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DISSIPATIVE BOUNDARY CONDITIONS FOR ONE-DIMENSIONAL NONLINEAR HYPERBOLIC SYSTEMS*

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Abstract. We give a new sufficient condition on the boundary conditions for the exponential stability of one-dimensional nonlinear hyperbolic systems on a bounded interval. Our proof relies on the construction of an explicit strict Lyapunov function. We compare our sufficient condition with other known sufficient conditions for nonlinear and linear one-dimensional hyperbolic systems.

Key words. nonlinear hyperbolic systems, boundary conditions, stability, Lyapunov function

AMS subject classifications. 35F30, 35F25, 93D20, 93D30

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1. Introduction.

We are concerned with the following one-dimensional $n \times n$ nonlinear hyperbolic system:

$$u_t + F(u)u_x = 0, \quad x \in [0,1], \quad t \in [0,+\infty), \tag{1.1}$$

where $u : [0,\infty) \times [0,1] \to \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathcal{M}_{n,n}(\mathbb{R})$, $\mathcal{M}_{n,n}(\mathbb{R})$ denoting, as usual, the set of $n \times n$ real matrices. We consider the case where, possibly after an appropriate state transformation, $F(0)$ is a diagonal matrix with distinct and nonzero eigenvalues:

$$\begin{aligned} F(0) &:= \text{diag }(\Lambda_1, \Lambda_2, \dots, \Lambda_n), \\ \Lambda_i &> 0 \quad \forall i \in \{1, \dots, m\}, \\ \Lambda_i &< 0 \quad \forall i \in \{m+1, \dots, n\}, \end{aligned} \tag{1.2}$$

$$\Lambda_i \neq \Lambda_j \quad \forall (i,j) \in \{1, \dots, n\}^2 \text{ such that } i \neq j. \tag{1.3}$$

In (1) and in what follows, diag $(\Lambda_1, \Lambda_2, \dots, \Lambda_n)$ denotes the diagonal matrix whose ith element on the diagonal is Λ_i.

Our concern is to analyze the asymptotic behavior of the classical solutions of the system under the following boundary condition:

$$\begin{pmatrix} u_+(t,0) \\ u_-(t,1) \end{pmatrix} = G \begin{pmatrix} u_+(t,1) \\ u_-(t,0) \end{pmatrix}, \qquad t \in [0,+\infty), \tag{1.4}$$

where the map $G : \mathbb{R}^n \to \mathbb{R}^n$ vanishes at 0, while $u_+ \in \mathbb{R}^m$, $u_- \in \mathbb{R}^{n-m}$ are defined by requiring that $u := (u_+^{\text{tr}}, u_-^{\text{tr}})^{\text{tr}}$. The problem is to find the map G such that

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the boundary condition (1.4) is dissipative, i.e., implies that the equilibrium solution $u \equiv 0$ of system (1.1) with the boundary condition (1.4) is exponentially stable.

This problem has been considered in the literature for more than 20 years. To our knowledge, the first results were published by Slemrod in [21] and Greenberg and Li in [9] for the special case of 2×2 (i.e., $u \in \mathbb{R}^2$) systems. A generalization to $n \times n$ systems was given by the Li school. Let us mention in particular [17] by Qin, [25] by Zhao, and [14, Theorem 1.3, page 173] by Li. All these results rely on a systematic use of direct estimates of the solutions and their derivatives along the characteristic curves. They give rise to sufficient dissipative boundary conditions which are kinds of "small gain conditions." The weakest sufficient condition [14, Theorem 1.3, page 173] is formulated as follows: $\rho(|G'(0)|) < 1$, where $\rho(A)$ denotes the spectral radius of $A \in \mathcal{M}_{n,n}(\mathbb{R})$ and $|A|$ denotes the matrix whose elements are the absolute values of the elements of $A \in \mathcal{M}_{n,n}(\mathbb{R})$.

In this paper we follow a different approach, which is based on a Lyapunov stability analysis. The special case of 2×2 systems and $F(u)$ diagonal has recently been treated in our previous paper [6]. In the present paper, by using a more general strict Lyapunov function (see section 4), we get a new weaker dissipative boundary condition, stated as follows:

$$\text{Inf }\{\|\Delta G'(0)\Delta^{-1}\|;\, \Delta \in \mathcal{D}_{n,+}\} < 1,$$

where $\| \ \|$ denotes the usual 2-norm of matrices in $\mathcal{M}_{n,n}(\mathbb{R})$ and $\mathcal{D}_{n,+}$ denotes the set of diagonal matrices whose elements on the diagonal are strictly positive.

Moreover, our proof is rather elementary, and the existence of a strict Lyapunov function may be useful for studying robustness issues.

Our paper is organized as follows. In section 2, after some mathematical preliminaries, a precise technical definition of our new dissipative boundary condition is followed by the statement of our exponential stability theorem. Section 3 is then devoted to a discussion of the optimality properties of our dissipative boundary condition and to a comparison of this condition with other stability criteria from the literature, namely the criterion [14, Theorem 1.3, p, 173] mentioned above and a stability criterion for *linear* hyperbolic systems due to Silkowski. The proof of our exponential stability theorem, including the Lyapunov stability analysis, is thoroughly given in section 4. The paper ends with two appendices, where some technical properties of our dissipative boundary condition are given.

2. A sufficient condition for exponential stability. For

$$x := (x_1, \dots, x_n)^{\mathrm{tr}} \in \mathbb{C}^n,$$

$|x|$ denotes the usual Hermitian norm of x:

$$|x| := \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

For $n \in \mathbb{N} \setminus \{0\}$ and $m \in \mathbb{N} \setminus \{0\}$, we denote by $\mathcal{M}_{n,m}(\mathbb{R})$ the set of $n \times m$ real matrices. We define, for $K \in \mathcal{M}_{n,m}(\mathbb{R})$,

$$\|K\| := \max\{|Kx|; x \in \mathbb{R}^n,\, |x| = 1\} = \max\{|Kx|; x \in \mathbb{C}^n,\, |x| = 1\},$$

and, if $n = m$,

$$\rho_1(K) := \text{Inf }\{\|\Delta K \Delta^{-1}\|;\, \Delta \in \mathcal{D}_{n,+}\}, \tag{2.1}$$

where $\mathcal{D}_{n,+}$ denotes the set of $n \times n$ real diagonal matrices with strictly positive diagonal elements.

For ε, let B_ε be the open ball of $\mathbb{R}^n$ of radius ε. We assume that, for some $\varepsilon_0 > 0$, $F : B_{\varepsilon_0} \to \mathcal{M}_{n,n}(\mathbb{R})$ is of class C^2 and that there exists $m \in \{0, \dots, n\}$ and n real numbers $\Lambda_1, \dots, \Lambda_n$ such that

$$\Lambda_i > 0 \quad \forall i \in \{1, \dots, m\} \text{ and } \Lambda_i < 0 \quad \forall i \in \{m+1, \dots, n\}, \tag{2.2}$$

$$F(0) = \operatorname{diag}(\Lambda_1, \dots, \Lambda_n), \tag{2.3}$$

$$\Lambda_i \neq \Lambda_j \quad \forall (i,j) \in \{1, \dots, n\}^2 \text{ such that } i \neq j. \tag{2.4}$$

For $u \in \mathbb{R}^n$, $u_+ \in \mathbb{R}^m$ and $u_- \in \mathbb{R}^{n-m}$ are defined by requiring

$$u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}.$$

As mentioned in the introduction, we are mainly interested in analyzing the asymptotic convergence of the classical solutions of the following Cauchy problem:

$$u_t + F(u)u_x = 0, \quad x \in [0,1], \quad t \in [0, +\infty), \tag{2.5}$$

$$\begin{pmatrix} u_+(t,0) \\ u_-(t,1) \end{pmatrix} = G \begin{pmatrix} u_+(t,1) \\ u_-(t,0) \end{pmatrix}, \qquad t \in [0, +\infty), \tag{2.6}$$

$$u(0,x) = u^0(x), \qquad x \in [0,1]. \tag{2.7}$$

Concerning G, we assume that $G : B_{\varepsilon_0} \to \mathbb{R}^n$ is of class C^2 and satisfies $G(0) = 0$. We define $F_+(u) \in \mathcal{M}_{m,n}(\mathbb{R})$, $F_-(u) \in \mathcal{M}_{(n-m),n}(\mathbb{R})$, $G_+(u) \in \mathbb{R}^m$, and $G_-(u) \in \mathbb{R}^{n-m}$ by requiring

$$F(u) = \begin{pmatrix} F_+(u) \\ F_-(u) \end{pmatrix}, \qquad G(u) = \begin{pmatrix} G_+(u) \\ G_-(u) \end{pmatrix}.$$

Regarding the existence of the solutions to the Cauchy problem (2.5)–(2.7), we have the following proposition.

PROPOSITION 2.1. *There exists $\delta_0 > 0$ such that, for every $u^0 \in H^2((0,1), \mathbb{R}^n)$ satisfying*

$$|u^0|_{H^2((0,1),\mathbb{R}^n)} \leqslant \delta_0$$

and the compatibility conditions

$$\begin{pmatrix} u^0_+(0) \\ u^0_-(1) \end{pmatrix} = G \begin{pmatrix} u^0_+(1) \\ u^0_-(0) \end{pmatrix}, \tag{2.8}$$

$$\begin{aligned} F_+(u^0(0))u^0_x(0) = {} & \left[G'_{+u_+} \begin{pmatrix} u^0_+(1) \\ u^0_-(0) \end{pmatrix} \right] F_+(u^0(1))u^0_x(1) \\ & + \left[G'_{+u_-} \begin{pmatrix} u^0_+(1) \\ u^0_-(0) \end{pmatrix} \right] F_-(u^0(0))u^0_x(0), \end{aligned} \tag{2.9}$$

$$(2.10)\quad F_-(u^0(1))u^0_x(1) = \left[G'_{-u_+}\begin{pmatrix}u^0_+(1)\\ u^0_-(0)\end{pmatrix}\right]F_+(u^0(1))u^0_x(1) + \left[G'_{-u_-}\begin{pmatrix}u^0_+(1)\\ u^0_-(0)\end{pmatrix}\right]F_-(u^0(0))u^0_x(0),$$

the Cauchy problem (2.5)–(2.7) *has a unique maximal classical solution*

$$u \in C^0([0,T), H^2((0,1),\mathbb{R}^n))$$

with $T \in [0,+\infty]$. *Moreover, if*

$$|u(t,\cdot)|_{H^2((0,1),\mathbb{R}^n)} \leqslant \delta_0 \quad \forall t \in [0,T),$$

then $T = +\infty$.

For a proof of this proposition, see, for instance, [12] by Kato, [13, pp. 2–3] by Lax, [16, pp. 35–43] by Majda, or [20, pp. 106–114] by Serre. Actually [12, 13, 16, 20] deal with $\mathbb{R}$ instead of $[0,1]$, but the proofs given there can be adapted to treat this new case. See also [15, pp. 96–107] by Li and Yu for the well-posedness of the Cauchy problem (2.5)–(2.7) in the framework of functions u of class C^1. Let us briefly explain how to adapt these proofs in order to get, for example, the existence of a solution $u \in C^0([0,T], H^2((0,1),\mathbb{R}^n))$ to the Cauchy problem (2.5)–(2.7) if $m = n$ (just to simplify the notation), for $T \in (0,+\infty)$ given, and for every $u^0 \in H^2((0,1),\mathbb{R}^n)$ satisfying the compatibility conditions (2.8)–(2.9) (when $m = n$, condition (2.10) disappears) and such that $|u^0|_{H^2((0,1),\mathbb{R}^n)}$ is small enough (the smallness depending on T in general). We first deal with the case where

$$T \in (0, \min\{\Lambda_1^{-1},\ldots,\Lambda_n^{-1}\}).$$

The basic ingredient is the following fixed point method, which is related to the one given in [15, page 97] (see also the pioneering works [12] and [13, pp. 2–3], where the authors deal with $\mathbb{R}$ instead of $[0,1]$). For $R > 0$ and for $u^0 \in H^2((0,1),\mathbb{R}^n)$ satisfying the compatibility conditions (2.8)–(2.9), let $C_R(u^0)$ be the set of

$$u \in L^\infty((0,T), H^2((0,1),\mathbb{R}^n)) \cap W^{1,\infty}((0,T), H^1((0,1),\mathbb{R}^n)) \cap W^{2,\infty}((0,T), L^2((0,1),\mathbb{R}^n))$$

such that

$$|u|_{L^\infty((0,T),H^2((0,1),\mathbb{R}^n))} \leqslant R,$$

$$|u|_{W^{1,\infty}((0,T),H^1((0,1),\mathbb{R}^n))} \leqslant R,$$

$$|u|_{W^{2,\infty}((0,T),L^2((0,1),\mathbb{R}^n))} \leqslant R,$$

$$u(\cdot,1) \in H^2((0,T),\mathbb{R}^n) \text{ and } |u(\cdot,1)|_{H^2((0,T),\mathbb{R}^n)} \leqslant R^2,$$

$$u(0,\cdot) = u^0,$$

$$u_t(0,\cdot) = -F(u^0)u^0_x.$$

The set $C_R(u^0)$ is a closed subset of $L^\infty((0,T),L^2((0,1),\mathbb{R}^n))$ (at least if $|u^0|_{H^2((0,1),\mathbb{R}^n)}$ is small enough so that $|u^0|_{C^0([0,1],\mathbb{R}^n)} < \varepsilon_0$). Given $R>0$, the set $C_R(u^0)$ is not empty if $|u^0|_{H^2((0,1),\mathbb{R}^n)}$ is small enough. Let $\mathcal{F} : C_R(u^0) \to L^\infty((0,T),H^2((0,1),\mathbb{R}^n)) \cap W^{1,\infty}((0,T),H^1((0,1),\mathbb{R}^n)) \cap W^{2,\infty}((0,T),L^2((0,1),\mathbb{R}^n))$ be defined by $\mathcal{F}(\tilde{u}) = u$, where u is the solution of the linear hyperbolic Cauchy problem

$$u_t + F(\tilde{u})u_x = 0, \quad u(t,0) = G(\tilde{u}(t,1)), \quad t \in [0,T], \tag{2.11}$$

$$u(0,x) = u^0(x), \qquad x \in [0,1].$$

The set $C_R(u^0)$ is a closed subset of $L^\infty((0,T),L^2((0,1),\mathbb{R}^n))$ (at least if $|u^0|_{H^2((0,1),\mathbb{R}^n)}$ is small enough so that $|u^0|_{L^\infty((0,1),\mathbb{R}^n)} \leqslant \varepsilon_0/2$). Moreover, given $R>0$, $C_R(u^0)$ is not empty if $|u^0|_{H^2((0,1),\mathbb{R}^n)}$ is small enough. Using standard energy estimates and the finite speed of propagation inherent in (2.11), one gets the existence of $M>0$ and $R_0>0$ such that, for every $R\in(0,R_0]$, there exists $\delta>0$ such that, for every $u^0 \in H^2((0,1),\mathbb{R}^n)$ such that $|u^0|_{H^2((0,1),\mathbb{R}^n)} \leqslant \delta$ and satisfying the compatibility conditions (2.8)–(2.9),

$$\mathcal{F}(C_R(u^0)) \subset C_R(u^0) \tag{2.12}$$

and

$$|\mathcal{F}(\tilde{u}_2) - \mathcal{F}(\tilde{u}_1)|_{L^\infty((0,T),L^2((0,1),\mathbb{R}^n))} + M|\mathcal{F}(\tilde{u}_2)(\cdot,1) - \mathcal{F}(\tilde{u}_1)(\cdot,1)|_{L^2((0,1),\mathbb{R}^n)}$$

$$\leqslant \frac{1}{2}|\tilde{u}_2-\tilde{u}_1|_{L^\infty((0,T),L^2((0,1),\mathbb{R}^n))} + \frac{M}{2}|\tilde{u}_2(\cdot,1)-\tilde{u}_1(\cdot,1)|_{L^2((0,1),\mathbb{R}^n)} \quad \forall(\tilde{u}_1,\tilde{u}_2) \in C_R(u^0).$$

This allows us to prove that $\mathcal{F}$ has a fixed point $u \in C_R(u^0)$; i.e., there exists a solution $u \in C_R(u^0)$ to the Cauchy problem (2.5)–(2.7). In order to get the extra regularity property $u \in C^0([0,T],H^2((0,1),\mathbb{R}^n))$, one can adapt [16, pp. 44–46] by noticing that, when one uses usual energy estimates to get (2.12), one also gets, for $u := \mathcal{F}(\tilde{u})$ with $\tilde{u} \in C_R(u^0)$, the "hidden regularity" $u_{xx}(\cdot,1) \in L^2((0,T),\mathbb{R}^n)$ together with estimates on $|u_{xx}(\cdot,1)|_{L^2((0,T),\mathbb{R}^n)}$ which are sufficient to take care of the boundary terms which now appear when one does integrations by parts. The case of general $T \in (0+\infty)$ follows by applying the above result to $[0,T_1],[T_1,2T_1],[2T_1,3T_1],\ldots$, with T_1 given in $(0,\min\{\Lambda_1^{-1},\ldots,\Lambda_n^{-1}\})$. This concludes our sketch of the proof of Proposition 2.1.

We adopt the following definition of the exponential stability of the equilibrium solution $u \equiv 0$.

DEFINITION 2.2. *The equilibrium solution $u \equiv 0$ of the nonlinear hyperbolic system* (2.5)–(2.6) *is exponentially stable (for the H^2-norm) if there exist $\varepsilon>0$, $\nu>0$, and $C>0$ such that, for every $u^0 \in H^2((0,1),\mathbb{R}^n)$ satisfying $|u^0|_{H^2((0,1),\mathbb{R}^n)} \leqslant \varepsilon$ and the compatibility conditions* (2.8)–(2.10)*, the classical solution u to the Cauchy problem* (2.5)–(2.7) *is defined on* $[0,+\infty)$ *and satisfies*

$$|u(t,\cdot)|_{H^2((0,1),\mathbb{R}^n)} \leqslant Ce^{-\nu t}|u_0|_{H^2((0,1),\mathbb{R}^n)} \quad \forall t \in [0,+\infty). \tag{2.13}$$

Our main result is the following theorem.

THEOREM 2.3. *If $\rho_1(G'(0)) < 1$, then the equilibrium $u \equiv 0$ of the quasi-linear hyperbolic system* (2.5)–(2.6) *is exponentially stable.*

The proof of this theorem is given in section 4.

As mentioned in the introduction, the next section is devoted to a comparison of our dissipative boundary condition (i.e., $\rho_1(G'(0)) < 1$) with other stability criteria from the literature, namely the criterion given in [14, Theorem 1.3, page 173] and a stability criterion for *linear* hyperbolic systems.

3. Comparison with other stability conditions. In this section, we first compare our condition $\rho_1(G'(0)) < 1$ for exponential stability (Theorem 2.3) with a prior condition found by Li [14, Theorem 1.3, page 173]. In the second part of this section we shall compare our condition to conditions for the stability of *linear* hyperbolic systems.

3.1. Comparison with the Li condition. Let us first introduce some notation and definitions. For $K \in \mathcal{M}_{n,m}(\mathbb{R})$, we denote by K_{ij} the term on the ith line and jth column of the matrix K and denote by $|K|$ the matrix in $\mathcal{M}_{n,m}(\mathbb{R})$ defined by

$$|K|_{ij} := |K_{ij}| \quad \forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, m\}.$$

We define, for $K \in \mathcal{M}_{n,n}(\mathbb{R})$,

$$R_2(K) := \text{Max} \left\{ \sum_{j=1}^{n} |K_{ij}|; \, i \in \{1, \dots, n\} \right\},$$

$$\rho_2(K) := \text{Inf} \, \{R_2(\Delta K \Delta^{-1}); \, \Delta \in \mathcal{D}_{n,+}\}.$$

Note that, by [14, Lemma 2.4, page 146],

$$\rho_2(K) = \rho(|K|), \tag{3.1}$$

where, for $A \in \mathcal{M}_{n,n}(\mathbb{R})$, $\rho(A)$ is the spectral radius of A. In the following theorem, we recall the sufficient condition for exponential stability introduced by Li.

Theorem 3.1 (see [14, Theorem 1.3, page 173]). *Assume that* $\rho_2(G'(0)) < 1$*; then* $0 \in C^1([0,1], \mathbb{R}^n)$ *is locally exponentially stable in the* $C^1([0,1])$*-norm for the hyperbolic system* (2.5)–(2.6)*; i.e., there exist* $\varepsilon > 0$*,* $\nu > 0$*, and* $C > 0$ *such that, for every* $u^0 \in C^1([0,1], \mathbb{R}^n)$ *satisfying* $|u^0|_{C^1([0,1],\mathbb{R}^n)} \leqslant \varepsilon$ *and the compatibility conditions* (2.8)–(2.10)*, the Cauchy problem* (2.5)–(2.7) *has a unique solution* u *in* $C^1([0,+\infty) \times [0,1], \mathbb{R}^n)$*, and this solution satisfies*

$$|u(t,\cdot)|_{C^1([0,1],\mathbb{R}^n)} \leqslant Ce^{-\nu t} |u^0|_{C^1([0,1],\mathbb{R}^n)} \quad \forall t \in [0, +\infty).$$

The following proposition and (3.3) show that our new sufficient condition, namely $\rho_1(G'(0)) < 1$, is weaker than the previous one.

Proposition 3.2. *For every* $K \in \mathcal{M}_{n,n}(\mathbb{R})$*,*

$$\rho_1(K) \leqslant \rho_2(K). \tag{3.2}$$

Let us point out that there are matrices K such that inequality (3.2) is strict. For example, for $a > 0$, let

$$K_a := \begin{pmatrix} a & a \\ -a & a \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R}).$$

Then

$$\rho_1(K_a) = \sqrt{2}a < 2a = \rho_2(K_a). \tag{3.3}$$

Remark 3.3. In fact, in [14, Theorem 1.3, page 173], it is assumed that G_+ depends only on u_- and that G_- depends only on u_+. However, if one takes

$$K := \begin{pmatrix} 0 & K_a \\ K_a & 0 \end{pmatrix} \in \mathcal{M}_{4,4}(\mathbb{R}),$$

$n = 4$, $m = 2$, and $G(u) := Ku$, which are allowed by the type of boundary conditions considered in [14, Theorem 1.3, page 173], one again gets $\rho_1(K) = \sqrt{2}a < 2a = \rho_2(K)$.

Proof of Proposition 3.2. Let us first prove the following lemma.

LEMMA 3.4. *For every $K \in \mathcal{M}_{n,n}(\mathbb{R})$, for every $D \in \mathcal{D}_{n,+}$, for every $\Delta \in \mathcal{D}_{n,+}$, for every $X \in \mathbb{R}^n$, and for every $Y \in \mathbb{R}^n$,*

$$Y^{\mathrm{tr}}\Delta K\Delta^{-1}X \leqslant \frac{1}{2}R_2(D\Delta^{-1}K^{\mathrm{tr}}\Delta D^{-1})|X|^2 + \frac{1}{2}R_2(D\Delta K\Delta^{-1}D^{-1})|Y|^2. \tag{3.4}$$

Proof of Lemma 3.4. Replacing, if necessary, K by $\Delta K\Delta^{-1}$, we may assume without loss of generality that Δ is the identity map of $\mathbb{R}^n$. We write $X := (X_1, \dots, X_n)^{\mathrm{tr}} \in \mathbb{R}^n$, $Y := (Y_1, \dots, Y_n)^{\mathrm{tr}} \in \mathbb{R}^n$, $D := \mathrm{diag}\,(D_1, \dots, D_n)$. One has

$$Y^{\mathrm{tr}}KX = \sum_{i=1}^{n} Y_i \left(\sum_{j=1}^{n} K_{ij}X_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{K_{ij}}{D_iD_j} D_iY_iD_jX_j$$

$$\leqslant \frac{1}{2}Q_1 + \frac{1}{2}Q_2, \tag{3.5}$$

with

$$Q_1 := \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{|K_{ij}|}{D_iD_j} D_j^2X_j^2 \text{ and } Q_2 := \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{|K_{ij}|}{D_iD_j} D_i^2Y_i^2.$$

Note that

$$Q_1 = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} |K_{ij}|D_i^{-1}D_j\right)X_j^2 = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} |(D^{-1}KD)^{\mathrm{tr}})_{ji}|\right)X_j^2$$

$$\leqslant \sum_{j=1}^{n} R_2((D^{-1}KD)^{\mathrm{tr}})X_j^2 = R_2(DK^{\mathrm{tr}}D^{-1})|X|^2. \tag{3.6}$$

Similarly,

$$Q_2 = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} D_i|K_{ij}|D_j^{-1}\right)Y_i^2 = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} |(DKD^{-1})_{ij}|\right)Y_i^2$$

$$\leqslant \sum_{i=1}^{n} R_2(DKD^{-1})Y_i^2 = R_2(DKD^{-1})|Y|^2. \tag{3.7}$$

Inequality (3.4) follows from (3.5), (3.6), and (3.7). This concludes the proof of Lemma 3.4. □

Let us go back to the proof of Proposition 3.2. One easily sees that

$$\{(D\Delta^{-1}, D\Delta);\ D \in \mathcal{D}_{n,+},\ \Delta \in \mathcal{D}_{n,+}\} = \mathcal{D}_{n,+} \times \mathcal{D}_{n,+}. \tag{3.8}$$

Equality (3.8) implies that

$$\rho_2(K^{\mathrm{tr}}) + \rho_2(K) = \mathrm{Inf}\,\{R_2(D\Delta^{-1}K^{\mathrm{tr}}\Delta D^{-1}) + R_2(D\Delta K\Delta^{-1}D^{-1});\ D \in \mathcal{D}_{n,+},\ \Delta \in \mathcal{D}_{n,+}\}. \tag{3.9}$$

Using (3.1), we have

$$\rho_2(K^{\text{tr}}) = \rho(|K^{\text{tr}}|) = \rho(|K|^{\text{tr}}) = \rho(|K|) = \rho_2(K), \tag{3.10}$$

which, together with (3.9), gives

$$\begin{aligned}\operatorname{Inf}\{R_2(D\Delta^{-1}|K|^{\text{tr}}\Delta D^{-1}) + R_2(D\Delta|K|\Delta^{-1}D^{-1});\ D \in \mathcal{D}_{n,+},\ \Delta \in \mathcal{D}_{n,+}\}\\ = 2\rho_2(K).\end{aligned} \tag{3.11}$$

Finally, let us note that, for every Δ in $\mathcal{D}_{n,+}$,

$$\operatorname{Sup}\{Y^{\text{tr}}\Delta K\Delta^{-1}X;\ X \in \mathbb{R}^n,\ Y \in \mathbb{R}^n, |X| = |Y| = 1\} = \|\Delta K\Delta^{-1}\| \geqslant \rho_1(K). \tag{3.12}$$

Proposition 3.2 follows from (3.4), (3.11), and (3.12). ◻

3.2. Comparison with stability conditions for linear hyperbolic systems. Replacing, if necessary, $y(t,x)$ by

$$\begin{pmatrix} y_+(t,x) \\ y_-(t,1-x) \end{pmatrix},$$

it may be assumed, without loss of generality, that the speeds of propagation Λ_i are all positive. More precisely we consider the special case of linear hyperbolic systems

$$y_t + \Lambda y_x = 0, \qquad y(t,0) = Ky(t,1), \tag{3.13}$$

where

$$\Lambda := \operatorname{diag}(\Lambda_1, \ldots, \Lambda_n), \qquad \text{with } \Lambda_i > 0 \quad \forall i \in \{1, \ldots, n\}. \tag{3.14}$$

In order to avoid compatibility conditions, one can deal with the case where $y(t,\cdot) \in L^2((0,1),\mathbb{R}^n)$ (instead of $y(t,\cdot) \in H^2((0,1),\mathbb{R}^n)$, as we consider above for the nonlinear hyperbolic system (2.5)–(2.6)). It is well known that the Cauchy problem associated with (3.13) is well posed in $L^2((0,1),\mathbb{R}^n)$; that is, for every $y^0 \in L^2((0,1),\mathbb{R}^n)$, there exists a unique

$$y \in C^0([0,+\infty), L^2((0,1),\mathbb{R}^n))$$

solution of (3.13) satisfying the initial condition

$$y(0,\cdot) = y^0. \tag{3.15}$$

Of course, (3.13) has to be understood in the classical weak sense; i.e., for every $\varphi \in C^1([0,+\infty) \times [0,1]; \mathbb{R}^n)$ with compact support and satisfying

$$\varphi^{\text{tr}}(t,1)\Lambda - \varphi_+^{\text{tr}}(t,0)\Lambda K = 0 \quad \forall t \in [0,+\infty),$$

we have

$$\int_0^{+\infty}\int_0^1 (\varphi_t^{\text{tr}} + \varphi_x^{\text{tr}}\Lambda) y dx dt + \int_0^1 \varphi^{\text{tr}}(0,x) y^0(x) dx = 0.$$

See, for example, [5, section 2.1].

As usual, we say that $0 \in L^2((0,1),\mathbb{R}^n)$ is exponentially stable for (3.13) (for the norm of $L^2((0,1),\mathbb{R}^n)$) if there exist $\nu > 0$ and $C > 0$ such that, for every $y^0 \in L^2((0,1),\mathbb{R}^n)$, the solution of the Cauchy problem (3.13), (3.15) satisfies

$$|y(t,\cdot)|_{L^2((0,1),\mathbb{R}^n)} \leqslant Ce^{-\nu t}|y^0|_{L^2((0,1),\mathbb{R}^n)} \quad \forall t \in [0,+\infty).$$

One easily checks that (3.13) is equivalent to

$$\phi_i(t) = \sum_{j=1}^{n} K_{ij}\phi_j(t-r_j) \quad \forall i \in \{1,\dots,n\}, \tag{3.16}$$

with

$$\phi_j(t) := y_j(t,0), \quad r_j := \frac{1}{\Lambda_j}, \quad j \in \{1,\dots,n\}.$$

Hence (3.13) can be considered as a linear time-delay system. By a classical result on linear time-delay systems (see, e.g., [10, Theorem 3.5 page 275] by Hale and Verduyn Lunel), $0 \in L^2((0,1),\mathbb{R}^n)$ is exponentially stable for the system (3.13) if and only if there exists $\delta > 0$ such that

$$\Big(\det\,(\mathrm{Id}_n - (\mathrm{diag}\,(e^{-r_1 z},\dots,e^{-r_n z}))K) = 0,\ z \in \mathbb{C}\Big) \Rightarrow (\Re(z) \leqslant -\delta), \tag{3.17}$$

where Id_n is the identity map of $\mathbb{R}^n$ and $\Re(z)$ denotes the real part of the complex number z. Note that $\rho_1(K) < 1$ implies the existence of $\delta > 0$ such that (3.17) holds. Indeed, let us assume that $\rho_1(K) < 1$. Then, by (2.1), there exist $\mu \in (0,1)$ and $D \in \mathcal{D}_{n,+}$ such that

$$\|DKD^{-1}\| \leqslant \mu. \tag{3.18}$$

Let us assume that $z \in \mathbb{C}$ is such that

$$\det\,\big(\mathrm{Id}_n - (\mathrm{diag}\,(e^{-r_1 z},\dots,e^{-r_n z}))K\big) = 0.$$

Then

$$\begin{aligned}\det\,&\big(\mathrm{Id}_n - (\mathrm{diag}\,(e^{-r_1 z},\dots,e^{-r_n z}))DKD^{-1}\big)\\ &= \det\,\big(D(\mathrm{Id}_n - (\mathrm{diag}\,(e^{-r_1 z},\dots,e^{-r_n z}))K)D^{-1}\big)\\ &\qquad = \det\,\big(\mathrm{Id}_n - (\mathrm{diag}\,(e^{-r_1 z},\dots,e^{-r_n z}))K\big) = 0,\end{aligned}$$

which implies that

$$\|(\mathrm{diag}\,(e^{-r_1 z},\dots,e^{-r_n z}))DKD^{-1})\| \geqslant 1. \tag{3.19}$$

Since

$$\begin{aligned}\|(\mathrm{diag}\,(e^{-r_1 z},\dots,e^{-r_n z}))DKD^{-1})\| &\leqslant \|\mathrm{diag}\,(e^{-r_1 z},\dots,e^{-r_n z})\|\|DKD^{-1}\|\\ &\leqslant e^{-\min\{r_1\Re(z),\dots,r_n\Re(z)\}}\|DKD^{-1}\|,\end{aligned}$$

one has, also using (3.18) and (3.19),

$$e^{-\min\{r_1\Re(z),\dots,r_n\Re(z)\}}\mu \geqslant 1. \tag{3.20}$$

Inequality (3.20) implies that (3.17) holds with $\delta := \ln(\mu)/(\max\{r_1,\dots,r_n\}) < 0$.

The converse is false: the existence of $\delta > 0$ such that (3.17) holds does not imply that $\rho_1(K) < 1$. For example, let us choose $r_1 := 1$, $r_2 := 2$, and

$$K := \begin{pmatrix} a & a \\ a & a \end{pmatrix}, \qquad a \in \mathbb{R}.$$

(This example is borrowed from [10, page 285].) It is easily seen that $\rho_1(K) = 2|a|$. Hence $\rho_1(K) < 1$ is equivalent to $a \in (-1/2, 1/2)$. However, the existence of $\delta > 0$ such that (3.17) holds is equivalent to $a \in (-1, 1/2)$.

If we want to try to apply results on the stability of the linear hyperbolic system (3.13) in order to get the stability of our nonlinear hyperbolic system (2.5)–(2.6), since $F(u)$ depends on u, it is natural to ask for the robustness of the stability of the linear hyperbolic system (3.13) with respect to small changes on the Λ_i's, i.e., on the speeds of propagation. (One can easily sees that the stability is robust with respect to small changes on K.) Let us adopt the following definition.

Definition 3.5. *The linear system* (3.13) *is robustly exponentially stable with respect to the speeds of propagation if there exists $\varepsilon > 0$ such that, for every $\tilde{\Lambda} :=$ diag $(\tilde{\Lambda}_1,\dots,\tilde{\Lambda}_n) \in \mathcal{D}_{n,+}$ such that*

$$|\tilde{\Lambda}_i - \Lambda_i| \leqslant \varepsilon \quad \forall i \in \{1,\dots,n\},$$

$0 \in L^2((0,1),\mathbb{R}^n)$ *is exponentially stable for the perturbed linear hyperbolic system*

$$y_t + \tilde{\Lambda} y_x = 0, \qquad y(t,0) = Ky(t,1).$$

One has, then, the following theorem, which is due to Silkowski (see [10, Theorem 6.1, page 286]; see also [26, 11]).

Theorem 3.6. *Let*

$$\rho_0(K) := \max\{\rho(\text{diag }(e^{\iota\theta_1},\dots,e^{\iota\theta_n})K);\ (\theta_1,\dots,\theta_n)^{\text{tr}} \in \mathbb{R}^n\}, \tag{3.21}$$

with $\iota := \sqrt{-1}$. If the $(r_1,\dots,r_n)$ are rationally independent, the linear system (3.13) *is exponentially stable if and only if $\rho_0(K) < 1$. In particular (note that $\rho_0(K)$ depends continuously on K), whatever $(r_1,\dots,r_n) \in (0,+\infty)^n$ is, the linear system* (3.13) *is robustly exponentially stable with respect to the speeds of propagation if and only if $\rho_0(K) < 1$.*

From this theorem the interest of comparing $\rho_0(K)$ and $\rho_1(K)$ is clear. This is done in the following proposition.

Proposition 3.7. *For every $n \in \mathbb{N}$ and for every $K \in \mathcal{M}_{n,n}(\mathbb{R})$,*

$$\rho_0(K) \leqslant \rho_1(K). \tag{3.22}$$

For every $n \in \{1,2,3,4,5\}$ and for every $K \in \mathcal{M}_{n,n}(\mathbb{R})$,

$$\rho_0(K) = \rho_1(K). \tag{3.23}$$

For every $n \in \mathbb{N} \setminus \{1,2,3,4,5\}$, there exists $K \in \mathcal{M}_{n,n}(\mathbb{R})$ such that

$$\rho_0(K) < \rho_1(K). \tag{3.24}$$

The proof of Proposition 3.7 is given in Appendix B.

4. Proof of Theorem 2.3. For the clarity of the analysis, we first deal in detail with the case where $m = n$ and then give only the main modifications to deal with the case $m < n$. When $m = n$ the boundary condition (2.6) reads

$$u(t, 0) = G(u(t, 1)), \qquad t \in [0, +\infty), \tag{4.1}$$

and the compatibility conditions (2.8)–(2.10) become

$$u^0(0) = G(u^0(1)), \tag{4.2}$$

$$F(u^0(0))u^0_x(0) = G'(u^0(1))F(u^0(1))u^0_x(1). \tag{4.3}$$

Let us introduce some simplifying notation,

$$\Lambda := F(0) \in \mathcal{D}_{n,+}, \quad K := G'(0), \quad v := u_x, \quad w := v_x = u_{xx}, \tag{4.4}$$

and let us denote by $\mathcal{S}_n$ the set of $n \times n$ real symmetric matrices and by $\mathcal{S}_{n,+}$ the set of $n \times n$ real symmetric positive definite matrices.

We shall repeatedly use the following lemma.

LEMMA 4.1. *Let $\Lambda := \operatorname{diag}(\Lambda_1, \dots, \Lambda_n) \in \mathcal{D}_n$ be such that* (2.4) *holds. Let $\Delta \in \mathcal{D}_n$. Then there exist a positive real number η and a map $N : \{M \in \mathcal{M}_{n,n}(\mathbb{R}); \|M - \Lambda\| < \eta\} \to \mathcal{S}_n$ of class C^∞ such that*

$$N(\Lambda) = \Delta,$$

$$N(M)M - M^{\mathrm{tr}}N(M) = 0 \quad \forall M \in \mathcal{M}_{n,n}(\mathbb{R}) \textit{ such that } \|M - \Lambda\| < \eta.$$

Proof of Lemma 4.1. Let $\mathcal{A}_n$ be the set of matrices $A \in \mathcal{M}_{n,n}(\mathbb{R})$ such that $A^{\mathrm{tr}} = -A$. For $M \in \mathcal{M}_{n,n}(\mathbb{R})$, let us consider the following linear map:

$$\begin{array}{rccc} \mathcal{L}_M : & \mathcal{S}_n & \to & \mathcal{A}_n \times \mathcal{D}_n, \\ & S & \mapsto & (SM - M^{\mathrm{tr}}S, \operatorname{Diag}(S)), \end{array}$$

where $\operatorname{Diag}(S) := \operatorname{diag}(S_{11}, \dots, S_{nn})$. Noticing that

$$S\Lambda - \Lambda^{\mathrm{tr}}S = (\Lambda_j - \Lambda_i)S_{ij} \quad \forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, n\}, \forall S \in \mathcal{S}_n$$

and using (2.4), it is easily checked that $\mathcal{L}_\Lambda : \mathcal{S}_n \to \mathcal{A}_n \times \mathcal{D}_n$ is an isomorphism. Hence there exists $\eta > 0$ such that, for every $M \in \mathcal{M}_{n,n}(\mathbb{R})$ such that $\|M - \Lambda\| < \eta$, $\mathcal{L}_M$ is an isomorphism. It then suffices to define N by

$$N(M) = \mathcal{L}_M^{-1}(0, \Delta).$$

This conclude the proof of Lemma 4.1. □

For the stability analysis, we now introduce the Lyapunov function candidate

$$V(u, v, w) = V_1(u) + V_2(u, v) + V_3(u, v, w), \tag{4.5}$$

with

$$V_1(u) = \int_0^1 u^{\mathrm{tr}}Q(u)u \, e^{-\mu x} dx, \tag{4.6}$$

$$V_2(u, v) = \int_0^1 v^{\mathrm{tr}}R(u)v \, e^{-\mu x} dx, \tag{4.7}$$

$$V_3(u, v, w) = \int_0^1 w^{\mathrm{tr}}S(u)w \, e^{-\mu x} dx, \tag{4.8}$$

where $\mu > 0$, $Q(u)$, $R(u)$, and $S(u)$ are symmetric positive definite matrices which will be defined later.

Remark 4.2. The weight $e^{-\mu x}$ is essential to get a strict Lyapunov function. It is similar to the one introduced in [4] to stabilize the Euler equation of incompressible fluids (see the definition of V given on page 1886 of [4]). It has also been used by Xu and Sallet in [24] for quite general linear hyperbolic systems (see also [22] by Tchousso, Besson, and Xu).

Let us compute the time derivative $\dot{V}_1$ of V_1 along the classical C^1-solutions of system (2.5) with boundary conditions (4.1). One has

$$\dot{V}_1 = \int_0^1 \Big\{ 2u^{\rm tr} Q(u) u_t + u^{\rm tr} \big(Q(u)\big)_t u \Big\} e^{-\mu x} dx$$
$$= \int_0^1 \Big\{ -2u^{\rm tr} Q(u) F(u) u_x + u^{\rm tr} [Q'(u) u_t] u \Big\} e^{-\mu x} dx,$$

where $Q'(u)$ is the linear map from $\mathbb{R}^n$ to $\mathcal{S}_n$ which stands for the derivative of Q at the point u. Hence

(4.9)
$$\dot{V}_1 = \int_0^1 \Big\{ -\big(u^{\rm tr} Q(u) F(u) u\big)_x + u^{\rm tr} \big(Q(u)F(u)\big)_x u - u^{\rm tr} \big[Q'(u) F(u) v\big] u \Big\} e^{-\mu x} dx.$$

For $f \in C^0([0,1], \mathbb{R}^n)$, we denote by $|f|_0$ its C^0-norm: $|f|_0 := \max\{|f(x)|;\, x \in [0,1]\}$. From now on, V_1 and $\dot{V}_1$ are considered as functionals defined, respectively, by (4.6) and (4.9) on the set $\mathcal{V}_1$ of $u \in C^1([0,1], \mathbb{R}^n)$ satisfying $|u|_0 < \varepsilon_0$ and the compatibility condition

$$u_0 = G(u_1), \tag{4.10}$$

with $u_0 := u(0)$ and $u_1 := u(1)$.

Since $\rho_1(K) < 1$ by assumption, there exists $D \in \mathcal{D}_{n,+}$ such that $\|DKD^{-1}\| < 1$. The matrix $Q(u)$ is selected as the matrix $N(F(u))$ of Lemma 4.1 with $\Delta := D^2 \Lambda^{-1}$.

Our estimates on V_1 and $\dot{V}_1$ are in the following lemma.

LEMMA 4.3. *There exists $\mu_1 > 0$ such that, for every $\mu \in (0, \mu_1)$, there exist positive real constants $\alpha_1, \beta_1, \delta_1$ such that, for every $u \in \mathcal{V}_1$ such that $|u|_0 \leqslant \delta_1$,*

$$\frac{1}{\beta_1} \int_0^1 |u|^2 dx \leqslant V_1(u) \leqslant \beta_1 \int_0^1 |u|^2 dx, \tag{4.11}$$

$$\dot{V}_1(u) \leqslant -\alpha_1 V_1(u) + \beta_1 \int_0^1 |u|^2 |u_x| dx. \tag{4.12}$$

Proof of Lemma 4.3. Throughout this proof, u is assumed to be in $\mathcal{V}_1$. From the construction of Q,

$$Q(0)F(0) = Q(0)\Lambda = D^2 \in \mathcal{D}_{n,+}, \tag{4.13}$$

and there exists $\delta_{11} \in (0, \varepsilon_0/2)$ such that

$$Q(a) \in \mathcal{S}_{n,+} \text{ and } Q(a)F(a) \in \mathcal{S}_{n,+} \quad \forall a \in \mathbb{R}^n \text{ such that } |a| \leqslant \delta_{11}. \tag{4.14}$$

Clearly, from (4.14), we obtain that, for every $\mu > 0$, there exists $\beta_1 > 0$ such that (4.11) holds if $|u|_0 \leqslant \delta_{11}$.

Let us now deal with the estimate (4.12) on $\dot{V}_1 (= \dot{V}_1(u))$. Let us decompose $\dot{V}_1$ in the following way:

$$\dot{V}_1 = \mathcal{T}_{11} + \mathcal{T}_{12} + \mathcal{T}_{13}, \tag{4.15}$$

with

$$\mathcal{T}_{11} := -\mu \int_0^1 \Big(u^{\text{tr}} Q(u) F(u) u\Big) e^{-\mu x} dx, \tag{4.16}$$

$$\mathcal{T}_{12} := -\int_0^1 \left(u^{\text{tr}} Q(u) F(u) u e^{-\mu x}\right)_x dx, \tag{4.17}$$

$$\mathcal{T}_{13} := \int_0^1 \Big\{ u^{\text{tr}} \Big([Q'(u)v] F(u) + Q(u)[F'(u)v] - [Q'(u)F(u)v] \Big) u \Big\} e^{-\mu x} dx. \tag{4.18}$$

Analysis of the first term $\mathcal{T}_{11}$. By (4.11) and (4.14), for every $\mu > 0$, there exists a positive real constant $\alpha_1 > 0$ such that, if $|u|_0 \leqslant \delta_{11}$,

$$\mathcal{T}_{11} \leqslant -\alpha_1 V_1. \tag{4.19}$$

Analysis of the second term $\mathcal{T}_{12}$. One has

$$\begin{aligned} \mathcal{T}_{12} &= -\Big[u^{\text{tr}} Q(u) F(u) u e^{-\mu x}\Big]_0^1 \\ &= -\Big(u_1^{\text{tr}} Q(u_1) F(u_1) u_1 e^{-\mu} - u_0^{\text{tr}} Q(u_0) F(u_0) u_0\Big). \end{aligned}$$

Let us introduce a notation in order to deal with estimates on "higher order terms." We denote by $\mathcal{O}(X, Y)$, with $X \geqslant 0$ and $Y \geqslant 0$, quantities such that there exist $C > 0$ and $\varepsilon > 0$, independent of u, v and w, satisfying

$$(Y \leqslant \varepsilon) \Rightarrow (|\mathcal{O}(X, Y)| \leqslant CX).$$

Using the compatibility condition (4.10), we have

$$\begin{aligned} \mathcal{T}_{12} &= -\Big(u_1^{\text{tr}} Q(u_1) F(u_1) u_1 e^{-\mu} - (G(u_1))^{\text{tr}} Q\big(G(u_1)\big) F\big(G(u_1)\big) G(u_1)\Big) \\ &= -u_1^{\text{tr}} \Big(Q(0)\Lambda e^{-\mu} - K^{\text{tr}} Q(0) \Lambda K\Big) u_1 + \mathcal{O}(|u_1|^3; |u_1|). \end{aligned} \tag{4.20}$$

For $u_1 \in \mathbb{R}^n$, we define $\zeta := D u_1$. Then, using (4.13), we have, for every $u_1 \in \mathbb{R}^n$,

$$u_1^{\text{tr}} K^{\text{tr}} Q(0) \Lambda K u_1 = u_1^{\text{tr}} K^{\text{tr}} D D K u_1 = \left(\zeta^{\text{tr}} D^{-1} K^{\text{tr}} D\right) \left(D K D^{-1} \zeta\right) = |D K D^{-1} \zeta|^2.$$

Hence, using (4.13) once again, we have, for every $u_1 \in \mathbb{R}^n$,

$$u_1^{\text{tr}} K^{\text{tr}} Q(0) \Lambda K u_1 \leqslant \|D K D^{-1}\|^2 \zeta^{\text{tr}} \zeta = \|D K D^{-1}\|^2 u_1^{\text{tr}} Q(0) \Lambda u_1. \tag{4.21}$$

From this inequality and the fact that $\|DKD^{-1}\| < 1$, it follows that, taking $\mu > 0$ small enough (which is always implicitly assumed), $Q(0)\Lambda e^{-\mu} - K^{\text{tr}} Q(0) \Lambda K$ is a positive definite matrix. Then, using (4.20), there exists $\delta_{12} > 0$ such that, if $|u_1| \leqslant \delta_{12}$,

$$\mathcal{T}_{12} \leqslant 0. \tag{4.22}$$

Analysis of the third term $\mathcal{T}_{13}$. The integrand of $\mathcal{T}_{13}$ is linear with respect to v and, at least, quadratic with respect to u. It follows that, increasing the value of β_1 if necessary, there exists a real positive constant δ_{13} such that, for $|u|_0 \leqslant \delta_{13}$,

$$\mathcal{T}_{13} \leqslant \beta_1 \int_0^1 |u|^2 |v| dx. \tag{4.23}$$

Then, collecting inequalities (4.19), (4.22), and (4.23) together, if

$$|u|_0 \leqslant \delta_1 := \min\{\delta_{11}, \delta_{12}, \delta_{13}\},$$

we conclude that

$$\dot{V}_1 = \mathcal{T}_{11} + \mathcal{T}_{12} + \mathcal{T}_{13} \leqslant -\alpha_1 V_1 + \beta_1 \int_0^1 |u|^2 |v| dx. \tag{4.24}$$

This completes the proof of Lemma 4.3. ∎

From Lemma 4.3 it appears that it is clearly necessary to examine the dynamics of $v = u_x$ in order to carry out the Lyapunov stability analysis. This is the reason why the Lyapunov function (4.5) is extended with terms involving v. By time differentiation of the system equations (2.5) and (4.1), it may be shown that v satisfies the dynamics

$$v_t + F(u)v_x + [F'(u)v]v = 0, \quad x \in [0,1], \quad t \in [0,+\infty), \tag{4.25}$$

$$F(u(t,0))v(t,0) = G'\big(u(t,1)\big)F\big(u(t,1)\big)v(t,1), \qquad t \in [0,+\infty). \tag{4.26}$$

Let us compute the time derivative of V_2 along the classical C^1-solutions of system (4.25) with boundary conditions (4.26). One has

$$\begin{aligned}\dot{V}_2 &= \int_0^1 \Big\{2v^{\mathrm{tr}} R(u) v_t + v^{\mathrm{tr}} \big(R(u)\big)_t v\Big\} e^{-\mu x} dx \\ &= \int_0^1 \Big\{-2v^{\mathrm{tr}} R(u)F(u)v_x - 2v^{\mathrm{tr}} R(u)[F'(u)v]v - v^{\mathrm{tr}}[R'(u)F(u)v]v\Big\} e^{-\mu x} dx. \qquad (4.27)\end{aligned}$$

From now on, V_2 and $\dot{V}_2$ are considered as functionals defined, respectively, by (4.7) and (4.27) on the set $\mathcal{V}_2$ of $(u,v) \in C^2([0,1],\mathbb{R}^n) \times C^1([0,1],\mathbb{R}^n)$ such that

$$|u|_0 < \varepsilon_0, \tag{4.28}$$

$$u_x = v, \tag{4.29}$$

$$u_0 = G(u_1), \tag{4.30}$$

$$F(u_0)v_0 = G'\big(u_1\big)F\big(u_1\big)v_1, \tag{4.31}$$

where $u_0 := u(0)$, $u_1 := u(1)$ as above, and $v_0 := v(0)$, $v_1 := v(1)$.

The matrix $R(u)$ is selected as the matrix $N(F(u))$ of Lemma 4.1 now with $\Delta := \Lambda D^2$. Our estimates on V_2 and $\dot{V}_2$ are in the following lemma.

LEMMA 4.4. *There exists $\mu_2 > 0$ such that, for every $\mu \in (0, \mu_2)$, there exist positive real constants $\alpha_2, \beta_2, \delta_2$ such that, for every $(u, v) \in \mathcal{V}_2$ such that $|u|_0 \leqslant \delta_2$,*

$$\frac{1}{\beta_2}\int_0^1 |v|^2 dx \leqslant V_2(u,v) \leqslant \beta_2 \int_0^1 |v|^2 dx, \tag{4.32}$$

$$\dot{V}_2(u,v) \leqslant -\alpha_2(V_2(u,v) + |v_1|^2) + \beta_2 \int_0^1 |v|^3 dx. \tag{4.33}$$

Proof of Lemma 4.4. Throughout this proof, (u, v) is assumed to be in $\mathcal{V}_2$. By the construction of R, we have

$$F(0)^{-1}R(0) = \Lambda^{-1}R(0) = D^2 \in \mathcal{D}_{n,+} \tag{4.34}$$

and the existence of $\delta_{21} \in (0, \varepsilon_0/2)$ such that

$$R(a) \in \mathcal{S}_{n,+} \text{ and } R(a)F(a) \in \mathcal{S}_{n,+} \quad \forall a \in \mathbb{R}^n \text{ such that } |a| \leqslant \delta_{21}. \tag{4.35}$$

Clearly, from (4.35), for every $\mu > 0$, there exists $\beta_2 > 0$ such that (4.32) holds if $|u|_0 \leqslant \delta_{21}$.

Let us now deal with the estimate (4.33) on $\dot{V}_2 (= \dot{V}_2(u,v))$. Let us decompose $\dot{V}_2$ in the following way:

$$\dot{V}_2 = \mathcal{T}_{21} + \mathcal{T}_{22} + \mathcal{T}_{23}, \tag{4.36}$$

with

$$\mathcal{T}_{21} := -\mu \int_0^1 \big(v^{\mathrm{tr}} R(u)F(u)v\big) e^{-\mu x} dx,$$

$$\mathcal{T}_{22} := -\int_0^1 \big(v^{\mathrm{tr}} R(u)F(u)v e^{-\mu x}\big)_x dx,$$

$$\mathcal{T}_{23} := \int_0^1 \Big\{ v^{\mathrm{tr}} \Big([(R(u)F(u))_x v] - 2R(u)[F'(u)v] - [R'(u)F(u)v] \Big) v \Big\} e^{-\mu x} dx.$$

Analysis of the first term $\mathcal{T}_{21}$. By (4.32) and (4.35), for every $\mu > 0$, there exists a positive real constant $\alpha_{21} > 0$ such that, if $|u|_0 \leqslant \delta_{21}$,

$$\mathcal{T}_{21} \leqslant -\alpha_{21} V_2. \tag{4.37}$$

Analysis of the second term $\mathcal{T}_{22}$. One has

$$\begin{aligned}\mathcal{T}_{22} &= -\Big[v^{\mathrm{tr}} R(u)F(u)v e^{-\mu x}\Big]_0^1 \\ &= -\Big(v_1^{\mathrm{tr}} R(u_1)F(u_1)v_1 e^{-\mu} - v_0^{\mathrm{tr}} R(u_0)F(u_0)v_0\Big).\end{aligned}$$

Under the boundary condition (4.31), we have

$$\mathcal{T}_{22} = -v_1^{\mathrm{tr}} \Big(R(u_1)F(u_1)e^{-\mu} - F(u_1)^{\mathrm{tr}} \big(G'(u_1)\big)^{\mathrm{tr}} \\ (F(G(u_1))^{-1})^{\mathrm{tr}} R(G(u_1)) G'\big(u_1\big) F(u_1) \Big) v_1,$$

which implies that

$$\mathcal{T}_{22} = -v_1^{\mathrm{tr}}\Big(R(0)\Lambda e^{-\mu} - \Lambda K^{\mathrm{tr}}\Lambda^{-1}R(0)K\Lambda\Big)v_1 + \mathcal{O}(|v_1|^2|u_1|;|u_1|). \tag{4.38}$$

We define $\zeta := Dv_1$. Then, using (4.34), we have, for every $v_1 \in \mathbb{R}^n$,

$$v_1^{\mathrm{tr}}K^{\mathrm{tr}}\Lambda^{-1}R(0)Kv_1 = v_1^{\mathrm{tr}}K^{\mathrm{tr}}DDKv_1 = \left(\zeta^{\mathrm{tr}}D^{-1}K^{\mathrm{tr}}D\right)\left(DKD^{-1}\zeta\right) = |DKD^{-1}\zeta|^2. \tag{4.39}$$

Therefore, using (4.34) once again, we get that, for every $v_1 \in \mathbb{R}^n$,

$$v_1^{\mathrm{tr}}K^{\mathrm{tr}}\Lambda^{-1}R(0)Kv_1 \leqslant \|DKD^{-1}\|^2\zeta^{\mathrm{tr}}\zeta = \|DKD^{-1}\|^2 v_1^{\mathrm{tr}}\Lambda^{-1}R(0)v_1. \tag{4.40}$$

From (4.40) and the fact that $\|DKD^{-1}\| < 1$, it follows that, choosing $\mu > 0$ small enough, $\Lambda^{-1}R(0)e^{-\mu} - K^{\mathrm{tr}}\Lambda^{-1}R(0)K$ is a positive definite matrix, which, in turn, implies that the matrix $R(0)\Lambda e^{-\mu} - \Lambda K^{\mathrm{tr}}\Lambda^{-1}R(0)K\Lambda$ is also positive definite. Hence there exist $\alpha_{22} > 0$ and $\delta_{22} > 0$ such that, if $|u_1| \leqslant \delta_{22}$, we have

$$\mathcal{T}_{22} \leqslant -\alpha_{22}|v_1|^2. \tag{4.41}$$

Analysis of the third term $\mathcal{T}_{23}$. The integrand of $\mathcal{T}_{23}$ is, at least, cubic with respect to v. It follows that, increasing the value of β_2 if necessary, there exists $\delta_{23} \in (0, \varepsilon_0/2)$ such that, for $|u|_0 \leqslant \delta_{23}$,

$$\mathcal{T}_{23} \leqslant \beta_2 \int_0^1 |v|^3 dx. \tag{4.42}$$

Then, collecting inequalities (4.37), (4.41), and (4.42) together, we conclude that if $|u|_0 \leqslant \delta_2 := \min\{\delta_{21}, \delta_{22}, \delta_{23}\}$ and if $\alpha_2 := \min\{\alpha_{21}, \alpha_{22}\}$, then

$$\dot{V}_2 = \mathcal{T}_{21} + \mathcal{T}_{22} + \mathcal{T}_{23} \leqslant -\alpha_2(V_2 + |v_1|^2) + \beta_2 \int_0^1 |v|^3 dx. \tag{4.43}$$

This completes the proof of Lemma 4.4. ∎

Note that V_2 is not sufficient to get an upper bound on $\int_0^1 |v|^3 dx$. For that reason, we also need to consider the dynamics of w to complete the Lyapunov stability analysis. By a further time differentiation of the system equations (4.25)–(4.26), we obtain

$$w_t + F(u)w_x + [F'(u)w]v + 2[F'(u)v]w + [F''(u)(v,v)]v = 0, \qquad x \in [0,1], \qquad t \in [0, +\infty), \tag{4.44}$$

under the boundary condition

$$F(u_0)w_0 + [F'(u_0)v_0]v_0 = [H'(u_1)F(u_1)v_1]v_1 + H(u_1)F(u_1)w_1 + H(u_1)[F'(u_1)v_1]v_1, \tag{4.45}$$

with the notation $w_0 := w(0), w_1 := w(1)$, and $H(u) := F(G(u))^{-1}G'(u)F(u)$. Using the previous boundary conditions (4.30) and (4.31), this boundary condition (4.45) may be written in compact form as

$$w_0 = F(G(u_1))^{-1}H(u_1)F(u_1)w_1 + \mathcal{Z}(u_1, v_1), \tag{4.46}$$

where $\mathcal{Z}$ is continuous on a neighborhood of $0 \in \mathbb{R}^n \times \mathbb{R}^n$ and such that

$$\mathcal{Z}(u_1, v_1) = \mathcal{O}\big(|v_1|^2; |u_1|\big). \tag{4.47}$$

Let us compute the time derivative of V_3 along the classical C^1-solutions of system (4.44) with boundary conditions (4.46). One has

$$\begin{aligned}\dot{V}_3 &= \int_0^1 \Big\{2w^{\mathrm{tr}} S(u) w_t + w^{\mathrm{tr}} \big(S(u)\big)_t w\Big\} e^{-\mu x} dx \\ &= \int_0^1 \Big\{-2w^{\mathrm{tr}} S(u) F(u) w_x - 2w^{\mathrm{tr}} S(u)\Big([F'(u)w]v + 2[F'(u)v]w + [F''(u)(v,v)]v\Big) \\ &\qquad\qquad - w^{\mathrm{tr}}[S'(u)F(u)v]w\Big\} e^{-\mu x} dx.\end{aligned} \tag{4.48}$$

From now on, V_3 and $\dot{V}_3$ are considered as functionals defined, respectively, by (4.8) and (4.48) on the set $\mathcal{V}_3$ of $(u, v, w) \in C^3([0,1], \mathbb{R}^n) \times C^2([0,1], \mathbb{R}^n) \times C^1([0,1], \mathbb{R}^n)$ such that

$$|u|_0 < \varepsilon_0, \tag{4.49}$$

$$u_x = v, \qquad v_x = w, \tag{4.50}$$

$$u_0 = G(u_1), \tag{4.51}$$

$$F(u_0)v_0 = G'\big(u_1\big)F\big(u_1\big)v_1, \tag{4.52}$$

$$w_0 = F(G(u_1))^{-1} H(u_1) F(u_1) w_1 + \mathcal{Z}(u_1, v_1), \tag{4.53}$$

with $u_0 := u(0)$, $u_1 := u(1)$, $v_0 := v(0)$, and $v_1 := v(1)$ as above, and $w_0 := w(0)$ and $w_1 := w(1)$.

The matrix $S(u)$ is selected as the matrix $N(F(u))$ of Lemma 4.1 now with $\Delta := \Lambda^2 D^2 \Lambda$. Our estimates on V_3 and $\dot{V}_3$ are in the following lemma.

LEMMA 4.5. *There exists $\mu_3 > 0$ such that, for every $\mu \in (0, \mu_3)$, there exist positive real constants $\alpha_3, \beta_3, \delta_3$ such that, for every $(u, v, w) \in \mathcal{V}_3$ such that $|u|_0 + |v|_0 \leqslant \delta_3$, one has*

$$\frac{1}{\beta_3} \int_0^1 |w|^2 dx \leqslant V_3(u, v, w) \leqslant \beta_3 \int_0^1 |w|^2 dx, \tag{4.54}$$

$$\dot{V}_3(u, v, w) \leqslant -\alpha_3 V_3(u, v, w) + \beta_3 |v_1|^4 + \beta_3 \int_0^1 \Big(|v|^2|w| + |w|^2|v|\Big) dx. \tag{4.55}$$

Proof of Lemma 4.5. Throughout this proof, we assume that $(u, v, w) \in \mathcal{V}_3$. By the construction of S, we have

$$\Lambda^{-2} S(0) \Lambda^{-1} = D^2 \in \mathcal{D}_{n,+} \tag{4.56}$$

and the existence of $\delta_{31} \in (0, \varepsilon_0/2)$ such that

$$S(a) \in \mathcal{S}_{n,+} \text{ and } S(a)F(a) \in \mathcal{S}_{n,+} \quad \forall a \in \mathbb{R}^n \text{ such that } |u| \leqslant \delta_{31}. \tag{4.57}$$

Clearly, from (4.57), we obtain that, for every $\mu > 0$, there exists $\beta_3 > 0$ such that (4.54) holds if $|u|_0 \leqslant \delta_{31}$.

Let us now deal with the estimate (4.55) on $\dot{V}_3 (= \dot{V}_3(u,v,w))$. Let us decompose $\dot{V}_3$ in the following way:

$$\dot{V}_3 = \mathcal{T}_{31} + \mathcal{T}_{32} + \mathcal{T}_{33}, \tag{4.58}$$

with

$$\mathcal{T}_{31} := -\mu \int_0^1 \Big(w^{\mathrm{tr}} S(u) F(u) w \Big) e^{-\mu x} dx,$$

$$\mathcal{T}_{32} := - \int_0^1 \Big(w^{\mathrm{tr}} S(u) F(u) w e^{-\mu x} \Big)_x dx,$$

$$\begin{aligned}\mathcal{T}_{33} := - \int_0^1 \Big\{ -w^{\mathrm{tr}} \Big([(S(u)F(u))_x v] + [S'(u)F(u)v] \Big) w + 2w^{\mathrm{tr}} S(u)[F'(u)w]v \\ + 4w^{\mathrm{tr}} S(u)[F'(u)v]w + 2w^{\mathrm{tr}}[F''(u)(v,v)]v \Big\} e^{-\mu x} dx.\end{aligned}$$

Analysis of the first term $\mathcal{T}_{31}$. By (4.54) and (4.57), for every $\mu > 0$, there exists a positive real constant $\alpha_3 > 0$ such that, if $|u|_0 \leqslant \delta_{31}$,

$$\mathcal{T}_{31} \leqslant -\alpha_3 V_3. \tag{4.59}$$

Analysis of the second term $\mathcal{T}_{32}$.

$$\begin{aligned}\mathcal{T}_{32} &= -\Big[w^{\mathrm{tr}} S(u) F(u) w e^{-\mu x} \Big]_0^1 \\ &= -\Big(w_1^{\mathrm{tr}} S(u_1) F(u_1) w_1 e^{-\mu} - w_0^{\mathrm{tr}} S(u_0) F(u_0) w_0 \Big).\end{aligned}$$

Under the boundary conditions (4.51) and (4.53), we have, also using (4.47),

$$\begin{aligned}\mathcal{T}_{32} = -w_1^{\mathrm{tr}} \Big(& S(u_1)F(u_1)e^{-\mu} \\ & - F(u_1)^{\mathrm{tr}} H(u_1)^{\mathrm{tr}} \big(F(G(u_1))^{-1} \big)^{\mathrm{tr}} S\big(G(u_1)\big) H(u_1) F(u_1) \Big) w_1 \\ & + \mathcal{O}(|v_1|^4 + |v_1|^2 |w_1|; |u_1|) \\ = -w_1^{\mathrm{tr}} \Big(& S(0)\Lambda e^{-\mu} - \Lambda^2 K^{\mathrm{tr}} \Lambda^{-2} S(0) \Lambda^{-1} K \Lambda^2 \Big) w_1 \\ & + \mathcal{O}\big(|v_1|^4 + |v_1|^2|w_1| + |w_1|^2|u_1|; |u_1|\big).\end{aligned} \tag{4.60}$$

For $w_1 \in \mathbb{R}^n$, we define $\zeta := Dw_1$. Then, using (4.56), we have, for every $w_1 \in \mathbb{R}^n$,

$$\begin{aligned}w_1^{\mathrm{tr}} K^{\mathrm{tr}} \Lambda^{-2} S(0) \Lambda^{-1} K w_1 &= w_1^{\mathrm{tr}} K^{\mathrm{tr}} D D K w_1 \\ &= \big(\zeta^{\mathrm{tr}} D^{-1} K^{\mathrm{tr}} D \big) \big(D K D^{-1} \zeta \big) = |DKD^{-1}\zeta|^2.\end{aligned}$$

Therefore, for every $w_1 \in \mathbb{R}^n$, we have, using (4.56) once again,

$$w_1^{\mathrm{tr}} K^{\mathrm{tr}} \Lambda^{-2} S(0) \Lambda^{-1} K w_1 \leqslant \|DKD^{-1}\|^2 \zeta^{\mathrm{tr}} \zeta = \|DKD^{-1}\|^2 w_1^{\mathrm{tr}} \Lambda^{-2} S(0) \Lambda^{-1} w_1.$$

From this inequality and the fact that $\|DKD^{-1}\|^2 < 1$, it follows that, choosing $\mu > 0$ small enough, $\Lambda^{-2}S(0)\Lambda^{-1}e^{-\mu} - K^{\mathrm{tr}}\Lambda^{-2}S(0)\Lambda^{-1}K$ is a positive definite symmetric matrix, which, in turn, implies that the matrix

$$S(0)\Lambda e^{-\mu} - \Lambda^2 K^{\mathrm{tr}}\Lambda^{-2}S(0)\Lambda^{-1}K\Lambda^2 \tag{4.61}$$

is also positive definite. Moreover, for every $\eta > 0$ and for every $(v_1, w_1) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$|v_1|^2|w_1| \leqslant \frac{1}{4\eta}|v_1|^4 + \eta|w_1|^2.$$

Hence, taking $\eta > 0$ small enough and also using (4.60), one gets the existence of $\delta_{32} > 0$ and $\beta_{32} > 0$ such that, if $|u|_0 + |v|_0 \leqslant \delta_{32}$,

$$\mathcal{T}_{32} \leqslant \beta_{32}|v_1|^4. \tag{4.62}$$

Analysis of the third term $\mathcal{T}_{33}$. Note that

$$(F(u)G(u))_x = [F'(u)v]G(u) + F(u)G'(u)v.$$

It follows that there exist $\delta_{33} > 0$ and $\beta_{33} > 0$ such that, if $|u|_0 + |v|_0 \leqslant \delta_{33}$, then

$$\mathcal{T}_{33} \leqslant \beta_{33} \int_0^1 \Big(|v|^2|w| + |w|^2|v|\Big)dx. \tag{4.63}$$

Then, collecting inequalities (4.59), (4.62), and (4.63) together, we conclude that if $|u|_0 + |v|_0 \leqslant \delta_3 := \min\{\delta_{31}, \delta_{32}, \delta_{33}\}$ and $\beta_3 := \max\{\beta_{32}, \beta_{33}\}$, then

$$\dot V_3 = \mathcal{T}_{31} + \mathcal{T}_{32} + \mathcal{T}_{33} \leqslant -\alpha_3 V_3 + \beta_3|v_1|^4 + \beta_3 \int_0^1 \Big(|v|^2|w| + |w|^2|v|\Big)dx. \tag{4.64}$$

This completes the proof of Lemma 4.5. □

Finally, we deal with V (see (4.5)) and $\dot V$, which are now considered as functionals on the set $\mathcal{V}$ of $u \in C^3([0,1], \mathbb{R}^n)$ satisfying (4.49), (4.51), (4.52), and (4.53) with $v := u_x$ and $w := u_{xx}$, $u_0 := u(0)$, $u_1 := u(1)$, $v_0 := u_x(0)$, $v_1 := u_x(1)$, $w_0 := u_{xx}(0)$, and $w_1 := u_{xx}(1)$. Of course, we "define" $\dot V$ by $\dot V(u) := \dot V_1(u) + \dot V_2(u, u_x) + \dot V_3(u, u_x, u_{xx})$. The following lemma holds.

LEMMA 4.6. *Let* $\mu \in (0, \min\{\mu_1, \mu_2, \mu_3\})$. *There exist positive real constants* α, β, *and* δ *such that, for every* $u \in \mathcal{V}$ *such that* $|u|_0 + |u_x|_0 \leqslant \delta$, *we have*

$$\frac{1}{\beta}\int_0^1 (|u|^2 + |u_x|^2 + |u_{xx}|^2)dx \leqslant V(u) \leqslant \beta \int_0^1 (|u|^2 + |u_x|^2 + |u_{xx}|^2)dx, \tag{4.65}$$

$$\dot V \leqslant -\alpha V. \tag{4.66}$$

Proof of Lemma 4.6. Throughout this proof, u is assumed to be in $\mathcal{V}$. Let $\bar\delta := \min\{\delta_1, \delta_2, \delta_3\}$, $\bar\alpha := \min\{\alpha_1, \alpha_2, \alpha_3\}$, and $\bar\beta := \max\{\beta_1, \beta_2, \beta_3\}$. It readily follows from (4.11), (4.32), and (4.54) that if $|u|_0 + |u_x|_0 \leqslant \bar\delta$, then

$$\frac{1}{\bar\beta}\int_0^1 (|u|^2 + |u_x|^2 + |u_{xx}|^2)dx \leqslant V(u) \leqslant \bar\beta \int_0^1 (|u|^2 + |u_x|^2 + |u_{xx}|^2)dx. \tag{4.67}$$

In order to check (4.66) (for $\delta > 0$ small enough and $\beta > 0$ large enough), let us first point out that, for every $\eta > 0$,

$$\int_0^1 |u_x|^2 |u_{xx}| dx \leqslant \int_0^1 \left(\frac{1}{4\eta} |u_x|^4 + \eta |u_{xx}|^2 \right) dx$$

$$\leqslant \frac{1}{4\eta} |u_x|_0^2 \int_0^1 |u_x|^2 dx + \eta \int_0^1 |u_{xx}|^2 dx. \tag{4.68}$$

In order to get (4.66), it suffices to use (4.12), (4.33), (4.54), (4.55), (4.67), (4.68) with $\eta := \alpha_3/(2\beta_3)^2$ and to point out that

$$\int_0^1 |u|^2 |u_x| dx \leqslant |u_x|_0 \int_0^1 |u|^2 dx,$$

$$\int_0^1 |u_x|^3 dx \leqslant |u_x|_0 \int_0^1 |u_x|^2 dx,$$

$$\int_0^1 |u_{xx}|^2 |u_x| dx \leqslant |u_x|_0 \int_0^1 |u_{xx}|^2 dx.$$

This concludes the proof of Lemma 4.6. □

Finally, let us explain how to deduce Theorem 2.3 from Proposition 2.1 and Lemma 4.6. By the Sobolev inequality (see, for instance, [3, Théorème VII, page 129]), there exists $C > 0$ such that, for every u in the Sobolev space $H^2((0,1), \mathbb{R}^n)$,

$$|u|_0 + |u_x|_0 \leqslant C_0 |u|_{H^2((0,1),\mathbb{R}^n)}, \tag{4.69}$$

with

$$|u|_{H^2((0,1),\mathbb{R}^n)} := \left(\int_0^1 (|u|^2 + |u_x|^2 + |u_{xx}|^2) dx \right)^{1/2}.$$

We choose $\mu \in (0, \min\{\mu_1, \mu_2, \mu_3\}]$. Let us point out that a simple density argument shows that (4.65) and (4.66) hold for every $u \in H^2((0,1), \mathbb{R}^n)$ satisfying (4.51), (4.52), and $|u|_0 + |u_x|_0 \leqslant \delta$. Let

$$\varepsilon := \min \left\{ \frac{\delta}{2C_0\beta}, \frac{\delta_0}{\beta} \right\}. \tag{4.70}$$

Note that $\beta \geqslant 1$. Using Lemma 4.6, (4.69), and (4.70), the following implications hold for every $u \in H^2((0,1), \mathbb{R}^n)$ satisfying (4.51) and (4.52):

$$\left(|u|_{H^2((0,1),\mathbb{R}^n)} \leqslant \varepsilon\right) \Rightarrow \left(|u|_0 + |u_x|_0 \leqslant \frac{\delta}{2} \text{ and } V(u) \leqslant \beta\varepsilon^2 \right), \tag{4.71}$$

(4.72)

$$\left(|u|_0 + |u_x|_0 \leqslant \delta \text{ and } V(u) \leqslant \beta\varepsilon^2\right) \Rightarrow \left(|u|_0 + |u_x|_0 \leqslant \frac{\delta}{2} \text{ and } |u|_{H^2((0,1),\mathbb{R}^n)} \leqslant \delta_0 \right),$$

$$(|u|_0 + |u_x|_0 \leqslant \delta) \Rightarrow (\dot{V}(u) \leqslant 0). \tag{4.73}$$

Now let $u^0 \in H^2((0,1),\mathbb{R}^n)$ satisfying (4.2), (4.3), and

$$|u^0|_{H^2((0,1),\mathbb{R}^n)} \leqslant \varepsilon.$$

Let $u \in C^0([0,T), H^2((0,1),\mathbb{R}^n))$ be the maximal classical solution the Cauchy problem (2.5)–(2.7). Using implications (4.71) to (4.73), one gets that

$$|u(t,\cdot)|_{H^2((0,1),\mathbb{R}^n)} \leqslant \delta_0 \quad \forall t \in [0,T), \tag{4.74}$$

$$|u(t,\cdot)|_0 + |u_x(t,\cdot)|_0 \leqslant \delta \quad \forall t \in [0,T). \tag{4.75}$$

Using Proposition 2.1 and (4.74), one gets that $T = +\infty$. Using Lemma 4.6 and (4.75), one gets that

$$|u(t,\cdot)|^2_{H^2((0,1),\mathbb{R}^n)} \leqslant \beta V(u(t,\cdot)) \leqslant \beta V(u^0)e^{-\alpha t} \leqslant \beta^2 |u^0|^2_{H^2((0,1),\mathbb{R}^n)} e^{-\alpha t}.$$

This concludes the proof of Theorem 2.3 when $m = n$. ∎

Let us now explain the modifications we use in order to deal with the case $0 < m < n$ (of course, the case $m = 0$ can be reduced to the case $m = n$ by considering $\tilde{u}(t,x) := u(t,1-x)$).

One first needs the following parametric version of Lemma 4.1.

LEMMA 4.7. *Let $\Lambda := \text{diag}\,(\Lambda_1,\ldots,\Lambda_n) \in \mathcal{D}_n$ be such that* (2.4) *holds. There exist a positive real number η and a map $\mathcal{N} : \{M \times \Delta \in \mathcal{M}_{n,n}(\mathbb{R}) \times \mathcal{D}_n;\ \|M - \Lambda\| < \eta\} \to \mathcal{S}_n$ of class C^∞ such that*

$$\mathcal{N}(\Lambda,\Delta) = \Delta \quad \forall \Delta \in \mathcal{D}_n^\rho,$$

$$\mathcal{N}(M,\Delta)M - M^{\text{tr}}\mathcal{N}(M,\Delta) = 0 \quad \forall (M,\Delta) \in \mathcal{M}_{n,n}(\mathbb{R}) \times \mathcal{D}_n \textit{ such that } \|M - \Lambda\| < \eta.$$

Proof of Lemma 4.7. With the notation of the proof of Lemma 4.1, it suffices to define $\mathcal{N}(M,D)$ by $\mathcal{N}(M,D) := \mathcal{L}_M^{-1}(0,\Delta)$. ∎

The Lyapunov function V now has the following structure:

$$V(u,v,w) = V_1(u) + V_2(u,v) + V_3(u,v,w), \tag{4.76}$$

with

$$V_1(u) = \int_0^1 u^{\text{tr}} Q(x,u) u dx, \tag{4.77}$$

$$V_2(u,v) = \int_0^1 v^{\text{tr}} R(x,u) v dx, \tag{4.78}$$

$$V_3(u,v,w) = \int_0^1 w^{\text{tr}} S(x,u) w dx, \tag{4.79}$$

where $Q(x,u)$, $R(x,u)$, and $S(x,u)$ are symmetric positive definite matrices *depending on* $x \in [0,1]$ defined in the following way. We fix $D \in \mathcal{D}_{n,+}$ such that $\|DKD^{-1}\| < 1$. Let $\mu \in (0,+\infty)$, which will be chosen small enough later. Let us recall that $|\Lambda| = \text{diag}\,(|\Lambda_1|,\ldots,|\Lambda_n|) = \text{diag}\,(\Lambda_1,\ldots,\Lambda_m,|\Lambda_{m+1}|,\ldots,|\Lambda_n|)$.

(i) We define $Q(x,u)$ by

$$Q(x,u) := \mathcal{N}\big(F(u), D^2|\Lambda|^{-1}\text{diag}\,(e^{-\mu x},\ldots,e^{-\mu x},e^{\mu x},\ldots,e^{\mu x})\big).$$

(ii) We define $R(x,u)$ by

$$R(x,u) := \mathcal{N}\big(F(u), D^2|\Lambda|\mathrm{diag}\,(e^{-\mu x},\dots,e^{-\mu x},e^{\mu x},\dots,e^{\mu x})\big).$$

(iii) Finally, we define $R(x,u)$ by

$$S(x,u) := \mathcal{N}\big(F(u), D^2|\Lambda|^3\mathrm{diag}\,(e^{-\mu x},\dots,e^{-\mu x},e^{\mu x},\dots,e^{\mu x})\big).$$

(In the above equalities and in the following, in diag $(e^{-\mu x},\dots,e^{-\mu x},e^{\mu x},\dots,e^{\mu x})$, $e^{-\mu x}$ is repeated m times and $e^{\mu x}$ is repeated $(n-m)$ times.) In order to deal with the boundary conditions on u and v, let us define

$$a_0 := \begin{pmatrix} u_+(0) \\ u_-(1) \end{pmatrix}, \quad a_1 := \begin{pmatrix} u_+(1) \\ u_-(0) \end{pmatrix}, \quad b_0 := \begin{pmatrix} v_+(0) \\ v_-(1) \end{pmatrix}, \quad b_1 := \begin{pmatrix} v_+(1) \\ v_-(0) \end{pmatrix}. \tag{4.80}$$

The boundary condition (4.10) is now (see (2.6))

$$a_0 = G(a_1). \tag{4.81}$$

Now $\mathcal{V}_1$ is defined as the set of $u \in C^1([0,1],\mathbb{R}^n)$ such that (4.81) holds and $|u|_0 < \varepsilon_0$.

Clearly, the estimate on V_1 given in Lemma 4.3 still holds. Let us check that the estimate of this lemma on $\dot{V}_1$ also holds.

The decomposition (4.15)–(4.18) becomes

$$\dot{V}_1 = \mathcal{T}_{11} + \mathcal{T}_{12} + \mathcal{T}_{13},$$

with

$$\mathcal{T}_{11} := \int_0^1 u^{\mathrm{tr}} Q_x(x,u)F(u)u\,dx,$$

$$\mathcal{T}_{12} := -\int_0^1 \big(u^{\mathrm{tr}} Q(x,u)F(u)u\big)_x dx,$$

$$\mathcal{T}_{13} := \int_0^1 u^{\mathrm{tr}}\Big([Q'_u(x,u)v]F(u) + Q(x,u)[F'(u)v] - [Q'_u(x,u)F(u)v]\Big)u\,dx.$$

Noticing that

$$Q_x(x,0) = -\mu D^2\Lambda^{-1}\mathrm{diag}\,(e^{-\mu x},\dots,e^{-\mu x},e^{\mu x},\dots,e^{\mu x})),$$

the term $\mathcal{T}_{11}$ can be treated as above. Similarly the term $\mathcal{T}_{13}$ can also be treated as above. Concerning $\mathcal{T}_{12}$, one has

$$\begin{aligned}\mathcal{T}_{12} &= -u_1 Q(1,u_1)F(u_1)u_1 + u_0 Q(0,u_0)F(u_0)u_0 \\ &= -u_1^{\mathrm{tr}} D^2|\Lambda|^{-1}\Lambda u_1 + u_0^{\mathrm{tr}} D^2|\Lambda|^{-1}\Lambda u_0 + \mathcal{O}(|u_1|^3;|u_1|) + \mathcal{O}(\mu|u_1|^2;\mu).\end{aligned} \tag{4.82}$$

Let

$$K_{++} \in \mathcal{M}_{m,m}(\mathbb{R}), \qquad K_{+-} \in \mathcal{M}_{m,(n-m)}(\mathbb{R}),$$
$$K_{-+} \in \mathcal{M}_{(n-m),n}(\mathbb{R}), \qquad K_{--} \in \mathcal{M}_{(n-m),(n-m)}(\mathbb{R})$$

be such that

$$K = \begin{pmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{pmatrix}.$$

Using (4.80) and (4.81), one has

(4.83)
$$u_0 = \begin{pmatrix} K_{++} & K_{+-} \\ 0 & \mathrm{Id}_{n-m} \end{pmatrix} a_1 + \mathcal{O}(|a_1|^2; |a_1|), \qquad u_1 = \begin{pmatrix} \mathrm{Id}_m & 0 \\ K_{-+} & K_{--} \end{pmatrix} a_1 + \mathcal{O}(|a_1|^2; |a_1|).$$

Using (4.82) and (4.83), straightforward computations lead to

$$\mathcal{T}_{12} = -a_1^{\mathrm{tr}}(D^2 - K^{\mathrm{tr}} D^2 K) a_1 + \mathcal{O}(|a_1|^3; |a_1|) + \mathcal{O}(\mu |a_1|^2; \mu). \tag{4.84}$$

However, $\|DKD^{-1}\| < 1$ implies (and is in fact equivalent to) the property "the symmetric matrix ${}^{\mathrm{tr}}(D^2 - K^{\mathrm{tr}} D^2 K)$ is positive definite," which, together with (4.84), implies again the existence of $\delta_{12} > 0$ such that (4.22) holds if $|a_1| \leqslant \delta_{12}$. Hence Lemma 4.3 still holds.

Similarly it can be checked that Lemmas 4.4 and 4.5 also hold, except that in (4.33) and (4.55), $|v_1|$ has to be replaced by $|b_1|$ (and the definitions of $\mathcal{V}_2$ and $\mathcal{V}_3$ have to be modified in order to deal with the new compatibility conditions). The proof of Theorem 2.3 is then completed as in the case $m = n$.

Remark 4.8. One can give a lower bound on the exponential decay in Theorem 2.3. Indeed, it follows from our proof of this theorem that, if $\rho_1(G'(0)) < 1$, for every $\nu \in (0, -\min\{|\lambda_1|, \ldots, |\lambda_n|\} \ln(\rho_1(G'(0))))$, there exist $\varepsilon > 0$ and $C > 0$ such that, for every $u^0 \in H^2((0,1), \mathbb{R}^n)$ satisfying $|u^0|_{H^2((0,1),\mathbb{R}^n)} \leqslant \varepsilon$ and the compatibility conditions (2.8)–(2.10), the classical solution u to the Cauchy problem (2.5)–(2.7) is defined on $[0, +\infty)$ and satisfies (2.13).

5. Conclusion and final remarks. We have presented a new sufficient condition on the boundary conditions for the exponential stability of one-dimensional nonlinear hyperbolic systems on a bounded interval. Our analysis relies on the construction of an explicit strict Lyapunov function. Moreover, we have compared our sufficient condition with other known sufficient conditions for nonlinear and linear systems. We conclude the paper with two additional comments.

1. The Lyapunov stability analysis presented in this paper can be extended to nonlinear hyperbolic systems of the form

$$u_t + F(u) u_x = h(u), \tag{5.1}$$

   i.e., systems having a nonzero right-hand side $h(u)$ with the map $h : \mathbb{R}^n \to \mathbb{R}^n$ of class C^2 vanishing at zero ($h(0) = 0$). Our main theorem (Theorem 2.3) can be extended, in a straightforward way, to system (5.1) with boundary conditions (1.4), provided $\|h'(0)\|$ is sufficiently small.
2. For the sake of simplicity, we have assumed throughout the paper that the diagonal matrix $F(0)$ has *distinct* nonzero diagonal entries. It turns out that this assumption may be slightly relaxed when the matrix $F(u)$ is block-diagonal. Indeed, in such a case, it is sufficient to assume that the Λ_i values are different in each block, but different blocks may share identical Λ_i values. This situation typically occurs when the system $u_t + F(u) u_x = 0$ is a model

for a network of interconnected 2×2 hyperbolic systems. Typical examples are hydraulic networks modeled by Saint Venant equations [7], road networks modeled by Aw–Rascle equations [1, 8], or pipeline networks modeled by isentropic Euler equations [2].

Appendix A. Some properties of the function ρ_1. In this appendix we give some properties which are useful for estimating and computing ρ_1. Some of these properties are used to prove Proposition 3.7.

PROPOSITION A.1. *Let $l \in \{1, \dots, n-1\}$. Let $K_1 \in \mathcal{M}_{l,l}(\mathbb{R})$, $K_2 \in \mathcal{M}_{l,n-l}(\mathbb{R})$, $K_3 \in \mathcal{M}_{n-l,l}(\mathbb{R})$, $K_4 \in \mathcal{M}_{n-l,n-l}(\mathbb{R})$ and let $K \in \mathcal{M}_{n,n}(\mathbb{R})$ be defined by*

$$K := \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}.$$

Then

$$\rho_1(K) \geqslant \max\{\rho_1(K_1), \rho_1(K_4)\}. \tag{A.1}$$

Moreover, if $K_2 = 0$ or $K_3 = 0$, then

$$\rho_1(K) = \max\{\rho_1(K_1), \rho_1(K_4)\}. \tag{A.2}$$

Proof of Proposition A.1. Let $D \in \mathcal{D}_{n,+}$. Let $D_1 \in \mathcal{D}_{l,+}$ and $D_2 \in \mathcal{D}_{n-l,+}$ be such that

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}.$$

Let

$$M := DKD^{-1}.$$

We have

$$M^{\mathrm{tr}} M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

with

$$M_{11} := D_1^{-1} K_1^{\mathrm{tr}} D_1^2 K_1 D_1^{-1} + D_1^{-1} K_3^{\mathrm{tr}} D_2^2 K_3 D_1^{-1},$$

$$M_{12} := D_1^{-1} K_1^{\mathrm{tr}} D_1^2 K_2 D_2^{-1} + D_1^{-1} K_3^{\mathrm{tr}} D_2^2 K_4 D_2^{-1},$$

$$M_{21} := D_2^{-1} K_2^{\mathrm{tr}} D_1^2 K_1 D_1^{-1} + D_2^{-1} K_4^{\mathrm{tr}} D_2^2 K_3 D_1^{-1},$$

$$M_{22} := D_2^{-1} K_2^{\mathrm{tr}} D_1^2 K_2 D_2^{-1} + D_2^{-1} K_4^{\mathrm{tr}} D_2^2 K_4 D_2^{-1}.$$

For $X \in \mathbb{R}^l$, let $\tilde{X} \in \mathbb{R}^n$ be defined by

$$\tilde{X} := \begin{pmatrix} X \\ 0 \end{pmatrix}.$$

Note that $|\tilde{X}| = |X|$ and that

$$\begin{aligned}\tilde{X}^{\rm tr} M^{\rm tr} M \tilde{X} &= X^{\rm tr} D_1^{-1} K_1^{\rm tr} D_1^2 K_1 D_1^{-1} X + X^{\rm tr} D_1^{-1} K_3^{\rm tr} D_2^2 K_3 D_1^{-1} X \\ &\geqslant X^{\rm tr} D_1^{-1} K_1^{\rm tr} D_1^2 K_1 D_1^{-1} X.\end{aligned}$$

Hence

$$\begin{aligned}\max\{Z^{\rm tr} M^{\rm tr} Z;\, Z \in \mathbb{R}^n, |Z| = 1\} &\geqslant \max\{X^{\rm tr} D_1^{-1} K_1^{\rm tr} D_1^2 K_1 D_1^{-1} X;\, X \in \mathbb{R}^l, |X| = 1\} \\ &\geqslant \rho_1(K_1)^2,\end{aligned}$$

which implies that $\rho_1(K_1) \leqslant \rho_1(K)$. Similarly $\rho_1(K_4) \leqslant \rho_1(K)$. This proves (A.1).

Let us now prove (A.2). We deal only with the case $K_3 = 0$ (the case $K_2 = 0$ being similar). Let $\eta > 0$. Let $D_1 \in \mathcal{D}_{l,+}$ and $D_2 \in \mathcal{D}_{n-l,+}$ be such that

$$\|D_1 K_1 D_1^{-1}\| \leqslant \rho_1(K_1) + \eta, \qquad \|D_2 K_4 D_2^{-1}\| \leqslant \rho_1(K_4) + \eta. \tag{A.3}$$

Let $\varepsilon > 0$ and

$$D := \begin{pmatrix} \varepsilon D_1 & 0 \\ 0 & D_2 \end{pmatrix} \in \mathcal{D}_{n,+}, \qquad M := DKD^{-1} \in \mathcal{M}_{n,n}(\mathbb{R}).$$

Let $Z \in \mathbb{R}^n$ and let $X \in \mathbb{R}^l$ and $Y \in \mathbb{R}^{n-l}$ be such that

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}.$$

We have

$$\begin{aligned}Z^{\rm tr} M^{\rm tr} M Z = {} & X^{\rm tr} D_1^{-1} K_1^{\rm tr} D_1^2 K_1 D_1^{-1} X + 2\varepsilon X^{\rm tr} D_1^{-1} K_1^{\rm tr} D_1^2 K_2 D_2^{-1} Y \\ & + \varepsilon^2 Y^{\rm tr} D_2^{-1} K_2^{\rm tr} D_1^2 K_2 D_2^{-1} Y + Y^{\rm tr} D_2^{-1} K_4^{\rm tr} D_2^2 K_4 D_2^{-1} Y.\end{aligned}$$

Hence there exists a constant $C > 0$ independent of Z and $\varepsilon > 0$ such that

$$Z^{\rm tr} M^{\rm tr} M Z \leqslant (\|D_1 K_1 D_1^{-1}\| |X|)^2 + (\|D_2 K_4 D_2^{-1}\| |Y|)^2 + C\varepsilon |Z|^2. \tag{A.4}$$

From (2.1), (A.3), and (A.4), we obtain that

$$\rho_1(K)^2 \leqslant \max\{(\rho_1(K_1) + \eta)^2, (\rho_1(K_4) + \eta)^2\} + C\varepsilon. \tag{A.5}$$

Letting $\varepsilon \to 0$ and $\eta \to 0$ in (A.5), one gets that $\rho_1(K)^2 \leqslant \max\{\rho_1(K_1)^2, \rho_1(K_4)^2\}$. This concludes the proof of Proposition A.1. □

PROPOSITION A.2. *The map $\rho_1 : \mathcal{M}_{n,n}(\mathbb{R}) \to [0, +\infty)$ is continuous.*

Proof of Proposition A.2. We proceed by induction on n. For $n = 1$ the function ρ_1 satisfies $\rho(k) = |k|$ for every $k \in \mathbb{R} = \mathcal{M}_{1,1}(\mathbb{R})$ and is therefore continuous. We now assume that ρ_1 is continuous on $\mathcal{M}_{p,p}(\mathbb{R})$ for every $p \in \{1, \dots, n-1\}$ and prove that ρ_1 is continuous on $\mathcal{M}_{n,n}(\mathbb{R})$. Since, for every $D \in \mathcal{D}_{n,+}$, the function $K \in \mathcal{M}_{n,n}(\mathbb{R}) \mapsto \|K\| \in \mathbb{R}$ is continuous, it readily follows from (2.1) that ρ_1 is upper semicontinuous on $\mathcal{M}_{n,n}(\mathbb{R})$. It remains only to check that ρ_1 is lower semicontinuous.

We argue by contradiction: let $K \in \mathcal{D}_{n,n}(\mathbb{R})$ and let $(K_k)_{k\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}_{n,n}(\mathbb{R})$ such that

$$K_k \to K \text{ as } k \to +\infty, \tag{A.6}$$

$$\lim_{k\to+\infty} \rho_1(K_k) < \rho_1(K). \tag{A.7}$$

Let $(D_k)_{k\in\mathbb{N}}$ be a sequence of elements of $\mathcal{D}_{n,+}$ such that

$$\|D_k K_k D_k^{-1}\| \leqslant \rho_1(K_k) + k^{-1} \quad \forall k \in \mathbb{N} \setminus \{0\}. \tag{A.8}$$

Note that, denoting by $(e_1, \dots, e_n)$ the canonical basis of $\mathbb{R}^n$,

$$|A_{ij}| = |e_i^{\mathrm{tr}} A e_j| \leqslant \|A\| \quad \forall A \in \mathcal{M}_{n,n}(\mathbb{R}), \forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, n\}.$$

Hence, if we denote by K_{ijk} the term on the ith line and jth column of the matrix K_k,

$$|K_{ijk}| \frac{d_{ik}}{d_{jk}} \leqslant \|D_k K_k D_k^{-1}\| \quad \forall (i,j) \in \{1, \dots, n\}^2, \forall k \in \mathbb{N}, \tag{A.9}$$

where $(d_{ik})_{i\in\{1,\dots,n\}}$ is defined by $D_k = \mathrm{diag}\,(d_{1k}, \dots, d_{nk})$. After suitable reorderings (note that $\rho_1(\Sigma A \Sigma^{-1}) = \rho_1(A)$ for every $A \in \mathcal{M}_{n,n}(\mathbb{R})$ and for every permutation matrix Σ) and extracting subsequences if necessary, we may assume without loss of generality that

$$d_{1k} \leqslant d_{2k} \leqslant \cdots \leqslant d_{(n-1)k} \leqslant d_{nk} \quad \forall k \in \mathbb{N}. \tag{A.10}$$

A simple scaling argument also shows that we may assume without loss of generality that

$$d_{1k} = 1 \quad \forall k \in \mathbb{N}. \tag{A.11}$$

Extracting subsequences if necessary, there exist $l \in \{1, \dots, n\}$, $(d_1, \dots, d_l) \in [1, +\infty)^l$ such that

$$d_{ik} \to d_i \text{ as } k \to +\infty \quad \forall i \in \{1, \dots, l\}, \tag{A.12}$$

$$d_{ik} \to +\infty \text{ as } k \to +\infty \quad \forall i \in \{l+1, \dots, n\}. \tag{A.13}$$

We first treat the case where $l = n$. Let $D := \mathrm{diag}\,(d_1, \dots, d_n) \in \mathcal{D}_{n,+}$. From (A.12), we have

$$D_k \to D \text{ as } k \to +\infty. \tag{A.14}$$

From (2.1), we have

$$\rho_1(K) \leqslant \|DKD^{-1}\|, \tag{A.15}$$

which, together with (A.8) and (A.14), implies that

$$\liminf_{k\to+\infty} \rho_1(K_k) \geqslant \rho_1(K),$$

in contradiction with (A.7).

It remains to deal with the case where $l < n$. Let us denote K_{ij} the term on the ith line and jth column of the matrix K. From (A.6), (A.7), (A.8), (A.9), (A.12), and (A.13), one gets that

$$K_{ij} = 0 \quad \forall (i,j) \in \{l+1, \dots, n\} \times \{1, \dots, l\}. \tag{A.16}$$

Let $K^1 \in \mathcal{M}_{l,l}(\mathbb{R})$, $K^2 \in \mathcal{M}_{l,n-l}(\mathbb{R})$, $K^4 \in \mathcal{M}_{n-l,n-l}(\mathbb{R})$ be such that

$$K = \begin{pmatrix} K^1 & K^2 \\ 0 & K^4 \end{pmatrix}.$$

Similarly, for $k \in \mathbb{N}$, let $K_k^1 \in \mathcal{M}_{l,l}(\mathbb{R})$, $K_k^2 \in \mathcal{M}_{l,n-l}(\mathbb{R})$, $K_k^3 \in \mathcal{M}_{n-l,l}(\mathbb{R})$, $K_k^4 \in \mathcal{M}_{n-l,n-l}(\mathbb{R})$ be defined by

$$K := \begin{pmatrix} K_k^1 & K_k^2 \\ K_k^3 & K_k^4 \end{pmatrix}.$$

From (A.2), we have

$$\rho_1(K) = \max\{\rho_1(K^1), \rho_1(K^4)\}. \tag{A.17}$$

From (A.1), we have

$$\rho_1(K_k) \geqslant \max\{\rho_1(K_k^1), \rho_1(K_k^4)\} \quad \forall k \in \mathbb{N}. \tag{A.18}$$

From our induction hypothesis (the continuity of ρ_1 on $\mathcal{M}_{p,p}(\mathbb{R})$ for every $p \in \{1, \dots, n-1\}$) and (A.6), we get that

$$\lim_{k \to +\infty} \rho_1(K_k^1) = \rho_1(K^1), \qquad \lim_{k \to +\infty} \rho_1(K_k^4) = \rho_1(K^4),$$

which, together with (A.17) and (A.18), again leads to a contradiction with (A.7). This concludes the proof of Proposition A.2. □

Our next proposition shows a case where the value of $\rho_1(K)$ may be given directly. (For a converse of this proposition, see Proposition B.1.)

PROPOSITION A.3. *Let $l \in \{1, \dots, n\}$. Let $(A_j)_{j \in \{1,\dots,l\}}$ and $(B_j)_{j \in \{1,\dots,l\}}$ be two sequences of vectors in $\mathbb{R}^n$ such that*

$$A_j^{\text{tr}} A_k = B_j^{\text{tr}} B_k \quad \forall (j,k) \in \{1, \dots, l\}^2, \tag{A.19}$$

$$\sum_{j=1}^{l} A_{ij}^2 = \sum_{j=1}^{l} B_{ij}^2 \quad \forall i \in \{1, \dots, n\}, \tag{A.20}$$

where A_{ij} (resp., B_{ij}) is the element on the ith line of the vector A_j (resp., B_j). We assume that the l vectors $A_1, \dots, A_l$ are linearly independent. Let $R \geqslant 0$ and let $K \in \mathcal{M}_{n,n}(\mathbb{R})$ be such that

$$KA_j = RB_j \quad \forall j \in \{1, \dots, l\}, \tag{A.21}$$

$$|KX| \leqslant R|X| \quad \forall X \in \mathbb{R}^n \text{ such that } X^{\text{tr}} A_j = 0 \quad \forall j \in \{1, l\}. \tag{A.22}$$

Then $\rho_1(K) = R$.

Proof of Proposition A.3. It readily follows from the assumptions of this proposition that $\|K\| = R$. Hence it remains only to check that

$$\|DKD^{-1}\| \geqslant R \quad \forall D \in \mathcal{D}_{n,+}. \tag{A.23}$$

Let $D := \text{diag }(D_1, \dots, D_n) \in \mathcal{D}_{n,+}$. For $j \in \{1, \dots, l\}$, let us define

$$E_j := (E_{1j}, \dots, E_{nj})^{\text{tr}} \in \mathbb{R}^n \setminus \{0\}, \qquad F_j := (F_{1j}, \dots, F_{nj})^{\text{tr}} \in \mathbb{R}^n \setminus \{0\}$$

by

$$E_j := DA_j, \qquad F_j := DB_j.$$

We have, for every $j \in \{1, \dots, l\}$,

$$DKD^{-1}E_j = RF_j, \tag{A.24}$$

$$E_{ij} = D_i A_{ij}, \qquad F_{ij} = D_i B_{ij} \quad \forall i \in \{1, \dots, n\}. \tag{A.25}$$

Using (A.20), (A.24), and (A.25), we get

$$\begin{aligned}
\sum_{j=1}^{l} |DKD^{-1}E_j|^2 &= R^2 \sum_{j=1}^{l} \left(\sum_{i=1}^{n} F_{ij}^2 \right) \\
&= R^2 \sum_{i=1}^{n} D_i^2 \left(\sum_{j=1}^{l} B_{ij}^2 \right) \\
&= R^2 \sum_{i=1}^{n} D_i^2 \left(\sum_{j=1}^{l} A_{ij}^2 \right) \\
&= R^2 \sum_{j=1}^{l} |E_j|^2.
\end{aligned}$$

In particular, there exists $p \in \{1, \dots l\}$, such that

$$|DKD^{-1}E_p|^2 \geqslant R^2 |E_p|^2,$$

which, together with the fact that $E_p \neq 0$, implies that $\|DKD^{-1}\| \geqslant R$. This concludes the proof of Proposition A.3. ∎

Appendix B. Proof of Proposition 3.7. Inequality (3.22) is obvious: indeed, for every $(\theta_1, \dots, \theta_n)^{\text{tr}} \in \mathbb{R}^n$ and for every $D \in \mathcal{D}_{n,+}$,

$$\begin{aligned}
\rho(\text{diag }(e^{\iota\theta_1}, \dots, e^{\iota\theta_n})K) &= \rho(D\text{diag }(e^{\iota\theta_1}, \dots, e^{\iota\theta_n})KD^{-1}) \\
&= \rho(\text{diag }(e^{\iota\theta_1}, \dots, e^{\iota\theta_n})DKD^{-1}) \\
&\leqslant \|\text{diag }(e^{\iota\theta_1}, \dots, e^{\iota\theta_n})DKD^{-1}\| \\
&\leqslant \|\text{diag }(e^{\iota\theta_1}, \dots, e^{\iota\theta_n})\| \|DKD^{-1}\| = \|DKD^{-1}\|.
\end{aligned}$$

The proof of (3.23) for every $n \in \{1,2,3,4,5\}$ is more complicated and relies on various independent propositions. The first proposition provides the converse (up to the D) to Proposition A.3 for generic $K \in \mathcal{M}_{n,n}(\mathbb{R})$.

PROPOSITION B.1. *Let $K \in \mathcal{M}_{n,n}(\mathbb{R})$ be such that, for every $M > 0$, there exists $\delta > 0$ such that*

$$\left(D := (D_1, \dots, D_n) \in \mathcal{D}_{n,+}, \sum_{i=1}^{n} D_i = 1, \ \min\{D_1, \dots, D_n\} < \delta\right) \Rightarrow (\|DKD^{-1}\| > M). \tag{B.1}$$

(It is easily checked that this property holds, for example, if $K_{ij} \neq 0$, for every $(i,j) \in \{1,\dots,n\}^2$ such that $i \neq j$, which is a generic property.) Then there exist $D \in \mathcal{D}_{n,+}$, an integer $l \in \{1,\dots,n\}$, l vectors $A_j \in \mathbb{R}^n$, $j \in \{1,\dots,l\}$, and l vectors $B_j \in \mathbb{R}^n$, $j \in \{1,\dots,l\}$, such that (A.19) *and* (A.20) *hold and*

$$\text{the vectors } A_j \in \mathbb{R}^n,\ j \in \{1,\dots,l\}, \text{ are linearly independent}, \tag{B.2}$$

$$DKD^{-1}A_j = \rho_1(K) B_j \quad \forall j \in \{1, \dots l\}, \tag{B.3}$$

$$|DKD^{-1}X| \leqslant \rho_1(K)|X| \quad \forall X \in \mathbb{R}^n. \tag{B.4}$$

Remark B.2. Proposition B.1 is false if assumption (B.1) is removed. Indeed, let us take $n = 2$ and

$$K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $\rho_1(K) = 0$, and it is easily seen that the conclusion of Proposition B.1 does not hold.

Proof of Proposition B.1. From (B.1), one gets the existence of $\tilde{D} \in \mathcal{D}_{n,+}$ such that

$$\|\tilde{D}K\tilde{D}^{-1}\| = \rho_1(K). \tag{B.5}$$

Replacing K by $\tilde{D}K\tilde{D}^{-1}$, we may assume without loss of generality that $\tilde{D}$ is the identity map Id_n of $\mathbb{R}^n$. Then

$$\|K\| = \rho_1(K). \tag{B.6}$$

Clearly, (B.1) implies that $K \neq 0$, and therefore, by (B.6),

$$\rho_1(K) \neq 0. \tag{B.7}$$

(In fact, if $K = 0$, the conclusion of Proposition B.1 obviously holds.) Note that (B.6) implies (B.4) with $D := \mathrm{Id}_n$. Let $p \in \{1,\dots,n\}$ be the dimension of the kernel of $K^{\mathrm{tr}}K - \rho_1(K)^2 \mathrm{Id}_n$ and let $(X_1,\dots,X_p)$ be an orthonormal basis of this kernel. For $j \in \{1,\dots,p\}$, let $Y_j := KX_j$. One has

$$|Y_j|^2 = X_j^{\mathrm{tr}} K^{\mathrm{tr}} K X_j = \rho_1(K)^2 |X_j|^2 \quad \forall j \in \{1,\dots,p\}, \tag{B.8}$$

$$Y_k^{\mathrm{tr}} Y_j = X_k^{\mathrm{tr}} K^{\mathrm{tr}} K X_j = \rho_1(K)^2 X_k^{\mathrm{tr}} X_j = 0 \quad \forall (k,j) \in \{1,\dots,p\}^2 \text{ such that } k \neq j. \tag{B.9}$$

For $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,p\}$, let us denote by X_{ij} (resp., Y_{ij}) the ith component of X_j (resp., Y_j). For $j \in \{1,\dots,p\}$, let us denote by E_j the element of $\mathbb{R}^n$ whose ith component is

$$E_{ij} := Y_{ij}^2 - X_{ij}^2. \tag{B.10}$$

Let us assume, for the moment, that

$$\forall \tau \in \mathbb{R}^n, \text{there exists } j \in \{1,\dots,p\} \text{ such that } \tau^{\mathrm{tr}} E_j \geqslant 0. \tag{B.11}$$

Applying the separation principle for convex sets to $\{0\}$ and the convex hull of the vectors E_j, $j \in \{1,\dots,p\}$ (see, e.g., [19, Theorem 3.4(b), page 58]), it follows from (B.11) that $0 \in \mathbb{R}^n$ is in the convex hull of the vectors $E_1,\dots E_p$: there exist p nonnegative real numbers $t_1,\dots,t_p$ such that

$$\sum_{j=1}^{p} t_i = 1, \qquad \sum_{j=1}^{p} t_i E_i = 0.$$

Let $l \in \{1,\dots,p\}$ be the number of the t_i's which are not equal to 0. Reordering the X_i's if necessary, we may assume that

$$t_j > 0 \quad \forall j \in \{1,\dots,l\}, \qquad t_j = 0 \quad \forall j \in \{l+1,\dots p\}.$$

For $j \in \{1,\dots,l\}$, we define $A_j \in \mathbb{R}^n$ and $B_j \in \mathbb{R}^n$ by

$$A_j := \sqrt{t_j} X_j, \qquad B_j := \sqrt{t_j} Y_j. \tag{B.12}$$

Then it is easily checked that the vectors $A_1,\dots,A_l$ are linearly independent, that (A.19) and (A.20) hold (one even has $A_k^{\mathrm{tr}} A_j = B_k^{\mathrm{tr}} B_j = 0$ for every $(k,j) \in \{1,\dots,l\}^2$ such that $k \neq j$), and that (B.3) holds with $D := \mathrm{Id}_n$.

It remains only to prove (B.11). Let $\tau := (\tau_1,\dots,\tau_n)^{\mathrm{tr}} \in \mathbb{R}^n$. For $s \in \mathbb{R}$, let

$$D(s) := \mathrm{diag}\,(1 + s\tau_1,\dots,1 + s\tau_n) \in \mathcal{D}_n.$$

For s small enough, $D(s) \in \mathcal{D}_{n,+}$, and therefore, by (B.6),

$$\|D(s)KD(s)^{-1}\|^2 \geqslant \|K\|^2 = \|D(0)KD(0)^{-1}\|^2. \tag{B.13}$$

Let us estimate the left-hand side of (B.13). By a classical theorem due to Rellich (see, e.g., [18, Theorem XII.3, page 4]) on perturbations of the spectrum of self-adjoint operators, there exist $\varepsilon > 0$, p real functions $\lambda_1,\dots,\lambda_p$ of class C^1 from $(-\varepsilon,\varepsilon)$ into $\mathbb{R}$, and p maps $x_1,\dots,x_p$ of class C^1 from $(-\varepsilon,\varepsilon)$ into $\mathbb{R}^n$ such that

$$\lambda_j(0) = \rho_1(K)^2, \qquad x_j(0) = X_j \quad \forall j \in \{1,\dots,p\}, \tag{B.14}$$

$$D(s)^{-1}K^{\mathrm{tr}}D(s)^2KD(s)^{-1}x_j(s) = \lambda_j(s)x_j(s) \quad \forall s \in (-\varepsilon,\varepsilon), \forall j \in \{1,\dots,p\}, \tag{B.15}$$

$$x_j(s)^{\mathrm{tr}}x_j(s) = 1 \quad \forall s \in (-\varepsilon,\varepsilon), \forall j \in \{1,\dots,p\}, \tag{B.16}$$

$$x_j(s)^{\mathrm{tr}}x_k(s) = 0 \quad \forall s \in (-\varepsilon,\varepsilon), \forall (j,k) \in \{1,\dots,p\}^p \text{ such that } k \neq j, \tag{B.17}$$

$$\|D(s)KD(s)^{-1}\|^2 = \max\{\lambda_1(s),\dots,\lambda_p(s)\} \quad \forall s \in (-\varepsilon,\varepsilon). \tag{B.18}$$

Differentiating (B.15) with respect to s and using (B.10), (B.14), (B.16), and (B.17), one gets

$$\lambda_j'(0) = 2\rho_1(K)^2\tau^{\mathrm{tr}}E_j \quad \forall j \in \{1,\dots,p\}. \tag{B.19}$$

Property (B.11) follows from (B.7), (B.13), (B.18), and (B.19). This concludes the proof of Proposition B.1. □

The number l appearing in Proposition B.1 turns out to be important to compare ρ_0 and ρ_1: we have the following proposition.

PROPOSITION B.3. *Let $K \in \mathcal{M}_{n,n}(\mathbb{R})$, $D \in \mathcal{D}_{n,+}$, $l \in \{1,\dots,n\}$, l vectors $A_j \in \mathbb{R}^n$, $j \in \{1,\dots,l\}$, and l vectors $B_j \in \mathbb{R}^n$, $j \in \{1,\dots,l\}$, be such that* (A.20), (B.2), (B.3), *and* (B.4) *hold. If $l = 1$, there exist $X \in \mathbb{R}^n$ and $\Upsilon := \mathrm{diag}\,(\Upsilon_1,\dots,\Upsilon_n) \in \mathcal{D}_n$ such that*

$$|X| \neq 0, \tag{B.20}$$

$$\Upsilon_i \in \{1,-1\} \quad \forall i \in \{1,\dots,n\}, \tag{B.21}$$

$$KX = \rho_1(K)\Upsilon X. \tag{B.22}$$

If $l = 2$, there exist $X \in \mathbb{C}^n$ and $(\Upsilon_1,\dots,\Upsilon_n) \in \mathbb{C}^n$ such that

$$|X| \neq 0, \tag{B.23}$$

$$|\Upsilon_i| = 1 \quad \forall i \in \{1,\dots,n\}, \tag{B.24}$$

$$KX = \rho_1(K)\mathrm{diag}\,(\Upsilon_1,\dots,\Upsilon_n)X. \tag{B.25}$$

In both cases ($l = 1$ or $l = 2$), one has (3.23).

Proof of Proposition B.3. Let us first consider the case $l = 1$. Let $i \in \{1,\dots,n\}$. From (A.20), one has $|A_{i1}| = B_{i1}$, and therefore there exists $\Upsilon_i \in \{-1,1\}$ such that $B_{i1} = \varepsilon_i A_{i1}$. From (B.3), one gets (B.22) if one defines X by $X := D^{-1}A_1$. Let us check that (3.23) holds. Let $(\theta_1,\dots,\theta_n)^{\mathrm{tr}} \in \mathbb{R}^n$ be defined by

$$\theta_i = 0 \text{ if } \Upsilon_i = 1, \qquad \theta_i = -\pi \text{ if } \Upsilon_i = -1.$$

Then (B.22) implies that

$$\mathrm{diag}\,(e^{\iota\theta_1},\dots,e^{\iota\theta_n})KX = \rho_1(K)X. \tag{B.26}$$

From (3.21), (B.20), and (B.26), we get that

$$\rho_0(K) \geqslant \rho_1(K), \tag{B.27}$$

which, together with (3.22), gives (3.23).

Let us now turn to the case $l = 2$. Let $i \in \{1,\dots,n\}$. From (A.20), one has

$$|A_{i1} + \iota A_{i2}| = |B_{i1} + \iota B_{i2}|,$$

and therefore there exists $\Upsilon_i \in \mathbb{C}$ such that $|\Upsilon_i| = 1$ and $B_{i1} + \iota B_{i2} = \Upsilon_i(A_{i1} + \iota A_{i2})$. From (B.3), one gets (B.22) if one defines X by $X := D^{-1}(A_1 + \iota A_2)$. Finally, the proof of (3.23) is the same as in the case $l = 1$. This concludes the proof of Proposition B.3. □

The next proposition deals with the case $n = l$.

Proposition B.4. *Let $K \in \mathcal{M}_{n,n}(\mathbb{R})$, $D \in \mathcal{D}_{n,+}$, $l \in \{1,\dots,n\}$, l vectors $A_j \in \mathbb{R}^n$, $j \in \{1,\dots,l\}$, and l vectors $B_j \in \mathbb{R}^n$, $j \in \{1,\dots,l\}$, be such that* (A.20), (B.2), (B.3), *and* (B.4) *hold. If $l = n$, there exist $X \in \mathbb{C}^n$ satisfying* (B.23) *and $\theta \in \mathbb{R}$ such that*

$$KX = e^{-\iota\theta}\rho_1(K)X \tag{B.28}$$

and (3.23) *again holds.*

Proof of Proposition B.4. If $\rho_1(K) = 0$, then $K = 0$ and the conclusion of Proposition B.4 holds. If $\rho_1(K) > 0$, it follows from (A.19), (B.2), and (B.3) and the assumption $l = n$ that $\rho_1(K)^{-1}KDKD^{-1}$ is an isometry. Hence there exist $Y \in \mathbb{C}^n \setminus \{0\}$ and $\theta \in \mathbb{R}$ such that $\rho_1(K)^{-1}DKD^{-1}Y = e^{-\iota\theta}Y$, which implies (B.28) if $X := D^{-1}Y$. Finally, (3.23) again follows from (B.23) and (B.28). This concludes the proof of Proposition B.4. □

Note that ρ_0 is continuous. Hence, from Proposition A.2, Proposition B.1, Proposition B.3, and Proposition B.4, in order to get (3.23) (for every $n \in \{1,\dots,5\}$) of Proposition 3.7, it remains to address, with the notation of the conclusion of Proposition B.1, the cases $(l,n) = (3,4)$, $(l,n) = (3,5)$, and $(l,n) = (4,5)$. This is done in the following proposition.

Proposition B.5. *Let $K \in \mathcal{M}_{n,n}(\mathbb{R})$, $D \in \mathcal{D}_{n,+}$, $l \in \{1,\dots,n\}$, l vectors $A_j \in \mathbb{R}^n$, $j \in \{1,\dots,l\}$, and l vectors $B_j \in \mathbb{R}^n$, $j \in \{1,\dots,l\}$, be such that* (A.20), (B.2), (B.3), *and* (B.4) *hold. If $(l,n) \in \{(3,4),(3,5),(4,5)\}$, there exist $X \in \mathbb{C}^n$ and $(\Upsilon_1,\dots,\Upsilon_n) \in \mathbb{C}^n$ such that* (B.23), (B.24), *and* (B.25) *hold. In particular, one has* (3.23).

Proof of Proposition B.5. The fact that (3.23) is implied by the assumptions of Proposition B.5, (B.23), (B.24), and (B.25) has already been pointed out in the proof of Proposition B.3. The case $(l,n) = (3,4)$ follows from the case $(l,n) = (3,5)$ by replacing $K \in \mathcal{M}_{4,4}(\mathbb{R})$ by the matrix

$$\tilde{K} := \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence we may assume that $n = 5$. Taking $X := D^{-1}(Y_1A_1 + Y_2A_2 + \cdots + Y_lA_l)$ it suffices to prove the existence of $Y := (Y_1, Y_2, \dots, Y_l)^{\mathrm{tr}} \in \mathbb{C}^l \setminus \{0\}$ such that

$$|Y_1B_{i1} + Y_2B_{i2} + \cdots + Y_lB_{il}|^2 - |Y_1A_{i1} + Y_2A_{i2} + \cdots + Y_lA_{il}|^2 = 0 \quad \forall i \in \{1,2,3,4,5\}. \tag{B.29}$$

Let us recall that, for $p \in \mathbb{N} \setminus \{0\}$, $\mathcal{S}_p$ denotes the set of elements $Q \in \mathcal{M}_{p,p}$ such that $Q^{\mathrm{tr}} = Q$. For $i \in \{1,2,3,4,5\}$, there exists a unique $Q_i \in \mathcal{S}_l$ such that, for every $Y := (Y_1, Y_2, \dots, Y_l)^{\mathrm{tr}} \in \mathbb{C}^l$,

$$Y^{\mathrm{tr}}Q_i\bar{Y}^{\mathrm{tr}} = |Y_1B_{i1} + Y_2B_{i2} + \cdots + Y_lB_{il}|^2 - |Y_1A_{i1} + Y_2A_{i2} + \cdots + Y_lA_{il}|^2,$$

with $\bar{Y} := (\bar{Y}_1, \bar{Y}_2, \dots, \bar{Y}_l)$ ($\bar{z}$ denoting the complex conjugate of $z \in \mathbb{C}$). Then (B.29) is equivalent to

$$Y^{\mathrm{tr}}Q_i\bar{Y}^{\mathrm{tr}} = 0 \quad \forall i \in \{1,2,3,4,5\}. \tag{B.30}$$

For a matrix $M \in \mathcal{M}_{p,p}(\mathbb{C})$, let us denote by tr (M) its trace. Using (A.20) we have that

$$\mathrm{tr}\ (Q_i) = 0 \quad \forall i \in \{1,2,3,4,5\}.$$

Using (A.19), one gets that

$$Y^{\mathrm{tr}}Q_1\bar{Y}^{\mathrm{tr}} + Y^{\mathrm{tr}}Q_2\bar{Y} + Y^{\mathrm{tr}}Q_3\bar{Y} + Y^{\mathrm{tr}}Q_4\bar{Y} + Y^{\mathrm{tr}}Q_5\bar{Y} = 0 \quad \forall Y \in \mathbb{C}^l.$$

Hence (B.30) is equivalent to

$$Y^{\mathrm{tr}}Q_i\bar{Y}^{\mathrm{tr}} = 0 \quad \forall i \in \{1,2,3,4\}.$$

Therefore Proposition B.5 is a consequence of the following proposition due to Voisin [23].

PROPOSITION B.6. *Let $l \in \{3,4\}$. Let Q_1, Q_2, Q_3, and Q_4 be four elements of $\mathcal{S}_l$ such that*

$$\text{(B.31)} \qquad \mathrm{tr}\,(Q_i) = 0 \quad \forall i \in \{1,2,3,4\}.$$

Then there exists $Y \in \mathbb{C}^l \setminus \{0\}$ such that

$$\text{(B.32)} \qquad Y^{\mathrm{tr}}Q_i\bar{Y} = 0 \quad \forall i \in \{1,2,3,4\}.$$

Proof of Proposition B.6. We reproduce the proof of [23]. For $l \in \mathbb{N}$, let $\overline{\mathcal{S}_{l,+}}$ be the set of semidefinite positive $S \in \mathcal{S}_l$. The first step is the following lemma.

LEMMA B.7. *Let l, p, and n be three positive integers. Let Q_i, $i \in \{1,\dots,n\}$, be n elements of $\mathcal{S}_l$. Assume that*

$$\text{(B.33)} \qquad \mathrm{tr}\,(Q_i) = 0 \quad \forall i \in \{1,\dots,n\},$$

$$\text{(B.34)} \qquad n < \frac{(p+1)(p+2)}{2} - 1.$$

Then there exists $S \in \overline{\mathcal{S}_{l,+}} \setminus \{0\}$ such that

(B.35) *the rank of S is less than or equal to p,*

$$\text{(B.36)} \qquad \mathrm{tr}\,(SQ_i) = 0 \quad \forall i \in \{1,\dots,n\}.$$

Proof of Proposition B.7. Let

$$C := \{S \in \overline{\mathcal{S}_{l,+}};\, \mathrm{tr}\,(S) = l,\ \mathrm{tr}\,(SQ_i) = 0 \quad \forall i \in \{1,\dots,n\}\}.$$

The set C is a closed convex bounded subset of $\mathcal{M}_{l,l}(\mathbb{R})$. By (B.33), $\mathrm{Id}_l \in C$, and therefore C is not empty. Hence, by the Krein–Milman theorem (see, e.g., [19, Theorem 3.21, page 70]), the convex set C has at least an extreme point. Let S be an extreme point of C. Then $S \in \overline{\mathcal{S}_{l,+}} \setminus \{0\}$ and satisfies (B.36). It remains only to check that (B.35) holds. Let k be the rank of S. There exist an orthonormal matrix $O \in \mathcal{M}_{l,l}(\mathbb{R})$ and a definite positive matrix $S_0 \in \mathcal{S}_k$ such that

$$\text{(B.37)} \qquad S = O^{\mathrm{tr}} \begin{pmatrix} S_0 & 0 \\ 0 & 0 \end{pmatrix} O.$$

Let

$$\text{(B.38)} \qquad \Pi := \left\{ O^{\mathrm{tr}} \begin{pmatrix} S' & 0 \\ 0 & 0 \end{pmatrix} O;\ S' \in \mathcal{S}_k,\ \mathrm{tr}\,(S') = 0 \right\} \subset \mathcal{S}_l.$$

Let us assume that

$$n < \frac{k(k+1)}{2} - 1. \tag{B.39}$$

Since Π is a vector subspace of $\mathcal{S}_l$ of dimension $(k(k+1)/2)-1$, (B.39) implies that there exists $S_0 \in \Pi \setminus \{0\}$ such that

$$\text{tr }(S_0 Q_i) = 0 \quad \forall i \in \{1, \dots, n\}. \tag{B.40}$$

Then, for $\tau \in \mathbb{R}$ with $|\tau|$ small enough, $S + \tau S_0$ is in C, which contradicts the fact that S is an extreme point of C. Hence (B.39) does not hold, which, together with (B.34), implies that $k \leqslant p$. This concludes the proof of Lemma B.7. ◻

Let us go back to the proof of Proposition B.6. We apply Lemma B.7 with $n = 4$ and $p = 2$ (then (B.34) holds). We get the existence of $S \in \overline{\mathcal{S}_{l,+}} \setminus \{0\}$ satisfying

(B.41) the rank of S is less than or equal to 2,

$$\text{tr }(SQ_i) = 0 \quad \forall i \in \{1, \dots, 4\}. \tag{B.42}$$

Let $\lambda_1 > 0$, $\lambda_2 \geqslant 0$, and 0 be the eigenvalues of S. Let

$$S_0 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \overline{\mathcal{S}_{l,+}}.$$

There exists an orthonormal matrix O such that

$$S = O^{\text{tr}} S_0 O. \tag{B.43}$$

Let $Z := (\sqrt{\lambda_1}, \iota\sqrt{\lambda_2}, 0) \in \mathbb{C}^l \setminus \{0\}$ and $Y := O^{\text{tr}} Z \in \mathbb{C}^l \setminus \{0\}$. Then, using (B.42) and (B.43), one gets that, for every $i \in \{1, \dots, 4\}$,

$$2Y^{\text{tr}} Q_i \overline{Y} = \text{tr }((Y\overline{Y}^{\text{tr}} + \overline{Y}Y)^{\text{tr}} Q_i) = \text{tr }(O^{\text{tr}}(Z\overline{Z}^{\text{tr}} + \overline{Z}Z^{\text{tr}})OQ_i)$$

$$= 2\text{tr }(O^{\text{tr}} S_0^{\text{tr}} O Q_i) = \text{tr }(SQ_i) = 0,$$

which concludes the proof of Proposition B.6 and therefore the proof of Proposition B.5. ◻

Finally, in order to end the proof of Proposition 3.7, it remains only to check that, for $n = 6$ and therefore for every $n \geqslant 6$, there exists $K \in \mathcal{M}_{n,n}(\mathbb{R})$ such that $l = 3$ and (3.24) hold. This is done in the following example.

Example B.8. Let $(u_1, v_1, w_1)^{\text{tr}} \in \mathbb{R}^3$, $(u_2, v_2, w_2)^{\text{tr}} \in \mathbb{R}^3$, $(x_1, y_1, z_1)^{\text{tr}} \in \mathbb{R}^3$, and $(x_2, y_2, z_2)^{\text{tr}} \in \mathbb{R}^3$. We define $A_1 \in \mathbb{R}^6$, $A_2 \in \mathbb{R}^6$, $A_3 \in \mathbb{R}^6$, $B_1 \in \mathbb{R}^6$, $B_2 \in \mathbb{R}^6$, and

$B_3 \in \mathbb{R}^6$ by

$$A_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ u_1 \\ u_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ v_1 \\ v_2 \end{pmatrix}, \quad A_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ w_1 \\ w_2 \end{pmatrix},$$

$$B_1 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/\sqrt{2} \\ x_1 \\ x_2 \end{pmatrix}, \quad B_2 := \begin{pmatrix} 1 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ y_1 \\ y_2 \end{pmatrix}, \quad B_3 := \begin{pmatrix} 0 \\ 1 \\ 1/\sqrt{2} \\ 0 \\ z_1 \\ z_2 \end{pmatrix}.$$

One easily checks that (A.20) holds if (and only if)

$$u_1^2 + v_1^2 + w_1^2 = x_1^2 + y_1^2 + z_1^2, \tag{B.44}$$

$$u_2^2 + v_2^2 + w_2^2 = x_2^2 + y_2^2 + z_2^2. \tag{B.45}$$

Similarly (A.19) holds if (and only if)

$$\frac{3}{2} + u_1^2 + u_2^2 - x_1^2 - x_2^2 = 0, \tag{B.46}$$

$$-1 + v_1^2 + v_2^2 - y_1^2 - y_2^2 = 0, \tag{B.47}$$

$$-\frac{1}{2} + w_1^2 + w_2^2 - z_1^2 - z_2^2 = 0, \tag{B.48}$$

$$\frac{1}{2} + u_1 v_1 + u_2 v_2 - x_1 y_1 - x_2 y_2 = 0, \tag{B.49}$$

$$u_1 w_1 + u_2 w_2 - x_1 z_1 - x_2 z_2 = 0, \tag{B.50}$$

$$-\frac{1}{2} + v_1 w_1 + v_2 w_2 - y_1 z_1 - y_2 z_2 = 0. \tag{B.51}$$

Note that (B.44), (B.46), (B.47), and (B.48) imply (B.45).

We take $l := 3$ and $R := 1$. We define $K \in \mathcal{M}_{6,6}(\mathbb{R})$ by requiring (A.21) and

$$KX = 0 \quad \forall X \in \mathbb{R}^6 \text{ such that } X^{\text{tr}} A_1 = X^{\text{tr}} A_2 = X^{\text{tr}} A_3 = 0.$$

From Proposition A.3 we get that if (B.44) and (B.46) to (B.51) hold, then

$$\rho_1(K) = 1.$$

Let us assume, for the moment, that (B.46) to (B.51) hold. If (3.24) does not hold, we have $\rho_0(K) = \rho_1(K) = 1$, and therefore there exist $X \in \mathbb{C}^6$ and $(\Upsilon_1, \ldots, \Upsilon_6)^{\text{tr}} \in \mathbb{C}^n$ such that (B.23), (B.24), and (B.25) hold. Clearly,

$$|KX| = |X|. \tag{B.52}$$

Since

$$|K(Y+Z)| = |Y| \quad \forall Y \in \mathbb{C}A_1 + \mathbb{C}A_2 + \mathbb{C}A_3,$$
$$\forall Z \in \mathbb{C}^n \text{ such that } Z^{\mathrm{tr}}A_1 = Z^{\mathrm{tr}}A_2 = Z^{\mathrm{tr}}A_3 = 0,$$

it follows from (B.52) that $X \in \mathbb{C}A_1 + \mathbb{C}A_2 + \mathbb{C}A_3$. Hence, there exist $\xi_1 \in \mathbb{C}$, $\xi_2 \in \mathbb{C}$, and $\xi_3 \in \mathbb{C}$ such that

$$X = \xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3. \tag{B.53}$$

Using (B.25), one gets $(KX)_1 = \Upsilon_1 X_1$ and $(KX)_2 = \Upsilon_1 X_2$, which, together with (A.21) and (B.24), imply that

$$|\xi_1| = |\xi_2| = |\xi_3|. \tag{B.54}$$

Using (B.23) and (B.54) one sees that, without loss of generality, we may assume that

$$\xi_1 = 1, \qquad |\xi_2| = |\xi_3| = 1.$$

Hence there exist $\theta_2 \in \mathbb{R}$ and $\theta_3 \in \mathbb{R}$ such that

$$\xi_2 = e^{i\theta_2}, \qquad \xi_3 = e^{i\theta_3}. \tag{B.55}$$

Now using $|(KX)_3| = |X_3|$, one gets

$$|\xi_2 + \xi_3| = \sqrt{2},$$

which, together with (B.55), is equivalent to

$$\cos(\theta_3 - \theta_2) = 0;$$

i.e., there exists $\varepsilon_3 \in \{1, -1\}$ such that

$$\xi_3 = \varepsilon_3 \iota \xi_2. \tag{B.56}$$

Proceeding similarly with the fourth of KX, one gets the existence of $\varepsilon_2 \in \{1, -1\}$ such that

$$\xi_2 = \varepsilon_2 \iota. \tag{B.57}$$

Then $|(KX)_5| = |X_5|$ and $|(KX)_6| = |X_6|$ are equivalent to

$$(u_1 + \varepsilon_1 w_1)^2 + v_1^2 = (x_1 + \varepsilon_1 z_1)^2 + y_1^2, \tag{B.58}$$

$$(u_2 + \varepsilon_1 w_2)^2 + v_2^2 = (x_2 + \varepsilon_1 z_2)^2 + y_2^2 \tag{B.59}$$

with

$$\varepsilon_1 := -\varepsilon_2 \varepsilon_3 \in \{1, -1\}.$$

Let

$$\begin{array}{lccc} F: & \mathbb{R}^{12} & \rightarrow & \mathbb{R}^7, \\ & P := (u_1, v_1, w_1, x_1, y_1, z_1, u_2, v_2, w_2, x_2, y_2, z_2)^{\mathrm{tr}} & \mapsto & F(P) \end{array} \tag{B.60}$$

be defined by

$$F(P) := \begin{pmatrix} \frac{3}{2} + u_1^2 + u_2^2 - x_1^2 - x_2^2 \\ -1 + v_1^2 + v_2^2 - y_1^2 - y_2^2 \\ -\frac{1}{2} + w_1^2 + w_2^2 - z_1^2 - z_2^2 \\ \frac{1}{2} + u_1 v_1 + u_2 v_2 - x_1 y_1 - x_2 y_2 \\ u_1 w_1 + u_2 w_2 - x_1 z_1 - x_2 z_2 \\ -\frac{1}{2} + v_1 w_1 + v_2 w_2 - y_1 z_1 - y_2 z_2 \\ u_1^2 + v_1^2 + w_1^2 - x_1^2 - y_1^2 - z_1^2 \end{pmatrix}.$$

Let Σ be the subset of $\mathbb{R}^{12}$ defined by

$$\Sigma := \{P \in \mathbb{R}^{12};\ F(P) = 0 \text{ and the rank of } F'(P) \text{ is } 7\}.$$

Let

$$\tilde{P} := \left(0, 1, 0, 1, 0, 0, -\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}, -\frac{1}{4}\right)^{\mathrm{tr}} \in \mathbb{R}^{12}.$$

One easily checks that $F(\tilde{P}) = 0$. Straightforward computations give

$$F'(\tilde{P}) = \begin{pmatrix} 0 & 0 & 0 & -2 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{3}{2} & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & -1 & 0 & \frac{1}{2} & -\frac{1}{4} & 0 & -\frac{1}{2} & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \frac{3}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & -\frac{3}{4} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B.61}$$

In particular, the rank of $F'(\tilde{P})$ is 7. Hence $\tilde{P}$ is in Σ, and the set Σ is not empty and is a submanifold of $\mathbb{R}^{12}$ of dimension $12 - 7 = 5$. The tangent space to this manifold at $\tilde{P}$ is Ker $F'(\tilde{P})$. Let G_+ be the map

$$\begin{aligned} G_+ :\quad & \mathbb{R}^{12} & \to\quad & \mathbb{R}^2, \\ & P := (u_1, v_1, w_1, x_1, y_1, z_1, u_2, v_2, w_2, x_2, y_2, z_2)^{\mathrm{tr}} & \mapsto\quad & G_+(P) \end{aligned}$$

defined by

$$G_+(P) := \begin{pmatrix} (u_1 + w_1)^2 + v_1^2 - (x_1 + z_1)^2 - y_1^2 \\ (u_2 + w_2)^2 + v_2^2 - (x_2 + z_2)^2 - y_2^2 \end{pmatrix}.$$

Similarly, let G_- be the map

$$\begin{array}{rccc} G_- : & \mathbb{R}^{12} & \to & \mathbb{R}^2, \\ & P := (u_1, v_1, w_1, x_1, y_1, z_1, u_2, v_2, w_2, x_2, y_2, z_2)^{\mathrm{tr}} & \mapsto & G_-(P) \end{array}$$

defined by

$$G_-(P) := \begin{pmatrix} (u_1 - w_1)^2 + v_1^2 - (x_1 - z_1)^2 - y_1^2 \\ (u_2 - w_2)^2 + v_2^2 - (x_2 - z_2)^2 - y_2^2 \end{pmatrix}.$$

Let $S_+ \subset \mathbb{R}^{12}$ and $S_- \subset \mathbb{R}^{12}$ be defined by

$$S_+ := \{P \in \mathbb{R}^{12};\, G_+(P) = 0\},$$

$$S_- := \{P \in \mathbb{R}^{12};\, G_-(P) = 0\}.$$

It suffices to check that

$$\text{(B.62)} \qquad \Sigma \text{ is not a subset of } S_- \cup S_+.$$

Note that $\tilde{P} \in S_- \cap S_+$ and

$$\text{(B.63)} \qquad G'_-(\tilde{P}) = \begin{pmatrix} 0 & 2 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 2 & -2 & -1 & 2 \end{pmatrix},$$

$$\text{(B.64)} \qquad G'_+(\tilde{P}) = \begin{pmatrix} 0 & 2 & 0 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix}.$$

In particular the rank of $G'_-(\tilde{P})$ and the rank of $G'_+(\tilde{P})$ are both equal to 2. Hence, if $r > 0$ is small enough, the set $\{P \in S_-;\, |P - \tilde{P}| < r\}$ and the set $\{P \in S_+;\, |P - \tilde{P}| < r\}$ are submanifolds of $\mathbb{R}^{12}$ of dimension $12 - 2 = 10$ whose tangent spaces at $\tilde{P}$ are $\operatorname{Ker} G'_-(\tilde{P})$ and $\operatorname{Ker} G'_+(\tilde{P})$, respectively. Therefore (B.62) holds if

$$\text{(B.65)} \qquad \operatorname{Ker} F'(\tilde{P}) \text{ is not a subset of } \operatorname{Ker} G'_-(\tilde{P}) \cup \operatorname{Ker} G'_+(\tilde{P}).$$

Property (B.65) follows from (B.61), (B.63), and (B.64). This concludes the proof of Proposition 3.7. □

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