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At least at the undergraduate level, most lectures and textbooks about hydrodynamics make use of the so-called Eulerian picture, where pressure, temperature, velocity of the fluid are treated as continuous fields defined by the value they take at each geometrical point of the reference frame the fluid moves in. There nevertheless exists another possible description of the movement which consists in labelling the fluid element themselves, and keep this labelling in the course of the motion. This so-called Lagrangian picture is scarcely taught for it often brings in somehow involved mathematics, although it occurs to be more intuitive than the Eulerian picture. In this paper, we try to illustrate the latter feature on the exemple of the otherwise cumbersome problem of the Rayleigh acoustic radiation pressure, and we argue that dealing with the mathematical difficulties implied by the Lagrangian picture may be a good pedagogical opportunity to familiarize with tensorial calculus.

PACS numbers: 47.10.A-, 43.20.+g, 43.25.+y, 47.35.Rs

I. INTRODUCTION

Wave propagation in fluids is an important topic, generally considered as part of the widest domain of Fluid Mechanics (FM)[1,2]. As a consequence, FM tools are used to deal with this topic. At the very centre of the latter tools stands the so-called Eulerian picture[13]: to speak curtly, a (usely Galilean) reference frame is defined. At a given point $\vec{r}$ of this frame, and at time t, the physical state of the fluid is described by a set of parameters: mass density ρ, pressure P, fluid velocity (with respect to the frame) $\vec{v}$, temperature T and so on. So that one deals with a set of (coupled) continuous fields $\rho(\vec{r},t)$, $P(\vec{r},t)$, $\vec{v}(\vec{r},t)$, $T(\vec{r},t)$, *etc.* For instance, in absence of any external force (gravity or other), the movement of an inviscid[14] fluid is ruled by the well-known Euler equation

$$\rho(\vec{r},t)\left(\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\overrightarrow{\mathrm{grad}})\,\vec{v}\right) = -\overrightarrow{\mathrm{grad}}\,P. \tag{1}$$

Such a picture has many advantages. First, it is convenient to describe the dynamics of the fluid by means of *local* equations coupling fields, exactly as in the Electromagnetism (EM) domain. Second, the Eulerian picture is particularly well adapted to situations in which the fluid really *flows*: when studying the stream of a river passing at some bridge, we are interested in the very behaviour of the water under this bridge at a time t, whichever the origin or the past behaviour of this water.

Nevertheless, the Eulerian picture is not without a few drawbacks. First, let us glance at equation (1). The left-hand side is obviously nonlinear in field $\vec{v}$, due to the $(\vec{v}\cdot\overrightarrow{\mathrm{grad}})\,\vec{v}$ term. In addition to the latter *explicit* nonlinearity, an *implicit* further nonlinearity hides in the mass density term as soon as the fluid is compressible, since $\rho(\vec{r},t)$ is *a priori* $\vec{v}$-dependent. A second drawback of the Eulerian picture comes up when free boundary conditions between two fluids have to be taken into account. Let us consider the example illustrated in figure 1: two different fluids − say 1 and 2 − are separated *at rest* by the (infinite) plane $x=0$. Let us consider now a plane pressure wave propagating from $x=-\infty$ in fluid 1 towards the boundary. As well known, this incident wave splits at the interface in two parts: a reflected wave, travelling back to $x=-\infty$ through medium 1, and a transmitted wave, travelling towards $x=\infty$ through medium 2. It is a time-honoured undergraduate level exercice to determine the reflection and the transmission coefficients at the interface. In principle, the answer is easy: one equalizes the fluid pressures

$$P_1(\text{interface}) = P_2(\text{interface}) \tag{2a}$$

and fluid velocities

$$\vec{v}_1(\text{interface}) = \vec{v}_2(\text{interface}), \tag{2b}$$

which provides two equations enabling us to calculate both coefficients. The above process in undoubtedly correct, but raises a nontrivial difficulty: *where* is the interface? At $x=0$? Certainly not, since the interface itself moves back and forth, due to the wave motion. As a matter of fact, the $x = 0$ plane spends half time in medium 1, half time in medium 2. Of course, locating the interface at $x = 0$ is the best *approximation*, and it leads to the the correct values of the reflection and transmission coefficients, but it should be regarded as but a order-zero approximation.

In fact, the above-underlined difficulties of the Eulerian picture can be (up to a point) overcome in the framework of an alternative picture, known as the Lagrangian picture[3–5]. It is precisely the aim of the present paper to sketch the main features of the Lagrangian picture which is poorly taught in academic courses and scarcely used when studying acoustic propagation in fluids.

To begin with, what is it all about? The philosophy of the Lagrangian picture can be outlined as follows. Contrary to the Eulerian picture, which, as recalled above, labels the geometric points of the reference frame disregarding the origin of the fluid elements passing through these points at time t, the Lagrangian picture labels the fluid elements disregarding the exact position they occupy at time t. Let us detail hereafter how it works. With this aim, let us consider a fluid at some time t_0. We denote $\vec{r}_0$ the fluid element that occurs to stand at point $\vec{r}_0$ of the reference frame at time t_0. We shall henceforth keep this label $\vec{r}_0$ to denote this fluid element, whatever its further motion. Thus, at time t, the fluid element $\vec{r}_0$ will be found at some point $\vec{r}$ given by

$$\vec{r}(\vec{r}_0, t) = \vec{r}_0 + \vec{u}(\vec{r}_0, t), \tag{3}$$

where $\vec{u}(\vec{r}_0, t)$ is the displacement undergone by the fluid element $\vec{r}_0$ between times t_0 and t. The physical state of the fluid is still described by a set of continuous fields: mass density, pressure, velocity, temperature, *etc.* The correspondance between both pictures is very simple. Superscripts $\mathcal{E}$ and $\mathcal{L}$ respectively standing for "Euler" and "Lagrange", we have, quantity A standing for whichever parameter ρ, P, $\vec{v}$, T, *etc.*,

$$A^{\mathcal{E}}\big(\vec{r}(\vec{r}_0, t), t\big) = A^{\mathcal{L}}(\vec{r}_0, t), \tag{4}$$

with $\vec{r}(\vec{r}_0, t)$ given by (3). Concretely, the above equation (4) means that $A^{\mathcal{L}}(\vec{r}_0, t)$ denotes the actual value of parameter A as undergone at time t by the fluid element labelled $\vec{r}_0$ which is currently at point $\vec{r}_0 + \vec{u}(\vec{r}_0, t)$ of the reference frame, *i.e.* $A^{\mathcal{L}}(\vec{r}_0, t)$ is *numerically* equal to $A^{\mathcal{E}}\big(\vec{r}(\vec{r}_0, t), t\big)$.

We show in the present paper that the Lagrangian picture offers several advantages from a *technical* point of view. To begin with, this picture rids us of *spurious* nonlinearities (for instance, the pseudo nonlinearity associated with the $(\vec{v} \cdot \overrightarrow{\mathrm{grad}})\,\vec{v}$ term in the left-hand side of the Euler equation (1)), and enables us to index and classify *true* nonlinearities, allowing perturbative resolutions of the field equations. An example of such a simplification is provided by the otherwise cumbersome problem of the so-called acoustic radiation pressure. A second technical advantage of the Lagrangian picture appears when dealing with the afore-mentioned free boundary conditions between two fluids. Let us go back to the example illustrated in figure 1, where two media are separated *at rest* by a plane interface located at $x = 0$. Choosing the latter rest state of the system to implement the Lagrangian labelling of the fluid elements, we denote $x = 0$ the right-hand face of the last slice of the (left) medium 1, as well as the left-hand face of the first slice of the (right) medium 2. This labelling will "follow" the motion of the system and (provided of course that no mixing occurs between the two fluids) the Lagrangian labelling of the interface will remain $x = 0$ throughout the propagation of the acoustic wave, whatever the amplitude of the latter and without any approximation.

Moreover, and more generally, the Lagrangian labelling of the fluid particles can be easily visualized using passive markers[15]. Observing the trajectories of the latter markers provides an experimental check for mathematical solutions or simulations and, besides, is of interest for environmental problems (see for instance[5]).

Teaching the Lagrangian picture also offers some advantage from a pedagogical point of view. A very natural way indeed to introduce continuous media dynamics consists in starting from a discrete description of matter: using Newton's Second Law, the dynamics of every fluid element is established, then merged into a continuous description (typically, in lattice dynamics, this continuous description naturally emerges when the conditions of the centre of Brillouin zone propagation are fulfilled). In this respect, the bridge with the Eulerian picture is a bit more delicate (leading for instance to the spurious $(\vec{v} \cdot \overrightarrow{\mathrm{grad}})\,\vec{v}$ nonlinearity mentioned above).

Nevertheless, our promotion of the Lagrangian picture would be incomplete if we did not stress the following feature. Let us consider the position of a moving fluid at some initial time t_0. Labelling the different fluid elements by their position $\vec{r}_0$ with respect to the reference frame, we can of course choose a set of Cartesian coordinates: $\vec{r}_0(x_0, y_0, z_0)$. But, in the course of the subsequent motion, we will have to take the distortion of the fluid elements into account in order to write correctly the local balance equations (as (1) for instance, where both hand-sides have to be reconsidered in the framework of the Lagrangian picture). Now, due to the *a priori* complex motion of the fluid, our coordinate system is no longer Cartesian at any time t, but *curvilinear*. (Of course, we should never have the same problem with the Eulerian picture, since then the choice of a Cartesian coordinate system to label the geometric points of the reference frame can be made *once for all*.)

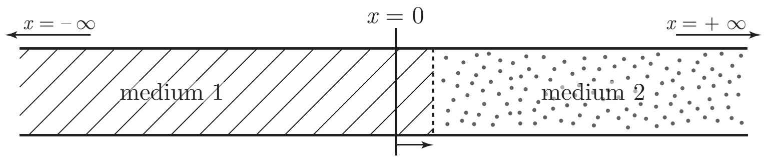

FIG. 1: Medium 1 and medium 2 are separated at rest by the plane $x = 0$. When a pressure wave propagates, the interface does not remain at $x = 0$, but moves back and forth on either side of the plane $x = 0$. In the Eulerian picture, locating the interface at $x = 0$ appears as an order-zero approximation.

In other words, *fully* mastering the Lagrangian picture involves being able to deal with any curvilinear coordinates system. This, of course, may be regarded as severely impairing the simplicity of the description. And it does. Nevertheless, it can be argued that this unavoidable drawback offers an opportunity to introduce, in a confortable 3-D flat Euclidian space, some concepts and notations that reveal essential in other domains, as for instance the theory of General Relativity (GR). We made the choice in the present paper to seize this opportunity to instil, as softy as possible, some notions about covariance, tensorial calculation, covariant derivation and so on.

With this general philosophy, the present paper is organized as follows. In section II, we focus on the one-dimension problem. We illustrate our purpose with the calculation (in subsection II A) of the exact solution of the sound propagation in a fluid with a linear extrapressure-to-strain thermodynamic relation, which reveals a surprising absence of frequency shift of the eigenmodes when the length of the fluid column is modified by the operator (in a way precised in the text). We next show (in subsection II B) that the so-called acoustic radiation pressure is entirely due to the nonlinearity of the extrapressure-to-strain relation, and we perform an exact (up to order 2) calculation of it, emphasizing the link with the frequency shift then associated with a fluid column length modification. In section III, we introduce a few geometrical concepts and quantities useful when handling curvilinear coordinate systems: coordinates bases in subsection III A, general notions about tensors in subsection III B; the metric tensor is presented in subsection III C and the metric coefficients used to pass from covariant to contravariant components and *vice versa*; the covariant derivation of these components is outlined in subsection III D and some basic elements of vector analysis are displayed in subsection III E.

In section IV, we extend the considerations of section II to the general three-dimension problem: geometrical displacement, dilatation, strain (subsection IV A). Then, we establish the 3-D motion equations, as well as their equivalence with those obtained with the Eulerian picture (subsection IV B). Next, we consider the energy balance in the Lagrangian picture: energy density and acoustic Poynting vector (subsection IV C). At last, we examine the well-known problem of the pulsating sphere in the framework of the Lagangian picture and we discuss the reflection/transmission coefficients at the interface in spherical geormetry (subsection IV D).

II. THE ONE-DIMENSION CASE

As announced in the introduction, we start the present study of the Lagrangian picture with the simplest situation we may have to face: the one-dimension problem. Let us therefore consider a fluid occupying *at rest* a cylindrical volume with axis Ox_0 and section S (figure 2a), at equilibrium pressure P_0 and mass density ρ_0. Both ends, labelled $x_0 = 0$ and $x_0 = L$, are made of pistons that are provisionally supposed to be fixed. As displayed in figure 2, the slice of fluid comprised between faces labelled x_0 and $x_0 + \mathrm{d}x_0$ has mass $\rho_0 S \mathrm{d}x_0$. At time t, its current thickness is

$$\big(x_0 + \mathrm{d}x_0 + u(x_0 + \mathrm{d}x_0, t)\big) - \big(x_0 + u(x_0, t)\big) = \left(1 + \frac{\partial u}{\partial x_0}\right) \mathrm{d}x_0, \tag{5a}$$

so that its current mass density is simply

$$\rho(x_0, t) = \frac{\rho_0}{1 + \dfrac{\partial u}{\partial x_0}}. \tag{5b}$$

Besides, within the framework of the Lagrangian picture, the pressure forces undergone by this slice of fluid are respectively $SP_0(x_0, t)$ (left end) and $-\,SP(x_0 + \mathrm{d}x_0, t)$ (right end). Consequently, applying Newton's Second Law to the later slice, we obtain

$$\rho_0 \frac{\partial^{\mathcal{L}2} u}{\partial t^2} = -\,\frac{\partial P}{\partial x_0}, \tag{6}$$

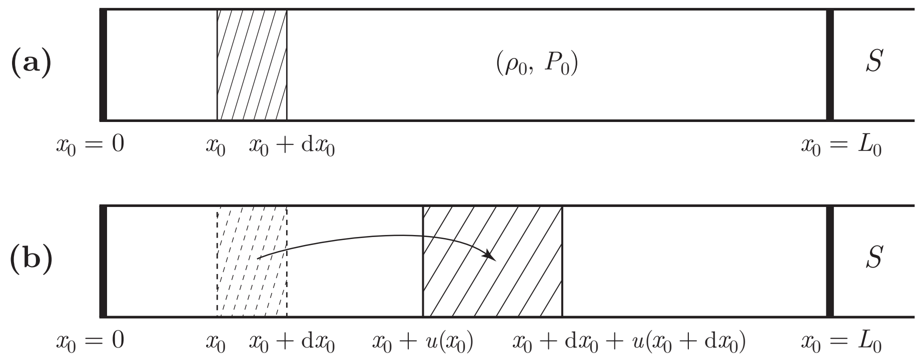

FIG. 2: (a). The fluid at rest, with equilibrium mass density ρ_0 and pressure P_0. (b). The fluid at time t: both ends, labelled $x_0 = 0$ and $x_0 = L$, are made of pistons that are provisionally supposed fixed.

where superscript $\mathcal{L}$ in the left-hand side recalls that the time-(second) derivative is understood at constant x_0 (even if the current position at time t of the face labelled "x_0" is miles away from the point with abscissa x_0 of the reference frame). The above equation (6) deserves at least a twofold comment: (i) it is strictly exact (we made no approximation); (ii) it is strictly linear in displacement u (or in velocity $v = \frac{\partial^{\mathcal{L}} u}{\partial t}$). Establishing equation (6) is half of the job. If we want now to get a propagation equation, we have to connect the pressure $P(x_0, t)$ with the expansion factor $\frac{\partial u}{\partial x_0}$ (or equivalently the mass density ρ). This is a thermodynamic affair. Throughout the present article, we shall assume, for the sake of simplicity, that any transformation undergone by the fluid is *isentropic*. In the framework of the Lagrangian picture, this means that the entropy of any fluid slice $[x_0, x_0 + \mathrm{d}x_0]$ is, at any time, equal to its equilibrium value, so that in the course of the motion, the pressure $P(x_0, t)$ can be expressed as a function of the sole[16] mass density $\rho(x_0, t)$. It will appear in the following discussion that it is most convenient to expand the extrapressure $P(x_0, t) - P_0$ in increasing powers of $\frac{\partial u}{\partial x_0}$:

$$P(x_0, t) - P_0 = -\,\kappa_1 \left(\frac{\partial u}{\partial x_0} \right) + \frac{1}{2} \kappa_2 \left(\frac{\partial u}{\partial x_0} \right)^2 + \cdots \,, \tag{7}$$

where $\kappa_1 > 0$ due to the Second law of thermodynamics. Combining the mechanical equation (6) (in which the superscript $\mathcal{L}$ for "Lagrange" is henceforth omitted) with the above thermodynamic relation (7), we get the sound propagation equation

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \kappa_1 \frac{\partial^2 u}{\partial x_0^2} \left(1 - \frac{\kappa_2}{\kappa_1} \frac{\partial u}{\partial x_0} + \cdots \right). \tag{8}$$

The above equation is nonlinear in displacement u, its nonlinearity originating exclusively in the κ_2, κ_3, *etc.* terms in the thermodynamic expansion (7).

A. The linear approximation

In this subsection, we deliberately linearize the above equation (7), *i.e.* we assume that $\kappa_2 = \kappa_3 = \cdots = 0$. The propagation equation (8) becomes also linear, and reads

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_0^2}, \tag{9a}$$

with

$$c = \sqrt{\frac{\kappa_1}{\rho_0}}. \tag{9b}$$

Let us look for the propagation eigenmodes, *i.e.* the monochromatic solutions of (9a). They necessarily read, owing for the boundary conditions we have choosen,

$$u_n(x_0, t) = \Re\big\{ A_n \sin(k_n x_0)\, \mathrm{e}^{-\mathrm{i}\omega_n t} \big\}, \tag{10a}$$

with

$$\omega_n = ck_n, \qquad k_n = \frac{n\pi}{L_0} \quad (n = 1, 2, \cdots), \tag{10b}$$

and A_n a (complex) amplitude. Note that, as far as the linearization of the *thermodynamic* relation (7) (*i.e.* $\kappa_2 = \kappa_3 = \cdots = 0$) is relevant, the above solution is *exact*, contrary to the solution generally proposed in the framework of the Eulerian picture, which requires *in addition* the linearization of the Euler equation (*i.e.* neglecting the $(\vec{v}\cdot\overrightarrow{\text{grad}})\,\vec{v}$ term in (1)[17]).

Let us now determine the overall acoutic energy associated with the wave, *i.e.* the variation (with respect to the rest state) of the sum of all the $[x_0, x_0+\mathrm{d}x_0]$ slices total energy. Since there is neither heat exchange between neighbouring slices nor external force, we just have to determine the (mechanical) work done by the pressure force to drive each fluid slice from its equilibrium state to its current state at time t. For the $[x_0, x_0+\mathrm{d}x_0]$ slice, the energy variation is exactly

$$\mathrm{d}E = \int_0^t \mathrm{d}t\left[-\,SP(x_0+\mathrm{d}x_0,t)\frac{\partial u(x_0+\mathrm{d}x_0,t)}{\partial t} + SP(x_0,t)\frac{\partial u(x_0,t)}{\partial t}\right] = S\mathrm{d}x_0\int_0^t \mathrm{d}t\left[-\,\frac{\partial P}{\partial x_0}\frac{\partial u}{\partial t} - P\frac{\partial^2 u}{\partial x_0\partial t}\right]. \tag{11a}$$

Owing to (6) and to (the linarized version of) (7), the above equation becomes

$$\mathrm{d}E = S\mathrm{d}x_0\left[\frac{1}{2}\rho_0\left(\frac{\partial u(x_0,t)}{\partial t}\right)^2 - P_0\frac{\partial u(x_0,t)}{\partial x_0} + \frac{1}{2}\kappa_1\left(\frac{\partial u(x_0,t)}{\partial x_0}\right)^2\right]. \tag{11b}$$

Integrating over the whole fluid, and accounting for (9b) and our boundary conditions, we finally get the overall acoustic energy E, which is a constant of the movement:

$$E = \frac{1}{2}\rho_0 S\int_0^{L_0}\mathrm{d}x_0\left[\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x_0}\right)^2\right]. \tag{11c}$$

Now, since any solution $u(x_0,t)$ of the wave equation (9a) is but a linear combination of eigenmodes of the type (10a), the above energy (11c) may also be written, all calculations carried out,

$$E = \frac{1}{4}\rho_0 SL_0\sum_{n=1}^{\infty}|A_n|^2\omega_n^2 = \sum_{n=1}^{\infty}N_n\hbar\omega_n \tag{11d}$$

where N_n is the so-called semi-classical quanta number in mode n. We would end this brief recall of the above results with the following (thought) experiment. Suppose that, while a given eigenmode (say mode n) is established in the cylindrical cavity delimited by the two pistons displayed in figure 2, we *slowly* move the piston at the end labelled "$x_0 = L_0$" at, say, a constant velocity V. By "slowly", we mean "adiabatically in the Ehrenfest sense". We have discussed at some length this issue in a foregoing paper[6], and shown that, in the course of such a kind of adiabatic parametric excitation of the system, the quanta number N_n is conserved. Let us recall that, in the framework of the Lagrangian picture, the label "$x_0 = L_0$" of the fluid in contact with the moving piston remains unchanged, although the total length of the fluid column is obviously $L(t) = L_0 + Vt$. In this respect, it is convenient to split the displacement $u(x_0,t)$ in two parts and set

$$u(x_0,t) = \frac{x_0}{L_0}Vt + w(x_0,t). \tag{12}$$

The first term in the right-hand side of the above equation (12) is the displacement of the slice labelled x_0, associated with a quasistatic expansion (or compression, according to the sign of V) of the fluid; the second term is the extradisplacement of the latter fluid slice due to the acoustic wave. Observe that the boundary conditions for $w(x_0,t)$ are the same as for $u(x_0,t)$: $w(x_0=0,t) = w(x_0=L_0,t) = 0$. Now, let us rewrite the wave equation (9a) in terms of w instead of u. We get, using (12),

$$\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x_0^2}, \tag{13}$$

i.e. exactly the same equation as for u. Solution (10a) is consequently unchanged: fascinating though it may be, the motion of the piston has strictly no influence upon eigenmode n. In particular, no frequency shift[18] occurs: the wave number $k_n = \frac{n\pi}{L_0}$ as well as the angular frequency $\omega_n = ck_n$ do keep their initial values even if $L(t)$ happens to become twice (or half) its initial value L_0. Moreover, glancing at equation (11d), one concludes that, since neither the quanta number N_n (Ehrenfest adiabaticity) nor the angular frequency ω_n (no frequency shift) are modified, the acoustic energy is unchanged; to move the piston, the operator has of course to take account of the quasistatic variation of the instantaneous equilibrium pressure $P_{\text{eq}}(t)$ of the fluid:

$$P(x_0,t) - P_0 = -\kappa_1 \frac{\partial u}{\partial x_0} = -\kappa_1 \left(\frac{Vt}{L_0} + \frac{\partial w}{\partial x_0} \right) \quad \rightsquigarrow \quad P(x_0,t) - P_{\text{eq}}(t) = -\kappa_1 \frac{\partial w}{\partial x_0} \tag{14a}$$

with

$$P_{\text{eq}}(t) = P_0 - \kappa_1 \frac{Vt}{L_0}, \tag{14b}$$

but he has no extrawork to supply, associated with the acoustic wave itself. In this sense, there is no radiation pressure corresponding to this acoustic wave. Before leaving this subsection, we should underline the following point: although equation (13) is exact whatever the value of velocity V, the Ehrenfest adiabaticity is required during the initial acceleration of the piston (from $V = 0$ to its cruising speed). Of course, the ideal fluid assumed in the calculations of this subsection does not exist. Nervertheless, the propagation of longitudinal expansion/compression waves through a mass-distributed spring (as those designed as decorative objects or toys for children for instance) is well described by the above equations of this subsection II A. In the next subsection, we consider a more realistic approximation of the thermodynamic relation (7), better adapted to real fluids.

B. Taking nonlinearity into account

Let us now approximate (7) by

$$P(x_0,t) - P_0 = -\kappa_1 \left(\frac{\partial u}{\partial x_0} \right) + \frac{1}{2} \kappa_2 \left(\frac{\partial u}{\partial x_0} \right)^2. \tag{15}$$

Taking account of the above nonlinearity ($\kappa_2 \neq 0$) involves the following consequence. Suppose that we move the right piston (labelled "$x_0 = L_0$") by an amount δL, and that we consider the motion of the fluid with respect to this new equilibrium position. Then we have, substituting δL for Vt in (12),

$$u(x_0,t) = \frac{x_0}{L_0} \delta L + w(x_0,t), \tag{16a}$$

so that (15) becomes

$$P(x_0,t) - P_{\text{eq}} = -\kappa_1' \left(\frac{\partial w}{\partial x_0} \right) + \frac{1}{2} \kappa_2 \left(\frac{\partial w}{\partial x_0} \right)^2, \tag{16b}$$

with

$$P_{\text{eq}} = P_0 - \kappa_1 \frac{\delta L}{L_0} + \frac{1}{2} \kappa_2 \left(\frac{\delta L}{L_0} \right)^2, \tag{16c}$$

$$\kappa_1' = \kappa_1 \left(1 - \frac{\kappa_2}{\kappa_1} \frac{\delta L}{L_0} \right). \tag{16d}$$

The above relations deserve the following comments: (i) the extrapressure-to-expansion factor thermodynamic relation keeps the same form; (ii) the equilibrium pressure is modified, as was already observed in the linear case ((16c) is but the generalization of (14b) with Vt substituted by δL); (iii) the *linear* compressibility coefficient κ_1 is changed in κ_1', due to the nonzero value of κ_2: in this change of the compressibility originates a frequency shift when moving piston L_0, as well as an acoustic radiation pressure. We will show below that frequency shift and radiation pressure are utterly entangled and, as it were, *consubstantial*.

How can we solve for $u(x_0,t)$ in the nonlinear propagation equation (8)? Unless an analytical solution can be found, a good approach is the perturbative method, provided that the condition $|\frac{\partial u}{\partial x_0}| \ll 1$ is fulfilled. For the sake of simplicity, let us start from an eigenmode of the linearized wave equation, say (see (10a))

$$u^{(1)}(x_0,t) = A \sin(kx_0) \cos(\omega t - \varphi), \tag{17a}$$

where we have deliberately ignored the eigenmode index n. Of course, $u^{(1)}(x_0,t)$ is not a solution of the full (*i.e.* nonlinearized) wave equation, but the solution having $u^{(1)}$ as linear approximation can be expanded in increasing powers of amplitude A:

$$u(x_0,t) = u^{(1)}(x_0,t) + u^{(2)}(x_0,t) + \cdots . \tag{17b}$$

Let us look for $u^{(2)}(x_0,t)$. Using (8) and (9b), we have

$$\frac{1}{c^2}\frac{\partial^2 u^{(2)}}{\partial t^2} - \frac{\partial^2 u^{(2)}}{\partial x_0^2} = -\frac{\kappa_2}{\kappa_1}\frac{\partial^2 u^{(1)}}{\partial x_0^2}\frac{\partial u^{(1)}}{\partial x_0}. \tag{18a}$$

The above equation means that the order-1 solution $u^{(1)}$ acts like a source-term with respect to the order-2 displacement $u^{(2)}$. Using (17a) we get this source-term:

$$-\frac{\kappa_2}{\kappa_1}\frac{\partial^2 u^{(1)}}{\partial x_0^2}\frac{\partial u^{(1)}}{\partial x_0} = \frac{\kappa_2}{\kappa_1}\frac{A^2 k^3}{2}\sin(2k_0x_0)\frac{1+\cos(2\omega t - 2\varphi)}{2}, \tag{18b}$$

which implies that solution $u^{(2)}$ is the sum of two contributions: one *static* and one oscillating at the angular frequency 2ω. Let us focus on the former contribution, which is, accounting for the boudary conditions ($u^{(2)}(x_0 = 0,t) = u^{(2)}(x_0 = L_0,t) = 0$),

$$u_{\mathrm{s}}^{(2)}(x_0) = \frac{\kappa_2}{\kappa_1}\frac{A^2 k}{16}\sin(2kx_0) \tag{19}$$

(index "s" for "static").

Now, it is interesting to calculate the static extrapressure $P^{[2]} - P_0$ associated with the acoustic mode, up to the second order in amplitude A. Using (15), we get

$$P^{[2]}(x_0,t) - P_0 = -\kappa_1\left(\frac{\partial u^{(1)}}{\partial x_0} + \frac{\partial u^{(2)}}{\partial x_0}\right) + \frac{1}{2}\kappa_2\left(\frac{\partial u^{(1)}}{\partial x_0}\right)^2. \tag{20}$$

As well known, the order-1 extrapressure term $-\kappa_1\frac{\partial u^{(1)}}{\partial x_0}$ oscillates at the angular frequency ω, and consequently averages to zero with time. The order-2 extrapressure term is the sum of a static (P_{s}) and a 2ω-oscillating contribution. Focusing, as above, on the former contribution, we find, all calculations carried out,

$$P_{\mathrm{s}} - P_0 = \frac{1}{8}\kappa_2 A^2 k^2. \tag{21}$$

The above result (21) deserves a few comments. First, it should be noticed that the extrapressure $P_{\mathrm{s}} - P_0$ is *homogeneous* (P_{s} does not depend on x_0), as expected for a static term. This static extrapressure is known as the Rayleigh radiation pressure[7,8]. It is noteworthy too that $P_{\mathrm{s}} - P_0$ is proportional to coefficient κ_2, and thus originates *exclusively* in the nonlinearity of the thermodynamic relation (15): this is the reason why we could not find such an extrapressure in the linear framework of our foregoing subsection II A. In this respect, it is interesting to link the Rayleigh radiation pressure and the (Lagrangian) acoustic energy density $\mathcal{E}$ that can be derived from (11d):

$$\mathcal{E} = \frac{E}{SL_0} = \frac{1}{4}\rho_0 A^2\omega^2 = \frac{1}{4}\kappa_1 A^2 k^2, \tag{22a}$$

so that, according to (21),

$$P_{\mathrm{s}} - P_0 = \frac{1}{2}\frac{\kappa_2}{\kappa_1}\mathcal{E}. \tag{22b}$$

As a last comment about result (21), we would show how radiation pressure and frequency shift are deeply entangled. With this aim, let us consider again the thought experiment we discussed above in subsection II A, *i.e.* let us move slowly the piston at the end labelled "$x_0 = L_0$". Performing on the displacement u the splitting displayed in (12), we are led to modify equations (14a-b) according to (16b-c), *i.e.*

$$P(x_0,t) - P_{\mathrm{eq}}(t) = -\kappa_1\left(1 - \frac{\kappa_2}{\kappa_1}\frac{Vt}{L_0}\right)\frac{\partial w}{\partial x_0} + \frac{1}{2}\kappa_2\left(\frac{\partial w}{\partial x_0}\right)^2, \tag{23a}$$

with

$$P_{\rm eq}(t) = P_0 - \kappa_1 \frac{Vt}{L_0} + \frac{1}{2}\left(\frac{Vt}{L_0}\right)^2. \tag{23b}$$

As already mentioned, the important issue is that, due to the *nonlinear* term in the right-hand side of (15) ($\kappa_2 \neq 0$), the *linear* term $-\kappa_1 \frac{\partial w}{\partial x_0}$ is changed in $-\kappa_1' \frac{\partial w}{\partial x_0}$, due to the variation $\delta L = Vt$ of the length of the cavity:

$$\kappa_1' = \kappa_1 \left(1 - \frac{\kappa_2}{\kappa_1}\frac{Vt}{L_0}\right), \tag{24}$$

Consequently, the wave equation ruling w becomes

$$\rho_0 \frac{\partial^2 w}{\partial t^2} = \kappa_1' \frac{\partial^2 w}{\partial x_0^2}\left(1 - \frac{\kappa_2}{\kappa_1}\frac{\partial w}{\partial x_0}\right), \tag{25}$$

i.e. the *same* equation as for u except that κ_1' should be substituted for κ_1 (compare for instance with (8)). Linearizing the above equation (25), we find a wave equation with the *modified* (see (9b)) sound velocity c' given by

$$c'^2 = \frac{\kappa_1'}{\rho_0} = c^2 \left(1 - \frac{\kappa_2}{\kappa_1}\frac{Vt}{L_0}\right). \tag{26a}$$

The above modification of the sound velocity, associated with an unchanged[19] wavevector $k_n = \frac{n\pi}{L_0}$, entails a change in the angular frequency:

$$\omega'^2 = \omega^2 \left(1 - \frac{\kappa_2}{\kappa_1}\frac{Vt}{L_0}\right). \tag{26b}$$

Observe by the way that the amplitude A of mode n is changed too. Nevertheless, since the piston is moved adiabatically (in the Ehrenfest sense), we have (see (11d) and the discussion thereafter)

$$A'^2 \omega' = A^2 \omega. \tag{26c}$$

As a consequence, the change in the acoustic energy of the wave is

$$\mathrm{d}E = \frac{1}{4}\rho_0 S L_0 A^2 \omega \delta\omega, \tag{27a}$$

or, using (26b), (9b) and (10b),

$$\mathrm{d}E = -\frac{1}{8}\kappa_2 A^2 k^2 S \delta L. \tag{27b}$$

As shown by the above equation, $\mathrm{d}E$ is therefore the work the operator has to supply to vary the volume of the cavity by an amount $\delta\mathcal{V} = S\delta L$, the (acoustic) pressure reigning in the cavity being $\frac{1}{8}\kappa_2 A^2 k^2$, *i.e.* precisely the extrapressure found in (21).

We would end this outline of the Lagrangian picture in the one-dimension case with a further remark. In the present section II, we have considered rigid boundary conditions, namely both pistons were fixed (or moved with a velocity imposed by the operator as concerns piston L_0). One may wonder how our results would be modified if, say, piston L_0 was not fixed, but imposed the external pressure P_0. Then the boundary condition on the fluid slice labelled L_0 would no longer be $u(x_0 = L_0) = 0$ ($\forall t$), but rather $P(x_0 = L_0) = P_0$. Of course equations (5a) through (10a) would hold unchanged, whereas the wavevector quantification relation (10b) would become $k_n = \left(n + \frac{1}{2}\right)\frac{\pi}{L_0}$. As concerns the energy balance, (11a) and (11b) would be unchanged while energy E should be substituted by the (conserved) quantity $E + P_0 S u(L_0, t)$ in (11c) and (11d). The shift δL of the boundary at $x_0 = L_0$ would be obtained by setting $P_0 = P_{\rm eq}(\delta L) + \frac{1}{8}\kappa_2 A^2 k^2$, as suggested by (21), with $P_{\rm eq}(\delta L)$ given by (16c). Observe by the way that such a shift of the overall length of the system would occur all the same if the acoustic vibration modes of the cavity, instead of being coherently activated, were thermally excited: the well known Grüneisen thermal expansion[9] would then be recovered. To sum up, save for the above slight necessary adaptations, our conclusions about the acoustic radiation pressure are unchanged: the latter does not depend on the boundary conditions we choose to implement our thought experiment.

In the next section, we sketch some geometric elements useful in the more general three-dimension problem. These notions, which involve curvilinear coordinates and tensorial notations, are not strictly speaking indispendable to grasp the true substance of the 3D generalization we propose in section IV, but they make the reading of this section more comfortable. We leave it to the reader either to work his way through the following section III, or to overfly it and just catch its notations.

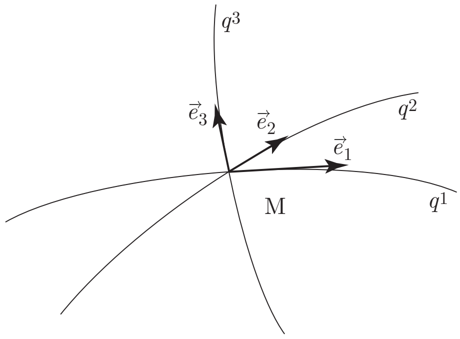

FIG. 3: The most general curvilinear coordinate system. The local basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ ($\vec{e}_i = \partial\vec{\mathrm{M}}/\partial q^i$) is *a priori* neither orthogonal nor normalized.

III. MANAGING CURVILINEAR COORDINATES

A. Coordinate bases

Let us consider now the most general Euclidean 3D reference frame. As displayed in figure 3, this frame is regarded as a set of geometrical points M. These points are labelled using a continuous set of curvilinear coordinates denoted (q^1, q^2, q^3). Note the upper position of the indices. The most familiar parametrization is the Cartesian one: $q^1 = x$, $q^2 = y$, $q^3 = z$. But we can also, for instance, use the spherical coordinate system: $q^1 = r$, $q^2 = \theta$, $q^3 = \varphi$; observe that, in the latter case, all coordinates have not the same dimension: r is a length, whereas θ and φ are angles. Now, let us fix two coordinates, say q^2 and q^3, and let q^1 vary: point M describes a so-called coordinate line, labelled by the couple (q^2, q^3). We can as well fix q^1 and q^2 and let q^3 vary or fix q^1 and q^3 and let q^2 vary: each point M is at the intersection of three coordinate lines. In the Cartesian coordinate system, the coordinate lines are (infinite) straight lines. In the spherical coordinate system, the (θ, φ) coordinate lines (r varying from 0 to infinity) are (semi-infinite) straight lines, the (r, θ) coordinates lines (φ varying from 0 to 2π) are circles – "parallels" in the terrestrian geographic description (constant latitude)– and the (r, φ) coordinate lines (θ varying from 0 to π) are semicircles – "meridians" (constant longitude). At each point of the space, one can define the set of vectors ($i = 1, 2, 3$)

$$\vec{e}_i = \frac{\partial\vec{\mathrm{M}}}{\partial q^i}, \tag{28}$$

where (as for any usual partial derivative), the derivation with respect to coordinate q^i is understood with the other two coordinates fixed. Note the lower position of index i in $\vec{e}_i$, which corresponds to the upper position of this index in the (symbolic) *denominator* ∂q^i. As a consequence of the above definition, each vector $\vec{e}_i$ is tangent to its corresponding coordinate line. In the Cartesian case, the $\vec{e}_i$ (*i.e.* $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$) are unit (dimensionless) vectors. This is no longer true with other curvilinear coordinate systems. For instance, in the spherical coordinate system, vector $\vec{e}_1 = \frac{\partial\vec{\mathrm{M}}}{\partial r}$ is a unit vector, but $\vec{e}_2 = \frac{\partial\vec{\mathrm{M}}}{\partial\theta}$ and $\vec{e}_3 = \frac{\partial\vec{\mathrm{M}}}{\partial\varphi}$ are not (length units). Extending the usual mathematical definitions of linear algebra, the set of (linearly independent) vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is commonly called a "coordinate basis". Observe that (except for the Cartesian case) the coordinate basis is *local*, in the sense that it varies from one point of the space to another. Using definition (28), the infinitesimal (vectorial) element from point $\mathrm{M}(q^1, q^2, q^3)$ to point $\mathrm{M}'(q^1 + \mathrm{d}q^1, q^2 + \mathrm{d}q^2, q^3 + \mathrm{d}q^3)$ is

$$\mathrm{d}\vec{\mathrm{M}} = \sum_{i=1}^{3} \vec{e}_i \mathrm{d}q^i. \tag{29a}$$

The $\mathrm{d}q^i$ are thus the component of vector $\mathrm{d}\vec{\mathrm{M}}$ on basis $\{\vec{e}_i\}$. In order to lighten formulas, and falling into step with a widely admitted (and applied) convention, we shall henceforth omit the $\sum_{i=1}^{3}$ summation symbol whenever a so-called "dumb" index is repeated in an expression, provided that it appears *once* in a lower position, *once* in an upper position. Thus the above equation (29a) will be simply written

$$\mathrm{d}\vec{\mathrm{M}} = \vec{e}_i \mathrm{d}q^i = \vec{e}_j \mathrm{d}q^j = \vec{e}_k \mathrm{d}q^k \ldots, \tag{29b}$$

the summation over i (or j, or k) being by convention understood.

It is noteworthy that, if we decide to change our coordinate system, say from (q^1, q^2, q^3) to (p^1, p^2, p^3), then the coordinate basis $\{\vec{e}_i\}$ will be changed in a new coordinate basis $\{\vec{f}_j\}$ according to

$$\vec{f}_j = \frac{\partial \overrightarrow{\mathrm{M}}}{\partial p^j} = \frac{\partial \overrightarrow{\mathrm{M}}}{\partial q^i}\frac{\partial q^i}{\partial p^j} = \vec{e}_i \Lambda^i_j, \tag{30}$$

where Λ is the so-called passage matrix, *i.e.* the matrix allowing to pass from the ancient basis to the new one (Λ^i_j is the component of the new basis vector $\vec{f}_j$ along the ancient basis vector $\vec{e}_i$). Writing the infinitisimal element $\mathrm{d}\overrightarrow{\mathrm{M}}$ again, we get, using (30),

$$\mathrm{d}\overrightarrow{\mathrm{M}} = \vec{f}_j \mathrm{d}p^j = \vec{e}_i \Lambda^i_j \mathrm{d}p^j = \vec{e}_i \mathrm{d}q^i, \tag{31}$$

which shows that the ancient components $\mathrm{d}q^i$ are expressed as a function of the new ones $\mathrm{d}p^j$ by means of Λ ($\mathrm{d}q^i = \Lambda^i_j \mathrm{d}p^j$). Expressing the $\mathrm{d}p^j$ as a function of the $\mathrm{d}q^i$ requires the *inverse* passage matrix Λ^{-1} ($\mathrm{d}p^j = (\Lambda^{-1})^j_i \mathrm{d}q^i$), hence the adjective "*contra*variant" coined to qualify the (usual) components of a vector on a basis.

B. Tensors

Now, in order to be as complete as possible, we should do some further linear algebra, and introduce notions about dual ($\mathcal{E}^*$) and bidual ($\mathcal{E}^{**}$) spaces of a given vector space $\mathcal{E}$. In addition to the fact that this presentation would provide a satisfactory mathematical frame, it would naturally lead to the notion of tensor. Unfortunately, any attempt to implement properly this programme invariably results in doubling the length of the present section, which should remain but a mathematical insert and should not overshadow our main goal, *i.e.* promoting the use of the Lagrangian picture. We consequently ask the reader either to admit part of the hereafter recalled results or to refer to some textbook[10] about tensorial calculus in order to find the proofs he needs. *Grosso modo* the reader is supposed to know that, being given any K-vector space $\mathcal{E}$ (K is the scalar corps; it will be $\mathbb{R}$ in the present paper), the set of all the linear mappings $\overleftarrow{X}$ (note the direction of the arrow) of $\mathcal{E}$ onto K, the so-called 1-forms, is itself a K-vector space, called $\mathcal{E}$'s dual and generally denoted $\mathcal{E}^*$. The 1-forms are often called "covectors". The same process can be iterated, and one defines $\mathcal{E}^{**}$ as the set of all the linear mappings of $\mathcal{E}^*$ onto K. One then easily shows that there exists a canonical isomorphism between $\mathcal{E}$ and its bidual $\mathcal{E}^{**}$ ("canonical" should here be understood in the sense of "intrinsic", *i.e.* utterly independent of any choice of basis). In other words, allowing for this biunivocal correspondence, $\mathcal{E}$ and $\mathcal{E}^{**}$ should be considered as identical. The vectors of $\mathcal{E}$ (and consequently $\mathcal{E}^{**}$) are called *contravariant* rank-1 tensors, prefix "contra" referring to the way their components transform under a basis change, as explained above (see (31)). The covectors (*i.e.* the vectors of $\mathcal{E}^*$) are called *covariant* rank-1 tensors, prefix "co" originating also in the way their components transform under a basis change in $\mathcal{E}$. It is then noteworthy that vectors and covectors play symmetrical roles, so that the former can be regarded as linear mappings (1-forms) of $\mathcal{E}^*$ onto K. Furthermore, the above considerations can be extended to multilinear mappings of ($\mathcal{E}$ or $\mathcal{E}^*$) $\times$ ($\mathcal{E}$ or $\mathcal{E}^*$) $\times \cdots$ onto K, thus defining rank-n tensors. An important exemple of bilinear form is introduced in the next subsection.

C. The metric tensor

So far, we have not equipped $\mathcal{E}$ with a dot product. Let us do it now. Our vector space $\mathcal{E}$ becomes pre-Hilbertian. Since it is the purpose of the present paper, let us focus on the case of the 3D-vector space $\mathcal{E}$ associated with our everyday life geometric space $\mathbb{R}^3$, and equip $\mathcal{E}$ with the usual Euclidean dot product. We then introduce the nondegenerate symmetrical bilinear form $\overleftarrow{\overleftarrow{g}}$:

$$\overleftarrow{\overleftarrow{g}}(\vec{u}, \vec{v}) = \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}, \qquad (\vec{u}, \vec{v}) \in \mathcal{E} \times \mathcal{E}. \tag{32}$$

(According to the terminology introduced in the foregoing subsection III B, $\overleftarrow{\overleftarrow{g}}$ is, by definition, a covariant rank-2 tensor.) Now, let us choice a basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, and build the three vectors (note the upper indices)

$$\vec{e}^{\,1} = \frac{\vec{e}_2 \wedge \vec{e}_3}{\mathcal{V}}, \quad \vec{e}^{\,2} = \frac{\vec{e}_3 \wedge \vec{e}_1}{\mathcal{V}}, \quad \vec{e}^{\,3} = \frac{\vec{e}_1 \wedge \vec{e}_2}{\mathcal{V}}, \tag{33a}$$

where

$$\mathcal{V} = (\vec{e}_1, \vec{e}_2, \vec{e}_3) \tag{33b}$$

is the mixed product of the basis vectors $\vec{e}_i$. Basis $\{\vec{e}^1, \vec{e}^2, \vec{e}^3\}$ will henceforth be called the cobasis[20] of basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. It is easy to check that

$$\vec{e}^k \cdot \vec{e}_i = \delta_i^k, \tag{34a}$$

$$(\vec{e}^1, \vec{e}^2, \vec{e}^3) = \frac{1}{\mathcal{V}}, \tag{34b}$$

$$\mathcal{V}\vec{e}^2 \wedge \vec{e}^3 = \vec{e}_1, \quad \mathcal{V}\vec{e}^3 \wedge \vec{e}^1 = \vec{e}_2, \quad \mathcal{V}\vec{e}^1 \wedge \vec{e}^2 = \vec{e}_3. \tag{34c}$$

Note, by the way, that the above relations show that the co-cobasis of basis $\{\vec{e}_i\}$ (*i.e.* the cobasis of basis $\{\vec{e}^k\}$) is basis $\{\vec{e}_i\}$ itself. This is of course a consequence of the afore-mentioned canonical isomorphism between $\mathcal{E}$ and $\mathcal{E}^{**}$. Furthermore, it can be shown that under the basis change $\{\vec{e}_i\} \rightsquigarrow \{\vec{f}_j\}$ implemented by means of the passage matrix Λ (see (30)), the cobasis $\{\vec{e}^k\}$ is changed into $\{\vec{f}^\ell\}$ with passage matrix Λ^{-1}. For this reason, for a given vector $\vec{V}$, we will call "contravariant", and note with upper indices, its components on basis $\{\vec{e}_i\}$ and "covariant" its components on basis $\{\vec{e}^k\}$, noted with lower indices. This point deserves the following development. Let us set

$$\overleftarrow{g}(\vec{e}_i, \vec{e}_j) = \vec{e}_i \cdot \vec{e}_j = g_{ij}. \tag{35}$$

The metric coefficients g_{ij} are a precious tool to lower or to lift tensorial indices, as explained below. Let us begin with the basis vectors themselves and expand the $\vec{e}_i$ on the cobasis $\{\vec{e}^k\}$:

$$\vec{e}_i = c_{ik}\vec{e}^k. \tag{36a}$$

Multiplying both hands by $\vec{e}_j$ (dot product), we get

$$\vec{e}_i \cdot \vec{e}_j = c_{ik}\vec{e}^k \cdot \vec{e}_j = c_{ik}\delta_j^k = c_{ij} = g_{ij}, \tag{36b}$$

so that (36a) reads *in fine*

$$\vec{e}_i = g_{ik}\vec{e}^k. \tag{36c}$$

The metric coefficients matrix (g_{ik}) therefore allows to expand the basis vectors on the cobasis. *A contrario*, expanding the cobasis vectors on basis $\{\vec{e}_i\}$:

$$\vec{e}^k = d^{ki}\vec{e}_i, \tag{37a}$$

multiplying both hands by $\vec{e}^\ell$ and setting

$$\vec{e}^k \cdot \vec{e}^\ell = g^{k\ell}, \tag{37b}$$

we get *in fine*

$$\vec{e}^k = g^{ki}\vec{e}_i, \tag{37c}$$

which means that (g^{ik}) is the passage matrix from basis $\{\vec{e}_i\}$ to its cobasis $\{\vec{e}^k\}$. Comparing (36c) and (37c), we observe that matrix (g^{ik}) is the *inverse* of matrix (g_{ik}). The latter matrices act like "index lifts" with respect to the basis and cobasis vectors. As a consequence, they act in like manner with tensorial indices. Let us indeed consider any vector $\vec{V}$, and let us expand it on basis $\{\vec{e}_i\}$:

$$\vec{V} = V^i\vec{e}_i. \tag{38a}$$

Using (36c), we get

$$\vec{V} = V^i g_{ik}\vec{e}^k = V_k\vec{e}^k. \tag{38b}$$

In the above expansion, the V^i are the *contravariant* components of $\vec{V}$ (upper index i), whereas $V_k = g_{ik}V^i$ are its *covariant* components (lower index k). Applying result (38a-b), we obtain

$$V^i = \vec{V} \cdot \vec{e}^i, \qquad V_k = \vec{V} \cdot \vec{e}_k. \tag{39}$$

The above expressions can be easily extended to higher-rank tensors. In the same connection, the metric coefficients allow various expressions for vectors products. Let us consider two vectors $\vec{U}$ and $\vec{V}$. Their dot product can be written

$$\vec{U}\cdot\vec{V} = g_{ij}U^iV^j = g^{ij}U_iV_j = U^iV_j = U_iV^j, \tag{40a}$$

and their wedge product

$$\vec{U}\wedge\vec{V} = \mathcal{V}\varepsilon_{ijk}U^iV^j\vec{e}^{\,k} = \frac{1}{\mathcal{V}}\varepsilon^{ijk}U_iV_j\vec{e}_k, \tag{40b}$$

where $\mathcal{V}$ is the elementary volume introduced in (33b) and $\varepsilon_{ijk}(=\varepsilon^{ijk})$ is the Levi-Civita symbol ($\varepsilon_{ijk} = +1$ if (i,j,k) is in clockwise order, $\varepsilon_{ijk} = -1$ if (i,j,k) is in anticlockwise order, $\varepsilon_{ijk} = 0$ if two indices are equal).

Before ending this quick overfly of the properties of the metric tensor, we would point out a useful relation between the elementary volume $\mathcal{V}$ and the determinant $\det(g_{ij})$ of matrix (g_{ij}). To derive it, let us consider three vectors $\vec{u}$, $\vec{v}$, $\vec{w}$. Their mixed product can be written

$$(\vec{u},\vec{v},\vec{w}) = \mathcal{V}\times\begin{vmatrix} u^1 & v^1 & w^1\\ u^2 & v^2 & w^2\\ u^3 & v^3 & w^3\end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}\times\frac{1}{\mathcal{V}}, \tag{41a}$$

hence

$$(\vec{u},\vec{v},\vec{w})^2 = \begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}\begin{vmatrix} u^1 & v^1 & w^1\\ u^2 & v^2 & w^2\\ u^3 & v^3 & w^3\end{vmatrix} = \begin{vmatrix} \vec{u}^{\,2} & \vec{u}\cdot\vec{v} & \vec{u}\cdot\vec{w}\\ \vec{v}\cdot\vec{u} & \vec{v}^{\,2} & \vec{c}\cdot\vec{w}\\ \vec{w}\cdot\vec{u} & \vec{w}\cdot\vec{v} & \vec{w}^{\,2}\end{vmatrix}. \tag{41b}$$

Applying (41b) in the particular case $(\vec{u}=\vec{e}_1, \vec{v}=\vec{e}_2, \vec{w}=\vec{e}_3)$, we get

$$\mathcal{V}^2 = (\vec{e}_1,\vec{e}_2,\vec{e}_3)^2 = \begin{vmatrix} g_{11} & g_{12} & g_{13}\\ g_{21} & g_{22} & g_{23}\\ g_{31} & g_{32} & g_{33}\end{vmatrix} = \det(g_{ij}). \tag{41c}$$

The fact that $\det(g_{ij})$ should be *positive* is linked to the *Euclidean* nature of the metric. Another type of metric would result in another sign for $\det(g_{ij})$. For instance, in the 4D spacetime of GR the $g_{\mu\nu}$ metric is locally Minkowskian, yielding a *negative* value for $\det(g_{\mu\nu})$.

D. Covariant derivation

Our brief overflight of the features of the curvilinear coordinates systems would be incomplete without a few words about the covariant derivation. The broad lines of this issue can be summarized as follows.

When using non Cartesian coordinates systems, the space-derivation of "scalar" fields raises no particular difficulty. On the other hand, when deriving "vector" or more generally "tensor" fields, one should take into acount not only the space-dependence of the components, but also that of the basis vectors themselves. There exists an elegant trick to do it. Let us define

$$\partial_i\vec{e}_j = \frac{\partial^2\overrightarrow{\mathrm{M}}}{\partial q^i\partial q^j} = \mathit{\Gamma}^k_{ij}\vec{e}_k, \tag{42}$$

where the (space-dependent) coefficients $\mathit{\Gamma}^k_{ij} = \mathit{\Gamma}^k_{ji}$ are known as the Christoffel symbols. They can be derived from the metric tensor. Considering a vector fileld $\vec{A}$, we have consequently

$$\partial_i\vec{A} = \partial_i(\vec{e}_jA^j) = \partial_iA^j\,\vec{e}_j + A^j\,\mathit{\Gamma}^k_{ij}\vec{e}_k. \tag{43a}$$

Permuting dumb indices k and j in the second term of the right-hand side of the above equation, we get

$$\partial_i\vec{A} = (\partial_iA^j + A^k\mathit{\Gamma}^j_{ik})\vec{e}_j = \vec{e}_j\mathrm{D}_iA^j, \tag{43b}$$

where

$$\mathrm{D}_iA^j = \partial_iA^j + A^k\mathit{\Gamma}^j_{ik} \tag{43c}$$

is the so-called "covariant derivative" of A^j. Conservely, using (34a) and (42), we have

$$0 = \partial_i(\vec{e}^j \cdot \vec{e}_k) = (\partial_i \vec{e}^j) \cdot \vec{e}_k + \vec{e}^j \cdot \Gamma^\ell_{ik}\vec{e}_l = (\partial_i \vec{e}^j) \cdot \vec{e}_k + \Gamma^j_{ik}, \tag{44a}$$

hence

$$\partial_i \vec{e}^j = -\,\Gamma^j_{ik}\vec{e}^k. \tag{44b}$$

Considering the same vector field $\vec{A}$ as above, but now expanded on the cobasis $\{\vec{e}^j\}$, we have

$$\partial_i \vec{A} = \partial_i(A_j \vec{e}^j) = \partial_i A_j\, \vec{e}^j - A_j\, \Gamma^j_{ik}\vec{e}^k. \tag{45a}$$

Permuting again dumb indices k and j in the second term of the right-hand side of the above equation, we are left with

$$\partial_i \vec{A} = (\partial_i A_j - A_k \Gamma^k_{ij})\vec{e}^j = (\mathrm{D}_i A_j)\vec{e}^j, \tag{45b}$$

where

$$D_i A_j = \partial_i A_j - A_k \Gamma^k_{ij}. \tag{45c}$$

As a conclusion, when using curvilinear coordinate bases, vector fields can be derived exactly as in Cartesian coordinate systems, provided that ordinary derivatives should be substituted by covariant derivatives. Observe the latter bring a Christoffel symbol with a plus sign when deriving contravariant components, and a minus sign when deriving covariant components. The above formulas (43a-b-c) and (45a-b-c) can be extended to any higher-order tensorial fields.

E. Vector analysis

Among the mathematical tools commonly used in electrodynamics as well as in fluid dynamics, the gradient, the curl and the divergence occupy an outstanding place. It is therefore important to be able to write them in any coordinate system. In the Cartesian coordinate system, a handy mnemonics is provided by the so-called "nabla vector"

$$\vec{\nabla} = \vec{e}_x \partial_x + \vec{e}_y \partial_y + \vec{e}_z \partial_z, \tag{46}$$

thanks to which the above three operators can be symbolically written

$$\overrightarrow{\mathrm{grad}}\, V = \vec{\nabla} V, \quad \vec{\mathrm{curl}}\, \vec{A} = \vec{\nabla} \wedge \vec{A}, \quad \mathrm{div}\, \vec{A} = \vec{\nabla} \cdot \vec{A}. \tag{47}$$

Unfortunately, the above mnemonics fails to provide the correct expression for the gradient, the curl and the divergence in other curvilinear coordinate systems. The aim of the present section is to propose another mnemonics, hardly more sophisticated than (46)-(47), which works in every case.

1. The gradient

For a scalar field V, $\overrightarrow{\mathrm{grad}}\, V$ is the vector such that $\mathrm{d}V = \overrightarrow{\mathrm{grad}}\, V \cdot \mathrm{d}\vec{M}$, whatever the elementary displacement $\mathrm{d}\vec{M}$. Since $\mathrm{d}V = \partial_i V \mathrm{d}q^i$, and owing to (28), we have $\partial_i V \mathrm{d}q^i = \overrightarrow{\mathrm{grad}}\, V \cdot \vec{e}_i \mathrm{d}q^i\ \forall \mathrm{d}q^i$, hence

$$\overrightarrow{\mathrm{grad}}\, V \cdot \vec{e}_i = \partial_i V \quad \rightsquigarrow \quad \overrightarrow{\mathrm{grad}}\, V = (\partial_i V)\vec{e}^i, \tag{48}$$

which means that the $\partial_i V$ are the *covariant* components of $\overrightarrow{\mathrm{grad}}\, V$. Observe that, for any "scalar" field, $\mathrm{D}_i V$ (covariant derivative) can be substituted for $\partial_i V$ in the above equation (48).

2. The curl

The curl of a vector field $\vec{A}$ is the vector $\vec{\mathrm{curl}}\, \vec{A}$ such that the circulation of $\vec{A}$ along a loop should be equal to the flux of $\vec{\mathrm{curl}}\, \vec{A}$ through any surface leaning upon this loop (Stokes's theorem). Let us choose, for instance, an elementary loop generated by vectors $\vec{e}_1 \mathrm{d}q^1$ and $\vec{e}_2 \mathrm{d}q^2$ (see figure 4). Our loop looks like a parallelogram the surface of which is

$$\vec{e}_1 \mathrm{d}q^1 \wedge \vec{e}_2 \mathrm{d}q^2 = \mathcal{V} \vec{e}^3 \mathrm{d}q^1 \mathrm{d}q^2. \tag{49}$$

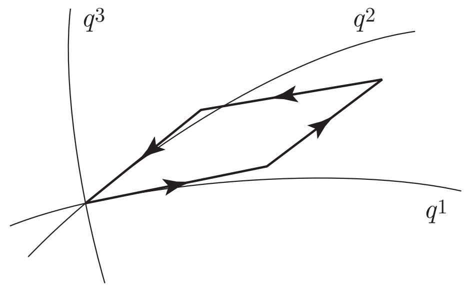

FIG. 4: Elementary loop generated by vectors $\vec{e}_1 \mathrm{d}q^1$ and $\vec{e}_2 \mathrm{d}q^2$.

The circulation of $\vec{A}$ along the four legs of our loop is

$$\begin{aligned}(\vec{A}\cdot\vec{e}_2)_{q^1+\mathrm{d}q^1}\mathrm{d}q^2 - (\vec{A}\cdot\vec{e}_1)_{q^2+\mathrm{d}q^2}\mathrm{d}q^1 - (\vec{A}\cdot\vec{e}_2)_{q^1}\mathrm{d}q^2 + (\vec{A}\cdot\vec{e}_2)_{q^2}\mathrm{d}q^1 &= (\partial_1 A_2 - \partial_2 A_1)\mathrm{d}q^1\mathrm{d}q^2 \\ &= \vec{\mathrm{curl}}\,\vec{A}\cdot(\mathcal{V}\vec{e}^{\,3}\mathrm{d}q^1\mathrm{d}q^2), \end{aligned} \tag{50a}$$

hence

$$(\vec{\mathrm{curl}}\,\vec{A})\cdot\vec{e}^{\,3} = (\vec{\mathrm{curl}}\,\vec{A})^{\,3} = \frac{1}{\mathcal{V}}(\partial_1 A_2 - \partial_2 A_1), \tag{50b}$$

and more generally

$$\vec{\mathrm{curl}}\,\vec{A} = \frac{1}{\mathcal{V}}\varepsilon^{ijk}\partial_i A_j \vec{e}_k. \tag{50c}$$

We leave it to the reader to check, using (45c) and $\Gamma^k_{ij} = \Gamma^k_{ji}$, that $\mathrm{D}_i A_j$ can be substituted for $\partial_i A_j$ in the above equation (50c).

3. The divergence

The divergence of a vector field $\vec{A}$ is the scalar field $\mathrm{div}\,\vec{A}$ such that the flux of $\vec{A}$ through a closed surface should be equal to the integral of $\mathrm{div}\,\vec{A}$ over the volume inside this surface (Ostrograsky's theorem). Let us choose, for instance, the integration domain generated by the triad $\vec{e}_1\mathrm{d}q^1$, $\vec{e}_2\mathrm{d}q^2$, $\vec{e}_3\mathrm{d}q^3$. Our domain looks like a parallelepiped, the volume of which is $(\vec{e}_1\mathrm{d}q^1, \vec{e}_2\mathrm{d}q^2, \vec{e}_3\mathrm{d}q^3) = \mathcal{V}\mathrm{d}q^1\mathrm{d}q^2\mathrm{d}q^3$. The flux of $\vec{A}$ through the two opposite faces labelled q^1 and $q^1 + \mathrm{d}q^1$ is

$$\big[\vec{A}\cdot(\vec{e}_2\wedge\vec{e}_3)\big]_{q^1+\mathrm{d}q^1}\mathrm{d}q^2\mathrm{d}q^3 - \big[\vec{A}\cdot(\vec{e}_2\wedge\vec{e}_3)\big]_{q^1}\mathrm{d}q^2\mathrm{d}q^3 = \partial_1(\vec{A}\cdot\mathcal{V}\vec{e}^{\,1})\mathrm{d}q^1\mathrm{d}q^2\mathrm{d}q^3. \tag{51a}$$

The total flux through the (whole) closed surface is therefore

$$\partial_i(\vec{A}\cdot\mathcal{V}\vec{e}^{\,i})\mathrm{d}q^1\mathrm{d}q^2\mathrm{d}q^3 = (\partial_i\vec{A})\cdot\mathcal{V}\vec{e}^{\,i}\mathrm{d}q^1\mathrm{d}q^2\mathrm{d}q^3 = \mathrm{div}\,\vec{A}(\mathcal{V}\mathrm{d}q^1\mathrm{d}q^2\mathrm{d}q^3) \tag{51b}$$

(we have used the property $\partial_i(\mathcal{V}\vec{e}^{\,i}) = 0$, easily checked from (28) and (33a)), hence

$$\mathrm{div}\,\vec{A} = \frac{1}{\mathcal{V}}\partial_i(\mathcal{V}A^i) = (\partial_i\vec{A})\cdot\vec{e}^{\,i}. \tag{51c}$$

Observe that, allowing for (43b), the above equation (51c) can also be written

$$\mathrm{div}\,\vec{A} = (\vec{e}_j\mathrm{D}_i A^j)\cdot\vec{e}^{\,i} = \delta^i_j\mathrm{D}_i A^j = \mathrm{D}_i A^i. \tag{51d}$$

It is noteworthy that, in order to calcultate $\overrightarrow{\mathrm{grad}}$, $\vec{\mathrm{curl}}$ and div, the mnemonics recalled in (47) *still holds* in curvilinear coordinates systems, provided that the "nabla vector" recalled in (46) should simply be substituted by

$$\vec{\nabla} = \vec{e}^{\,i}\mathrm{D}_i. \tag{52}$$

Our quick presentation of the mathematical gear relevant to the use of curvilinear coordinates is now complete. In the next section, we shall apply it to put the 3D fluid dynamics in equations in the framework of the Lagrange picture.

IV. EXTENSION TO THE THREE-DIMENSION CASE

The aim of the present section IV is to regard again the fluid considered in section II, but now in a 3D geometry. In this respect, we label each fluid element by its position M_0 at time t_0, using curvilinear coordinates q^1, q^2, q^3.

A. Displacement, expansion and deformation

Let $\{\vec{e}_{01}, \vec{e}_{02}, \vec{e}_{03}\}$ be the coordinate basis at point M_0, *i.e.*

$$\vec{e}_{0i} = \frac{\partial \overrightarrow{\mathrm{M}_0}}{\partial q^i}. \tag{53a}$$

Setting

$$\mathcal{V}_0 = (\vec{e}_{01}, \vec{e}_{02}, \vec{e}_{03}), \tag{53b}$$

the volume of the elementary "cube" of fluid is

$$\mathrm{d}\tau_0 = \mathcal{V}_0 \ \mathrm{d}q^1 \mathrm{d}q^2 \mathrm{d}q^3 \tag{53c}$$

and its mass is $\rho_0 \mathrm{d}\tau_0$.

1. Displacement

Let now the fluid move. As illustrated in figure 5, the elementary fluid element labelled (q^1, q^2, q^3), which was located at point M_0 at time t_0, will be found at time t at point M. Vector $\overrightarrow{\mathrm{M}_0\mathrm{M}}$ is referred to as the displacement of this fluid element, and we define the (Lagrangian) displacement field $\vec{u}(q^1, q^2, q^3, t)$ (henceforth denoted $\vec{u}(q^i, t)$).

The coordinate basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ at point M is consequently

$$\vec{e}_i = \frac{\partial \overrightarrow{\mathrm{M}}}{\partial q^i} = \vec{e}_{0i} + \partial_i \vec{u}. \tag{54}$$

2. Expansion

Setting

$$\mathcal{V} = (\vec{e}_1, \vec{e}_2, \vec{e}_3) = J \mathcal{V}_0, \tag{55a}$$

the volume of the elementary cube of the fluid is now

$$\mathrm{d}\tau = \mathcal{V} \mathrm{d}q^3 = J \, \mathrm{d}\tau_0 \tag{55b}$$

Coefficient J characterizes the fluid expansion (if $J > 1$) or contraction (if $J < 1$) between times t_0 and t. Observe that $J = J(q^i, t)$ is itself a (Lagrangian) scalar field. An incompressible flow corresponds to $J = 1 \ \forall (q^i, t)$. Using the above equation (55a), one can derive an interesting relation between the expansion rate $\frac{\partial J}{\partial t}$ and the velocity field $\vec{v}(q^i, t) = \frac{\partial \vec{u}(q^i, t)}{\partial t}$. Calculating $\frac{\partial \mathcal{V}}{\partial t}$, we find, remembering (33b),

$$\begin{aligned} \frac{\partial \mathcal{V}}{\partial t} &= \left(\frac{\partial \vec{e}_1}{\partial t}, \vec{e}_2, \vec{e}_3\right) + \left(\vec{e}_1, \frac{\partial \vec{e}_2}{\partial t}, \vec{e}_3\right) + \left(\vec{e}_1, \vec{e}_2, \frac{\partial \vec{e}_1}{\partial t}\right) \\ &= \left(\partial_1 \vec{v}, \vec{e}_2, \vec{e}_3\right) + \left(\vec{e}_1, \partial_2 \vec{v}, \vec{e}_3\right) + \left(\vec{e}_1, \vec{e}_2, \partial_3 \vec{v}\right) \\ &= \mathcal{V}\big(\partial_1 \vec{v} \cdot \vec{e}^{\,1} + \partial_2 \vec{v} \cdot \vec{e}^{\,2} + \partial_3 \vec{v} \cdot \vec{e}^{\,3}\big) \\ &= \mathcal{V} \, \partial_i \vec{v} \cdot \vec{e}^{\,i}, \end{aligned} \tag{56a}$$

i.e., allowing for result (51c),

$$\frac{\partial \mathcal{V}}{\partial t} = \mathcal{V}\,\mathrm{div}\,\vec{v}, \quad \text{or equivalently} \quad \frac{\partial J}{\partial t} = J\,\mathrm{div}\,\vec{v}. \tag{56b}$$

Therefore, an incompressible flow corresponds to $\mathrm{div}\,\vec{v} = 0$, exactly like in the framework of the Eulerian picture. The above result (56b) is particularly simple. Nevertheless, we would draw the reader's attention onto the following trap to be avoided. Integrating (56b) over time is straightforward and yields

$$J(q^i, t) = J(q^i, t_0)\,\exp\left[\int_{t_0}^{t} \mathrm{d}t'\;\mathrm{div}\,\vec{v}(q^i, t')\right], \tag{57a}$$

which is correct. Nevertheless, tempting though it may be, one should *not* permute the $\partial/\partial t$ and div operators in the time-integral in the right-hand side of the above equation (57a), which would lead to the nice-but-false result $J = J_0 \exp(\mathrm{div}\,\vec{u})$. Within the Lagrangian picture indeed

$$\frac{\partial^{\mathcal{L}} \mathrm{div}\,\vec{u}}{\partial t} \neq \mathrm{div}\left(\frac{\partial^{\mathcal{L}} \vec{u}}{\partial t}\right), \tag{57b}$$

contrary to the Eulerian picture where both operators do commute.

3. Deformation

The passage from the undeformed basis $\{\vec{e}_{01}, \vec{e}_{02}, \vec{e}_{03}\}$ to the deformed basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ characterizes by itself the deformation undergone by the (fluid, under the circumstances) matter associated with the displacement field $\vec{u}$. Let us consider indeed point $\mathrm{M}_0(q^1, q^2, q^3)$ and, in the vicinity of M_0, another point M'_0 with coordinates $(q^1 + \mathrm{d}q^1, q^2 + \mathrm{d}q^2, q^3 + \mathrm{d}q^3)$. In the basis $\{\vec{e}_{0i}\}$ defined in (53a), the (infinitesimal) vector $\vec{\mathrm{M}_0\mathrm{M}'_0}$ is expanded as

$$\vec{\mathrm{M}_0\mathrm{M}'_0} = \vec{e}_{0j}\,\mathrm{d}q^j. \tag{58a}$$

After displacement, the matter element labelled (q^1, q^2, q^3) has moved from point M_0 to point M, whereas the matter element labelled $(q^1 + \mathrm{d}q^1, q^2 + \mathrm{d}q^2, q^3 + \mathrm{d}q^3)$ has moved from M'_0 to M'. The situation is illustrated in figure 5.

According with the definition (54) of basis$\{\vec{e}_i\}$, the infinitesimal vector $\vec{\mathrm{MM}'}$ is expanded as

$$\vec{\mathrm{MM}'} = \vec{e}_j\,\mathrm{d}q^j, \tag{58b}$$

where the $\mathrm{d}q^j$ are the *same* as in (58a). As a consequence of the above expansions (58a) and (58b), the correspondence between $\vec{\mathrm{M}_0\mathrm{M}'_0}$ and $\vec{\mathrm{MM}'}$ is linear:

$$\vec{\mathrm{MM}'} = f(\vec{\mathrm{M}_0\mathrm{M}'_0}), \tag{59a}$$

with

$$\vec{e}_i = f(\vec{e}_{0i}) = \vec{e}_{0i} + \partial_i \vec{u} = (\delta_i^k + D_i u_0^k)\vec{e}_{0k} = f_i^k \vec{e}_{0k}. \tag{59b}$$

B. Motion equations

Let us now come back to equation (6) and see how it can be generalized in the three-dimension case. Considering a fluid amount with (rest) volume $\int \mathcal{V}_0 \mathrm{d}^3 q$ and (rest) mass density ρ_0, Newton's Second Law reads

$$\int \mathrm{d}^3 q\, \mathcal{V}_0 \rho_0 \frac{\partial^2 \vec{u}}{\partial t^2} = \iint_{\Sigma(t)} -P\,\mathrm{d}\vec{s}, \tag{60a}$$

where $\Sigma(t)$ is the external surface of the fluid amount at time t (see figure 6), and P the actual pressure exerted onto this fluid amount by its surroundings. Using Ostrogradsky's theorem, the surface integral in the right-hand side of the above equation is changed in a volume integral

$$\iint_{\Sigma(t)} -P\,\mathrm{d}\vec{s} = \int \mathrm{d}^3 q\, \mathcal{V}(-\overrightarrow{\mathrm{grad}}\,P) \tag{60b}$$

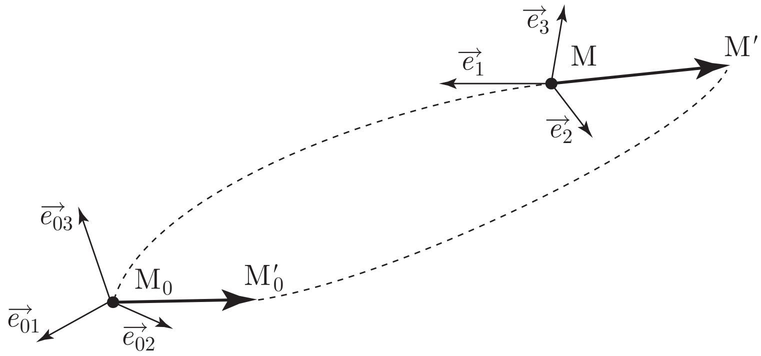

FIG. 5: Transformations associated with the displacement field $\vec{u}$: $f(\vec{e}_{0i}) = \vec{e}_i$ and $f(\mathrm{M_0\vec{M}'_0}) = \mathrm{M\vec{M}'}$

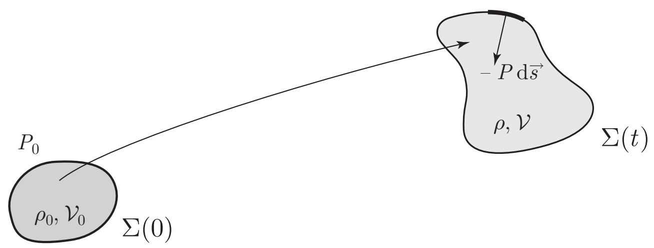

FIG. 6: An amount of fluid with rest mass density ρ_0 and wrapped in its (external) surface $\Sigma(0)$ under the (rest) pressure P_0 is displaced and deformed in the course of the motion. At time t, it undergoes the outside pressure P exerted onto its (external) surface $\Sigma(t)$.

(observe that the volume of the fluid element labelled d^3q is $\mathcal{V}\mathrm{d}^3q$ at time t, and no longer $\mathcal{V}_0\mathrm{d}^3q$ as was the case at rest). Since the above equations (60a) and (60b) hold whatever the integration volume, we are left with the local motion equation

$$\rho_0 \frac{\partial^2 \vec{u}}{\partial t^2} = -\,J\,\overrightarrow{\mathrm{grad}}\,P, \tag{60c}$$

where the superscript "$\mathcal{L}$" (for "Lagrange") is understood in the left hand-side time-derivative.

The above motion equation obtained in the Lagrangian picture is interestingly compared to the Euler equation recalled in (1). To begin with, let us observe that the fluid velocity $\vec{v}$ is nothing else than the (Lagrangian) time-derivative of the displacement $\vec{u}$. Restoring provisionally superscripts "$\mathcal{L}$" and "$\mathcal{E}$", we have indeed

$$\frac{\partial^{\mathcal{L}} \vec{u}}{\partial t} = \vec{v} \tag{61a}$$

and

$$\frac{\partial^{\mathcal{L}2} \vec{u}}{\partial t^2} = \frac{\partial^{\mathcal{L}} \vec{v}}{\partial t} = \frac{\partial^{\mathcal{E}} \vec{v}}{\partial t} + \big(\vec{v}\cdot\overrightarrow{\mathrm{grad}}\big)\vec{v}. \tag{61b}$$

Moreover, observing that the fluid mass density ρ is (as a consequence of (55a) for instance) equal to ρ_0/J, we conclude that equations (1) and (60c) are perfectly equivalent.

Interesting too is the comparison between the three-dimension motion equation (60c) and its one-dimension reduced expression (6). A (too) quick glance may instil the uncomfortable –and misleading– feeling that the J factor has disappeared from the right-hand side of (6). In fact, in the one-dimension problem considered in section II, if x_0 denotes the usual abscissa, then the vector $\vec{e}_{01} = \partial_1\vec{M}_0$ is unitary, but the vector $\vec{e}_1 = \partial_1\vec{M}$ *is not* unitary. We have indeed, owing to (54),

$$\vec{e}_1 = \partial_1\vec{M} = \vec{e}_{01} + \partial_1(u\vec{e}_{01}) = \vec{e}_{01}\left(1 + \frac{\partial u}{\partial x_0}\right). \tag{62}$$

Consequently one should be careful not confusing co- and contravariant components. Now, the $\frac{\partial P}{\partial x_0} = \partial_1 P$ term in the right-hand side of (6) is the *covariant* component of vector $\overrightarrow{\text{grad}}\, P$ on the basis vector $\vec{e}_1$. In other words, we have

$$\partial_1 P = \frac{\partial P}{\partial x_0} = \vec{e}_1 \cdot \overrightarrow{\text{grad}}\, P = J\, \vec{e}_{01} \cdot \overrightarrow{\text{grad}}\, P, \tag{63}$$

so that, since $\vec{u} = u\vec{e}_{01}$, result (6) is recovered.

We would end the present section with a last remark. Owing to (33a), (48) and (55a), the motion equation (60c) reads

$$\rho_0 \frac{\partial^2 \vec{u}}{\partial t^2} = -\frac{1}{\mathcal{V}_0} \partial_i P\, \frac{1}{2} \varepsilon^{ijk} \vec{e}_j \wedge \vec{e}_k. \tag{64a}$$

Using (59b), we get

$$\rho_0 \frac{\partial^2 \vec{u}}{\partial t^2} = -\frac{1}{2\mathcal{V}_0} \partial_i P\, \varepsilon^{ijk} f_j^\ell f_k^m \vec{e}_{0\ell} \wedge \vec{e}_{0m}. \tag{64b}$$

Projecting both hand-sides of the above equation on the basis vector $\vec{e}_{0n}$, and allowing for (53b), we are left with

$$\rho_0 \frac{\partial^2 u_{0n}}{\partial t^2} = -\frac{1}{2} \partial_i P\, \varepsilon^{ijk} f_j^\ell f_k^m \varepsilon_{\ell mn} = -\, T_n^i \partial_i P, \tag{64c}$$

(where $T_n^i = \frac{1}{2} \varepsilon^{ijk} f_j^\ell f_k^m \varepsilon_{\ell mn}$) which can be regarded as the covariant form (*i.e.* true whichever of the coordinate system $\{q^i\}$ is chosen) of the motion equation. Observe that the latter is clearly non linear. It can be linearized by substituting $f_j^\ell f_k^m$ by $\delta_j^\ell \delta_k^m$, entailing then $T_n^i = \delta_n^i$, which leads to the simple form

$$\rho_0 \frac{\partial^2 u_{0n}}{\partial t^2} = -\partial_n P. \tag{64d}$$

The above simple form is *exact* in the one-dimension case, as recalled above.

C. Energy balance in the Lagrangian picture

In the present subsection, we aim at revisiting the set of results (11a-d) and extending them in 3D geometry. Let us consider the fluid amount displayed in figure 6, and let E be its total energy. For the sake of simplicity, we assume here that there is no external field (gravity or other) acting on the fluid. Since the transformations of the latter fluid are supposed to be isentropic, the time variations of E are only due to the work of the outer pressure forces exerted upon surface $\Sigma(t)$:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \iint_{\Sigma(t)} -P\, \mathrm{d}\vec{s} \cdot \frac{\partial \vec{u}}{\partial t}, \tag{65a}$$

entailing (through Ostrogradsky's theorem)

$$\frac{\mathrm{d}E}{\mathrm{d}t} + \int \mathrm{d}^3 q\, \mathcal{V}\, \mathrm{div}\, \vec{\mathcal{G}} = 0, \tag{65b}$$

where $\vec{\mathcal{G}} = P \dfrac{\partial \vec{u}}{\partial t}$ is the so-called acoustic Poynting vector. Detailing the above energy conservation equation, we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int \mathrm{d}^3 q\, \mathcal{V} \Big[\overrightarrow{\text{grad}}\, P \cdot \frac{\partial \vec{u}}{\partial t} + P\, \mathrm{div}\, \frac{\partial \vec{u}}{\partial t} \Big]. \tag{66a}$$

Owing to (60c) we have $\mathcal{V}\, \overrightarrow{\text{grad}}\, P = -\, \rho_0 \mathcal{V}_0 \frac{\partial^2 \vec{u}}{\partial t^2}$, and owing to (56b) we have $\mathcal{V} \mathrm{div}\, \frac{\partial \vec{u}}{\partial t} = \frac{\partial \mathcal{V}}{\partial t}$, so that (66a) is equivalent to

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E_{\mathrm{k}}}{\mathrm{d}t} + \int \mathrm{d}^3 q \Big(-\, P \frac{\partial \mathcal{V}}{\partial t} \Big), \tag{66b}$$

where $E_{\rm k} = \displaystyle\int {\rm d}^3q\,\frac{1}{2}\rho_0\mathcal{V}_0\Big(\frac{\partial\vec{u}}{\partial t}\Big)^2$ is the overall kinetic energy of the fluid and $\mathcal{P}_{\rm c} = \displaystyle\int {\rm d}^3q\Big(-\,P\frac{\partial\mathcal{V}}{\partial t}\Big)$ is the overall compression power of the pressure forces undergone by the fluid. As a matter of fact, due to the absence of any thermal exchange, equation (66b) is but the expression of the First Principle of Thermodynamics. As was already the case in 1D geometry, calculating the latter compression power is a thermodynamic affair which requires the P-versus-J relation. We have indeed

$$\mathcal{P}_{\rm c} = \int {\rm d}^3q\Big(-\,P\frac{\partial\mathcal{V}}{\partial t}\Big) = \frac{\rm d}{{\rm d}t}\int {\rm d}^3q\,\mathcal{V}_0\int_1^J -\,P(J'){\rm d}J'. \tag{67}$$

In the particular case of a linear acoustic response of the type

$$p = P - P_0 = -\,\kappa_1(J-1), \tag{68a}$$

we are left, all calculations carried out, with, setting $\kappa_1 = \rho_0 c^2$,

$$E + P_0(V - V_0) = \frac{1}{2}\int {\rm d}^3q\,\rho_0\mathcal{V}_0\bigg[\Big(\frac{\partial\vec{u}}{\partial t}\Big)^2 + c^2(J-1)^2\bigg], \tag{68b}$$

where V and V_0 are the overall volume of the fluid, respectively at current time t and at rest. Observe that, in the framework of the *small* deformation approximation (68a), the expansion term $J-1$ should be substituted by ${\rm div}\,\vec{u}$. In the framework of the latter approximation, the above result (68b) thus generalizes (11c). Observe too that, in the case where the fluid undergoes an outerpressure P_0, the quantity $E + P_0(V - V_0)$ is conserved in the course of time.

D. The puslating sphere

The well known problem of the acoustic wave generated by a pulsating sphere will illustrate the convenience of the Lagrangian picture. Let us consider an infinite homogeneous elastic fluid. A pulsating sphere, with centre at the origin of a spherical coordinate system and with current radius $R(t) = R_0 + \xi(t)$, excitates spherical waves in the fluid. For the sake of simplicity, let us restrict our study to the small deformation limit. Due to the spherical symmetry of the motion, all relevant quantities (pressure, temperature) are independent of angles θ and φ; the displacement field $\vec{u}$ reads

$$\vec{u}(\vec{r},t) = u(r,t)\widehat{e}_r, \tag{69a}$$

where $\widehat{e}_r = \frac{\partial\overrightarrow{\rm M_0}}{\partial r}$ is the (normalized, under the circumstances) usual radial vector of the spherical coordinate system. Observe that, in order to lighten notations, we have deliberately omitted indices "0" and "r" in the component u_{0r} of the displacement field $\vec{u}$. Thus, using the linearized form (64d), we get the simple motion equation

$$\rho_0\frac{\partial^2 u}{\partial t^2} = -\,\frac{\partial P}{\partial r}. \tag{69b}$$

Assuming a linear p-*versus*-$(J{-}1)$ thermodynamic relation of the type (68a) and linearizing the expansion term $J-1$, we obtain the extrapressure

$$p = -\,\kappa_1{\rm div}\,\vec{u} = -\,\frac{\kappa_1}{r^2}\frac{\partial(r^2u)}{\partial r}. \tag{70a}$$

From the above equation, the d'Alembert wave equation

$$\Big(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\Big)\big(rp(r,t)\big) = 0 \tag{70b}$$

is obtained. The mathematical treatment of the above equations is well known from textbooks. The general solution is conveniently written

$$p(r,t) = \frac{1}{r}\bigg[f\Big(t - \frac{r-R_0}{c}\Big) + g\Big(t + \frac{r-R_0}{c}\Big)\bigg], \tag{71}$$

where f and g are two functions the meaning of which is obvious. If no incoming spherical wave comes from infinity, g should be taken equal to zero. Choosing the Lagrangian picture reveals its convenience *in fine* when one has to

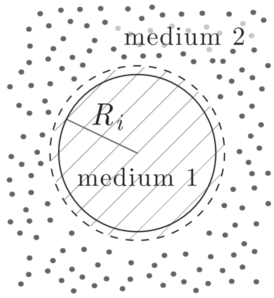

FIG. 7: Medium 1 and medium 2 are separated at rest by a spherical interface with radius R_i and centre at the origin. An outgoing spherical pressure wave, generated in the $r < R_i$ area, is reflected/transmitted at the interface.

determine function f knowing the motion $\xi(t)$ of the pulsating sphere, as shown below. Using (71) to rewrite (69b), we get

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{1}{r^2} f\left(t - \frac{r - R_0}{c}\right) + \frac{1}{rc} f'\left(t - \frac{r - R_0}{c}\right). \tag{72a}$$

Simply equalizing $u(r,t)$ with $\xi(t)$ for $r = R_0$ we obtain

$$\rho_0 \ddot{\xi} = \frac{1}{R_0^2} f(t) + \frac{1}{R_0 c} f'(t) = \frac{1}{R_0^2}\left(1 + \frac{R_0}{c}\frac{\partial}{\partial t}\right) f(t). \tag{72b}$$

The above equation can be symbolically solved for f:

$$\begin{aligned} f(t) &= \rho_0 R_0^2 \left(1 + \frac{R_0}{c}\frac{\partial}{\partial t}\right)^{(-1)} \ddot{\xi}(t) \\ &= \rho_0 R_0^2 \left(\ddot{\xi}(t) - \frac{R_0}{c}\xi^{(3)}(t) + \left(\frac{R_0}{c}\right)^2 \xi^{(4)}(t) + \cdots\right). \end{aligned} \tag{72c}$$

As a consequence, using (71) and integrating (72a), the extrapressure p and the displacement u respectively read

$$\begin{aligned} p(r,t) &= \rho_0 \frac{R_0^2}{r}\left(1 + \frac{R_0}{c}\frac{\partial}{\partial t}\right)^{(-1)} \ddot{\xi}\left(t - \frac{r - R_0}{c}\right) \\ &= \rho_0 \frac{R_0^2}{r}\left(\ddot{\xi} - \frac{R_0}{c}\xi^{(3)}(t) + \left(\frac{R_0}{c}\right)^2 \xi^{(4)}(t) + \cdots\right)\left(t - \frac{r - R_0}{c}\right), \end{aligned} \tag{73a}$$

$$\begin{aligned} u(r,t) &= \frac{R_0^2}{r^2}\left(1 + \frac{R_0}{c}\frac{\partial}{\partial t}\right)^{(-1)}\left(1 + \frac{r}{c}\frac{\partial}{\partial t}\right)\xi\left(t - \frac{r - R_0}{c}\right) \\ &= \frac{R_0^2}{r^2}\xi\left(t - \frac{r - R_0}{c}\right) + \frac{R_0^2}{rc}\left(1 - \frac{R_0}{r}\right)\left(\dot{\xi} - \frac{R_0}{c}\ddot{\xi} + \left(\frac{R_0}{c}\right)^2 \xi^{(3)} + \cdots\right)\left(t - \frac{r - R_0}{c}\right). \end{aligned} \tag{73b}$$

Last, calculating the flux of the acoustic Poynting vector through a sphere with arbitrarily large radius $\mathcal{R}$, one gets the acoustic power radiated by the pulsating sphere

$$\mathcal{P}_{\rm rad} \overset{\mathcal{R}\to\infty}{=} 4\pi\rho_0 \frac{R_0^4}{c}\left[\left(1 + \frac{R_0}{c}\frac{\partial}{\partial t}\right)^{(-1)} \ddot{\xi}\right]^2 \left(t - \frac{\mathcal{R} - R_0}{c}\right). \tag{73c}$$

The above formalism offers the opportunity to determine in spherical geometry the reflection and transmission coefficients of a pressure wave at the interface between two media 1 and 2, as illustrated in figure 7.

Let us consider an outgoing spherical pressure wave generated in medium 1. This wave reaches the interface and is then split in a reflected wave and a transmitted wave. Neglecting further possible reflection at the origin, the pressure field is described (in the linear approximation) by

$$r < R_i \ : \quad p(r,t) = \frac{1}{r}\left[f_1\left(t - \frac{r-R_i}{c_1}\right) + g_1\left(t + \frac{r-R_i}{c_1}\right)\right] \tag{74a}$$

$$r > R_i \ : \quad p(r,t) = \frac{1}{r} f_2\left(t - \frac{r-R_i}{c_2}\right), \tag{74b}$$

where c_1 (resp. c_2) stands for the sound velocity in medium 1 (resp. medium 2). Equalizing the pressures at the interface $r = R_i$ we naturally get

$$f_1(t) + g_1(t) = f_2(t). \tag{75a}$$

Equalizing the accelerations of the fluid at the interface $r = R_i$, we also get (see (72a) for instance)

$$\frac{1}{\rho_{01}}\left[\frac{1}{R_i}\big(f_1(t)+g_1(t)\big) + \frac{1}{c_1}\big(f_1'(t) - g_1'(t)\big)\right] = \frac{1}{\rho_{02}}\left[\frac{1}{R_i}f_2(t) + \frac{1}{c_2}f_2'(t)\right], \tag{75b}$$

where ρ_{01} (resp. ρ_{02}) stands for the *rest* mass density of medium 1 (resp. medium 2). Introducing the acoustic impedances

$$Z_1 = \rho_{01}c_1 \quad \text{and} \quad Z_2 = \rho_{02}c_2, \tag{76}$$

and assuming a monochromatic regime with angular frequency ω, the above system (75a-b) is easily solved for $r_p = \frac{g_1}{f_1}$ and $t_p = \frac{f_2}{f_1}$ (r_p and t_p are the pressure reflection and transmission coefficients). In the case of a plane interface (*i.e.* $R_i \to \infty$), we are then left with

$$r_\text{p} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \quad \text{and} \quad t_\text{p} = 1 + r_\text{p}. \tag{77}$$

Observe that, in this plane-wave case, reflection and transmission are non dispersive. If the radius of curvature R_i of the interface is *finite*, things are not that simple: the system (75a-b) yields the complex frequency-dependent reflection and transmission coefficients

$$r_\text{p} = \frac{1 - \dfrac{Z_1}{Z_2} + \mathrm{i}\dfrac{c_1}{\omega R_i}(1 - \dfrac{\rho_{01}}{\rho_{02}})}{1 + \dfrac{Z_1}{Z_2} - \mathrm{i}\dfrac{c_1}{\omega R_i}(1 - \dfrac{\rho_{01}}{\rho_{02}})} \quad \text{and} \quad t_\text{p} = 1 + r_\text{p}. \tag{78}$$

It can be remarked that, if $\rho_{01} = \rho_{02}$, the 1-D result (77) holds. Otherwise, reflection and transmission are dispersive. The 1-D result is recovered provided that $c_1/\omega R_i \ll 1$, *i.e.* provided that the acoustic wavelength is small compared to the radius of curvature of the interface, not surprisingly.

V. CONCLUSION

Scarcely taught in undergraduate level courses on fluid dynamics, the Lagrangian picture offers nevertheless a wealth of appreciable advantages. First, since each element of matter is given once for all a (fixed) label, it is well adapted to the discrete-to-continuous description passage and, in this sense, particularly *intuitive*. Second, when two moving fluids – or a moving fluid and some other material device acting as a source – keep in contact, it provides a very handy framework to write exact boundary conditions; it is the case for instance for the surface waves on the sea or for the pulsating sphere. Third, the Lagrangian picture, as it deals with (by definition) *closed* systems, is well designed to implement thermodynamics laws (in the present paper we focussed on isentropic transformations, but other situations may be considered as well). Fourth, there are no *spurious* non linearities left, and the remaining (true) non linearities can be addressed perturbatively, should the occasion arise.

Of course, the necessity of manipulating curvilinear coordinates and tensorial calculus may appear a bit daunting. But is it a prohibitive drawback or rather an opportunity to familiarize softly, in our flat 3D Euclidean everyday life space, with notions that, in the much more involved domain of GR[11,12], turn out to be an absolute must?

We acknowledge Professor Jean-Claude Bacri for simulating conversations during the preparation of this work.

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[13] We deliberately use here the terms “Eulerian picture” and “Lagrangian picture” for they are the exact transposition to fluid mechanics of the so-called “Schrödinger picture” and “Heisenberg picture” of quantum mechanics.

[14] We shall neglect viscosity throughout the present paper, for the sake of simplicity.

[15] Passive markers are designed to follow the flow without perturbing it in any way. They behave exactly as would behave a “painted” fluid particle.

[16] Let us recall that, in the Lagrangian picture, the slice $[x_0, x_0 + \mathrm{d}x_0]$ does constitute a thermodynamically speaking *closed* system: no particle exchange with the outside occurs, except a possible (and neglected in this paper) matter diffusion.

[17] Such a linearization is justified here as soon as the fluid particle velocity is small compared to the wave phase velocity.

[18] We deliberately choose this term instead of “Doppler shift” which, in the framework of the presently discussed thought experiment, would be inappropriate.

[19] unchanged, because our calculation is performed in the Lagrangian picture, of course.

[20] Strictly speaking, this is a misuse of language, since covectors are supposed to belong to $\mathcal{E}^*$, not to $\mathcal{E}$. Nevertheless, it can be shown that $\overleftarrow{\overleftarrow{g}}$ induces a (basis-dependent) biunivocal correspondence between $\mathcal{E}$ and $\mathcal{E}^*$.