Higher index focus-focus singularities in the Jayne-Cummings-Gaudin model : symplectic invariants and monodromy.

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Abstract

We study the symplectic geometry of the Jaynes-Cummings-Gaudin model with $n = 2m - 1$ spins. We show that there are focus-focus singularities of maximal Williamson type $(0, 0, m)$. We construct the linearized normal flows in the vicinity of such a point and show that soliton type solutions extend them globally on the critical torus. This allows us to compute the leading term in the Taylor expansion of the symplectic invariants and the monodromy associated to this singularity.

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1   Introduction

The theory of integrable systems started with the work of Liouville [1] where he defined the general concept of integrable systems and their integration by "quadratures". Let $\mathcal{M}$ be a symplectic space of dimension $2n$, the system is Liouville integrable if we can define n Hamiltonians H_i in involution such that "generically" $dH_1 \wedge dH_2 \wedge \cdots \wedge dH_n$ has maximal rank $r = n$. The remarkable Arnold-Liouville theorem states that the space $\mathcal{M}$ is then fibered by invariant n-dimensional Lagrangian tori. This is a *semi global* result as it gives a global information on the fiber, which is a torus, and it is sufficient to assert that the motion is quasi periodic almost everywhere. This fibration however contains singular fibers i.e. fibers containing points where the rank r is not maximal, which play a very important role for the global properties of the system, see e.g. [11]. In this work, we will concentrate on special singularities where the rank $r = 0$. They correspond to equilibrium points, which can be stable or unstable, and have also a very rich physical content.

An equilibrium point $x^{(0)}$ is a simultaneous critical point for all the H_i, in other words, it is a point at which the differential of the moment map μ vanishes. Expanding the H_i around such a point, we get n Poisson commuting quadratic forms Q_i.

We shall consider the case of a purely focus-focus equilibrium point of Williamson type ($m_e = m_h = 0$ and $m_{ff} = m$). In this case, the dimension of $\mathcal{M}$ is equal to $4m$, and there exist canonical coordinates (p_i, q_i) such that the quadratic forms Q_i are linear combinations of the quadratic normal forms

$$\begin{aligned}
K_j &= p_{2j}q_{2j} + p_{2j+1}q_{2j+1}, \quad j = 1, ..., m \\
L_j &= -p_{2j}q_{2j+1} + p_{2j+1}q_{2j}, \quad j = 1, ..., m
\end{aligned}$$

By Eliasson theorem [8], this description extends to a local neighborhood of the point x_0. There exists a *symplectic* diffeomorphism $\Phi : U \subset \mathbb{R}^{2n} \longrightarrow \mathcal{V} \subset \mathcal{M}$ mapping the neighborhood U of the origin to the neighborhood $\mathcal{V}$ of $x^{(0)}$, and a local diffeomorphism $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ in a neighborhood of the origin such that:

$$\mu \circ \Phi = \psi \circ \mu^{(0)}$$

where $\mu^{(0)}$ is the quadratic moment map $(p, q) \to (K, L)$. An important consequence of this theorem is that it allows to extend the flows associated to the quadratic generators K_j and L_j from a neighborhood of $x^{(0)}$ (*local* level) to a family of fibers containing the level set of $x^{(0)}$ (*semi global* level). These flows will be called *normal flows* and will play an important role in this paper.

Using the complex variables

$$w_j = p_{2j} + ip_{2j+1}, \quad z_j = q_{2j} + iq_{2j+1}, \quad j = 1, ..., m \tag{1}$$

we have

$$w_j \bar{z}_j = K_j + iL_j$$

hence, the level set $\mathcal{S}$ of the equilibrium point is the image, by the diffeomorphism Φ, of the product of m two dimensional components, themselves the union of two complex planes intersecting transversely

$$\mathcal{C}_j : w_j = 0 \text{ or } z_j = 0, \quad j = 1, ..., m$$

It is a $2m$-dimensional cone $\mathcal{C}$

$$\mathcal{C} = \prod_{j=1}^{m} \mathcal{C}_j$$

We see that in a neighborhood of the equilibrium point, the phase-space fibration induced by the moment map is symplectically equivalent to the fibration of a direct product of m independent integrable four dimensional dynamical systems with a focus-focus singularity. An important question is to see whether such a simple description holds also at the semi-global level.

In the Jaynes-Cummings-Gaudin model that we will consider in this paper, the level set of the equilibrium point is a compact pinched torus of dimension $2m$. A result of Tien Zung [2] asserts that, at the topological level, this pinched torus is equivalent to a product of m two dimensional pinched tori. But this is not true at the symplectic level, and it is the purpose of this work to compute symplectic invariants defined in [3, 4] preventing this simple product decomposition of the singular fiber.

To achieve this goal, we will combine two informations. The first one is a very simple description of the normal modes around the singularity, which provide a complete *local* description of the system. The second one, of a *semi global* nature, is provided by the explicit solitonic solutions of the equations of motion on the singular fibre. It turns out that the two objects, normal modes and solitons, match perfectly : solitons are just global extensions of normal modes to the full singular fiber. The $2m$ real parameters describing the initial conditions of the solitonic solution may be seen as coordinates on the singular fiber which precisely realize its decomposition into a product of m two dimensional singular tori, thereby extending to the whole fiber the decomposition provided by the normal coordinates in the vicinity of the critical point.
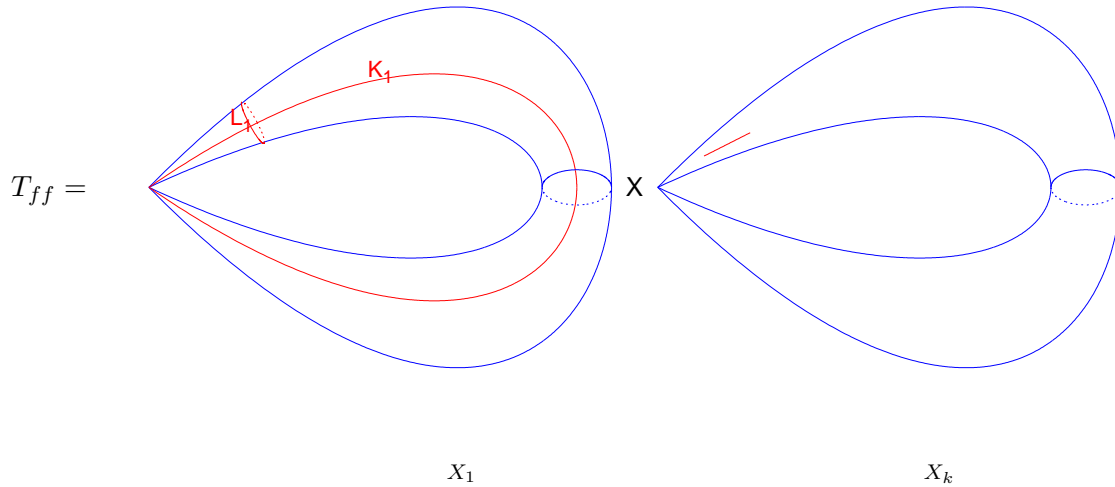
$$T_{ff} =$$

K_1

L

X

X_1

X_k

Figure 1: The singular torus seen as a product of two dimensional pinched tori whose coordinates are the parameters X_k entering the solitonic solution of the equations of motion. A big motion (K_1) on one of the components of the torus induces a motion on the other components of the torus as well.

The plan of the paper is as follows. In Section 2 we recall the definition of the Jaynes-Cummings-Gaudin model and basic facts about its integrability. In Section 3 we study the vicinity of an unstable point of Williamson type ($m_e = m_h = 0, m_{ff} = m$). In particular we give a simple description of normal coordinates around that point. In Section 4 we relate these normal modes to solitonic solutions on the pinched torus, the level set of the unstable point. In Section 5 we construct closed periodic trajectories on Liouville tori close to the critical one. This requires to take into account both diagonal [4] and non diagonal [5] components of the motion which signal the obstruction to Tien Zung decomposition at the symplectic level. The action variables associated to these periodic motion provide the symplectic invariants in the sense of San Vũ Ngoc. We use them to determine Duistermaat's monodromy [9]. A number of technical details are provided in the Appendices for the sake of completeness.

Acknowlegment. We wish to thank San Vũ Ngoc for very helpful discussions.

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2 The Jaynes-Cummings-Gaudin model

2.1 Basic definitions and integrability

This model describes a collection of n spins coupled to a single harmonic oscillator. It derives from the Hamiltonian:

$$H = \sum_{j=1}^{n} (2\epsilon_j + \omega)s_j^z + \omega \bar{b}b + \sum_{j=1}^{n} \left(\bar{b}s_j^- + bs_j^+ \right) \tag{2}$$

The $\vec{s}_j$ are spin variables, and $b, \bar{b}$ is a harmonic oscillator. The Poisson brackets read

$$\{s_j^a, s_j^b\} = -\epsilon_{abc}s_j^c, \quad \{b, \bar{b}\} = i \tag{3}$$

The $\vec{s}_j$ brackets are degenerate. We fix the value of the Casimir functions

$$||\vec{s}_j||^2 = \sum_{a=1}^{3} s_j^a s_j^a = s^2$$

Phase space has dimension $2(n+1)$. In the Hamiltonian we have used $s_j^{\pm} = s_j^1 \pm is_j^2$ which have Poisson brackets $\{s_j^z, s_j^{\pm}\} = \pm is_j^{\pm}$, $\{s_j^+, s_j^-\} = 2is_j^z$. The equations of motion read

$$\dot{b} = -i\omega b - i\sum_{j=1}^{n} s_j^- \tag{4}$$

$$\dot{s_j^z} = i(\bar{b}s_j^- - bs_j^+) \tag{5}$$

$$\dot{s_j^+} = i(2\epsilon_j + \omega)s_j^+ - 2i\bar{b}s_j^z \tag{6}$$

$$\dot{s_j^-} = -i(2\epsilon_j + \omega)s_j^- + 2ibs_j^z \tag{7}$$

Introducing the Lax matrices

$$L(\lambda) = 2\lambda\sigma^z + 2(b\sigma^+ + \bar{b}\sigma^-) + \sum_{j=1}^{n} \frac{\vec{s}_j \cdot \vec{\sigma}}{\lambda - \epsilon_j} \tag{8}$$

$$M(\lambda) = -i\lambda\sigma^z - i\frac{\omega}{2}\sigma^z - i(b\sigma^+ + \bar{b}\sigma^-) \tag{9}$$

where σ^a are the Pauli matrices, $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm i\sigma^y)$,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

it is not difficult to check that the equations of motion are equivalent to the Lax equation

$$\dot{L}(\lambda) = [M(\lambda), L(\lambda)] \tag{10}$$

Letting

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}$$

we have

$$A(\lambda) = 2\lambda + \sum_{j=1}^{n} \frac{s_j^z}{\lambda - \epsilon_j} \tag{11}$$

$$B(\lambda) = 2b + \sum_{j=1}^{n} \frac{s_j^-}{\lambda - \epsilon_j} \tag{12}$$

$$C(\lambda) = 2\bar{b} + \sum_{j=1}^{n} \frac{s_j^+}{\lambda - \epsilon_j} \tag{13}$$

The non vanishing Poisson brackets of these functions are simple :

$$\{A(\lambda), B(\mu)\} = \frac{i}{\lambda - \mu}(B(\lambda) - B(\mu)) \tag{14}$$

$$\{A(\lambda), C(\mu)\} = -\frac{i}{\lambda - \mu}(C(\lambda) - C(\mu)) \tag{15}$$

$$\{B(\lambda), C(\mu)\} = \frac{2i}{\lambda - \mu}(A(\lambda) - A(\mu)) \tag{16}$$

It follows immediately that $\mathrm{Tr}\,(\mathrm{L}^2(\lambda)) = 2\mathrm{A}^2(\lambda) + 2\mathrm{B}(\lambda)\mathrm{C}(\lambda)$ Poisson commute for different values of the spectral parameter:

$$\{\mathrm{Tr}\,(\mathrm{L}^2(\lambda_1)), \mathrm{Tr}\,(\mathrm{L}^2(\lambda_2))\} = 0$$

Hence $\Lambda(\lambda) \equiv \frac{1}{2}\mathrm{Tr}\,(\mathrm{L}^2(\lambda))$ generates Poisson commuting quantities. One has

$$\Lambda(\lambda) = \frac{Q_{2n+2}(\lambda)}{\prod_j(\lambda - \epsilon_j)^2} = 4\lambda^2 + 4H_{n+1} + 2\sum_{j=1}^{n} \frac{H_j}{\lambda - \epsilon_j} + \sum_{j=1}^{n} \frac{s^2}{(\lambda - \epsilon_j)^2} \tag{17}$$

where the $(n+1)$ commuting Hamiltonians H_j, $j = 1, \cdots, n+1$ read

$$H_{n+1} = b\bar{b} + \sum_j s_j^z \tag{18}$$

and

$$H_j = 2\epsilon_j s_j^z + (bs_j^+ + \bar{b}s_j^-) + \sum_{k \neq j} \frac{s_j \cdot s_k}{\epsilon_j - \epsilon_k}, \quad j = 1, \cdots, n \tag{19}$$

The physically interesting Hamiltonian eq.(2) is

$$H = \omega H_{n+1} + \sum_{j=1}^{n} H_j \tag{20}$$

2.2 Critical points

The critical points are equilibrium points for all the Hamiltonians H_j, $j = 1, \cdots, n+1$. At such points the derivatives with respect of all coordinates on phase space vanish. In particular, since

$$\frac{\partial H_{n+1}}{\partial \bar{b}} = b, \quad \frac{\partial H_j}{\partial \bar{b}} = s_j^-$$

we see that the critical points must be located at

$$b = \bar{b} = 0, \quad s_j^\pm = 0, \quad s_j^z = e_j s, \quad e_j = \pm 1 \tag{21}$$

When we expand around a configuration eq.(21), all the quantities $(b, \bar{b}, s_j^+, s_j^-)$ are first order, but s_j^z is second order because

$$s_j^z = e_j \sqrt{s^2 - s_j^+ s_j^-} = s e_j - \frac{e_j}{2s} s_j^+ s_j^- + \cdots, \quad e_j = \pm 1$$

It is then simple to see that all first order terms in the expansions of the Hamiltonians H_j vanish. Hence we have found 2^n critical points.

3 The vicinity of an unstable fixed point

3.1 Normal Forms

In the following discussion, we are considering the system with an odd number of spins $n = 2m-1$ with $m \geq 1$, and we are assuming that the equilibrium point defined by eq. (21) has the Williamson type $m_e = m_h = 0$ and $m_{ff} = m$, i.e. it consists locally of a direct product of m elementary focus-focus singularities. It has been shown before that such points can be obtained by choosing the Zeeman energies ϵ_j in a suitable region in parameter space [14, 13]. The quadratic normal form around an equilibrium point can be easily obtained from the Lax representation [10, 14] and this is recalled in Section 6 below. The above assumption on the Williamson type is equivalent to the statement that the classical Bethe equation (52) has no real root (hence the choice n odd). Since the equation is real, it has m pairs of complex conjugated roots $E_j, \bar{E}_j$, for $1 \leq j \leq m$. To first order in small deviations around the equilibrium point, the normal coordinates are $B(E_j)$, $B(\bar{E}_j)$, $C(E_j)$, and $C(\bar{E}_j)$ where the functions $B(\lambda), C(\lambda)$ are those defined in eqs.(12,13). The precise relation with the canonical coordinates eq.(1) is:

$$B(E_j) = -ia'(E_j)z_j, \quad B(\bar{E}_j) = w_j \tag{22}$$
$$C(E_j) = \bar{w}_j, \quad C(\bar{E}_j) = ia'(\bar{E}_j)\bar{z}_j \tag{23}$$

These expressions for z_j and w_j as functions of the dynamical variables of the Gaudin model should be understood as giving the differential of the function Φ^{-1} at the equilibrium point. An important consequence of Eliasson's theorem is that the knowledge of this differential is sufficient to identify the generators of the normal flows, as long as we are interested in their restriction to the singular torus.

The symplectic form in terms of normal coordinates reads:

$$\omega^{(0)} = \frac{1}{2} \sum_j dz_j \wedge d\bar{w}_j + d\bar{z}_j \wedge dw_j$$

As explained in the Introduction, the components of the moment map $\mu^{(0)}$ are the quadratic Hamiltonians:

$$L_j = \frac{1}{2i}(\bar{z}_j w_j - z_j \bar{w}_j), \quad K_j = \frac{1}{2}(\bar{z}_j w_j + z_j \bar{w}_j)$$

The Hamilton equations of motion associated to $\omega^{(0)}$ are:

$$\dot{z}_j = 2\frac{\partial H}{\partial \bar{w}_j}, \quad \dot{w}_j = -2\frac{\partial H}{\partial \bar{z}_j}, \quad \dot{\bar{z}}_j = 2\frac{\partial H}{\partial w_j}, \quad \dot{\bar{w}}_j = -2\frac{\partial H}{\partial z_j}$$

From them, we see that the flows associated to K_j, L_j are

$$K_j : (z_j \to e^{a_j} z_j, w_j \to e^{-a_j} w_j), \quad L_j : (z_j \to e^{i\theta_j} z_j, w_j \to e^{i\theta_j} w_j)$$

3.2 Generators of normal flows

We begin by expanding the function $\mu^{(0)} \circ \Phi^{-1}$ to second order in small deviations from the equilibrium point, recalling that $B(E_j)$, $B(\bar{E}_j)$, $C(E_j)$, and $C(\bar{E}_j)$ are first order in these deviations. This gives:

$$K_j = \frac{i}{2}\left(\frac{B(E_j)C(E_j)}{a'(E_j)} - \frac{B(\bar{E}_j)C(\bar{E}_j)}{a'(\bar{E}_j)}\right) \tag{24}$$

$$L_j = -\frac{1}{2}\left(\frac{B(E_j)C(E_j)}{a'(E_j)} + \frac{B(\bar{E}_j)C(\bar{E}_j)}{a'(\bar{E}_j)}\right) \tag{25}$$

But Eliasson's theorem implies that $\mu^{(0)} \circ \Phi^{-1} = \psi^{-1} \circ \mu$, so we should be able to express these quantities in terms of the conserved Hamiltonians $H_1,...,H_{n+1}$. In Section 6 below, we show that:

$$B(E_j)C(E_j) = 4E_j^2 + 4H_{n+1} + \sum_{i=1}^{n} \frac{2H_i}{E_j - \epsilon_i} + \sum_{k=1}^{n} \frac{s^2}{(E_j - \epsilon_i)^2} + \dots \tag{26}$$

where the neglected terms are of order 3 and higher in small deviations from the equilibrium point. This formula enables to construct the Taylor expansion of the function ψ^{-1} up to first order. We get:

$$K_j = i\left(\frac{2}{a'(E_j)} - \frac{2}{a'(\bar{E}_j)}\right)\delta H_{n+1} + i\sum_{i=1}^{n}\left(\frac{1}{a'(E_j)(E_j - \epsilon_i)} - \frac{1}{a'(\bar{E}_j)(\bar{E}_j - \epsilon_i)}\right)\delta H_i + \tag{27}$$

$$L_j = -\left(\frac{2}{a'(E_j)} + \frac{2}{a'(\bar{E}_j)}\right)\delta H_{n+1} - \sum_{i=1}^{n}\left(\frac{1}{a'(E_j)(E_j - \epsilon_i)} + \frac{1}{a'(\bar{E}_j)(\bar{E}_j - \epsilon_i)}\right)\delta H_i + \tag{28}$$

Here $\delta H_i = H_i - H_{i,\text{eq}}$, where $H_{i,\text{eq}}$ denotes the value of H_i at the equilibrium point. The neglected term are of order at least two in the δH_i's. When we consider fibers of the moment

map closer and closer to the fiber containg the equilibrium point, the flows generated by the linear terms in the δH_i's in equations (27) and (28) tend towards the normal flows defined in the Introduction. In particular, they coincide with these normal flows on the fiber containing the equilibrium point.

4 Normal flows on a singular torus

4.1 Normal flows and soliton solutions

From the previous discussion, the normal flows on the singular torus are given by the Hamiltonian flows associated to the generators K_j and L_j defined in terms of the conserved Hamiltonians in eqs. (27) and (28). On the singular torus, these flows take the form of solitonic trajectories. For the reader's convenience a complete derivation of the soliton formulae, which were constructed in [13], is given in Section 9 below. Here, we summarize the main features of these solitonic solutions.

We recall that $n = 2m - 1$, and we assume that $\mathcal{E}^0 = \emptyset$ in the construction of Section 9. We define column vectors $(E^j)_l = E_l^j$ and $(XE^j)_l = X_l E_l^j$, $l = 1 \cdots 2m$, and two polynomials

$$\mathcal{P}^-(\lambda) = \frac{1}{D_{n+1}} \det \begin{pmatrix} 1 & \lambda & \cdots & \lambda^{n-} & 0 & 0 & \cdots & 0 \\ 1 & E & \cdots & E^{n-} & X & XE & \cdots & XE^{n+} \end{pmatrix}$$

and

$$\mathcal{P}^+(\lambda) = \frac{1}{D_0} \det \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & \lambda & \cdots & \lambda^{n+} \\ 1 & E & \cdots & E^{n-} & X & XE & \cdots & XE^{n+} \end{pmatrix}$$

where D_0 and D_{n+1} are the determinants:

$$D_0 = \det \begin{pmatrix} 1 & E & \cdots & E^{n-} & X & XE & \cdots & XE^{n+-1} \end{pmatrix} \tag{29}$$

$$D_{n+1} = \det \begin{pmatrix} 1 & E & \cdots & E^{n--1} & X & XE & \cdots & XE^{n+} \end{pmatrix} \tag{30}$$

The indices $n_\pm$ are defined as

$$n_+ = \mathrm{degree}(\mathcal{P}^+(\lambda)) = \frac{1}{2}(n-1) = m - 1, \quad n_- = \mathrm{degree}(\mathcal{P}^-(\lambda)) = \frac{1}{2}(n+1) = m$$

The X_l are given by:

$$X_l(\{t\}) = X_l(0) e^{\sqrt{-1}\left(\sum_i \frac{se_i}{E_l - \epsilon_i} t_i - t_{n+1}\right)}, \tag{31}$$

where t_i are the times flows associated to the Hamiltonians H_i and $e_i = \pm 1$ characterize the critical point $s_i^z = se_i$. The constants $X_l(0)$ are subjected to the reality conditions:

$$\overline{X_l(\{t\})} X_{\bar{l}}(\{t\}) = \overline{X_l(0)} X_{\bar{l}}(0) = -\frac{1}{4} \tag{32}$$

where $X_{\bar{l}}$ is associated to $E_{\bar{l}} \equiv \overline{E_l}$. The function $C(\lambda)$ is given by :

$$C(\lambda) = \tag{33}$$

$$2\frac{\det \begin{pmatrix} 1 & \lambda & \cdots & \lambda^{n-} & 0 & 0 & \cdots & 0 \\ 1 & E & \cdots & E^{n-} & X & XE & \cdots & XE^{n+} \end{pmatrix} \det \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & \lambda & \cdots & \lambda^{n+} \\ 1 & E & \cdots & E^{n-} & X & XE & \cdots & XE^{n+} \end{pmatrix}}{\prod_j (\lambda - \epsilon_j) \det \begin{pmatrix} 1 & E & \cdots & E^{n--1} & X & XE & \cdots & XE^{n+} \end{pmatrix}^2}$$

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Let us now specialize the evolution equation (31) to the normal flows defined through their generators (27) and (28). Let us denote by a_l (resp. θ_l) the time variable of the normal flow generated by K_l (resp. L_l). For an observable $X(t_1, \cdots, t_{n+1})$, we have

$$X(a_l) = X\left(t_i = i\left(\frac{1}{a'(E_l)(E_l - \epsilon_i)} - \frac{1}{a'(\bar{E}_l)(\bar{E}_l - \epsilon_i)}\right)a_l, \; t_{n+1} = i\left(\frac{2}{a'(E_l)} - \frac{2}{a'(\bar{E}_l)}\right)a_l\right)$$

$$X(\theta_l) = X\left(t_i = -\left(\frac{1}{a'(E_l)(E_l - \epsilon_i)} + \frac{1}{a'(\bar{E}_l)(\bar{E}_l - \epsilon_i)}\right)\theta_l, \; t_{n+1} = -\left(\frac{2}{a'(E_l)} + \frac{2}{a'(\bar{E}_l)}\right)\theta_l\right)$$

Let us apply this formula to $X_k(\{t\})$ in eq.(31). We have to evaluate:

$$\frac{1}{a'(E_l)}\sum_{i=1}^{n}\frac{se_i}{(E_k - \epsilon_i)(E_l - \epsilon_i)}$$

Suppose first $k \neq l$. We have:

$$\frac{1}{a'(E_l)}\sum_{i=1}^{n}\frac{se_i}{(E_k - \epsilon_i)(E_l - \epsilon_i)} = \frac{1}{a'(E_l)}\sum_{i=1}^{n}\frac{1}{E_k - E_l}\left(\frac{se_i}{E_l - \epsilon_i} - \frac{se_i}{E_k - \epsilon_i}\right)$$

$$= \frac{1}{a'(E_l)}\sum_{i=1}^{n}\frac{1}{E_k - E_l}(-2E_l + 2E_k) = \frac{2}{a'(E_l)}$$

where we have used the classical Bethe equation:

$$a(E) = 2E + \sum_{i}\frac{se_i}{E - \epsilon_i} = 0$$

If $k = l$, we have directly:

$$\frac{1}{a'(E_l)}\sum_{i=1}^{n}\frac{se_i}{(E_l - \epsilon_i)^2} = \frac{1}{a'(E_l)}(2 - a'(E_l))$$

Using these sums in eq.(31), we arrive at the very simple result:

$$X_k(a_l) = X_k(0)e^{a_l(\delta_{kl} - \delta_{k\bar{l}})}, \quad X_k(\theta_l) = X_k(0)e^{i\theta_l(\delta_{kl} + \delta_{k\bar{l}})} \tag{34}$$

It is clear that the reality condition (32) is preserved by these flows. The meaning of this formula is that the solitons can be viewed as a non-linear global extension of the normal modes on the critical torus. The a_l trajectories give m non compact cycles, while the θ_l trajectories give the remaining m compact cycles. In view of eq.(31) we see that the soliton solution depends on $2m$ complex parameters $X_l(0)$ submitted to the m complex conditions eq.(32). Hence there remains m complex (or $2m$ real) parameters which may be viewed as coordinates on the singular torus. These solitonic coordinates can be interpreted as generalizing at the semi global level the decomposition of the singular fiber into a product of two dimensional singular tori that is performed by normal coordinates in the vicinity of the critical point.

4.2   Solitonic picture of the tangent cone

We would like now to precise the connection between these solitonic formulae and the linearized flows in the vicinity of the equilibrium point. As discussed in the Introduction, the tangent cone to the singular torus at the equilibrium point is the product of m two dimensional components $\mathcal{C}_k$. There is an important relation between the components $\mathcal{C}_k$ of the cone $\mathcal{C}$ and the flows associated to the Hamiltonians K_j on the critical manifold. In normal coordinates the orbits of generators L_j are closed circles, which become closed orbits when mapped by Φ on phase-space $\mathcal{M}$. The m other generators K_j induce flows which become unbounded when a_j goes to $\pm\infty$ and the trajectories eventually leave the open neighborhood U in which the diffeomorphism Φ is defined. Notice however that when $a_j \to +\infty$, $z_j = 0$ is a contracting manifold for the flow K_j and $w_j = 0$ is an expanding manifold. When $a_j \to -\infty$ the situation is opposite. Hence a pattern $\mathcal{S}$ of m signs $a_j \to \pm\infty$ identifies a unique contracting manifold $\mathcal{C}_{\mathcal{S}}^{contracting}$.

Because there is only one critical point on the critical manifold, when we send all the a_j to $\pm\infty$ we end up into a neighborhood of the critical point. The contracting manifold $\mathcal{C}_{\mathcal{S}}^{contracting}$ is therefore the asymptotic manifold when all the a_j's are sent to infinity with the associated sign pattern.

From eq. (34), the large a_j limit corresponds to:

$$|X_j| \to \infty \text{ and } |X_{\bar{j}}| \to 0 \text{ if } a_j \to +\infty$$

$$|X_{\bar{j}}| \to \infty \text{ and } |X_j| \to 0 \text{ if } a_j \to -\infty$$

To connect the soliton solutions with the definition of the tangent cone, we have to take this limit in the soliton formulae. Let $\mathcal{I}$ the set of m indices $(i_1, i_2, \cdots i_m) \subset \{1 \cdots m, \bar{1} \cdots \bar{m}\}$ such that $X_{i_j} \to \infty$, specifying a contracting component $\mathcal{C}_{\mathcal{S}(\mathcal{I})}^{contracting}$ of the tangent cone. If $j \in \mathcal{I}$ we still denote by $\bar{j}$ the label of the complex conjugate root $\overline{E_j} \equiv E_{\bar{j}}$ (so that $\bar{\bar{j}} = j$). Then because of the reality condition eq.(32), $\bar{j} \notin \mathcal{I}$ and the indices $\bar{j}$ belong to the complementary set $\overline{\mathcal{I}}$ of $\mathcal{I}$ in $\{1 \cdots m, \bar{1} \cdots \bar{m}\}$ (see Fig. 2).

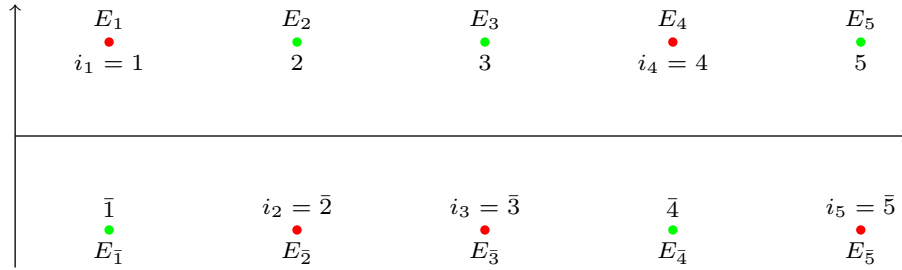

Figure 2: The red dots represent the subset $\mathcal{I} = \{i_1 = 1, i_2 = \bar{2}, i_3 = \bar{3}, i_4 = 4, i_5 = \bar{5}\}$. The green dots represent the complementary (and conjugate) subset $\bar{\mathcal{I}} = \{\bar{i}_1 = \bar{1}, \bar{i}_2 = 2, \bar{i}_3 = 3, \bar{i}_4 = \bar{4}, \bar{i}_5 = 5\}$.

It is useful to relate this definition of the components $\mathcal{C}_{\mathcal{S}(\mathcal{I})}^{contracting}$ of the tangent cone with the one based on the normal coordinates introduced in Section 3. As we have said, either $j \in \mathcal{I}$ or $\bar{j} \in \mathcal{I}$. If $j \in \mathcal{I}$, $z_j \to 0$, so according to eqs. (22),(23), $|C(E_{\bar{j}})| << |C(E_j)|$. If $\bar{j} \in \mathcal{I}$, the converse holds and $|C(E_j)| << |C(E_{\bar{j}})|$. To check that this is indeed true, we need to know the asymptotic expression for $C(\lambda)$ when $|X_j| \to \infty$ for any $j \in \mathcal{I}$. Using eq. (33) and the asymptotic expressions eqs. (95), (96), (97) given in Section 10 below, we see that on the component $\mathcal{C}_{\mathcal{I}}$ of tangent cone, $C(\lambda)$ behaves as:

$$C(\lambda) \simeq \frac{2}{\prod_i (\lambda - \epsilon_i)} \prod_{i \in \mathcal{I}} (\lambda - E_{\bar{i}}) \sum_{j \in \mathcal{I}} \left(\frac{E_{\bar{j}} - E_j}{X_j} \prod_{\substack{k \in \mathcal{I} \\ k \neq j}} \frac{E_{\bar{k}} - E_j}{E_k - E_j} (\lambda - E_k) \right) \tag{35}$$

This expression shows that $C(\lambda)$ goes to zero when all the X_j's go to infinity, for $j \in \mathcal{I}$. This shows that we are indeed describing a $2m$-dimensional subspace (over $\mathbb{R}$) of the tangent space at the equilibrium point. Next, we see that, at this leading order, $C(\lambda)$ vanishes at $\lambda = E_{\bar{j}}$. In reality, $C(E_{\bar{j}})$ is not exactly zero, because of the subleading terms included in the determinant D_1 defined in Section 10. If the typical value of the X_j's scales like X, the ratio of these subleading terms over the dominant one scales like X^{-2}. From this, we conclude that $C(E_{\bar{j}})/C(E_j) \to 0$ when $X \to \infty$, as expected. More precisely, if $j \in \mathcal{I}$, to leading order in the scale X, we have from eq. (35):

$$C(E_j, X_{i \in \mathcal{I}} \to \infty) \simeq -\frac{2}{X_j \prod_i (E_j - \epsilon_i)} \prod_{k \in \mathcal{I}} (E_j - E_{\bar{k}})^2, \quad C(E_{\bar{j}}) = O(X^{-3}) \tag{36}$$

In the limit $a_j \to -\infty$, the dominant variables become the $X_{\bar{j}}$'s. The leading behavior for $C(\lambda)$ is now given by eq. (35) where the set $\mathcal{I}$ is replaced by the complementary set $\overline{\mathcal{I}}$ (in which all indices are replaced by $i \to \bar{i}$). The state of the system is now labelled by the coordinates $B(E_j) = \overline{C(E_{\bar{j}})}$. Using the reality condition (32) and replacing E_j's by $E_{\bar{j}}$'s in eq. (36) gives for $j \in \mathcal{I}$:

$$B(E_j, X_{i \in \mathcal{I}} \to 0) \simeq \frac{8X_j}{\prod_i (E_j - \epsilon_i)} \prod_{k \in \mathcal{I}} (E_j - E_{\bar{k}})^2, \quad B(E_{\bar{j}}) = O(X^3) \tag{37}$$

where the scale X goes to zero.

5 Periodic flows, symplectic invariants and monodromy

5.1 Motivation

In an integrable system, the action integrals $S_\gamma = \int_\gamma \alpha$, where the symplectic form $\omega = d\alpha$ and γ denotes any closed cycle on regular Arnold–Liouville tori, are smooth functions of the conserved quantities in any open set containing only regular values of the moment map. These functions are also the generators of periodic flows on Arnold-Liouville tori. Inspired by previous works [3, 4, 5], we wish to examine the construction of periodic flows in the vicinity of a singular torus. We know already that m such flows are generated by the L_j's. As we have seen, on the singular torus, the flow generated by K_j corresponds to a large motion, moving away from the

critical point $x^{(0)}$ along the corresponding expanding manifold in the remote past and returning to $x^{(0)}$ along the contracting manifold in the far future. As suggested by Fig. 3, it is possible to use these large solitonic trajectories to construct periodic orbits, at least for regular tori close to the singular one.
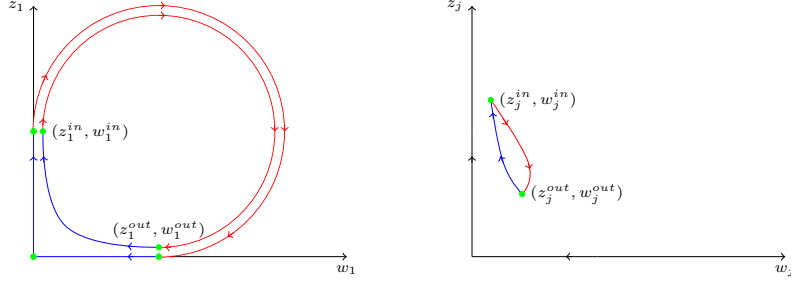

Figure 3: The red curves represent the large motion of the $\alpha_1 K_1 + \beta_1 L_1$ flow projected on the z_1, w_1 manifold in the soliton case (outer curve) and on a nearby Liouville torus (inner curve). The motion starts from a position $(z_1^{in}, w_1^{in} \simeq 0)$ on the expanding manifold and returns to a position $(z_1^{out} \simeq 0, w_1^{out})$ on the contracting manifold. At the same time on the $j^{th} > 1$ component, there is an induced motion from (z_j^{in}, w_j^{in}) to (z_j^{out}, w_j^{out}). One can close the trajectory by adding an extra motion $\alpha_1 K_1 + \beta_1 L_1$ from (z_1^{out}, w_1^{out}) to (z_1^{in}, w_1^{in}) and a motion $\alpha_j K_j + \beta_j L_j$ from (z_j^{out}, w_j^{out}) to (z_j^{in}, w_j^{in}) (blue segments). These extra motions lie entirely in the domain of validity of the normal form.

5.2 Periodic flows.

The Hamiltonian generating the periodic flow is a function of the K_l, L_l:

$$\mathcal{H} = \mathcal{H}(K_1, L_1, \cdots, K_m, L_m)$$

This function however is not defined globally and this gives rise to the *monodromy* phenomenon. The flow itself is a linear combinations of the flows generated by K_l, L_l

$$\partial_t = \sum_l \alpha_l \partial_{a_l} + \beta_l \partial_{\theta_l}, \quad \alpha_l = \partial\mathcal{H}/\partial K_l, \quad \beta_l = \partial\mathcal{H}/\partial L_l, \tag{38}$$

The X's become functions of time

$$X_k = X_k(0)e^{\alpha_k t + i\beta_k t}, \quad X_{\bar{k}} = X_{\bar{k}}(0)e^{-\alpha_k t + i\beta_k t} \tag{39}$$

We want to find conditions on α_k, β_k in order to get a closed trajectory.

To simplify notations, we will consider the large motion generated by $\alpha_1 K_1 + \beta_1 L_1$. Fig. 3 shows that we first have to study the corresponding symplectic map for an initial condition (z_j^{in}, w_j^{in}) close to the expanding manifold associated to K_1 i.e. such that w_1^{in} is small. The initial conditions and the time lapse $2T$ are chosen in such a way that the image (z_j^{out}, w_j^{out}) is close to the contracting manifold, i.e. z_1^{out} is small. In general, (z_j^{out}, w_j^{out}) is different from (z_1^{in}, w_1^{in}) for $j \neq 1$, reflecting the impossibility to express our dynamical system in the vicinity of the pinched torus as a product of m independent integrable systems with 2 degrees of freedom, each of them exhibiting an isolated focus-focus singularity. Although such factorization holds at the topological level [2], it does not hold at the level of symplectic manifolds. The obstruction to such a symplectic product decomposition is encoded in the non-diagonal symplectic invariants, which we will now compute. For this purpose we will use the fact recalled in Section 7, that the map $(z_j^{in}, w_j^{in}) \rightarrow (z_j^{out}, w_j^{out})$ is strongly constrained by the condition that K_1 and L_1 commute with all the generators of the moment map.

It is first necessary to identify the components $\mathcal{C}_{\mathcal{S}(\mathcal{I})}^{contracting}$ of the cone corresponding to the *in* and *out* states for a large motion generated by $\alpha_1 K_1 + \beta_1 L_1$. This flow sends the solitonic coordinate X_1 into $e^{\alpha_1 a_1 + i\beta_1 \theta_1} X_1$ and $X_{\bar{1}}$ into $e^{-\alpha_1 a_1 + i\beta_1 \theta_1} X_{\bar{1}}$, and leaves unchanged all the other coordinates X_k, $X_{\bar{k}}$. Let us choose an initial condition on an expanding subspace for this flow. This means that $|X_{\bar{1}}^{in}| >> |X_1^{in}|$. In principle, the other coordinates can be chosen arbitrarily, but it is convenient to assume also that $|X_{\bar{k}}^{in}| >> |X_k^{in}|$ for $k > 1$, so that the initial point is close to the critical point. Let us denote by $\mathcal{I}_0$ the set of indices $\{2 \cdots m\}$. The manifold $\mathcal{C}^{in}$ is obtained by sending a_1 to $-\infty$ and the manifold $\mathcal{C}^{out}$ by sending a_1 to $+\infty$, the other $j \in \bar{\mathcal{I}}_0$ being spectators. Then $\mathcal{C}^{in} = \mathcal{C}_{\mathcal{S}(\{\bar{1}\} \cup \bar{\mathcal{I}}_0)}^{contracting}$ and $\mathcal{C}^{out} = \mathcal{C}_{\mathcal{S}(\{1\} \cup \bar{\mathcal{I}}_0)}^{contracting}$.

The symplectic invariants are smooth functions of the conserved quadratic Hamiltonians K_j, L_j $(1 \leq j \leq m)$ defined in the vicinity of the singular torus [3, 4, 5]. The knowledge of solitonic trajectories allows us to compute them exactly on the singular torus, that is for $K_j = L_j = 0$.

5.2.1 Diagonal case.

As apparent of Fig.[3], the trajectory is composed of two pieces. The first one is the large solitonic motion generated by $\alpha_1 K_1 + \beta_1 L_1$ during a time lapse $2T$ which yields a relation between $(z_1^{in}, w_1^{in}) \rightarrow (z_1^{out}, w_1^{out})$ as explained in Section 7. This part of the trajectory is complemented by a further action of the flow $\alpha_1 K_1 + \beta_1 L_1$ during a time lapse τ relating $(z_1^{out}, w_1^{out}) \rightarrow (z_1^{in}, w_1^{in})$ so as to get a closed trajectory. This last part of the analysis lies entirely in the domain of validity of the normal forms. One should therefore have

$$z_1^{in} = e^{\alpha_1 \tau + i\beta_1 \tau} z_1^{out}, \quad w_1^{in} = e^{-\alpha_1 \tau + i\beta_1 \tau} w_1^{out} \tag{40}$$

Using the leading order expressions for z_1^{out}, w_1^{out}, eqs.(63,64) these two equations are equivalent to

$$\bar{z}_1^{in} w_1^{in} = K_1 + iL_1 = e^{-\alpha_1 \tau + i\beta_1 \tau} \Phi_1^0(T) \tag{41}$$

As shown in Section 7, one has, again to leading order

$$w_1^{out} = \frac{\Phi_1^0(T)}{K_1 + iL_1} w_1^{in} = \frac{\Phi_1^0(T)}{\bar{z}_1^{in}}$$

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The variables z_1^{in}, w_1^{in} are obtained from $C(E_1), C(E_{\bar{1}})$ evaluated at $t = -T$ and the variables z_1^{out}, w_1^{out} are obtained from $C(E_1), C(E_{\bar{1}})$ evaluated at $t = T$. Explicitely we have:

$$\begin{cases} C(E_1)|_{-T} = \bar{w}_1^{in} \simeq 0 \\ C(E_{\bar{1}})|_{-T} = ia'(\bar{E}_1)\bar{z}_1^{in} \end{cases}, \quad \begin{cases} C(E_1)|_T = \bar{w}_1^{out} \\ C(E_{\bar{1}})|_T = ia'(\bar{E}_1)\bar{z}_1^{out} \simeq 0 \end{cases}$$

so that:

$$\Phi_1^0 = w_1^{out} \bar{z}_1^{in} = \frac{1}{ia'(\bar{E}_1)} C(E_{\bar{1}})|_{-T} \, \overline{C(E_1)}|_T$$

When $t \to -T$ the set of X's tending to ∞ is $\mathcal{I} = \overline{\mathcal{I}}_0 \cup \{\bar{1}\}$, then:

$$C(E_{\bar{1}})|_{X_{\bar{1}} \to \infty} \simeq -2 \frac{(E_{\bar{1}} - E_1)^2}{\prod_{i=1}^n (E_{\bar{1}} - \epsilon_i)} \frac{\prod_{k \in \mathcal{I}_0}(E_{\bar{1}} - E_k)^2}{X_{\bar{1}}}, \quad C(E_1) \simeq 0,$$

Similarly, when $t \to T$ the set of X's tending to ∞ is $\mathcal{I} = \overline{\mathcal{I}}_0 \cup \{1\}$, then:

$$C(E_1)|_{X_1 \to \infty} \simeq -2 \frac{\prod_{k \in \mathcal{I}_0}(E_1 - E_k)^2}{\prod_{i=1}^n (E_1 - \epsilon_i)} \frac{(E_1 - E_{\bar{1}})^2}{X_1}, \quad C(E_{\bar{1}}) \simeq 0,$$

Hence, we get:

$$\Phi_1^0 = \frac{4}{ia'(E_{\bar{1}})} \frac{(E_{\bar{1}} - E_1)^4 \prod_{k \in \mathcal{I}_0}(E_{\bar{1}} - E_{\bar{k}})^2 (E_{\bar{1}} - E_k)^2}{\prod_i (E_{\bar{1}} - \epsilon_i)^2} \frac{1}{X_{\bar{1}}(-T)\overline{X_1}(T)}$$

Next, we have

$$\begin{cases} X_1(T) = X_1(0)e^{\alpha_1 T + i\beta_1 T} \\ X_{\bar{1}}(-T) = X_{\bar{1}}(0)e^{\alpha_1 T - i\beta_1 T} \end{cases} \implies X_{\bar{1}}(-T)\overline{X_1}(T) = -\frac{1}{4}e^{2\alpha_1 T - 2i\beta_1 T}$$

so that eq.(41) becomes

$$c_1 \equiv \bar{z}_1 w_1 = e^{-\alpha_1 \tau + i\beta_1 \tau} \Phi_1^0(T) = \rho_1 e^{i\gamma_1} \, e^{-\alpha_1(2T + \tau) + i\beta_1(2T + \tau)}$$

where we have set

$$\rho_1 e^{i\gamma_1} = -\frac{16}{ia'(E_{\bar{1}})} \frac{(E_{\bar{1}} - E_1)^4 \prod_{k>1}(E_{\bar{1}} - E_{\bar{k}})^2 (E_{\bar{1}} - E_k)^2}{\prod_k (E_{\bar{1}} - \epsilon_k)^2} \tag{42}$$

The quantity $2T + \tau$ is the total time it takes to run once along the closed trajectory, that is the period of the motion. Normalizing it to 2π, the above equation determines α_1 and β_1:

$$\alpha_1 = -\frac{1}{2\pi} \log \frac{|c_1|}{\rho_1}, \quad \beta_1 = \frac{1}{2\pi}(\arg(c_1) - \gamma_1) \tag{43}$$

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5.2.2 Non diagonal case.

Similarly, one can compensate the non-diagonal motion induced by the large $\alpha_1 K_1 + \beta_1 L_1$ flow by adding flows $\alpha_j K_j + \beta_j L_j$. The evolution of z_j is now the composition of this new flow and the indirect evolution due to the $\alpha_1 K_1 + \beta_1 L_1$ flow. Hence

$$z_j^{out} = e^{(\alpha_j + i\beta_j)(2T+\tau)} \rho_{zj}^{(1)} z_j^{in}, \quad j \neq 1$$

Periodicity requires

$$e^{-2\pi(\alpha_j + i\beta_j)} = \rho_{zj}^{(1)} \tag{44}$$

where we have set the period $2T + \tau$ to 2π. This determines the remaining constants α_j, β_j in eq.(38), once we have computed $\rho_{zj}^{(1)}$ and $\rho_{wj}^{(1)}$ ($j \neq 1$) as defined in Section 7.

The initial conditions were chosen such that when $t = -T$ the system lies on a subspace of the tangent cone whose natural coordinates are the $B(E_j)^{in}$'s given by eq. (37) with $\mathcal{I} = \mathcal{I}_0 \cup \{1\}$. In the long time limit $t = T$, i.e. when $a_1 = \alpha_1 t$ is positive and large, we get $|X_{\bar{1}}^{out}| << |X_1^{out}|$, and the system lies on a different subspace of the tangent cone. The $B(E_j)^{out}$ are again given by eq. (37) with $\mathcal{I} = \mathcal{I}_0 \cup \{\bar{1}\}$. From the discussion in Section 7, we expect that:

$$B(E_j)^{out} = \rho_{zj}^{(1)} B(E_j)^{in}, \quad j \neq 1$$

By using eq.(37), this is indeed the case and we find :

$$\rho_{zj}^{(1)} = \left(\frac{E_j - E_1}{E_j - E_{\bar{1}}} \right)^2 \tag{45}$$

As explained in Section 7, we expect the general relation $\rho_{zj}^{(1)} \bar{\rho}_{wj}^{(1)} = 1$ between $\rho_{zj}^{(1)}$ and $\rho_{wj}^{(1)}$. As a consistency check, we may evaluate $\rho_{wj}^{(1)}$ directly. Using that $w_j = \overline{C(E_j)}$, we have

$$C(E_j)^{out} = \bar{\rho}_{wj}^{(1)} C(E_j)^{in}, \quad j \neq 1$$

Now $C(E_j)^{in}$ is obtained by eq.(36) with the index set $\mathcal{I} = \overline{\mathcal{I}_0} \cup \{\bar{1}\}$ and $C(E_j)^{out}$ is obtained from the same formula with an index set $\mathcal{I} = \overline{\mathcal{I}_0} \cup \{1\}$. So we have:

$$\bar{\rho}_{wj}^{(1)} = \frac{C(E_j)^{out}}{C(E_j)^{in}} = \left(\frac{E_j - E_{\bar{1}}}{E_j - E_1} \right)^2 \tag{46}$$

Clearly, the relation $\rho_{zj}^{(1)} \bar{\rho}_{wj}^{(1)} = 1$ is satisfied. Finally the constants α_j, β_j are determined by:

$$e^{-2\pi(\alpha_j + i\beta_j)} = \left(\frac{E_j - E_1}{E_j - E_{\bar{1}}} \right)^2 \tag{47}$$

5.3 Action variables and symplectic invariants.

Let us define the 1-form on the base (i.e. the image of the moment map)

$$\Omega^{(1)} = \sum_j \Omega_j^{(1)}, \quad \Omega_j^{(1)} = \alpha_j dK_j + \beta_j dL_j$$

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Remembering that

$$c_1 = \bar{z}_1^{in} w_1^{in} = \bar{z}_1^{out} w_1^{out} = K_1 + iL_1$$

we see that $\Omega^{(1)}$ is singular when $c_1 \to 0$. Isolating the singular part, we may write

$$\Omega^{(1)} = -\frac{1}{4\pi}(\log c_1 \, dc_1 + \log \bar{c}_1 \, d\bar{c}_1) + \Omega^{(1)reg}$$

where

$$\Omega^{(1)reg} = \frac{1}{2\pi}\log\rho_1 \, dK_1 - \frac{1}{2\pi}\gamma_1 \, dL_1 + \sum_{j>1}\alpha_j dK_j + \beta_j dL_j$$

The form Ω^{reg} contains all the regularized symplectic invariants of San Vũ Ngoc [4], computed here on the singular fiber. In the case of one spin, we find

$$\rho_1 e^{i\gamma_1} = -\frac{8}{s}(2s - \epsilon_1)^{3/2}(\epsilon_1 + i\sqrt{2s - \epsilon_1^2})$$

Setting $s = 1$ and $\epsilon_1 = 0$ yields

$$\Omega^{(1)reg} = \frac{1}{2\pi}\left(5\log 2 \, dK_1 + \frac{\pi}{2} \, dL_1\right)$$

and we recover the invariants computed in [6] (up to the normalization of the period to 2π).

To define the form $\Omega^{(1)}$, we need to introduce a cut in the K_1, L_1 plane. In this cut plane, the form $\Omega^{(1)}$ is closed and remembering eq.(38) we have

$$\Omega^{(1)} = d\mathcal{H}^{(1)}$$

The Hamiltonian $\mathcal{H}^{(1)}$ is the action associated to the closed trajectory we have constructed. It is defined on the cut plane K_1, L_1.

We have computed the action $\mathcal{H}^{(1)}$ to leading order. However following [5] we can give a more global interpretation using the spectral curve. First the normal actions K_j, L_j are just the integrals over the vanishing cycles A_j (encircling pairs of roots of the spectral polynomial which coalesce into double roots on the critical fiber) as shown in Fig. 5.

$$K_j + iL_j = \frac{1}{2i\pi} \oint_{A_j} \sqrt{\Lambda(\lambda)} d\lambda \tag{48}$$

This can already be seen at the quadratic level

$$\frac{1}{2i\pi} \oint_{A_j} \sqrt{\Lambda} d\lambda = \frac{1}{2i\pi} \oint_{A_j} \sqrt{a^2(\lambda) + \sum_k \frac{a(\lambda)}{a'(E_k)(\lambda - E_k)} B_k C_k} \, d\lambda$$

$$= \frac{1}{2i\pi} \oint_{A_j} \left(a(\lambda) + \frac{1}{2}\sum_j \frac{1}{a'(E_k)(\lambda - E_k)} B_k C_k\right) d\lambda = \frac{1}{2a'(E_j)} B_j C_j = K_j + iL_j$$

Eq.(48) has a meaning beyond the quadratic approximation and can be viewed as the inverse of the function ψ in Eliasson normal form [10, 5]. It is a non linear extension of eqs.(27, 28).
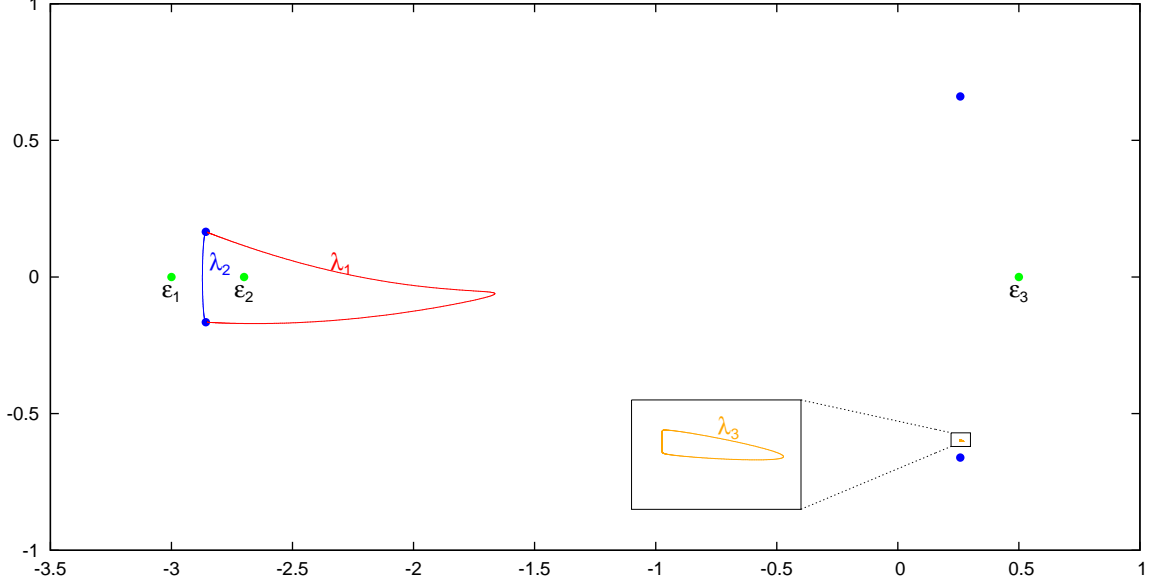
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Figure 4: The motion of the three separated variables for the soliton solution in the 3-spins model. The flow is given by eq.(39) with α_k, β_k given by eqs.(43,47). The limiting flow is obtained by setting $X_j \to X_j(\frac{t}{2T})$ where $T \simeq -\frac{1}{4\pi} \log |c_1|$ is large, and $-2\pi T \le t \le 2\pi T$.

Concerning the action $\mathcal{H}^{(1)}$ we see that it has the unique property of being singular when $c_1 = K_1 + iL_1 \to 0$ but it remains regular when $c_j = K_j + iL_j \to 0, j \ne 1$. To get an indication of its global definition, we may consider the example of the 3-spins model and the motion of the separated variables of the soliton solution (see Appendix 9) corresponding to the limiting motion generated by $\mathcal{H}^{(1)}$ on the singular fiber. On Fig.[4] we show the motion of the separated coordinates λ_k defined in eq.(69). Here λ_1 is associated to the sign $(+)$ in eq.(78), while λ_2, λ_3 are associated to the sign $(-)$. Hence the corresponding points (μ_k, λ_k) belong to different sheets in the representation of the spectral curve as a covering of the complex plane. On a fiber close to the singular one, the action integral is

$$\mathcal{H}^{(1)} = \frac{1}{2i\pi} \sum_k \oint \mu_k d\lambda_k = \frac{1}{2i\pi} \sum_k \oint \sqrt{\Lambda(\lambda_k)} d\lambda_k$$

where the λ_k describe trajectories close to the critical ones. When $k = 3$ we integrate over a trivial cycle and the contribution vanishes. The $k = 1, 2$ contributions combine to make a non trivial cycle B_1 as indicated on Fig.[5].

$$\mathcal{H}^{(1)} = \frac{1}{2i\pi} \oint_{B_1} \sqrt{\Lambda(\lambda)} d\lambda \tag{49}$$

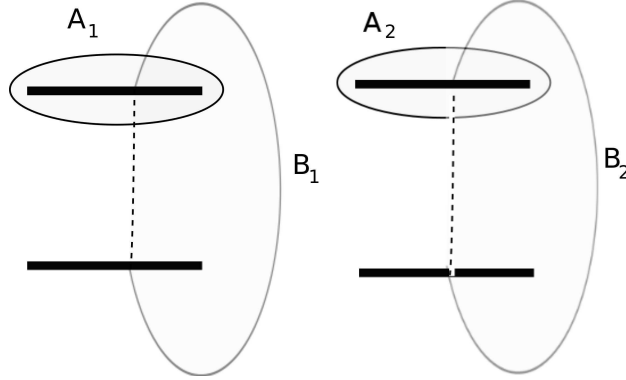We can eliminate the H_i between eq.(48) and eq.(49) thereby obtaining the full Taylor expansion

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Figure 5: The spectral curve with the system of cuts and cycles corresponding to the limiting motion of the separated variables λ_k.

of $\mathcal{H}^{(1)}$ in terms of the K_j, L_j.

We believe that this can be generalized to the n spin model. We have $2m$ vanishing cycles A_j and $2m$ cycles B_j. The action integrals on the cycles A_j give directly the $c_j = K_j + iL_j$ and the action integral on the B_j give the periodic Hamiltonian $\mathcal{H}^{(j)}$ (see section 11) which become singular when c_j tends to zero but remain regular when $c_i, i \neq j$ tends to zero. The important ingredient here is the identification of the B_j cycles through the motion of the divisor of the proper soliton solution. A complete analysis of this motion has not yet been achieved, however using the asymptotic expressions for determinants given in section 10, it is possible to show that as the time t_1 goes to $-\infty$, we have one separated variable λ_1^+ located at $E_{\bar{1}}$, and another one, $\lambda_{\bar{1}}^-$, at E_1. When t_1 goes to $+\infty$, there is one separated variable λ_i^+ located at E_1 and one λ_j^+ at $E_{\bar{1}}$, The other remaining separated variables, $m-2$ of type λ^+ and $m-1$ of type λ^- are found to occupy the same sets of points in the complex plane in both limits $t_1 \to -\infty$ and $t_1 \to \infty$ due to the periodicity conditions eq.(47). The difficult task remains to show that this possibly complicated motion of the separated variables can be continuously deformed into one of the B_j cycles. If one succeeds in doing this, the periodic Hamiltonians can then be obtained more globally by the construction of section 11. An alternative approach would be to use Picard-Fuchs equations for the action variables as in [7].

5.4 Monodromy

The orbits of the periodic motions defined in the previous section provide a basis of cycles on the Liouville tori close to the singular fiber. It is well known that when we follow these cycles by continuity when we perform a non trivial closed contour in the image of regular values of the moment map, the initial and final basis of cycles may be different and are related by an element of $GL(n, \mathbb{Z})$.

In order to compute the monodromy, we have first to determine the fundamental group π_1 of the manifold of regular values of the moment map. The manifold of singular values of the moment map is a stratified manifold by the rank of the moment map. At the equilibrium point

the rank is zero, however the equilibrium point belongs to bigger strata where the rank is $r \leq 2m$. Eliasson's theorem allows an easy description of this stratification in a local neigborhood U of the equilibrium point. The moment map is given by

$$(x, p) \in U \rightarrow (K_1, L_1, K_2, L_2, \cdots, K_m, L_m)$$

The derivatives of each pair K_j, L_j generically span a two dimensional subspace of the tangent space of the image of the moment map near the equilibrium point and these spaces are linearly independent for two different pairs. For each pair the only way to span a space of dimension smaller than 2 is when the two tangent vectors are identically zero. In this case the rank of the momentum map drops to $r = 2m - 2$ and the equation of this stratum is $K_j = L_j = 0$. The singular values of the momentum map in a neighborhood of the equilibrium point consists therefore of m strata $\mathcal{O}_j$ of dimension $2m - 2$ given by the equations $K_j = L_j = 0$, for $i \in \{1, \cdots m\}$. Of course each one of these strata is itself stratified by the intersection of p such manifolds $\mathcal{O}_{j_1} \cap \mathcal{O}_{j_2} \cap \cdots \mathcal{O}_{j_p}$ where the rank is $r = 2m - 2p$. The equilibrium point itself is obtained when $p = m$.

Because the co-dimension of $\mathcal{O}_j$ is 2, we can draw non trivial circles S_1 around it. Indeed $\mathcal{O}_j$ intersects the plane K_j, L_j precisely at the origin and such an S_1 can be taken to be $K_j + iL_j \rightarrow e^{i\alpha_j}(K_j + iL_j)$. The fundamental group of the image of the momentum map minus the $\mathcal{O}_j$'s is a product of m factors S_1: $\pi_1 = S_1 \times S_1 \times \cdots S_1$.

We can now compute the monodromy. Locally we have $\Omega^{(1)} = d\mathcal{H}^{(1)}$. Recall that

$$\bar{z}_1 w_1 = K_1 + iL_1 = \rho_1 e^{i\gamma_1}$$

The regular values of the moment map are such that $\rho_1 \neq 0$. Monodromy is defined in the open set of the regular values of the moment map. Let us take as a non trivial closed path a circle around the origin in the K_1, L_1 plane: $c_1(\alpha_1) = e^{i\alpha_1}\rho_1$, $\gamma_1 \leq \alpha_1 \leq \gamma_1 + 2\pi$. Integrating the form $\Omega^{(1)}$ over this circle one gets a non trivial contribution from $\Omega_1^{(1)}$

$$\oint_C \Omega^{(1)} = \oint_C \Omega_1^{(1)} = -\frac{i}{4\pi} \oint_{\gamma_1}^{2\pi + \gamma_1} \left[c_1(\alpha_1) \log c_1(\alpha_1) - \bar{c}_1(\alpha_1) \log \bar{c}_1(\alpha_1) \right] d\alpha_1$$

$$= \frac{1}{2\pi} \rho_1 \oint_{\gamma_1}^{2\pi + \gamma_1} \alpha_1 \cos \alpha_1 d\alpha_1 = \rho_1 \sin \gamma_1 = L_1$$

This means that when we go around that circle, the Hamiltonian $\mathcal{H}^{(1)}$ becomes $\mathcal{H}^{(1)} + L_1$. The new periodic trajectory contains an extra S^1 factor generated by L_1. This is exactly the monodromy phenomenon. Note that when the image of the moment map winds once around the stratum $\mathcal{O}_j$, $\mathcal{H}^{(1)}$ remains single-valued when $j \neq 1$, so for $\mathcal{H}^{(1)}$ only paths which wind around the stratum $\mathcal{O}_1$ give rise to a non-trivial monodromy.

Following up the transformation of the action integrals $\mathcal{H}^{(j)}$ defined in the previous section along such a non trivial path would also lead to an interpretation of the monodromy in the framework of the Picard-Lefschetz theory as in [12].

6   Appendix : Normal Forms.

Normal coordinates simultaneously diagonalize all the flows H_j in the quadratic approximation.

$$\{H_j, a_\alpha\} = E_{j,\alpha} a_\alpha \tag{50}$$

To construct them, we return to eqs.(14 – 16). They imply

$$\left\{\frac{1}{2}\mathrm{Tr}\, L^2(\lambda), C(\mu)\right\} = \frac{2i}{\lambda - \mu}\Big(A(\lambda)C(\mu) - A(\mu)C(\lambda)\Big)$$

When we expand around a configuration given by eq.(21), all the quantities $(b, \bar{b}, s_j^+, s_j^-)$ are first order, but s_j^z is second order because $s_j^z = se_j - \frac{e_j}{2s}s_j^+ s_j^- + \cdots, \quad e_j = \pm 1$. In this expansion, the Hamiltonians are quadratic while $C(\mu)$ is first order. Therefore the Poisson bracket in the left-hand side is linear. Now $A(\lambda)$ is constant plus second order, so that in the right-hand side we can replace $A(\lambda)$ and $A(\mu)$ by their zeroth order expression :

$$A(\lambda) \simeq a(\lambda) = 2\lambda + \sum_{j=1}^{n} \frac{se_j}{\lambda - \epsilon_j}$$

and we arrive at

$$\left\{\frac{1}{2}\mathrm{Tr}\, L^2(\lambda), C(\mu)\right\} = \frac{2i}{\lambda - \mu}\Big(a(\lambda)C(\mu) - a(\mu)C(\lambda)\Big) \tag{51}$$

Eq.(51) will be precisely of the form of eq.(50) if we can kill the unwanted term $C(\lambda)$. This is achieved by imposing the condition

$$a(\mu) = 0, \quad \text{``Classical Bethe Equation''} \tag{52}$$

This is an equation of degree $n + 1$ for μ. For non degenerate singularities it has only simple roots. Calling E_i its solutions, we construct in this way $n + 1$ variables $C(E_i)$. Clearly, they all Poisson commute

$$\{C(E_i), C(E_j)\} = 0 \tag{53}$$

Since phase space has dimension $2(n + 1)$ this is half what we need. To construct the conjugate variables, we consider eq.(16). In our linear approximation it reads

$$\{B(E_i), C(E_j)\} = \frac{2i}{E_i - E_j}(a(E_i) - a(E_j))$$

If E_i and E_j are *different* solutions of eq.(52), then obviously

$$\{B(E_i), C(E_j)\} = 0, \quad E_i \neq E_j \tag{54}$$

If however $E_j = E_i$ then

$$\{B(E_i), C(E_i)\} = 2ia'(E_i) \tag{55}$$

Finally, we also have

$$\{B(E_i), B(E_j)\} = 0 \tag{56}$$

Up to normalisation, we have indeed constructed canonical coordinates.

It is simple to express the quadratic Hamiltonians in theses coordinates:

$$\frac{1}{2}\mathrm{Tr}\, L^2(\lambda) = a^2(\lambda) + \sum_j \frac{a(\lambda)}{a'(E_j)(\lambda - E_j)} B(E_j)C(E_j) \qquad (57)$$

This has the correct analytical properties in λ and together with the Poisson brackets eqs.(53,54, 55, 56) we reproduce eq.(51). Note that there is no pole at $\lambda = E_j$ because $a(E_j) = 0$. Expanding around $\lambda = \infty$ we get

$$H_{n+1} = s\sum_k e_k + \sum_i \frac{1}{2a'(E_i)}B(E_i)C(E_i)$$

and computing the residue at $\lambda = \epsilon_j$, we find

$$H_j = se_j\left[2\epsilon_j + \sum_k \frac{se_k}{\epsilon_j - \epsilon_k}\right] + \sum_i \frac{1}{2a'(E_i)}\frac{se_j}{\epsilon_j - E_i}B(E_i)C(E_i)$$

We can invert these formulae: devide eq.(57) by $\lambda - E_j$ and take the residue at $\lambda = E_j$. Since $a(E_j) = 0$ we get

$$\frac{1}{2}\mathrm{Tr}\, L^2(E_j) = B(E_j)C(E_j)$$

or explicitly

$$B(E_j)C(E_j) = 4E_j^2 + 4H_{n+1} + \sum_{i=1}^n \frac{2H_i}{E_j - \epsilon_i} + \sum_{k=1}^n \frac{s^2}{(E_j - \epsilon_i)^2}$$

We can now make contact with the Williamson classification theorem.

If E_j is *real* we have $B(E_j) = \overline{C(E_j)}$ and we set

$$C(E_j) = \sqrt{|a'(E_j)|}(p_j + i\eta_j q_j), \quad B(E_j) = \sqrt{|a'(E_j)|}(p_j - i\eta_j q_j)$$

where $\eta_j = -\mathrm{sign}(a'(E_j))$ and p_j, q_j are *real* canonical coordinates. Then eq.(55) is satisfied. Moreover

$$B(E_j)C(E_j) = |a'(E_j)|(p_j^2 + q_j^2)$$

i.e. we have an *elliptic* singularity.

If E_j is *complex*, there is another root $E_{j+1} = \bar{E}_j$ which is its complex conjugate. Then $\overline{B(E_j)} = C(E_{j+1})$. We introduce *real* canonical coordinates $p_j, q_j, p_{j+1}, q_{j+1}$ and set

$$B(E_j) = -ia'(E_j)(q_j + iq_{j+1}), \quad B(E_{j+1}) = p_j + ip_{j+1}$$
$$C(E_j) = p_j - ip_{j+1}, \quad C(E_{j+1}) = ia'(E_{j+1})(q_j - iq_{j+1})$$

then eqs.(53,54,55,56) are satisfied and

$$\begin{aligned} B(E_j)C(E_j) &= -ia'(E_j)(K_j + iL_j) \\ B(E_{j+1})C(E_{j+1}) &= ia'(E_{j+1})(K_j - iL_j) \end{aligned}$$

i.e. we have a *focus-focus* singularity.

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7 Appendix: In-Out transformation.

We consider a flow generated by the Hamiltonian $\alpha_1 K_1 + \beta_1 L_1$. Such linear combinations were needed in Section 5, although here only the K_1 part really matters. We assume $\alpha_1 > 0$. Let us fix some $\delta > 0$ with δ small enough that Eliasson map Φ is well defined in the ball of radius 2δ around the origin. Consider the point $U : (z_1 = \delta, w_1 = 0)$ on the expanding manifold and the trajectory of the flow starting at that point. Since the trajectory stays on the *global* unstable manifold, which is equal to the *global* stable manifold, it will reappear after a time lapse $2T$ on the contracting manifold at a point $S : (z_1 = 0, |w_1| = \delta)$. The flow evaluated at time $2T$ is well defined in a neighborhood Σ_U of the point U and defines a symplectic mapping $F_T^{(1)}$ of Σ_U, $|w_1^{\text{in}}| << |z_1^{\text{in}}|$, to a neighborhood Σ_S of the point S in the contracting subspace $|z_1^{\text{out}}| << |w_1^{\text{out}}|$:

$$\Sigma_U : (z_1^{in}, w_1^{in}, z_2^{in}, w_2^{in}, \cdots, z_m^{in}, w_m^{in}) \xrightarrow{F_T^{(1)}} \Sigma_S : (z_1^{out}, w_1^{out}, z_2^{out}, w_2^{out}, \cdots, z_m^{out}, w_m^{out})$$

We also assume that the other components z_i^{in}, w_i^{in}, z_i^{out}, w_i^{out}, $i = 2, \cdots, m$ are small (this will be satisfied in our case) so that the initial and final points lie in a small neighborhood of the critical point. One can find the precise form of this mapping by writing that it commutes with the flows generated by K_j, L_j, $j = 1 \cdots m$. Let us fix the level set by imposing:

$$\bar{z}_j^{in} w_j^{in} = \bar{z}_j^{out} w_j^{out} = K_j + iL_j \tag{58}$$

Let us also assume that $K_j + iL_j \neq 0$. We may then choose $(w_1^{in}, \cdots, w_m^{in})$ as coordinates on the intersection of Σ_U with the level set and write:

$$w_j^{out} = F_j^{(1)}(K_1, L_1, \cdots K_m, L_m, w_1^{in}, \bar{w}_1^{in}, \cdots, w_m^{in}, \bar{w}_m^{in})$$

The condition that the transformation $F^{(1)}$ commutes with the flows $K_1, L_1, \cdots, K_m, L_m$ means:

$$e^{-a_j + i\theta_j} F_j^{(1)}(K_1, L_1, \cdots K_m, L_m, w_1, \bar{w}_1, \cdots, w_m, \bar{w}_m) =$$
$$F_j^{(1)}(K_1, L_1, \cdots K_m, L_m, e^{-a_1+i\theta_1}w_1, e^{-a_1-i\theta_1}\bar{w}_1, \cdots, e^{-a_m+i\theta_m}w_m, e^{-a_m-i\theta_m}\bar{w}_m)$$

To solve this equation we choose the parameters a_k, θ_k such that

$$e^{-a_k + i\theta_k} w_k = \eta_k, \quad w_k, \eta_k \neq 0, \quad 1 \leq k \leq m$$

where η_k are constants. Then the equation becomes:

$$F_j^{(1)}(K_1, L_1, \cdots K_m, L_m, w_1, \bar{w}_1, \cdots, w_m, \bar{w}_m) = \rho_{wj}^{(1)}(K_1, L_1, \cdots K_m, L_m)w_j$$

where $\rho_{wj}^{(1)}(K_1, L_1, \cdots K_m, L_m) = \eta_j^{-1} F_j^{(1)}(K_1, L_1, \cdots K_m, L_m, \eta_1, \bar{\eta}_1, \cdots, \eta_m, \bar{\eta}_m)$. Hence, we have shown that

$$w_j^{out} = \rho_{wj}^{(1)}(K_1, L_1, \cdots K_m, L_m)w_j^{in} \tag{59}$$

where the super script (1) is to remind that this is a transformation under the $\alpha_1 K_1 + \beta_1 L_1$ flow. In exactly the same way, we show

$$z_j^{out} = \rho_{zj}^{(1)}(K_1, L_1, \cdots K_m, L_m)z_j^{in} \tag{60}$$

The conditions eq.(58) then imply

$$\rho_{wj}^{(1)}(K_1, L_1, \cdots K_m, L_m)\overline{\rho_{zj}^{(1)}(K_1, L_1, \cdots K_m, L_m)} = 1 \tag{61}$$

When $j \neq 1$, the functions $\rho_{wj}^{(1)}(K_1, L_1, \cdots K_m, L_m)$ do not depend on the choice of the time lapse $2T$ for the flow, nor on the choice of the parameters α_1, β_1. They are smooth in a neighborhood of the critical torus. As shown before [4, 5] they are symplectic invariants attached to the Liouville foliation in the vicinity of the singular torus. Let us now show that for $j = 1$, $\rho_{w1}^{(1)}(K_1, L_1, \cdots K_m, L_m)$ is singular as $K_1 + iL_1 \to 0$.

To see this, we first consider the critical torus, on which the dilating stratum is given by $w_1 = 0$ and the contracting stratum by $z_1 = 0$. We start with an initial condition on the dilating stratum $(w_{1,sol}^{in} = 0, z_{1,sol}^{in})$. We let the system evolve with respect to the flow . After a large motion during the time lapse $2T$, the point reappears on the contracting stratum $(w_{1,sol}^{out}, z_{1,sol}^{out} = 0)$. The same reasoning as before on the invariance under the flows K_j, L_j shows that:

$$w_{1,sol}^{out} = \frac{\Phi_1^0}{\bar{z}_{1,sol}^{in}} \tag{62}$$

where Φ_1^0 is a constant, which depends only on the combination $\alpha_1 T$. Let us now consider a torus close to the critical torus. The point $(z_{1,sol}^{in}, w_{1,sol}^{in} = 0)$ is displaced to (z_1^{in}, w_1^{in}) and $(z_{1,sol}^{out} = 0, w_{1,sol}^{out})$ is displaced to (z_1^{out}, w_1^{out}). The small parameters can be chosen to be w_j^{in} $(1 \leq j \leq m)$ and we can write:

$$z_1^{out} = \left(\frac{z_1^{in}\bar{w}_1^{in}}{\overline{\Phi_1^0}}\right) z_1^{in} + O(w^2) \tag{63}$$

$$w_1^{out} = \frac{\Phi_1^0}{\bar{z}_1^{in}} + O(1)w_1^{in} + O(w^2) = \left(\frac{\Phi_1^0}{\bar{z}_1^{in}w_1^{in}} + O(1)\right) w_1^{in} + O(w^2) \tag{64}$$

where we have imposed the condition $\bar{z}_1^{in}w_1^{in} = \bar{z}_1^{out}w_1^{out} = K_1 + iL_1$. Connecting these equations with the definition (59) we see that:

$$\rho_{w1}^{(1)} \simeq \frac{\Phi_1^0}{K_1 + iL_1} + O(1)$$

in the vicinity of the singular torus. As explained in [4], a regularization procedure is needed in order to define the diagonal symplectic invariant.

8 Appendix: Symplectic invariance.

In this Appendix we explain why the quantity Ω^{reg} computed in Section 5.3 has an intrinsic meaning and is invariant under symplectic transformations preserving the singular Liouville foliation. We extend to higher dimensions the arguments developed in [4].

Recall that two singular foliations $\mathcal{F}$ and $\mathcal{F}'$ in the symplectic manifolds M, ω and M', ω' are equivalent if there exists a symplectic diffeomorphism $\varphi : \mathcal{F} \to \mathcal{F}'$ sending leaves to leaves.

Having two such foliations, by Eliasson theorem, we can define for each of them normal coordinates (z_j, w_j) and (z'_j, w'_j), $j = 1 \cdots m$, valid in neighborhoods U and U' of the focus-focus point. We can also define the moment maps $\mu^{(0)} : (z_j, w_j) \to (K_j, L_j)$ and $\mu^{(0)'} : (z'_j, w'_j) \to (K'_j, L'_j)$, $j = 1 \cdots m$ where the Hamiltonians K_i, L_i and K'_i, L'_i are as in Section 3. In the neighborhoods U and U', the study of the symplectic diffeomorphism φ therefore reduces to the study of the commutative diagram

$$
\begin{array}{ccc}
\mathbb{R}^{4m} & \xrightarrow{\Phi} & \mathbb{R}^{4m} \\
\downarrow \mu^{(0)} & & \downarrow \mu^{(0)'} \\
\mathbb{R}^{2m} & \xrightarrow{\Psi} & \mathbb{R}^{2m}
\end{array}
\tag{65}
$$

where Φ is a symplectomorphism and Ψ is a diffeomorphism. We will show that the diffeomorphism Ψ is extremely constrained and such that $L_i = \zeta_i L'_i$ with $\zeta_i = \pm 1$ and $K_i = K'_i$, up to a flat function (vanishing at the origin together with all its derivatives) so that the Taylor expansion of Ω^{reg} has an invariant meaning.

The first observation is that the rank of the moment map is preserved under Φ. Manifolds where the rank of $\mu^{(0)}$ is r are mapped to manifolds where the rank of $\mu^{(0)'}$ is r. Manifolds of rank 2 are the (complex) coordinate planes z_i, w_i, characterized by the equations $c_i \equiv K_i + iL_i \neq 0$ and $c_j = 0$ for $j \neq i$. Therefore the map Ψ sends the coordinate axis (in complex notation) $c_i \neq 0$ to a coordinate axis $c'_{i'} \neq 0$. This defines a map $\sigma : i' = \sigma(i)$ of the set $\{1, 2 \cdots m\}$ to itself which is a permutation. We can assume that this permutation is the identity.

One can generalize this argument. Manifolds of rank 4 are the characterized by $c_i \neq 0, c_j \neq 0$ and all other $c_k = 0$. Therefore the map Ψ sends the plane $c_i \neq 0, c_j \neq 0$ and all other $c_k = 0$ to a similar plane $c'_r \neq 0, c'_s \neq 0$ and all other $c'_l = 0$. But that plane must contain the axes $c'_i \neq 0$ and $c'_j \neq 0$, hence it is the whole plane $c'_i \neq 0, c'_j \neq 0$ and all other $c'_k = 0$. Repeating the argument, manifolds of rank $2m - 2$ are hyperplanes $z_i = 0, w_i = 0$ characterized by the equations $c_i = K_i + iL_i = 0$ and $c_j \neq 0$ for $j \neq i$ and the map Ψ sends the hyperplane $c_i = 0$ to the hyperplane $c'_i = 0$.

The symplectomorphism Φ sends a periodic flow to a periodic flow of the same period. In particular, the flow generated by L_i is periodic. Let us denote by $(K'_i(K, L), L'_i(K, L))$ the coordinates of the image by Ψ of the point of coordinates (K_i, L_i). The Hamiltonian vector field of L'_i is

$$
X_{L'_i} = \sum_k \frac{\partial L'_i}{\partial L_k} X_{L_k} + \frac{\partial L'_i}{\partial K_k} X_{K_k}
$$

However, *on the open sets U and U'*, the only periodic flows are those generated by the L_k and L'_k. Hence $\frac{\partial L'_i}{\partial K_k} = 0$ and $\frac{\partial L'_i}{\partial L_k} = A_{ik}$ where A_{ik} is an invertible matrix of $GL_m(\mathbb{Z})$. Therefore

$$
L'_i = \sum_k A_{ik} L_k
$$

On the other hand, we have just seen that if only c_i is different from zero, then only c'_i is different from zero. This implies that the matrix A is diagonal, and since it is an element of $GL_m(\mathbb{Z})$, it is of the form $A_{ik} = \zeta_i \delta_{ij}, \zeta_i = \pm 1$. We have shown that

$$
L'_i = \zeta_i L_i, \quad \zeta_i = \pm 1
\tag{66}
$$

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Next we want to analyse the functions

$$K'_i = K'_i(K_1, L_1 \cdots K_m, L_m)$$

The remarks on the preservation of the rank of the moment map put constraints on the form of the Taylor expansions of these functions. First, because the manifold $c_i = 0, c_j \neq 0, j \neq i$ is mapped to the manifold $c'_i = 0, c'_j \neq 0, j \neq i$, we can write

$$K'_i = A_i(K_1, L_1 \cdots K_m, L_m)K_i + B_i(K_1, L_1 \cdots K_m, L_m)L_i \qquad (67)$$

where the functions A_i, B_i are regular at 0. Moreover, since the axis $c_i \neq 0, c_j = 0, j \neq i$ is mapped on the axis $c'_i \neq 0, c'_j = 0, j \neq i$ we have

$$A_i(K_1, L_1 \cdots K_m, L_m) = \alpha_i + O(K, L), \quad B_i(K_1, L_1 \cdots K_m, L_m) = \beta_i + O(K, L)$$

where α_i, β_i are constants such that $\alpha_i \beta_i \neq 0$. We show below the much stronger results $\alpha_i + O(K, L) = 1 + \text{flat}$ and $\beta_i + O(K, L) = 0 + \text{flat}$.

Let us assume for simplicity that $i = 1$. Let us consider, in the plane (z'_1, w'_1), the points on the critical torus $m'^{in}_0 : (w'_1 = \delta, z'_1 = 0)$ and $m'^{out}_0 : (w'_1 = 0, z'_1 = \delta)$ (all other coordinates vanish). The points $m^{in}_0 = \Phi^{-1}(m'^{in}_0)$ and $m^{out}_0 = \Phi^{-1}(m'^{out}_0)$ are in the plane (z_1, w_1) on the critical torus and we can assume $m^{in}_0 : (w_1 = a, z_1 = 0)$, $m^{out}_0 : (w_1 = 0, z_1 = b)$ (all other coordinate vanish).

Let us now consider a small neighborhood of m'^{in}_0 and let us apply at each of its points the flow generated by (K'_1, L'_1) during times (a'_1, θ'_1) chosen in such a way that the point $m'^{in} : (z'^{in}_1 = \bar{c}'_1/\delta, w'^{in}_1 = \delta, z'^{in}_j, w'^{in}_j)$ is sent to $m'^{out} : (z'^{out}_1 = \delta, w'^{out}_1 = c'_1/\delta, z'^{out}_j = z'^{in}_j, w'^{out}_j = w'^{in}_j)$ i.e.

$$e^{-a'_1 + i\theta'_1} = \frac{c'_1}{\delta^2}$$

In particular, when $|c'_1| \to 0$, we find

$$a'_1 \to -\log|c'_1| \simeq -\log|c_1|$$

where the last relation comes from eqs.(66,67). This map sends m'^{in}_0 to m'^{out}_0 and extends to a smooth map F' from a neighborhood of m'^{in}_0 to a neighborhood of m'^{out}_0. The map $F = \Phi^{-1} \circ F' \circ \Phi$ sends a neighborhood of m^{in}_0 to a neighborhood of m^{out}_0 and this map, being a composition of C^∞ maps, should be C^∞. On the other hand, the map F is obtained by applying the flow

$$\sum_{j=1}^m a'_1 \frac{\partial K'_1}{\partial K_j} X_{K_j} + \theta'_1 X_{L_1} + \sum_{j=1}^m a'_1 \frac{\partial K'_1}{\partial L_j} X_{L_j}$$

On (z_1, w_1), this gives

$$z^{out}_1 = \exp\left[a'_1 \frac{\partial K'_1}{\partial K_1} + i\left(\theta'_1 + a'_1 \frac{\partial K'_1}{\partial L_1}\right)\right] z^{in}_1$$

$$w^{out}_1 = \exp\left[-a'_1 \frac{\partial K'_1}{\partial K_1} + i\left(\theta'_1 + a'_1 \frac{\partial K'_1}{\partial L_1}\right)\right] w^{in}_1$$

When $|c_1| \to 0$, we should have $z^{in} \to \frac{\bar{c}_1}{\bar{a}}$, $z^{out} \to b$. Inserting into the first equation, we find

$$\bar{a}b = \exp\left[\log \bar{c}_1 + a_1'\frac{\partial K_1'}{\partial K_1} + i\left(\theta_1' + a_1'\frac{\partial K_1'}{\partial L_1}\right)\right]$$

(for the other equation $w^{in} \to a$ $w^{out} \to \frac{c_1}{b}$, we obtain the same condition). Using the values of a_1', θ_1', we get in the limit $|c_1| \to 0$

$$\bar{a}b = \exp\left[\log |c_1|\left(1 - \frac{\partial K_1'}{\partial K_1} - i\frac{\partial K_1'}{\partial L_1}\right)\right]$$

Finiteness when $|c_1| \to 0$ requires $\alpha_1 = 1, \beta_1 = 0$. One can push the argument and show that smoothness of the map in the limit $c_1 \to 0$ gives

$$\frac{\partial K_1'}{\partial K_1} = 1 + \text{flat}, \quad \frac{\partial K_1'}{\partial L_1} = \text{flat}$$

the argument is the same as in [4].

On $(z_j, w_j), j \neq 1$, we get

$$z_j^{out} = \exp\left[a_1'\frac{\partial K_1'}{\partial K_j} + i\left(a_1'\frac{\partial K_1'}{\partial L_j}\right)\right]z_j^{in}$$

$$w_j^{out} = \exp\left[-a_1'\frac{\partial K_1'}{\partial K_j} + i\left(a_1'\frac{\partial K_1'}{\partial L_j}\right)\right]w_j^{in}$$

We require that (z_j^{out}, w_j^{out}) be finite when c_1 tends to zero with (z_j^{in}, w_j^{in}) finite. This gives

$$\frac{\partial K_1'}{\partial K_j} = \text{flat}, \quad \frac{\partial K_1'}{\partial L_j} = \text{flat}$$

9   Appendix: Solitons.

In this appendix we recall the construction of solitons in the Jaynes-Cummings-Gaudin model [13]. We present it in some details because we need to generalize it to include all times t_i and we wish to present in a self contained way the formulae we use in our calculation of the symplectic invariants.

The Lax form eq. (10) of the equation of motion implies that the so-called spectral curve Γ, defined by $\det(L(\lambda) - \mu) = 0$ is a constant of motion. Specifically:

$$\Gamma: \ \mu^2 - A^2(\lambda) - B(\lambda)C(\lambda) = 0, \ \text{i.e. } \mu^2 = \frac{Q_{2n+2}(\lambda)}{\prod_j(\lambda - \epsilon_j)^2} \tag{68}$$

Defining $y = \mu \prod_j(\lambda - \epsilon_j)$, the equation of the curve becomes $y^2 = Q_{2n+2}(\lambda)$ which is an hyperelliptic curve. Since the polynomial $Q_{2n+2}(\lambda)$ has degree $2n+2$, the genus of the curve in n.

The separated variables are $g = n$ points on the curve whose coordinates (λ_k, μ_k) can be taken as coordinates on phase space. They are defined as follows. Let us write

$$C(\lambda) = 2\bar{b} + \sum_{j=1}^{n} \frac{s_j^+}{\lambda - \epsilon_j} \equiv 2\bar{b} \frac{\prod_{k=1}^{n}(\lambda - \lambda_k)}{\prod_{j=1}^{n}(\lambda - \epsilon_j)} \tag{69}$$

the separated coordinates are the collection $(\lambda_k, \mu_k = A(\lambda_k))$. They have canonical Poisson brackets

$$\{\lambda_k, \mu_{k'}\} = -i\delta_{k,k'} \tag{70}$$

Notice however that if $\bar{b} = 0$ theses variables are not well defined.

There are only $2n$ such coordinates which turn out to be invariant under the global $U(1)$ rotation generated by H_{n+1}:

$$b \to e^{i\theta} \, b, \quad \bar{b} \to e^{-i\theta} \, \bar{b}, \quad s_j^- \to e^{i\theta} s_j^-, \quad s_j^+ \to e^{-i\theta} s_j^+$$

So they describe the *reduced* model obtained by fixing the value of H_{n+1} and taking into consideration only the dynamical variables invariant under this $U(1)$ action. The initial dynamical model can be recovered by adding the phase of the oscillator coordinates $\bar{b}, b$ to the separated variables.

We show now how to reconstruct spin coordinates from the separated variables (λ_k, μ_k) and the oscillator coordinates $(\bar{b}, b)$. For $C(\lambda)$ we have eq.(69). It is a rational fraction of λ which has simple poles at $\lambda = \epsilon_j$ whose residue is s_j^+. For $A(\lambda)$, we write:

$$A(\lambda) = \frac{P_{n+1}(\lambda)}{\prod_{j=1}^{n}(\lambda - \epsilon_j)}$$

The polynomial $P_{n+1}(\lambda) - 2\lambda \prod_{j=1}^{n}(\lambda - \epsilon_j)$ is of degree $n - 1$, and we know its value at the n points λ_j because:

$$P_{n+1}(\lambda_j) = \mu_j \prod_{k=1}^{n}(\lambda_j - \epsilon_k) \tag{71}$$

Therefore, we can write:

$$P_{n+1}(\lambda) = 2\lambda \prod_{j=1}^{n}(\lambda - \epsilon_j) + \sum_i (\mu_i - 2\lambda_i) \prod_{k=1}^{n}(\lambda_i - \epsilon_k) \frac{\prod_{l \neq i}(\lambda - \lambda_l)}{\prod_{l \neq i}(\lambda_i - \lambda_l)} \tag{72}$$

Once $A(\lambda)$ and $C(\lambda)$ are known, $B(\lambda)$ is determined by:

$$B(\lambda) = \frac{Q_{2n+2}(\lambda) - P_{n+1}^2(\lambda)}{C(\lambda) \prod_k (\lambda - \epsilon_k)^2} = \frac{1}{2\bar{b}} \frac{Q_{2n+2}(\lambda) - P_{n+1}^2(\lambda)}{\prod_i (\lambda - \lambda_i) \prod_k (\lambda - \epsilon_k)} \tag{73}$$

The polynomial in the numerator is of degree $2n$, and moreover it is divisible by $\prod_i(\lambda - \lambda_i)$ because $P_{n+1}^2(\lambda_i) = Q_{2n+2}(\lambda_i)$, so we can write

$$B(\lambda) = 2b \frac{\prod_i (\lambda - \bar{\lambda}_i)}{\prod_k (\lambda - \epsilon_k)} \tag{74}$$

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In this construction, the variables $\bar{\lambda}_i$ and the corresponding $\bar{\mu}_i$ are complicated functions of the (λ_i, μ_i). The set of physical configurations (often refered to here as the *real slice*) is obtained by writing that the set $(\bar{\lambda}_i, \bar{\mu}_i)$ is the *complex conjugate* of the set (λ_i, μ_i). This leads to a set of complicated relations whose solution is not known in general, but that we will solve in the soliton case.

The Hamiltonians H_j of the reduced model are obtained by writing that the points (λ_k, μ_k) belong to the spectral curve. We get a linear system of equations $\sum_j \mathcal{B}_{kj} H_j = V_k$ where

$$\mathcal{B}_{kj} = \frac{1}{\lambda_k - \epsilon_j}, \quad V_k = \frac{1}{2}\left(\mu_k^2 - \lambda_k^2 - 4H_{n+1} - \sum_j \frac{s^2}{(\lambda_k - \epsilon_j)^2} \right)$$

Its solution is

$$H_i = \sum_k (\mathcal{B}^{-1})_{ik} V_k, \quad (\mathcal{B}^{-1})_{jp} = \frac{\prod_{l \neq p}(\epsilon_j - \lambda_l) \prod_i (\lambda_p - \epsilon_i)}{\prod_{i \neq j}(\epsilon_j - \epsilon_i) \prod_{l \neq p}(\lambda_p - \lambda_l)}$$

Using the Poisson bracket eq.(70), the equation of motion of the variable λ_k with respect to H_i

is then (no summation over k):

$$\partial_{t_i} \lambda_k = i\mu_k (\mathcal{B}^{-1})_{ik}, \tag{75}$$

We are interested in the system with prescribed real values of the conserved quantities, $H_1,...,H_{n+1}$, i.e. we take as coordinates the (λ_j, H_j) instead of the (λ_j, μ_j). This amounts to fixing the Liouville torus we work with, or equivalently the spectral polynomial $Q_{2n+2}(\lambda)$. In this setting, the μ_j are determined by the equations

$$\prod_{k=1}^n (\lambda_j - \epsilon_k)\mu_j = (\pm)_j \sqrt{Q_{2n+2}(\lambda_j)}$$

The equations of motion eq.(75) must be complemented by the equation of motion for the phase of b.

We consider now the level set containing the critical point eq.(21). It corresponds to a maximally degenerate spectral curve:

$$Q_{2n+2}(\lambda) = 4 \prod_{l=1}^{n+1} (\lambda - E_l)^2 \tag{76}$$

where the zeroes of $Q_{2n+2}(\lambda)$ are the roots of the classical Bethe equation:

$$2E + \sum_{j=1}^n \frac{se_j}{E - \epsilon_j} = 0 \tag{77}$$

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The separated variables λ_i, μ_i now satisfy

$$\mu_i = 2(\pm)_i \frac{\prod_l(\lambda_i - E_l)}{\prod_j(\lambda_i - \epsilon_j)} \tag{78}$$

The choice of sign here plays a crucial role in the description of the various strata of the level set.

We first examine the case where a λ_i is frozen at a root E_l. Let us assume that $Q_{2n+2}(\lambda)$ has a *real* root at $\lambda = E$. This means that $A^2(\lambda) + B(\lambda)C(\lambda)$ vanishes when $\lambda = E$. But for real λ, $A(\lambda)$ is real and one has $C(\lambda) = \overline{B(\lambda)}$ so that $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ must all vanish at $\lambda = E$. In particular, recalling eq.(69), this means that one of the separated variables, say λ_1, is frozen at the value E, *provided* $\bar{b}b \neq 0$. This implies also that $\lambda - E$ divides simultaneously $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$, and therefore $(\lambda - E)^2$ divides $Q_{2n+2}(\lambda)$. A real root of $Q_{2n+2}(\lambda)$ is necessarily a double root and one λ_i must be frozen at E.

Nothing simple can be said in the case $Q_{2n+2}(\lambda)$ has a simple *complex* root. So, let us assume that it has a *double* complex root E. Of course the complex conjugate $\bar{E}$ is also a double root. Since $\Lambda(\lambda) = A^2(\lambda) + B(\lambda)C(\lambda)$ we have

$$\Lambda'(\lambda) = 2A(\lambda)A'(\lambda) + B'(\lambda)C(\lambda) + B(\lambda)C'(\lambda)$$

So, if $\Lambda(\lambda)$ has a double zero E we have

$$\begin{aligned} 0 &= A^2(E) + B(E)C(E) \\ 0 &= 2A(E)A'(E) + B'(E)C(E) + B(E)C'(E) \end{aligned}$$

In contrast to the real case, we cannot infer from these equations that $C(E) = 0$. But if $C(E) = 0$ that is if one λ_i freezes at E, the first equation implies $A(E) = 0$ and the second equation implies $B(E)C'(E) = 0$. Therefore if the zero of $C(\lambda)$ at $\lambda = E$ is simple, then necessarily $B(E) = 0$. But $B(E)$ is the complex conjugate of $C(\bar{E})$ which must therefore vanish *provided* $\bar{b}b \neq 0$. From this we conclude that another separated variable λ_k freezes at $\bar{E}$. In contrast to the real case, freezing is not compulsory, but it is the possibility to freeze the λ_i by complex conjugated pairs that leads to the description of the real slice and the stratification of the level set.

Coming back to eq.(78), we see from eqs.(71,76) that:

$$P_{n+1}(\lambda_i) = (\pm)_i\sqrt{Q_{2n+2}(\lambda_i)} = 2(\pm)_i \prod_l(\lambda_i - E_l), \quad P_{n+1}(\lambda) = 2(\lambda^{n+1} - \sigma_1(\epsilon)\lambda^n + \cdots)$$

If we take the $+$ sign for all i, then obviously $P_{n+1}(\lambda) = 2\prod_l(\lambda - E_l)$, and taking the residue at $\lambda = \epsilon_j$ in $P_{n+1}(\lambda)/\prod_i(\lambda - \epsilon_i)$, we find $s_j^z = se_j$. This is the static solution corresponding to the critical point.

To go beyond this trivial solution, we divide the λ_i into three sets $\lambda_i^+ \in \mathcal{E}^+$ and $\lambda_i^- \in \mathcal{E}^-$ depending on the sign in this formula, and $\lambda_i^0 \in \mathcal{E}^0$ is the set of λ_i frozen at some E_l.

We define the polynomials:

$$\mathcal{P}(\lambda, t) = \prod_j(\lambda - \lambda_j(t)) = \mathcal{P}^{(+)}(\lambda)\mathcal{P}^{(0)}(\lambda)\mathcal{P}^{(-)}(\lambda)$$

The equations of motion eq.(75) read in the soliton case

$$\partial_{t_i}\lambda_k = \sqrt{-1}\mu_k(\mathcal{B}^{-1})_{ik} = 2\sqrt{-1}(\pm)_k\frac{\prod_{l'}(\lambda_k - E_{l'})\prod_{l\neq k}(\epsilon_i - \lambda_l)}{\prod_{j\neq i}(\epsilon_i - \epsilon_j)\prod_{l\neq k}(\lambda_k - \lambda_l)}$$

hence, for $E_l \notin \mathcal{E}^0$,

$$\sum_k{}'(\pm)_k\frac{1}{\lambda_k - E_l}\partial_{t_i}\lambda_k = -2\sqrt{-1}\frac{\prod_l(\epsilon_i - \lambda_l)}{\prod_{j\neq i}(\epsilon_i - \epsilon_j)}\sum_k{}'\frac{\prod_{l'\neq l}'(\lambda_k - E_{l'})}{(\lambda_k - \epsilon_i)\prod_{l\neq k}(\lambda_k - \lambda_l)}$$

where $\sum_k{}'$ means that the frozen $\lambda_k \in \mathcal{E}^0$ are excluded. We can rewrite this equation as

$$
\begin{aligned}
\partial_{t_i}\log\frac{\mathcal{P}^{(+)}(E_l)}{\mathcal{P}^{(-)}(E_l)} &= -2\sqrt{-1}\frac{\prod_l(\epsilon_i - \lambda_l)}{\prod_{j\neq i}(\epsilon_i - \epsilon_j)}\sum_k{}'\mathrm{Res}_{\lambda_k}\frac{\prod_{l'\neq l}(z - E_{l'})}{(z - \epsilon_i)\prod_l(z - \lambda_l)} \\
&= 2\sqrt{-1}\frac{\prod_l(\epsilon_i - \lambda_l)}{\prod_{j\neq i}(\epsilon_i - \epsilon_j)}(\mathrm{Res}_\infty + \mathrm{Res}_{\epsilon_i})\frac{\prod_{l'\neq l}'(z - E_{l'})}{(z - \epsilon_i)\prod_l'(z - \lambda_l)} \\
&= 2\sqrt{-1}\frac{\prod_l(\epsilon_i - \lambda_l)}{\prod_{j\neq i}(\epsilon_i - \epsilon_j)}\left(-1 + \frac{\prod_{l'\neq l}'(\epsilon_i - E_{l'})}{\prod_l'(\epsilon_i - \lambda_l)}\right)
\end{aligned}
$$

so that

$$\partial_{t_i}\log\frac{\mathcal{P}^{(+)}(E_l)}{\mathcal{P}^{(-)}(E_l)} = -2\sqrt{-1}\frac{\prod_l(\epsilon_i - \lambda_l)}{\prod_{j\neq i}(\epsilon_i - \epsilon_j)} + 2\frac{\sqrt{-1}}{\epsilon_i - E_l}\frac{\prod_{l'}(\epsilon_i - E_{l'})}{\prod_{j\neq i}(\epsilon_i - \epsilon_j)}$$

Remembering the identities:

$$2\frac{\prod_k(\epsilon_j - E_k)}{\prod_{k\neq j}(\epsilon_j - \epsilon_k)} = se_j, \quad \sum_l E_l = \sigma_1(\epsilon) \tag{79}$$

we can rewrite

$$\partial_{t_i}\log\frac{\mathcal{P}^{(+)}(E_l)}{\mathcal{P}^{(-)}(E_l)} = -\sqrt{-1}\frac{s_i^+}{\bar{b}} + \sqrt{-1}\frac{se_i}{\epsilon_i - E_l}$$

But the equation of motion for $\bar{b}$ is

$$\partial_{t_i}\bar{b} = \{H_i, \bar{b}\} = \sqrt{-1}s_i^+$$

so that finally

$$\partial_{t_i}\log\frac{\mathcal{P}^{(+)}(E_l)}{\mathcal{P}^{(-)}(E_l)} = -\partial_{t_i}\log\bar{b} + \sqrt{-1}\frac{se_i}{\epsilon_i - E_l}, \quad E_l \notin \mathcal{E}^0$$

Integrating these equations we get

$$\mathcal{P}^{(-)}(E_l) = \bar{b}(\{t\})X_l(\{t\})\mathcal{P}^{(+)}(E_l), \quad X_l(\{t\}) = X_l(0)e^{\sqrt{-1}\left(\sum_i\frac{se_i}{E_l - \epsilon_i}t_i - t_{n+1}\right)}, \quad E_l \notin \mathcal{E}^0 \tag{80}$$

The t_{n+1} dependence of X_l is fixed by noticing that $\mathcal{P}^{(\pm)}(E_l)$ are independent of t_{n+1}, while $\partial_{t_{n+1}}\bar{b} = \sqrt{-1}\,\bar{b}$.

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Denoting $n_\pm, n_0$ the number of elements in $\mathcal{E}^{(\pm)}, \mathcal{E}^0$ respectively, we get a set of $n - n_0 + 1$ *linear* equations for the $n_+ + n_- + 1 = n - n_0 + 1$ unknown coefficients of the polynomials $\mathcal{P}_-(\lambda)$ and $\bar{b}(t)\mathcal{P}_+(\lambda)$.

These real soliton solutions live on the pre-images of focus-focus singularities These are composed of various strata, depending on the number of n_0 of separated variables frozen by pairs at the double roots $(E_l, E_{\bar{l}} = E_l^*)$ of the spectral polynomial. Separated variables are necessarily frozen at the real double roots of the spectral polynomial. These strata are themselves products of $(n + 1 - n_0)/2$ two dimensional pinched tori.

We then consider the remaining complex roots of the spectral polynomial where no separated variables are frozen. This is a self conjugate set. We write eq.(80) as a linear system for the symmetric functions $\sigma_i^{(-)}(\{\lambda_j^{(-)}\})$ and $\sigma_i^{(+)}(\{\lambda_j^{(+)}\})$ and $\bar{b}$. We introduce the vector

$$V = \begin{pmatrix} (-1)^{n_-}\sigma_{n_-}^{(-)}/\bar{b} \\ \vdots \\ -\sigma_1^{(-)}/\bar{b} \\ 1/\bar{b} \\ (-1)^{1+n_+}\sigma_{n_+}^{(+)} \\ \vdots \\ \sigma_1^{(+)} \end{pmatrix}$$

then eq.(80) reads

$$(1, E, \cdots, E^{n_-}, X, XE, \cdots, XE^{n_+ - 1})V = XE^{n_+}$$

where we have defined the column vectors $(E^j)_l = E_l^j$ and $(XE^j)_l = X_l E_l^j$ where X_l is given by eq.(80). The formulae used in the text are straightforward consequences Cramer's solution of this linear system.

In the vector X we should incorporate the reality conditions

$$\overline{X_l}\, X_{\bar{l}} = -\frac{1}{4}, \quad E_{\bar{l}} \equiv \overline{E_l}$$

which we derive below.

9.1   Reality conditions.

It remains to impose the condition that the set of zeros $\{\bar{\lambda}_k\}$ of $B(\lambda)$ in eq.(74) is the complex conjugate of the set of zeros $\{\lambda_k\}$ of $C(\lambda)$. Going back to eq.(73), the first step is to build the polynomial $P_{n+1}(\lambda)$. Using the fact that $P_{n+1}(\lambda_i) = 2(\pm)_i \prod_l(\lambda_i - E_l)$ we may write :

$$P_{n+1}(\lambda) = 2\prod_l(\lambda - E_l) - 4\sum_{\lambda_i^-\, E_l \notin \mathcal{E}^0} \prod (\lambda_i^- - E_l)\frac{\prod_{\lambda_l^- \neq \lambda_i^-}(\lambda - \lambda_l^-)}{\prod_{\lambda_l^- \neq \lambda_i^-}(\lambda_i^- - \lambda_l^-)}\frac{\mathcal{P}_+(\lambda)\mathcal{P}_0(\lambda)}{\mathcal{P}_+(\lambda_i^-)}$$

To derive this formula, we used the fact that the polynomials $P_{n+1}(\lambda)$ and $2\prod_l(\lambda - E_l)$ have the same terms of degrees $n + 1$ and n; the last statement comes from the relation $\sum_j \epsilon_j = \sum_l E_l$

which is a direct consequence of the classical Bethe equation. Alternatively, we can also write:

$$P_{n+1}(\lambda) = -2\prod_l (\lambda - E_l) + 4\sum_{\lambda_i^+}\prod_{E_l \notin \mathcal{E}^0}(\lambda_i^+ - E_l)\frac{\prod_{\lambda_l^+ \neq \lambda_i^+}(\lambda - \lambda_l^+)}{\prod_{\lambda_l^+ \neq \lambda_i^+}(\lambda_i^+ - \lambda_l^+)}\frac{\mathcal{P}_-(\lambda)\mathcal{P}_0(\lambda)}{\mathcal{P}_-(\lambda_i^+)}$$
$$+4(\lambda + \Sigma_1' - \sigma_1'(E))\mathcal{P}_+(\lambda)\mathcal{P}_-(\lambda)\mathcal{P}_0(\lambda)$$

Note that the last term in the right hand side is necessary to adjust the terms of degree $n+1$ and n in λ between the two sides of the equation. Again, we used the relation $\sum_j \epsilon_j = \sum_l E_l$. These two expressions for $P_{n+1}(\lambda)$ motivate the following definitions of polynomials $\mathcal{S}_\pm(\lambda)$:

$$\mathcal{S}_+(\lambda) = \sum_{\lambda_i^-}\prod_{E_l \notin \mathcal{E}^0}(\lambda_i^- - E_l)\frac{\prod_{\lambda_l^- \neq \lambda_i^-}(\lambda - \lambda_l^-)}{\mathcal{P}_-'(\lambda_i^-)\mathcal{P}_+(\lambda_i^-)}$$

$$\mathcal{S}_-(\lambda) = \sum_{\lambda_i^+}\prod_{E_l \notin \mathcal{E}^0}(\lambda_i^+ - E_l)\frac{\prod_{\lambda_l^+ \neq \lambda_i^+}(\lambda - \lambda_l^+)}{\mathcal{P}_+'(\lambda_i^+)\mathcal{P}_-(\lambda_i^+)} + (\lambda + \Sigma_1' - \sigma_1'(E))\mathcal{P}_+(\lambda)$$

so that we can write:

$$P_{n+1}(\lambda) = 2\prod_l(\lambda - E_l) - 4\mathcal{S}_+(\lambda)\mathcal{P}_+(\lambda)\mathcal{P}_0(\lambda) \tag{81}$$

$$P_{n+1}(\lambda) = -2\prod_l(\lambda - E_l) + 4\mathcal{S}_-(\lambda)\mathcal{P}_-(\lambda)\mathcal{P}_0(\lambda) \tag{82}$$

Note that $\mathcal{S}_+(\lambda)$ has degree $n_- - 1$ and $\mathcal{S}_-(\lambda)$ has degree $n_+ + 1$. Now, we have:

$$4\prod_l(\lambda - E_l)^2 - P_{n+1}^2(\lambda) = 16\,\mathcal{S}_-(\lambda)\mathcal{S}_+(\lambda)\mathcal{P}_0(\lambda)\mathcal{P}(\lambda) = 4\bar{b}b\mathcal{P}(\lambda)\bar{\mathcal{P}}(\lambda)$$

where $\mathcal{P}(\lambda) = \prod_j(\lambda - \lambda_j) = \mathcal{P}_-(\lambda)\mathcal{P}_0(\lambda)\mathcal{P}_+(\lambda)$ and $\bar{\mathcal{P}}(\lambda) = \prod_j(\lambda - \bar{\lambda}_j)$ is the complex conjugate of $\mathcal{P}(\lambda)$, that is $\bar{\mathcal{P}}(\lambda) = \overline{\mathcal{P}(\bar{\lambda})}$. Therefore:

$$\bar{b}b\,\bar{\mathcal{P}}(\lambda) = 4\,\mathcal{S}_-(\lambda)\mathcal{S}_+(\lambda)\mathcal{P}_0(\lambda) \tag{83}$$

So the zeroes $\bar{\lambda}_i$ of $\bar{\mathcal{P}}(\lambda)$ split into the zeroes $\bar{\lambda}_i^+$, $\bar{\lambda}_i^-$ and $\bar{\lambda}_i^0$ of $\mathcal{S}_+(\lambda)$, $\mathcal{S}_-(\lambda)$ and $\mathcal{P}_0(\lambda)$ respectively. A direct consequence of these definitions and of eqs. (81) and (82) is that:

$$P_{n+1}(\bar{\lambda}_i^+) = +2\prod_l(\bar{\lambda}_i^+ - E_l) \tag{84}$$

$$P_{n+1}(\bar{\lambda}_i^-) = -2\prod_l(\bar{\lambda}_i^- - E_l) \tag{85}$$

By the definition of the $\bar{\lambda}_i^0$'s, we see that the set $\mathcal{E}^0$ is self conjugate and that $\mathcal{P}_0(\lambda) = \bar{\mathcal{P}}_0(\lambda)$. The above definition of $\mathcal{S}_-(\lambda)$ shows that the coefficient of its term of highest degree is equal to one (this is *not* the case for $\mathcal{S}_+(\lambda)$). Because of this:

$$\mathcal{S}_-(\lambda) = \bar{\mathcal{P}}_-(\lambda) \tag{86}$$

Combining this with eq. (83) we get also:

$$\mathcal{S}_+(\lambda) = \frac{1}{4}\bar{b}b\,\bar{\mathcal{P}}_+(\lambda) \tag{87}$$

Comparing the terms of highest degrees in $\mathcal{S}_+(\lambda)$ and $\bar{\mathcal{P}}_+(\lambda)$ gives:

$$\bar{b}b = 4\sum_{\lambda_i^-}\frac{\prod_{E_l\notin\mathcal{E}^0}(\lambda_i^- - E_l)}{\mathcal{P}'_-(\lambda_i^-)\mathcal{P}_+(\lambda_i^-)} \tag{88}$$

It is expressed only in terms of λ_i^-'s. So, if $n_- = 0$, we recover the fact already mentioned that $\bar{b}b = 0$ and the system remains at the critical point.

At this stage, we are ready to enforce the reality condition. As discussed above, the real slice is obtained by imposing that the set $\{\bar{\lambda}_i\}$ be the same as the set $\{\lambda_i^*\}$, where in the rest of this section we denote by z^* the complex conjugate of z. Equivalently:

$$\bar{\mathcal{P}}(\lambda^*) = \mathcal{P}(\lambda)^*$$

for any λ. From the discussion at the beginning of this section, we know that the frozen variables λ_i^0 appear in complex conjugate pairs so that $\mathcal{P}_0(\lambda^*) = \mathcal{P}_0(\lambda)^*$. We also know that $\mathcal{P}_0(\lambda) = \bar{\mathcal{P}}_0(\lambda)$ so that $\bar{\mathcal{P}}_0(\lambda^*) = \mathcal{P}_0(\lambda)^*$. The above reality condition becomes then:

$$\bar{\mathcal{P}}_-(\lambda^*)\bar{\mathcal{P}}_+(\lambda^*) = \mathcal{P}_-(\lambda)^*\mathcal{P}_+(\lambda)^*$$

It is clearly sufficient to impose simultaneously:

$$\bar{\mathcal{P}}_-(\lambda^*) = \mathcal{P}_-(\lambda)^*, \quad \bar{\mathcal{P}}_+(\lambda^*) = \mathcal{P}_+(\lambda)^* \tag{89}$$

but we claim that this condition is also necessary. This comes from the fact already noted that the sign of $P_{n+1}(\lambda)/\prod_l(\lambda - E_l)$ is positive for $\lambda = \lambda_i^+$ or $\lambda = \bar{\lambda}_i^+$ and negative for $\lambda = \lambda_i^-$ or $\lambda = \bar{\lambda}_i^-$. So the roots $\bar{\lambda}_i^+$ have to be complex conjugates of λ_i^+ and likewise, the roots $\bar{\lambda}_i^-$ have to be complex conjugates of λ_i^-. An interesting and useful consequence of this is that we must have deg $\mathcal{P}_\pm =$ deg $\mathcal{S}_\pm =$ deg $\bar{\mathcal{P}}_\pm$, which requires $n_+ = n_- - 1$. Since $n_+ + n_0 + n_- = n$ we find:

$$n_+ = \frac{1}{2}(n - 1 - n_0), \quad n_- = \frac{1}{2}(n + 1 - n_0)$$

Because the coefficients of highest degrees of $\mathcal{P}_-(\lambda)$ and $\mathcal{P}_+(\lambda)$ are set equal to one, the above constraints (89) are equivalent to (assuming of course that $\bar{\mathcal{P}}_+(\lambda)$ and $\bar{\mathcal{P}}_-(\lambda)$ are mutually prime and similarly for $\mathcal{P}_+(\lambda)$ and $\mathcal{P}_-(\lambda)$).

$$\frac{\bar{\mathcal{P}}_-(\lambda^*)}{\bar{\mathcal{P}}_+(\lambda^*)} = \left(\frac{\mathcal{P}_-(\lambda)}{\mathcal{P}_+(\lambda)}\right)^*$$

We also note that we should add the constraint $\bar{b} = b^*$. This plus the fact that these two polynomials involve a total of $n_+ + n_-$ unknown coefficients, shows that it is necessary and sufficient to enforce the following conditions for the $n + 1 - n_0 = n_+ + n_- + 1$ roots E_l which don't belong to $\mathcal{E}^0$:

$$\frac{\bar{\mathcal{P}}_-(E_l^*)}{b\bar{\mathcal{P}}_+(E_l^*)} = \left(\frac{\mathcal{P}_-(E_l)}{b\mathcal{P}_+(E_l)}\right)^* \tag{90}$$

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As we have seen, the general solution of the Hamiltonian evolution on the critical torus, eq. (80) implies:

$$\frac{\mathcal{P}_-(E_l)}{\bar{b}\mathcal{P}_+(E_l)} = X_l \tag{91}$$

To evaluate the left-hand side of the conditions (90), we set $\lambda = E_l$ in eqs.(81, 82), which gives:

$$P_{n+1}(E_l) = -4\mathcal{S}_+(E_l)\mathcal{P}_+(E_l)\mathcal{P}_0(E_l), \quad P_{n+1}(E_l) = 4\mathcal{S}_-(E_l)\mathcal{P}_-(E_l)\mathcal{P}_0(E_l), \quad E_l \notin \mathcal{E}^0$$

and therefore:

$$\mathcal{S}_+(E_l)\mathcal{P}_+(E_l) = -\mathcal{S}_-(E_l)\mathcal{P}_-(E_l) \tag{92}$$

Since $\mathcal{P}_+(E_l) \neq 0$, and remembering eq.(80) this implies:

$$\frac{\mathcal{S}_-(E_l)}{\mathcal{S}_+(E_l)} = -\frac{1}{\bar{b}X_l}$$

But using eqs. (86) and (87), we get:

$$\frac{\bar{\mathcal{P}}_-(E_l^*)}{\bar{b}\bar{\mathcal{P}}_+(E_l^*)} = -\frac{1}{4X_{\bar{l}}} \tag{93}$$

where we define the index $\bar{l}$ to be such that $E_{\bar{l}} = E_l^*$. Given eqs. (91) and (93), the reality conditions (90) and the condition $\bar{b} = b^*$ are satisfied if and only if:

$$X_l^* \, X_{\bar{l}} = -\frac{1}{4} \tag{94}$$

As expected, times disappears from these conditions so that they reduce to constraints on the integration constants:

$$X_l(0)^* \, X_{\bar{l}}(0) = -\frac{1}{4}$$

This characterizes a stratum of dimension $n - n_0 + 1$ on the real slice.

10 Appendix : Asymptotic behavior of solitonic formulae

Let us assume that the m solitonic amplitudes X_j become large while keeping fixed ratios. Because of the reality condition (94), the m remaining amplitudes $X_{\bar{j}}$ go to zero with fixed ratios. With no loss of generality, let us assume that j runs from 1 to m and $\bar{j}$ from $m+1$ to $2m$. The complete expression (33) for $C(\lambda)$ involves three determinants D_1, D_2 and D defined by:

$$D_1 = \det \begin{pmatrix} 1 & \lambda & \cdots & \lambda^m & 0 & 0 & \cdots & 0 \\ 1 & E & \cdots & E^{n-} & X & XE & \cdots & XE^{m-1} \end{pmatrix}$$

$$D_2 = \det \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & \lambda & \cdots & \lambda^{m-1} \\ 1 & E & \cdots & E^m & X & XE & \cdots & XE^{m-1} \end{pmatrix}$$

$$D = \det \begin{pmatrix} 1 & E & \cdots & E^{m-1} & X & XE & \cdots & XE^{m-1} \end{pmatrix}$$

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In the limit where $|X_j| \to \infty$ and $X_{\bar{j}} \to 0$ for $1 \le j \le m$, the dominant term in D_1 is given by:

$$D_1 \simeq \det \begin{pmatrix} 1 & \lambda & \cdots & \lambda^m & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & X_1 & X_1 E_1 & \cdots & X_1 E_1^{m-1} \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ 0 & 0 & \cdots & 0 & X_m & X_m E_m & \cdots & X_m E_m^{m-1} \\ 1 & E_{\bar{1}} & \cdots & E_{\bar{1}}^m & 0 & 0 & \cdots & 0 \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ 1 & E_{\bar{m}} & \cdots & E_{\bar{m}}^m & 0 & 0 & \cdots & 0 \end{pmatrix}$$

So, we get:

$$D_1 \simeq \prod_{i<j} |E_j - E_i|^2 \prod_{j=1}^{m} (\lambda - E_{\bar{j}}) \, X_1 \cdots X_m \tag{95}$$

It is also important to note that the subleading terms of D_1 vanish. We obtain such terms either by removing one of the X_j's or by adding one of the $X_{\bar{j}}$'s. In the first case, the m last columns to the right have only $m-1$ non-vanishing lines, so they are linearly dependent. Likewise, in the second case, the first $m+1$ columns on the left have only m non-vanishing lines. So the first corrections to D_1 originate from the terms where we simultaneously remove one X_j and add one $X_{\bar{k}}$.

The leading term in D_2 has the form:

$$\det \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & \lambda & \cdots & \lambda^{m-1} \\ 0 & 0 & \cdots & 0 & X_1 & X_1 E_1 & \cdots & X_1 E_1^{m-1} \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ 0 & 0 & \cdots & 0 & X_m & X_m E_m & \cdots & X_m E_m^{m-1} \\ 1 & E_{\bar{1}} & \cdots & E_{\bar{1}}^m & 0 & 0 & \cdots & 0 \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ 1 & E_{\bar{m}} & \cdots & E_{\bar{m}}^m & 0 & 0 & \cdots & 0 \end{pmatrix}$$

which is equal to zero. One has therefore to consider subleading terms. They can be obtained by removing one of the X_j's or by adding one of the $X_{\bar{j}}$'s. In fact, only the first option leads to

non-zero terms. Therefore, we may write:

$$D_2 \simeq \sum_{j=1}^{m} \det \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & \lambda & \cdots & \lambda^{m-1} \\ 0 & 0 & \cdots & 0 & X_1 & X_1 E_1 & \cdots & X_1 E_1^{m-1} \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ 0 & 0 & \cdots & 0 & X_{j-1} & X_{j-1} E_{j-1} & \cdots & X_{j-1} E_{j-1}^{m-1} \\ 1 & E_j & \cdots & E_j^m & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & X_{j+1} & X_{j+1} E_{j+1} & \cdots & X_{j+1} E_{j+1}^{m-1} \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ 0 & 0 & \cdots & 0 & X_m & X_m E_m & \cdots & X_m E_m^{m-1} \\ 1 & E_{\bar{1}} & \cdots & E_{\bar{1}}^m & 0 & 0 & \cdots & 0 \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ 1 & E_{\bar{m}} & \cdots & E_{\bar{m}}^m & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Explicitely:

$$D_2 \simeq \prod_{i<j} |E_j - E_i|^2 \prod_{j=1}^{m} (\lambda - E_j) \sum_{j=1}^{m} \left(\frac{1}{X_j} \frac{E_{\bar{j}} - E_j}{\lambda - E_j} \prod_{k \neq j} \frac{E_{\bar{k}} - E_j}{E_k - E_j} \right) X_1 \cdots X_m \qquad (96)$$

The same reasoning as for D_1 shows that the next corrections to this estimate of D_2 are obtained by removing one more of the X_j's and by adding simultaneously one of the $X_{\bar{k}}$'s.

The determinant D has a non-vanishing leading order term:

$$D \simeq \det \begin{pmatrix} 0 & 0 & \cdots & 0 & X_1 & X_1 E_1 & \cdots & X_1 E_1^{m-1} \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ 0 & 0 & \cdots & 0 & X_m & X_m E_m & \cdots & X_m E_m^{m-1} \\ 1 & E_{\bar{1}} & \cdots & E_{\bar{1}}^m & 0 & 0 & \cdots & 0 \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ 1 & E_{\bar{m}} & \cdots & E_{\bar{m}}^m & 0 & 0 & \cdots & 0 \end{pmatrix}$$

so that:

$$D \simeq (-1)^m \prod_{i<j} |E_j - E_i|^2 \, X_1 \cdots X_m \qquad (97)$$

11 Appendix: Periodic flows.

There is a general procedure in the algebro-geometric setting to construct Hamiltonians generating periodic flows. From the equation of the spectral curve we have

$$\delta\mu \, d\lambda = \sum_i \frac{d\lambda}{\mu(\lambda - \epsilon_i)} \delta H_i = \sum_i \sigma_i(\lambda) d\lambda \delta H_i$$

where

$$\sigma_i(\lambda)d\lambda = \frac{d\lambda}{\mu(\lambda - \epsilon_i)} = \frac{\prod_{k \neq i}(\lambda - \epsilon_k)}{\sqrt{Q_{2n+2}(\lambda)}}d\lambda$$

form a basis of holomorphic differentials. Recalling the flows generated by H_i, eq.(75), we obtain the important relation

$$\sigma_j(\lambda_k) \sum_k \partial_{t_i}\lambda_k = i \sum_k \mu_k(\mathcal{B}^{-1})_{ik}\frac{\mathcal{B}_{kj}}{\mu_k} = i\delta_{ij}$$

To find the Hamiltonian generating the periodic flows, we should need the description of the periodic trajectory of the n coordinates λ_k of the divisor as n ovals on the spectral curve. However only the homotopy class of the trajectory will be important in the following discussion.

Let us assume for the time being that it is homotopic to n independent non intersecting ovals B_i. Let us define

$$\mathcal{H}^{(i)} = \oint_{B_i} \sqrt{\Lambda(\lambda)}d\lambda$$

To the B_i-cycles we associate a basis of normalized Abelian differentials $\omega_i(\lambda)d\lambda$. Of course we can expand it on the $\sigma_j(\lambda)d\lambda$ basis

$$\omega_i(\lambda)d\lambda = \sum_j \mathcal{N}_{ij} \sigma_j(\lambda)d\lambda$$

Equipped with these Abelian differentials, we define the angles through the Abel transformation

$$\theta_j = \sum_k \int^{\lambda_k} \omega_j(\lambda)d\lambda$$

Let us compute the equation of motion of these angles under the Hamiltonians $\mathcal{H}^{(i)}$.

$$
\begin{aligned}
\partial_{\tau_i}\theta_j &= \sum_k \omega_j(\lambda_k)\partial_{\tau_i}\lambda_k = \sum_{k,l} \omega_j(\lambda_k)\frac{\partial\mathcal{H}^{(i)}}{\partial H_l}\partial_{t_l}\lambda_k \\
&= \sum_{l,n} \frac{\partial\mathcal{H}^{(i)}}{\partial H_l}\mathcal{N}_{jn}\sum_k \sigma_n(\lambda_k)\partial_{t_l}\lambda_k \\
&= i\sum_{l,n} \frac{\partial\mathcal{H}^{(i)}}{\partial H_l}\mathcal{N}_{jn}\delta_{ln} = \oint_{B_i}\mathcal{N}_{jl}\frac{\partial}{\partial H_l}\sqrt{\Lambda}d\lambda \\
&= i\oint_{B_i}\mathcal{N}_{jl}\sigma_l(\lambda)d\lambda = i\oint_{B_i}\omega_j(\lambda)d\lambda = 2i\pi\delta_{ij}
\end{aligned}
$$

Hence

$$\partial_{\tau_i}\theta_j = 2i\pi\delta_{ij}$$

and the flows of the Hamiltonians $\mathcal{H}^{(i)}$ are 2π-periodic. So the problem of finding the Hamiltonians generating the periodic flows reduces to the description of the homotopy class of the n non intersecting ovals of the periodic trajectories.

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