K_1-injectivity for properly infinite C*-algebras

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K_1-INJECTIVITY FOR PROPERLY INFINITE C^*-ALGEBRAS

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Dedicated to Alain Connes on the occasion of his 60th birthday.

1. Introduction

One of the main tools to classify C*-algebras is the study of its projections and its unitaries. It was proved by J. Cuntz in [Cun81] that if A is a *purely infinite* simple C*-algebra, then the kernel of the natural map for the unitary group $\mathcal{U}(A)$ to the K-theory group $K_1(A)$ is reduced to the connected component $\mathcal{U}^0(A)$, *i.e.* A is K_1*-injective* (see §3). We study in this note a finitely generated C*-algebra, the K_1-injectivity of which would imply the K_1-injectivity of all unital *properly infinite* C*-algebras.

Note that such a question was already considered in [Blac07], [BRR08].

The author would like to thank H. Lin, R. Nest, M. Rørdam and W. Winter for helpful comments.

2. Preliminaries

Let us first review briefly the theory introduced by J. Cuntz ([Cun78]) of comparison of positive elements in a C*-algebra.

Definition 2.1. ([Cun78], [Rør92]) Given two positive elements a, b in a C*-algebra A, one says that:
– a is *dominated* by b (written $a \precsim b$) if and only if there is a sequence $\{d_k; k \in \mathbb{N}\}$ in A such that $\|d_k^* b d_k - a\| \to 0$ when $k \to \infty$,
– a is *properly infinite* if $a \neq 0$ and $a \oplus a \precsim a \oplus 0$ in the C*-algebra $M_2(A) := M_2(\mathbb{C}) \otimes A$.

This leads to the following notions of infiniteness for C*-algebras.

Definition 2.2. ([Cun78], [Cun81], [KR00]) A unital C*-algebra A is said to be:
– *properly infinite* if its unit 1_A is properly infinite in A,
– *purely infinite* if all the non zero positive elements in A are properly infinite in A.

Remark 2.3. E. Kirchberg and M. Rørdam have proved in [KR00, Theorem 4.16] that a C*-algebra A is purely infinite (in the above sense) if and only if there is no character on the C*-algebra A and any positive element a in A which lies in the closed two-sided ideal generated by another positive element b in A satisfies $a \precsim b$.

2010 *Mathematics Subject Classification.* Primary: 46L80; Secondary: 46L06, 46L35.
Key words and phrases. K_1-injectivity, Proper Infiniteness, C*-algebras.

The first examples of such C*-algebras were given by J. Cuntz in [Cun81]: For any integer $n \geq 2$, the C*-algebra $\mathcal{T}_n$ is the universal unital C*-algebra generated by n isometries $s_1, \ldots, s_n$ satisfying the relation

$$s_1 s_1^* + \ldots + s_n s_n^* \leq 1 \tag{2.1}$$

Then, the closed two sided ideal in $\mathcal{T}_n$ generated by the *minimal* projection $p_{n+1} := 1 - s_1 s_1^* - \ldots - s_n s_n^*$ is isomorphic to the C*-algebra $\mathcal{K}$ of compact operators on an infinite dimension separable Hilbert space and one has an exact sequence

$$0 \to \mathcal{K} \to \mathcal{T}_n \xrightarrow{\pi} \mathcal{O}_n \to 0\,, \tag{2.2}$$

where the quotient $\mathcal{O}_n$ is a purely infinite *simple* unital nuclear C*-algebra ([Cun81]).

Remark 2.4. A unital C*-algebra A is properly infinite if and only if there exists a unital $*$-homomorphism $\mathcal{T}_2 \to A$.

3. K_1-injectivity of $\mathcal{T}_n$

Given a unital C*-algebra A with unitary group $\mathcal{U}(A)$, denote by $\mathcal{U}^0(A)$ the connected component of 1_A in $\mathcal{U}(A)$. For each strictly positive integer $k \geq 1$, the upper diagonal embedding $u \in \mathcal{U}(M_k(A)) \mapsto (u \oplus 1_A) \in \mathcal{U}(M_{k+1}(A))$ sends the connected component $\mathcal{U}^0(M_k(A))$ into $\mathcal{U}^0(M_{k+1}(A))$, whence a canonical homomorphism Θ_A from $\mathcal{U}(A)/\mathcal{U}^0(A)$ to $K_1(A) := \lim\limits_{k\to\infty} \mathcal{U}(M_k(A))/\mathcal{U}^0(M_k(A))$. As noticed by B. Blackadar in [Blac07], this map is (1) neither injective, (2) nor surjective in general:

(1) If $\mathfrak{U}_2$ denotes the compact unitary group of the matrix C*-algebra $M_2(\mathbb{C})$, $A := C(\mathfrak{U}_2 \times \mathfrak{U}_2, M_2(\mathbb{C}))$ and $u, v \in \mathcal{U}(A)$ are the two unitaries $u(x,y) = x$ and $v(x,y) = y$, then $z := uvu^*v^*$ is not unitarily homotopic to 1_A in $\mathcal{U}(A)$ but the unitary $z \oplus 1_A$ belongs to $\mathcal{U}^0(M_2(A))$ ([AJT60]).
(2) If $A = C(\mathbb{T}^3)$, then $\mathcal{U}(A)/\mathcal{U}^0(A) \cong \mathbb{Z}^3$ but $K_1(A) \cong \mathbb{Z}^4$.

Definition 3.1. The unital C*-algebra A is said to be K_1-*injective* if the map Θ_A is injective.

J. Cuntz proved in [Cun81] that Θ_A is surjective as soon as the C*-algebra A is properly infinite and that it is also injective if the C*-algebra A is simple and purely infinite. Now, the K-theoretical six-term cyclic exact sequence associated to the exact sequence (2.2) implies that $K_1(\mathcal{T}_n) = 0$ since $K_1(\mathcal{K}) = K_1(\mathcal{O}_n) = 0$. Thus, the map $\Theta_{\mathcal{T}_n}$ is zero.

Proposition 3.2. *For all $n \geq 2$, the C*-algebra $\mathcal{T}_n$ is K_1-injective, i.e. any unitary $u \in \mathcal{U}(\mathcal{T}_n)$ is unitarily homotopic to $1_{\mathcal{T}_n}$ in $\mathcal{U}(\mathcal{T}_n)$ (written $u \sim_h 1_{\mathcal{T}_n}$).*

Proof. The C*-algebras $\mathcal{T}_n$ have real rank zero by Proposition 2.3 of [Zha90]. And Lin proved that any unital C*-algebra of real rank zero is K_1-injective ([Lin01, Corollary 4.2.10]). □

Corollary 3.3. *If $\alpha : \mathcal{T}_3 \to \mathcal{T}_3$ is a unital $*$-endomorphism, then its restriction to the unital copy of $\mathcal{T}_2$ generated by the two isometries s_1, s_2 is unitarily homotopic to $id_{\mathcal{T}_2}$ among all unital $*$-homomorphisms $\mathcal{T}_2 \to \mathcal{T}_3$.*

Proof. The isometry $\sum_{k=1,2} \alpha(s_k)s_k^*$ extends to a unitary $u \in \mathcal{U}(\mathcal{T}_3)$ such that $\alpha(s_k) = us_k$ for $k = 1, 2$ ([BRR08, Lemma 2.4]). But Proposition 3.2 yields that $\mathcal{U}(\mathcal{T}_3) = \mathcal{U}^0(\mathcal{T}_3)$, whence a homotopy $u \sim_h 1$ in $\mathcal{U}(\mathcal{T}_3)$, and so the desired result holds. □

Remark 3.4. The unital map $\iota : \mathbb{C} \to \mathcal{T}_2$ induces an isomorphism in K-theory. Indeed, $[1_{\mathcal{T}_2}] = [s_1 s_1^*] + [s_2 s_2^*] + [p_3] = 2\,[1_{\mathcal{T}_2}] + [p_3]$ and so $[1_{\mathcal{T}_2}] = -[p_3]$ is invertible in $K_0(\mathcal{T}_2)$.

4. K_1-injectivity of properly infinite C*-algebras

Denote by $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ the universal unital free product with amalgamation over $\mathbb{C}$ (in the sequel called full unital free product) of two copies of $\mathcal{T}_2$ amalgamated over $\mathbb{C}$ and let $\jmath_0$, $\jmath_1$ be the two canonical unital inclusions of $\mathcal{T}_2$ in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$. We show in this section that the K_1-injectivity of $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ is equivalent to the K_1-injectivity of all the unital properly infinite C*-algebras. The proof is similar to that of the universality of the full unital free product $\mathcal{O}_\infty *_{\mathbb{C}} \mathcal{O}_\infty$ (see Theorem 5.5 of [BRR08]).

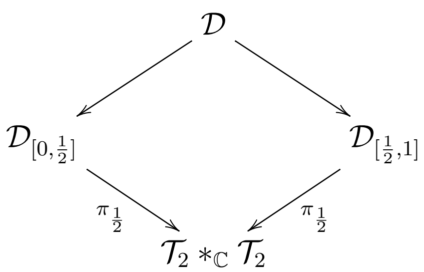
Definition 4.1. ([Blan09], [BRR08, §2]) If X is a compact Hausdorff space, a unital $C(X)$-algebra is a unital C*-algebra A endowed with a unital $*$-homomorphism from the C*-algebra $C(X)$ of continuous functions on X to the centre of A.

For any non-empty closed subset Y of X, we denote by π_Y^A (or simply by π_Y if no confusion is possible) the quotient map from A to the quotient A_Y of A by the (closed) ideal $C_0(X \setminus Y) \cdot A$. For any point $x \in X$, we also denote by A_x the quotient $A_{\{x\}}$ and by π_x the quotient map $\pi_{\{x\}}$.

Proposition 4.2. *The following assertions are equivalent.*

(i) $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ *is* K_1*-injective.*
(ii) $\mathcal{D} := \{ f \in C([0,1], \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)\,;\ f(0) \in \jmath_0(\mathcal{T}_2) \text{ and } f(1) \in \jmath_1(\mathcal{T}_2) \}$ *is properly infinite.*
(iii) *There exists a unital* $*$*-homomorphism* $\theta : \mathcal{T}_2 \to \mathcal{D}$.
(iv) *There exists a projection* $q \in \mathcal{D}$ *with* $\pi_0(q) = \jmath_0(s_1 s_1^*)$ *and* $\pi_1(q) = \jmath_1(s_1 s_1^*)$.
(v) *Any unital properly infinite C***-algebra* A *is* K_1*-injective.*

Proof. (i)⇒(ii) We have a pull-back diagram



and the two C*-algebras $\mathcal{D}_{[0,\frac{1}{2}]}$ and $\mathcal{D}_{[\frac{1}{2},1]}$ are properly infinite (Remark 2.4). Hence, the implication follows from [BRR08, Proposition 2.7].

(ii)⇒(iii) is Remark 2.4 applied to the C*-algebra $\mathcal{D}$.

(iii)⇒(iv) The two full, properly infinite projections $\jmath_0(s_1 s_1^*)$ and $\pi_0 \circ \theta(s_1 s_1^*)$ are unitarily equivalent in $\jmath_0(\mathcal{T}_2)$ by [LLR00, Lemma 2.2.2] and [BRR08, Proposition 2.3].

Thus, they are homotopic among the projections in the C*-algebra $\jmath_0(\mathcal{T}_2)$ (written $\jmath_0(s_1s_1^*) \sim_h \pi_0 \circ \theta(s_1s_1^*)$) by Proposition 3.2. Similarly, $\pi_1 \circ \theta(s_1s_1^*) \sim_h \jmath_1(s_1s_1^*)$ in $\jmath_1(\mathcal{T}_2)$. Further, $\pi_0 \circ \theta(s_1s_1^*) \sim_h \pi_1 \circ \theta(s_1s_1^*)$ in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ by hypothesis, whence the result by composition.

(iv)⇒(v) By [BRR08, Proposition 5.1], it is equivalent to prove that if p and p' are two properly infinite full projections in A, then there exist full properly infinite projections p_0, and p_0' in A such that $p_0 \leq p$, $p_0' \leq p'$ and $p_0 \sim_h p_0'$.

Fix two such projections p and p' in A. Then, there exist unital $*$-homomorphisms $\sigma : \mathcal{T}_2 \to pAp$, $\sigma' : \mathcal{T}_2 \to p'Ap'$ and isometries $t, t' \in A$ such that $1_A = t^*pt = (t')^*p't'$. Now, one thoroughly defines unital $*$-homomorphisms $\alpha_0 : \mathcal{T}_2 \to A$ and $\alpha_1 : \mathcal{T}_2 \to A$ by

$$\alpha_0(s_k) := \sigma(s_k) \cdot t \quad \text{and} \quad \alpha_1(s_k) := \sigma'(s_k) \cdot t' \qquad \text{for } k = 1, 2\,,$$

whence a unital $*$-homomorphism $\alpha := \alpha_0 * \alpha_1 : \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \to A$ such that $\alpha \circ \jmath_0 = \alpha_0$ and $\alpha \circ \jmath_1 = \alpha_1$.

The two full properly infinite projections $p_0 = \alpha_0(s_1s_1^*)$ and $p_0' = \alpha_1(s_1s_1^*)$ satisfy $p_0 \leq p$ and $p_0' \leq p'$. Further, the projection $(id \otimes \alpha)(q)$ gives a continuous path of projections in A from p_0 to p_0' . □

Remark 4.3. The C*-algebra $M_2(\mathcal{D})$ is properly infinite by [BRR08, Proposition 2.7].

Lemma 4.4. *$K_0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) = \mathbb{Z}$ and $K_1(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) = 0$*

Proof. The commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\imath_1} & \mathcal{T}_2 \\ {\scriptstyle \imath_0}\downarrow & & \downarrow{\scriptstyle \jmath_1} \\ \mathcal{T}_2 & \xrightarrow{\jmath_0} & \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \end{array}$$

yields by [Ger97, Theorem 2.2] a six-term cyclic exact sequence

$$\begin{array}{ccccc} K_0(\mathbb{C}) = \mathbb{Z} & \overset{(\imath_0 \oplus \imath_1)_*}{\longrightarrow} & K_0(\mathcal{T}_2 \oplus \mathcal{T}_2) = \ \mathbb{Z} \oplus \mathbb{Z} & \overset{(\jmath_0)_* - (\jmath_1)_*}{\longrightarrow} & K_0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) & \longleftarrow & K_1(\mathcal{T}_2 \oplus \mathcal{T}_2) = 0 \oplus 0 & \longleftarrow & K_1(\mathbb{C}) = 0 \end{array}$$

Now, Remark 3.4 implies that the map $(\imath_0 \oplus \imath_1)_*$ is injective, whence the equalities. □

Remark 4.5. G. Skandalis noticed that the C*-algebra $\mathcal{T}_2$ is KK-equivalent to $\mathbb{C}$ and so $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ is KK-equivalent to $\mathbb{C} *_{\mathbb{C}} \mathbb{C} = \mathbb{C}$.

This Lemma entails that the K_1-injectivity question for unital properly infinite C*-algebras boils down to knowing whether $\mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) = \mathcal{U}^0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$. Note that Proposition 3.2 already yields that $\mathcal{U}(\mathcal{T}_2) *_{\mathbb{T}} \mathcal{U}(\mathcal{T}_2) \subset \mathcal{U}^0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$.

But the following holds.

Proposition 4.6. *Set $p_3 = 1 - s_1s_1^* - s_2s_2^*$ in the Cuntz algebra $\mathcal{T}_2$ and let u be the canonical unitary generating $C^*(\mathbb{Z})$.*
(i) The relations $\jmath_0(s_k) \mapsto s_k$ and $\jmath_1(s_k) \mapsto u\, s_k$ ($k = 1, 2$) uniquely define a unital $*$-homomorphism $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \to \mathcal{T}_2 *_{\mathbb{C}} C^*(\mathbb{Z})$ which is injective but not K_1-surjective.

(ii) The two projections $\jmath_0(p_3)$ and $\jmath_1(p_3)$ satisfy $\jmath_1(p_3) \not\sim \jmath_0(p_3)$ in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$.
(iii) There is no $v \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ such that $\jmath_1(s_1s_1^* + s_2s_2^*) = v\,\jmath_0(s_1s_1^* + s_2s_2^*)\,v^*$.
(iv) There is a unitary $v \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ such that $\jmath_1(s_1s_1^*) = v\,\jmath_0(s_1s_1^*)\,v^*$.

Proof. (i) The unital C*-subalgebra of $\mathcal{O}_3$ generated by the two isometries s_1 and s_2 is isomorphic to $\mathcal{T}_2$, whence a unital C*-embedding $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \subset \mathcal{O}_3 *_{\mathbb{C}} \mathcal{O}_3$ ([ADEL04]). Let Φ be the $*$-homomorphism from $\mathcal{O}_3 *_{\mathbb{C}} \mathcal{O}_3$ to the free product $\mathcal{O}_3 *_{\mathbb{C}} C^*(\mathbb{Z}) = C^*\big(s_1, s_2, s_3, u\big)$ fixed by the relations

$$\Phi(\jmath_0(s_k)) = s_k \quad \text{and} \quad \Phi(\jmath_1(s_k)) = u\,s_k \quad \text{for } k = 1, 2, 3$$

and let $\Psi : \mathcal{O}_3 *_{\mathbb{C}} C^*(\mathbb{Z}) \to \mathcal{O}_3 *_{\mathbb{C}} \mathcal{O}_3$ be the only $*$-homomorphism such that

$$\Psi(u) = \sum_{l=1}^{3} \jmath_1(s_l)\jmath_0(s_l)^* \quad \text{and} \quad \Psi(s_k) = \jmath_0(s_k) \quad \text{for } k = 1, 2, 3.$$

For all $k = 1, 2, 3$, we have the equalities:
– $\Psi \circ \Phi(\jmath_0(s_k)) = \Psi(s_k) = \jmath_0(s_k)$,
– $\Psi \circ \Phi(\jmath_1(s_k)) = \Psi(us_k) = \jmath_1(s_k)$,
– $\Phi \circ \Psi(s_k) = \Phi(\jmath_0(s_k)) = s_k$.

Also, $\Psi(u)^*\Psi(u) = \sum_{l,l'} \jmath_0(s_{l'})\jmath_1(s_{l'})^*\jmath_1(s_l)\jmath_0(s_l)^* = 1_{\mathcal{O}_3 *_{\mathbb{C}} \mathcal{O}_3} = \Psi(u)\Psi(u)^*$, *i.e.* $\Psi(u)$ is a unitary in $\mathcal{O}_3 *_{\mathbb{C}} \mathcal{O}_3$ which satisfies:
– $\Phi \circ \Psi(u) = \sum_{l=1,2,3} \Phi(\jmath_1(s_l))\Phi(\jmath_0(s_l)^*) = u$.

Thus, Φ is an invertible unital $*$-homomorphism with inverse Ψ ([Blac07]), and the restriction of Φ to the C*-subalgebra $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ takes values in $\mathcal{T}_2 *_{\mathbb{C}} C^*(\mathbb{Z}) \subset \mathcal{O}_3 *_{\mathbb{C}} C^*(\mathbb{Z})$.

Now, there is (see [Ger97]) a six-term cyclic exact sequence

$$\begin{array}{ccccc} K_0(\mathbb{C}) = \mathbb{Z} & \hookrightarrow & K_0\big(\mathcal{T}_2 \oplus C^*(\mathbb{Z})\big) = \mathbb{Z} \oplus \mathbb{Z} & \to & K_0(\mathcal{T}_2 *_{\mathbb{C}} C^*(\mathbb{Z})\,) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{T}_2 *_{\mathbb{C}} C^*(\mathbb{Z})\,) & \leftarrow & K_1\big(\mathcal{T}_2 \oplus C^*(\mathbb{Z})\big) = 0 \oplus \mathbb{Z} & \leftarrow & K_1(\mathbb{C}) = 0 \end{array}$$

and so, $K_1(\mathcal{T}_2 *_{\mathbb{C}} C^*(\mathbb{Z})\,) = \mathbb{Z}$, whereas $K_1(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) = 0$ by Lemma 4.4.

(ii) Let $\pi_0 : \mathcal{T}_2 \to L(H)$ be a unital $*$-representation on a separable Hilbert space H such that $\pi_0(p_3)$ is a rank one projection, let $\pi_1 : \mathcal{T}_2 \to L(H)$ be a unital $*$-representation such that $\pi_1(p_3)$ is a rank two projection and consider the induced unital $*$-representation $\pi = \pi_0 * \pi_1$ of the unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$.

Then the two projections $\pi[\jmath_0(p_3)] = \pi_0(p_3)$ and $\pi[\jmath_1(p_3)] = \pi_1(p_3)$ have distinct ranks and so cannot be equivalent in $L(H)$. Hence, $\jmath_0(p_3) \not\sim \jmath_1(p_3)$ in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$.

(iii) This is just a rewriting of the previous assertion since $s_1s_1^* + s_2s_2^* = 1 - p_3$. Indeed, the partial isometry $b = \jmath_1(s_1)\jmath_0(s_1)^* + \jmath_1(s_2)\jmath_0(s_2)^*$ defines a Murray-von Neumann equivalence in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ between the projections $\jmath_0(s_1s_1^* + s_2s_2^*) = 1 - \jmath_0(p_3)$ and $\jmath_1(s_1s_1^* + s_2s_2^*) = 1 - \jmath_1(p_3)$. Thus, they are unitarily equivalent in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ if and only if $\jmath_0(p_3) \sim \jmath_1(p_3)$ in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ ([LLR00, Proposition 2.2.2]).

(iv) There exists a unitary $v \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ (which is necessarily K_1-trivial by Lemma 4.4) such that $\jmath_1(s_1s_1^*) = v\,\jmath_0(s_1s_1^*)\,v^*$. Indeed, we have the inequalities

$$1 > s_2s_2^* + p_3 > s_2s_2^* > s_2s_1(s_2s_2^* + p_3)s_1^*s_2^* + s_2s_2(s_2s_2^* + p_3)s_2^*s_2^* \qquad \text{in } \mathcal{T}_2\,.$$

Thus, if we set $w := \jmath_1(s_1)\jmath_0(s_1)^*$, then $1 - w^*w = \jmath_0(s_2s_2^* + p_3)$ and $1 - ww^* = \jmath_1(s_2s_2^* + p_3)$ are two properly infinite and full K_0-equivalent projections in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$. Thus, there is a partial isometry $a \in \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ with $a^*a = 1 - w^*w$ and $aa^* = 1 - ww^*$ ([Cun81]). The sum $v = a + w$ has the required properties ([BRR08, Lemma 2.4]). $\square$

Remarks 4.7. (i) The equivalence (iv)$\Leftrightarrow$(v) in Proposition 4.2 implies that all unital properly infinite C*-algebras are K_1-injective if and only if the unitary $v \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ constructed in Proposition 4.6.(iv) belongs to the connected component $\mathcal{U}^0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$.

Note that $v \oplus 1 \sim_h 1 \oplus 1$ in $\mathcal{U}(M_2(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2))$ by [LLR00, Exercice 8.11].

(ii) Let $\sigma \in \mathcal{U}(\mathcal{T}_2)$ be the symmetry $\sigma = s_1s_2^* + s_2s_1^* + p_3$, let $v \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ be a unitary such that $\jmath_1(s_1s_1^*) = v\jmath_0(s_1s_1^*)v^*$ (Proposition 4.6.(iv)) and set $z := v^*\jmath_1(\sigma)v\jmath_0(\sigma)$.

Then, $q_1 = \jmath_0(s_1s_1^*)$, $q_2 = \jmath_0(s_2s_2^*)$ and $q_3 = z\jmath_0(s_2s_2^*)z^*$ are three properly infinite full projections in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ which satisfy:

– $q_1q_3 = \jmath_0(s_1s_1^*)\,v^*\,\jmath_1(s_2s_2^*)\,v = v^*\,\jmath_1(s_1s_1^*)\jmath_1(s_2s_2^*)\,v = 0 = q_1q_2$,
– $q_2 \sim_h q_1 \sim_h q_3$ in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ since $\sigma \in \mathcal{U}^0(\mathcal{T}_2)$ and so $z \sim_h v^*v = 1$ in $\mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$,
– $q_1 + q_3 = v^*\jmath_1(s_1s_1^* + s_2s_2^*)v \not\sim_u \jmath_0(s_1s_1^* + s_2s_2^*) = q_1 + q_2$ in $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ by Proposition 4.6.(iii).

Addendum

(iii) Let $\alpha = \alpha_0 * \alpha_1$ be the unital $*$-endomorphism of the free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ defined by $\alpha_0(s_k) = \jmath_0(s_k)$ and $\alpha_1(s_k) = v^*\jmath_1(s_k)$ for $k = 1, 2$. Then $\alpha_0(s_2s_2^* + p_3) = 1 - \alpha_0(s_1s_1^*) = 1 - \alpha_1(s_1s_1^*) = \alpha_1(s_2s_2^* + p_3)$ and $\alpha_0(s_2s_2^*) \sim_h \alpha_0(s_1s_1^*) = \alpha_1(s_1s_1^*) \sim_h \alpha_1(s_2s_2^*)$ among the projections in $\alpha(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$. But $\alpha_0(p_3) \not\sim \alpha_1(p_3)$ in $\alpha(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$.

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