# Scaling limit of the path leading to the leftmost particle in a branching random walk

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***Summary.*** We consider a discrete-time branching random walk defined on the real line, which is assumed to be supercritical and in the boundary case. It is known that its leftmost position of the $n$-th generation behaves asymptotically like $\frac{3}{2}\ln n$, provided the non-extinction of the system. The main goal of this paper, is to prove that the path from the root to the leftmost particle, after a suitable normalizatoin, converges weakly to a Brownian excursion in $D([0,1],\mathbb{R})$.

***Keywords.*** Branching random walk; spinal decomposition.

# 1 Introduction

We consider a branching random walk, which is constructed according to a point process $\mathcal{L}$ on the line. Precisely speaking, the system is started with one initial particle at the origin. This particle is called the root, denoted by $\varnothing$. At time 1, the root dies and gives birth to some new particles, which form the first generation. Their positions constitute a point process distributed as $\mathcal{L}$. At time 2, each of these particles dies and gives birth to new particles whose positions – relative to that of their parent – constitute a new independent copy of $\mathcal{L}$. The system grows according to the same mechanism.

We denote by $\mathbb{T}$ the genealogical tree of the system, which is clearly a Galton-Watson tree rooted at $\varnothing$. If a vertex $u \in \mathbb{T}$ is in the $n$-th generation, we write $|u| = n$ and denote its position by $V(u)$. Then $\{V(u),\ |u| = 1\}$ follows the same law as $\mathcal{L}$. The family of positions $(V(u);\ u \in \mathbb{T})$ is viewed as our branching random walk.

Throughout the paper, the branching random walk is assumed to be in the boundary case (Biggins and Kyprianou [5]):

$$\mathbf{E}\Big[\sum_{|u|=1} 1\Big] > 1, \qquad \mathbf{E}\Big[\sum_{|x|=1} \mathrm{e}^{-V(x)}\Big] = 1, \qquad \mathbf{E}\Big[\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}\Big] = 0. \tag{1.1}$$

For any $y \in \mathbb{R}$, let $y_+ := \max\{y, 0\}$ and $\log_+ y := \log(\max\{y, 1\})$. We also assume the following integrability conditions:

$$\mathbf{E}\Big[\sum_{|u|=1} V(u)^2\mathrm{e}^{-V(u)}\Big] < \infty, \tag{1.2}$$

$$\mathbf{E}[X(\log_+ X)^2] < \infty, \qquad \mathbf{E}[\widetilde{X}\log_+ \widetilde{X}] < \infty, \tag{1.3}$$

where

$$X := \sum_{|u|=1} \mathrm{e}^{-V(u)}, \qquad \widetilde{X} := \sum_{|u|=1} V(u)_+\mathrm{e}^{-V(u)}.$$

We define $I_n$ to be the leftmost position in the $n$-th generation, i.e.

$$I_n := \inf\{V(u),\ |u| = n\}, \tag{1.4}$$

with $\inf\emptyset := \infty$. If $I_n < \infty$, we choose a vertex uniformly in the set $\{u : |u| = n,\ V(u) = I_n\}$ of leftmost particles at time $n$ and denote it by $m^{(n)}$. We let $[\![\varnothing, m^{(n)}]\!] = \{\varnothing =: m_0^{(n)}, m_1^{(n)}, \ldots, m_n^{(n)} := m^{(n)}\}$ be the shortest path in $\mathbb{T}$ relating the root $\varnothing$ to $m^{(n)}$, and introduce the path from the root to $m^{(n)}$ as follows

$$(I_n(k);\ 0 \le k \le n) := (V(m_k^{(n)});\ 0 \le k \le n).$$

In particular, $I_n(0) = 0$ and $I_n(n) = I_n$. Let $\sigma$ be the positive real number such that $\sigma^2 = \mathbf{E}\Big[\sum_{|u|=1} V(u)^2\mathrm{e}^{-V(u)}\Big]$. Our main result is as follows.

**Theorem 1.1** *The rescaled path* $(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}};\ 0 \le s \le 1)$ *converges in law in* $D([0,1],\mathbb{R})$*, to a normalized Brownian excursion* $(e_s;\ 0 \le s \le 1)$.

**Remark 1.2** *It has been proved in [1], [11] and [2] that* $I_n$ *is around* $\frac{3}{2}\ln n$. *In [3], the authors proved that, for the model of branching Brownian motion, the time reversed path followed by the leftmost particle converges in law to a certain stochastic process.*

Let us say a few words about the proof of Theorem 1.1. We first consider the path leading to $m^{(n)}$, by conditioning that its ending point $I_n$ is located atypically below $\frac{3}{2}\ln n - z$

with large $z$. Then we apply the well-known spinal decomposition to show that this path, conditioned to $\{I_n \leq \frac{3}{2}\ln n - z\}$, behaves like a simple random walk staying positive but tied down at the end. Such a random walk, being rescaled, converges in law to the Brownian excursion (see [9]). We then prove our main result by removing the condition of $I_n$. The main strategy is borrowed from [2], but with appropriate refinements.

The rest of the paper is organized as follows. In Section 2, we recall the spinal decomposition by a change of measures, which implies the useful many-to-one lemma. We prove a conditioned version of Theorem 1.1 in Section 3. In Section 4, we remove the conditioning and prove the theorem.

Throughout the paper, we use $a_n \sim b_n$ $(n \to \infty)$ to denote $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$; and let $(c_i)_{i\geq 0}$ denote finite and positive constants. We write $\mathbf{E}[f;\, A]$ for $\mathbf{E}[f\mathbf{1}_A]$. Moreover, $\sum_{\varnothing} := 0$ and $\prod_{\varnothing} := 1$.

# 2 Lyons' change of measures and spinal decomposition

For any $a \in \mathbb{R}$, let $\mathbf{P}_a$ be the probability measure such that $\mathbf{P}_a((V(u),\ u \in \mathbb{T}) \in \cdot) = \mathbf{P}((V(u)+a,\ u \in \mathbb{T}) \in \cdot)$. The corresponding expectation is denoted by $\mathbf{E}_a$. Let $(\mathcal{F}_n,\ n \geq 0)$ be the natural filtration generated by the branching random walk and let $\mathcal{F}_\infty := \vee_{n\geq 0}\mathcal{F}_n$. We introduce the following random variables:

$$W_n := \sum_{|u|=n} e^{-V(u)}, \qquad n \geq 0. \tag{2.1}$$

It follows immediately from (1.1) that $(W_n,\ n \geq 0)$ is a non-negative martingale with respect to $(\mathcal{F}_n)$. It is usually referred as the additive martingale. We define a probability measure $\mathbf{Q}_a$ on $\mathcal{F}_\infty$ such that for any $n \geq 0$,

$$\left.\frac{d\mathbf{Q}_a}{d\mathbf{P}_a}\right|_{\mathcal{F}_n} := e^a W_n. \tag{2.2}$$

For convenience, we write $\mathbf{Q}$ for $\mathbf{Q}_0$.

Let us give the description of the branching random walk under $\mathbf{Q}_a$ in an intuitive way, which is known as the spinal decomposition. We introduce another point process $\widehat{\mathcal{L}}$ with Radon-Nykodin derivative $\sum_{x\in\mathcal{L}} e^{-x}$ with respect to the law of $\mathcal{L}$. Under $\mathbf{Q}_a$, the branching random walk evolves as follows. Initially, there is one particle $w_0$ located at $V(w_0) = a$. At each step $n$, particles at generation $n$ die and give birth to new particles independently according to the law of $\mathcal{L}$, except for the particle $w_n$ which generates its children according to the law of $\widehat{\mathcal{L}}$. The particle $w_{n+1}$ is chosen proportionally to $e^{-V(u)}$ among the children $u$ of $w_n$. We still call $\mathbb{T}$ the genealogical tree of the process, so that $(w_n)_{n\geq 0}$ is a ray in $\mathbb{T}$, which is called the spine. This change of probabilities was presented in various forms; see, for example [15], [11] and [8].

It is convenient to use the following notation. For any $u \in \mathbb{T} \setminus \{\varnothing\}$, let $\overleftarrow{u}$ be the parent of $u$, and

$$\Delta V(u) := V(u) - V(\overleftarrow{u}).$$

Let $\Omega(u)$ be the set of brothers of $u$, i.e. $\Omega(u) := \{v \in \mathbb{T} : \overleftarrow{v} = \overleftarrow{u},\ v \neq u\}$. Let $\delta$ denote the Dirac measure. Then under $\mathbf{Q}_a$, $\sum_{|u|=1} \delta_{\Delta V(u)}$ follows the law of $\widehat{\mathcal{L}}$. Further, We recall the following proposition, from [11] and [15].

**Proposition 2.1** *(1) For any $|u| = n$, we have*

$$\mathbf{Q}_a[w_n = u | \mathcal{F}_n] = \frac{e^{-V(u)}}{W_n}. \tag{2.3}$$

*(2) Under $\mathbf{Q}_a$, the random variables $\left(\sum_{v\in\Omega(w_n)} \delta_{\Delta V(v)},\ \Delta V(w_n)\right)$, $n \geq 1$ are i.i.d..*

As a consequence of this proposition, we get the many-to-one lemma as follows:

**Lemma 2.2** *There exists a centered random walk $(S_n;\ n \geq 0)$ with $\mathbf{P}_a(S_0 = a) = 1$ such that for any $n \geq 1$ and any measurable function $g : \mathbb{R}^n \to [0, \infty)$, we have*

$$\mathbf{E}_a\Big[\sum_{|u|=n} g(V(u_1), \ldots, V(u_n))\Big] = \mathbf{E}_a[e^{S_n - a} g(S_1, \ldots, S_n)], \tag{2.4}$$

*where we denote by $[\![\varnothing,\, u]\!] = \{\varnothing =: u_0,\, u_1 \ldots, u_{|u|} := u\}$ the ancestral line of $u$ in $\mathbb{T}$.*

Note that by (1.3), $S_1$ has the finite variance $\sigma^2 = \mathbf{E}[S_1^2] = \mathbf{E}[\sum_{|u|=1} V(u)^2 e^{-V(u)}]$.

## 2.1 Convergence in law for the one-dimensional random walk

Let us introduce some results about the centered random walk $(S_n)$ with finite variance, which will be used later. For any $0 \leq m \leq n$, we define $\underline{S}_{[m,n]} := \min_{m\leq j\leq n} S_j$, and $\underline{S}_n = \underline{S}_{[0,n]}$. We denote by $R(x)$ the renewal function of $(S_n)$, which is defined as follows:

$$R(x) = \mathbf{1}_{\{x=0\}} + \mathbf{1}_{\{x>0\}} \sum_{k\geq 0} \mathbf{P}(-x \leq S_k < \underline{S}_{n-1}). \tag{2.5}$$

For the random walk $(-S_n)$, we define $\underline{S}^-_{[m,n]}$, $\underline{S}^-_n$ and $R_-(x)$ similarly. It is known (see [10] p. 360) that there exists $c_0 > 0$ such that

$$\lim_{x\to\infty} \frac{R(x)}{x} = c_0. \tag{2.6}$$

Moreover, it is shown in [13] that there exist $C_+,\ C_- > 0$ such that for any $a \geq 0$,

$$\mathbf{P}_a\Big(\underline{S}_n \geq 0\Big) \sim \frac{C_+}{\sqrt{n}} R(a); \tag{2.7}$$

$$\mathbf{P}_a\Big(\underline{S}^-_n \geq 0\Big) \sim \frac{C_-}{\sqrt{n}} R_-(a). \tag{2.8}$$

We also state the following inequalities (see Lemmas 2.2 and 2.4 in [4], respectively).

**Fact 2.3** **(i)** *There exists a constant $c_1 > 0$ such that for any $b \geq a \geq 0$, $x \geq 0$ and $n \geq 1$,*

$$\mathbf{P}\big(\underline{S}_n \geq -x;\ S_n \in [a-x, b-x]\big) \leq c_1(1+x)(1+b-a)(1+b)n^{-3/2}. \tag{2.9}$$

**(ii)** *Let $0 < \lambda < 1$. There exists a constant $c_2 > 0$ such that for any $b \geq a \geq 0$, $x,\ y \geq 0$ and $n \geq 1$,*

$$\mathbf{P}_x(S_n \in [y+a, y+b], \underline{S}_n \geq 0, \underline{S}_{[\lambda n, n]} \geq y) \leq c_2(1+x)(1+b-a)(1+b)n^{-3/2}. \tag{2.10}$$

Before we give the next lemma, we recall the definition of lattice distribution (see [10], p. 138). The distribution of a random variable $X_1$ is lattice, if it is concentrated on a set of points $\alpha + \beta\mathbb{Z}$, with $\alpha$ arbitrary. The largest $\beta$ satisfying this property is called the span of $X_1$. Otherwise, the distribution of $X_1$ is called non-lattice.

**Lemma 2.4** *Let $(r_n)_{n\geq 0}$ be a sequence of real numbers such that $\lim_{n\to\infty} \frac{r_n}{\sqrt{n}} = 0$. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a Riemann integrable function. We suppose that there exists a non-increasing function $\overline{f} : \mathbb{R}_+ \to \mathbb{R}$ such that $|f(x)| \leq \overline{f}(x)$ for any $x \geq 0$ and $\int_{x\geq 0} x\overline{f}(x)dx < \infty$. For $0 < \Delta < 1$, let $F : D([0, \Delta],\ \mathbb{R}) \to [0, 1]$ be continuous. Let $a \geq 0$.*

**(I) Non-lattice case.** *If the distribution of $(S_1 - S_0)$ is non-lattice, then there exists a constant $C_1 > 0$ such that*

$$\lim_{n\to\infty} n^{3/2}\mathbf{E}\Big[F\Big(\frac{S_{\lfloor sn\rfloor}}{\sigma\sqrt{n}}; 0 \leq s \leq \Delta\Big) f(S_n - y);\ \underline{S}_n \geq -a,\ \underline{S}_{[\Delta n, n]} \geq y\Big] \\ = C_1 R(a) \int_{x\geq 0} f(x) R_-(x) dx \mathbf{E}[F(e_s; 0 \leq s \leq \Delta)], \tag{2.11}$$

*uniformly in $y \in [0, r_n]$.*

**(II) Lattice case.** *If the distribution of $(S_1 - S_0)$ is supported in $(\alpha + \beta\mathbb{Z})$ with span $\beta$, then for any $d \in \mathbb{R}$,*

$$
\begin{aligned}
(2.12) \quad \lim_{n\to\infty} n^{3/2}\mathbf{E}\Big[F\Big(\frac{S_{\lfloor sn\rfloor}}{\sigma\sqrt{n}}; 0 \le s \le \Delta\Big) f(S_n - y + d);\ \underline{S}_n \ge -a,\ \underline{S}_{[\Delta n,n]} \ge y - d\Big] \\
= C_1 R(a)\beta \sum_{j\ge\lceil -\frac{d}{\beta}\rceil} f(\beta j + d) R_-(\beta j + d)\mathbf{E}[F(e_s; 0 \le s \le \Delta)].
\end{aligned}
$$

*uniformly in $y \in [0, r_n] \cap \{\alpha n + \beta\mathbb{Z}\}$.*

*Proof of Lemma 2.4.* The lemma is a refinement of Lemma 2.3 in [2], which proved the convergence in the non-lattice case when $a = 0$ and $F \equiv 1$. We consider the non-lattice case first. We denote the expectation on the left-hand side of (2.11) by $\chi(F, f)$. Observe that for any $K \in \mathbb{N}_+$,

$$\chi(F, f) = \chi\big(F, f(x)1_{(0\le x\le K)}\big) + \chi\big(F, f(x)1_{(x>K)}\big).$$

Since $0 \le F \le 1$, we have $\chi\big(F, f(x)1_{(x>K)}\big) \le \chi\big(1, f(x)1_{(x>K)}\big)$, which is bounded by

$$\sum_{j\ge K} \mathbf{E}_a\Big[f(S_n - y - a);\ \underline{S}_n \ge 0,\ \underline{S}_{[\Delta n,n]} \ge y + a,\ S_n \in [y + a + j, y + a + j + 1]\Big].$$

Recall that $|f(x)| \le \overline{f}(x)$ with $\overline{f}$ non-increasing. We get that

$$\chi\big(1, f(x)1_{(x>K)}\big) \le \sum_{j\ge K} \overline{f}(j)\mathbf{P}_a\Big[\underline{S}_n \ge 0,\ \underline{S}_{[\Delta n,n]} \ge y + a,\ S_n \in [y + a + j, y + a + j + 1]\Big].$$

It then follows from (2.10) that

$$\chi\big(1, f(x)1_{(x>K)}\big) \le 2c_2(1 + a)\Big(\sum_{j\ge K} \overline{f}(j)(2 + j)\Big)n^{-3/2}. \tag{2.13}$$

Since $\int_0^\infty x\overline{f}(x)dx < \infty$, the sum $\sum_{j\ge K} \overline{f}(j)(2 + j)$ decreases to zero as $K \uparrow \infty$. We thus only need to estimate $\chi\big(F, f(x)1_{(0\le x\le K)}\big)$. Note that $f$ is Riemann integrable. It suffices to consider $\chi\Big(F, 1_{(0\le x\le K)}\Big)$ with $K$ a positive constant.

Applying the Markov property at time $\lfloor \Delta n\rfloor$ shows that

$$
\begin{aligned}
\chi\Big(F, 1_{(0\le x\le K)}\Big) &= \mathbf{E}_a\Big[F\Big(\frac{S_{\lfloor sn\rfloor} - a}{\sigma\sqrt{n}}; 0 \le s \le \Delta\Big);\ S_n \le y + a + K, \underline{S}_n \ge 0, \underline{S}_{[\Delta n,n]} \ge y + a\Big] \\
(2.14) \quad &= \mathbf{E}_a\Big[F\Big(\frac{S_{\lfloor sn\rfloor} - a}{\sigma\sqrt{n}}; 0 \le s \le \Delta\Big)\Psi_K(S_{\lfloor \Delta n\rfloor});\ \underline{S}_{\lfloor \Delta n\rfloor} \ge 0\Big],
\end{aligned}
$$

where $\Psi_K(x) := \mathbf{P}_x\Big[S_{n-\lfloor \Delta n\rfloor} \le y+a+K,\ \underline{S}_{n-\lfloor \Delta n\rfloor} \ge y+a\Big]$. By reversing time, we obtain that $\Psi_K(x) = \mathbf{P}\Big[\underline{S}_m^- \ge (-S_m) + (y+a-x) \ge -K\Big]$ with $m := n - \lfloor \Delta n\rfloor$.

We define $\tau_n$ as the first time when the random walk $(-S)$ hits the minimal level during $[0,n]$, namely, $\tau_n := \inf\{k \in [0,n] : -S_k = \underline{S}_n^-\}$. Define also $\varkappa(z,\zeta;n) := \mathbf{P}(-S_n \in [z, z+\zeta],\ \underline{S}_n^- \ge 0)$ for any $z,\ \zeta \ge 0$. Then,

$$\begin{aligned}\Psi_K(x) &= \sum_{k=0}^{m} \mathbf{P}\Big[\tau_m = k;\ \underline{S}_m^- \ge (-S_m) + (y+a-x) \ge -K\Big]\\ &= \sum_{k=0}^{m} \mathbf{P}\Big[-S_k = \underline{S}_k^- \ge -K;\ \varkappa(x-y-a, \underline{S}_k^- + K; m-k)\Big],\end{aligned} \tag{2.15}$$

where the last equality follows from the Markov property.

Let $\psi(x) := xe^{-x^2/2}\mathbf{1}_{(x\ge 0)}$. Combining Theorem 1 of [6] with (2.7) yields that

$$\varkappa(z,\zeta;n) = \mathbf{P}_0\Big[-S_n \in [z,z+\zeta];\ \underline{S}_n \ge 0\Big] = \frac{C_-\zeta}{\sigma n}\psi\Big(\frac{z}{\sigma\sqrt{n}}\Big) + o(n^{-1}), \tag{2.16}$$

uniformly in $z \in \mathbb{R}_+$ and $\zeta$ in compact sets of $\mathbb{R}_+$. Note that $\psi$ is bounded on $\mathbb{R}_+$. Therefore, there exists a constant $c_3 > 0$ such that for any $\zeta \in [0,K]$, $z \ge 0$ and $n \ge 0$,

$$\varkappa(z,\zeta;n) \le c_3\frac{(1+K)}{n+1}. \tag{2.17}$$

Let $k_n := \lfloor \sqrt{n}\rfloor$. We divide the sum on the right-hand side of (2.15) into two parts:

$$\Psi_K(x) = \sum_{k=0}^{k_n} + \sum_{k=k_n+1}^{m} \mathbf{P}\big[-S_k = \underline{S}_k^- \ge -K;\ \varkappa(x-y-a, \underline{S}_k^- + K; m-k)\big]. \tag{2.18}$$

By (2.16), under the assumption that $y = o(\sqrt{n})$, the first part becomes that

$$\begin{aligned}&\frac{C_-}{\sigma m}\psi\Big(\frac{x-a}{\sigma\sqrt{m}}\Big)\sum_{k=0}^{k_n}\mathbf{E}\big[\underline{S}_k^- + K; -S_k = \underline{S}_k^- \ge -K\big] + o(n^{-1})\sum_{k=0}^{k_n}\mathbf{P}\big[-S_k = \underline{S}_k^- \ge -K\big]\\ =\ &\frac{C_-}{\sigma m}\psi\Big(\frac{x-a}{\sigma\sqrt{m}}\Big)\int_0^K R_-(u)du + o(n^{-1}),\end{aligned} \tag{2.19}$$

where the last equation comes from the fact that $\sum_{k\ge 0}\mathbf{E}\big[\underline{S}_k^- + K; -S_k = \underline{S}_k^- \ge -K\big] = \int_0^K R_-(u)du$. On the other hand, using (2.17) for $\varkappa(x-y-a, \underline{S}_k^- + K; m-k)$ and then applying (i) of Fact 2.3 imply that for $n$ large enough, the second part of (2.18) is bounded by

$$\sum_{k=k_n+1}^{m} c_3 \frac{1+K}{m+1-k}\mathbf{P}\big(\underline{S}_k^- \geq -K,\ -S_k \in [-K,0]\big)$$

(2.20)

$$\leq c_4 \sum_{k=k_n+1}^{m} \frac{(1+K)^3}{(m+1-k)k^{3/2}} = o(n^{-1}).$$

By (2.19) and (2.20), we obtain that as $n$ goes to infinity,

$$\Psi_K(x) = o(n^{-1}) + \frac{C_-}{\sigma(n-\lfloor\Delta n\rfloor)}\psi\Big(\frac{x-a}{\sigma\sqrt{n-\lfloor\Delta n\rfloor}}\Big)\int_0^K R_-(u)du, \tag{2.21}$$

uniformly in $x \geq 0$ and $y \in [0, r_n]$. Plugging it into (2.14) and then combining with (2.7) yield that

$$\begin{aligned}\chi(F, 1_{(0\leq x\leq K)}) &= o(n^{-3/2}) + \frac{C_-}{\sigma(1-\Delta)n}\int_0^K R_-(u)du \\ &\times \frac{C_+R(a)}{\sqrt{\Delta n}}\mathbf{E}_a\Big[F\Big(\frac{S_{\lfloor sn\rfloor}-a}{\sigma\sqrt{n}}; 0\leq s\leq\Delta\Big)\psi\Big(\frac{S_{\Delta n}-a}{\sigma\sqrt{(1-\Delta)n}}\Big)\Big|\underline{S}_{\Delta n}\geq 0\Big].\end{aligned}$$

Theorem 1.1 of [7] says that under the conditioned probability $\mathbf{P}_a\Big(\cdot\Big|\underline{S}_{\Delta n}\geq 0\Big)$, $(\frac{S_{\lfloor r\Delta n\rfloor}}{\sigma\sqrt{\Delta n}}; 0\leq r\leq 1)$ converges in law to a Brownian meander, denoted by $(\mathcal{M}_r; 0\leq r\leq 1)$. Therefore,

$$\chi(F, 1_{(0\leq x\leq K)}) \sim \frac{C_-C_+R(a)}{\sigma n^{3/2}(1-\Delta)\sqrt{\Delta}}\int_0^K R_-(u)du\mathbf{E}\Big[F\Big(\sqrt{\Delta}\mathcal{M}_{s/\Delta}; 0\leq s\leq\Delta\Big)\psi\Big(\frac{\sqrt{\Delta}\mathcal{M}_1}{\sqrt{1-\Delta}}\Big)\Big].$$

It remains to check that

$$\frac{1}{(1-\Delta)\sqrt{\Delta}}\mathbf{E}\Big[F\Big(\sqrt{\Delta}\mathcal{M}_{s/\Delta}; 0\leq s\leq\Delta\Big)\psi\Big(\frac{\sqrt{\Delta}\mathcal{M}_1}{\sqrt{1-\Delta}}\Big)\Big] = \sqrt{\frac{\pi}{2}}\mathbf{E}\Big[F(e_s; 0\leq s\leq\Delta)\Big]. \tag{2.22}$$

Let $(R_s; 0\leq s\leq 1)$ be a standard three-dimensional Bessel process. Then, as is shown in [12],

$$\begin{aligned}&\frac{1}{(1-\Delta)\sqrt{\Delta}}\mathbf{E}\Big[F\Big(\sqrt{\Delta}\mathcal{M}_{s/\Delta}; 0\leq s\leq\Delta\Big)\psi\Big(\frac{\sqrt{\Delta}\mathcal{M}_1}{\sqrt{1-\Delta}}\Big)\Big] \\ =\ &\sqrt{\frac{\pi}{2}}\frac{1}{(1-\Delta)\sqrt{\Delta}}\mathbf{E}\Big[\frac{1}{R_1}F\Big(\sqrt{\Delta}R_{s/\Delta}; 0\leq s\leq\Delta\Big)\psi\Big(\frac{\sqrt{\Delta}R_1}{\sqrt{1-\Delta}}\Big)\Big], \\ =\ &\sqrt{\frac{\pi}{2}}\mathbf{E}\Big[\frac{1}{(1-\Delta)^{3/2}}e^{-\frac{R_\Delta^2}{2(1-\Delta)}}F\Big(R_s; 0\leq s\leq\Delta\Big)\Big],\end{aligned}$$

where the last equation follows from the scaling property of Bessel process. Let $(r_s; 0 \le s \le 1)$ be a standard three-dimensional Bessel bridge. Note that for any $\Delta < 1$, $(r_s; 0 \le s \le \Delta)$ is equivalent to $(R_s; 0 \le s \le \Delta)$, with density $(1-\Delta)^{-3/2} \exp(-\frac{R_\Delta^2}{2(1-\Delta)})$ (see p. 468 (3.11) of [16]). Thus,

$$\frac{1}{(1-\Delta)\sqrt{\Delta}} \mathbf{E}\Big[F\Big(\sqrt{\Delta}\mathcal{M}_{s/\Delta}; 0 \le s \le \Delta\Big)\psi\Big(\frac{\sqrt{\Delta}\mathcal{M}_1}{\sqrt{1-\Delta}}\Big)\Big] = \sqrt{\frac{\pi}{2}}\mathbf{E}\Big[F(r_s; 0 \le s \le \Delta)\Big].$$

Since a normalized Brownian excursion is exactly a standard three-dimensional Bessel bridge, this yields (2.22). Therefore, (2.11) is proved by taking $C_1 = \sqrt{\frac{\pi}{2}}\frac{C_- C_+}{\sigma}$.

The proof of the lemma in the lattice case is along the same lines, except that we use Theorem 2 (instead of Theorem 1) of [6]. □

# 3 Conditioning on the event $\{I_n \le \frac{3}{2}\ln n - z\}$

On the event $\{I_n \le \frac{3}{2}\ln n - z\}$, we analyze the sample path leading to a particle located at the leftmost position at the $n$th generation. For $z \ge 0$ and $n \ge 1$, let $a_n(z) := \frac{3}{2}\ln n - z$ if the distribution of $\mathcal{L}$ is non-lattice and let $a_n(z) := \alpha n + \beta\lfloor\frac{\frac{3}{2}\ln n - \alpha n}{\beta}\rfloor - z$ if the distribution of $\mathcal{L}$ is supported by $\alpha + \beta\mathbb{Z}$. This section is devoted to the proof of the following proposition.

**Proposition 3.1** *For any $\Delta \in (0,1]$ and any continuous functional $F : D([0,\Delta], \mathbb{R}) \to [0,1]$,*

$$\text{(3.1)} \quad \lim_{z\to\infty}\limsup_{n\to\infty}\bigg|\mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}}; 0 \le s \le \Delta\Big)\Big| I_n \le a_n(z)\Big] - \mathbf{E}\Big[F(e_s; 0 \le s \le \Delta)\Big]\bigg| = 0.$$

We begin with some preliminary results.

For any $0 < \Delta < 1$ and $L$, $K \ge 0$, we denote by $J^{\Delta}_{z,K,L}(n)$ the following collection of particles:

$$\text{(3.2)} \quad \Big\{u \in \mathbb{T} : |u| = n,\ V(u) \le a_n(z),\ \min_{0\le k\le n} V(u_k) \ge -z+K,\ \min_{\Delta n\le k\le n} V(u_k) \ge a_n(z+L)\Big\}.$$

**Lemma 3.2** *For any $\varepsilon > 0$, there exists $L_\varepsilon > 0$ such that for any $L \ge L_\varepsilon$, $n \ge 1$ and $z \ge K \ge 0$,*

$$\text{(3.3)} \qquad \mathbf{P}\Big(m^{(n)} \notin J^{\Delta}_{z,K,L}(n),\ I_n \le a_n(z)\Big) \le \Big(e^K + \varepsilon(1+z-K)\Big)e^{-z}.$$

*Proof.* It suffices to show that for any $\varepsilon \in (0,1)$, there exists $L_\varepsilon \geq 1$ such that for any $L \geq L_\varepsilon$, $n \geq 1$ and $z \geq K \geq 0$,

$$\mathbf{P}\Big(\exists |u| = n : V(u) \leq a_n(z),\ u \notin J^{\Delta}_{z,K,L}(n)\Big) \leq \Big(e^K + \varepsilon(1+z-K)\Big)e^{-z}. \tag{3.4}$$

We observe that

$$\begin{aligned} \mathbf{P}\Big(\exists |u| = n : V(u) \leq a_n(z),\ u \notin J^{\Delta}_{z,K,L}(n)\Big) &\leq \mathbf{P}\Big(\exists u \in \mathbb{T} : V(u) \leq -z+K\Big) \\ + \mathbf{P}\Big(\exists |u| = n : V(u) \leq a_n(z),\ &\min_{0\leq k\leq n} V(u_k) \geq -z+K,\ \min_{\Delta n \leq k \leq n} V(u_k) \leq a_n(z+L)\Big). \end{aligned} \tag{3.5}$$

On the one hand, by (2.4),

$$\begin{aligned} \mathbf{P}\Big(\exists u \in \mathbb{T} : V(u) \leq -z+k\Big) &\leq \sum_{n\geq 0} \mathbf{E}\Big[\sum_{|u|=n} \mathbf{1}_{\{V(u)\leq -z+K<\min_{k<n} V(u_k)\}}\Big] \\ &= \sum_{n\geq 0} \mathbf{E}[e^{S_n};\ S_n \leq -z+K < \underline{S}_{n-1}] \leq e^{-z+K}. \end{aligned} \tag{3.6}$$

On the other hand, denoting $A_n(z) := [a_n(z)-1, a_n(z)]$ for any $z \geq 0$,

$$\begin{aligned} &\mathbf{P}\Big(\exists |u| = n : V(u) \leq a_n(z),\ \min_{0\leq k\leq n} V(u_k) \geq -z+K,\ \min_{\Delta n\leq k\leq n} V(u_k) \leq a_n(z+L)\Big) \\ = \ &\mathbf{P}_{z-K}\Big(\exists |u| = n : V(u) \leq a_n(K),\ \min_{0\leq k\leq n} V(u_k) \geq 0,\ \min_{\Delta n\leq k\leq n} V(u_k) \leq a_n(K+L)\Big) \\ \leq \ &\sum_{\ell \geq L+K} \sum_{j=K}^{j=K+\ell} \mathbf{P}_{z-K}\Big(\exists |u| = n : V(u) \in A_n(j),\ \min_{0\leq k\leq n} V(u_k) \geq 0,\ \min_{\Delta n\leq k\leq n} V(u_k) \in A_n(\ell)\Big). \end{aligned}$$

According to Lemma 3.3 in [2], there exist constants $1 > c_5 > 0$ and $c_6 > 0$ such that for any $n \geq 1$, $L \geq 0$ and $x$, $z \geq 0$,

$$\begin{aligned} &\mathbf{P}_x\Big(\exists u \in \mathbb{T} : |u| = n,\ V(u) \in A_n(z),\ \min_{0\leq k\leq n} V(u_k) \geq 0,\ \min_{\Delta n\leq k\leq n} V(u_k) \in A_n(z+L)\Big) \\ \leq \ &c_6(1+x)e^{-c_5 L}e^{-x-z}. \end{aligned} \tag{3.7}$$

Hence, combining (3.6) with (3.5) yields that

$$\begin{aligned} &\mathbf{P}\Big(\exists |u| = n : V(u) \leq a_n(z),\ u \notin J^{\Delta}_{z,K,L}(n)\Big) \\ \leq \ &e^{-z+K} + \sum_{\ell \geq L} \sum_{0\leq j\leq \ell} c_6(1+z-K)e^{-c_5(\ell-j)}e^{-z-j} \\ \leq \ &\Big(e^K + c_7 \sum_{\ell\geq L} e^{-c_5 \ell}(1+z-K)\Big)e^{-z}, \end{aligned}$$

where the last inequality comes from the fact that $\sum_{j\ge 0} e^{-(1-c_5)j} < \infty$. We take $L_\varepsilon = -c_8 \ln \varepsilon$ so that $c_7 \sum_{\ell\ge L} e^{-c_5 \ell} \le \varepsilon$ for all $L \ge L_\varepsilon$. Therefore, for any $L \ge L_\varepsilon$, $n \ge 1$ and $z \ge K \ge 0$,

$$\mathbf{P}\Big(\exists |u| = n : V(u) \le a_n(z),\ u \notin J^{\Delta}_{z,K,L}(n)\Big) \le \Big(e^K + \varepsilon(1+z-K)\Big)e^{-z}, \tag{3.8}$$

which completes the proof. □

For $b \in \mathbb{Z}_+$, we define

$$\mathcal{E}_n = \mathcal{E}_n(z,b) := \{\forall k \le n-b, \min_{u \ge w_k, |u|=n} V(u) > a_n(z)\}. \tag{3.9}$$

We note that on the event $\mathcal{E}_n \cap \{I_n \le a_n(z)\}$, any particle located at the leftmost position must be separated from the spine after time $n-b$.

**Lemma 3.3** *For any $\eta > 0$ and $L > 0$, there exist $K(\eta) > 0$, $B(L,\eta) \ge 1$ and $N(\eta) \ge 1$ such that for any $b \ge B(L,\eta)$, $n \ge N(\eta)$ and $z \ge K \ge K(\eta)$,*

$$\mathbf{Q}\Big(\mathcal{E}_n^c,\ w_n \in J^{\Delta}_{z,K,L}(n)\Big) \le \eta(1+L)^2(1+z-K)n^{-3/2}. \tag{3.10}$$

We feel free to omit the proof of Lemma 3.3 since it is just a slightly stronger version of Lemma 3.8 in [2]. It follows from the same arguments.

Let us turn to the proof of Proposition 3.1. We break it up into 3 steps.

*Step (I) (The conditioned convergence of $(\frac{I_n(\lfloor sn \rfloor)}{\sigma\sqrt{n}}; 0 \le s \le \Delta)$ for $\Delta < 1$ in the non-lattice case)*

Assume that the distribution of $\mathcal{L}$ is non-lattice in this step. Recall that $a_n(z) = \frac{3}{2}\ln n - z$. The tail distribution of $I_n$ has been given in Propositions 1.3 and 4.1 of [2], recalled as follows.

**Fact 3.4 ([2])** *There exists a constant $C > 0$ such that*

$$\lim_{z\to\infty} \limsup_{n\to\infty} \Big| \frac{e^z}{z} \mathbf{P}(I_n \le a_n(z)) - C \Big| = 0. \tag{3.11}$$

*Furthermore, for any $\varepsilon > 0$, there exist $N_\varepsilon \ge 1$ and $\Lambda_\varepsilon > 0$ such that for any $n \ge N_\varepsilon$ and $\Lambda_\varepsilon \le z \le \frac{3}{2}\ln n - \Lambda_\varepsilon$,*

$$\Big| \frac{e^z}{z} \mathbf{P}(I_n \le a_n(z)) - C \Big| \le \varepsilon. \qquad \Box \tag{3.12}$$

For any continuous functional $F : D([0,\Delta],\ \mathbb{R}) \to [0,1]$, it is convenient to write that

$$\Sigma_n(F,z) := \mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn \rfloor)}{\sigma\sqrt{n}}; 0 \le s \le \Delta\Big)\mathbf{1}_{\{I_n \le a_n(z)\}}\Big]. \tag{3.13}$$

In particular, if $F \equiv 1$, $\Sigma_n(1,z) = \mathbf{P}(I_n \le a_n(z))$. Thus,

$$\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} = \mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}}; 0 \le s \le \Delta\Big)\Big| I_n \le a_n(z)\Big]. \tag{3.14}$$

Let us prove the following convergence for $0 < \Delta < 1$,

$$\lim_{z\to\infty}\limsup_{n\to\infty}\Big|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} - \mathbf{E}[F(e_s,\ 0 \le s \le \Delta)]\Big| = 0. \tag{3.15}$$

*Proof of (3.15).* For any $n \ge 1$, $L \ge 0$ and $z \ge K \ge 0$, let

$$\Pi_n(F) = \Pi_n(F,z,K,L) := \mathbf{E}\Big[F\Big(\frac{I_n(sn)}{\sigma\sqrt{n}}; 0 \le s \le \Delta\Big)\mathbf{1}_{\{m^{(n)} \in J^{\Delta}_{z,K,L}(n)\}}\Big]. \tag{3.16}$$

By Lemma 3.2, we obtain that for $L \ge L_\varepsilon$, $n \ge 1$ and $z \ge K \ge 0$,

$$\Big|\Sigma_n(F,z) - \Pi_n(F)\Big| \le \Big(e^K + \varepsilon(1+z-K)\Big)e^{-z}. \tag{3.17}$$

Note that $m^{(n)}$ is chosen uniformly among the particles located at the leftmost position. Thus,

$$\begin{aligned}\Pi_n(F) &= \mathbf{E}\Big[\sum_{|u|=n}\mathbf{1}_{(u=m^{(n)},\ u\in J^{\Delta}_{z,K,L}(n))}F\Big(\frac{V(u_{\lfloor sn\rfloor})}{\sigma\sqrt{n}}; 0 \le s \le \Delta\Big)\Big]\\ &= \mathbf{E}\Big[\frac{1}{\sum_{|u|=n}\mathbf{1}_{(V(u)=I_n)}}\sum_{|u|=n}\mathbf{1}_{(V(u)=I_n,\ u\in J^{\Delta}_{z,K,L}(n))}F\Big(\frac{V(u_{\lfloor sn\rfloor})}{\sigma\sqrt{n}}; 0 \le s \le \Delta\Big)\Big].\end{aligned}$$

Applying the change of measures given in (2.2), it follows from Proposition 2.1 that

$$\Pi_n(F) = \mathbf{E_Q}\Big[\frac{e^{V(w_n)}}{\sum_{|u|=n}\mathbf{1}_{(V(u)=I_n)}}\mathbf{1}_{(V(w_n)=I_n,\ w_n\in J^{\Delta}_{z,K,L}(n))}F\Big(\frac{V(w_{\lfloor sn\rfloor})}{\sigma\sqrt{n}}; 0 \le s \le \Delta\Big)\Big]. \tag{3.18}$$

In order to estimate $\Pi_n$, we restrict ourselves to the event $\mathcal{E}_n$. Define

$$\Lambda_n(F) := \mathbf{E_Q}\Big[\frac{e^{V(w_n)}}{\sum_{|u|=n}\mathbf{1}_{(V(u)=I_n)}}\mathbf{1}_{(V(w_n)=I_n,\ w_n\in J^{\Delta}_{z,K,L}(n))}F\Big(\frac{V(w_{\lfloor sn\rfloor})}{\sigma\sqrt{n}}; 0 \le s \le \Delta\Big);\ \mathcal{E}_n\Big].$$

In view of Lemma 3.3, for any $b \ge B(L,\eta)$, $n \ge N(\eta)$ and $z \ge K \ge K(\eta)$,

$$\begin{aligned}\Big|\Pi_n(F) - \Lambda_n(F)\Big| &\le \mathbf{E_Q}\Big[e^{V(w_n)};\ w_n \in J^{\Delta}_{z,K,L}(n),\ \mathcal{E}_n^c\Big]\\ &\le e^{-z}n^{-3/2}\mathbf{Q}\Big(\mathcal{E}_n^c,\ w_n \in J^{\Delta}_{z,K,L}(n)\Big)\\ &\le \eta(1+L)^2(1+z-K)e^{-z}.\end{aligned} \tag{3.19}$$

On the event $\mathcal{E}_n \cap \{I_n \le a_n(z)\}$, $\Lambda_n(F)$ equals

$$\mathbf{E}_{\mathbf{Q}}\Big[\frac{e^{V(w_n)}}{\sum_{u>w_{n-b},|u|=n}\mathbf{1}_{(V(u)=I_n)}}\mathbf{1}_{(V(w_n)=I_n,\ w_n\in J^{\Delta}_{z,K,L}(n))}F\Big(\frac{V(w_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\le s\le\Delta\Big);\ \mathcal{E}_n\Big].$$

Let, for $x\ge 0$, $L>0$, and $b\ge 1$,

$$\begin{aligned} f_{L,b}(x) &:= \mathbf{E}_{\mathbf{Q}_x}\Big[\frac{e^{V(w_b)-L}\mathbf{1}_{\{V(w_b)=I_b\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=I_b\}}},\ \min_{0\le k\le b}V(w_k)\ge 0, V(w_b)\le L\Big] \\ &\le \mathbf{Q}_x\Big(\min_{0\le k\le b}V(w_k)\ge 0, V(w_b)\le L\Big). \end{aligned} \tag{3.20}$$

We choose $n$ large enough so that $\Delta n\le n-b$. Thus, applying the Markov property at time $n-b$ yields that

$$\begin{aligned} \Lambda_n(F) = n^{3/2}e^{-z}\mathbf{E}_{\mathbf{Q}}\Big[F\Big(\frac{V(w_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\le s\le\Delta\Big)f_{L,b}(V(w_{n-b})-a_n(z+L)); \\ \min_{0\le k\le n-b}V(w_k)\ge -z+K,\ \min_{\Delta n\le k\le n-b}V(w_k)\ge a_n(z+L),\ \mathcal{E}_n\Big]. \end{aligned} \tag{3.21}$$

Let us introduce the following quantity by removing the restriction to $\mathcal{E}_n$:

$$\begin{aligned} \Lambda_n^I(F) := n^{3/2}e^{-z}\mathbf{E}_{\mathbf{Q}}\Big[F\Big(\frac{V(w_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\le s\le\Delta\Big)f_{L,b}(V(w_{n-b})-a_n(z+L)); \\ \min_{0\le k\le n-b}V(w_k)\ge -z+K,\ \min_{\Delta n\le k\le n-b}V(w_k)\ge a_n(z+L)\Big]. \end{aligned} \tag{3.22}$$

We immediately observe that

$$\begin{aligned} \Big|\Lambda_n(F)-\Lambda_n^I(F)\Big| \le n^{3/2}e^{-z}\mathbf{Q}\Big(f_{L,b}(V(w_{n-b})-a_n(z+L)), \\ \min_{0\le k\le n-b}V(w_k)\ge -z+K,\ \min_{\Delta n\le k\le n-b}V(w_k)\ge a_n(z+L);\ (\mathcal{E}_n)^c\Big). \end{aligned} \tag{3.23}$$

By (3.20), we check that $\Big|\Lambda_n(F)-\Lambda_n^I(F)\Big| \le n^{3/2}e^{-z}\mathbf{Q}(w_n\in J^{\Delta}_{z,K,L}(n),\ (\mathcal{E}_n)^c)$. Applying Lemma 3.3 again implies that

$$\Big|\Lambda_n(F)-\Lambda_n^I(F)\Big| \le \eta(1+L)^2(1+z-K)e^{-z}. \tag{3.24}$$

Combining with (3.19), we obtain that for any $b\ge B(L,\eta)$, $z\ge K\ge K(\eta)$ and $n$ large enough,

$$\Big|\Pi_n(F)-\Lambda_n^I(F)\Big| \le 2\eta(1+L)^2(1+z-K)e^{-z}. \tag{3.25}$$

Note that $(V(w_k);\ k \geq 1)$ is a centered random walk under $\mathbf{Q}$ and that it is proved in [2] that $f_{L,b}$ satisfies the conditions of Lemma 2.4. By (I) of Lemma 2.4, we get that

$$\lim_{n\to\infty} \Lambda_n^I(F) = \alpha_{L,b}^I R(z-K)e^{-z}\mathbf{E}[F(e_s,\ 0\leq s\leq \delta)], \tag{3.26}$$

where $\alpha_{L,b}^I := C_1\int_{x\geq 0} f_{L,b}(x)R_-(x)dx \in [0,\infty)$. Thus, by (3.25), one sees that for any $b \geq B(L,\eta)$ and $z \geq K \geq K(\eta)$,

$$\limsup_{n\to\infty}\left|\Pi_n(F) - \alpha_{L,b}^I R(z-K)e^{-z}\mathbf{E}[F(e_s,\ 0\leq s\leq \Delta)]\right| \leq 2\eta(1+L)^2(1+z-K)e^{-z}. \tag{3.27}$$

Going back to (3.17), we deduce that for any $L \geq L_\varepsilon$, $b \geq B(L,\eta)$ and $z \geq K \geq K(\eta)$,

$$\begin{aligned} &\limsup_{n\to\infty}\left|\Sigma_n(F,z) - \alpha_{L,b}^I R(z-K)e^{-z}\mathbf{E}[F(e_s,\ 0\leq s\leq \Delta)]\right| \\ \leq\ & 2\eta(1+L)^2(1+z-K)e^{-z} + \Big(e^K + \varepsilon(1+z-K)\Big)e^{-z}. \end{aligned}$$

Recall that $\lim_{z\to\infty}\frac{R(z)}{z} = c_0$. We multiply each term by $\frac{e^z}{z}$, and then let $z$ go to infinity to conclude that

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\left|\frac{e^z}{z}\Sigma_n(F,z) - \alpha_{L,b}^I c_0\mathbf{E}[F(e_s,\ 0\leq s\leq \Delta)]\right| \leq 2\eta(1+L)^2+\varepsilon. \tag{3.28}$$

In particular, taking $F \equiv 1$ gives that

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\left|\frac{e^z}{z}\mathbf{P}(I_n \leq a_n(z)) - \alpha_{L,b}^I c_0\right| \leq 2\eta(1+L)^2+\varepsilon. \tag{3.29}$$

It follows from Fact 3.4 that $|C - \alpha_{L,b}^I c_0| \leq 2\eta(1+L)^2+\varepsilon$. We thus choose $0<\varepsilon<C/10$ and $0<\eta\leq\frac{\varepsilon}{2(1+L_\varepsilon)^2}$ so that $2C > \alpha_{L_\varepsilon,b}^I c_0 > C/2 > 0$.

Therefore, for any $\varepsilon \in (0, C/10)$, $0<\eta\leq\frac{\varepsilon}{2(1+L_\varepsilon)^2}$, $L = L_\varepsilon$ and $b \geq B(L_\varepsilon,\eta)$,

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\left|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} - \mathbf{E}[F(e_s, 0\leq s\leq\Delta)]\right| \leq \frac{4\varepsilon}{C/2-2\varepsilon}, \tag{3.30}$$

which completes the proof of (3.15) in the non-lattice case.

*Step (II) (The conditioned convergence of $(\frac{I_n(sn)}{\sigma\sqrt{n}};\ 0\leq s\leq\Delta)$ for $\Delta<1$ in the lattice case)* Assume that the law of $\mathcal{L}$ is supported by $\alpha+\beta\mathbb{Z}$ with span $\beta$. Recall that $a_n(0) = \alpha n + \beta\lfloor\frac{\frac{3}{2}\ln n - \alpha n}{\beta}\rfloor$ and that $a_n(z) = a_n(0) - z$. We use the same notation of Step (I). Let us prove

$$\lim_{\beta\mathbb{Z}\ni z\to\infty}\limsup_{n\to\infty}\left|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} - \mathbf{E}[F(e_s,\ 0\leq s\leq\Delta)]\right| = 0. \tag{3.31}$$

Suppose that $z \in \beta\mathbb{Z}$. Whereas the arguments of Step (I), we obtain that for any $L \geq L_\varepsilon$, $b \geq B(L,\eta)$, $z \geq K \geq K(\eta)$ and $n$ sufficiently large,

$$\Big|\Sigma_n(F,z) - \Lambda_n^{II}(F)\Big| \leq 2\eta(1+L)^2(1+z-K)e^{-z} + \Big(e^K + \varepsilon(1+z-K)\Big)e^{-z}, \tag{3.32}$$

where

$$\Lambda_n^{II}(F) = \Lambda^{II}(F,z,K,L,b) := e^{a_n(0)}e^{-z}\mathbf{E}_{\mathbf{Q}}\Big[F\Big(\frac{V(w_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\leq s\leq\Delta\Big)\times$$
$$f_{L,b}\big(V(w_{n-b} - a_n(z+L))\big);\ \min_{0\leq k\leq n-b} V(w_k) \geq -z+K,\ \min_{\Delta n\leq k\leq n-b} V(w_k) \geq a_n(z+L)\Big].$$

Under $\mathbf{Q}$, the distribution of $V(w_1) - V(w_0)$ is also supported by $\alpha + \beta\mathbb{Z}$. Let $d = d(L,b) := \beta\lceil\frac{\alpha b - L}{\beta}\rceil - \alpha b + L$ and $\lambda_n := n^{3/2}e^{-a_n(0)}$. Recall that $f_{L,b}$ is well defined in (3.20), it follows from (II) of Lemma 2.4 that

$$\lim_{n\to\infty} \lambda_n\Lambda_n^{II}(F) = \alpha_{L,b}^{II}R(z-K)e^{-z}\mathbf{E}[F(e_s,\ 0\leq s\leq\Delta)]. \tag{3.33}$$

where $\alpha_{L,b}^{II} := C_1\beta\sum_{j\geq 0} f_{L,b}(\beta j + d)R_-(\beta j + d) \in [0,\infty)$. Observe that $1 \leq \lambda_n \leq e^\beta$. Combining with (3.32), we conclude that

$$\limsup_{\beta\mathbb{Z}\ni z\to\infty}\limsup_{n\to\infty}\Big|\frac{e^z}{z}\lambda_n\Sigma_n(F,z) - \alpha_{L,b}^{II}c_0\mathbf{E}[F(e_s,\ 0\leq s\leq\Delta)]\Big| \leq e^\beta(2\eta(1+L)^2+\varepsilon). \tag{3.34}$$

We admit for the moment that there exist $0 < c_9 < c_{10} < \infty$ such that $\alpha_{L,b}^{II} \in [c_9, c_{10}]$ for all $L$, $b$ large enough. Then take $\varepsilon < \frac{c_9c_0}{4e^\beta}$, $L = L_\varepsilon$, $\eta = \frac{\varepsilon}{2(1+L_\varepsilon)^2}$ and $b \geq B(L_\varepsilon,\eta)$ so that $e^\beta(2\eta(1+L)^2+\varepsilon) < c_9c_0/2 \leq \alpha_{L_\varepsilon,b}^{II}c_0/2 \leq 2c_{10}c_0$. Note that $\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} = \frac{\frac{e^z}{z}\lambda_n\Sigma_n(F,z)}{\frac{e^z}{z}\lambda_n\Sigma_n(1,z)}$. We thus deduce from (3.34) that

$$\limsup_{\beta\mathbb{Z}\ni z\to\infty}\limsup_{n\to\infty}\Big|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} - \mathbf{E}[F(e_s,\ 0\leq s\leq\Delta)]\Big| \leq \frac{4\varepsilon}{c_9c_0/e^\beta - 2\varepsilon}, \tag{3.35}$$

which tends to zero as $\varepsilon \downarrow 0$.

It remains to prove that $\alpha_{L,b}^{II} \in [c_9, c_{10}]$ for all $L$, $b$ large enough. Instead of investigating the entire system, we consider the branching random walk killed at 0. Define

$$I_n^{kill} := \inf\{V(u) : |u| = n, V(u_k) \geq 0,\ \forall 0\leq k\leq n\}, \tag{3.36}$$

and we get the following fact from Corollary 3.4 and Lemma 3.6 of [2].

**Fact 3.5 ([2])** *There exists a constant $c_{11} > 0$ such that for any $n \ge 1$ and $x,\ z \ge 0$,*

$$\mathbf{P}_x(I_n^{kill} \le a_n(z)) \le c_{11}(1+x)e^{-x-z}. \tag{3.37}$$

*Moreover, there exists $c_{12} > 0$ such that for any $n \ge 1$ and $z \in [0, a_n(1)]$,*

$$\mathbf{P}(I_n^{kill} \le a_n(z)) \ge c_{12}e^{-z}. \tag{3.38}$$

Even though Fact 3.5 is proved in [2] under the assumption that the distribution of $\mathcal{L}$ is non-lattice, the lattice case is actually recovered from that proof.

Analogically, let $m^{kill,(n)}$ be the particle chosen uniformly in the set $\{u : |u| = n,\ V(u) = I_n^{kill},\ \min_{0\le k\le n} V(u_k) \ge 0\}$. Moreover, let $\Sigma_n^{kill}(1,z) := \mathbf{P}\Big[I_n^{kill} \le a_n(z)\Big]$ and $\Pi_n^{kill}(1,z,z,L) := \mathbf{P}\Big[I_n^{kill} \le a_n(z),\ m^{kill,(n)} \in J^{\Delta}_{z,z,L}(n)\Big]$. By (3.7) again, we check that for all $L \ge L_\varepsilon$,

$$\begin{aligned} &\Big|\Sigma_n^{kill}(1,z) - \Pi_n^{kill}(1,z,z,L)\Big| \\ \le\ & \mathbf{P}\Big[\exists |u| = n : V(u) \le a_n(z);\ \min_{0\le k\le n} V(u_k) \ge 0;\ \min_{\Delta n\le k\le n} V(u_k) \le a_n(z+L)\Big] \\ \le\ & \varepsilon e^{-z}. \end{aligned} \tag{3.39}$$

Recounting the arguments of Step (1), one sees that for any $L \ge L_\varepsilon$, $b \ge B(L,\eta)$, $z \ge K(\eta)$ and $n$ sufficiently large,

$$\Big|\Pi_n^{kill}(1,z,z,L) - \Lambda_n^{kill}\Big| \le 2\eta(1+L)^2 e^{-z}, \tag{3.40}$$

where

$$\Lambda_n^{kill} := \mathbf{E}_{\mathbf{Q}}\Big[f^{kill}(V(w_{n-b}));\ \min_{0\le k\le n-b} V(w_k) \ge 0,\ \min_{\Delta n\le k\le n-b} V(w_k) \ge a_n(z+L)\Big], \tag{3.41}$$

with $f^{kill}(x) := \mathbf{E}_{\mathbf{Q}_x}\Big[\frac{e^{V(w_b)}\mathbf{1}_{\{V(w_b)=I_b^{kill}\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=I_b^{kill},\ \min_{0\le j\le b} V(u_j)\ge 0\}}};\ \min_{0\le k\le b} V(w_k) \ge a_n(z+L), V(w_b) \le a_n(z)\Big]$. For $\varepsilon > 0$ and $n$ sufficiently large, it has been proved in [2] that

$$\Big|e^z \Lambda_n^{II}(1,z,z,L,b) - \Lambda_n^{kill}\Big| \le \varepsilon. \tag{3.42}$$

Recalling the convergence (3.33) with $K = z$ and $F \equiv 1$, we deduce from (3.39), (3.40) and (3.42) that for any $L \ge L_\varepsilon$, $b \ge B(L,\eta)$ and $z \ge K(\eta)$,

$$\limsup_{n\to\infty} \Big|\lambda_n \Sigma_n^{kill}(1,z) - \alpha_{L,b}^{II} e^{-z}\Big| \le e^{\beta}\Big(2\eta(1+L)^2 + 2\varepsilon\Big)e^{-z}, \tag{3.43}$$

since $R(0) = 1$ and $1 \le \lambda_n \le e^\beta$. Fact 3.5 implies that $c_{12} \le e^z \lambda_n \mathbf{P}(I_n^{kill} \le a_n(z)) \le c_{11} e^\beta$. Hence, we obtain that

$$c_{12} - e^\beta\Big(2\eta(1+L)^2 + 2\varepsilon\Big) \le \alpha_{L,b}^{II} \le e^\beta c_{11} + e^\beta\Big(2\eta(1+L)^2 + 2\varepsilon\Big). \tag{3.44}$$

Let $c_{10} := c_{11}e^\beta + c_{12}$ and $c_9 := 3c_{12}/4 > 0$. For any $\varepsilon < e^{-\beta}c_{12}/12$, we take $L = L_\varepsilon$ and $\eta \le \varepsilon/2(1+L_\varepsilon)^2$. Then $c_{10} > \alpha_{L,b}^{II} \ge c_9 > 0$ for $b \ge B(L_\varepsilon, \eta)$. This completes the second step.

*Step (III)(The tightness)* Actually, it suffices to prove the following proposition.

**Proposition 3.6** *For any $\eta > 0$,*

$$\lim_{\delta\to 0} \limsup_{z\to\infty} \limsup_{n\to\infty} \mathbf{P}\Big(\sup_{0\le k\le \delta n} |I_n(n-k) - I_n| \ge \eta\sigma\sqrt{n}\Big| I_n \le a_n(z)\Big) = 0. \tag{3.45}$$

The first two steps allow us to obtain the following fact whether the distribution is lattice or non-lattice.

**Fact 3.7** *There exist constants $c_{13}, c_{14} \in (0,\infty)$ such that*

$$c_{13} \le \liminf_{z\to\infty} \liminf_{n\to\infty} \frac{e^z}{z}\mathbf{P}(I_n \le a_n(z)) \le \limsup_{z\to\infty} \limsup_{n\to\infty} \frac{e^z}{z}\mathbf{P}(I_n \le a_n(z)) \le c_{14}. \tag{3.46}$$

*Proof of Proposition 3.6.* First, we observe that for any $M \ge 1$ and $\delta \in (0, 1/2)$,

$$\begin{aligned} &\mathbf{P}\Big(\sup_{0\le k\le \delta n} |I_n(n-k) - I_n| \ge \delta\sigma\sqrt{n},\ I_n \le a_n(z)\Big) \\ \le\ &\mathbf{P}\Big(m_n^{(n)} \notin J_{z,0,L}^{1/2}(n),\ I_n \le a_n(z)\Big) + \mathbf{P}\Big(I_n(n - \lfloor \delta n\rfloor) \ge M\sigma\sqrt{\delta n},\ I_n \le a_n(z)\Big) + \chi(\delta, z, n). \end{aligned}$$

where $\chi(\delta,z,n) := \mathbf{P}\Big(m_n^{(n)} \in J_{z,0,L}^{1/2}(n),\ I_n(n - \lfloor\delta n\rfloor) \le M\sigma\sqrt{\delta n},\ \sup_{0\le k\le\delta n} |I_n(n-k) - I_n| \ge \eta\sigma\sqrt{n}\Big)$.

It follows from Lemma 3.2 that for any $\varepsilon > 0$, if $L \ge L_\varepsilon$, $n \ge 1$ and $z \ge 0$,

$$\mathbf{P}\Big(m_n^{(n)} \notin J_{z,0,L}^{1/2}(n),\ I_n \le a_n(z)\Big) \le (1 + \varepsilon(1+z))e^{-z}. \tag{3.47}$$

Then dividing each term of (3.47) by $\mathbf{P}(I_n \le a_n(z))$ yields that

$$\begin{aligned} &\mathbf{P}\Big(\sup_{0\le k\le\delta n} |I_n(n-k) - I_n| \ge \eta\sigma\sqrt{n}\Big| I_n \le a_n(z)\Big) \\ \le\ &\frac{(1+\varepsilon(1+z))e^{-z}}{\mathbf{P}(I_n \le a_n(z))} + \mathbf{P}\Big(I_n(n - \lfloor\delta n\rfloor) \ge M\sigma\sqrt{\delta n}\Big| I_n \le a_n(z)\Big) + \frac{\chi(\delta,z,n)}{\mathbf{P}(I_n \le a_n(z))}. \end{aligned} \tag{3.48}$$

On the one hand, by Fact 3.7,

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\frac{(1+\varepsilon(1+z))e^{-z}}{\mathbf{P}(I_n\le a_n(z))}\le\frac{\varepsilon}{c_{13}}. \tag{3.49}$$

On the other hand, Steps (I) and (II) tell us that for any $1>\delta>0$ and $M\ge 1$,

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\mathbf{P}\Big[I_n(n-\lfloor\delta n\rfloor)\ge M\sigma\sqrt{\delta n}\Big|I_n\le a_n(z)\Big]=\mathbf{P}[e_{1-\delta}\ge M\sqrt{\delta}], \tag{3.50}$$

which, by Chebyshev's inequality, is bounded by $\frac{\mathbf{E}[e_{1-\delta}]}{M\sqrt{\delta}}=\frac{4\sqrt{1-\delta}}{M\sqrt{2\pi}}$. Consequently,

$$\begin{aligned}&\limsup_{z\to\infty}\limsup_{n\to\infty}\mathbf{P}\Big(\sup_{0\le k\le\delta n}|I_n(n-k)-I_n|\ge\eta\sigma\sqrt{n}\Big|I_n\le a_n(z)\Big)\\ \le\ &\frac{\varepsilon}{c_{13}}+\frac{2}{M}+\limsup_{z\to\infty}\limsup_{n\to\infty}\frac{\chi(\delta,z,n)}{\mathbf{P}(I_n\le a_n(z))}.\end{aligned} \tag{3.51}$$

Let us estimate $\chi(\delta,z,n)$. One sees that

$$\chi(\delta,z,n)\ \le\ \mathbf{E}\Big[\sum_{|u|=n}\mathbf{1}_{\{u\in J^{1/2}_{z,L}(n);\ \sup_{0\le k\le\delta n}|V(u_{n-k})-V(u)|\ge\eta\sigma\sqrt{n};\ V(u_{n-\lfloor\delta n\rfloor})\le M\sigma\sqrt{\delta n}\}}\Big].$$

By Lemma 2.4, it becomes that

$$\begin{aligned}\chi(\delta,z,n)\ &\le\ \mathbf{E}\Big[e^{S_n};S_n\le a_n(z),\underline{S}_n\ge -z,\underline{S}_{[n/2,n]}\ge a_n(z+L),\\ &\qquad S_{n-\lfloor\delta n\rfloor}\le M\sigma\sqrt{\delta n},\ \sup_{0\le k\le\delta n}|S_{n-k}-S_n|\ge\eta\sigma\sqrt{n}\Big]\\ &\le\ n^{3/2}e^{-z}\Upsilon(\delta,z,n),\end{aligned}$$

where $\Upsilon(\delta,z,n):=\mathbf{P}\Big(S_n\le a_n(z),\ \underline{S}_n\ge -z,\ \underline{S}_{[n/2,n]}\ge a_n(z+L),\ S_{n-\lfloor\delta n\rfloor}\le M\sigma\sqrt{\delta n},$ $\sup_{0\le k\le\delta n}|S_{n-k}-S_n|\ge\eta\sigma\sqrt{n},\ S_{n-\lfloor\delta n\rfloor}\le M\sigma\sqrt{\delta n}\Big)$.

Reversing time yields that

$$\begin{aligned}\Upsilon(\delta,z,n)\le\mathbf{P}\Big(&\underline{S}^-_n\ge -a_n(0),\ \underline{S}^-_{n/2}\ge -L,\ -S_n\in[-a_n(z),-a_n(z+L)],\\ &\sup_{0\le k\le\delta n}|-S_k|\ge\eta\sigma\sqrt{n},-S_{\lfloor\delta n\rfloor}\le M\sigma\sqrt{\delta n}-a_n(z+L)\Big).\end{aligned} \tag{3.52}$$

Applying the Markov property at time $\lfloor\delta n\rfloor$, we obtain that

$$\Upsilon(\delta,z,n)=\mathbf{E}\Big[\Theta(-S_{\lfloor\delta n\rfloor});\ \underline{S}^-_{\delta n}\ge -L,\sup_{0\le k\le\delta n}|-S_k|\ge\eta\sigma\sqrt{n}\Big], \tag{3.53}$$

where $\Theta(x) := \mathbf{1}_{\{x\le M\sigma\sqrt{\delta n}-a_n(z+L)\}}\mathbf{P}_x\Big(\underline{S}^-_{(1/2-\delta)n} \ge -L, \underline{S}^-_{(1-\delta)n} \ge -a_n(0), -S_{n-\lfloor\delta n\rfloor} \in [-a_n(z), -a_n(z+L)]\Big)$. Reversing time again implies that

$$\Theta(x) \le \mathbf{1}_{\{x\le M\sigma\sqrt{\delta n}\}}\mathbf{P}\Big(\underline{S}_{(1-\delta)n} \ge -z-L, \underline{S}_{[n/2,(1-\delta)n]} \ge a_n(z+2L), S_{n-\lfloor\delta n\rfloor} \in [x+a_n(z+L), x+a_n(z)]\Big).$$

By (2.10), $\Theta(x) \le c_{15}(1+z+L)(1+L)(1+M\sigma\sqrt{\delta n}+2L)n^{-3/2}$. Plugging it into (3.53) and taking $n$ large enough so that $1+2L < \eta\sigma\sqrt{\delta n}$, we get that

$$\Upsilon(\delta,z,n) \le c_{15}(1+z)(1+L)^2 n^{-3/2}(M+\eta)\sigma\sqrt{\delta n}\mathbf{E}\Big[\underline{S}^-_{\delta n} \ge -L, \sup_{0\le k\le\delta n}|-S_k| \ge \eta\sigma\sqrt{n}\Big].$$

Recall that $\chi(\delta,z,n) \le e^{-z}n^{3/2}\Upsilon(\delta,z,n)$. We check that

$$\chi(\delta,z,n) \le c_{15}e^{-z}(1+z)(1+L)^2(M+\eta)\sigma \times \mathbf{E}_L\Big[\sup_{0\le k\le\delta n}(-S_k) \ge \eta\sigma\sqrt{n}\Big|\underline{S}^-_{\delta n} \ge 0\Big]\Big(\sqrt{\delta n}\mathbf{P}_L\Big[\underline{S}^-_{\delta n} \ge 0\Big]\Big). \tag{3.54}$$

On the one hand, by Theorem 1.1 of [7], $\mathbf{E}_L\Big[\sup_{0\le k\le\delta n}(-S_k) \ge \eta\sigma\sqrt{n}\Big|\underline{S}^-_{\delta n} \ge 0\Big]$ converges to $\mathbf{P}(\sup_{0\le s\le 1}\mathcal{M}_s \ge \eta/\sqrt{\delta})$ as $n\to\infty$. On the other hand, (2.7) shows that $\sqrt{\delta n}\mathbf{P}_L\Big[\underline{S}^-_{\delta n} \ge 0\Big]$ converges to $C_-R_-(L)$ as $n\to\infty$. Therefore,

$$\limsup_{n\to\infty}\chi(\delta,z,n) \le c_{15}e^{-z}(1+z)(1+L)^2(M+\eta)\sigma C_-R_-(L) \times \mathbf{P}(\sup_{0\le s\le 1}\mathcal{M}_s \ge \eta/\sqrt{\delta}).$$

Going back to (3.51) and letting $z\to\infty$, we deduce from Fact 3.7 that

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\mathbf{P}\Big(\sup_{0\le k\le\delta n}|I_n(n-k)-I_n| \ge \eta\sigma\sqrt{n}\Big|I_n \le a_n(z)\Big) \le \frac{\varepsilon}{c_{13}} + \frac{2}{M} + \frac{c_{15}(1+L)^2(M+\eta)\sigma C_-R_-(L)\times\mathbf{P}(\sup_{0\le s\le 1}\mathcal{M}_s \ge \eta/\sqrt{\delta})}{c_{13}}. \tag{3.55}$$

Notice that $\mathbf{P}(\sup_{0\le s\le 1}\mathcal{M}_s \ge \eta/\sqrt{\delta})$ decreases to 0 as $\delta\downarrow 0$. Take $M \ge 2/\varepsilon$. We conclude that for any $0<\varepsilon<c_{13}$,

$$\limsup_{\delta\to 0}\limsup_{z\to\infty}\limsup_{n\to\infty}\mathbf{P}\Big(\sup_{0\le k\le\delta n}|I_n(n-k)-I_n| \ge \eta\sigma\sqrt{n}\Big|I_n \le a_n(z)\Big) \le \frac{\varepsilon}{c_{13}} + \varepsilon, \tag{3.56}$$

which completes the proof of Proposition 3.6. And Proposition 3.1 is thus proved. □

# 4 Proof of Theorem 1.1

Let us prove the main theorem now. It suffices to prove that for any continuous functional $F : D([0,1],\mathbb{R}) \to [0,1]$, we have

$$\lim_{n\to\infty} \Big| \mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn \rfloor)}{\sigma\sqrt{n}}; 0 \le s \le 1\Big)\Big] - \mathbf{E}\Big[F(e_s,\ 0 \le s \le 1)\Big]\Big| = 0. \tag{4.1}$$

*Proof of (4.1).* Define for $A \ge 0$,

$$\mathcal{Z}[A] := \{u \in \mathbb{T} : V(u) \ge A > \max_{k<|u|} V(u_k)\}. \tag{4.2}$$

For any particle $u \in \mathcal{Z}[A]$, there is a subtree rooted at $u$. If $|u| \le n$, let

$$I_n(u) := \min_{v \ge u, |v|=n} V(v).$$

Moreover, assume $m_n^u$ is the particle uniformly chosen in the set $\{|v| = n : v \ge u, V(v) = I_n(u)\}$. Similarly, we write $[\![\varnothing, m_n^u]\!] := \{\varnothing =: m_0^u, m_1^u, \cdots, m_n^u\}$. The trajectory leading to $m_n^u$ is denoted by $\{V(m_k^u); 0 \le k \le n\}$. Let $\omega_A$ be the particle uniformly chosen in $\{u \in \mathcal{Z}[A] : |u| \le n,\ I_n(u) = I_n\}$.

Let $\mathcal{Y}_A := \{\max_{u\in\mathcal{Z}[A]} |u| \le M,\ \max_{u\in\mathcal{Z}[A]} V(u) \le M\}$. Then for any $\varepsilon > 0$, there exist $M := M(A,\varepsilon)$ large enough such that $\mathbf{P}(\mathcal{Y}_A^c) \le \varepsilon$. It follows that

$$\begin{aligned} &\Big|\mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn \rfloor)}{\sigma\sqrt{n}}; 0 \le s \le 1\Big)\Big] - \mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn \rfloor)}{\sigma\sqrt{n}}; 0 \le s \le 1\Big); \mathcal{Y}_A, |I_n - a_n(0)| \le A/2\Big]\Big| \\ \le\ & \varepsilon + \mathbf{P}[|I_n - a_n(0)| \ge A/2]. \end{aligned} \tag{4.3}$$

We then check that for $n \ge M$,

$$\begin{aligned} &\mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn \rfloor)}{\sigma\sqrt{n}}; 0 \le s \le 1\Big); \mathcal{Y}_A, |I_n - a_n(0)| \le A/2\Big] \\ =\ & \mathbf{E}\Big[\sum_{u\in\mathcal{Z}[A]} \mathbf{1}_{(u=\omega_A)} F\Big(\frac{V(m^u_{\lfloor sn \rfloor})}{\sigma\sqrt{n}}; 0 \le s \le 1\Big); \mathcal{Y}_A, |I_n - a_n(0)| \le A/2\Big]. \end{aligned} \tag{4.4}$$

Define another trajectory $\{\tilde{V}(m_k^u); 0 \le k \le n\}$ as follows.

$$\tilde{V}(m_k^u) := \begin{cases} V(u) & \text{if } k < |u|; \\ V(m_k^u) & \text{if } |u| \le k \le n. \end{cases} \tag{4.5}$$

It follows that

$$\begin{aligned}(4.6)\qquad & \mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}};0\le s\le 1\Big);\mathcal{Y}_A,|I_n-a_n(0)|\le A/2\Big]\\ = \ & \mathbf{E}\Big[\sum_{u\in\mathcal{Z}[A]}\mathbf{1}_{(u=\omega_A)}F\Big(\frac{\tilde V(m^u_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\le s\le 1\Big);\mathcal{Y}_A,|I_n-a_n(0)|\le A/2\Big]+o_n(1),\end{aligned}$$

where $o_n(1)\to 0$ as $n$ goes to infinity.

Define the sigma-field $\mathcal{G}_A := \sigma\{(u,V(u),I_n(u));u\in\mathcal{Z}[A]\}$. Note that on $\mathcal{Y}_A$, $I_n = \min_{u\in\mathcal{Z}[A]} I_n(u)$ as long as $n\ge M$. One sees that $\mathcal{Y}_A\cap\{|I_n-a_n(0)|\le A/2\}$ is $\mathcal{G}_A$-measurable for all $n$ large enough. Thus,

$$\begin{aligned}(4.7)\qquad & \mathbf{E}\Big[\sum_{u\in\mathcal{Z}[A]}\mathbf{1}_{(u=\omega_A)}F\Big(\frac{\tilde V(m^u_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\le s\le 1\Big);\mathcal{Y}_A,|I_n-a_n(0)|\le A/2\Big]\\ = \ & \mathbf{E}\Big[\sum_{u\in\mathcal{Z}[A]}\mathbf{1}_{(u=\omega_A)}\mathbf{E}\Big[F\Big(\frac{\tilde V(m^u_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\le s\le 1\Big)\Big|\mathcal{G}_A,\ u=\omega_A\Big];\mathcal{Y}_A,|I_n-a_n(0)|\le A/2\Big].\end{aligned}$$

Further, we notice by the branching property that conditioned on $\{(u,V(u));u\in\mathcal{Z}[A]\}$, the subtrees generated by $u\in\mathcal{Z}[A]$ are independent copies of the original one, started from $V(u)$, respectively. Therefore, given $\mathcal{Y}_A\cap\{|I_n-a_n(0)|\le A/2\}$,

$$\begin{aligned}& \mathbf{1}_{(u=\omega_A)}\mathbf{E}\Big[F\Big(\frac{\tilde V(m^u_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\le s\le 1\Big)\Big|\mathcal{G}_A,\ u=\omega_A\Big]\\ = \ & \mathbf{1}_{(u=\omega_A)}\mathbf{E}\Big[F\Big(\frac{I(\lfloor s(n-|u|)\rfloor)}{\sigma\sqrt{n-|u|}};0\le s\le 1\Big)\Big|I_{n-|u|}\le a_n(-r_u)\Big]+o_n(1),\end{aligned}$$

where $r_u := \min\{\min_{v\in\mathcal{Z}[A]\setminus\{u\}} I_n(v)-a_n(0),\ A/2\}-V(u)$ is independent of $I_{n-|u|}$. Thus, (4.6) becomes that

$$\begin{aligned}(4.8)\qquad & \mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}};0\le s\le 1\Big);\mathcal{Y}_A,|I_n-a_n(0)|\le A/2\Big]\\ = \ & \mathbf{E}\Big[\sum_{u\in\mathcal{Z}[A]}\mathbf{1}_{(u=\omega_A)}\mathbf{E}\Big[F\Big(\frac{I(\lfloor s(n-|u|)\rfloor)}{\sigma\sqrt{n-|u|}};0\le s\le 1\Big)\Big|I_{n-|u|}\le a_n(-r_u)\Big];\\ & \qquad \mathcal{Y}_A,|I_n-a_n(0)|\le A/2\Big]+o_n(1).\end{aligned}$$

The event $\mathcal{Y}_A\cap\{|I_n-a_n(0)|\le A/2\}$ ensures that $A/2+M\ge -r_u\ge A/2$. The conditioned convergence has been given in Proposition 3.1. We need a slightly stronger version here.

According to Proposition 3.1, for any $\varepsilon > 0$, there exists $z_\varepsilon > 0$ such that for all $z \geq z_\varepsilon$,

$$\text{(4.9)} \quad \limsup_{n\to\infty} \Big|\mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}}; 0 \leq s \leq 1\Big)\Big| I_n \leq a_n(z)\Big] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)]\Big| < \varepsilon.$$

Thus, for any $z \geq z_\varepsilon$, there exists $N_z \geq 1$ such that for any $n \geq N_z$,

$$\text{(4.10)} \quad \Big|\mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}}; 0 \leq s \leq 1\Big)\Big| I_n \leq a_n(z)\Big] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)]\Big| < 2\varepsilon.$$

Take $A = 2z_\varepsilon$ and $K = M$. We say that for $n$ sufficiently large,

$$\text{(4.11)} \quad \sup_{z\in[z_\varepsilon, z_\varepsilon+K]} \Big|\mathbf{E}\Big[F\Big(\frac{I(\lfloor s(n)\rfloor)}{\sigma\sqrt{n}}; 0 \leq s \leq 1\Big)\Big| I_n \leq a_n(z)\Big] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)]\Big| \leq 3\varepsilon.$$

In the lattice case, (4.11) follows immediately. We only need to prove it in the non-lattice case.

Recall that $\Sigma_n(F, z) = \mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}}; 0 \leq s \leq 1\Big);\ I_n \leq a_n(z)\Big]$ with $0 \leq F \leq 1$. Then, for any $\ell > 0$ and $z \geq 0$,

$$\begin{aligned}
\text{(4.12)} \quad & \Big|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} - \frac{\Sigma_n(F,z+\ell)}{\Sigma_n(1,z+\ell)}\Big| \\
\leq\ & \Big|\frac{\Sigma_n(F,z) - \Sigma_n(F,z+\ell)}{\Sigma_n(1,z)}\Big| + \Big|\frac{\Sigma_n(F,z+\ell)}{\Sigma_n(1,z)} - \frac{\Sigma_n(F,z+\ell)}{\Sigma_n(1,z+\ell)}\Big| \\
=\ & \frac{1}{\Sigma_n(1,z)}\Big(\Big|\Sigma_n(F,z) - \Sigma_n(F,z+\ell)\Big| + \frac{\Sigma_n(F,z+\ell)}{\Sigma_n(1,z+\ell)}\Big|\Sigma_n(1,z+\ell) - \Sigma_n(1,z)\Big|\Big).
\end{aligned}$$

Since $0 \leq F \leq 1$, the two following inequalities

$$\begin{aligned}
\Big|\Sigma_n(F,z) - \Sigma_n(F,z+\ell)\Big| &= \mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}}; 0 \leq s \leq 1\Big);\ a_n(z+\ell) < I_n \leq a_n(z)\Big] \\
&\leq \mathbf{P}(a_n(z+\ell) < I_n \leq a_n(z)),
\end{aligned}$$

and $\frac{\Sigma_n(F,z+\ell)}{\Sigma_n(1,z+\ell)} \leq 1$ hold. Note also that $|\Sigma_n(1,z+\ell) - \Sigma_n(1,z)| = \mathbf{P}(a_n(z+\ell) < I_n \leq a_n(z))$. It follows that

$$\begin{aligned}
\text{(4.13)} \quad \Big|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} - \frac{\Sigma_n(F,z+\ell)}{\Sigma_n(1,z+\ell)}\Big| &\leq 2\frac{\mathbf{P}(a_n(z+\ell) < I_n \leq a_n(z))}{\mathbf{P}(I_n \leq a_n(z))} \\
&= 2 - 2\frac{\mathbf{P}(I_n \leq a_n(z+\ell))}{\mathbf{P}(I_n \leq a_n(z))}.
\end{aligned}$$

In view of Fact 3.4, we take $\frac{3}{2}\ln n - \Lambda_{\varepsilon'} \geq \ell + z > z \geq \Lambda_{\varepsilon'}$ so that for any $n \geq N_{\varepsilon'}$,

$$\text{(4.14)} \quad \frac{\mathbf{P}(I_n \leq a_n(z+\ell))}{\mathbf{P}(I_n \leq a_n(z))} \geq \frac{(C-\varepsilon')(z+\ell)e^{-z-\ell}}{(C+\varepsilon')ze^{-z}} \geq \frac{C-\varepsilon'}{C+\varepsilon'}e^{-\ell}.$$

For $\varepsilon' = C\varepsilon/8 > 0$, we choose $\zeta = \frac{\varepsilon}{4}$ so that $\frac{C-\varepsilon'}{C+\varepsilon'}e^{-\zeta} \geq 1 - \frac{\varepsilon}{2}$. As a consequence, for any $\Lambda_{\varepsilon'} \leq z \leq \frac{3}{2}\ln n - \Lambda_{\varepsilon'} - \zeta$, $0 \leq \ell \leq \zeta$ and $n \geq N_{\varepsilon'}$,

$$\Big|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} - \frac{\Sigma_n(F,z+\ell)}{\Sigma_n(1,z+\ell)}\Big| \leq 2\Big(1 - \frac{C-\varepsilon'}{C+\varepsilon'}e^{-\ell}\Big) \leq \varepsilon. \tag{4.15}$$

For $\varepsilon > 0$, $z_\varepsilon$ can be chosen so that $[z_\varepsilon, z_\varepsilon + K] \subset [\Lambda_{\varepsilon'}, \frac{3}{2}\ln n - \Lambda_{\varepsilon'}]$ for $n \geq e^K N_{\varepsilon'}$. For any integer $0 \leq j \leq \lceil K/\zeta \rceil$, let $z_j := z_\varepsilon + j\zeta$. Then $[z_\varepsilon, z_\varepsilon + K] \subset \cup_{0\leq j\leq \lceil K/\zeta\rceil}[z_j, z_{j+1}]$. Take $N'_\varepsilon = \max_{0\leq j\leq\lceil K/\zeta\rceil}\{N_{z_j}, e^K N_{\varepsilon'}\}$. By (4.10) and (4.15), we conclude that for any $n \geq N'_\varepsilon$,

$$\begin{aligned}
&\sup_{z\in[z_\varepsilon, z_\varepsilon+K]} \Big|\mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}}; 0 \leq s \leq 1\Big)\Big| I_n \leq a_n(z)\Big] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)]\Big| \\
\leq\ & \sup_{0\leq j\leq\lceil K/\zeta\rceil} \Big|\frac{\Sigma_n(F,z_j)}{\Sigma_n(1,z_j)} - \mathbf{E}[F(e_s,\ 0 \leq s \leq 1)]\Big| + \sup_{0\leq j<\lceil K/\zeta\rceil}\ \sup_{z_j\leq z\leq z_{j+1}} \Big|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} - \frac{\Sigma_n(F,z_j)}{\Sigma_n(1,z_j)}\Big| \\
\leq\ & 3\varepsilon.
\end{aligned}$$

We continue to prove the main theorem. Since $\sum_{u\in\mathcal{Z}[A]} \mathbf{1}_{(u=\omega_A)} = 1$, we deduce from (4.8) and (4.11) that for $n$ sufficiently large,

$$\begin{aligned}
&\Big|\mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}}; 0 \leq s \leq 1\Big); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2\Big] - \mathbf{E}[F(e_s,\ 0 \leq s \leq 1)]\Big| \\
\leq\ & 3\varepsilon\mathbf{P}(\mathcal{Y}_A; |I_n - a_n(0)| \leq A/2) + o_n(1) + \mathbf{P}(\mathcal{Y}_A^c) + \mathbf{P}(|I_n - a_n(0)| \geq A/2) \\
\leq\ & 4\varepsilon + o_n(1) + \mathbf{P}(|I_n - a_n(0)| \geq A/2).
\end{aligned}$$

Going back to (4.3), we conclude that for $n$ large enough,

$$\Big|\mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}}; 0 \leq s \leq 1\Big)\Big] - \mathbf{E}[F(e_s,\ 0 \leq s \leq 1)]\Big| \leq 5\varepsilon + 2\mathbf{P}(|I_n - a_n(0)| \geq A/2) + o_n(1).$$

Let $n$ go to infinity and then make $\varepsilon \downarrow 0$. Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty} \Big|\mathbf{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}}; 0 \leq s \leq 1\Big)\Big] - \mathbf{E}[F(e_s,\ 0 \leq s \leq 1)]\Big| \\
\leq\ & \limsup_{z\to\infty}\limsup_{n\to\infty} 2\mathbf{P}(|I_n - a_n(0)| \geq z).
\end{aligned} \tag{4.16}$$

It remains to show that $\limsup_{z\to\infty}\limsup_{n\to\infty}\mathbf{P}(|I_n - a_n(0)| \geq z) = 0$. Because of Fact (3.7), it suffices to prove that

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\mathbf{P}(I_n \geq a_n(0) + z) = 0. \tag{4.17}$$

In the non-lattice case, Theorem 1.1 of [2] implies it directly. In the lattice case, we see that for $n$ large enough,

$$\mathbf{P}(I_n \geq a_n(0) + z) \leq \mathbf{E}\Big[\prod_{u \in \mathcal{Z}[A]} (1 - \Phi_u(z,n)); \mathcal{Y}_A\Big] + \varepsilon, \tag{4.18}$$

with $\Phi_u(z,n) := \mathbf{P}(I_{n-|u|} \leq a_n(V(u) - z))$. Take $A = 2z$ here. Then it follows from Fact 3.7 that for $n$ large enough and for any particle $u \in \mathcal{Z}[A]$,

$$\Phi_u(z,n) \geq c_{13}/2(V(u) - z)e^{z - V(u)} \geq \frac{c_{13}}{4} V(u) e^{z - V(u)}. \tag{4.19}$$

(4.18) hence becomes that

$$\begin{aligned} \limsup_{n \to \infty} \mathbf{P}(I_n \geq a_n(0) + z) &\leq \mathbf{E}\Big[\prod_{u \in \mathcal{Z}[A]} (1 - \frac{c_{13}}{4} V(u) e^{z - V(u)}); \mathcal{Y}_A\Big] + \varepsilon \\ &\leq \mathbf{E}\Big[\exp\Big(-\frac{c_{13}}{4} e^z \sum_{u \in \mathcal{Z}[A]} V(u) e^{-V(u)}\Big)\Big] + \varepsilon. \end{aligned}$$

It has been proved that as $A$ goes to infinity, $\sum_{u \in \mathcal{Z}[A]} V(u) e^{-V(u)}$ converges almost surely to some limit $D_\infty$, which is strictly positive on the set of non-extinction of $\mathbb{T}$, (see (5.2) in [2]). We end up with

$$\limsup_{z \to \infty} \limsup_{n \to \infty} \mathbf{P}(I_n \geq a_n(0) + z) \leq \varepsilon, \tag{4.20}$$

which completes the proof of Theorem 1.1. □

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