5)-(4.6) we find
$$\tau(\alpha_- + \beta x_n) - m_-(x',\xi')_{|x_n=0^-} \leq -C\lambda,$$
if $|x_n|$ is sufficiently small. It follows that $\mathfrak{F}_-$ is elliptic negative, if $\alpha_+/\|\kappa'\|_{L^\infty}$ is sufficiently large. In this region we may thus argue as we did in Section 4.3.

(2) When $\dfrac{\lambda_0^+|\xi'|}{2\lambda_1^+\|\partial_{x'}\kappa\|_{L^\infty}} \geq \tau \geq \dfrac{\sigma_0 m_+(x',\xi')}{\alpha_+}$, the factorization is valid. Arguing as in (4.3)-(4.4) we find that
$$\tau(\alpha_+ + \beta x_n) - m_+(x',\xi') \geq C\lambda,$$
if $|x_n|$ is sufficiently small. It follows that $\mathfrak{F}_+$ is elliptic positive, if $\alpha_+/\|\kappa'\|_{L^\infty}$ is sufficiently large. In this region we may thus argue as we did in Section 4.2.

It is important to note that for β large and $\|\kappa'\|_{L^\infty}$ and $\|\kappa''\|_{L^\infty}$ sufficiently small the weight functions $\Phi_\pm$ satisfy the (necessary and sufficient) sub-ellipticity condition (2.26) with a loss of a half derivative. Then the counterpart of Lemma 2.8 becomes, for $\|\kappa'\|_{L^\infty}$ sufficiently small,
$$|\mathfrak{f}_\pm| \leq \delta\lambda \quad\Longrightarrow\quad C^{-1}\tau \leq |\xi'| \leq C\tau \ \text{ and } \ \{\xi_n + s_\pm + \operatorname{Im}(\mathfrak{m}_\pm), \mathfrak{f}_\pm\} \geq C'\lambda,$$
for some $\delta > 0$ chosen sufficiently small. This allows us to then obtain the same results as that of Lemma 3.7 for the first-order factors $\mathcal{P}_{\mathfrak{F}\pm}$.

(3) Finally we consider the region $\tau \geq |\xi'|\dfrac{\lambda_0^+}{4\lambda_1^+\|\partial_{x'}\kappa\|_{L^\infty}}$. There the roots are no longer smooth, but we are well-inside an elliptic region; with a perturbation argument, we may in fact disregard the contribution of κ.

From (4.18) we may write
$$\mathcal{P}_\pm \equiv \underbrace{a_{nn}^\pm\Big(\big[D_n + S_\pm(x,D') + i\tau\partial_n\varphi_\pm\big]^2 + \frac{b_{jk}^\pm}{a_{nn}^\pm}D_jD_k\Big)}_{P_\pm^0} + R_\pm, \tag{4.20}$$
with $R_\pm = R_{1,\pm}(x,D',\tau)D_n + R_{2,\pm}(x,D',\tau)$, where $R_{j,\pm} \in \operatorname{op}^w(\mathcal{S}_\tau^j)$, $j=1,2$, that satisfy
$$\|R_{j,\pm}(x,D',\tau)u\| \leq C\|\kappa'\|_{L^\infty}\|u\|_{L^2(\mathbb{R};\mathcal{H}^j)} \tag{4.21}$$
The first term $P_\pm^0$ in (4.20) corresponds to the conjugated operator in the sections above, where the weight function only depended on the x_n variable. This term can be factored in two pseudo-differential first-order terms:
$$\mathcal{P}_+^0 \equiv \mathcal{P}_{E+}a_{nn}^+\mathcal{P}_{F+}, \qquad \mathcal{P}_-^0 \equiv \mathcal{P}_{F-}a_{nn}^-\mathcal{P}_{E-}, \tag{4.22}$$
with the notation we introduced in Section 2.3. In this third region we have $f_\pm \geq C\lambda$ by (4.19). Let $\chi_2 \in \mathcal{S}_\tau^0$ be a symbol that localizes in this region and set $\Xi_2 = \operatorname{op}^w(\chi_2)$.

For $\|\kappa'\|_{L^\infty}$ bounded with (4.23) we have
$$\|H_\pm R_{1,\pm}D_n\Xi_2 v_\pm\| \lesssim \tau^k\|\kappa'\|_{L^\infty}\|H_\pm D_n\Xi_2 v_\pm\|_{L^2(\mathbb{R};\mathcal{H}^{1-k}} + C(\kappa)\|H_\pm D_n v_\pm\|, \tag{4.23}$$
$$\|H_\pm R_{2,\pm}D_n\Xi_2 v_\pm\| \lesssim \tau^k\|\kappa'\|_{L^\infty}\|H_\pm\Xi_2 v_\pm\|_{L^2(\mathbb{R};\mathcal{H}^{2-k}} + C(\kappa)\|H_\pm v_\pm\|, \tag{4.24}$$

for $k = 0$ or $\frac{1}{2}$.

On the one hand, arguing as in Section 4.2 we have (see (4.14))

$$\|H_+\mathcal{P}^0_+\Xi_2 v_+\| + \|H_+ v_+\| \gtrsim |\mathcal{V}_{2,+}|_{\mathcal{H}^{\frac{1}{2}}} + |\Xi_2 v_{+|x_n=0^+}|_{\mathcal{H}^{3/2}} + \|H_+\Xi_2 v_+\|_{L^2(\mathbb{R};\mathcal{H}^2)} + \|H_+\Xi_2 D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^1)}, \tag{4.25}$$

where $\mathcal{V}_{2,\pm}$ is given as in (4.10).

On the other hand, with Lemma 3.4 we have

$$\|H_-\mathcal{P}^0_-\Xi_2 v_-\|_{L^2(\mathbb{R};\mathcal{H}^{-k})} + \|H_- v_-\| + \|H_- D_n v_-\| + |\mathcal{V}_{2,-} + i a^-_{nn} M_-\Xi_2 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{1}{2}-k}} \gtrsim \|H_-\mathcal{P}_{E-}\Xi_2 v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})},$$

for $k = 0$ or $\frac{1}{2}$, which gives

$$\|H_-\mathcal{P}^0_-\Xi_2 v_-\| + \tau^k\|H_- v_-\| + \tau^k\|H_- D_n v_-\| + \tau^k|\mathcal{V}_{2,-} + i a^-_{nn} M_-\Xi_2 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{1}{2}-k}} \gtrsim \tau^k\|H_-\mathcal{P}_{E-}\Xi_2 v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}.$$

Combined with Lemma 3.2 we obtain

$$\|H_-\mathcal{P}^0_-\Xi_2 v_-\| + \tau^k\Big(\|H_- v_-\| + \|H_- D_n v_-\| + |\mathcal{V}_{2,-}|_{\mathcal{H}^{\frac{1}{2}-k}} + |\Xi_2 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{3}{2}-k}}\Big) \gtrsim \tau^k\|H_-\Xi_2 v_-\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \tau^k\|H_+\Xi_2 D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} \tag{4.26}$$

Now, from the transmission conditions (4.9)–(4.11), by adding ε(4.26)+(4.25) we obtain, for ε small,

$$\begin{aligned}
&\|H_+\mathcal{P}^0_+\Xi_2 v_+\| + \|H_-\mathcal{P}^0_-\Xi_2 v_-\| + \tau^k\Big(|\theta_\varphi|_{\mathcal{H}^{\frac{3}{2}-k}} + |\Theta_\varphi|_{\mathcal{H}^{\frac{1}{2}-k}} + |v_{|x_n=0^+}|_{\mathcal{H}^{\frac{1}{2}-k}}\Big)\\
&\quad + \tau^k\big(\|H_- v_-\| + \|H_- D_n v_-\|\big) + \|H_+ v_+\| + \|H_+ D_n v_+\|\\
&\quad \gtrsim \tau^k\Big(|\Xi_2 D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{1}{2}-k}} + |\Xi_2 D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{1}{2}-k}}\\
&\quad + |\Xi_2 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{3}{2}-k}} + |\Xi_2 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{3}{2}-k}} + \|\Xi_2 v\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})}\\
&\quad + \|H_-\Xi_2 D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \|H_+\Xi_2 D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big).
\end{aligned} \tag{4.27}$$

With (4.23)–(4.24) we see that the same estimate holds for $\mathcal{P}_\pm$ in place of $\mathcal{P}^0_\pm$ for $\|\kappa'\|_{L^\infty}$ chosen sufficiently small. This estimate is of the same quality as those obtained in the two other regions.

Summing up, we have obtained three microlocal overlapping regions and estimates in each of them. The three regions are illustrated in Figure 5. As we did above we make sure that the preliminary cut-off region of Section 3.1 does not interact with the overlapping zones by choosing ϵ sufficiently small (see (3.5) and Lemma 3.1).

The overlap of the regions allows us to use a partition of unity argument and we can conclude as in Section 4.4.

5. Necessity of the geometric assumption on the weight function

Considering the operator $\mathcal{L}_\tau$ given by (1.23), we may wonder about the relevance of conditions (1.28) to derive a Carleman estimate. In the simple model and weight used here, it turns out that we can show that condition (1.28) is necessary for an estimate to hold. For simplicity, we consider a *piecewise constant* case $c = H_+c_+ + H_-c_-$ as in Section 1.5.

Theorem 5.1. *Let us assume that (1.29) is violated, i.e.,*

$$\exists \xi_0' \in \mathbb{R}^{n-1} \setminus 0, \quad \frac{\alpha_+}{\alpha_-} < \frac{m_+(\xi_0')}{m_-(\xi_0')}. \tag{5.1}$$

Then, for any neighborhood V of the origin, $C > 0$, and $\tau_0 > 0$, there exists

$$v = H_+v_+ + H_-v_-, \quad v_\pm \in \mathscr{C}_c^\infty(\mathbb{R}^n),$$

satisfying the transmission conditions (1.21)–(1.22) at $x_n = 0$, and $\tau \geq \tau_0$, such that

$$\operatorname{supp}(v) \subset V \quad \textit{and} \quad C\|\mathcal{L}_\tau v\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \leq \|v\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})},$$

To prove Theorem 5.1 we wish to construct a function v, depending on the parameter τ, such that $\|\mathcal{L}_\tau v\|_{L^2} \ll \|v\|_{L^2}$ as τ becomes large. The existence of such a quasi-mode v obviously ruins any hope to obtain a Carleman estimate for the operator $\mathcal{L}$ with a weight function satisfying (5.1). The remainder of this section is devoted to this construction.

We set

$$\begin{aligned}(\mathcal{M}_\tau u)(\xi', x_n) = H_+(x_n)c_n^+\big(D_n + ie_+\big)\big(D_n + if_+\big)u_+ \\ + H_-(x_n)c_n^-\big(D_n + ie_-\big)\big(D_n + if_-\big)u_-,\end{aligned} \tag{5.2}$$

that is, the action of the operator $\mathcal{L}_\tau$ given in (1.23) in the Fourier domain with respect to x'. Observe that the terms in each product commute here. We start by constructing a quasi-mode for $\mathcal{M}_\tau$, i.e., functions $u_\pm(\xi', x_n)$ compactly supported in the x_n variable and in a conic neighborhood of ξ_0' in the variable ξ' with $\|\mathcal{M}_\tau u\|_{L^2} \ll \|u\|_{L^2}$, so that u is nearly an eigenvector of $\mathcal{M}_\tau$ for the eigenvalue 0.

Condition 5.1 implies that there exists $\tau_0 > 0$ such that

$$\frac{m_-(\xi_0')}{\alpha_-} < \tau_0 < \frac{m_+(\xi_0')}{\alpha_+} \Longrightarrow \tau_0\alpha_+ - m_+(\xi_0') < 0 < \tau_0\alpha_- - m_-(\xi_0').$$

By homogeneity we may in fact choose (τ_0, ξ_0') such that $\tau_0^2 + |\xi_0'|^2 = 1$. We have thus, using the notation in (1.23),

$$f_+(x_n = 0) = \tau\alpha_+ - m_+(\xi') < 0 < f_-(x_n = 0) = \tau\alpha_- - m_-(\xi'),$$

for (τ, ξ') in a conic neighborhood Γ of (τ_0, ξ_0') in $\mathbb{R} \times \mathbb{R}^{n-1}$. Let $\chi_1 \in \mathscr{C}_c^\infty(\mathbb{R})$, $0 \leq \chi_1 \leq 1$, with $\chi_1 \equiv 1$ in a neighborhood of 0, such that $\operatorname{supp}(\psi) \subset \Gamma$ with

$$\psi(\tau, \xi') = \chi_1\Big(\frac{\tau}{(\tau^2 + |\xi'|^2)^{\frac12}} - \tau_0\Big)\chi_1\Big(\Big|\frac{\xi'}{(\tau^2 + |\xi'|^2)^{\frac12}} - \xi_0'\Big|\Big).$$

We thus have

$$f_+(x_n = 0) \leq -C\tau, \quad C'\tau \leq f_-(x_n = 0) \quad \text{in } \operatorname{supp}(\psi).$$

Let $(\tau, \xi') \in \operatorname{supp}(\psi)$. We can solve the equations

$$\begin{aligned}
&\big(D_n + i f_+(x_n,\xi'))\big) q_+ = 0 \quad \text{on } \mathbb{R}_+, \quad f_+(x_n,\xi') = \tau\varphi'(x_n) - m_+(\xi') = f_+(0) + \tau\beta x_n,\\
&\big(D_n + i f_-(x_n,\xi'))\big) q_- = 0 \quad \text{on } \mathbb{R}_-, \quad f_-(x_n,\xi') = \tau\varphi'(x_n) - m_-(\xi') = f_-(0) + \tau\beta x_n,\\
&\big(D_n + i e_-(x_n,\xi'))\big) \tilde{q}_- = 0 \quad \text{on } \mathbb{R}_-, \quad e_-(x_n,\xi') = \tau\varphi'(x_n) + m_-(\xi') = e_-(0) + \tau\beta x_n,
\end{aligned}$$

that is

$$\begin{aligned}
q_+(\xi',x_n) &= Q_+(\xi',x_n) q_+(\xi',0), \quad Q_+(\xi',x_n) = e^{x_n\left(f_+(0) + \frac{\tau\beta x_n}{2}\right)},\\
q_-(\xi',x_n) &= Q_-(\xi',x_n) q_-(\xi',0), \quad Q_-(\xi',x_n) = e^{x_n\left(f_-(0) + \frac{\tau\beta x_n}{2}\right)},\\
\tilde{q}_-(\xi',x_n) &= \tilde{Q}_-(\xi',x_n) \tilde{q}_-(\xi',0), \quad \tilde{Q}_-(\xi',x_n) = e^{x_n\left(e_-(0) + \frac{\tau\beta x_n}{2}\right)}.
\end{aligned}$$

Since $f_+(0) < 0$ a solution of the form of q_+ is a good idea on $x_n \geq 0$ as long as $\tau\beta x_n + 2f_+(0) \leq 0$, i.e., $x_n \leq 2|f_+(0)|/\tau\beta$. Similarly as $f_-(0) > 0$ (resp. $e_-(0) > 0$) a solution of the form of q_- (resp. $\tilde{q}_-$) is a good idea on $x_n \leq 0$ as long as $\tau\beta x_n + 2f_-(0) \geq 0$ (resp. $\tau\beta x_n + 2e_-(0) \geq 0$). To secure this we introduce a cut-off function $\chi_0 \in \mathscr{C}_c^\infty((-1,1);[0,1])$, equal to 1 on $[-\frac{1}{2},\frac{1}{2}]$ and for $\gamma \geq 1$ we define

$$u_+(\xi',x_n) = Q_+(\xi',x_n)\psi(\tau,\xi')\chi_0\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big), \tag{5.3}$$

and

(5.4)
$$u_-(\xi',x_n) = aQ_-(\xi',x_n)\psi(\tau,\xi')\chi_0\Big(\frac{\tau\beta\gamma x_n}{f_-(0)}\Big) + b\tilde{Q}_-(\xi',x_n)\psi(\tau,\xi')\chi_0\Big(\frac{\tau\beta\gamma x_n}{e_-(0)}\Big),$$

with $a, b \in \mathbb{R}$, and

$$u(\xi',x_n) = H_+(x_n)u_+(\xi',x_n) + H_-(x_n)u_-(\xi',x_n)$$

The factor γ is introduced to control the size of the support in the x_n direction. Observe that we can satisfy the transmission condition (1.21)–(1.22) by choosing the coefficients a and b. Transmission condition (1.21) implies

$$a + b = 1. \tag{5.5}$$

Transmission condition (1.22) and the equations satisfied by Q_+, Q_- and $\tilde{Q}_-$ imply

$$c_+ m_+ = c_-(a-b)m_-. \tag{5.6}$$

In particular note that $a - b \geq 0$ which gives $a \geq \frac{1}{2}$.

We have the following lemma.

Lemma 5.2. *For τ sufficiently large we have*

$$\|\mathcal{M}_\tau u\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \leq C(\gamma^2 + \tau^2)\gamma\tau^{n-1}e^{-C'\tau/\gamma}$$

and

$$\|u\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \geq C\tau^{n-2}\big(1 - e^{-C'\tau/\gamma}\big).$$

See Section 6.2.3 for a proof.

We now introduce

$$v_\pm(x', x_n) = (2\pi)^{-(n-1)} \chi_0(|\tau^{\frac{1}{2}} x'|) \check{\hat{u}}_\pm(x', x_n) = (2\pi)^{-(n-1)} \chi_0(|\tau^{\frac{1}{2}} x'|) \hat{u}_\pm(-x', x_n),$$

that is, a localized version of the inverse Fourier transform (in x') of $u_\pm$. The functions $v_\pm$ are smooth and compactly supported in $\mathbb{R}^{n-1}_\pm \times \mathbb{R}$ and they satisfy transmission conditions (1.21)–(1.22). We set $v(x', x_n) = H_+(x_n) v_+(x', x_n) + H_-(x_n) v_-(x', x_n)$. In fact we have the following estimates.

Lemma 5.3. *Let $N \in \mathbb{N}$. For τ sufficiently large we have*

$$\|\mathcal{L}_\tau v\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \leq C(\gamma^2 + \tau^2)\gamma\tau^{n-1} e^{-C'\tau/\gamma} + C_{\gamma,N}\tau^{-N}$$

and

$$\|v\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \geq C\tau^{n-2}\big(1 - e^{-C'\tau/\gamma}\big) - C_{\gamma,N}\tau^{-N}.$$

See Section 6.2.4 for a proof.

We may now conclude the proof of Theorem 5.1. In fact, if V is an arbitrary neighborhood of the origin, we choose τ and γ sufficiently large so that $\operatorname{supp}(v) \subset V$. We then keep γ fixed. The estimates of Lemma 5.3 show that

$$\|\mathcal{L}_\tau v\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \|v\|^{-1}_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \underset{\tau\to\infty}{\longrightarrow} 0.$$

Remark 5.4. As opposed to the analogy we give at the beginning of Section 1.6, the construction of this quasi-mode does not simply rely on one of the first-order factor. The transmission conditions are responsible for this fact. The construction relies on the factor $D_n + if_+$ in $x_n \geq 0$, i.e., a one-dimensional space of solutions (see (5.3)), and on both factors $D_n + if_-$ and $D_n + ie_-$ in $x_n \geq 0$, i.e., a two-dimensional space of solutions (see (5.4)). See also (5.5) and (5.6).

6. Appendix

6.1. A few facts on pseudo-differential operators.

6.1.1. *Standard classes and Weyl quantization.* We define for $m \in \mathbb{R}$ the class of tangential symbols $\mathcal{S}^m$ as the smooth functions on $\mathbb{R}^n \times \mathbb{R}^{n-1}$ such that, for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^{n-1}$,

$$N_{\alpha\beta}(a) = \sup_{(x,\xi')\in\mathbb{R}^n\times\mathbb{R}^{n-1}} \langle\xi'\rangle^{-m+|\beta|} |(\partial_x^\alpha \partial_{\xi'}^\beta a)(x, \xi')| < \infty, \tag{6.1}$$

with $\langle\xi'\rangle^2 = 1 + |\xi'|^2$. The quantities on the l.h.s. above are called the semi-norms of the symbol a. For $a \in \mathcal{S}^m$, we define $\operatorname{op}(a)$ as the operator defined on $\mathscr{S}(\mathbb{R}^n)$ by

$$(\operatorname{op}(a)u)(x', x_n) = a(x, D')u(x', x_n) = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} a(x', x_n, \xi') \hat{u}(\xi', x_n) d\xi' (2\pi)^{1-n}, \tag{6.2}$$

with $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, where $\hat{u}$ is the partial Fourier transform of u with respect to the variable x'. For all $(k, s) \in \mathbb{Z} \times \mathbb{R}$ we have

$$\operatorname{op}(a) : H^k(\mathbb{R}_{x_n}; H^{s+m}(\mathbb{R}^{n-1}_{x'})) \to H^k(\mathbb{R}_{x_n}; H^s(\mathbb{R}^{n-1}_{x'})) \quad \text{continuously}, \tag{6.3}$$

and the norm of this mapping depends only on $\{N_{\alpha\beta}(a)\}_{|\alpha|+|\beta|\le\mu(k,s,m,n)}$, where $\mu : \mathbb{Z}\times\mathbb{R}\times\mathbb{R}\times\mathbb{N}\to\mathbb{N}$.

We shall also use the *Weyl quantization* of a denoted by $\mathrm{op}^w(a)$ and given by the formula

$$(6.4)\quad \begin{aligned}(\mathrm{op}^w(a)u)(x',x_n) &= a^w(x,D')u(x',x_n)\\ &= \iint_{\mathbb{R}^{2n-2}} e^{i(x'-y')\cdot\xi'} a\big(\frac{x'+y'}{2},x_n,\xi'\big)u(y',x_n)dy'd\xi'(2\pi)^{1-n}.\end{aligned}$$

Property (6.3) holds as well for $\mathrm{op}^w(a)$. A nice feature of the Weyl quantization that we use in this article is the simple relationship with adjoint operators with the formula

$$(6.5)\qquad \big(\mathrm{op}^w(a)\big)^* = \mathrm{op}^w(\bar a),$$

so that for a real-valued symbol $a\in\mathcal{S}^m$ $(\mathrm{op}^w(a))^* = \mathrm{op}^w(a)$. We have also for $a_j\in\mathcal{S}^{m_j}$, $j=1,2$,

$$(6.6)\qquad \mathrm{op}^w(a_1)\mathrm{op}^w(a_2)=\mathrm{op}^w(a_1\sharp a_2),\quad a_1\sharp a_2\in\mathcal{S}^{m_1+m_2},$$

with, for any $N\in\mathbb{N}$,

$$(6.7)\quad \big(a_1\sharp a_2\big)(x,\xi) - \sum_{j<N}\big(i\sigma(D_{x'},D_{\xi'};D_{y'},D_{\eta'})/2\big)^j a_1(x,\xi)a_2(y,\eta)/j!\big|_{(y,\eta)=(x,\xi)}\in\mathcal{S}^{m-N},$$

where σ is the symplectic two-form, i.e., $\sigma(x,\xi;y,\eta)=y\cdot\xi - x\cdot\eta$. In particular,

$$(6.8)\qquad \mathrm{op}^w(a_1)\mathrm{op}^w(a_2)=\mathrm{op}^w(a_1a_2)+\mathrm{op}^w(r_1),\quad r_1\in\mathcal{S}^{m_1+m_2-1},$$

$$(6.9)\qquad \text{with } r_1=\frac{1}{2i}\{a_1,a_2\}+r_2,\quad r_2\in\mathcal{S}^{m_1+m_2-2},$$

$$(6.10)\qquad [\mathrm{op}^w(a_1),\mathrm{op}^w(a_2)]=\mathrm{op}^w\big(\frac{1}{i}\{a_1,a_2\}\big)+\mathrm{op}^w(r_3),\quad r_3\in\mathcal{S}^{m_1+m_2-3},$$

where $\{a_1,a_2\}$ is the Poisson bracket. Moreover, for $b_j\in\mathcal{S}^{m_j}$, $j=1,2$, both real-valued, we have

$$(6.11)\quad [\mathrm{op}^w(b_1),i\mathrm{op}^w(b_2)]=\mathrm{op}^w(\{b_1,b_2\})+\mathrm{op}^w(s_3),\quad s_3\text{ real-valued}\in\mathcal{S}^{m_1+m_2-3}.$$

Lemma 6.1. *Let $a\in\mathcal{S}^1$ such that $a(x,\xi')\ge\mu\langle\xi'\rangle$, with $\mu\ge 0$. Then there exists $C>0$ such that*

$$\mathrm{op}^w(a)+C\ge\mu\langle D'\rangle,\qquad (\mathrm{op}^w(a))^2+C\ge\mu^2\langle D'\rangle^2.$$

Proof. The first statement follows from the sharp Gårding inequality [19, Chap. 18.1 and 18.5] applied to the nonnegative first-order symbol $a(x,\xi')-\mu\langle\xi'\rangle$; moreover $(\mathrm{op}^w(a))^2=\mathrm{op}^w(a^2)+\mathrm{op}^w(r)$ with $r\in\mathcal{S}^0$, so that the Fefferman-Phong inequality [19, Chap. 18.5] applied to the second-order $a^2-\mu^2\langle\xi'\rangle^2$ implies the result. ■

6.1.2. *Semi-classsical pseudo-differential calculus with a large parameter.* We let $\tau \in \mathbb{R}$ be such that $\tau \geq \tau_0 \geq 1$. We set $\lambda^2 = 1 + \tau^2 + |\xi'|^2$. We define for $m \in \mathbb{R}$ the class of symbols $\mathcal{S}_\tau^m$ as the smooth functions on $\mathbb{R}^n \times \mathbb{R}^{n-1}$, depending on the parameter τ, such that, for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^{n-1}$,

$$N_{\alpha\beta}(a) = \sup_{\substack{(x,\xi') \in \mathbb{R}^n \times \mathbb{R}^{n-1} \\ \tau \geq \tau_0}} \lambda^{-m+|\beta|} |(\partial_x^\alpha \partial_{\xi'}^\beta a)(x, \xi', \tau)| < \infty. \tag{6.12}$$

Note that $\mathcal{S}_\tau^0 \subset \mathcal{S}^0$. The associated operators are defined by (6.2). We can introduce Sobolev spaces and Sobolev norms which are adapted to the scaling large parameter τ. Let $s \in \mathbb{R}$; we set

$$\|u\|_{\mathcal{H}^s} := \|\Lambda^s u\|_{L^2(\mathbb{R}^{n-1})}, \quad \text{with } \Lambda^s := \operatorname{op}(\lambda^s)$$

and

$$\mathcal{H}^s = \mathcal{H}^s(\mathbb{R}^{n-1}) := \{u \in \mathscr{S}'(\mathbb{R}^{n-1});\ \|u\|_{\mathcal{H}^s} < \infty\}.$$

The space $\mathcal{H}^s$ is algebraically equal to the classical Sobolev space $H^s(\mathbb{R}^{n-1})$, which norm is denoted by $\|.\|_{H^s}$. For $s \geq 0$ note that we have

$$\|u\|_{\mathcal{H}^s} \sim \tau^s \|u\|_{L^2(\mathbb{R}^{n-1})} + \|\langle D' \rangle^s u\|_{L^2(\mathbb{R}^{n-1})}.$$

If $a \in \mathcal{S}_\tau^m$ then, for all $(k, s) \in \mathbb{Z} \times \mathbb{R}$, we have

$$\operatorname{op}(a) : H^k(\mathbb{R}_{x_n}; \mathcal{H}^{s+m}) \to H^k(\mathbb{R}_{x_n}; \mathcal{H}^s(\mathbb{R}_{x'}^{n-1})) \quad \text{continuously,} \tag{6.13}$$

and the norm of this mapping depends only on $\{N_{\alpha\beta}(a)\}_{|\alpha|+|\beta| \leq \mu(k,s,m,n)}$, where $\mu : \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times \mathbb{N} \to \mathbb{N}$.

For the calculus with a large parameter we shall also use the Weyl quantization of (6.4). All the formulæ listed in (6.5)–(6.11) hold as well, with $\mathcal{S}^m$ everywhere replaced by $\mathcal{S}_\tau^m$. We shall often use the Gårding inequality as stated in the following lemma.

Lemma 6.2. *Let $a \in \mathcal{S}_\tau^m$ such that $\operatorname{Re} a \geq C\lambda^m$. Then*

$$\operatorname{Re}(\operatorname{op}^w(a)u, u) \gtrsim \|u\|^2_{L^2(\mathbb{R}; \mathcal{H}^{\frac{m}{2}})},$$

for τ sufficiently large.

Proof. The proof follows from the Sharp Gårding inequality [19, Chap. 18.1 and 18.5] applied to the nonnegative symbol $a - C\lambda^m$. ∎

We recall the following definition.

Definition 6.3. The essential support of a symbol $a \in \mathcal{S}_\tau^m$, denoted by $\operatorname{esssupp}(a)$, is the complement of the largest open set of $\mathbb{R} \times \mathbb{R}^{n-1} \times \{\tau \geq 1\}$ where the estimates for $\mathcal{S}_\tau^{-\infty} = \cap_{m \in \mathbb{R}} \mathcal{S}_\tau^m$ hold.

For technical reasons we shall often need the following result.

Lemma 6.4. *Let $m, m' \in \mathbb{R}$ and $a_1(x, \xi') \in \mathcal{S}^m$ and $a_2(x, \xi', \tau) \in \mathcal{S}_\tau^{m'}$ such that the essential support of a_2 is contained in a region where $\langle \xi' \rangle \gtrsim \tau$. Then*

$$\operatorname{op}^w(a_1)\operatorname{op}^w(a_2) = \operatorname{op}^w(b_1), \quad \operatorname{op}^w(a_2)\operatorname{op}^w(a_1) = \operatorname{op}^w(b_2),$$

with $b_1, b_2 \in \mathcal{S}_\tau^{m+m'}$. Moreover the asympotic series of (6.7) is also valid for these cases (with $\mathcal{S}^m$ replaced by $\mathcal{S}_\tau^m$).

Proof. As the essential support is invariant when we change quantization, we may simply use the standard quantization in the proof. With a_1 and a_2 satisfying the assumption listed above we thus consider $\mathrm{op}(a_1)\mathrm{op}(a_2)$. For fixed τ the standard composition formula applies and we have (see [19, Section 18.1] or [2])

$$\big(a_1 \circ a_2\big)(x, \xi', \tau) = (2\pi)^{1-n} \iint e^{-iy'\cdot\eta'} a_1(x, \xi' - \eta')a_2(x' - y', x_n, \xi', \tau)dy'd\eta'.$$

Properties of oscillatory integrals (see e.g. [2, Appendices I.8.1 and I.8.2]) give, for some $k \in \mathbb{N}$,

$$|\big(a_1 \circ a_2\big)(x, \xi', \tau)| \leq C \sup_{\substack{|\alpha|+|\beta|\leq k \\ (y',\eta')\in R^{2n-2}}} \langle (y', \eta')\rangle^{-|m|} |\partial_{y'}^\alpha \partial_{\eta'}^\beta a_1(x, \xi' - \eta')a_2(x' - y', x_n, \xi', \tau)|,$$

In a region $\langle \xi' \rangle \gtrsim \tau$ that contains the essential support of a_2 we have $\langle \xi' \rangle \sim \lambda$. With the so-called Peetre inequality, we thus obtain

$$|\big(a_1 \circ a_2\big)(x, \xi', \tau)| \lesssim \langle \eta' \rangle^{-|m|} \langle \xi' - \eta' \rangle^m \lambda^{m'} \lesssim \langle \xi' \rangle^m \lambda^{m'} \lesssim \lambda^{m+m'}.$$

In a region $\langle \xi' \rangle \lesssim \tau$ outside of the essential support of a_2 we find, for any $\ell \in \mathbb{N}$,

$$|\big(a_1 \circ a_2\big)(x, \xi', \tau)| \lesssim \langle \eta' \rangle^{-|m|} \langle \xi' - \eta' \rangle^m \lambda^{-\ell} \lesssim \langle \xi' \rangle^m \lambda^{-\ell} \lesssim \lambda^{m-\ell}.$$

In the whole phase space we thus have $|\big(a_1 \circ a_2\big)(x, \xi')| \lesssim \lambda^{m+m'}$. The estimation of $|\partial_x^\alpha \partial_{\xi'}^\beta \big(a_1 \circ a_2\big)(x, \xi', \tau)|$ can be done similarly to give

$$|\partial_x^\alpha \partial_\xi^\beta \big(a_1 \circ a_2\big)(x, \xi', \tau)| \lesssim \lambda^{m+m'-|\beta|}.$$

Hence $a_1 \circ a_2 \in \mathcal{S}_\tau^{m+m'}$. Moreover, we also obtain the asymptotic series (following the references cited above)

$$\big(a_1 \circ a_2\big)(x, \xi', \tau) - \sum_{j<N} \big(iD_\xi \cdot D_y\big)^j a_1(x, \xi)a_2(y, \eta, \tau)/j!\big|_{(y,\eta)=(x,\xi)} \in \mathcal{S}_\tau^{m+m'-N},$$

where each term is respectively in $\mathcal{S}_\tau^{m+m'-j}$ be the arguments given above. From this series the corresponding Weyl-quantization series follows.

For the second result, considering the adjoint operator $\big(\mathrm{op}(a_2)\mathrm{op}(a_1)\big)^*$ yields a composition of operators as in the first case. The second result thus follows from the first one. ■

Remark 6.5. The symbol class and calculus we have introduced in this section can be written as $\mathcal{S}_\tau^m = \mathcal{S}(\lambda^m, g)$ in the sense of the Weyl-Hörmander calculus [19, Sec 18.4–18.6] with the phase-space metric $g = |dx|^2 + |d\xi|^2/\lambda^2$.

6.2. **Proofs of some intermediate results.**

6.2.1. *Proof of Lemma 2.8.* For simplicity we remove the $\pm$ notation here. We first prove that there exist $C > 0$ and $\eta > 0$ such that

$$|q_2| \leq \eta\tau^2 \ \text{ and } \ |q_1| \leq \eta\tau^2 \implies \{q_2, q_1\} \geq C\tau^3. \tag{6.14}$$

We set

$$\tilde q_2 = (\xi_n + s)^2 + \frac{b_{jk}}{a_{nn}}\xi_j\xi_k - (\varphi')^2, \qquad \tilde q_1 = \varphi'(\xi_n + s).$$

We have $q_j(x,\xi) = \tau^2 \tilde q_j(x, \xi/\tau)$. Observe next that we have $\{q_2, q_1\}(x,\xi) = \tau^3\{\tilde q_2, \tilde q_1\}(x, \xi/\tau)$. We thus have $\tilde q_2 = 0$ and $\tilde q_1 = 0 \ \Rightarrow\ \{\tilde q_2, \tilde q_1\} > 0$. As $\tilde q_2(x,\xi) = 0$ and $\tilde q_1(x,\xi) = 0$ yields a compact set for (x,ξ) (recall the x lays in a compact set K here), for some $C > 0$, we have

$$\tilde q_2 = 0 \text{ and } \tilde q_1 = 0 \quad \Longrightarrow \quad \{\tilde q_2, \tilde q_1\} > C.$$

This remain true locally, i.e., for some $C' > 0$ and $\eta > 0$,

$$|\tilde q_2| \le \eta \text{ and } |\tilde q_1| \le \eta \quad \Longrightarrow \quad \{\tilde q_2, \tilde q_1\} > C'.$$

Then (6.14) follows.

We note that $q_2^\pm = 0$ and $q_1^\pm = 0$ implies $\tau \sim |\xi'|$. Hence, for τ sufficiently large we have (2.25). We thus obtain

$$q_2^\pm = 0 \text{ and } q_1^\pm = 0 \quad \Leftrightarrow \quad \xi_n + s_\pm = 0 \text{ and } \tau\varphi'_\pm = m_\pm.$$

Let us assume that $|f| \le \delta\lambda$ with δ small and $\lambda^2 = 1 + \tau^2 + |\xi'|^2$. Then

$$\tau \lesssim |\xi'| \lesssim \tau. \tag{6.15}$$

We set $\xi_n = -s$, i.e., we choose $q_1 = 0$. A direct computation yields

$$\{q_2, q_1\} = \tau e \varphi'\{\xi_n + s, f\} + \tau f \varphi'\{\xi_n + s, e\} \quad \text{if } \ \xi_n + s = 0.$$

With (2.25) we have $|q_2| \le C\delta\tau^2$. For δ small, by (6.14) we have $\{q_2, q_1\} \ge C\tau^3$. Since $f\tau\varphi'\{\xi_n + s, e\} \le C\delta\tau^3$ we obtain $e\tau\varphi'\{\xi_n + s, f\} \ge C\tau^3$, with $C > 0$, for δ sufficiently small. With (6.15) we have $\tau \lesssim e \lesssim \tau$ and the result follows. ∎

6.2.2. *Proof of Lemma 3.1.* We set $s = 2\ell + 1$ and $\omega_1 = \mathrm{op}(\psi_\epsilon)\omega$. We write

$$\begin{aligned}
&2\,\mathrm{Re}(\mathcal{P}_{F+}\omega_1, iH_+\tau^s\omega_1)\\
&\quad = (i[D_n, H_+]\omega_1, \tau^s\omega_1) + 2(F_+\omega_1, H_+\tau^s\omega_1)\\
&\quad = \tau^s|\omega_{1|x_n=0^+}|^2_{L^2(\mathbb{R}^{n-1})} + 2(\tau^{s+1}\varphi'\omega_1, H_+\omega_1) - 2(\tau^s M_+\omega_1, H_+\omega_1)\\
&\quad \ge \tau^s|\omega_{1|x_n=0^+}|^2_{L^2(\mathbb{R}^{n-1})} + 2(\tau^{s+1}C_0\omega_1, H_+\ \omega_1) - 2C_1\tau^s\|H_+\omega_1\|^2_{L^2(\mathbb{R};H^{\frac12}(\mathbb{R}^{n-1}))},
\end{aligned}$$

by (3.4). We have

$$\begin{aligned}
&2(\tau^{s+1}C_0\omega_1, H_+\ \omega_1) - 2C_1\tau^s\|H_+\omega_1\|^2_{L^2(\mathbb{R};H^{\frac12}(\mathbb{R}^{n-1}))}\\
&\qquad = 2\tau^s(2\pi)^{1-n}\int_0^\infty\int_{\mathbb{R}^{n-1}} \big(C_0\tau - C_1\langle\xi'\rangle\big)|\psi_\epsilon(\tau,\xi')\hat\omega(\xi', x_n)|^2 d\xi' dx_n
\end{aligned}$$

As $\tau \ge C\langle\xi\rangle/\epsilon$ in $\mathrm{supp}(\psi_\epsilon)$, for ϵ sufficiently small we have

$$\begin{aligned}
&2(\tau^{s+1}C_0\omega_1, H_+\ \omega_1) - 2C_1\tau^s\|H_+\omega_1\|^2_{L^2(\mathbb{R};H^{\frac12}(\mathbb{R}^{n-1}))}\\
&\qquad \gtrsim \int_0^\infty\int_{\mathbb{R}^{n-1}} \lambda^{s+1}|\psi_\epsilon(\tau,\xi')\hat\omega(\xi', x_n)|^2 d\xi' dx_n \gtrsim \|H_+\omega_1\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.
\end{aligned}$$

Similarly we find $\tau^s|\omega_{1|x_n=0^+}|^2_{L^2(\mathbb{R}^{n-1})} \gtrsim |\omega_{1|x_n=0^+}|^2_{\mathcal{H}^{\ell+\frac12}}$. The result for $\mathcal{P}_{E+}$ follows from the Young inequality. The proof is identical for $\mathcal{P}_{F+}$.

On the other side of the interface we write

$$\begin{aligned} &2\operatorname{Re}(H_-\mathcal{P}_{F-}\omega_1, iH_-\tau^s\omega_1)\\ &\quad = (i[D_n, H_-]\omega_1, \tau^s\omega_1) + 2(F_-\omega_1, H_-\tau^s\omega_1)\\ &\quad = -\tau^s|\omega_{1|x_n=0^-}|^2_{L^2(\mathbb{R}^{n-1})} + 2(\tau^{s+1}\varphi'\omega_1, H_-\omega_1) - 2(\tau^s M_-\omega_1, H_-\omega_1), \end{aligned}$$

which yields a boundary contribution with the opposite sign. ■

6.2.3. *Proof of Lemma 5.2.* Let $(\tau, \xi') \in \operatorname{supp}(\psi)$. We choose τ sufficiently large so that, through $\operatorname{supp}(\psi)$, $|\xi'|$ is itself sufficiently large, so as to have the symbol $m_\pm$ homogeneous –see (2.15).

We set

$$\begin{aligned} y_+(\xi', x_n) &= Q_+(\xi', x_n)\chi_0\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big),\\ y_-(\xi', x_n) &= aQ_-(\xi', x_n)\chi_0\Big(\frac{\tau\beta\gamma x_n}{f_-(0)}\Big) + b\tilde{Q}_-(\xi', x_n)\chi_0\Big(\frac{\tau\beta\gamma x_n}{e_-(0)}\Big). \end{aligned}$$

On the one hand we have $i(D_n + if_+)y_+ = \frac{\tau\beta\gamma}{|f_+(0)|}Q_+(\xi', x_n)\chi_0'\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big)$, and

$$\begin{aligned} (\mathcal{M}_\tau y_+)(\xi', x_n) = 2\tau\beta\gamma c_+ m_+ &\frac{Q_+(\xi', x_n)}{|f_+(0)|}\chi_0'\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big)\\ &- (\tau\beta\gamma)^2 c_+ \frac{Q_+(\xi', x_n)}{|f_+(0)|^2}\chi_0''\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big), \end{aligned}$$

as $D_n + ie_+ = D_n + i(f_+ + 2m_+)$, so that

$$\begin{aligned} \int_0^{+\infty} |(\mathcal{M}_\tau y_+)(\xi', x_n)|^2 dx_n \le 8c_+^2 m_+^2 &\Big(\frac{\tau\beta\gamma}{f_+(0)}\Big)^2 \int_0^{+\infty} \chi_0'\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big)^2 e^{x_n(2f_+(0)+\tau\beta x_n)} dx_n\\ &+ 2c_+^2\Big(\frac{\tau\beta\gamma}{f_+(0)}\Big)^4 \int_0^{+\infty} \chi_0''\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big)^2 e^{x_n(2f_+(0)+\tau\beta x_n)} dx_n. \end{aligned}$$

On the support of $\chi_0^{(j)}\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big)$, $j = 1, 2$, we have $|f_+(0)|/(2\tau\beta\gamma) \le x_n \le |f_+(0)|/(\tau\beta\gamma)$ and in particular $2f_+(0) + \tau\beta\gamma x_n \le -|f_+(0)|$ and which gives

$$\begin{aligned} &\int_0^{+\infty} |(\mathcal{M}_\tau y_+)(\xi', x_n)|^2 dx_n\\ &\quad\le c_+^2\Big(\frac{\tau\beta\gamma}{f_+(0)}\Big)^2\bigg(8m_+^2\|\chi_0'\|^2_{L^\infty} + 2\Big(\frac{\tau\beta\gamma}{f_+(0)}\Big)^2\|\chi_0''\|^2_{L^\infty}\bigg) \int_{\frac{|f_+(0)|}{2\tau\beta\gamma}\le x_n\le\frac{|f_+(0)|}{\tau\beta\gamma}} e^{-|f_+(0)|x_n} dx_n\\ &\quad\le c_+^2\frac{\tau\beta\gamma}{|f_+(0)|}\bigg(4m_+^2\|\chi_0'\|^2_{L^\infty} + \Big(\frac{\tau\beta\gamma}{f_+(0)}\Big)^2\|\chi_0''\|^2_{L^\infty}\bigg) e^{-\frac{f_+(0)^2}{2\tau\beta\gamma}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} (\mathcal{M}_\tau y_-)(\xi', x_n) = 2\tau\beta\gamma c_- m_- &\bigg(\frac{aQ_-(\xi', x_n)}{f_-(0)}\chi_0'\Big(\frac{\tau\beta\gamma x_n}{f_-(0)}\Big) - b\frac{\tilde{Q}_-(\xi', x_n)}{e_-(0)}\chi_0'\Big(\frac{\tau\beta\gamma x_n}{e_-(0)}\Big)\bigg)\\ &- c_-(\tau\beta\gamma)^2\bigg(a\frac{Q_-(\xi', x_n)}{f_-(0)^2}\chi_0''\Big(\frac{\tau\beta\gamma x_n}{f_-(0)}\Big) + b\frac{\tilde{Q}_-(\xi', x_n)}{e_-(0)^2}\chi_0''\Big(\frac{\tau\beta\gamma x_n}{e_-(0)}\Big)\bigg), \end{aligned}$$

and because of the support of $\chi_0^{(j)}\Big(\frac{\tau\beta\gamma x_n}{f_-(0)}\Big)$, resp. $\chi_0^{(j)}\Big(\frac{\tau\beta\gamma x_n}{e_-(0)}\Big)$, $j=1,2$, for $x_n \leq 0$, we obtain

$$\int_{-\infty}^{0} |(\mathcal{M}_\tau y_-)(\xi', x_n)|^2 dx_n \leq 2c_-^2 \frac{\tau\beta\gamma a^2}{f_-(0)}\Big(4m_-^2\|\chi_0'\|_{L^\infty}^2 + \|\chi_0''\|_{L^\infty}^2\Big(\frac{\tau\beta\gamma}{f_-(0)}\Big)^2\Big)e^{-\frac{f_-(0)^2}{2\tau\beta\gamma}}$$
$$+ 2c_-^2 \frac{\tau\beta\gamma b^2}{e_-(0)}\Big(4m_-^2\|\chi_0'\|_{L^\infty}^2 + \|\chi_0''\|_{L^\infty}^2\Big(\frac{\tau\beta\gamma}{e_-(0)}\Big)^2\Big)e^{-\frac{e_-(0)^2}{2\tau\beta\gamma}}.$$

Now we have $(\mathcal{M}_\tau u)(\xi', x_n) = \psi(\tau,\xi')(\mathcal{M}_\tau y)(\xi', x_n)$. As $|\xi'| \sim \tau$ in $\operatorname{supp}(\psi)$ we obtain

$$\|\mathcal{M}_\tau u\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})}^2 \leq C(\gamma^2+\tau^2)\gamma e^{-C'\tau/\gamma} \int_{\mathbb{R}^{n-1}} \psi(\tau,\xi')^2 d\xi'.$$

With the change of variable $\xi' = \tau\eta$ we find

$$\int_{\mathbb{R}^{n-1}} \psi(\tau,\xi')^2 d\xi' = C\tau^{n-1}, \tag{6.16}$$

which gives the first result.

On the other hand observe now that

$$\|y_+\|_{L^2(\mathbb{R}_+)}^2 = \int_0^{+\infty} Q_+(\xi', x_n)^2 \chi_0\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big)^2 dx_n$$
$$\geq \int_{0\leq \frac{\tau\beta\gamma x_n}{|f_+(0)|}\leq\frac12} e^{x_n(2f_+(0)+\tau\beta x_n)} dx_n = \frac{|f_+(0)|}{\tau\beta\gamma}\int_0^{\frac12} e^{2t\frac{|f_+(0)|}{\tau\beta\gamma}(f_+(0)+t\frac{|f_+(0)|}{2\gamma})}dt$$
$$\geq \frac{|f_+(0)|}{\tau\beta\gamma}\int_0^{\frac12} e^{-2t\frac{|f_+(0)|^2}{\tau\beta\gamma}}dt = \frac{1}{2|f_+(0)|}\Big(1-e^{-\frac{|f_+(0)|^2}{\tau\beta\gamma}}\Big).$$

We also have

$$\|y_-\|_{L^2(\mathbb{R}_-)}^2 = \int_{-\infty}^{0}\Big(aQ_-(\xi', x_n)\chi_0\Big(\frac{\tau\beta\gamma x_n}{f_-(0)}\Big) + b\tilde{Q}_-(\xi', x_n)\chi_0\Big(\frac{\tau\beta\gamma x_n}{e_-(0)}\Big)\Big)^2 dx_n$$
$$\geq \int_{-\frac12\leq\frac{\tau\beta\gamma x_n}{f_-(0)}\leq 0} e^{x_n(2f_-(0)+\tau\beta x_n)}\Big(a+be^{x_n(e_-(0)-f_-(0))}\Big)^2 dx_n,$$

and as $e_-(0)-f_-(0) = 2m_- \geq 0$, $a+b=1$ and $a\geq\frac12$, we have $a+be^{x_n(e_-(0)-f_-(0))}\geq\frac12$, and thus obtain

$$\|y_-\|_{L^2(\mathbb{R}_-)}^2 \geq \frac14 \int_{-\frac12\leq\frac{\tau\beta\gamma x_n}{f_-(0)}\leq 0} e^{x_n(2f_-(0)+\tau\beta x_n)}dx_n \geq \frac{1}{8f_-(0)}\Big(1-e^{-\frac{|f_-(0)|^2}{\tau\beta\gamma}}\Big),$$

arguing as above. As a result, using (6.16), we have

$$\|u\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})}^2 \geq C\tau^{n-2}\big(1-e^{-C'\tau/\gamma}\big).$$

∎

6.2.4. *Proof of Lemma 5.3.* We start with the second result. We set

$$z_+ = \big(1 - \chi_0(|\tau^{\frac{1}{2}} x'|)\big)\check{u}_+(x', x_n), \qquad \text{for } x_n \geq 0.$$

We shall prove that for all $N \in \mathbb{N}$ we have $\|z_+\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R}_+)} \leq C_{\gamma,N}\tau^{-N}$.

From the definition of χ_0 we find

$$\|z_+\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R}_+)} \leq \int\limits_{|\tau^{\frac{1}{2}} x'|\geq \frac{1}{2}} \int\limits_{\mathbb{R}_+} |\hat{u}_+(x', x_n)|^2 dx' dx_n.$$

Recalling the definition of u_+ and performing the change of variable $\xi' = \tau\eta$ we obtain

$$\hat{u}_+(x', x_n) = \tau^{n-1} \int\limits_{\mathbb{R}^{n-1}} e^{i\tau\phi}\tilde{\psi}(\eta)\chi_0\Big(\frac{\beta\gamma x_n}{|\tilde{f}_+(\eta)|}\Big) d\eta,$$

where the complex phase function is given by

$$\phi = -x'\cdot\eta - i x_n\big(\tilde{f}_+(\eta) + \frac{\beta x_n}{2}\big), \quad \text{with} \quad \tilde{f}_+(\eta) = \alpha_+ - m_+(\eta),$$

and

$$\tilde{\psi}(\eta) = \chi_1\Big(\frac{1}{(1+|\eta|^2)^{\frac{1}{2}}} - \tau_0\Big)\chi_1\Big(\Big|\frac{\eta}{(1+|\eta|^2)^{\frac{1}{2}}} - \xi_0'\Big|\Big).$$

Here τ is chosen sufficiently large so that m_+ is homogeneous. Observe that $\tilde{\psi}$ has a compact support independent of τ and that $\tilde{f}_+(\eta) + \frac{\beta x_n}{2} \leq -C < 0$ in the support of the integrand.

We place ourselves in the neighborhood of a point x' such that $|\tau^{\frac{1}{2}} x'| \geq \frac{1}{2}$. Up to a permutation of the variables we may assume that $|\tau^{\frac{1}{2}} x_1| \geq C$. We then introduce the following differential operator

$$L = \tau^{-1}\frac{\partial_{\eta_1}}{-ix_1 - x_n\partial_{\eta_1} m_+(\eta)},$$

that satisfies $Le^{i\tau\phi} = e^{i\tau\phi}$. We thus have

$$\hat{u}_+(x', x_n) = \tau^{n-1} \int\limits_{\mathbb{R}^{n-1}} e^{i\tau\phi}(L^t)^N\Big(\tilde{\psi}(\eta)\chi_0\Big(\frac{\beta\gamma x_n}{|\tilde{f}_+(\eta)|}\Big)\Big) d\eta,$$

and we find

$$|\hat{u}_+(x', x_n)| \leq C_N \frac{\tau^{n-1}\gamma^N}{|\tau x_1|^N} e^{-C\tau x_n}.$$

More generally for $|\tau^{\frac{1}{2}} x'| \geq \frac{1}{2}$ we have

$$|\hat{u}_+(x', x_n)| \leq C_N \frac{\tau^{n-1}\gamma^N}{|\tau x'|^N} e^{-C\tau x_n}.$$

Then we obtain

$$\int_{|\tau^{\frac{1}{2}}x'|\geq\frac{1}{2}}\int_{\mathbb{R}_+}|\hat{u}_+(x',x_n)|^2dx'dx_n$$
$$\leq C_N^2\gamma^{2N}\tau^{2n-2}\Big(\int_{|\tau^{\frac{1}{2}}x'|\geq\frac{1}{2}}\frac{1}{|\tau x'|^{2N}}dx'\Big)\Big(\int_{\mathbb{R}_+}e^{-2C\tau x_n}dx_n\Big)$$
$$\leq C'_N\gamma^{2N}\tau^{\frac{3}{2}n-N-\frac{5}{2}}\int_{|x'|\geq\frac{1}{2}}\frac{1}{|x'|^{2N}}dx'.$$

Similarly, setting $z_-=\big(1-\chi_0(|\tau^{\frac{1}{2}}x'|)\big)\check{\hat{u}}_-(x',x_n)$ for $x_n\leq 0$ we obtain $\|z_-\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R}_-}\leq C_{\gamma,N}\tau^{-N}$. The second result thus follows from Lemma 5.2.

For the first result we write

$$\mathcal{L}_\tau v_\pm=(2\pi)^{-(n-1)}\chi_0(|\tau^{\frac{1}{2}}x'|)\mathcal{L}_\tau\check{\hat{u}}_\pm+(2\pi)^{-(n-1)}\big[\mathcal{L}_\tau,\chi_0(|\tau^{\frac{1}{2}}x'|)\big]\check{\hat{u}}_\pm$$

The first term is estimated using Lemma 5.2 as

$$(2\pi)^{-\frac{(n-1)}{2}}\|\mathcal{L}_\tau\check{\hat{u}}_\pm\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R}_\pm)}=\|\mathcal{M}_\tau u_\pm\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R}_\pm)}.$$

Observing that $\mathcal{L}_\tau$ is a *differential* operator the commutator is thus a first-order differential operator in x' with support in a region $|\tau^{\frac{1}{2}}x'|\geq C$, because of the behavior of χ_1 near 0. The coefficients of this operator depend on τ polynomially. The zero-order terms can be estimated as we did for z_+ above with an additional $\tau^{\frac{3}{2}}$ factor.

For the first-order term observe that we have

$$\partial_{x'_j}\check{\hat{u}}_+(x',\tau)=\tau^n\int_{\mathbb{R}^{n-1}}\eta_j e^{i\tau\left(x'\cdot\eta-ix_n(\tilde{f}_+(\eta)+\frac{\beta x_n}{2})\right)}\tilde{\psi}(\eta)\chi_0\Big(\frac{\beta\gamma x_n}{|\tilde{f}_+(\eta)|}\Big)d\eta.$$

We thus obtain similar estimates as above with an additional $\tau^{\frac{3}{2}}$ factor. This concludes the proof. ∎

References

[1] S. Alinhac, *Non-unicité pour des opérateurs différentiels à caractéristiques complexes simples*, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 3, 385–393.

[2] S. Alinhac and P. Gérard, *Pseudo-differential operators and the Nash-Moser theorem*, Graduate Studies in Mathematics, vol. 82, American Mathematical Society, Providence, RI, 2007, Translated from the 1991 French original by Stephen S. Wilson.

[3] H. Bahouri, *Dépendance non linéaire des données de Cauchy pour les solutions des équations aux dérivées partielles*, J. Math. Pures Appl. (9) **66** (1987), no. 2, 127–138.

[4] V. Barbu, *Exact controllability of the superlinear heat equation*, Appl. Math. Optim. **42** (2000), 73–89.

[5] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim. **30** (1992), no. 5, 1024–1065.

[6] A. Benabdallah, Y. Dermenjian, and J. Le Rousseau, *Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient and applications to controllability and an inverse problem*, J. Math. Anal. Appl. **336** (2007), 865–887.

[7] ———, *Carleman estimates for stratified media*, J. Funct. Anal. **260** (2011), 3645–3677.

[8] A. L. Bukhgeim and M. V. Klibanov, *Global uniqueness of class of multidimensional inverse problems*, Soviet Math. Dokl. **24** (1981), 244–247.

[9] P. Buonocore and P. Manselli, *Nonunique continuation for plane uniformly elliptic equations in Sobolev spaces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** (2000), no. 4, 731–754.

[10] N. Burq, *Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel*, Acta Math. **180** (1998), no. 1, 1–29.

[11] A.-P. Calderón, *Uniqueness in the Cauchy problem for partial differential equations.*, Amer. J. Math. **80** (1958), 16–36.

[12] T. Carleman, *Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes*, Ark. Mat., Astr. Fys. **26** (1939), no. 17, 9.

[13] A. Doubova, A. Osses, and J.-P. Puel, *Exact controllability to trajectories for semilinear heat equations with discontinuous diffusion coefficients*, ESAIM Control Optim. Calc. Var. **8** (2002), 621–661, A tribute to J. L. Lions.

[14] E. Fernández-Cara and E. Zuazua, *Null and approximate controllability for weakly blowing up semilinear heat equations*, Ann. Inst. H. Poincaré, Analyse non lin. **17** (2000), 583–616.

[15] N. Filonov, *Second-order elliptic equation of divergence form having a compactly supported solution*, J. Math. Sci. (New York) **106** (2001), no. 3, 3078–3086, Function theory and phase transitions.

[16] A. Fursikov and O. Yu. Imanuvilov, *Controllability of evolution equations*, vol. 34, Seoul National University, Korea, 1996, Lecture notes.

[17] L. Hörmander, *On the uniqueness of the Cauchy problem*, Math. Scand. **6** (1958), 213–225.

[18] ———, *Linear partial differential operators*, Die Grundlehren der mathematischen Wissenschaften, Bd. 116, Academic Press Inc., Publishers, New York, 1963.

[19] ———, *The analysis of linear partial differential operators. III*, Grundlehren der Mathematischen Wissenschaften, vol. 274, Springer-Verlag, Berlin, 1985, Pseudodifferential operators.

[20] ———, *The analysis of linear partial differential operators. IV*, Grundlehren der Mathematischen Wissenschaften, vol. 275, Springer-Verlag, Berlin, 1994, Fourier integral operators.

[21] O Yu. Imanuvilov, V. Isakov, and M. Yamamoto, *An inverse problem for the dynamical Lamé system with two sets of boundary data*, Comm. Pure Appl. Math. **56** (2003), 1366–1382.

[22] O. Yu. Imanuvilov and J.-P. Puel, *Global carleman estimates for weak solutions of elliptic nonhomogeneous Dirichlet problems*, Int. Math. Res. Not. **16** (2003), 883–913.

[23] V. Isakov, *Inverse problems for partial differential equations*, Springer Verlag, Berlin, 1998.

[24] D. Jerison and C. E. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Ann. of Math. (2) **121** (1985), no. 3, 463–494, With an appendix by E. M. Stein.

[25] D. Jerison and G. Lebeau, *Harmonic analysis and partial differential equations (Chicago, IL, 1996)*, Chicago Lectures in Mathematics, ch. Nodal sets of sums of eigenfunctions, pp. 223–239, The University of Chicago Press, Chicago, 1999.

[26] F. John, *Continuous dependence on data for solutions of partial differential equations with a presribed bound*, Comm. Pure Appl. Math. **13** (1960), 551–585.

[27] C. E. Kenig, J. Sjöstrand, and G. Uhlmann, *The Calderón problem with partial data*, Ann. of Math. **165** (2007), 567–591.

[28] H. Koch and D. Tataru, *Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients*, Comm. Pure Appl. Math. **54** (2001), no. 3, 339–360.

[29] ———, *Sharp counterexamples in unique continuation for second order elliptic equations*, J. Reine Angew. Math. **542** (2002), 133–146.

[30] J. Le Rousseau, *Carleman estimates and controllability results for the one-dimensional heat equation with BV coefficients*, J. Differential Equations **233** (2007), 417–447.

[31] J. Le Rousseau and G. Lebeau, *On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations*, ESAIM Control Optim. Calc. Var. **18** (2012), 712–747.

[32] J. Le Rousseau and L. Robbiano, *Carleman estimate for elliptic operators with coefficents with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations*, Arch. Rational Mech. Anal. **195** (2010), 953–990.

[33] ———, *Local and global Carleman estimates for parabolic operators with coefficients with jumps at interfaces*, Inventiones Math. **183** (2011), 245–336.

[34] G. Lebeau, *Équation des ondes amorties*, Algebraic and geometric methods in mathematical physics (Kaciveli, 1993), Math. Phys. Stud., vol. 19, Kluwer Acad. Publ., Dordrecht, 1996, pp. 73–109.

[35] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur*, Comm. Partial Differential Equations **20** (1995), 335–356.

[36] ———, *Stabilisation de l'équation des ondes par le bord*, Duke Math. J. **86** (1997), 465–491.

[37] G. Lebeau and E. Zuazua, *Null-controllability of a system of linear thermoelasticity*, Arch. Rational Mech. Anal. **141** (1998), 297–329.

[38] N. Lerner, *Metrics on the phase space and non-selfadjoint pseudo-differential operators*, Pseudo-Differential Operators, Vol. 3, Birkhäuser, Basel, 2010.

[39] N. Mandache, *On a counterexample concerning unique continuation for elliptic equations in divergence form*, Math. Phys. Anal. Geom. **1** (1998), no. 3, 273–292.

[40] K. Miller, *Stabilized quasi-reversibility and other nearly-best-possible methods for non-well-posed problems*, Symposium on Non-Well-Posed Problems and Logarithmic Convexity (Heriot-Watt Univ., Edinburgh, 1972), Springer, Berlin, 1973, pp. 161–176. Lecture Notes in Math., Vol. 316.

[41] ———, *Nonunique continuation for uniformly parabolic and elliptic equations in self-adjoint divergence form with Hölder continuous coefficients*, Arch. Rational Mech. Anal. **54** (1974), 105–117.

[42] A. Pliś, *On non-uniqueness in Cauchy problem for an elliptic second order differential equation*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 95–100.

[43] J. Rauch and M. Taylor, *Exponential decay of solutions to hyperbolic equations in bounded domains*, Indiana Univ. Math. J. **24** (1974), 79–86.

[44] F. Schulz, *On the unique continuation property of elliptic divergence form equations in the plane*, Math. Z. **228** (1998), no. 2, 201–206.

[45] C. D. Sogge, *Oscillatory integrals and unique continuation for second order elliptic differential equations*, J. Amer. Math. Soc. **2** (1989), no. 3, 491–515.

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