Breaking the curse of dimensionality in sparse polynomial approximation of parametric PDEs

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Abstract

The numerical approximation of parametric partial differential equations $\mathcal{D}(u, y) = 0$ is a computational challenge when the dimension d of of the parameter vector y is large, due to the so-called *curse of dimensionality*. It was recently shown in [5, 6] that, for a certain class of elliptic PDEs with diffusion coefficients depending on the parameters in an affine manner, there exist polynomial approximations to the solution map $y \mapsto u(y)$ with an algebraic convergence rate that is independent of the parametric dimension d. The analysis in [5, 6] used, however, the affine parameter dependence of the operator. The present paper proposes a strategy for establishing similar results for some classes parametric PDEs that do not necessarily fall in this category. Our approach is based on building an analytic extension $z \mapsto u(z)$ of the solution map on certain tensor product of ellipses in the complex domain, and using this extension to estimate the Legendre coefficients of u. The varying radii of the ellipses in each coordinate z_j reflect the anisotropy of the solution map with respect to the corresponding parametric variables y_j. This allows us to derive algebraic convergence rates for tensorized Legendre expansions in the case $d = \infty$. We also show that such rates are preserved when using certain interpolation procedures, which is an instance of a non-intrusive method. As examples of parametric PDE's that are covered by this approach, we consider (i) elliptic diffusion equations with coefficients that depend on the parameter vector y in a not necessarily affine manner, (ii) parabolic diffusion equations with similar dependence of the coefficient on y, (iii) nonlinear, monotone parametric elliptic PDE's, and (iv) elliptic equations set on a domain that is parametrized by the vector y. We give general strategies that allows us to derive the analytic extension in a unified abstract way for all these examples, in particular based on the holomorphic version of the implicit function theorem in Banach spaces, generalizing recent results in [13, 15]. We expect that this approach can be applied to a large variety of parametric PDEs, showing that the curse of dimensionality can be overcome under mild assumptions.

1 Introduction

1.1 High dimensional parametric PDE's

This paper is concerned with the numerical approximation of parametric partial differential equations of the general form

$$\mathcal{D}(u, y) = 0 \tag{1.1}$$

where $u \mapsto \mathcal{D}(u, y)$ is a partial differential linear or nonlinear operator that depends on a parameter vector $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, and therefore so does the solution $u(y)$. We assume that the y_j vary in finite intervals.

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Up to a change of variable, we may assume for simplicity that all these intervals are $[-1,1]$ and therefore y ranges in the hypercube

$$U = [-1,1]^d \subset \mathbb{R}^d, \tag{1.2}$$

Assuming that for any $y \in U$, the above problem is well posed in a certain Banach space X, we may introduce the *solution map*

$$y \in U \mapsto u(y) \in X \ . \tag{1.3}$$

Such PDEs occur in both contexts of deterministic and stochastic modelling. In the first case, the parameter sequence y is known or controlled by the user, and a typical goal is to optimize an output of the equation with respect to y. In the second case, the parameters y_j are random variables which take upon rescaling, values in $[-1,1]$. This reflects the uncertainty in the model, and the goal is the resulting statistical properties of the solution u.

In both settings, a typical challenge is to *simultaneously* approximate solutions to the entire family of equations up to some prescribed accuracy, at reasonable computational cost. This may be viewed as building a cheaply computable numerical approximation $\tilde{u}$ to the solution map u, for example based on the knowledge of only a few *instances* of solutions associated to particular choices of y. This task is difficult, since, in contrast to the standard problem of approximating a real-valued function $u : \mathbb{R} \to \mathbb{R}$, the solution map u

(i) takes its value in an infinite dimensional space X, or in a finite dimensional subspace $X_h \subset X$ when using a given numerical solver.

(ii) is defined on a multidimensional domain $U \subset \mathbb{R}^d$ where the parametric dimension d can be large, or even infinite.

The second item refers to the exponential blow up of complexity occuring in discretization methods, as the number d of variables grows, often refered to as *the curse of dimensionality*. Another expression of this phenomenon is the deterioration of approximation rates as d grows, for functions of a given smoothness: for example the accuracy in the L^∞ (or uniform) metric of reconstructing an arbitrary function with continuous derivatives up to order m by piecewise polynomials from h-spaced grid samples is at best of order h^m and therefore, in terms of the number of degrees of freedom n, equal to $n^{-m/d}$, which is a very poor convergence rate when d is large. A deeper investigation in terms of nonlinear width theory [11, 18] reveals that this poor convergence rate cannot be improved by *any* other discretization method.

A typical setting for high dimensional parametric PDEs occurs for problems with are parametrized by a *function* h varying over a certain class, according to

$$\mathcal{P}(u,h) = 0. \tag{1.4}$$

The function h may for example describe (i) a spatially variable diffusion coefficient, (ii) a source term, or (iii) the shape of the physical domain. Using a given basis $(\psi_j)_{j\geq 1}$ for expanding h into

$$h = h(y) := \sum_{j\geq 1} y_j \psi_j, \tag{1.5}$$

results in the parametric model (1.1), where

$$\mathcal{D}(u,y) := \mathcal{P}(u,h(y)) = \mathcal{P}\Big(u, \sum_{j\geq 1} y_j \psi_j\Big), \tag{1.6}$$

and where the number of variables is now countably infinite, that is $d = \infty$, or very large if the above expansion has been truncated with high accuracy. This situation is for example typical in the case of diffusion equations with ramdom coefficients expanded in the Karhunen-Loeve basis.

In view of the above mentionned obstructions, numerical approximation of the resulting solution map requires non-standard discretization tools and a description of the smoothness of this map which differs from the classical description in terms of C^m spaces. A key idea is to introduce more subtle models which reflect the *anisotropy* of this map in the sense that it has a weaker or smoother dependence on certain variables than others. Intuitively this is due the fact that the convergence of the series (1.5) for all $y \in U$ should typically be reflected by a certain form of decay in the size of ψ_j as $j \to +\infty$, resulting into weaker dependence in the corresponding variables y_j. As a consequence the discretization tools should also reflect this anisotropy.

1.2 Sparse polynomial approximation

The effectiveness of the previously described paradigm was demonstrated in [5, 6], using *sparse polynomials approximations* in the parametric variables. The considered problem was the elliptic diffusion equation

$$-\operatorname{div}(a\nabla u) = f, \tag{1.7}$$

set on a physical domain $D \subset \mathbb{R}^m$ with homogeneous Dirichlet boundary conditions and right-hand side $f \in H^{-1}(D)$, with the diffusion coefficient function a depending on a parameter vector in an affine manner

$$a = a(x,y) = \bar{a}(x) + \sum_{j\geq 1} y_j\psi_j(x), \quad x \in D, \quad y \in U, \tag{1.8}$$

with $U = [-1,1]^{\mathbb{N}}$. The functions $\bar{a}$ and $(\psi_j)_{j\geq 1}$ belong to $L^\infty(D)$ and one assumes that the ellipticity assumption

$$0 < r \leq a(x,y) \leq R < \infty, \tag{1.9}$$

holds for all $x \in D$ and $y \in U$, so that the solution map is well-defined and bounded from U to $X := H_0^1(D)$. The approach consists in studying the summability properties of the formal Taylor expansion

$$u(y) = \sum_{\nu\in\mathcal{F}} t_\nu y^\nu, \tag{1.10}$$

where

$$y^\nu := \prod_{j\geq 1} y_j^{\nu_j}, \quad t_\nu = \frac{1}{\nu!}\partial^\nu u(0) \in X, \quad \nu! := \prod_{j\geq 1} \nu_j!, \tag{1.11}$$

and where $\mathcal{F}$ is the set of all finitely supported sequences $\nu = (\nu_1, \nu_2, \ldots, 0, 0, \ldots) \in \mathbb{N}_0^{\mathbb{N}}$. The main result, Theorem 1.2 in [6], is the following.

Theorem 1.1 *If* $(\|\psi_j\|_{L^\infty(D)})_{j\geq 1} \in \ell^p(\mathbb{N})$ *for some* $0 < p < 1$ *and if* (1.9) *holds, then the sequence* $(\|t_\nu\|_X)_{\nu\in\mathcal{F}}$ *belongs to* $\ell^p(\mathcal{F})$*, and one has*

$$u(y) = \sum_{\nu\in\mathcal{F}} t_\nu y^\nu, \tag{1.12}$$

in the sense of unconditional convergence in $L^\infty(U, X)$.

This result has some important consequences regarding the convergence of approximations u_n of u obtained by restriction of its Taylor series to the indices corresponding to the n largest $\|t_\nu\|_X$. Generally speaking, to any sequence $(a_\nu)_{\nu\in\mathcal{F}}$ of real numbers indexed by $\mathcal{F}$ and any $n \geq 1$, we associate the sets

$\Lambda_n := \Lambda_n((a_\nu)_{\nu\in\mathcal{F}})$ of indices ν corresponding to the n largest $|a_\nu|$ (with an arbitrary choice in case of non-uniqueness). Then, an elementary observation is that if $(a_\nu)_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F})$ and $q > p$, one has

$$\Big(\sum_{\nu\notin\Lambda_n} |a_\nu|^q\Big)^{1/q} \le \|(a_\nu)\|_{\ell^p(\mathcal{F})}(n+1)^{-\frac{1}{p}+\frac{1}{q}}. \tag{1.13}$$

This is proved by introducing the decreasing rearrangement $(a_j^*)_{j>0}$ of the sequence $(|a_\nu|)_{\nu\in\mathcal{F}}$ and by combining the two observations

$$\Big(\sum_{\nu\notin\Lambda_n} |a_\nu|^q\Big)^{1/q} = \Big(\sum_{j>n}(a_j^*)^q\Big)^{1/q} \le \Big(\sum_{j>n}(a_{n+1}^*)^{q-p}(a_j^*)^p\Big)^{1/q} \le (a_{n+1}^*)^{1-p/q}\|(a_\nu)\|_{\ell^p(\mathcal{F})}^{p/q}, \tag{1.14}$$

and

$$(n+1)(a_{n+1}^*)^p \le \sum_{j\le n+1}(a_j^*)^p \le \|(a_\nu)\|_{\ell^p(\mathcal{F})}^p. \tag{1.15}$$

Working under the assumptions of the above theorem, and denoting by $\Lambda_n \subset \mathcal{F}$ the set of indices $\nu \in \mathcal{F}$ corresponding to the n largest $\|t_\nu\|_X$, we thus have

$$\sup_{y\in U}\|u(y) - \sum_{\nu\in\Lambda_n} t_\nu y^\nu\|_X \le \sum_{\nu\notin\Lambda_n}\|t_\nu\|_X \le \|(\|t_\nu\|_X)\|_{\ell^p(\mathcal{F})}(n+1)^{-s}, \ \ s := \frac{1}{p} - 1\,. \tag{1.16}$$

The polynomials $\sum_{\nu\in\Lambda_n} t_\nu y^\nu$ therefore provide approximations to the solution map u which converge in $L^\infty(U,X)$ with rate n^{-s} despite the fact that $d = \infty$. This shows that one can in principle overcome the curse of dimensionality in the approximation of $u(y)$ by a proper choice of sparse polynomial spaces.

The proof of Theorem 1.1 is based on the analysis of the anisotropic smoothness of the solution map, in the sense of extending it to the complex domain and making a fine study of its region of holomorphy in several complex variables. Unfortunately this latter aspect is heavily tied to the affine dependence of the coefficients with respect to the parameters in (1.8) and to the linear nature of the equation (1.7).

Many practically relevant parametric PDEs are nonlinear and depend on the parameters y in a non-affine manner. The objective of the present paper is to propose a general strategy in order to derive similar polynomial approximation results for such PDEs. Here are a few examples, among many others, that can be treated by our approach:

(i) Operator equations such as (1.7), with non-affine, yet holomorphic, dependence in y of the diffusion coefficients and such that the problem is well posed uniformly in $y \in U$. Typical instances are

$$a(x,y) := \bar{a} + \Big(\sum_{j\ge1} y_j\psi_j\Big)^2, \tag{1.17}$$

with $\bar{a}$ a strictly positive function which satisfies $\bar{a}(x) \ge r > 0$ for any $x \in D$, or

$$a(x,y) = \exp\Big(\sum_{j\ge1} y_j\psi_j\Big), \ \ y \in U \tag{1.18}$$

so that the solution $u(y)$ of (1.7) is uniquely defined in $X = H_0^1(D)$.

(ii) Linear parabolic evolution equations with spatial operators as in (i). Specifically, for a coefficient a as in (i), we consider in the Gel'fand evolution triple $V \subset H \simeq H^* \subset V^*$ the parabolic problem

$$\partial_t u - \mathrm{div}(a\nabla u) - f = 0 \quad \text{in} \quad (0,T)\times D, \tag{1.19}$$

where $f \in L^2(0,T;V^*)$, with initial and boundary conditions

$$u|_{\partial D} = 0 \quad \text{for} \quad 0 < t < T \ , \quad \text{and} \quad u|_{t=0} = u_0 \in H \ , \quad \text{for} \quad y \in U \ . \tag{1.20}$$

Here $V = H_0^1(D)$ and $H = L^2(D)$. A solution space (see [9]) is

$$X := L^2(0,T;V) \cap H^1(0,T;V^*). \tag{1.21}$$

Other boundary conditions can be accomodated with other choices of the space V.

(iii) Nonlinear operator equations, with analytic dependence of $\mathcal{D}$ on u and on y, and such that the problem is uniformly well posed in $y \in U$. One typical instance is the monotone, elliptic problem

$$u^{2q+1} - \mathrm{div}(a\nabla u) - f = 0, \tag{1.22}$$

which is set on a physical domain $D \subset \mathbb{R}^m$ of dimension $m \geq 2$ and with homogeneous Dirichlet boundary conditions on ∂D and right-hand side $f \in H^{-1}(D)$, where a depends on y as in (1.8), and where $q \geq 0$ is an integer such that $q < \frac{m}{m-2}$. These conditions ensure existence and uniqueness of the solution $u(y)$ in $X = H_0^1(D)$, for every $y \in U$, by the theory of monotone operators (see Chapter 6 of [16]).

(iv) Operator equations on domains whose shape depends on a parameter sequence y. As a simple example, consider the Laplace equation

$$-\Delta v = f, \tag{1.23}$$

with homogeneous Dirichlet boundary conditions set on a physical domain $D(y) \subset \mathbb{R}^2$ that depends on y in the following manner

$$D(y) := \{(x_1, x_2) \ : \ 0 \leq x_1 \leq 1, \ 0 \leq x_2 \leq \phi(x_1, y)\}, \tag{1.24}$$

where $\phi(t,y) := \bar{\phi} + \sum_{j \geq 1} y_j \psi_j(t)$ satisfies a condition of the same type as (1.9) ensuring that the boundary of $D(y)$ is not self-intersecting. Using the map $\Phi(y)(x_1,x_2) := (x_1, x_2\phi(x_1))$ one can transport back the solution $v(y) \in H^1(D(y))$ into the reference domain $D = [0,1]^2$ according to $u(y) := v(y) \circ \Phi(y) \in H^1(D)$. The functions $u(y)$ are now solutions to an elliptic PDE set on D with diffusion coefficients and source term that both depend on the parameter sequence y in an holomorphical, but non-affine manner.

The strategy developed in [5, 6] for proving Theorem 1.1 for the model equation (1.7) with coefficients given by (1.8) does not carry over for the above problems. In fact, this theorem will generally fail to hold, in the sense that $(\|\psi_j\|_{L^\infty(D)})_{j \geq 1} \in \ell^p(\mathbb{N})$ for some $p < 1$ and yet the Taylor series of u does not converge in $L^\infty(U,X)$. This is due to the fact that, for the above models, the solution map does not generally admit an holomorphic extension in a neighbourhood of the whole unit polydisc

$$\mathcal{U} := \bigotimes_{j \geq 1} \{|z_j| \leq 1\} \ . \tag{1.25}$$

As a simple example, consider model (i) or (ii) with $a(x,y) = 1 + by_1^2$, as a particular case of (1.17) where b is a constant strictly larger than 1. Then holomorphy in the first variable on an open disc $\{|z_1| < \rho_1\}$ may hold only if $\rho_1 \leq b^{-1/2} < 1$. A more elaborate inspection of models (iii) and (iv) reveals similar problems. A different approach is therefore needed for the construction and convergence analysis of sparse polynomial approximation.

2 Main results and outline

2.1 Sparse Legendre series

In this paper, we consider sparse approximations constructed by truncation of the tensorized Legendre series

$$u(y) = \sum_{\nu \in \mathcal{F}} u_\nu P_\nu(y), \tag{2.1}$$

where $P_\nu(y) := \prod_{j\geq 1} P_{\nu_j}(y_j)$, with P_n denoting the univariate Legendre polynomial of degree n for the interval $[-1,1]$ normalized according to $\|P_n\|_{L^\infty([-1,1])} = |P_n(1)| = 1$. This series may be rewritten into

$$u(y) = \sum_{\nu \in \mathcal{F}} v_\nu L_\nu(y), \tag{2.2}$$

where $L_\nu(y) := \prod_{j\geq 1} L_{\nu_j}(y_j)$, with L_n denoting the version of P_n normalized in $L^2([-1,1], \frac{dt}{2})$ so that

$$u_\nu = \Big(\prod_{j\geq 1}(1+2\nu_j)\Big)^{1/2} v_\nu. \tag{2.3}$$

If the solution map is uniformly bounded in U in the sense that

$$\|u(y)\|_X \leq C_0, \ \ y \in U, \tag{2.4}$$

then the convergence of the above series is ensured in the space $L^2(U,X)$ of square integrable, X-valued map with respect to the uniform product probability measure

$$d\mu(y) := \bigotimes_{j\geq 1} \frac{dy_j}{2} \ . \tag{2.5}$$

The use of Legendre series in place of Taylor series allows us to obtain similar sparse approximation results under weaker assumptions on the domains of holomorphic extension of the solution map, which turn out to be valid for models such as (i), (ii) and (iii).

To be more specific, for $s > 1$, we introduce the Bernstein ellipse in the complex plane

$$\mathcal{E}_s := \Big\{ \frac{w + w^{-1}}{2} : |w| \leq s \Big\}, \tag{2.6}$$

which has semi axes of length $\frac{s+s^{-1}}{2}$ and $\frac{s-s^{-1}}{2}$ and denote

$$\mathcal{E}_\rho := \bigotimes_{j\geq 1} \mathcal{E}_{\rho_j}, \tag{2.7}$$

the tensorized poly-ellipse when $\rho := (\rho_j)_{j\geq 1}$ is a sequence. Our analysis of the sparsity of Legendre coefficients relies on holomorphic extensions of u over domains of this type. Note that when s is close to 1, the ellipse $\mathcal{E}_s$ concentrates near the real interval $[-1,1]$ and does not contain the unit disc if $s < s^* = 1+\sqrt{5}/2$. Therefore the polydisc $\mathcal{U}$ is not contained in $\mathcal{E}_\rho$ if $\rho_j < s^*$ for at least one value of j. As it will be established, models (i), (ii), (iii) and (iv) are particular instances where the following general assumption holds for the operator $\mathcal{D}$ in (1.1).

Definition 2.1 *For $\varepsilon > 0$ and $0 < p < 1$, we say that $\mathcal{D}$ satisfies the* (p, ε)-holomorphy assumption $\mathbf{HA}(p,\varepsilon)$ *if and only if*

(i) *For each $y \in U$, there exists a unique solution $u(y) \in X$ of the problem* (1.1) *and the map $y \mapsto u(y)$ from U to X is uniformly bounded, i.e.*

$$\sup_{y \in U} \|u(y)\|_X \leq C_0 \ , \tag{2.8}$$

for some finite constant $C_0 > 0$.

(ii) *There exists a positive sequence $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ and a constant $C_\varepsilon > 0$ such that for any sequence $\rho := (\rho_j)_{j \geq 1}$ of numbers strictly larger than 1 that satisfies*

$$\sum_{j=1}^{\infty} (\rho_j - 1) b_j \leq \varepsilon, \tag{2.9}$$

the map u admits a complex extension $z \mapsto u(z)$ that is holomorphic with respect to each variable z_j on a set of the form $\mathcal{O}_\rho := \bigotimes_{j \geq 1} \mathcal{O}_{\rho_j}$, $\mathcal{O}_{\rho_j} \subset \mathbb{C}$ is an open set containing $\mathcal{E}_{\rho_j}$. This extension is bounded on $\mathcal{E}_\rho := \bigotimes_{j \geq 1} \mathcal{E}_{\rho_j}$, according to

$$\sup_{z \in \mathcal{E}_\rho} \|u(z)\|_X \leq C_\varepsilon. \tag{2.10}$$

Our first result is that such assumptions ensure ℓ^p summability of the Legendre coefficients. For the purpose of further numerical implementation we do actually establish a stronger result. To any sequence $c := (c_\nu)_{\nu \in \mathcal{F}}$, we associate its monotone envelope $\mathbf{c} := (\mathbf{c}_\nu)_{\nu \in \mathcal{F}}$ defined by

$$\mathbf{c}_\nu := \sup_{\mu \geq \nu} |c_\nu|, \quad \nu \in \mathcal{F}, \tag{2.11}$$

where $\mu \geq \nu$ means that $\mu_j \geq \nu_j$ for all j. We also say that a set $\Lambda \subset \mathcal{F}$ is monotone if and only if

$$\nu \in \Lambda \quad \text{and} \quad \mu \leq \nu \Rightarrow \mu \in \Lambda. \tag{2.12}$$

For $p > 0$, we introduce the space $\ell^p_m(\mathcal{F})$ of sequences that have their monotone envelope in $\ell^p(\mathcal{F})$.

Theorem 2.2 *If the differential operator $\mathcal{D}$ is such that $\mathbf{HA}(p, \varepsilon)$ holds for some $0 < p < 1$ and $\varepsilon > 0$, then the sequences $(\|u_\nu\|_X)_{\nu \in \mathcal{F}}$ and $(\|v_\nu\|_X)_{\nu \in \mathcal{F}}$ belong to $\ell^p_m(\mathcal{F})$, and*

$$u(y) = \sum_{\nu \in \mathcal{F}} u_\nu P_\nu = \sum_{\nu \in \mathcal{F}} v_\nu L_\nu, \tag{2.13}$$

holds in the sense of unconditional convergence in $L^\infty(U, X)$.

Using (1.13), we can translate the conclusion of the above theorem in terms of convergence rates for sparse Legendre approximations: if $\Lambda^P_n \subset \mathcal{F}$ and $\Lambda^L_n \subset \mathcal{F}$ are the sets of indices $\nu \in \mathcal{F}$ corresponding respectively to the n largest terms in the monotone envelopes $\mathbf{u} = (\mathbf{u}_\nu)_{\nu \in \mathcal{F}}$ and $\mathbf{v} = (\mathbf{v}_\nu)_{\nu \in \mathcal{F}}$ of the sequences $(\|u_\nu\|_V)_{\nu \in \mathcal{F}}$ and $(\|v_\nu\|_V)_{\nu \in \mathcal{F}}$, then

$$\|u - \sum_{\nu \in \Lambda^P_n} u_\nu P_\nu\|_{L^\infty(U,X)} \leq \sum_{\nu \notin \Lambda^P_n} \mathbf{u}_\nu \leq \|\mathbf{u}\|_{\ell^p(\mathcal{F})} (n+1)^{-s}, \ \ s := \frac{1}{p} - 1, \tag{2.14}$$

and

$$\|u - \sum_{\nu \in \Lambda^L_n} v_\nu L_\nu\|_{L^2(U,X,d\mu)} = \Big(\sum_{\nu \notin \Lambda^L_n} \mathbf{v}_\nu^2 \Big)^{\frac{1}{2}} \leq \|\mathbf{v}\|_{\ell^p(\mathcal{F})} (n+1)^{-s}, \ \ s := \frac{1}{p} - \frac{1}{2} \ . \tag{2.15}$$

In consequence, the n-term truncated Legendre series provide approximations to the solution map u in $L^\infty(U,X)$ with similar convergence rates as the Taylor series and provide approximations with better decay rate in the least square sense. The interest of using the monotone envelope is that the sets $\Lambda_n^P \subset \mathcal{F}$ and $\Lambda_n^L \subset \mathcal{F}$ can be chosen to be monotone in the sense of (2.12), a property that appears to be useful for numerical computation [4, 7]. In the present paper, we shall make use of this property to show that the convergence rate n^{-s} in (2.14) can be preserved when the Legendre projections are replaced by properly defined polynomial interpolations of u at certain points.

2.2 Establishing assumptions $\mathbf{HA}(p,\varepsilon)$

In the case of models (i), (ii) and (iii), we verify $\mathbf{HA}(p,\varepsilon)$ using $b_j := \|\psi_j\|_{L^\infty(D)}$, under the assumption that $(\|\psi_j\|_{L^\infty(D)})_{j\geq1}$ belongs to $\ell^p(\mathbb{N})$. In the case of model (iv), we establish the validity of $\mathbf{HA}(p,\varepsilon)$ using $b_j := \|\psi_j\|_{L^\infty(D)} + \|\psi_j'\|_{L^\infty(D)}$, and therefore under the additional assumption that $(\|\psi_j'\|_{L^\infty(D)})_{j\geq1}$ belongs to $\ell^p(\mathbb{N})$. Here, we propose two general frameworks that allow us to establish $\mathbf{HA}(p,\varepsilon)$ for such models, as well as for many other potential models of parametric PDEs.

The first framework is when the parametric PDE has the general variational form

$$u \in X: \quad B(u,v,y) = F(v,y), \;\; v \in Y, \tag{2.16}$$

where X, Y are Hilbert spaces over $\mathbb{C}$ and where, for every fixed $y \in U$, the maps $(u,v) \mapsto B(u,v,y)$ and $v \mapsto F(v,y)$ are continuous sesquilinear and antilinear forms on $X \times Y$ respectively on Y. In this setting, the operator $\mathcal{D}$ of (1.1) is defined from $X \times U$ into the antidual Y^* of Y, according to

$$\mathcal{D}(u,y) := B(u,\cdot,y) - F(\cdot,y)\ . \tag{2.17}$$

In many practical instances, the two spaces X and Y coincide, however $X \neq Y$ is relevant for the treatment of parabolic evolution problems. We use the same notations B and F to denote the corresponding maps from U into the spaces of sesquilinear and antilinear continuous forms on $X \times Y$ and on Y, respectively, defined by

$$B(y)(v,w) := B(v,w,y) \quad \text{and} \quad F(y)(w) := F(w,y), \quad v \in X,\ w \in Y,\ y \in U\ . \tag{2.18}$$

The following result shows that the validity of $\mathbf{HA}(p,\varepsilon)$ expressing the analytic behaviour of the solution map $y \mapsto u(y)$ follows from a similar analytic behaviour of the maps B and F.

Theorem 2.3 *For $\varepsilon > 0$ and $0 < p < 1$, assume that there exists a positive sequence $(b_j)_{j\geq1} \in \ell^p(\mathbb{N})$, and two constants $0 < r \leq R < \infty$ and a constant $M < \infty$ such that the following holds:*

(i) For any sequence $\rho := (\rho_j)_{j\geq1}$ of numbers strictly greater than 1 that satisfies

$$\sum_{j=1}^{\infty} (\rho_j - 1) b_j \leq \varepsilon, \tag{2.19}$$

the maps B and F admit extensions that are holomorphic with respect to every variable on a set of the form $\mathcal{O}_\rho = \bigotimes_{j\geq1} \mathcal{O}_{\rho_j}$, where $\mathcal{O}_{\rho_j} \subset \mathbb{C}$ is an open set containing $\mathcal{E}_{\rho_j}$.

(ii) These extensions satisfy for all $z \in \mathcal{O}_\rho$ the uniform continuity conditions

$$\sup_{w \in Y\setminus\{0\}} \frac{|F(w,z)|}{\|w\|_Y} \leq M, \qquad \sup_{v \in X\setminus\{0\}, w \in Y\setminus\{0\}} \frac{|B(v,w,z)|}{\|v\|_X \|w\|_Y} \leq R, \tag{2.20}$$

and the uniform inf-sup conditions: there exists $r > 0$ such that

$$\inf_{v \in X \setminus \{0\}} \sup_{w \in Y \setminus \{0\}} \frac{|B(v,w,z)|}{\|v\|_X \|w\|_Y} \geq r \quad \textit{and} \quad \inf_{w \in Y \setminus \{0\}} \sup_{v \in X \setminus \{0\}} \frac{|B(v,w,z)|}{\|v\|_X \|w\|_Y} > r. \tag{2.21}$$

Then, $\mathcal{D}$ satisfies the assumptions in $\mathbf{HA}(p,\varepsilon)$ with the same p and ε and with the same sequence b.

Our second framework is concerned with parametric PDEs of the form (1.4), where $\mathcal{P}$ is a linear or nonlinear operator defined from the product of two Banach spaces X and L over $\mathbb{C}$ into a third Banach space W over $\mathbb{C}$. The parameter function h is expanded in terms of the parameter sequence $y \in U$ according to (1.5), where the ψ_j are functions in L and we assume that the expansion in (1.5) converges in L for all $y \in U$. In the particular case of (1.7), we have $X = H_0^1(D)$, $L = L^\infty(D)$ and $W = H^{-1}(D)$. We introduce the set

$$h(U) = \{h(y) \;:\; y \in U\} \subset L\ . \tag{2.22}$$

The following theorem shows that the validity of $\mathbf{HA}(p,\varepsilon)$ is ensured provided that $(\|\psi_j\|_L)_{j\geq 1} \in \ell^p(\mathbb{N})$ and that $\mathcal{P}$ satisfies certain smoothness properties, in addition to the well-posedness of the problem (1.4) over $h(U)$.

Theorem 2.4 *Assume that:*

- *One has $(\|\psi_j\|_L)_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $0 < p \leq 1$.*
- *The problem (1.4) is well-posed in X for all $h \in h(U)$.*
- *The map $(u,h) \mapsto \mathcal{P}(u,h)$ is continuously differentiable from $X \times L$ into W.*
- *For every $h \in h(U)$, the partial differential $\frac{\partial \mathcal{P}}{\partial u}(u(h),h)$ is an isomorphism from X onto W.*

Then there exists an $\varepsilon > 0$, for which $\mathcal{D}$ satisfies the assumptions $\mathbf{HA}(p,\varepsilon)$.

The rest of this paper is organized as follows. In §3, we prove Theorem 2.2 by deriving upper bounds for the X-norms of Legendre coefficients $\|u_\nu\|_X$ and showing the $\ell^p(\mathcal{F})$ summability of the corresponding sequences of coefficient bounds. Our approach may be viewed as a variant of the technique developped in [6] in a special case. In §4, we show in addition that under the assumptions of Theorem (2.2), similar convergence rates $\mathcal{O}(n^{-s})$ with $s = \frac{1}{p} - 1$ in $L^\infty(U,X)$ can be obtained by certain interpolation processes introduced in [4], despite the possible growth of the Lebesgue constant. The proofs of Theorems 2.3 and 2.4 are given in §5. Finally, we discuss in §6 the application of the two frameworks to models (i) to (iv). We show that (iv) and (iii) can be treated in the framework of Theorem 2.3 and Theorem 2.4, respectively, and that both frameworks may be used to treat (i) and (ii).

3 Sparse Legendre expansions

In this section, we prove a slightly stronger version of Theorem 2.2, as explained further. Note that

$$\|v_\nu\|_X = \frac{\|u_\nu\|_X}{\left(\prod_{j\geq 1}(1+2\nu_j)\right)^{1/2}}, \tag{3.1}$$

so that it suffices to prove the $\ell^p(\mathcal{F})$ summability of the sequence $(\|u_\nu\|_X)_{\nu\in\mathcal{F}}$. We first give upper estimates for the $\|u_\nu\|_X$. These estimates are a generalization of those established in [6, Lemma 4.2] for the particular problem (1.7) with coefficients given by (1.8). Since the proof is very similar, we only sketch it.

Lemma 3.1 *Let $\rho := (\rho_j)_{j\geq 1}$ be any sequence of numbers strictly larger than 1, such that that u has an extension that is holomorphic in each variable on a domain of the form $\mathcal{O}_\rho = \bigotimes_{j\geq} \mathcal{O}_{\rho_j}$, where $\mathcal{O}_{\rho_j} \subset \mathbb{C}$ is an open neighbourhood of $\mathcal{E}_{\rho_j}$, with uniform bound*

$$\sup_{z\in\mathcal{E}_\rho} \|u(z)\|_V \leq C. \tag{3.2}$$

Then, the Legendre coefficients satisfy

$$\|u_\nu\|_X \leq C \prod_{j\geq 1:\nu_j\neq 0} (2\nu_j+1)\phi(\rho_j)\rho_j^{-\nu_j}, \tag{3.3}$$

where $\phi(t) := \frac{\pi t}{2(t-1)}$ for $t > 1$, with in the case $\nu = (0,0,\ldots)$ the convention that the empty product equals 1.

Proof: For $\nu \in \mathcal{F}$, the coefficient u_ν is given by

$$u_\nu = \prod_{j\geq 1}(2\nu_j+1)\int_U u(y)P_\nu(y)d\mu(y)\ . \tag{3.4}$$

The estimate for $\nu = (0,0,\ldots)$ is trivial since $d\mu$ is a probability measure. We now prove, for $\nu \in \mathcal{F}\backslash\{0\}$,

$$\left\|\int_U u(y)P_\nu(y)d\mu(y)\right\|_X \leq C \prod_{j\geq 1:\nu_j\neq 0} \phi(\rho_j)\rho_j^{-\nu_j}\ . \tag{3.5}$$

To this end, we use induction on $J_\nu := \#(\mathrm{supp}(\nu))$, the (finite) number of non-zero coordinates in ν. Let $J \in \mathbb{N}$ and assume that (3.3) holds for any $\mu \in \mathcal{F}$ such that $J_\mu \leq J$. Let $\nu \in \mathcal{F}$ with $J_\nu = J+1$. Without loss of generality, we suppose that $\nu_1 \neq 0$. We introduce the notation $\hat{y} := (y_2, y_3, \ldots)$ and $\hat{\nu} := (\nu_2, \nu_3, \ldots)$. Let us note that $\hat{\nu} \in \mathcal{F}$ and $J_{\hat{\nu}} = J$. Now, we have

$$\int_U u(y)P_\nu(y)d\mu(y) = \int_{-1}^{1}\int_U u(y_1,\hat{y})P_{\nu_1}(y_1)P_{\hat{\nu}}(\hat{y})\frac{dy_1}{2}d\mu(\hat{y}) = \int_U w(\hat{y})P_{\hat{\nu}}(\hat{y})d\mu(\hat{y}), \tag{3.6}$$

where

$$w(\hat{y}) := \int_{-1}^{1} u(y_1,\hat{y})P_{\nu_1}(y_1)\frac{dy_1}{2}. \tag{3.7}$$

The induction hypothesis applied to w on the poly-ellipse $\mathcal{E}_{\hat{\rho}}$ with $\hat{\rho} = (\rho_2, \rho_3, \ldots)$ implies

$$\left\|\int_U u(y)P_\nu(y)d\mu(y)\right\|_X \leq \sup_{\hat{z}\in\mathcal{E}_{\hat{\rho}}} \|w(\hat{z})\|_X \prod_{j\geq 2:\nu_j\neq 0} \phi(\rho_j)\rho_j^{-\nu_j} \tag{3.8}$$

It remains to show that there exists a constant $C > 0$ such that for any $\hat{z} \in \mathcal{E}_{\hat{\rho}}$

$$\|w(\hat{z})\|_X \leq C\phi(\rho_1)\rho_1^{-\nu_1}\ . \tag{3.9}$$

For any $\hat{z} \in \mathcal{E}_{\hat{\rho}}$ fixed, the map $z_1 \mapsto u(z_1, \hat{z})$ is holomorphic on an open neighborhood $\mathcal{O}_{\rho_1}$ of the ellipse $\mathcal{E}_{\rho_1}$. Therefore, Cauchy's integral formula applied with respect to z_1 yields

$$u(y_1,\hat{z}) = \frac{1}{2i\pi}\int_{\partial\mathcal{E}_{\rho_1}} \frac{u(z_1,\hat{z})}{(z_1-y_1)}dz_1, \tag{3.10}$$

for any $y_1 \in [-1,1]$, hence

$$w(\hat{z}) = \frac{1}{2i\pi} \int_{\partial \mathcal{E}_{\rho_1}} u(z_1,\hat{z}) \frac{Q_{\nu_1}(z_1)}{2} dz_1, \tag{3.11}$$

where Q_n is the function of a single complex variable $t \notin [-1,1]$ defined by

$$Q_n(t) := \int_{-1}^{1} \frac{P_n(s)}{t-s} ds. \tag{3.12}$$

With C as in (3.2) it follows that

$$\|w(\hat{z})\|_X \le C \frac{\rho_1}{2} \max_{z_1 \in \mathcal{E}_{\rho_1}} |Q_{\nu_1}(z_1)| \tag{3.13}$$

where we have used the fact that the ellipse $\mathcal{E}_{\rho_1}$ has perimeter of length less than $2\pi\rho_1$. We conclude by using the estimate

$$\max_{z \in \mathcal{E}_t} |Q_n(z)| \le \frac{\pi\ t^{-n}}{t-1}, \tag{3.14}$$

established at the bottom of page 313 in [8]. □

We now turn to the proof of Theorem 2.2. The previous lemma shows that if $\mathbf{HA}(p,\varepsilon)$ holds, then for any sequence $\rho := (\rho_j)_{j\ge1}$ of real numbers strictly larger than 1 such that $\sum_{j=1}^{\infty}(\rho_j-1)b_j \le \varepsilon$, we have

$$\|u_\nu\|_X \le C_\varepsilon \prod_{j\ge1:\nu_j\neq0} (2\nu_j+1)\phi(\rho_j)\rho_j^{-\nu_j}, \quad \nu \in \mathcal{F} - \{(0,0,\dots)\}, \tag{3.15}$$

where $(b_j)_{j\ge1}$ and C_ε are as in Definition 2.1. We use this estimate in order to establish the $\ell^p(\mathcal{F})$ summability of the monotone envelope $\mathbf{u}$ of $(\|u_\nu\|_X)_{\nu\in\mathcal{F}}$. To this end, we use a specific design of the sequence ρ that depends on the index ν, in a similar spirit as in §4.3 of [6].

Let $B > 0$ be arbitrary but fixed, and $J \ge 1$ be an integer such that $\sum_{j>J}|b_j| \le \frac{\varepsilon}{4B}$. We introduce $F := \{j \in \mathbb{N} \ :\ j > J\}$ and define $\nu_F := (\nu_{J+1},\nu_{J+2},\dots)$ for any $\nu \in \mathcal{F}$. We introduce the sequence $\rho(\nu) := (\rho_j)_{j\ge1}$ that depends on ν according to

$$\rho_j := 1 + \frac{\varepsilon}{4\|b\|_{\ell^1(\mathbb{N})}} \ \text{ for } \ j \le J \ \text{ and } \ \rho_j := 1 + \frac{\varepsilon}{4\|b\|_{\ell^1(\mathbb{N})}} + B + \frac{\varepsilon}{2|b_j|}\frac{\nu_j}{1+|\nu_F|} \ \text{ for } \ j > J, \tag{3.16}$$

where $|\nu_F| := \sum_{j>J}\nu_j$. It is easily checked that $\sum_{j\ge1}(\rho_j-1)|b_j| \le \varepsilon$, so that the estimate (3.15) holds with $\rho = \rho(\nu)$. We introduce the notation $\kappa = 1 + \frac{\varepsilon}{4\|(b_j)\|_{\ell^1}}$ and $C_\kappa = \phi(\kappa) > 1$. Since ϕ is a decreasing function and $\rho_j \ge \kappa$ for any $j \ge 1$, then $\phi(\rho_j) \le C_\kappa$ for any $j \ge 1$. Consequently, for $\nu \neq 0$,

$$\|u_\nu\|_X \le C_\varepsilon \prod_{j\le J:\nu_j\neq0} C_\kappa(2\nu_j+1)\kappa^{-\nu_j} \prod_{j>J:\nu_j\neq0} C_\kappa(2\nu_j+1)\rho_j^{-\nu_j}\ . \tag{3.17}$$

Using the crude estimates $(2n+1)\kappa^{-n} \le c_\kappa \kappa^{-\frac{n}{2}}$ for some constant $c_\kappa > 1$ and $C_\kappa(2n+1) \le (3C_\kappa)^n$ for any $n \ge 1$, we have

$$\|u_\nu\|_X \le q_\nu := C_\varepsilon \beta_E(\nu)\beta_F(\nu), \quad \nu \neq 0 \tag{3.18}$$

where

$$\beta_E(\nu) := (c_\kappa C_\kappa)^J \prod_{j\le J} \kappa^{-\nu_j/2} \quad \text{and} \quad \beta_F(\nu) := \prod_{j>J} (3C_\kappa)^{\nu_j}\rho_j^{-\nu_j}. \tag{3.19}$$

We denote $\mathcal{F}_E$ the multi-indices in $\mathcal{F}$ supported in $E := \{j \le J\}$ and $\mathcal{F}_F$ the multi-indices in $\mathcal{F}$ supported in F, with convention that $\nu = 0$ belongs to both sets. Observe that the estimate (3.18) remains valid for $\nu = 0$. The separable form of this estimate implies that

$$\sum_{\nu\in\mathcal{F}} \|u_\nu\|^p \le \sum_{\nu\in\mathcal{F}} q_\nu^p = C_\varepsilon^p A_E A_F \quad \text{where} \quad A_E := \sum_{\nu\in\mathcal{F}_E} \beta_E(\nu)^p \quad \text{and} \quad A_F := \sum_{\nu\in\mathcal{F}_F} \beta_F(\nu)^p. \tag{3.20}$$

Now, on the one hand, we have

$$A_E = (c_\kappa C_\kappa)^{pJ} \sum_{\nu\in\mathcal{F}_E} \prod_{j\le J} \kappa^{-p\nu_j/2} = (c_\kappa C_\kappa)^{pJ} S^J \quad \text{where} \quad S := \sum_{n=0}^{\infty} \kappa^{-pn/2} < +\infty. \tag{3.21}$$

On the other hand, defining $d_j := \frac{6C_\kappa b_j}{\varepsilon}$ for $j > J$ and using the inequality $\rho_j \ge \frac{\varepsilon}{2|b_j|}\frac{\nu_j}{1+|\nu_F|}$, we obtain

$$\beta_F(\nu) \le \prod_{j>J} \Big(\frac{1+|\nu_F|}{\nu_j} d_j\Big)^{\nu_j} = \frac{(1+|\nu_F|)^{|\nu_F|}}{\nu_F^{\nu_F}} \prod_{j>J} d_j^{\nu_j}, \tag{3.22}$$

Using the bounds $(1+n)^n \le n! e^{n+1}$ and $n! e^n \le \max\{1, e\sqrt{n}\} n^n$ which holds for any $n \ge 1$, it follows that

$$\beta_F(\nu) \le e \frac{|\nu_F|!}{\nu_F!} \bar{d}^{\nu_F}, \tag{3.23}$$

where $\bar{d}$ defined by $\bar{d}_j = e d_j$ for $j > J$. Therefore

$$A_F \le e^p \sum_{\nu\in\mathcal{F}_F} \Big(\frac{|\nu_F|!}{\nu_F!} \bar{d}^{\nu_F}\Big)^p = \sum_{\nu\in\mathcal{F}} \Big(\frac{|\nu|!}{\nu!} \tilde{d}^{\nu}\Big)^p \tag{3.24}$$

where $\tilde{d} := (\bar{d}_{J+1}, \bar{d}_{J+2}, \dots)$. Up to possibly choosing a larger value of J, we may assume that

$$\|\tilde{d}\|_{\ell^1} = \sum_{j>J} \bar{d}_j \le \frac{6C_\kappa}{\varepsilon} \sum_{j>J} |b_j| < 1.$$

We then invoke [5, Theorem 7.2] which says that the sequence $(\frac{|\nu|!}{\nu!}\tilde{d}^\nu)_{\nu\in\mathcal{F}}$ belongs to $\ell^p(\mathcal{F})$ if and only if $\tilde{d} \in \ell^p(\mathbb{N})$ and $\|\tilde{d}\|_{\ell^1} \le 1$. This shows that $A_F < +\infty$. As a result $(q_\nu)_{\nu\in\mathcal{F}}$ and $(\|u_\nu\|_V)_{\nu\in\mathcal{F}}$ belongs to $\ell^p(\mathcal{F})$.

Finally, denoting by

$$e_j := (0, \dots, 0, 1, 0, \dots), \tag{3.25}$$

the Kronecker sequence with 1 at position j, we observe that $(q_\nu)_{\nu\in\mathcal{F}}$ satisfies

$$\frac{q_{\nu+e_j}}{q_\nu} = \frac{1}{\sqrt{\kappa}}, \quad \nu \in \mathcal{F}, \quad j \le J, \tag{3.26}$$

and for any $j > J$,

$$\frac{q_{e_j}}{q_0} = 3C_\kappa \frac{1}{\Big(\kappa + B + \frac{\varepsilon}{2|b_j|}\frac{1}{2}\Big)} \quad \text{and} \quad \frac{q_{\nu+e_j}}{q_\nu} = 3C_\kappa \frac{\Big(\kappa + B + \frac{\varepsilon}{2|b_j|}\frac{\nu_j}{1+|\nu_F|}\Big)^{\nu_j}}{\Big(\kappa + B + \frac{\varepsilon}{2|b_j|}\frac{\nu_j+1}{2+|\nu_F|}\Big)^{\nu_j+1}}, \quad \nu \ne 0\ . \tag{3.27}$$

If B is chosen large enough, the quotient $\frac{q_{\nu+e_j}}{q_\nu}$ is smaller than 1 for any $\nu \in \mathcal{F}$ and any $j \ge 1$. Therefore the sequence $(q_\nu)_{\nu\in\mathcal{F}}$ is monotone decreasing in the sense that

$$\mu \le \nu \Rightarrow q_\nu \le q_\mu\ . \tag{3.28}$$

This implies that $(q_\nu)_{\nu\in\mathcal{F}}$ coincides with its monotone envelope. As a result $(q_\nu)_{\nu\in\mathcal{F}} \in \ell^p_m(\mathcal{F})$. Therefore $(\|u_\nu\|_X)_{\nu\in\mathcal{F}} \in \ell^p_m(\mathcal{F})$, which concludes the proof of Theorem 2.2.

4 Sparse high dimensional interpolation

The conclusion of Theorem 2.2 shows that, under its assumptions, there exists a sequence of nested monotone sets $(\Lambda_n)_{n\geq 1}$, with $\#(\Lambda_n) = n$, such that

$$\inf_{v\in X_{\Lambda_n}} \|u-v\|_{L^\infty(U,X)} \leq C(n+1)^{-s}, \quad s=\frac{1}{p}-1, \tag{4.1}$$

where we have used for any finite set $\Lambda \subset \mathcal{F}$ the notation

$$X_\Lambda := \text{span}\left\{\sum_{\nu\in\Lambda} v_\nu y^\nu : v_\nu \in X\right\}. \tag{4.2}$$

We remark that $X_\Lambda := \mathbb{P}_\Lambda \otimes X$, where

$$\mathbb{P}_\Lambda := \text{span}\{y^\nu \ : \ \nu \in \Lambda\}. \tag{4.3}$$

One way to compute an approximation of the solution map $y \mapsto u(y)$ in the polynomial spaces X_{Λ_n} is by interpolation. Polynomial interpolation processes on the spaces X_Λ for arbitrary monotone sets Λ have been introduced and studied in [4]. Given $z := (z_j)_{j\geq 1}$, a sequence of pairwise distinct points of $[-1,1]$, we associate with any finite subset $\Lambda \subset \mathcal{F}$ the following finite set of points in U,

$$\Gamma_\Lambda := \{z_\nu : \nu \in \Lambda\} \quad \text{where} \quad z_\nu := (z_{\nu_j})_{j\geq 1}. \tag{4.4}$$

It is shown in [4] that if $\Lambda \subset \mathcal{F}$ is monotone, then the set Γ_Λ is unisolvent for $\mathbb{P}_\Lambda$, i.e. for any function g defined in Γ_Λ and taking values in $\mathbb{C}$, there exists a unique polynomial $I_\Lambda g$ in $\mathbb{P}_\Lambda$ that coincides with g on Γ_Λ. The interpolant can be expressed and computed in a simple manner: if we write $\Lambda := \{\nu^1, \cdots, \nu^N\}$ such that for any $k = 1\cdots, N$, the set $\Lambda_k := \{\nu^1, \cdots, \nu^k\}$ is monotone, then

$$I_\Lambda g = \sum_{i=1}^N g_{\nu^i} H_{\nu^i}, \tag{4.5}$$

where the polynomials $(H_\nu)_{\nu\in\Lambda}$ are a hierarchical basis of $\mathbb{P}_\Lambda$ given by

$$H_\nu(y) := \prod_{j\geq 1} h_{\nu_j}(y_j) \quad \textit{where} \quad h_0(t) = 1 \ \text{ and } \ h_k(t) = \prod_{j=0}^{k-1} \frac{t-z_j}{z_k - z_j}, \quad k \geq 1, \tag{4.6}$$

and where the coefficients g_{ν^k} are recursively defined by

$$g_{\nu^1} := g(z_0), \quad g_{\nu^{k+1}} := g(z_{\nu^{k+1}}) - I_{\Lambda_k} g(z_{\nu^{k+1}}) = g(z_{\nu^{k+1}}) - \sum_{i=1}^k g_{\nu^i} H_{\nu^i}(z_{\nu^{k+1}}). \tag{4.7}$$

A standard vectorization technique yields that Γ_Λ is also unisolvent for X_Λ. The interpolation operator, that here maps functions defined from U to X into X_Λ can also be computed by the recursion (4.5) where the coefficient g_ν now belongs to the Banach space X. We use the same notation I_Λ for this interpolation operator.

One way to relate the accuracy of the interpolation operator I_Λ to the error of best polynomial approximation is via the Lebesgue constant, which is defined by

$$\mathbb{L}_\Lambda := \sup_{g\in B(U)} \frac{\|I_\Lambda g\|_{L^\infty(U)}}{\|g\|_{L^\infty(U)}}, \tag{4.8}$$

where $B(U)$ is the set of bounded functions g on U which are defined everywhere on U. We indeed have the classical inequality

$$\|g - I_\Lambda g\|_{L^\infty(U)} \leq (1 + \mathbb{L}_\Lambda) \inf_{h \in \mathbb{P}_\Lambda} \|g - h\|_{L^\infty(U)}, \tag{4.9}$$

from which it follows that

$$\|u - I_\Lambda u\|_{L^\infty(U)} \leq (1 + \mathbb{L}_\Lambda) \inf_{v \in v_\Lambda} \|u - v\|_{L^\infty(U,X)}, \tag{4.10}$$

for a function u defined from U taking values in X.

In [4], algebraic bounds have been derived for $\mathbb{L}_\Lambda$ given that algebraic bounds are available for the Lebesgue constants λ_k of the interpolation on the set of $k+1$ points $\{z_0, \dots, z_k\}$. Namely, if there exists $\theta > 0$ such that $\lambda_k \leq (k+1)^\theta$, for any $k \geq 0$, then for any finite monotone set Λ,

$$\mathbb{L}_\Lambda \leq (\#(\Lambda))^{\theta+1}. \tag{4.11}$$

Let us stress that this bound is independent of the shape of the monotone set Λ, it only depends on its cardinality. Sequences $z = (z_j)_{j \geq 0}$ for which it can be proved that $\lambda_k \leq (k+1)^\theta$, are available in the literature, see [3], with $2 < \theta < 3$.

Using such sequences and the same monotone sets $(\Lambda_n)_{n \geq 1}$ that give the estimate (4.1), under the assumptions of Theorem 2.2, it follows from (4.10) and (4.11) that a first estimate for the interpolation error is of the form

$$\|u - I_{\Lambda_n} u\|_{L^\infty(U,X)} \leq C(n+1)^{-s+\theta+1}, \quad s = \frac{1}{p} - 1 \, . \tag{4.12}$$

The following result recovers the best n-term approximation rate $\mathcal{O}(n^{-s})$ for the interpolation based on a different choice of monotone sets. A similar analysis was developed in [4] in the particular case of the solution map u of (1.7) and under the assumptions of Theorem 1.1. It is based on the fact that the algebraic growth of the univariate Lebesgue constants λ_k can be absorbed inside the estimates obtained for Legendre coefficients based on analyticity.

Theorem 4.1 *Under the assumptions of Theorem 2.2, there exists a constant $C > 0$ and a nested sequence of monotone sets $(\Lambda_n)_{n \geq 1}$ with $\#(\Lambda_n) = n$ for which*

$$\|u - I_{\Lambda_n} u\|_{L^\infty(U,X)} \leq C(n+1)^{-s}, \quad s = \frac{1}{p} - 1. \tag{4.13}$$

Proof: The unconditional convergence in $L^\infty(U, X)$ of the Legendre series yields: for any finite monotone set Λ,

$$I_\Lambda u = I_\Lambda \Big(\sum_{\nu \in \mathcal{F}} u_\nu P_\nu \Big) = \sum_{\nu \in \mathcal{F}} u_\nu I_\Lambda P_\nu = \sum_{\nu \in \Lambda} u_\nu I_\Lambda P_\nu + \sum_{\nu \notin \Lambda} u_\nu I_\Lambda P_\nu \, .$$

The univariate Legendre polynomial P_k is of degree k, therefore for any $\nu \in \mathcal{F}$, the polynomial P_ν belongs to $\mathbb{P}_{\mathcal{R}_\nu}$ where $\mathcal{R}_\nu := \{\mu \in \mathcal{F} : \mu \leq \nu\}$. The monotonicity of Λ implies then that $P_\nu \in \mathbb{P}_\Lambda$, hence $I_\Lambda P_\nu = P_\nu$, for any $\nu \in \Lambda$. From the recursive expression (4.5) of the interpolation operator, it is also easily checked that for any given $\nu \in \mathcal{F}$ and monotone set Λ,

$$P \in \mathbb{P}_{\mathcal{R}_\nu} \Rightarrow I_\Lambda P = I_{\Lambda \cap \mathcal{R}_\nu} P \, .$$

The two previous observations imply

$$(I - I_\Lambda) u = \sum_{\nu \notin \Lambda} u_\nu (I - I_{\Lambda \cap \mathcal{R}_\nu}) P_\nu,$$

where I denotes the identity operator. Therefore

$$\|(I - I_\Lambda)u\|_{L^\infty(U,X)} \leq \sum_{\nu\notin\Lambda} \|u_\nu\|_X (1 + \mathbb{L}_{\Lambda\cap\mathcal{R}_\nu}) \|P_\nu\|_{L^\infty(U)} \leq 2 \sum_{\nu\notin\Lambda} \|u_\nu\|_X \mathbb{L}_{\Lambda\cap\mathcal{R}_\nu} \ .$$

If the univariate sequence is such that $\lambda_k \leq (k+1)^\theta$ for some $\theta > 0$, then we have

$$\mathbb{L}_{\Lambda\cap\mathcal{R}_\nu} \leq \#(\Lambda \cap \mathcal{R}_\nu)^{\theta+1} \leq \#(\mathcal{R}_\nu)^{\theta+1} = \Big(\prod_{j\geq 1}(1+\nu_j)\Big)^{\theta+1} =: p_\theta(\nu),$$

so that

$$\|(I - I_\Lambda)u\|_{L^\infty(U,X)} \leq 2 \sum_{\nu\notin\Lambda} \|u_\nu\|_X p_\theta(\nu) \ .$$

In order to prove (4.13), it is thus sufficient to prove that the sequence $w = (w_\nu)_{\nu\in\mathcal{F}}$ with $w_\nu := p_\theta(\nu)\|u_\nu\|_X$ belongs to $\ell^p_m(\mathcal{F})$. Indeed, the nested sequence of monotone sets Λ_n of indices $\nu \in \mathcal{F}$ corresponding to the n largest terms in the monotone envelope $\mathbf{w}$ of w then provide the estimate

$$\|u - I_{\Lambda_n}u\|_{L^\infty(U,X)} \leq 2\|\mathbf{w}\|_{\ell^p(\mathcal{F})}(n+1)^{-s}, \quad s = \frac{1}{p} - 1. \tag{4.14}$$

Since we work under the assumptions of Theorem 2.2, we have by (3.17)

$$w_\nu \leq C_\varepsilon \prod_{j\leq J:\nu_j\neq 0} C_\kappa(\nu_j+1)^{\theta+1}(2\nu_j+1)\kappa^{-\nu_j} \prod_{j>J:\nu_j\neq 0} C_\kappa(\nu_j+1)^{\theta+1}(2\nu_j+1)\rho_j^{-\nu_j}, \tag{4.15}$$

where J, κ, C_κ are defined in the proof of Theorem 2.2 given in the previous section. Using the crude estimates, $(n+1)^{\theta+1}(2n+1)\kappa^{-n} \leq c_{\theta,\kappa}\kappa^{-\frac{n}{2}}$ for some constant $c_{\theta,\kappa} > 1$ and $C_\kappa(n+1)^{\theta+1}(2n+1) \leq (mC_\kappa)^n$ for some $m \geq 1$, we infer

$$w_\nu \leq q_\nu := C_\varepsilon \beta^w_E(\nu)\beta^w_F(\nu), \quad \nu \neq 0 \tag{4.16}$$

where

$$\beta^w_E(\nu) := (c_{\theta,\kappa}C_\kappa)^J \prod_{j\leq J} \kappa^{-\nu_j/2} \quad \text{and} \quad \beta^w_F(\nu) := \prod_{j>J}(mC_\kappa)^{\nu_j}\rho_j^{-\nu_j}. \tag{4.17}$$

These estimates are of similar type as those given in (3.19) for the sequence $(\|u_\nu\|_X)_{\nu\in\mathcal{F}}$, and the $\ell^p_m(\mathcal{F})$ summability of w may thus be derived by the exact same arguments as those given at the end of the previous section. □

5 Holomorphic extension on poly-ellipses

In this section, we provide the proofs of Theorem 2.3 and Theorem 2.4.

5.1 Proof of Theorem 2.3

Let p, ε, b, $\rho := (\rho_j)_{j\geq 1}$ and $\mathcal{O}_\rho$ be as in the assumptions of Theorem 2.3. First, using the continuity and inf-sup conditions (2.20) and (2.21), a standard functional analytic argument similar to the proof of the Lax-Milgram lemma, shows that for any $z \in \mathcal{O}_\rho$, the parametric, variational problem

$$\mathcal{D}(u,z) := B(z)(u,\cdot) - L(z)(\cdot) = 0 \quad \text{in} \quad Y^* \tag{5.1}$$

is well posed in X, uniformly with respect to z. Accordingly, the solution map $z \in \mathcal{O}_\rho \mapsto u(z) \in X$ is well-defined and uniformly bounded in $\mathcal{O}_\rho$ in the sense that

$$\sup_{z\in\mathcal{O}_\rho} \|u(z)\|_X \le \frac{M}{r}, \tag{5.2}$$

where r and M are given in the condition (2.20). To complete the proof of Theorem 2.3, we only need to prove that u is holomorphic in $\mathcal{O}_\rho$ with respect to each variable. We first observe that u is continuous on $\mathcal{O}_\rho$: indeed, for $z, \tilde{z} \in \mathcal{O}_\rho$, we have from the equations $\mathcal{D}(u(z), z) = 0$ and $\mathcal{D}(u(\tilde{z}), \tilde{z}) = 0$ in Y^* that

$$B(\tilde{z})\Big(u(\tilde{z}) - u(z), v\Big) = -\Big(B(\tilde{z}) - B(z)\Big)(u(z), v) + \Big(F(\tilde{z}) - F(z)\Big)(v), \quad v \in Y\ . \tag{5.3}$$

Therefore, taking $v = u(\tilde{z}) - u(z)$ and using the continuity and inf-sup conditions (2.20) and (2.21), we obtain

$$r\|u(\tilde{z}) - u(z)\|_X^2 \le \|B(\tilde{z}) - B(z)\|_{\mathcal{L}(X\times Y,\mathbb{C})}\|u(z)\|_X\|u(\tilde{z}) - u(z)\|_X + \|F(\tilde{z}) - F(z)\|_{Y^*}\|u(\tilde{z}) - u(z)\|_X, \tag{5.4}$$

which combined with (5.2) implies

$$\|u(\tilde{z}) - u(z)\|_X \le \frac{1}{r}\Big(\|B(\tilde{z}) - B(z)\|_{\mathcal{L}(X\times Y,\mathbb{C})}\frac{M}{r} + \|F(\tilde{z}) - F(z)\|_{Y^*}\Big), \tag{5.5}$$

so that the holomorphy of B and F implies the continuity of u. Now let $z \in \mathcal{O}_\rho$, $j \ge 1$ and $\delta \in \mathbb{C}$ such that $z + \delta e_j \in \mathcal{O}_\rho$, where e_j is the j-th Kronecker sequence in $\mathbb{C}^{\mathbb{N}}$ and introduce $w_\delta = \frac{1}{\delta}(u(z + \delta e_j) - u(z))$. Taking $\tilde{z} = z + \delta e_j$ in (5.3), we obtain, for every $v \in Y$

$$B(z)(w_\delta, v) = -\frac{B(z+\delta e_j) - B(z)}{\delta}(u(z+\delta e_j), v) + \frac{F(z+\delta e_j) - F(z)}{\delta}(v), \qquad v \in Y\ . \tag{5.6}$$

By the holomorphic dependence of B and L on z,

$$\left\|\frac{F(z+\delta e_j) - F(z)}{\delta} - \frac{\partial F}{\partial z_j}(z)\right\|_{Y^*} = o_\delta(1) \quad \text{and} \quad \left\|\frac{B(z+\delta e_j) - B(z)}{\delta} - \frac{\partial B}{\partial z_j}(z)\right\|_{\mathcal{L}(X\times Y,\mathbb{C})} = o_\delta(1), \tag{5.7}$$

where we use the generic notation $o_\delta(1)$ for a positive quantity that tends to 0 as $\mathbb{C} \ni \delta \to 0$. Hence for any $v \in Y$

$$\left|B(z)(w_\delta, v) - \frac{\partial F}{\partial z_j}(z)(v) + \frac{\partial B}{\partial z_j}(z)(u(z+\delta e_j), v)\right| = \|v\|_Y o_\delta(1), \tag{5.8}$$

where we have used (5.2) to get the bound $\|u(z+\delta e_j)\|_X \le \frac{M}{r}$ for any δ such that $z + \delta e_j \in \mathcal{O}_\rho$. This, combined with the continuous dependence of u on z, implies

$$\left\|B(z)(w_\delta, \cdot) - \frac{\partial F}{\partial z_j}(z)(\cdot) + \frac{\partial B}{\partial z_j}(z)(u(z), \cdot)\right\|_{Y^*} = o_\delta(1)\ . \tag{5.9}$$

Finally, if $w_0 \in X$ is the unique solution of the variational problem

$$B(z)(w_0, v) = \frac{\partial F}{\partial z_j}(z)(v) - \frac{\partial B}{\partial z_j}(z)(u(z), v), \quad v \in Y, \tag{5.10}$$

then

$$\|B(z)(w_\delta - w_0, \cdot)\|_{Y^*} = o_\delta(1)\ . \tag{5.11}$$

Using again the inf-sup condition in (2.21), we deduce that $\|w_\delta - w_0\|_X \to 0$. This shows that the map $z \mapsto u(z)$ from $\mathbb{C}$ to X admits a partial complex derivative $\frac{\partial u}{\partial z_j}(z) \in X$ with respect to the complex extension z_j of each coordinate variable y_j. In addition, this derivative is the unique solution of the variational problem

$$B(z)(\frac{\partial u}{\partial z_j}(z), v) = \frac{\partial F}{\partial z_j}(z)(v) - \frac{\partial B}{\partial z_j}(z)(u(z), v), \quad v \in Y \ . \tag{5.12}$$

The proof of the holomorphy of u with respect to every variable on $\mathcal{O}_\rho$ is then complete. □

Remark 5.1 *Inspection of the proof of Theorem 2.3 reveals that it remains valid verbatim when X and Y are reflexive Banach spaces.*

5.2 Proof of Theorem 2.4

We recall that the framework for Theorem 2.4 is as follows: the operator $\mathcal{D}$ depends on the parameter $y \in U$ through the functions $h(y) = \sum_{j\geq 1} y_j\psi_j$ where the ψ_j belong to some Banach space L over $\mathbb{C}$, according to

$$\mathcal{D}(u, y) = \mathcal{P}\Big(u, h(y)\Big), \tag{5.13}$$

where $\mathcal{P}$ is a linear or nonlinear operator from $X \times L$ into a Banach space W over $\mathbb{C}$. We set $b := (b_j)_{j\geq 1}$ with $b_j := \|\psi_j\|_L$, and propose to use this sequence to show that $\mathcal{D}$ satisfies the assumptions $\mathbf{HA}(p, \varepsilon)$ of Defintion 2.1. It is already assumed that $b \in \ell^p(\mathbb{N})$ for some $p < 1$. Therefore, in order to prove the theorem, we only need to show the existence of some $\varepsilon > 0$ for which the point (ii) in Definition 2.1 is satisfied.

Before proving Theorem 2.4, we give two simple, yet useful observations. The first observation is that we can use a simple open neighbourhood $\mathcal{O}_s$ for the complex ellipse $\mathcal{E}_s$.

Lemma 5.2 *Let $s > 1$ and introduce the open set in $\mathbb{C}$*

$$\mathcal{O}_s := \bigcup_{t\in[-1,1]} \{\xi \in \mathbb{C} : |\xi - t| < s - 1\} = \{\xi \in \mathbb{C} \ : \ \mathrm{dist}(\xi, [-1, 1]) < s - 1\} \ . \tag{5.14}$$

Then $\mathcal{O}_s$ is an open neighborhood of $\mathcal{E}_s$.

Proof: It is sufficient to prove that $\partial\mathcal{E}_s \subset \mathcal{O}_s$. Since the ellipse $\partial\mathcal{E}_s$ has half-axes $\frac{s+s^{-1}}{2}$ and $\frac{s-s^{-1}}{2}$ and foci ± 1, for any $\xi \in \partial\mathcal{E}_s$ we have

(i) If $\Re(\xi) \in [-1, 1]$, then since $|\Im(\xi)| \leq \frac{s-s^{-1}}{2} < s - 1$, we have $|\xi - \Re(\xi)| < s - 1$.

(ii) If $\Re(\xi) > 1$ then $|\xi + 1| > 2$, but since $|\xi - 1| + |\xi + 1| = s + s^{-1}$, we have $|\xi - 1| < s + s^{-1} - 2 < s - 1$.

(iii) If $\Re(\xi) < -1$, then by symmetry with (ii), we have $|\xi + 1| < s - 1$.

This shows that in the three cases $|\xi - t| < s - 1$ for some $t \in [-1, 1]$ and completes the proof. □

Our second observation is concerned with the topology of the set $h(U) := \{h(y) \ : \ y \in U\} \subset L$ introduced in (2.22).

Lemma 5.3 *Assume that the the sequence $(\|\psi_j\|_L)_{j\geq 1}$ belongs to $\ell^1(\mathbb{N})$. Then $h(U)$ is compact in L.*

Proof: Let $(h_n)_{n\geq 1}$ be a sequence in $h(U)$. Since $(\|\psi_j\|_L)_{j\geq 1} \in \ell^1(\mathbb{N})$, the sequence $(h_n)_{n\geq 1}$ is bounded in L. Each h_n is of the form $h_n = \sum_{j\geq 1} y_{n,j}\psi_j$. Using a Cantor diagonal argument, we infer that there exists $y = (y_j)_{j\geq 1} \in U$ such that

$$\lim_{n\to+\infty} y_{\sigma(n),j} = y_j, \quad j \geq 1, \tag{5.15}$$

where $(\sigma(n))_{n\geq 1}$ is a monotone sequence of positive integers. Defining $h := \sum_{j\geq 1} y_j\psi_j \in h(U)$, we may write for any $k \geq 1$,

$$\|h_{\sigma(n)} - h\|_L \leq \|\sum_{j=1}^{k}(y_j - y_{\sigma(n),j})\psi_j\|_L + 2\sum_{j\geq k+1}\|\psi_j\|_L. \tag{5.16}$$

It follows that $h_{\sigma(n)}$ converges towards h in L and therefore $h(U)$ is compact. □

We now consider an arbitrary $y \in U$ and the corresponding $h(y) \in h(U)$. The assumptions of Theorem 2.4 say that $\mathcal{P}$ is continuously differentiable as a mapping from $X \times L$ into W, that $\mathcal{P}(u(y), h(y)) = 0$ in W and that the partial differential $\frac{\partial\mathcal{P}}{\partial u}(u(y), h(y))$ is an isomorphism from X onto W. Therefore, by the holomorphic version of the implicit function theorem on complex Banach spaces, see [12, Theorem 10.2.1], there exists an $\varepsilon > 0$, and a mapping G from $\mathring{\mathcal{B}}(h(y), \varepsilon)$ the open ball of L with center $h(y)$ and radius ε into X such that $G(h(y)) = u(y)$ and $\mathcal{P}(G(h), h) = 0$ for any h in $\mathring{\mathcal{B}}(h(y), \varepsilon)$. In addition, the map G is uniformly bounded and holomorphic on $\mathring{\mathcal{B}}(h(y), \varepsilon)$ with

$$dG(h) = -\Big(\frac{\partial\mathcal{P}}{\partial u}(G(h), h)\Big)^{-1} \circ \frac{\partial\mathcal{P}}{\partial h}(G(h), h), \quad h \in \mathring{\mathcal{B}}(h(y), \varepsilon)\ . \tag{5.17}$$

Let us note that $\varepsilon = \varepsilon(y)$ depends actually on y. We claim that ε can be made independent of $y \in U$. Since $\bigcup_{y\in U} \mathring{\mathcal{B}}(h(y), \frac{\varepsilon(y)}{2})$ is an infinite open covering of $h(U)$ and since $h(U)$ is compact in L, there exists a a finite subcover, ie. a finite number M and $y^1, \cdots, y^M$ in U such that

$$h(U) \subset \bigcup_{j=1}^{M} \mathring{\mathcal{B}}\Big(h(y^j), \frac{\varepsilon(y^j)}{2}\Big)\ . \tag{5.18}$$

We introduce $\varepsilon := \min_{1\leq j\leq M} \frac{\varepsilon(y^j)}{2}$. Let $y \in U$ and $h \in L$ such that $\|h - h(y)\|_L < \varepsilon$. According to (5.18), $h(y)$ belongs to some $\mathring{\mathcal{B}}(h(y^j), \frac{\varepsilon(y^j)}{2})$, therefore, for $j = 1, ..., M$,

$$\|h - h(y^j)\|_L \leq \|h - h(y)\|_L + \|h(y) - h(y^j)\|_L < \varepsilon + \frac{\varepsilon(y^j)}{2} \leq \frac{\varepsilon(y^j)}{2} + \frac{\varepsilon(y^j)}{2} = \varepsilon(y^j).$$

This shows that $\mathring{\mathcal{B}}(h(y), \varepsilon) \subset \mathring{\mathcal{B}}(h(y^j), \varepsilon(y^j))$ and it implies that

$$h^{\varepsilon}(U) := \bigcup_{y\in U} \mathring{\mathcal{B}}(h(y), \varepsilon) \subset \bigcup_{j=1}^{M} \mathring{\mathcal{B}}(h(y^j), \varepsilon(y^j))\ . \tag{5.19}$$

In particular the map G is well defined and is continuously differentiable as a mapping from $h^{\varepsilon}(U)$ into the complex Banach space X.

To conclude the proof of Theorem 2.4, we verify assumption $\mathbf{HA}(p, \varepsilon)$. Let $\rho := (\rho_j)_{j\geq 1}$ a sequence of numbers strictly greater than 1 such that $\sum_{j\geq 1}(\rho_j - 1)b_j \leq \varepsilon$ and $\mathcal{O}_\rho := \bigotimes_{j\geq 1} \mathcal{O}_{\rho_j}$, where for $s > 1$, $\mathcal{O}_s$ is the open domain in $\mathbb{C}$ defined in (5.14). For any $z := (z_j)_{j\geq 1} \in \mathcal{O}_\rho$, we define $h(z) := \sum_{j\geq 1} z_j\psi_j \in L$. If $y = (y_j)_{j\geq 1} \in U$ satisfies $|z_j - y_j| < \rho_j - 1$, for every $j \geq 1$, we then have

$$\|h(z) - h(y)\|_L = \Big\|\sum_{j\geq 1}(z_j - y_j)\psi_j\Big\|_L \leq \sum_{j\geq 1}|z_j - y_j|\|\psi_j\|_L < \sum_{j\geq 1}(\rho_j - 1)b_j \leq \varepsilon, \tag{5.20}$$

therefore $h(z) \in h^\varepsilon(U)$ and $G(h(z))$ is well defined. We extend the solution map u on the domain $\mathcal{O}_\rho$ by $u(z) := G(h(z))$. By holomorphy of G on $h^\varepsilon(U)$ and affine dependence of $h(z)$ on z, it follows that

$$z \mapsto h(z) \mapsto u(z) = G(h(z)),$$

is holomorphic with respect to every variable on $\mathcal{O}_\rho$. Moreover

$$\sup_{z\in\mathcal{O}_\rho} \|u(z)\|_X = \sup_{z\in\mathcal{O}_\rho} \|G(h(z))\|_X \leq \sup_{h\in h^\varepsilon(U)} \|G(h)\|_X \leq \max_{i=1,\dots,M} \sup_{h\in\mathring{\mathcal{B}}(h(y^j),\varepsilon(y^j))} \|G(h)\|_X < \infty\,. \tag{5.21}$$

This completes the proof of Theorem 2.4. ☐

Remark 5.4 *Inspection of the above proof reveals that we can weaken the assumption in the sense that holomorphy of the map $\mathcal{P}$ is required only over a set of the form $X \times h_\eta(U)$ for some $\eta > 0$ instead of $X \times L$, where $h_\eta(U) := \{h \in L \;:\; \mathrm{dist}_L(h, h(U)) < \eta\}$.*

6 Applications

In this section, we show that the models (i)-(ii)-(iii)-(iv) discussed in the introduction are covered by at least one of the two frameworks of Theorem 2.3 or Theorem 2.4. Specifically, we check the assumptions of Theorem 2.3 for models (i)-(ii)-(iv) and of Theorem 2.4 for models (i)-(ii)-(iii).

6.1 Models (i) and (ii): Linear elliptic and parabolic PDEs with parametric coefficients

We recall that model (i) is the parametric elliptic diffusion equation (1.7) with the typical instances of the diffusion coefficient a

$$a(x,y) := \bar a(x) + \Big(\sum_{j\geq 1} y_j\psi_j(x)\Big)^2 \quad \text{or} \quad a(x,y) = \exp\Big(\sum_{j\geq 1} y_j\psi_j\Big), \quad x \in D, \;\; y \in U. \tag{6.1}$$

In both examples, we assume that the sequence $b := (\|\psi_j\|_{L^\infty(D)})_{j\geq 1}$ belongs to $\ell^p(\mathbb{N})$ for some $p < 1$, and for $z = (z_j)_{j\geq 1} \in \mathbb{C}^\mathbb{N}$ we define $a(x,z)$ by replacing y_j by z_j in the above expressions. We shall use the domains $\mathcal{O}_\rho := \otimes_{j\geq 1}\mathcal{O}_{\rho_j}$ given in (5.14) to verify the assumptions of Theorem (2.3). Here, the inf-sup condition (2.21) is implied by the usual coercivity condition.

We begin with the first example, assuming that $\bar a$ is in $L^\infty(D)$ and is uniformly bounded from below by some $r_0 > 0$. This implies that a satisfies a uniform ellipticity assumption of type (1.9) with r_0 and $R_0 = \|\bar a\|_{L^\infty(D)} + \|b\|^2_{\ell^1(\mathbb{N})}$ and establishes the well-posedness of (1.7) in $X = H^1_0(D)$ for any $y \in U$. Now given $\varepsilon = \sqrt{\frac{r_0}{2}}$ and $\rho := (\rho_j)_{j\geq 1}$ a sequence satisfying $\rho_j > 1$ for every j and $\sum_{j\geq 1}(\rho_j - 1)b_j \leq \varepsilon$, we have for $z \in \mathcal{O}_\rho$ and $x \in D$

$$\begin{aligned} \Re(a(x,z)) &= \bar a(x) + \Big(\textstyle\sum_{j\geq 1}\Re(z_j)\psi_j(x)\Big)^2 - \Big(\textstyle\sum_{j\geq 1}\Im(z_j)\psi_j(x)\Big)^2 \\ &\geq r_0 - \Big(\textstyle\sum_{j\geq 1}|\Im(z_j)|b_j\Big)^2 \geq r_0 - \Big(\textstyle\sum_{j\geq 1}(\rho_j-1)b_j\Big)^2 \geq r, \end{aligned} \tag{6.2}$$

with $r := \frac{r_0}{2} > 0$, where we have used that for $s > 1$ the domain $\mathcal{O}_s$ is contained in the strip $\{t \in \mathbb{C} : |\Im(t)| \leq s - 1\}$. We also have the upper bound

$$|a(x,z)| \leq \bar a(x) + \Big(\sum_{j\geq 1}|z_j||\psi_j(x)|\Big)^2 \leq R_0 + \Big(\sum_{j\geq 1}\rho_j b_j\Big)^2 \leq R_0 + 2\Big(\sum_{j\geq 1}(\rho_j-1)b_j\Big)^2 + 2\Big(\sum_{j\geq 1}b_j\Big)^2 \leq R,$$

with $R := R_0 + 2\varepsilon^2 + 2\|b\|_{\ell^1}^2$. Using in addition the fact that $z \mapsto a(z)$ is holomorphic in each variable in $\mathcal{O}_\rho$, we conclude that the sesquilinear and antilinear form

$$B(u, v, z) = \int_D a(x, z)\nabla u(x)\overline{\nabla v(x)}dx \quad \text{and} \quad F(v, z) = F(v) = \int_D f(x)\overline{v(x)}dx \tag{6.3}$$

satisfy the assumptions of Theorem 2.3 with $X = Y = H_0^1(D)$, p, ε, r, R and $M = \|f\|_{H^{-1}(D)}$.

For the second example, the uniform ellipticity assumption is satisfied with $r_0 := \exp\Big(-\|b\|_{\ell^1}\Big) > 0$ and $R_0 = 1/r_0$. Now given an $0 < \varepsilon < \frac{\pi}{2}$ and a sequence ρ with the usual assumption, we have for $z \in \mathcal{O}_\rho$ and $x \in D$,

$$\Re\Big(\exp\Big(\sum_{j\geq 1} z_j\psi_j\Big)\Big) = \exp\Big(\sum_{j\geq 1}\Re(z_j)\psi_j\Big)\cos\Big(\sum_{j\geq 1}\Im(z_j)\psi_j\Big) \geq \exp\Big(-\sum_{j\geq 1}|\Re(z_j)|b_j\Big)\cos(\varepsilon) \geq r \tag{6.4}$$

where $r = \exp\Big(-\varepsilon - \|b\|_{\ell^1}\Big)\cos(\varepsilon) > 0$ and the upper bound

$$\Big|\exp\Big(\sum_{j\geq 1} z_j\psi_j\Big)\Big| = \exp\Big(\sum_{j\geq 1}\Re(z_j)\psi_j\Big) \leq R := \exp(\varepsilon + \|b\|_{\ell^1})\ . \tag{6.5}$$

Similar to the first example, Theorem 2.3 applies for this second model.

For the parabolic equation (1.19) in model (ii), again with the spatial differential operator as in (1.7) with coefficient a as in (6.1), and with the choice of spaces $X = L^2(0, T; H_0^1(D)) \cap H^1(0, T; H^{-1}(D))$ and $Y = L^2(0, T; H_0^1(D)) \times L^2(D)$, the sesquilinear and antilinear forms corresponding to the parabolic problem (1.19) read for $v \in X$ and $w = (w_1, w_2) \in Y$ as

$$B(v, w, z) = \int_0^T\int_D \Big(\partial_t v(x, t)\overline{w_1(x, t)} + a(x, z)\nabla_x v(x, t)\overline{\nabla_x w_1(x, t)}\Big)\,dxdt + \int_D v(x, 0)\overline{w_2(x)}dx, \tag{6.6}$$

and

$$F(w) = \int_0^T\int_D f(x, t)\overline{w_1(x, t)}dxdt + \int_D u_0(x)\overline{w_2(x)}dx \tag{6.7}$$

with all integrals to be understood as the corresponding duality pairings. The boundedness (2.20) of these forms is readily verified with the above choices of spaces. The verification of the inf-sup conditions (2.21) for the parametric coefficients (1.8) or (6.1), on the parameter domain $\mathcal{O}_\rho$ follows from the fact that

$$0 < r < \Re(a(x, z)) \leq |a(x, z)| \leq R,\ x \in D,\ z \in \mathcal{O}_\rho, \tag{6.8}$$

and using the general arguments given in [17, Appendix].

The application of the previous arguments is tied to the simple formula of the diffusion coefficient a and may be tedious when applied to diffusion coefficients with complicated formulas. One can overcome this difficulty using the second framework, that is, Theorem 2.4. Let us consider a diffusion coefficient a that depends on y according to

$$a(y) = \mathcal{A}(h(y)), \quad h(y) := \sum_{j\geq 1} y_j\psi_j(x), \tag{6.9}$$

where $\mathcal{A}$ is a map from $L^\infty(D)$ into itself such that

$$0 < r \leq \mathcal{A}(h) \leq R < \infty, \tag{6.10}$$

for all $h \in h(U)$, and such that $\mathcal{A}$ is continuously differentiable over $L^\infty(D)$ viewed as a Banach space over $\mathbb{C}$. We also assume that $(\|\psi_j\|_{L^\infty})_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $0 < p < 1$. The two examples (6.1) correspond to $\mathcal{A}(h) = \bar{a} + h^2$ and $\mathcal{A}(t) = \exp(h)$. To cast model (i) into the second framework, we introduce the operator

$$\mathcal{P}(u,h) = -\text{ div } (\mathcal{A}(h)\nabla u) - f, \tag{6.11}$$

This operator is well defined and continuously differentiable from $X \times L$ into W where

$$(X, L, W) := (H_0^1(D), L^\infty(D), H^{-1}(D)), \tag{6.12}$$

viewed as complex Banach spaces. For any $u \in X$ and $h \in L$,

$$\frac{\partial \mathcal{P}}{\partial u}(u,h)(v) = -\text{ div } (\mathcal{A}(h)\nabla v), \tag{6.13}$$

and therefore the uniform ellipticity assumption (6.10) implies that $\frac{\partial \mathcal{P}}{\partial u}(u(h(y)), h(y))$ is an isomorphism from X onto W, for all $y \in U$. Therefore, all the assumptions of Theorem 2.4 hold.

Similar arguments apply for the parabolic problem of model (ii) with

$$\mathcal{P}(u,h) = (\partial_t u - \text{ div } (\mathcal{A}(h)\nabla u) - f, u(\cdot, 0)), \tag{6.14}$$

with the choices $X := L^2(0,T;V) \cap H^1(0,T;V^*)$, $L := L^\infty(D)$ and $W := L^2(0,T;V^*) \times H$, where $V = H_0^1(D)$ and $H = L^2(D)$.

6.2 Model (iii): non linear, elliptic PDE

The nonlinear equation (1.22) is associated to the operator,

$$\mathcal{D}(u,y) := u^{2q+1} - \text{div}(a(y)\nabla u) - f, \tag{6.15}$$

where $f \in H^{-1}(D)$ is a given, real-valued function, D is a bounded Lipschitz subdomain of $\mathbb{R}^m$. Here $a(y)$ is as in (1.8) and satisfies (1.9), and $q \geq 0$ is an integer such that $q < \frac{m}{m-2}$ so that $u^{2q+1} \in H^{-1}(D)$. Thus $X = H_0^1(D)$ and $\mathcal{D}$ maps $X \times U$ into $X^* = H^{-1}(D)$. More generally, we consider equations (1.1) associated with an operator of the form

$$\mathcal{D}(u,y) := g(u) - \text{div}(\mathcal{A}(h(y))\nabla u) - f, \tag{6.16}$$

where $f \in X^*$ and $h(y)$ and $\mathcal{A}$ are as in the previous section §6.1, and with $(\|\psi_j\|_{L^\infty})_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $0 < p < 1$. In addition, we assume that g is a function defined on $\mathbb{C}$, such that

1) g is holomorphic on $\mathbb{C}$.

2) $g(0) = 0$ and, for $t \in \mathbb{R}$, $g'(t) \geq 0$.

3) g maps continuously X into X^*.

4) For any $u \in X$, the sesquilinear form $v, w \mapsto \int_D g'(u) v \overline{w}$ is continuous over $X \times X$.

These assumptions are in particular fulfilled by the polynomial nonlinearity $g : t \mapsto t^{2q+1}$ when $q < \frac{m}{m-2}$. Let us now verify the assumptions of Theorem 2.4.

First, we establish for every $y \in U$, the well posedness of the nonlinear problem on X understood as a Banach space over $\mathbb{R}$. It follows from the above items 2) and 3) that, for any fixed $y \in U$, the nonlinear operator

$$T(y) : u \mapsto g(u) - \operatorname{div}(\mathcal{A}(h(y))\nabla u), \tag{6.17}$$

is continuous, strongly monotone and coercive from X into X^*. By the theory of monotone operators on Banach spaces X over the coefficient field $\mathbb{R}$, see for example Theorem 1 in Chapter 6 of [16], for every $y \in U$, the problem (1.1) admits a unique (real-valued) solution $u(y) \in X$.

We next view the spaces (X, L, W) defined as in (6.12) as Banach spaces over $\mathbb{C}$ and observe that the map

$$(v, h) \mapsto \mathcal{P}(v, h) := g(v) - \operatorname{div}(\mathcal{A}(h)\nabla v) - f, \tag{6.18}$$

is continuously differentiable over $X \times L$, thanks to the assumptions on g and $\mathcal{A}$. For every $(v, h) \in X \times L$, the first partial differential is given by

$$\frac{\partial \mathcal{P}}{\partial u}(v, h)(w) = g'(v)w - \operatorname{div}(\mathcal{A}(h)\nabla w) \in W\ . \tag{6.19}$$

In particular, for any $h \in h(U)$, we have

$$\frac{\partial \mathcal{P}}{\partial u}(u(h), h)(w) = g'(u(h))w - \operatorname{div}(\mathcal{A}(h)\nabla w)\ . \tag{6.20}$$

This operator is associated to the sesquilinear form

$$b(v, w) = \int_D g'(u(h))v\bar{w} + \int_D \mathcal{A}(h)\nabla v \cdot \overline{\nabla w}. \tag{6.21}$$

which is continuous by the upper inequality in (6.10) and item 4). In addition it satisfies the coercivity condition

$$b(v, v) \geq r\|v\|_X^2, \quad v \in X, \tag{6.22}$$

by the lower inequality in (6.10) and item 2). Therefore, by Lax-Milgram theory, it is an isomorphism from X onto W. All the assumptions in Theorem 2.4 are thus fullfilled.

Remark 6.1 *In the case of the nonlinear equation* (1.22)*, a possible way to extend the solution for complex valued parameter z would be to rather consider the equation*

$$|u|^{2q}u - \operatorname{div}(a(z)\nabla u) = f. \tag{6.23}$$

It is easily seen that monotone operator theory applied to the equation verified by the vector (v, w) where $u = v + iw$ allows us to uniquely define the solution $u(z)$ of the above equation under the ellipticity condition $0 < r \leq \Re(a(z)) \leq |a(z)| \leq R$. However the presence of the modulus $|u|$ in the equation obstructs holomorphic dependence on the z_j variable. In our approach, we maintain the original equation (1.22)*. In this case the existence and holomorphy of the solution $u(z)$ for the complex argument z does* not *follow from monotone operator theory, but rather from the implicit function theorem argument used in Theorem 4.2.*

6.3 Model (iv): Parametrized domain

As a simple example of PDE set on a parametrized domain, we consider the Laplace equation

$$-\Delta v = f \tag{6.24}$$

with homogeneous Dirichlet boundary condition set on a physical domain $D(y) \subset \mathbb{R}^2$ that depends on $y \in U$ in the following manner

$$D(y) := \{(x_1, x_2) \ : \ 0 \le x_1 \le 1, \ 0 \le x_2 \le \phi(x_1, y)\}, \tag{6.25}$$

with

$$\phi(t, y) := \bar{\phi}(t) + \sum_{j \ge 1} y_j \psi_j(t), \tag{6.26}$$

where the functions $\bar{\phi}$ and ψ_j belong to $W^{1,\infty}([0,1])$, that is, are Lipschitz continuous on $[0,1]$. We assume that ϕ satisfies a condition of the same type as (1.9), namely

$$0 < r \le \bar{\phi}(t) + \sum_{j \ge 1} y_j \psi_j(t) \le R < \infty, \quad t \in [0,1], \ y \in U. \tag{6.27}$$

The lower inequality ensures that the boundary of $D(y)$ is not self-intersecting. We also assume that the above series converges in $W^{1,\infty}([0,1])$, uniformly in $y \in U$, that is

$$\delta := \Big\| |\phi'| + \sum_{j \ge 1} |\psi_j'| \Big\|_{L^\infty([0,1])} < \infty \, . \tag{6.28}$$

In the above model, the source term f is fixed independently of y and should therefore be defined on the union of all domains $D(y)$ for $y \in U$. For simplicity, we assume that f is defined over $\tilde{D} := [0,1] \times [0,R]$ and that $f \in L^2(\tilde{D})$. It follows that $f \in L^2(D(y))$, with $\|f\|_{L^2(D(y))} \le \|f\|_{L^2(\tilde{D})}$, for all $y \in U$.

Our strategy for treating this model is the following. We use the bijective map

$$\Phi(y) : \ x := (x_1, x_2) \mapsto \Phi(x, y) := (x_1, x_2 \phi(x_1, y)), \tag{6.29}$$

to transport back the solution $v(y) \in H_0^1(D(y))$ into the reference domain $D := [0,1]^2$ according to

$$u(y) := v(y) \circ \Phi(y), \tag{6.30}$$

meaning that $u(x, y) = v(\Phi(x, y), y)$ for all $x \in D$. We then study the linear elliptic PDE satisfied by $u(y)$ on D. This PDE has matricial diffusion coefficients and source term that depends on y. We then show that under certain conditions on the functions ψ_j, one can establish the $\mathbf{HA}(p, \varepsilon)$ for the solution map $y \mapsto u(y)$, using the framework of Theorem 2.3.

6.3.1 A change of variables

Having fixed a parameter $y \in U$, we use in what follows the simpler notation u, v and Φ for $u(y)$, $v(y)$ and $\Phi(y)$. The transformation Φ maps the domain D into $D(y)$ and the boundary ∂D into $\partial D(y)$. The function $v \in H_0^1(D(y))$ is the unique solution of the variational problem:

$$\int_{D(y)} \nabla v \cdot \nabla w = \int_{D(y)} f w, \quad w \in H_0^1(D(y)). \tag{6.31}$$

The function $u = v \circ \Phi$ is defined on D, and we have

$$\nabla u(x) = (D_\Phi(x))^t \nabla v(\Phi(x)), \tag{6.32}$$

where, for $x = (x_1, x_2) \in D$,

$$D_\Phi(x) = \begin{bmatrix} 1 & 0 \\ x_2 \phi'(x_1, y) & \phi(x_1, y) \end{bmatrix}, \tag{6.33}$$

where the derivative in ϕ' is meant with respect to the variable x_1. Since Φ is Lipschitz continuous on D, it follows that $u \in X := H_0^1(D)$. Pulling back the variational formula (6.31) to the reference domain D using the bijective map Φ, one obtains that u is the unique solution to the variational problem

$$\int_D \Big((D_\Phi^{-1})^t \nabla u\Big) \cdot \Big((D_\Phi^{-1})^t \nabla w\Big)\ J_\Phi = \int_D (f \circ \Phi)\ w\ J_\Phi, \qquad w \in V, \tag{6.34}$$

where J_Φ is the Jacobian of the transformation Φ which is given by $J_\Phi(x) = \phi(x_1, y)$ for any $x \in D$. We introduce the maps A and g defined on $D \times U$ by

$$A(x,y) := \phi(x_1,y)(D_\Phi^{-1})(D_\Phi^{-1})^t = \begin{bmatrix} \phi(x_1,y) & -x_2\phi'(x_1,y) \\ \\ -x_2\phi'(x_1,y) & \frac{1+(x_2\phi'(x_1,y))^2}{\phi(x_1,y)} \end{bmatrix}, \tag{6.35}$$

and

$$g(x,y) := \phi(x_1,y)(f \circ \Phi)(x) = \phi(x_1,y) f\Big(x_1, x_2\phi(x_1,y)\Big) \tag{6.36}$$

and the sesquilinear and antilinear forms $B(y)$ and $F(y)$ defined on X by

$$B(y)(w_1,w_2) := \int_D \Big(A(x,y)\nabla w_1(x)\Big) \cdot \overline{\nabla w_2(x)} dx \tag{6.37}$$

and

$$F(y)(w) := \int_D g(x,y)\overline{w(x)} dx. \tag{6.38}$$

To be consistent with our previous notations, we use the notations $B(w_1, w_2, y)$ instead of $B(y)(w_1, w_2)$ and $F(w, y)$ instead of $F(y)(w)$. From (6.34), we deduce that $u(y) \in X$ is the unique solution to the variational problem

$$B(u(y), w, y) = F(w, y), \quad w \in X. \tag{6.39}$$

This is a linear elliptic PDE with parametric matricial diffusion coefficients and parametric source terms. Our next goal is to discuss under which circumstances the assumptions of Theorem 2.3 are satisfied for this problem, with $X = Y = H_0^1(D)$. We introduce the sequence $b := (b_j)_{j \geq 1}$, with

$$b_j := \|\psi_j\|_{L^\infty([0,1])} + \|\psi_j'\|_{L^\infty([0,1])} \tag{6.40}$$

and assume that $b \in \ell^p(\mathbb{N})$ for some $p < 1$. We propose to use this sequence for the verification of the assumptions of Theorem 2.3.

6.3.2 Analyticity of the map F

We first study the antilinear forms $w \mapsto F(w, y)$. The assumption that $f \in L^2(\tilde{D})$ ensures a uniform bound of the form

$$|F(w,y)| \leq C\|w\|_Y, \quad w \in Y,\ y \in U \tag{6.41}$$

where

$$C := C_P \sup_{y \in U} \|g(y)\|_{L^2(D)} \leq C_P R \|f\|_{L^2(\tilde{D})}, \tag{6.42}$$

with C_P the Poincaré constant for D. More assumptions on f are needed in order to define an holomorphic extension of F in a neighbourhood of U. One sufficient assumption is that the map

$$x_2 \mapsto f(\cdot, x_2), \tag{6.43}$$

from $[0,R]$ to $L^2([0,1])$ is analytic on $[0,R]$. Note that this assumption imposes smooth dependence of f on the second variable. It holds of course if f is analytic in both variables, for example if f is a constant. Since $[0,R]$ is compact, there exists $\varepsilon_1 > 0$ such that the previous map has an holomorphic and uniformly bounded extension on the domain

$$\mathcal{C}_{\varepsilon_1} := \Big\{ \xi \in \mathbb{C} : \ \ \mathrm{dist}(\xi,[0,R]) < \varepsilon_1 \Big\}. \tag{6.44}$$

Let now $\rho := (\rho_j)_{j\geq 1}$ a sequence of numbers strictly greater than 1 satisfying

$$\sum_{j=1}^{\infty} (\rho_j - 1) b_j \leq \varepsilon_1. \tag{6.45}$$

We consider the domain $\mathcal{O}_\rho = \otimes_{j\geq 1}\mathcal{O}_{\rho_j}$ where the definition of the open complex domains $\mathcal{O}_s$ is given in (5.14). For $z \in \mathcal{O}_\rho$ and $y \in U$ such that $|z_j - y_j| < \rho_j - 1$ for any $j \geq 1$, one has for any $t \in [0,1]$

$$|\phi(t,z) - \phi(t,y)| = \Big|\sum_{j\geq 1}(z_j - y_j)\psi_j(t)\Big| < \sum_{j\geq 1}(\rho_j - 1)b_j \leq \varepsilon_1. \tag{6.46}$$

Since by (6.27), $\phi(t,y) \in [0,R]$, then one has $\phi(t,z) \in \mathcal{C}_{\varepsilon_1}$. It follows that the map $y \mapsto g(y)$ defined from U into $L^2(D)$ admits an holomorphic extension $z \mapsto g(z)$ on the domain $\mathcal{O}_\rho$, defined by

$$g(x,z) := \phi(x_1,z) f(x_1, x_2\phi(x_1,z)). \tag{6.47}$$

Consequently, the map $y \mapsto F(y)$ from U to Y^* admits a uniformly bounded holomorphic extension on the domain $\mathcal{O}_\rho$, defined by

$$F(z)(w) := \int_D g(x,z)\overline{w(x)}dx. \tag{6.48}$$

6.3.3 Analyticity of the map B

The map $y \mapsto A(y)$ defined by (6.35) is a rational function of the components y_j of $y \in U$ taking values in the space of 2×2 symmetric matrices. Let $0 < \varepsilon \leq \frac{r}{2}$ where r is the lower bound in (6.27) and where $\rho := (\rho_j)_{j\geq 1}$ is a sequence of numbers strictly greater than 1 satisfying

$$\sum_{j=1}^{\infty} (\rho_j - 1) b_j \leq \varepsilon\ . \tag{6.49}$$

For $z \in \mathcal{O}_\rho$ and $y \in U$ such that $|z_j - y_j| < \rho_j - 1$ for every $j \geq 1$, one has by (6.46) that $|\phi(t,z) - \phi(t,y)| \leq \varepsilon$ for any $t \in [0,1]$, therefore

$$\Re\Big(\phi(t,z)\Big) \geq \phi(t,y) - \varepsilon \geq r - \varepsilon \geq \frac{r}{2}, \quad t \in [0,1]. \tag{6.50}$$

In addition, we have for all $x \in D$,

$$\frac{r}{2} \leq |\phi(x_1,z)| = \left|\phi(x_1,y) + \sum_{j\geq 1}(z_j - y_j)\psi_j(x_1)\right| \leq R + \varepsilon \tag{6.51}$$

and

$$|\phi'(x_1,z)| = \left|\phi'(x_1,y) + \sum_{j\geq 1}(z_j - y_j)\psi_j'(x_1)\right| \leq \delta + \varepsilon, \tag{6.52}$$

It follows that the map $y \mapsto A(y)$ admits a uniformly bounded holomorphic extension $z \mapsto A(z)$ on $\mathcal{O}_\rho$ defined by

$$A(x,z) = \begin{bmatrix} \phi(x_1,z) & -x_2\phi'(x_1,z) \\ \\ -x_2\phi'(x_1,z) & \frac{1+(x_2\phi'(x_1,z))^2}{\phi(x_1,z)} \end{bmatrix}, \quad x \in D. \tag{6.53}$$

As a consequence, the map $y \mapsto B(y)$ from U to $\mathcal{B}(X \times X)$, the space of continuous sesquilinear forms over X, admits a uniformly bounded holomorphic extension on $\mathcal{O}_\rho$, defined by

$$B(w_1,w_2,z) := \int_D \Big(A(x,z)\nabla w_1\Big) \cdot \overline{\nabla w_2}, \qquad w_1,w_2 \in X. \tag{6.54}$$

Note that the uniform bound is independent of the choice of ρ that satisfies (6.49).

Concerning the uniform inf-sup condition, we establish the stronger property that the sesquilinear forms $B(z)$ are uniformly coercive on the domains $\mathcal{O}_\rho$, up to restricting the range of ε to a smaller interval than $]0, r/2]$.

We introduce the notation $y := \Re(z)$ and $s := \Im(z)$. Using (6.50), (6.51) and (6.52), we have for any $t \in [0,1]$ and any $z \in \mathcal{O}_\rho$ that

$$\phi(t,y) = \Re(\phi(t,z)) \geq \frac{r}{2} \quad \text{and} \quad |\phi(t,y)| \leq |\phi(t,z)| \leq R + \frac{r}{2} \quad \text{and} \quad |\phi'(t,y)| \leq |\phi'(t,z)| \leq \delta + \frac{r}{2}. \tag{6.55}$$

The symmetric real matrices $A(x,y)$ have determinants equal to 1 and, from the above inequalities, their traces are positive and bounded by

$$C_1 := R + \frac{r}{2} + \frac{2}{r}\Big(1 + (\delta + \frac{r}{2})^2\Big). \tag{6.56}$$

Therefore these matrices are positive definite with coercivity constant $\tilde{r} := 1/C_1$. This implies in particular that

$$|B(w,w,y)| \geq \tilde{r}\|w\|_X^2, \quad w \in X,\ y = \Re(z),\ z \in \mathcal{O}_\rho. \tag{6.57}$$

To prove the uniform coercivity of the bilinear forms $B(z)$ on $\mathcal{O}_\rho$, it is therefore sufficient to prove that the parametric sesquilinear forms $B(z) - B(y)$ have norms strictly smaller than $\tilde{r}/2$, uniformly on $\mathcal{O}_\rho$. To verify this, we note that the three entries in the symmetric matrices $(A(x,z) - A(x,y))$ are $\phi(x_1,s)$, $-x_2\phi'(x_1,s)$ and

$$\xi(x,z) := \frac{1 + (x_2\phi'(x_1,z))^2}{\phi(x_1,z)} - \frac{1 + (x_2\phi'(x_1,y))^2}{\phi(x_1,y)}. \tag{6.58}$$

Since $\mathcal{O}_\rho$ is contained in the tensorized strip $\otimes_{j\geq 1}\{|\Im(z_j)| \leq \rho_j - 1\}$, the condition (6.49) readily implies that the two first entries are bounded by ε. Concerning the third entry, we have

$$\xi(x,z) = \Big(1 + (x_2\phi'(x_1,y))^2\Big)\Big(\frac{1}{\phi(x_1,y) + i\phi(x_1,s)} - \frac{1}{\phi(x_1,y)}\Big) + \frac{2x_2^2\phi'(x_1,y)\phi'(x_1,s) - \phi'(x_1,s)^2}{\phi(x_1,z)}. \tag{6.59}$$

Therefore, combining the previous inequalities, we obtain

$$|\xi(x,z)| \leq \Big(1 + (\delta + \frac{r}{2})^2\Big)\frac{\varepsilon}{(\frac{r}{2})^2} + \frac{2(R + \frac{r}{2})\varepsilon + \varepsilon^2}{\frac{r}{2}}. \tag{6.60}$$

We conclude that the norms of the matrices $(A(t,z) - A(t,y))$ are uniformly bounded by $C_2\varepsilon$ for some constant C_2 depending on R, r and δ. Up to choosing ε small enough, we have $C_2\varepsilon < \frac{\tilde{r}}{2}$, in which case, we have for any $w \in V$

$$|B(w,w,z) - B(w,w,y)| \leq \int_D \Big|\Big((A(x,z) - A(x,y))\nabla w\Big) \cdot \overline{\nabla w}\Big| \leq \frac{\tilde{r}}{2}\int_D |\nabla w|^2, \tag{6.61}$$

Therefore, with this value of $r > 0$, for any $z \in \mathcal{O}_\rho$ and for any $w \in X$ holds

$$|B(w, w, z)| \geq \frac{\tilde{r}}{2} \|w\|_X^2 \ . \tag{6.62}$$

This uniform coercivity implies both inf-sup conditions (2.21) with $X = Y = H_0^1(D)$. To complete the verification of the assumptions of Theorem 2.3, we only need to possibly reduce the value of ε so that $\varepsilon \leq \varepsilon_1$ where ε_1 was used in the proof of the analyticity of the antilinear form $F(z)$. $\square$

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