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Abstract. Motivated by the Broué conjecture on blocks with abelian defect groups for finite reductive groups, we study "parabolic" Deligne-Lusztig varieties and construct on those which occur in the Broué conjecture an action of a braid monoid, whose action on their ℓ-adic cohomology will conjecturally factor through a cyclotomic Hecke algebra. In order to construct this action, we need to enlarge the set of varieties we consider to varieties attached to a "ribbon category"; this category has a *Garside family*, which plays an important role in our constructions, so we devote the first part of our paper to the necessary background on categories with Garside families.

1. Introduction

In this paper, we study "parabolic" Deligne-Lusztig varieties, one of the main motivations being the Broué conjecture on blocks with abelian defect groups for finite reductive groups.

Let $\mathbf{G}$ be a connected reductive algebraic group over an algebraic closure $\overline{\mathbb{F}}_p$ of the prime field $\mathbb{F}_p$ of characteristic p. Let F be an isogeny on $\mathbf{G}$ such that some power F^δ is a Frobenius endomorphism attached to a split structure over the finite field $\mathbb{F}_{q^\delta}$; this defines a positive real number q such that q^δ is an integral power of p. When $\mathbf{G}$ is quasi-simple, any isogeny F such that the group of fixed points $\mathbf{G}^F$ is finite is of the above form; such a group $\mathbf{G}^F$ is called a "finite reductive group" or a "finite group of Lie type".

Let $\mathbf{L}$ be an F-stable Levi subgroup of a (non necessarily F-stable) parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Then, for ℓ a prime number different from p, Lusztig has constructed a "cohomological induction" $R_{\mathbf{L}}^{\mathbf{G}}$ which associates with any $\overline{\mathbb{Q}}_\ell\mathbf{L}^F$-module a virtual $\overline{\mathbb{Q}}_\ell\mathbf{G}^F$-module. We study the particular case $R_{\mathbf{L}}^{\mathbf{G}}(\mathrm{Id})$, which is given by the alternating sum of the ℓ-adic cohomology groups of the variety

$$\mathbf{X}_{\mathbf{P}} = \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g\mathbf{P} \cap F(g\mathbf{P}) \neq \emptyset\}$$

on which $\mathbf{G}^F$ acts on the left. We will construct a monoid of endomorphisms M of $\mathbf{X}_{\mathbf{P}}$ related to the braid group, which conjecturally will induce in some cases an action of a cyclotomic Hecke algebra on the cohomology of $\mathbf{X}_{\mathbf{P}}$. To construct M we need to enlarge the set of varieties we consider, to include varieties attached to morphisms in a "ribbon category" — the "parabolic Deligne-Lusztig varieties" of this paper; M corresponds to the endomorphisms in the "conjugacy category" of this ribbon category of the object attached to $\mathbf{X}_{\mathbf{P}}$.

The relationship with Broué's conjecture for the principal block comes as follows: assume, for some prime number $\ell \neq p$, that a Sylow ℓ-subgroup S of $\mathbf{G}^F$ is abelian. Then Broué's conjecture [Br1] predicts in this special case an equivalence

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of derived categories between the principal block of $\overline{\mathbb{Z}}_\ell \mathbf{G}^F$ and that of $\overline{\mathbb{Z}}_\ell N_{\mathbf{G}^F}(S)$. Now $\mathbf{L} := C_{\mathbf{G}}(S)$ is a Levi subgroup of a (non F-stable unless $\ell|q-1$) parabolic subgroup $\mathbf{P}$; restricting to unipotent characters and discarding an eventual torsion by changing coefficients from $\overline{\mathbb{Z}}_\ell$ to $\overline{\mathbb{Q}}_\ell$, this translates after refinement (see [BM]) into conjectures about the cohomology of $\mathbf{X}_\mathbf{P}$ (see 9.1); these conjectures predict that the image in the cohomology of our monoid M is a cyclotomic Hecke algebra.

The main feature of the ribbon categories we consider is that they have *Garside families*. This concept has appeared in recent work to understand the ordinary and dual monoids attached to the braid groups; in the first part of this paper, we recall its basic properties and go as far as computing the centralizers of "periodic elements", which is what we need in the applications. The reader who wants to avoid the general theory of Garside families can try to read only Section 5 where we spell out the results in the case of Artin monoids.

In the second part, we first define the parabolic Deligne-Lusztig varieties which are the aim of our study, and then go on to establish their properties. We extend to this setting in particular all the material in [BM] and [BR2].

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After this paper was written, we received a preprint from Xuhua He and Sian Nie (see [HN]) where, amidst other interesting results, they also prove Theorem 8.1.

I. Garside families

This part collects some prerequisites on categories with Garside families. It is mostly self-contained apart from the next section where the proofs are omitted; we refer for them to [DDM] or the book [DDGKM] in preparation.

2. Basic results on Garside families

Given a category $\mathcal{C}$, we write $f \in \mathcal{C}$ to say that f is a morphism of $\mathcal{C}$, and we write $\mathcal{C}(x,y)$ (resp. $\mathcal{C}(x,\text{-})$, resp. $\mathcal{C}(\text{-},y)$) for the set of morphisms from $x \in \operatorname{Obj}\mathcal{C}$ to $y \in \operatorname{Obj}\mathcal{C}$ (resp. the set of morphisms with source x, resp. the set of morphisms with target y). We write fg for the composition of $f \in \mathcal{C}(x,y)$ and $g \in \mathcal{C}(y,z)$, and $\mathcal{C}(x)$ for $\mathcal{C}(x,x)$. By $\mathcal{S} \subset \mathcal{C}$ we mean that $\mathcal{S}$ is a set of morphisms in $\mathcal{C}$.

Recall that a category is cancellative if each one of the relations $hf = hg$ or $fh = gh$ implies $f = g$; equivalently every morphism is a monomorphism and an epimorphism. We say that f left-divides g, or equivalently that g is a right-multiple of f, written $f \preccurlyeq g$, if there exists h such that $g = fh$; in this situation since the category is cancellative h is uniquely defined by g and f and we write $h = f^{-1}g$. Similarly we say that f right-divides g, or that g is a left-multiple of h and write $g \succcurlyeq f$ if there exists h such that $g = hf$.

We denote by $\mathcal{C}^\times$ the set of invertible morphisms of $\mathcal{C}$, and write $f =^\times g$ if there exists $h \in \mathcal{C}^\times$ such that $fh = g$ (or equivalently there exists $h \in \mathcal{C}^\times$ such that $f = gh$).

Definition 2.1. *In a cancellative category $\mathcal{C}$ a Garside family is a subset $\mathcal{S} \subset \mathcal{C}$ such that;*

(i) *$\mathcal{S}$ together with $\mathcal{C}^\times$ generates $\mathcal{C}$, and $\mathcal{C}^\times\mathcal{S} \subset \mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$.*

(ii) *For every product fg with $f, g \in \mathcal{S}$, we can write $fg = f_1 g_1$ with $f_1, g_1 \in \mathcal{S}$ such that if for $k \in \mathcal{C}$ and $h \in \mathcal{S}$ we have $h \preccurlyeq kfg$ then $h \preccurlyeq kf_1$.*

If item (ii) of the above definition holds we say that the 2-term sequence (f_1, g_1) is an $\mathcal{S}$-normal decomposition of fg. We extend this notion first to the case where $f, g \in \mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ by requiring the same condition but with $f_1, g_1 \in \mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$; we extend then $\mathcal{S}$-normal decompositions to longer lengths by saying that $(x_1, \ldots, x_n)$ is an $\mathcal{S}$-normal decomposition of $x = x_1 \ldots x_n$ if for each i the sequence (x_i, x_{i+1}) is an $\mathcal{S}$-normal decomposition. We finally extend it to elements $x \in \mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ by saying that (x) is an $\mathcal{S}$-normal decomposition.

In a cancellative category with a Garside family every element x admits an $\mathcal{S}$-normal decomposition. We will just say "normal decomposition" if $\mathcal{S}$ is clear from the context. A normal decomposition $(x_1, \ldots, x_n)$ is *strict* if no entry is invertible and all entries excepted possibly x_n are in $\mathcal{S}$. In a cancellative category with a Garside family every non-invertible element admits a strict $\mathcal{S}$-normal decomposition.

Normal decompositions are unique up to invertible elements, precisely

Lemma 2.2 ([DDM, 2.11])**.** *If $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_{n'})$ with $n \leq n'$ are two normal decompositions of x then for any $i \leq n$ we have $x_1 \cdots x_i =^\times x'_1 \cdots x'_i$ and for $i > n$ we have $x'_i \in \mathcal{C}^\times$.*

Head functions.

Definition 2.3. *Let $\mathcal{C}$ be a cancellative category and let $\mathcal{S} \subset \mathcal{C}$. Then we say that a function $\mathcal{C} - \mathcal{C}^\times \xrightarrow{H} \mathcal{S}$ is an $\mathcal{S}$-head function if for any $h \in \mathcal{S}$, we have $h \preccurlyeq g \Leftrightarrow h \preccurlyeq H(g)$.*

We say that a subset $\mathcal{S} \subset \mathcal{C}$ is closed under right-divisor if $f \succcurlyeq g$ with $f \in \mathcal{S}$ implies $g \in \mathcal{S}$. We have the following criterion to be Garside:

Proposition 2.4 (see [DDM, 3.10 and 3.34])**.** *Assume that $\mathcal{C}$ is a cancellative category and that $\mathcal{S} \subset \mathcal{C}$ together with $\mathcal{C}^\times$ generates $\mathcal{C}$. Consider the following property for an $\mathcal{S}$-head function:*

$$(\mathcal{H}) \qquad \forall f \in \mathcal{C}, \forall g \in \mathcal{C} - \mathcal{C}^\times, H(fg) =^\times H(fH(g)).$$

Then $\mathcal{S}$ is Garside if there exists an $\mathcal{S}$-head function satisfying $(\mathcal{H})$ or there exists an $\mathcal{S}$-head function and $\mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ is closed under right-divisor. Conversely if $\mathcal{S}$ is Garside then $\mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ is closed under right-divisor and any $\mathcal{S}$-head function satisfies $(\mathcal{H})$.

An $\mathcal{S}$-head function H computes the first term of a normal decomposition in the sense that if $(x_1, \ldots, x_n)$ is a normal decomposition of $x \in \mathcal{C} - \mathcal{C}^\times$ then $H(x) =^\times x_1$. Further any $x \in \mathcal{C} - \mathcal{C}^\times$ has a strict normal decomposition $(x_1, \ldots, x_n)$ with $H(x) = x_1$.

Let $\mathcal{C}$ be a cancellative category with a Garside family $\mathcal{S}$. For $f \in \mathcal{C}$ we define $\lg_{\mathcal{S}}(f)$ to be the minimum number k of morphisms $s_1, \ldots, s_k \in \mathcal{S}$ such that $s_1 \cdots s_k =^\times f$, thus $\lg_{\mathcal{S}}(f) = 0$ if $f \in \mathcal{C}^\times$; if $f \notin \mathcal{C}^\times$ then $\lg_{\mathcal{S}}(f)$ is also the number of terms in a strict normal decomposition of f.

The following shows that $\mathcal{S}$ "determines" $\mathcal{C}$ up to invertible elements; we say that a subset $\mathcal{C}_1$ of $\mathcal{C}$ is closed under right-quotient if an equality $f = gh$ with $f, g \in \mathcal{C}_1$ implies $h \in \mathcal{C}_1$.

Lemma 2.5 ([DDGKM, VII 2.13]). *Let $\mathcal{C}_1$ be a subcategory of $\mathcal{C}$ closed under right-quotient which contains $\mathcal{S}$. Then $\mathcal{C} = \mathcal{C}_1\mathcal{C}^\times \cup \mathcal{C}^\times$ and $\mathcal{S}$ is a Garside family in $\mathcal{C}_1$.*

Categories with automorphism. Most categories we want to consider will have no non-trivial invertible element, which simplifies Definition 2.1(i) to "$\mathcal{S}$ generates $\mathcal{C}$". The only source of invertible elements will be the following construction.

An automorphism of a category $\mathcal{C}$ is a functor $F : \mathcal{C} \to \mathcal{C}$ which has an inverse. Given such an automorphism we define

Definition 2.6. *The semi-direct product category $\mathcal{C} \rtimes \langle F\rangle$ is the category whose objects are the objects of $\mathcal{C}$ and whose morphisms are the pairs (g, F^i), which will be denoted by gF^i, where $g \in \mathcal{C}$ and i is an integer. The source of gF^i is* $\operatorname{source}(g)$ *and the target of gF^i is $F^{-i}(\operatorname{target}(g))$. The composition rule is given by $gF^i \cdot hF^j = gF^i(h)F^{i+j}$ when* $\operatorname{source}(h) = \operatorname{target}(gF^i)$.

Note that we do *not* identify (g, F^i) and (g, F^j) even when F^{i-j} is the identity functor — it will be convenient in our semi-direct products to have the cyclic group generated by F to be infinite even though F acts via a finite order automorphism.

The conventions on F are such that the composition rule is natural. However, they imply that the morphism (Id, F) of the semi-direct product category represents the functor F^{-1}: it is a morphism from the object $F(A)$ to the object A and we have the commutative diagram:

$$
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
\downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\
A & \xrightarrow{\quad f \quad} & B
\end{array}
$$

$\mathcal{C}$ embeds in $\mathcal{C} \rtimes \langle F\rangle$ by identifying g and (g, F^0).

Lemma 2.7 ([DDGKM, VIII 1.34 (ii)]). *If $\mathcal{S}$ is a Garside family in the cancellative category $\mathcal{C}$, and F an automorphism of $\mathcal{C}$ preserving $\mathcal{S}$, then $\mathcal{S}$ is also a Garside family in $\mathcal{C} \rtimes \langle F\rangle$.*

If $(f_1, \dots f_k)$ is an $\mathcal{S}$-normal decomposition of $f \in \mathcal{C}$ then $(f_1, \dots, f_kF^i)$ is an $\mathcal{S}$-normal decomposition of $fF^i \in \mathcal{C} \rtimes \langle F\rangle$. Note that if $\mathcal{C}$ has no non-trivial invertible element, then the only invertible elements in $\mathcal{C} \rtimes \langle F\rangle$ are $\{F^i\}_{i\in\mathbb{Z}}$. In general, if $a, b \in \mathcal{C}$ then $aF^i \preccurlyeq bF^j$ if and only if $a \preccurlyeq b$.

We have the following property

Proposition 2.8 ([DDGKM, VII 4.4]). *Assume that the cancellative category $\mathcal{C}$ has a Garside family $\mathcal{S}$ and has no non-trivial invertible morphisms. Let F be an automorphism of $\mathcal{C}$ preserving $\mathcal{S}$. Then the subcategory of fixed objects and morphisms $\mathcal{C}^F$ has a Garside family which consists of the fixed points $\mathcal{S}^F$.*

Gcds and lcms, Noetherianity. We call *right-lcm* of a family $\mathcal{C}_1 \subset \mathcal{C}$ a right-multiple f of all morphisms in $\mathcal{C}_1$ such that for any other common right-multiple f' we have $f \preccurlyeq f'$; this corresponds to the categorical notion of a pullback. Similarly a *left-gcd* of the family $\mathcal{C}_1$ is a common left-divisor f such that for any other common left-divisor f' we have $f' \preccurlyeq f$; it corresponds to the notion of a pushout. Left-lcms and right-gcds are defined in the same way exchanging left and right.

The existence of left-gcds and right-lcms are related when the cancellative category $\mathcal{C}$ is right-Noetherian, which means that there is no infinite sequence $f_0 \succcurlyeq f_1 \succcurlyeq \cdots \succcurlyeq f_n \succcurlyeq \cdots$ where f_{i+1} is a *proper* right-divisor of f_i, that is we do not have $f_i =^\times f_{i+1}$. It means equivalently since $\mathcal{C}$ is cancellative that there is no infinite sequence $g_0 \preccurlyeq g_1 \preccurlyeq \cdots \preccurlyeq g_n \preccurlyeq \cdots \preccurlyeq g$ where g_i is a proper left-divisor of g_{i+1}. The equivalence is obtained by $f_i = g_i^{-1} g$ and $g = f_0$. In a right-Noetherian category any element is right-divisible by an *atom*, which is an element which cannot be written as the product of two non-invertible elements. If the category is Noetherian (that is, both left and right-Noetherian) we have:

Proposition 2.9 ([DDGKM, II 2.64]). *A cancellative and Noetherian category is generated by its atoms and its invertible elements.*

We say that $\mathcal{C}$ admits conditional right-lcms if, whenever f and g have a common right-multiple, they have a right-lcm. We then have:

Proposition 2.10 ([DDGKM, II 2.41]). *If $\mathcal{C}$ is cancellative, right-Noetherian and admits conditional right-lcms, then any family of morphisms of $\mathcal{C}$ with the same source has a left-gcd.*

If $\mathcal{C}$ admits conditional right-lcms we say that a subset $X \subset \mathcal{C}$ is closed (resp. weakly closed) under right-lcm if whenever two elements of X have a right-lcm in $\mathcal{C}$ this lcm is in X (resp. in $X\mathcal{C}^\times$). If further X is closed under right-quotient an lcm in $\mathcal{C}$ which is in X is also an lcm in X. The following is proved in [DDM, Proposition 3.25] (where there is a Noetherianity assumption not used in the direct part of the proof).

Lemma 2.11. *If $\mathcal{S}$ is a Garside family in a category which admits conditional right-lcms then $\mathcal{S}\mathcal{C}^\times$ is closed under right-lcm.*

Here is a general situation when a Garside family of a subcategory can be determined.

Lemma 2.12 ([DDGKM, VII 1.10]). *Let $\mathcal{S}$ be a Garside family in $\mathcal{C}$ assumed cancellative, right-Noetherian and having conditional right-lcms. Let $\mathcal{S}_1 \subset \mathcal{S}$ be a subfamily such that $\mathcal{S}_1\mathcal{C}^\times \cup \mathcal{C}^\times$ is as a subset of $\mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ closed under right-lcm and right-quotient; then $\mathcal{S}_1$ is a Garside family in the subcategory $\mathcal{C}_1$ generated by $\mathcal{S}_1\mathcal{C}^\times$. Moreover $\mathcal{C}_1$ is a subcategory closed under right-quotient.*

Lemma 2.13 ([DDGKM, VII 1.18]). *Let M be a cancellative right-Noetherian monoid which admits conditional right-lcms and let M' be a submonoid of M closed under right-quotient and weakly closed under right-lcm. Then any $u \in M$ has a unique (up to right-multiplication by $M'^\times$) maximal left-divisor in M'.*

Garside maps. An important special case is when a Garside family $\mathcal{S}$ is attached to a Garside map. A Garside map is a map $\operatorname{Obj}\mathcal{C} \xrightarrow{\Delta} \mathcal{C}$ where $\Delta(x) \in \mathcal{C}(x, \text{-})$ such that the map $x \mapsto \operatorname{target}(\Delta(x))$ is injective and such that $\mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ is both the set of elements that left-divide some $\Delta(x)$ and the set of elements that right-divide some $\Delta(x)$.

This definition of a Garside map agrees with [DDGKM, V 2.30] if we take in account that, using the notation of loc. cit., the fact that $\mathcal{S}^\#$ is the set of left- and right-divisors of Δ implies that the Garside family $\mathcal{S}$ is bounded.

A Garside map allows to define a functor Φ, first on objects by taking for $\Phi(x)$ the target of $\Delta(x)$, then on morphisms, first on morphisms $s \in \mathcal{S}$ by, if $s \in \mathcal{C}(x, \text{-})$

defining s' by $ss' = \Delta$ (we omit the source of Δ if it is clear from the context) and then $\Phi(s)$ by $s'\Phi(s) = \Delta$. We then extend Φ by using normal decompositions; it can be shown that this is well-defined and defines a functor such that for any $f \in \mathcal{C}$ we have $f\Delta = \Delta\Phi(f)$. It can also be shown that the cancellativity of $\mathcal{C}$ implies that Φ is an automorphism.

The automorphism Φ is a typical automorphism of $\mathcal{C}$ preserving $\mathcal{S}$ that we will call the *Garside automorphism*.

If $\mathcal{S}$ is attached to a Garside map, we then have the following properties:

Proposition 2.14. (i) *If $f \preccurlyeq g$ then $\lg_{\mathcal{S}}(f) \le \lg_{\mathcal{S}}(g)$.*
(ii) *Assume $f, g, h \in \mathcal{S}$ and (f, g) is $\mathcal{S}$-normal; then $\lg_{\mathcal{S}}(fgh) \le 2$ implies $gh \in \mathcal{S}\mathcal{C}^\times$.*
(iii) *For $f \in \mathcal{C}(x, \text{-})$, the first term of an $\mathcal{S}$-normal decomposition of x is a left-gcd of f and $\Delta(x)$.*

Proof. (i) is [DDGKM, V 2.39 (v)], (iii) is [DDGKM, V 1.14]. (ii) is [DDGKM, IV 1.38] using [DDGKM, 2.15] which says, with the notation as in loc. cit., that $\mathcal{S}^\#$ est left-comultiple-closed. □

We will write Δ^p for the map which associates with an object x the morphism $\Delta(x)\Delta(\Phi(x))\cdots\Delta(\Phi^{p-1}(x))$. For any $f \in \mathcal{C}(x, \text{-})$ there exists p such that $f \preccurlyeq \Delta^p(x)$.

Proposition 2.15 ([DDGKM, III 1.37 and V 2.14])**.** *If $\mathcal{S}$ is a Garside family attached to a Garside map Δ then for any positive integer p, Δ^p is a Garside map and $\{f_1 f_2 \cdots f_p \mid f_i \in \mathcal{S}\}$ is a Garside family attached to Δ^p.*

3. The conjugacy category

The context for this section is a cancellative category $\mathcal{C}$.

Definition 3.1. *Given a category $\mathcal{C}$, we define the* conjugacy category $\operatorname{Conj}\mathcal{C}$ *of $\mathcal{C}$ as the category whose objects are the endomorphisms of $\mathcal{C}$ and where, for $w \in \mathcal{C}(A)$ and $w' \in \mathcal{C}(B)$ we set $\operatorname{Conj}\mathcal{C}(w, w') = \{x \in \mathcal{C}(A, B) \mid xw' = wx\}$. We say that x* conjugates *w to w' and call* centralizer *of w the set $\operatorname{Conj}\mathcal{C}(w)$. The composition of morphisms in $\operatorname{Conj}\mathcal{C}$ is given by the composition in $\mathcal{C}$, which is compatible with the defining relation for $\operatorname{Conj}\mathcal{C}$.*

Note that it is the formula for $\operatorname{Conj}\mathcal{C}(w, w')$ that forces the objects of $\operatorname{Conj}\mathcal{C}$ to be endomorphisms of $\mathcal{C}$.

Since $\mathcal{C}$ is cancellative, the data x and w determine w' (resp. x and w' determine w). This allows us to write w^x for w' (resp. ${}^x w'$ for w); this illustrates that our category $\operatorname{Conj}\mathcal{C}$ is a right-conjugacy category; we call left-conjugacy category the opposed category.

A proper notation for an element of $\operatorname{Conj}\mathcal{C}(w, \text{-})$ is a triple $w \xrightarrow{x} w^x$ (that we will abbreviate often to $x \xrightarrow{w} \text{-}$), since x by itself does not specify its source; but we will use just x when the context makes clear which source w is meant (or which target is meant). The forgetful functor which sends $w \in \operatorname{Obj}(\operatorname{Conj}\mathcal{C})$ to $\operatorname{source}(w)$ and $w \xrightarrow{x} \text{-}$ to x is faithful, though not injective on objects; it allows us to identify $\operatorname{Conj}\mathcal{C}(w, \text{-})$ with the subset $\{x \in \mathcal{C}(\operatorname{source}(w), \text{-}) \mid x \preccurlyeq wx\}$; similarly we may identify $\operatorname{Conj}\mathcal{C}(\text{-}, w)$ with the subset $\{x \in \mathcal{C}(\text{-}, \operatorname{source}(w)) \mid xw \succcurlyeq x\}$.

It follows that the category $\operatorname{Conj}\mathcal{C}$ inherits automatically from $\mathcal{C}$ properties such as cancellativity or Noetherianity. The forgetful functor maps $(\operatorname{Conj}\mathcal{C})^\times$ surjectively to $\mathcal{C}^\times$, so in particular the subset $\operatorname{Conj}\mathcal{C}(w,\text{-})$ of $\mathcal{C}(\operatorname{source}(w),\text{-})$ is closed under multiplication by $\mathcal{C}^\times$. In the proofs and statements which follow we identify $\operatorname{Conj}\mathcal{C}$ with a subset of $\mathcal{C}$ and $(\operatorname{Conj}\mathcal{C})^\times$ to $\mathcal{C}^\times$; for the statements obtained about $\operatorname{Conj}\mathcal{C}$ to make sense, the reader has to check that the sources and target of morphisms viewed as morphisms in $\operatorname{Conj}\mathcal{C}$ make sense.

Lemma 3.2. (i) *The subset* $\operatorname{Conj}\mathcal{C}$ *of* $\mathcal{C}$ *is closed under right-quotient.*
(ii) *The subset* $\operatorname{Conj}\mathcal{C}(w,\text{-})$ *of* $\mathcal{C}(\operatorname{source}(w),\text{-})$ *is closed under right-lcm. In particular if* $\mathcal{C}$ *admits conditional right-lcms then so does* $\operatorname{Conj}\mathcal{C}$.

Similarly $\operatorname{Conj}\mathcal{C}(\text{-},w)$ *is a subset of* $\mathcal{C}(\text{-},\operatorname{source}(w))$ *closed under left-lcm and left-quotient.*

Proof. We show (i). If $y = xz$ with $y \in \operatorname{Conj}\mathcal{C}(w,w')$, $x \in \operatorname{Conj}\mathcal{C}(w,\text{-})$ and $z \in \mathcal{C}(\text{-},\operatorname{source}(w'))$ we have $x \preccurlyeq wx$ and $yw' = wy$. By cancellation, let us define w'' by $xw'' = wx$. Now since $y = xz$ the equality $yw' = wy$ gives $xzw' = wxz = xw''z$ which gives by cancellation that $zw' = w''z$ showing that $z \in \operatorname{Conj}\mathcal{C}(\text{-},w')$.

We now show (ii). If $x, y \in \operatorname{Conj}\mathcal{C}(w,\text{-})$ then $x \preccurlyeq wx$ and $y \preccurlyeq wy$. Suppose now that x and y have a right-lcm z in $\mathcal{C}$. Then $x \preccurlyeq wz$ and $y \preccurlyeq wz$ from which it follows that $z \preccurlyeq wz$, that is $z \in \operatorname{Conj}\mathcal{C}(w,\text{-})$, thus z is the image by the forgetful functor of a right-lcm of x and y in $\operatorname{Conj}\mathcal{C}$.

The proof of the second part is just a mirror symmetry of the above proof. □

Proposition 3.3. *Assume that* $\mathcal{S}$ *is a Garside family in* $\mathcal{C}$*; then* $\operatorname{Conj}\mathcal{C} \cap \mathcal{S}$ *is a Garside family in* $\operatorname{Conj}\mathcal{C}$ *and* $\mathcal{S}$*-normal decompositions of an element of* $\operatorname{Conj}\mathcal{C}$ *are* $\operatorname{Conj}\mathcal{C} \cap \mathcal{S}$*-normal decompositions.*

Proof. We will use Proposition 2.4 by showing that $(\operatorname{Conj}\mathcal{C} \cap \mathcal{S}) \cup \mathcal{C}^\times$ generates $\operatorname{Conj}\mathcal{C}$ and exhibiting a $\operatorname{Conj}\mathcal{C} \cap \mathcal{S}$-head function $H : \operatorname{Conj}\mathcal{C} - \mathcal{C}^\times \to \operatorname{Conj}\mathcal{C} \cap \mathcal{S}$ satisfying $(\mathcal{H})$.

Let H be a $\mathcal{S}$-head function in $\mathcal{C}$. We first show that the restriction of H to $\operatorname{Conj}\mathcal{C}$ takes its values in $\operatorname{Conj}\mathcal{C} \cap \mathcal{S}$. Indeed if $x \preccurlyeq wx$ then $H(x) \preccurlyeq H(wx) =^\times H(wH(x)) \preccurlyeq wH(x)$ where the middle $=^\times$ is by $(\mathcal{H})$.

We now deduce by induction on $\lg_{\mathcal{S}}$ that $(\operatorname{Conj}\mathcal{C} \cap \mathcal{S}) \cup \mathcal{C}^\times$ generates $\operatorname{Conj}\mathcal{C}$. The induction starts with elements of length 0 which are exactly the elements of $\mathcal{C}^\times$. Assume now that $x \in \operatorname{Conj}\mathcal{C}$ is such that $\lg_{\mathcal{S}}(x) = n > 0$ and define x' by $x = H(x)x'$; since $H(x)$ can be taken as the first term of a strict normal decomposition we have $\lg_{\mathcal{S}}(x') = n-1$. Since we proved $H(x) \in \operatorname{Conj}\mathcal{C}$, we deduce by Lemma 3.2(i) that $x' \in \operatorname{Conj}\mathcal{C}$, whence the result by induction.

It is straightforward that the restriction of H to $\operatorname{Conj}\mathcal{C} - \mathcal{C}^\times$ is still a head function satisfying $(\mathcal{H})$, which proves that $\operatorname{Conj}\mathcal{C} \cap \mathcal{S}$ is a Garside family. The assertion about normal decompositions follows. □

Simultaneous conjugacy. A straightforward generalization of the conjugacy category is the "simultaneous conjugacy category", where objects are families of morphisms $w_1, \ldots, w_n$ with same source and target, and morphisms verify $x \preccurlyeq w_i x$ for all i. Most statements have a straightforward generalization to this case.

F-conjugacy. We want to consider "twisted conjugation" by an automorphism, which will be useful for applications to Deligne-Lusztig varieties, but also for internal applications, with the automorphism being the one induced by a Garside map.

Definition 3.4. *Let F be an automorphism of finite order of the category $\mathcal{C}$. We define the F-conjugacy category of $\mathcal{C}$, denoted by $F\text{-}\operatorname{Conj}\mathcal{C}$, as the category whose objects are the morphisms in some $\mathcal{C}(A,F(A))$ and where, for $w\in\mathcal{C}(A,F(A))$ and $w'\in\mathcal{C}(B,F(B))$ we set $F\text{-}\operatorname{Conj}\mathcal{C}(w,w')=\{x\in\mathcal{C}\mid xw'=wF(x)\}$. We say that x* F-conjugates *w to w' and we call* F-centralizer *of a morphism w of $\mathcal{C}$ the set $F\text{-}\operatorname{Conj}\mathcal{C}(w)$.*

Note that F-conjugacy specializes to conjugacy when $F=\mathrm{Id}$; again, it is the formula for $F\text{-}\operatorname{Conj}\mathcal{C}(w,w')$ which forces the objects of $F\text{-}\operatorname{Conj}\mathcal{C}$ to lie in some $\mathcal{C}(A,F(A))$.

The notion of F-conjugacy turns out to be a particular form of conjugacy in the semi-direct product category $\mathcal{C}\rtimes\langle F\rangle$; this is the same as the relation between twisted conjugacy classes in a group and conjugacy classes in cosets.

Consider the application which sends $w\in\mathcal{C}(A,F(A))\subset\operatorname{Obj}(F\text{-}\operatorname{Conj}\mathcal{C})$ to $wF\in(\mathcal{C}\rtimes\langle F\rangle)(A)\subset\operatorname{Obj}(\operatorname{Conj}(\mathcal{C}\rtimes\langle F\rangle))$. Since $x(w'F)=(wF)x$ is equivalent to $xw'=wF(x)$, this extends to a functor ι from $F\text{-}\operatorname{Conj}\mathcal{C}$ to $\operatorname{Conj}(\mathcal{C}\rtimes\langle F\rangle)$. This functor is clearly an isomorphism onto its image.

The image $\iota(\operatorname{Obj}(F\text{-}\operatorname{Conj}\mathcal{C}))$ is the subset of $\mathcal{C}\rtimes\langle F\rangle$ which consists of endomorphisms which lie in $\mathcal{C}F$; and $\iota(F\text{-}\operatorname{Conj}\mathcal{C})$ identifies via the forgetful functor with the subset $\operatorname{Conj}(\mathcal{C}\rtimes\langle F\rangle)\cap\mathcal{C}$ of $\mathcal{C}\rtimes\langle F\rangle$.

Remark that, since in $\operatorname{Conj}(\mathcal{C}\rtimes\langle F\rangle)$ there is no morphism between gF^i and $g'F^j$ when $i\neq j$, the full subcategory that we will denote by $\operatorname{Conj}(\mathcal{C}F)$ of $\operatorname{Conj}(\mathcal{C}\rtimes\langle F\rangle)$ whose objects are in $\mathcal{C}F$ is a union of connected components of $\operatorname{Conj}(\mathcal{C}\rtimes\langle F\rangle)$; thus many properties will transfer automatically from $\operatorname{Conj}(\mathcal{C}\rtimes\langle F\rangle)$ to $\operatorname{Conj}(\mathcal{C}F)$.

In particular, if $\mathcal{C}$ has a Garside family $\mathcal{S}$ and F is a Garside automorphism, then $\mathcal{S}$ is still a Garside family for $\mathcal{C}\rtimes\langle F\rangle$ by 2.7, and by Proposition 3.3 and the above remark gives rise to a Garside family $\mathcal{S}\cap\operatorname{Conj}(\mathcal{C}F)$ of $\operatorname{Conj}(\mathcal{C}F)$. The image $\iota(F\text{-}\operatorname{Conj}\mathcal{C})$ is the subcategory of $\operatorname{Conj}(\mathcal{C}F)$ consisting (via the forgetful functor) of the morphisms in $\mathcal{C}$, thus satisfies the assumptions of Lemma 2.5: it is closed under right-quotient, because in a relation $fg=h$ if f and h do not involve F the same must be true for g, and contains the Garside family $\mathcal{S}\cap\operatorname{Conj}(\mathcal{C}F)$ of $\operatorname{Conj}(\mathcal{C}F)$.

This will allow to generally translate statements about conjugacy categories to statements about F-conjugacy categories. For example, $\iota^{-1}(\mathcal{S}\cap\operatorname{Conj}(\mathcal{C}F))$ is a Garside family for $F\text{-}\operatorname{Conj}\mathcal{C}$; this last family is just $F\text{-}\operatorname{Conj}\mathcal{C}\cap\mathcal{S}$ when identifying $F\text{-}\operatorname{Conj}\mathcal{C}$ with a subset of morphisms of $\mathcal{C}$ by the forgetful functor.

The assumption that F acts through an automorphism of finite order is used as follows: since $(xF)^x=Fx=(xF)^{F^{-1}}$ and the action of F has finite order, two morphisms in $\mathcal{C}F$ are conjugate in $\mathcal{C}\rtimes\langle F\rangle$ if and only if they are conjugate by a morphism of $\mathcal{C}$.

The cyclic conjugacy category. A restricted form of conjugation called "cyclic conjugacy" will be important in applications. In particular, it turns out (a particular case of Proposition 3.9) that two periodic braids are conjugate if and only if they are cyclically conjugate. The context for this subsection is again a cancellative category $\mathcal{C}$.

Definition 3.5. *We define the cyclic conjugacy category* $\mathrm{cyc}\,\mathcal{C}$ *of* $\mathcal{C}$ *as the subcategory of* $\mathrm{Conj}\,\mathcal{C}$ *generated by* $\mathcal{S}' = \cup_w\{x \in \mathrm{Conj}\,\mathcal{C}(w,\text{-}) \mid x \preccurlyeq w\}$.

That is, $\mathrm{cyc}\,\mathcal{C}$ has the same objects as $\mathrm{Conj}\,\mathcal{C}$ but contains only the products of elementary conjugations of the form $w = xy \xrightarrow{x} yx$. Note that since $\mathcal{C}$ is cancellative $\cup_w\{x \in \mathrm{Conj}\,\mathcal{C}(w,w') \mid x \preccurlyeq w\} = \{x \in \mathrm{Conj}\,\mathcal{C}(\text{-},w') \mid w' \succcurlyeq x\}$ so cyclic conjugacy "from the left" and "from the right" are the same. To be more precise, the functor which is the identity on objects, and when $w = xy$ and $w' = yx$, sends $x \in \mathrm{cyc}\,\mathcal{C}(w,w')$ to $y \in \mathrm{cyc}\,\mathcal{C}(w',w)$, is an isomorphism between $\mathrm{cyc}\,\mathcal{C}$ and its opposed category.

Proposition 3.6. *Assume* $\mathcal{C}$ *is right-Noetherian and admits conditional right-lcms; if* $\mathcal{S}$ *is a Garside family in* $\mathcal{C}$ *then* $\mathcal{S}' \cap \mathcal{S}$ *is a Garside family in* $\mathrm{cyc}\,\mathcal{C}$.

Proof. Set $\mathcal{S}_1 = \mathcal{S}' \cap \mathcal{S}$. We first observe that $\mathcal{S}_1\mathcal{C}^\times \cup \mathcal{C}^\times$ generates $\mathrm{cyc}\,\mathcal{C}$. Indeed if $x \preccurlyeq w$ and we choose a decomposition $x = s_1 \cdots s_n$ as a product of morphisms in $\mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ it is clear that each s_i is in $\mathrm{cyc}\,\mathcal{C}$, so is in $\mathcal{S}_1\mathcal{C}^\times \cup \mathcal{C}^\times$.

The proposition then results from Lemma 2.12, which applies to $\mathrm{cyc}\,\mathcal{C}$ since $\mathcal{S}_1\mathcal{C}^\times \cup \mathcal{C}^\times$ is closed under right-divisor and right-lcm; this is obvious for right-divisor and for right-lcm results from the facts that $\mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ is closed under right-lcm by Lemma 2.11 and that a right-lcm of two divisors of w is a divisor of w. □

We see by Lemma 2.12 that $\mathrm{cyc}\,\mathcal{C}$ is closed under right-quotient in $\mathrm{Conj}\,\mathcal{C}$.

We now prove that $\mathcal{S}'$ — which does not depend on the existence of a Garside family $\mathcal{S}$ in $\mathcal{C}$ — is a Garside family attached to a Garside map; $\mathcal{S}'$ is usually larger than the Garside family $\mathcal{S}' \cap \mathcal{S}$ of Proposition 3.6, since it contains all left-divisors of w even if w is not in $\mathcal{S}$.

Proposition 3.7. *Assume* $\mathcal{C}$ *is right-Noetherian and admits conditional right-lcms; then* $\mathcal{S}'$ *is a Garside family in* $\mathrm{cyc}\,\mathcal{C}$ *attached to the Garside map* Δ *such that* $\Delta(w) = w \in \mathrm{cyc}\,\mathcal{C}(w)$*; the corresponding Garside automorphism* Φ *is the identity functor.*

Proof. The set $\mathcal{S}'$ generates $\mathrm{cyc}\,\mathcal{C}$ by definition of $\mathrm{cyc}\,\mathcal{C}$. It is closed under right-divisors since $xy \preccurlyeq w$ implies $x \preccurlyeq w$ so that w^x is defined and $y \preccurlyeq w^x$; since $\mathcal{C}$ is right-Noetherian and admits conditional right-lcms, any two morphisms of $\mathcal{C}$ with same source have a gcd by Proposition 2.10. We define a function $H : \mathrm{cyc}\,\mathcal{C} - \mathcal{C}^\times \to \mathcal{S}'$ by letting $H(x)$ be an arbitrarily chosen left-gcd of x and w if $x \in \mathrm{cyc}\,\mathcal{C}(w,\text{-})$. It is readily checked that H is an $\mathcal{S}'$-head function. We conclude by Proposition 2.4 that $\mathcal{S}'$ is a Garside family for $\mathrm{cyc}\,\mathcal{C}$. The set $\mathcal{S}'(w,\text{-})$ is the set of left-divisors of $w = \Delta(w)$; similarly $\mathcal{S}'(\text{-},w)$ is the set of right-divisors of $w = \Delta(w)$. Hence Δ is a Garside map in $\mathrm{cyc}\,\mathcal{C}$. The equation $xw^x = wx$ shows that Φ is the identity. □

We say that a subset $X \subset \mathcal{C}$ is closed under left-gcd if whenever two elements of X have a left-gcd in $\mathcal{C}$ this gcd is in X.

Proposition 3.8. *Assume* $\mathcal{C}$ *is right-Noetherian and admits conditional right-lcms; then the subcategory* $\mathrm{cyc}\,\mathcal{C}$ *of* $\mathrm{Conj}\,\mathcal{C}$ *is closed under left-gcd.*

Proof. Let $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_m)$ be $\mathcal{S}'$-normal decompositions respectively of $x \in \mathrm{cyc}\,\mathcal{C}(w,\text{-})$ and $y \in \mathrm{cyc}\,\mathcal{C}(w,\text{-})$.

We first prove that if $\gcd(x_1,y_1) \in \mathcal{C}^\times$ then $\gcd(x,y) \in \mathcal{C}^\times$ (here we consider left-gcds in $\mathrm{Conj}\,\mathcal{C}$). We proceed by induction on $\inf\{m,n\}$. We write Δ for $\Delta(w)$

when there is no ambiguity on the source w. Since x_n and y_m divide Δ, we get that $\gcd(x,y)$ divides

$$\begin{aligned}\gcd(x_1\cdots x_{n-1}\Delta, y_1\cdots y_{m-1}\Delta) &=^\times \gcd(\Delta x_1\cdots x_{n-1}, \Delta y_1\cdots y_{m-1})\\ &=^\times \Delta\gcd(x_1\cdots x_{n-1}, y_1\cdots y_{m-1}) =^\times \Delta = w,\end{aligned}$$

where the first equality uses that Φ is the identity and the third results from the induction hypothesis. So we get that $\gcd(x,y)$ divides w thus is in $\mathcal{S}'$; by the property of normal decompositions it thus divides x_1 and y_1, thus is in $\mathcal{C}^\times$.

We now prove the proposition. If $\gcd(x_1,y_1)\in\mathcal{C}^\times$ then $\gcd(x,y)\in\mathcal{C}^\times$ thus is in $\operatorname{cyc}\mathcal{C}$ and we are done. Otherwise let d_1 be a gcd of x_1 and y_1 and let $x^{(1)}, y^{(1)}$ be defined by $x = d_1x^{(1)}$, $y = d_1y^{(1)}$. Similarly let d_2 be a gcd of the first terms of a normal decomposition of $x^{(1)}$, $y^{(1)}$ and let $x^{(2)}$, $y^{(2)}$ be the remainders, etc... Since $\mathcal{C}$ is right-Noetherian the sequence $d_1, d_1d_2, \ldots$ of increasing divisors of x must stabilize at some stage k, which means that the corresponding remainders $x^{(k)}$ and $y^{(k)}$ have first terms of their normal decomposition coprime, so by the first part are themselves coprime. Thus $\gcd(x,y) =^\times d_1\cdots d_k \in \operatorname{cyc}\mathcal{C}$. □

We now give a quite general context where cyclic conjugacy coincides with conjugacy.

Proposition 3.9. *Let $\mathcal{C}$ be a right-Noetherian category with a Garside map Δ, and let x be an endomorphism of $\mathcal{C}$ such that for n large enough we have $\Delta \preccurlyeq x^n$. Then we have* $\operatorname{cyc}\mathcal{C}(x,\text{-}) = \operatorname{Conj}\mathcal{C}(x,\text{-})$.

Proof. We first show that the property $\exists n,\ \Delta\preccurlyeq x^n$ is stable by conjugacy. Indeed, if $u\in\operatorname{Conj}\mathcal{C}(x,\text{-})$ then there exists k such that $u\preccurlyeq\Delta^k$. Since $\Delta^{k+1}\preccurlyeq x^{n(k+1)}$, we have $u^{-1}\Delta^k\cdot\Delta\preccurlyeq u^{-1}x^{n(k+1)}$. If Φ is the Garside automorphism attached to Δ, we have $u^{-1}\Delta^k\cdot\Delta = \Delta\cdot\Phi(u^{-1}\Delta^k)$ thus $\Delta\preccurlyeq u^{-1}x^{n(k+1)}$. We deduce that $(x^u)^{n(k+1)} = (u^{-1}x\cdot u)^{n(k+1)} = u^{-1}x^{n(k+1)}\cdot u$ is divisible by Δ.

We prove then by Noetherian induction on f that $f\in\operatorname{Conj}\mathcal{C}(x,\text{-})$ implies $f\in\operatorname{cyc}\mathcal{C}(x,\text{-})$. This is true if f is invertible. Otherwise, write $f = u_1f_1$ with $u_1 = \gcd(f,x)$; then $u_1\in\operatorname{cyc}\mathcal{C}(x,x^{u_1})$. If we can prove that if $f\in\operatorname{Conj}\mathcal{C}(x,\text{-})$, $f\notin\mathcal{C}^\times$, then $\gcd(f,x)\notin\mathcal{C}^\times$, we will be done by Noetherian induction since we can write similarly $f_1 = u_2f_2,\ldots$ and the sequence $u_1,u_2,\ldots$ has to exhaust f.

Since as observed any $u\in\operatorname{Conj}\mathcal{C}(x,\text{-})$ divides some power of x (x^{nk} if $u\preccurlyeq\Delta^k$) it is enough to show that if $u\in\operatorname{Conj}\mathcal{C}(x,\text{-})$, $u\notin\mathcal{C}^\times$ and $u\preccurlyeq x^n$, then $\gcd(u,x)\notin\mathcal{C}^\times$. We do this by induction on n. From $u\in\operatorname{Conj}\mathcal{C}(x,\text{-})$ we have $u\preccurlyeq xu$, and from $u\preccurlyeq x^n$ we deduce $u\preccurlyeq x\gcd(u,x^{n-1})$. If $\gcd(u,x^{n-1})\in\mathcal{C}^\times$ then $u\preccurlyeq x$ and we are done: $\gcd(x,u) = u$. Otherwise let $u_1 = \gcd(u,x^{n-1})$. We have $u_1\preccurlyeq xu_1$, $u_1\notin\mathcal{C}^\times$ and $u_1\preccurlyeq x^{n-1}$ thus we are done by induction. □

The F-cyclic conjugacy. Let F be a finite order automorphism of the category $\mathcal{C}$. We define $F\text{-}\operatorname{cyc}\mathcal{C}$ as the subcategory of $F\text{-}\operatorname{Conj}\mathcal{C}$ generated by $\cup_w\{x\in F\text{-}\operatorname{Conj}\mathcal{C}(w,\text{-}) \mid x\preccurlyeq w\}$, or equivalently, since $\mathcal{C}$ is cancellative, by $\cup_{w'}\{x\in\operatorname{Conj}\mathcal{C}(\text{-},w') \mid w'\succcurlyeq F(x)\}$. By the functor ι, the morphisms in $F\text{-}\operatorname{cyc}\mathcal{C}(w,w')$ identify with the morphisms in $\operatorname{cyc}(\mathcal{C}\rtimes\langle F\rangle)(wF,w'F)$ which lie in $\mathcal{C}$. To simplify notation, we will denote by $\operatorname{cyc}\mathcal{C}(wF,w'F)$ this last set of morphisms. If $\mathcal{C}$ is right-Noetherian and admits conditional right-lcms, then so does $\mathcal{C}\rtimes\langle F\rangle$. If $\mathcal{S}$ is a Garside family in $\mathcal{C}$ and F is an automorphism preserving $\mathcal{S}$, and we translate Proposition 3.6 to the image of ι and then to $F\text{-}\operatorname{cyc}\mathcal{C}$, we get that $\cup_w\{x \in F\text{-}\operatorname{Conj}\mathcal{C}(w,\text{-}) \mid x \preccurlyeq w \text{ and } x \in \mathcal{S}\}$ is a Garside family in $F\text{-}\operatorname{cyc}\mathcal{C}$.

Similarly Proposition 3.7 says that the set $\cup_w\{x \in F\text{-}\operatorname{Conj}\mathcal{C}(w,\text{-}) \mid x \preccurlyeq w\}$ is a Garside family in $F\text{-}\operatorname{cyc}\mathcal{C}$ attached to the Garside map Δ which sends the object w to the morphism $w \in F\text{-}\operatorname{cyc}\mathcal{C}(w, F(w))$; the associated Garside automorphism is the functor F.

Finally Proposition 3.8 says that under the assumptions of Proposition 3.7 the subcategory $F\text{-}\operatorname{cyc}\mathcal{C}$ of $F\text{-}\operatorname{Conj}\mathcal{C}$ is closed under left-gcd.

Periodic elements.

Definition 3.10. *Let $\mathcal{C}$ be a cancellative category with a Garside family $\mathcal{S}$ attached to Garside map Δ; then an endomorphism f of $\mathcal{C}$ is said to be (d,p)-*periodic *if $f^d \in \Delta^p\mathcal{C}^\times$ for some positive integers d,p.*

Note that if f is (d,p)-periodic it is also (nd,np)-periodic for any non-zero integer n; conversely, up to cyclic conjugacy, a (nd,np)-periodic element is (d,p)-periodic, see 3.12. We call d/p the *period* of f. If the Garside automorphism Φ given by Δ is of finite order, then a conjugate of a periodic element is periodic of the same period, though the minimal pair (d,p) may change. Our interest in periodic elements comes mainly from the fact that one can describe their centralizers, which is related to the fact that by Proposition 3.9 two periodic morphisms are conjugate if and only if they are cyclically conjugate.

We deal first with the case $p=2$, where we show by elementary computations the following:

Lemma 3.11. *Let f be a $(d,2)$-periodic element of $\mathcal{C}$ and let $e = \lfloor \frac{d}{2} \rfloor$. Then f is cyclically conjugate to a $(d,2)$-periodic element g such that $g^e \in \mathcal{S}\mathcal{C}^\times$.*

Further, if g is a $(d,2)$-periodic element such that $g^e \in \mathcal{S}\mathcal{C}^\times$, then

- *if d is even g is $(d/2,1)$-periodic.*
- *if d is odd and we define $h \in \mathcal{S}\mathcal{C}^\times$ by $g^e h = \Delta$ and $\varepsilon \in \mathcal{C}^\times$ by $g^d = \Delta^2\varepsilon$ then $g = h\Phi(h)\varepsilon$.*

Proof. We will prove by increasing induction on i that for $i \leq d/2$ there exists $v \in \operatorname{cyc}\mathcal{C}$ such that $(f^v)^i \in \mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ and $(f^v)^d \in \Delta^2\mathcal{C}^\times$. We start the induction with $i=0$ where the result holds trivially with $v=1$.

We consider now the general step: assuming the result for i such that $i+1 \leq d/2$, we will prove it for $i+1$. We thus have a v for step i, thus replacing f by f^v we may assume that $f^i \in \mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ and $f^d \in \Delta^2\mathcal{C}^\times$; we will conclude by finding $v \in \mathcal{S}$ such that $v \preccurlyeq f$, $(f^v)^{i+1} \in \mathcal{S}\mathcal{C}^\times$ and $(f^v)^d \in \Delta^2\mathcal{C}^\times$. If $f^{i+1} \preccurlyeq \Delta$ we have the desired result with $v=1$. We may thus assume that $\lg_{\mathcal{S}}(f^{i+1}) \geq 2$. Since $f^{i+1} \preccurlyeq \Delta^2$ we have actually $\lg_{\mathcal{S}}(f^{i+1}) = 2$ by Proposition 2.14(i); since f^i is in $\mathcal{S}\mathcal{C}^\times$ and divides f^{i+1}, a normal decomposition of f^{i+1} can be written (f^iv, w) with $f^iv, w \in \mathcal{S}\mathcal{C}^\times$. As $f^ivw \cdot f^iv \preccurlyeq f^ivw \cdot f^ivw = f^{2(i+1)} \preccurlyeq f^d =^\times \Delta^2$, we still have $2 = \lg_{\mathcal{S}}(f^iv \cdot w \cdot f^iv) = \lg_{\mathcal{S}}(f^iv \cdot w)$. By Proposition 2.14(ii) we thus have $w \cdot f^iv \in \mathcal{S}\mathcal{C}^\times$. Then $\mathcal{S}\mathcal{C}^\times \ni w \cdot f^iv = w(vw)^iv = (f^v)^{i+1}$ and $v \preccurlyeq f$.

So v will do if $(f^v)^d \in \Delta^2\mathcal{C}^\times$. Write $f^d = \Delta^2\varepsilon$ with $\varepsilon \in \mathcal{C}^\times$; then f commutes with $\Delta^2\varepsilon$, thus f^{i+1} also, which can be written $\Phi^2(f^{i+1})\varepsilon = \varepsilon f^{i+1}$ or equivalently $\Phi^2(f^iv)\Phi^2(w)\varepsilon = \varepsilon f^ivw$. Now since Φ preserves normal decompositions $(\Phi^2(f^iv), \Phi^2(w)\varepsilon)$ is a normal decomposition thus comparing with (f^iv, w) by Lemma 2.2 there exists $\varepsilon' \in \mathcal{C}^\times$ such that $\Phi^2(f^iv)\varepsilon' = \varepsilon f^iv$. Thus $f^i\Delta^2\Phi^2(v)\varepsilon' =$

$\Delta^2\Phi^2(f^iv)\varepsilon' = \Delta^2\varepsilon f^iv = f^i\Delta^2\varepsilon v$, the last equality using again that f commutes with $\Delta^2\varepsilon$. Canceling $f^i\Delta^2$ we get $\Phi^2(v)\varepsilon' = \varepsilon v$, whence $v(f^v)^d = f^dv = \Delta^2\varepsilon v = \Delta^2\Phi^2(v)\varepsilon' = v\Delta^2\varepsilon'$ whence the result by canceling v.

We prove now the second part. Since $g^e \in \mathcal{SC}^\times$ the element h defined by $g^eh = \Delta$ is in $\mathcal{SC}^\times \cup \mathcal{C}^\times$. Defining $\varepsilon \in \mathcal{C}^\times$ by $g^d = \Delta^2\varepsilon$ we get $g^eh\Delta\varepsilon = \Delta^2\varepsilon = g^d$, whence by cancellation $h\Delta\varepsilon = g^eg^a$ with $a = 1$ if d is odd and $a = 0$ if d is even. Using $h\Delta\varepsilon = \Delta\Phi(h)\varepsilon = g^eh\Phi(h)\varepsilon$ and canceling g^e we get $h\Phi(h)\varepsilon = g^a$.

If d is odd we get the statement of the lemma, and if d is even we get $h\Phi(h) \in \mathcal{C}^\times$, so $h \in \mathcal{C}^\times$, so $g^e \in \Delta\mathcal{C}^\times$. □

We will need at one stage the following more general statement (see [DDGKM, VIII, 3.33]) whose proof uses an interpretation by Bestvina of normal decompositions as geodesics.

Theorem 3.12. *Let f_1 be a (d_1,k_1)-periodic element of $\mathcal{C}$; let $d = d_1/\gcd(d_1,k_1)$ and $k = k_1/\gcd(d_1,k_1)$; then f_1 is cyclically conjugate to a (d,k)-periodic element f. Further, write an equality $dk' = 1 + kd'$ in positive integers. Then f is cyclically conjugate to a (d,k)-periodic element g such that $g^{d'} \preccurlyeq \Delta^{k'}$. If we then define $g_1 \in \mathcal{C}$ by $g^{d'}g_1 = \Delta^{k'}$ then $(g_1\Phi^{k'})^d =^\times \Delta$ and $(g_1\Phi^{k'})^k =^\times g$ in $\mathcal{C} \rtimes \langle\Phi\rangle$.*

***F*-periodic elements.** Let us apply Lemma 3.11 to the case of a semi-direct product category $\mathcal{C} \rtimes \langle F\rangle$ where $\mathcal{C}$ is a cancellative category with a Garside family $\mathcal{S}$ attached to a Garside map Δ and F is an automorphism of finite order of $\mathcal{C}$ preserving $\mathcal{S}$; then $\mathcal{S}$ is still a Garside family of $\mathcal{C} \rtimes \langle F\rangle$. We assume further that $\mathcal{C}$ has no non-trivial invertible elements. Then a morphism $yF \in \mathcal{C}F$ is (d,p)-periodic if and only if $\operatorname{target}(y) = F(\operatorname{source}(y))$ and $(yF)^d = \Delta^pF^d$.

From Lemma 3.11 we can deduce:

Corollary 3.13. *Let $yF \in \mathcal{C}F$ be $(d,2)$-periodic and let $e = \lfloor\frac{d}{2}\rfloor$ and $\Lambda = \Phi F^{-e}$. Then*

(i) *If d is even, there exists an $(e,1)$-periodic element $xF \in \mathcal{C}F$ cyclically conjugate to yF. The centralizer of xF in $\mathcal{C}$ identifies with $\operatorname{cyc}\mathcal{C}(xF)$. Further, we may compute this centralizer in the category of fixed points $(\operatorname{cyc}\mathcal{C})^\Lambda$ since the morphisms in $\operatorname{cyc}\mathcal{C}(xF)$ are Λ-stable.*

(ii) *If d is odd, there exists a $(d,2)$-periodic element $xF \in \mathcal{C}F$ cyclically conjugate to yF such that $(xF)^e \preccurlyeq \Delta F^e$. The element s defined by $(xF)^es = \Delta F^e$ is such that, in the category $\mathcal{C} \rtimes \langle\Lambda\rangle$, we have $x\Lambda^2 = (s\Lambda)^2$ and $(s\Lambda)^d = \Delta\Lambda^d$. The centralizer of xF in $\mathcal{C}$ identifies with the $F^d\Phi^{-2}$-fixed points of $\operatorname{cyc}\mathcal{C}(s\Lambda)$.*

Note that 2.8 describes Garside families for the fixed point categories mentioned above.

Proof. Lemma 3.11 shows that yF is cyclically conjugate to a $(d,2)$-periodic element xF such that $(xF)^e \in \mathcal{S}F^e$.

If d is even Lemma 3.11 says that xF is $(e,1)$-periodic, and Proposition 3.9 says that the centralizer of xF is $\operatorname{cyc}\mathcal{C}(xF)$. The elements of this centralizer, commuting to xF, commute to $(xF)^e = \Delta F^e$ thus are $\Phi^{-1}F^e$-stable.

If d is odd Lemma 3.11 says that if $(xF)^eh = \Delta$ then $xF = h\Phi(h)F^d$. Since $h = sF^{-e}$ we get $x = sF^{-e}\Phi(sF^{-e})F^{d-1} = s\Lambda(s)$. This can be rewritten $x\Lambda^2 = (s\Lambda)^2$. Now since $\Delta F^es^{-1} = (xF)^e$ we get $(\Delta F^es^{-1})^d = \Delta^{2e}F^{de}$ which gives

$(\Lambda^{-1}s^{-1})^d\Delta^d = \Delta^{2e}\Lambda^{-d}$ and finally $(s\Lambda)^d = \Delta\Lambda^d$. The elements of $\operatorname{Conj}\mathcal{C}(xF)$ commute to $(xF)^e = \Delta F^e s^{-1}$ thus commute to $s\Lambda$ thus $\operatorname{Conj}\mathcal{C}(xF) \subset \operatorname{Conj}\mathcal{C}(s\Lambda)$. Note that the elements of $\operatorname{Conj}\mathcal{C}(xF)$ commute to $(xF)^d$ thus to $F^d\Phi^{-2}$. Using $x\Lambda^2 = (s\Lambda)^2$ we get $\operatorname{Conj}\mathcal{C}(s\Lambda) \subset \operatorname{Conj}\mathcal{C}(x\Lambda^2)$; but $x\Lambda^2 = xF(F^d\Phi^{-2})^{-1}$ so $\operatorname{Conj}\mathcal{C}(x\Lambda^2)^{F^d\Phi^{-2}} \subset \operatorname{Conj}\mathcal{C}(xF)$, whence the result using that by Proposition 3.9 we have $\operatorname{Conj}\mathcal{C}(s\Lambda) = \operatorname{cyc}\mathcal{C}(s\Lambda)$. □

We will apply 3.12 in the following particular form

Corollary 3.14. *Assume that F is of finite order and that $\Phi = \mathrm{Id}$. Then any periodic element of $\mathcal{C}F$ is conjugate to a (d,k)-periodic element $yF \in \mathcal{C}F$ where k is prime to d. Further for any choice of positive integers d' and k' with $dk' = 1 + kd'$, the element yF is cyclically conjugate to a (d,k)-periodic element xF satisfying $(xF)^{d'} \preccurlyeq \Delta^{k'}$. If we then define $x_1 \in \mathcal{C}$ by $(xF)^{d'}x_1F^{-d'} = \Delta^{k'}$ then $(x_1F^{-d'})^d = \Delta F^{-dd'}$ and $(x_1F^{-d'})^k = (xF)F^{-k'd}$.*

We have a partial converse:

Lemma 3.15. *Assume that F is of finite order and that $\Phi = \mathrm{Id}$. Let d, k, d', k' be positive integers such that $dk' = 1 + kd'$ with d' prime to the order of F. If $x_1 \in \mathcal{C}$ satisfies $(x_1F^{-d'})^d = \Delta F^{-dd'}$ then the element $xF \in \mathcal{C}F$ defined by $(x_1F^{-d'})^k = (xF)F^{-k'd}$ satisfies $(xF)^d = \Delta^k F^d$.*

Proof. The element $x_1F^{-d'}$ is $F^{-dd'}$-stable since $\Phi = \mathrm{Id}$ and $(x_1F^{-d'})^d = \Delta F^{-dd'}$. Since d' is prime to the order of F an element $F^{-dd'}$-stable is F^d-stable. Thus, raising the equality $(x_1F^{-d'})^k = (xF)F^{-k'd}$ to the d-th power we get $(xF)^d = (x_1F^{-d'})^{dk}F^{k'd^2} = (\Delta F^{-dd'})^k F^{k'd^2} = \Delta^k F^d$. □

The following lemma shows that we can always choose d' satisfying the assumption of lemma 3.15.

Lemma 3.16. *Given k and d coprime natural integers, and an integer δ, there exists natural integers d', k' such that $dk' = 1 + kd'$ with d' prime to δ.*

Proof. k' and d' exist since k and d are coprime; we may change d' by any multiple of d. Thus it is sufficient to show that given coprime integers d and d', we may choose a such that $d'+ad$ is prime to any given δ. Let $p_1, \ldots, p_n$ be the prime factors of δ. We have to choose a such that $d' + ad$ is nonzero mod each p_i. If $p_i|d$ this is automatic. If p_i is prime to d we have to avoid $a \equiv -d'/d \pmod{p_i}$; by the Chinese remainder theorem we can choose a to avoid this finite set of congruences. □

4. An example: ribbon categories

An example of a category with a Garside family is a *Garside monoid*, which is just the case where $\mathcal{C}$ has one object. In this case we will say Garside element instead of Garside map.

Example 4.1. A classical example is given by the *Artin monoid* $(B^+, \mathbf{S})$ associated with a Coxeter system (W, S). If the presentation of W is

$$W = \langle S \mid s^2 = 1, \underbrace{sts\cdots}_{m_{s,t}} = \underbrace{tst\cdots}_{m_{s,t}} \text{ for } s,t \in S\rangle$$

then B^+ is defined by the presentation $B^+ = \langle \mathbf{S} \mid \underbrace{\mathbf{sts}\cdots}_{m_{s,t}} = \underbrace{\mathbf{tst}\cdots}_{m_{s,t}} \text{ for } \mathbf{s},\mathbf{t} \in \mathbf{S}\rangle$ where $\mathbf{S}$ is a copy of S; the group with the same presentation is the Artin group B. There is an obvious quotient $B^+ \to W$ since the relations of B^+ hold in W. Matsumoto's lemma stating that two reduced expressions for an element of W can be related by using only braid relations implies that there is a well-defined section $W \mapsto \mathbf{W}$ of the quotient $B^+ \to W$ which maps a reduced expression $s_1 \cdots s_n$ to the product $\mathbf{s}_1 \cdots \mathbf{s}_n \in B^+$. The monoid B^+ is cancellative, Noetherian, admits conditional left-lcms and right-lcms; the set $\mathbf{S}$ is the set of atoms of B^+ and $\mathbf{W}$ is a Garside family in B^+ (for details, see [DDM, 6.27]). The Garside family $\mathbf{W}$ is attached to a Garside element if and only if W is finite. In this case we call B^+ *spherical*. The Garside element is the lift to $\mathbf{W}$ of the longest element w_0 of W; it will be written $\mathbf{w}_0$ or Δ depending on the context.

Finally, an automorphism ϕ of (W, S) (that is, an automorphism of W which preserves S) extends naturally to an automorphism of $(B^+, \mathbf{S})$ given by $\mathbf{s} \mapsto \phi(\mathbf{s})$ which preserves the Garside family $\mathbf{W}$.

Example 4.2. Another example, attached to the same Artin braid group B as the above example, is the *dual braid monoid* introduced by David Bessis (see [B1]), whose construction can be extended to well-generated finite complex reflection groups.

The constructions of this section apply to the study, in the semi-direct product of an Artin monoid $(B^+, \mathbf{S})$ by an automorphism stabilizing $\mathbf{S}$, of the conjugates and normalizer of a "parabolic" submonoid — the submonoid generated by a subset of the atoms $\mathbf{S}$. The "ribbon category" that we consider occurs, when the automorphism is the identity, in the work of Paris [Pa] and Godelle [G] on this topic. In Section 7 we will attach parabolic Deligne-Lusztig varieties to elements of the ribbon category and endomorphisms of these varieties to elements of the conjugacy category of this ribbon category.

The next proposition gives a list of properties that spherical Artin monoids satisfy; the rest of the section describes ribbons in an arbitrary monoid satisfying the same properties, which includes the case of the dual braid monoid; this is a motivation for giving the results in a more general context. Before stating this proposition, we need a definition.

Definition 4.3. *We say that a set* $\mathbf{I}$ *of atoms of a cancellative monoid* M *is* parabolic *if the submonoid* $M_{\mathbf{I}}$ *of* M *generated by* $\mathbf{I}$ *is closed under right-quotient and weakly closed under right-lcm.*

Note that a monoid generated by a set $\mathbf{I}$ of atoms has no non-trivial invertible elements, since such an element would be a product of atoms and an atom is not invertible. Similarly, since an atom cannot be a product of several atoms, we see that $\mathbf{I}$ is the whole set of atoms of the monoid.

Proposition 4.4. *Let* $M = B^+ \rtimes \langle\phi\rangle$ *be the semi-direct product of a spherical Artin monoid by a diagram automorphism (see 4.1); then*

(i) M *is cancellative, right-Noetherian and admits conditional right-lcms.*
(ii) *There exists a finite set* $\mathbf{S} \subset M$ *which is a transversal of the* $=^\times$*-classes of atoms in* M*, and together with* $M^\times$ *generates* M*.*
(iii) *Any conjugate in* M *of an element of* $\mathbf{S}$ *is in* $\mathbf{S}$*.*

(iv) *M has a Garside family $\mathcal{S}$ attached to a Garside element Δ.*
(v) *For any parabolic subset $\mathbf{I}$ of $\mathbf{S}$, the maximal divisor $\Delta_{\mathbf{I}}$ of Δ given by Lemma 2.13 (which is unique since $M_{\mathbf{I}}^{\times} = \{1\}$) is a Garside element in $M_{\mathbf{I}}$, and $\mathcal{S} \cap M_{\mathbf{I}}$ is a Garside family attached to $\Delta_{\mathbf{I}}$.*
(vi) *For any parabolic subset $\mathbf{I} \subset \mathbf{S}$ and any $\mathbf{s} \in \mathbf{S} - \mathbf{I}$ there exists a parabolic subset $\mathbf{J}$ such that $\Delta_{\mathbf{J}}$ is the right-lcm of $\mathbf{s}$ and $\Delta_{\mathbf{I}}$.*

Proof. Let us prove (i). The monoid M is cancellative since it embeds in the semi-direct product of the Artin group by ϕ. Similarly it inherits from B^+ Noetherianity and the Garside family $\mathbf{W}$, which implies that it admits conditional right-lcms.

We prove (ii). Take for $\mathbf{S}$ the set of atoms of M. An invertible element must have length 0, hence the powers of ϕ are the only invertible elements. The atoms are the elements of length 1 that is the elements of $\mathbf{S}\langle\phi\rangle$, thus $\mathbf{S}$ is indeed a transversal of the atoms.

For (iii), we have to check that if we have $\mathbf{s}f = f\mathbf{t}$ with $\mathbf{s} \in \mathbf{S}$ and f and $\mathbf{t}$ in M then $\mathbf{t} \in \mathbf{S}$. Taking lengths we see that the length of $\mathbf{t}$ is 1 so that $\mathbf{t} = \mathbf{s}'\phi^k$ for some integer k and some $\mathbf{s}' \in \mathbf{S}$. Looking then at the powers of ϕ on both sides we get $k = 0$.

For (iv), take $\Delta = \mathbf{w}_0$. We have seen in Example 4.1 that (using the notation of loc. cit.) the lift $\mathbf{w}_0$ to $\mathbf{W}$ of the longest element w_0 of W is a Garside element in B^+. Hence $\Delta = \mathbf{w}_0$ is a Garside element in M by Lemma 2.7. We take $\mathcal{S} = \mathbf{W}$; it is a Garside family attached to Δ.

For (v) we notice first that $M_{\mathbf{I}}$, being generated by atoms, has no non-trivial invertible elements.

Before proving the rest, let us state the following (the fact that this fails in dual braid monoids is a motivation for defining parabolic subsets).

Lemma 4.5. *Any subset of $\mathbf{S}$ is parabolic.*

Proof. We show that $M_{\mathbf{I}}$ is closed under right-quotient. Since both sides of each defining relation for an Artin monoid involve the same elements of $\mathbf{S}$, two equivalent words for an element $v \in M$ involve the same subset of the generating set $\mathbf{S}$; we call this subset the *support* of v. Hence if $xy = z$ with $x, z \in M_{\mathbf{I}}$ then the power of ϕ in y is 0 and the support of y is a subset of that of z, thus a subset of $\mathbf{I}$, thus y is in $M_{\mathbf{I}}$.

We now show that $M_{\mathbf{I}}$ is weakly closed under right-lcms. Keeping the notations of 4.1, B^+ is associated to the Coxeter system (W, S). Since $M_{\mathbf{I}}$ is a spherical Artin monoid associated with the Coxeter subgroup W_I of W generated by the image in W of $\mathbf{I}$ (see for example [Pa, 3.1]) two elements of $M_{\mathbf{I}}$ have a right-lcm in $M_{\mathbf{I}}$. This right-lcm is left-divisible by any of their right-lcms in M, so has to be equal to one of these lcms since $M_{\mathbf{I}}$ is obviously stable by left-divisor. □

Since by $M_{\mathbf{I}}$ is a spherical Artin monoid it has a Garside element $\mathbf{w}_{\mathbf{I}}$, the lift of the longest element of W_I. The corresponding Garside family is $\mathbf{W}_{\mathbf{I}} = \mathbf{W} \cap M_{\mathbf{I}}$, that is the set of divisors in $M_{\mathbf{I}}$ of Δ which by definition of $\Delta_{\mathbf{I}}$ are the left-divisors of $\Delta_{\mathbf{I}}$. We get that $\mathbf{w}_{\mathbf{I}}$ and $\Delta_{\mathbf{I}}$ have the same set of left-divisors, so are equal since $M_{\mathbf{I}}^{\times} = \{1\}$.

We finally show (vi). We take $\mathbf{J} = \mathbf{I} \cup \{\mathbf{s}\}$. The following lemma applied with $\mathbf{S} = \mathbf{I}$ (resp. $\mathbf{S} = \mathbf{J}$) gives that $\Delta_{\mathbf{I}}$ is a right-lcm of $\mathbf{I}$ (resp. $\Delta_{\mathbf{J}}$ is a right-lcm of $\mathbf{J}$). We thus get the result by associativity of the lcm.

Lemma 4.6. *The Garside element* $\Delta = \mathbf{w}_0$ *of* B^+ *is the right-lcm of* $\mathbf{S}$.

Proof. By [DDM, 6.27] a common multiple of $\mathbf{S}$ in $\mathbf{W}$ corresponds to an element $w \in W$ such that $l(sw) < l(w)$ for all $s \in S$. It is well known that only w_0 satisfies this, so $\Delta = \mathbf{w}_0$ is the only element of $\mathbf{W}$ multiple of all the atoms. □

□

The category $\operatorname{Conj}(M, \mathcal{I})$**.** Until the end of Section 4, we fix a monoid M and a transversal $\mathbf{S}$ of its set of atoms; we assume that M has a Garside family $\mathcal{S}$ associated with a Garside element Δ so that these data satisfy properties (i) to (vi) of Proposition 4.4.

The reader only interested in internal applications to this paper can assume that we are in the case $M = B^+ \rtimes \langle\phi\rangle$, the semi-direct product of a spherical Artin monoid with a diagram automorphism (with $\mathbf{S}$ the usual atoms and Garside family $\mathcal{S} = \mathbf{W}$). Our results apply also to the case of dual Artin monoids, but this will not be used in this paper.

We fix also the conjugacy class $\mathcal{I}$ under M of a subset of $\mathbf{S}$. By property 4.4(iii) any element of $\mathcal{I}$ is a subset of $\mathbf{S}$. We assume all elements of this class are parabolic subsets (which is automatic in the ordinary Artin monoid case where all subsets are parabolic).

Let $\operatorname{Conj}(M, \mathcal{I})$ be the connected component of the simultaneous conjugacy category of M whose objects are the elements of $\mathcal{I}$. A morphism in $\operatorname{Conj}(M, \mathcal{I})$ with source $\mathbf{I} \in \mathcal{I}$ is given by $\mathbf{b} \in M$ such that for each $\mathbf{s} \in \mathbf{I}$ we have $\mathbf{s}^{\mathbf{b}} \in M$, which by property 4.4(iii) implies $\mathbf{s}^{\mathbf{b}} \in \mathbf{S}$. We denote such a morphism in $\operatorname{Conj}(M, \mathcal{I})(\mathbf{I}, \text{-})$ by $\mathbf{I} \xrightarrow{\mathbf{b}} \text{-}$, or if we want to specify the target we denote it by $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$ where $\mathbf{J} = \{\mathbf{s}^{\mathbf{b}} \mid \mathbf{s} \in \mathbf{I}\}$, and in this situation we write $\mathbf{J} = \mathbf{I}^{\mathbf{b}}$.

By Proposition 3.3 the set $\{\mathbf{I} \xrightarrow{\mathbf{b}} \text{-} \mid \mathbf{b} \in \mathcal{S}\} \cap \operatorname{Conj}(M, \mathcal{I})$ is a Garside family in $\operatorname{Conj}(M, \mathcal{I})$.

The ribbon category. For $\mathbf{b} \in M$ we denote by $\alpha_{\mathbf{I}}(\mathbf{b})$ the maximal left-divisor of $\mathbf{b}$ in $M_{\mathbf{I}}$ given by Lemma 2.13, which is unique since $M_{\mathbf{I}}^{\times} = \{1\}$. We denote by $\omega_{\mathbf{I}}(\mathbf{b})$ the element defined by $\mathbf{b} = \alpha_{\mathbf{I}}(\mathbf{b})\omega_{\mathbf{I}}(\mathbf{b})$. We say that $\mathbf{b} \in M$ is $\mathbf{I}$-*reduced* if it is left-divisible by no element of $\mathbf{I}$, or equivalently if $\alpha_{\mathbf{I}}(\mathbf{b}) = 1$.

Definition 4.7. *We define the ribbon category* $M(\mathcal{I})$ *as the subcategory of* $\operatorname{Conj}(M, \mathcal{I})$ *obtained by restricting the morphisms to the* $\mathbf{I} \xrightarrow{\mathbf{b}} \text{-}$ *such that* $\mathbf{b}$ *is* $\mathbf{I}$*-reduced.*

This makes sense since the above class of morphisms is stable by composition by (ii) in the next proposition; assertion (i) of the next proposition is a motivation for restricting to such morphisms by showing that we "lose nothing" in doing so.

Proposition 4.8.
(i) *Given* $\mathbf{I} \in \mathcal{I}$ *and* $\mathbf{b} \in M$ *then* $\mathbf{I} \xrightarrow{\mathbf{b}} \text{-} \in \operatorname{Conj}(M, \mathcal{I})$ *if and only if* $\mathbf{I}^{\alpha_{\mathbf{I}}(\mathbf{b})} = \mathbf{I}$ *and* $\mathbf{I} \xrightarrow{\omega_{\mathbf{I}}(\mathbf{b})} \text{-} \in M(\mathcal{I})$.
(ii) *If* $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \in M(\mathcal{I})$ *then for any* $\mathbf{b}' \in M$ *we have* $\alpha_{\mathbf{J}}(\mathbf{b}') = \alpha_{\mathbf{I}}(\mathbf{b}\mathbf{b}')^{\mathbf{b}}$. *In particular if* $(\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}) \in M(\mathcal{I})$ *and* $(\mathbf{J} \xrightarrow{\mathbf{b}'} \mathbf{K}) \in \operatorname{Conj}(M, \mathcal{I})$ *then* $(\mathbf{I} \xrightarrow{\mathbf{b}\mathbf{b}'} \mathbf{K}) \in M(\mathcal{I})$ *if and only if* $(\mathbf{J} \xrightarrow{\mathbf{b}'} \mathbf{K}) \in M(\mathcal{I})$.
(iii) *If two morphisms in* $M(\mathcal{I})$ *admit a right-lcm in* $\operatorname{Conj}(M, \mathcal{I})$*, then this lcm is in* $M(\mathcal{I})$.

Note that if $\mathbf{I}\xrightarrow{\mathbf{c}}$ - is the right-lcm of two morphisms $\mathbf{I}\xrightarrow{\mathbf{b}}$ - and $\mathbf{I}\xrightarrow{\mathbf{b}'}$ - as in (iii) then by Lemma 3.2 $\mathbf{c}$ is the right-lcm in M of $\mathbf{b}$ and $\mathbf{b}'$.

Proof. Let us prove (i). We prove that if $\mathbf{s}\in\mathbf{I}$ and $\mathbf{s}^{\mathbf{b}}\in M$ then $\mathbf{s}^{\alpha_{\mathbf{I}}(\mathbf{b})}\in\mathbf{I}$. This will prove (i) in one direction —we use that $\mathbf{I}$ is finite, see 4.4(ii), so that $\mathbf{I}^{\alpha_{\mathbf{I}}(\mathbf{b})}\subset\mathbf{I}$ implies $\mathbf{I}^{\alpha_{\mathbf{I}}(\mathbf{b})}=\mathbf{I}$. The converse is obvious.

By property 4.4(iii) we have $\mathbf{sb}=\mathbf{bt}$ for some $\mathbf{t}\in\mathbf{S}$. If $\mathbf{s}\preccurlyeq\mathbf{b}$ we write $\mathbf{b}=\mathbf{s}^k\mathbf{b}'$ for some k and $\mathbf{b}'$ such that $\mathbf{s}$ does not left-divide $\mathbf{b}'$. We have $\mathbf{sb}'=\mathbf{b}'\mathbf{t}$ and $\alpha_{\mathbf{I}}(\mathbf{b})=\mathbf{s}^k\alpha_{\mathbf{I}}(\mathbf{b}')$ and we are reduced to the case where $\mathbf{s}$ does not left-divide $\mathbf{b}$. Then any right-lcm of $\mathbf{s}$ and $\alpha_{\mathbf{I}}(\mathbf{b})$ left-divides $\mathbf{sb}=\mathbf{bt}$ and there is such a right-lcm in $M_{\mathbf{I}}$ since $M_{\mathbf{I}}$ is weakly closed under right-lcm (4.4(v)). We write this lcm $\mathbf{sv}=\alpha_{\mathbf{I}}(\mathbf{b})\mathbf{u}$, with $\mathbf{v}$ and $\mathbf{u}$ in $M_{\mathbf{I}}$ since $M_{\mathbf{I}}$ is closed under right-quotient (4.4(v)) and $\mathbf{v},\mathbf{u}\neq 1$ since $\mathbf{s}\not\preccurlyeq\mathbf{b}$. Since $\mathbf{sv}\preccurlyeq\mathbf{sb}$ we get that $\mathbf{v}$ left-divides $\mathbf{b}$, so left-divides $\alpha_{\mathbf{I}}(\mathbf{b})$, thus $\alpha_{\mathbf{I}}(\mathbf{b})=\mathbf{va}$ for some $\mathbf{a}\in M_{\mathbf{I}}$. We get $\mathbf{sv}=\alpha_{\mathbf{I}}(\mathbf{b})\mathbf{u}=\mathbf{vau}$. By property 4.4(iii) we have $\mathbf{au}\in\mathbf{S}$, thus $\mathbf{u}$ is an atom which is in $M_{\mathbf{I}}$, hence $\mathbf{u}\in\mathbf{I}$ and $\mathbf{a}=1$ since $\mathbf{S}$ is a transversal for $=^{\times}$. We get $\mathbf{s}^{\alpha_{\mathbf{I}}(\mathbf{b})}=\mathbf{s}^{\mathbf{v}}=\mathbf{u}\in\mathbf{I}$, which gives the result.

Let us prove (ii). For $\mathbf{s}\in\mathbf{I}$ let $\mathbf{s}'=\mathbf{s}^{\mathbf{b}}\in\mathbf{J}$. Since $\mathbf{I}\xrightarrow{\mathbf{b}}\mathbf{J}\in M(\mathcal{I})$ we have $\mathbf{s}\not\preccurlyeq\mathbf{b}$. Then $\mathbf{bs}'=\mathbf{sb}$ is a common multiple of $\mathbf{s}$ and $\mathbf{b}$ which has to be an lcm since $\mathbf{s}'$ is an atom. So for $\mathbf{s}\in\mathbf{I}$ we have $\mathbf{s}\preccurlyeq\mathbf{bb}'$ if and only if $\mathbf{bs}'\preccurlyeq\mathbf{bb}'$, that is, $\mathbf{s}^{\mathbf{b}}\preccurlyeq\mathbf{b}'$ whence the result.

To prove (iii) we show first the statement that if for $\mathbf{b},\mathbf{c}\in M$ we have $\mathbf{b}\preccurlyeq\mathbf{c}$ and $\mathbf{I}\xrightarrow{\mathbf{b}}$ - $\in M(\mathcal{I})$, then $\mathbf{b}\preccurlyeq\omega_{\mathbf{I}}(\mathbf{c})$. We write $\mathbf{c}=\mathbf{bb}'$ and $\mathbf{J}=\mathbf{I}^{\mathbf{b}}$. By (ii) we have $\alpha_{\mathbf{I}}(\mathbf{c})^{\mathbf{b}}=\alpha_{\mathbf{J}}(\mathbf{b}')$, whence $\alpha_{\mathbf{I}}(\mathbf{c})\mathbf{b}=\mathbf{b}\alpha_{\mathbf{J}}(\mathbf{b}')\preccurlyeq\mathbf{bb}'=\mathbf{c}=\alpha_{\mathbf{I}}(\mathbf{c})\omega_{\mathbf{I}}(\mathbf{c})$. Left-canceling $\alpha_{\mathbf{I}}(\mathbf{c})$ we get $\mathbf{b}\preccurlyeq\omega_{\mathbf{I}}(\mathbf{c})$.

Now (iii) is a particular case of the above statement since if $\mathbf{c}$ is the right-lcm of $\mathbf{b}$ and $\mathbf{b}'$ where $\mathbf{I}\xrightarrow{\mathbf{b}}$ - and $\mathbf{I}\xrightarrow{\mathbf{b}'}$ - are in $M(\mathcal{I})$, we get that $\omega_{\mathbf{I}}(\mathbf{c})$ is a common right-multiple of $\mathbf{b}$ and $\mathbf{b}'$, thus $\mathbf{c}\preccurlyeq\omega_{\mathbf{I}}(\mathbf{c})$, which implies $\alpha_{\mathbf{I}}(\mathbf{c})=1$. □

Note that by Proposition 4.8(i) a morphism in $M(\mathcal{I})$ with source $\mathbf{I}$ corresponds by the forgetful functor to an element $\mathbf{b}\in M$ such that $\alpha_{\mathbf{I}}(\mathbf{b})=1$ and such that for each $\mathbf{s}\in\mathbf{I}$ we have $\mathbf{s}^{\mathbf{b}}\in M$. We will thus sometimes just denote by $\mathbf{b}$ such a morphism in $M(\mathcal{I})$ when the context makes its source clear.

The next proposition shows that $(\mathcal{S}\cap M(\mathcal{I}))\cup M^{\times}$ generates $M(\mathcal{I})$. Note any element of $M^{\times}$ gives rise to an element of $M(\mathcal{I})$.

Proposition 4.9. *All the terms of a normal decomposition in* $\operatorname{Conj}(M,\mathcal{I})$ *of a morphism of* $M(\mathcal{I})$ *are in* $M(\mathcal{I})$.

Proof. Let $\mathbf{I}\xrightarrow{\mathbf{b}}$ - $\in M(\mathcal{I})$ and let $\mathbf{b}=\mathbf{b}_1\cdots\mathbf{b}_k$ be a normal decomposition in M, which gives a normal decomposition of $\mathbf{I}\xrightarrow{\mathbf{b}}$ - in $\operatorname{Conj}(M,\mathcal{I})$ by Proposition 3.3. We proceed by induction on k. We have $\alpha_{\mathbf{I}}(\mathbf{b}_1)\preccurlyeq\alpha_{\mathbf{I}}(\mathbf{b})=1$ thus $\alpha_{\mathbf{I}}(\mathbf{b}_1)=1$ and $\mathbf{I}\xrightarrow{\mathbf{b}_1}\mathbf{I}^{\mathbf{b}_1}\in M(\mathcal{I})$. This is the first step of the induction. Now, by 4.8(ii) we get $\mathbf{I}^{\mathbf{b}_1}\xrightarrow{\mathbf{b}_2\cdots\mathbf{b}_k}$ - $\in M(\mathcal{I})$ which concludes by induction. □

Corollary 4.10. *The set* $\mathcal{S}\cap M(\mathcal{I})=\{\mathbf{I}\xrightarrow{\mathbf{w}}$ - $\in\operatorname{Conj}(M,\mathcal{I})\mid\mathbf{w}\in\mathcal{S}$ *and* $\alpha_{\mathbf{I}}(\mathbf{w})=1\}$ *is a Garside family in* $M(\mathcal{I})$.

Proof. By 4.8(ii) and 4.8(iii) the subcategory $M(\mathcal{I})$ of $\operatorname{Conj}(M,\mathcal{I})$ is closed under right-quotient and right-lcm, hence the subfamily $\mathcal{S}\cap M(\mathcal{I})$ is closed under right-quotient and right-lcm in $\mathcal{S}\cap\operatorname{Conj}(M,\mathcal{I})$. Thus Lemma 2.12 gives the result since $(\mathcal{S}\cap M(\mathcal{I}))\cup M^\times$ generates $M(\mathcal{I})$ by Proposition 4.9. □

Our aim now is Proposition 4.15 which gives a description of the atoms of $M(\mathcal{I})$, and a convenient criterion to decide whether $\mathbf{b}\in M$ gives rise to an element of $M(\mathcal{I})$.

For $\mathbf{I}\subset\mathbf{S}$ let $\Phi_\mathbf{I}$ be the Garside automorphism of $M_\mathbf{I}$ associated with the Garside element $\Delta_\mathbf{I}$ (see 4.4(iv)). Since $\mathbf{I}$ is finite (see 4.4(ii)) and is the whole set of atoms of $M_\mathbf{I}$, we have $\Phi_\mathbf{I}(\mathbf{I})=\mathbf{I}$.

We denote by Φ the Garside automorphism of M associated to Δ. Since Φ is an automorphism which preserves $\mathcal{S}$, for $\mathbf{I}\subset\mathbf{S}$, it sends the Garside family $\mathcal{S}\cap M_\mathbf{I}$ to the Garside family $\mathcal{S}\cap M_{\Phi(\mathbf{I})}$ thus $\Phi(\Delta_\mathbf{I})=\Delta_{\Phi(\mathbf{I})}$.

Proposition 4.11. *$M(\mathcal{I})$ has a Garside map defined by the collection of morphisms $\mathbf{I}\xrightarrow{\Delta_\mathbf{I}^{-1}\Delta}\Phi(\mathbf{I})$ for $\mathbf{I}\in\mathcal{I}$.*

Proof. We have $\Phi_\mathbf{I}(\mathbf{I})=\mathbf{I}$ and $\omega_\mathbf{I}(\Delta)=\Delta_\mathbf{I}^{-1}\Delta$. Thus by Proposition 4.8(i) $\mathbf{I}\xrightarrow{\Delta_\mathbf{I}^{-1}\Delta}\Phi(\mathbf{I})\in\mathcal{S}\cap M(\mathcal{I})$. We need two lemmas.

Lemma 4.12. *Any morphism $\mathbf{I}\xrightarrow{\mathbf{b}}\text{-}\in M(\mathcal{I})\cap\mathcal{S}$ left-divides $\mathbf{I}\xrightarrow{\Delta_\mathbf{I}^{-1}\Delta}\Phi(\mathbf{I})$.*

Proof. The divisibility we seek is equivalent to $\Delta_\mathbf{I}\mathbf{b}$ left-dividing Δ. Since $\Delta_\mathbf{I}$ and $\mathbf{b}$ left-divide Δ, a right-lcm δ of these elements divides Δ. We claim that $\delta=^\times\Delta_\mathbf{I}\mathbf{b}$ which will show the lemma. Since $\mathbf{I}^\mathbf{b}\subset\mathbf{S}$ we have $\Delta_\mathbf{I}^\mathbf{b}\in M$ thus $\delta\preccurlyeq\mathbf{b}\Delta_\mathbf{I}^\mathbf{b}=\Delta_\mathbf{I}\mathbf{b}$. Notice that $\alpha_\mathbf{I}(\delta)=\Delta_\mathbf{I}$ since $\Delta_\mathbf{I}\preccurlyeq\delta$ and $\alpha_\mathbf{I}(\delta)\preccurlyeq\alpha_\mathbf{I}(\Delta)=\Delta_\mathbf{I}$. Now write $\delta=\mathbf{bx}$; by Proposition 4.8(ii) we have $\alpha_{\mathbf{I}^\mathbf{b}}(\mathbf{x})=\alpha_\mathbf{I}(\delta)^\mathbf{b}=\Delta_\mathbf{I}^\mathbf{b}$. Thus $\Delta_\mathbf{I}^\mathbf{b}\preccurlyeq\mathbf{x}$ thus $\mathbf{b}\Delta_\mathbf{I}^\mathbf{b}\preccurlyeq\mathbf{bx}=\delta$, whence our claim. □

Lemma 4.13. *If $\mathbf{I}\xrightarrow{\mathbf{b}}\mathbf{J}$ is in $M(\mathcal{I})$ we have $\Delta_\mathbf{J}=\Delta_\mathbf{I}^\mathbf{b}$; conjugation by $\mathbf{b}$ induces an isomorphism of Garside monoids $M_\mathbf{I}\xrightarrow{\sim}M_\mathbf{J}$ which preserves normal forms.*

Proof. It is sufficient to prove the lemma for elements of the generating set $(\mathcal{S}\cap M(\mathcal{I}))\cup M^\times$. So we assume $\mathbf{b}\in\mathcal{S}\cup M^\times$. If $\mathbf{b}\in\mathcal{S}$, in the proof of Lemma 4.12 we have $\Delta_\mathbf{I}^\mathbf{b}\preccurlyeq\mathbf{x}$ where $\mathbf{x}$ is a right-divisor hence a left-divisor of Δ, thus $\Delta_\mathbf{I}^\mathbf{b}\preccurlyeq\Delta$. This is also clearly true if $\mathbf{b}\in M^\times$. Since $\Delta_\mathbf{I}^\mathbf{b}\in M_\mathbf{J}$ we get $\Delta_\mathbf{I}^\mathbf{b}\preccurlyeq\Delta_\mathbf{J}$. We show by contradiction that this divisibility cannot be strict. By Lemma 4.12 we can write $\Delta_\mathbf{I}^{-1}\Delta=\mathbf{bb}'$; then by 4.8(ii) we have $\mathbf{J}\xrightarrow{\mathbf{b}'}\Phi(\mathbf{I})\in M(\mathcal{I})$ and by the same argument as above $\Delta_\mathbf{J}^{\mathbf{b}'}\preccurlyeq\Delta_{\Phi(\mathbf{I})}$. Now $\mathbf{b}'$ induces by conjugation a morphism $M_\mathbf{J}\to M_{\Phi(\mathbf{I})}$ so we can transport the strict divisibility $\Delta_\mathbf{I}^\mathbf{b}\prec\Delta_\mathbf{J}$ to $\Delta_\mathbf{I}^{\mathbf{bb}'}\prec\Delta_\mathbf{J}^{\mathbf{b}'}$. Composing we get $\Phi(\Delta_\mathbf{I})=\Delta_\mathbf{I}^{\mathbf{bb}'}\prec\Delta_\mathbf{J}^{\mathbf{b}'}\preccurlyeq\Delta_{\Phi(\mathbf{I})}=\Phi(\Delta_\mathbf{I})$, a contradiction.

The second part of the statement follows from the first since the first term of a normal form of an element $\mathbf{x}$ in a monoid with a Garside element Δ is a left-gcd of $\mathbf{x}$ and Δ (see Proposition 2.14(iii)), and the conjugation by $\mathbf{b}$ preserves gcds since it is an isomorphism. □

We now show the proposition. We know by Lemma 4.12 that any $\mathbf{I}\xrightarrow{\mathbf{b}}\mathbf{J}$ in $\mathcal{S}\cap M(\mathcal{I})$ left-divides $\mathbf{I}\xrightarrow{\Delta_\mathbf{I}^{-1}\Delta}\Phi(\mathbf{I})$. It remains to show that such a morphism right-divides $\Delta_{\Phi^{-1}(\mathbf{J})}^{-1}\Delta$, which is equivalent to $\mathbf{b}\Delta_\mathbf{J}$ right-dividing Δ since $\Phi(\Delta_{\Phi^{-1}(\mathbf{J})})=$

$\Delta_{\mathbf{J}}$. This in turn is equivalent to $\mathbf{b}\Delta_{\mathbf{J}}$ left-dividing Δ since Δ is a Garside element. The result is then a consequence of the fact that $\Delta_{\mathbf{I}}\mathbf{b}$ divides Δ as we have seen in Lemma 4.12 and of the equality $\mathbf{b}\Delta_{\mathbf{J}} = \Delta_{\mathbf{I}}\mathbf{b}$ which is given by Lemma 4.13. □

Proposition 4.14. *Let $\mathbf{I} \in \mathcal{I}$ and let $\mathbf{J}$ be a parabolic subset of $\mathbf{S}$ such that $M_{\mathbf{I}} \subsetneq M_{\mathbf{J}}$. Then $\Delta_{\mathbf{I}} \preccurlyeq \Delta_{\mathbf{J}}$ (see 4.4(vi)) and $\mathbf{I} \xrightarrow{v(\mathbf{J},\mathbf{I})} \Phi_{\mathbf{J}}(\mathbf{I})$, where $v(\mathbf{J},\mathbf{I}) = \Delta_{\mathbf{I}}^{-1}\Delta_{\mathbf{J}}$, is a morphism in $M(\mathcal{I})$.*

Proof. As noted after Proposition 4.8 we have to show that $\alpha_{\mathbf{I}}(v(\mathbf{J},\mathbf{I})) = 1$ and that any $\mathbf{t} \in \mathbf{I}$ is conjugate by $v(\mathbf{J},\mathbf{I})$ to an element of M. Since $\Delta_{\mathbf{I}}^{-1}\Delta_{\mathbf{J}}$ left-divides $\Delta_{\mathbf{I}}^{-1}\Delta$, and $\alpha_{\mathbf{I}}(\Delta_{\mathbf{I}}^{-1}\Delta) = 1$, by definition of $\Delta_{\mathbf{I}}$, we get the first property. The second is clear since by definition $v(\mathbf{J},\mathbf{I})$ conjugates $\mathbf{t}$ to $\Phi_{\mathbf{J}}(\Phi_{\mathbf{I}}^{-1}(\mathbf{t}))$. □

(i) of the next proposition is due to Paris [Pa, 5.6] in the case of Artin monoids.

Proposition 4.15. (i) *Let $\mathbf{I} \in \mathcal{I}$ and $\mathbf{b} \in M$ such that $\alpha_{\mathbf{I}}(\mathbf{b}) = 1$ and such that there exists $p > 0$ such that $(\Delta_{\mathbf{I}}^{p})^{\mathbf{b}} \in M$. Then $\mathbf{I} \xrightarrow{\mathbf{b}} \text{-} \in M(\mathcal{I})$.*

(ii) *The atoms of $M(\mathcal{I})$ are the $v(\mathbf{J},\mathbf{I})$ not strictly divisible by another $v(\mathbf{J}',\mathbf{I})$ for $\mathbf{I} \in \mathcal{I}$.*

Proof. Since M is right-Noetherian, for (i) it suffices to prove that under our assumption $\mathbf{b}$ is either invertible or left-divisible by some non-invertible $\mathbf{v} \in M$ giving rise to an element of $M(\mathcal{I})$; indeed if $\mathbf{b} = \mathbf{v}\mathbf{b}'$ where $\mathbf{I} \xrightarrow{\mathbf{v}} \mathbf{I}' \in M(\mathcal{I})$ then by 4.8(ii) we have $\alpha_{\mathbf{I}'}(\mathbf{b}') = 1$ and since $\mathbf{I}^{\mathbf{v}} = \mathbf{I}'$ we have $(\Delta_{\mathbf{I}'}^{p})^{\mathbf{b}'} \in M$ by Lemma 4.13, so by Noetherian induction we have $\mathbf{I}' \xrightarrow{\mathbf{b}'} \text{-} \in M(\mathcal{I})$, whence $\mathbf{I} \xrightarrow{\mathbf{b}} \text{-} \in M(\mathcal{I})$. We will prove that $\mathbf{b}$ is left-divisible by $v(\mathbf{J},\mathbf{I})$ for some parabolic $\mathbf{J} \supsetneq \mathbf{I}$ which will imply (i). We proceed by decreasing induction on p. We show that if for $i > 0$ we have $\mathbf{s} \preccurlyeq \Delta_{\mathbf{I}}^{i}\mathbf{b}$ for some atom $\mathbf{s}$ not in $M_{\mathbf{I}}$, $v(\mathbf{J},\mathbf{I}) \preccurlyeq \Delta_{\mathbf{I}}^{i-1}\mathbf{b}$ where $\mathbf{J}$ is as prescribed in 4.4(vi) from $\mathbf{I}$ and $\mathbf{s}$. Indeed, the right-lcm of $\mathbf{s}$ and $\Delta_{\mathbf{I}}$ is $\Delta_{\mathbf{J}}$ by property 4.4(vi) thus from $\mathbf{s} \preccurlyeq \Delta_{\mathbf{I}}^{i}\mathbf{b}$ and $\Delta_{\mathbf{I}} \preccurlyeq \Delta_{\mathbf{I}}^{i}\mathbf{b}$ we deduce $\Delta_{\mathbf{J}} \preccurlyeq \Delta_{\mathbf{I}}^{i}\mathbf{b}$. Since $\Delta_{\mathbf{J}} = \Delta_{\mathbf{I}}v(\mathbf{J},\mathbf{I})$ we get as claimed $v(\mathbf{J},\mathbf{I}) \preccurlyeq \Delta_{\mathbf{I}}^{i-1}\mathbf{b}$. The induction starts at $i = p$ by taking for $\mathbf{s}$ any atom left-dividing $\mathbf{b}$, thus not in $M_{\mathbf{I}}$ since $\alpha_{\mathbf{I}}(\mathbf{b}) = 1$. Such an atom satisfies $\mathbf{s} \preccurlyeq \mathbf{b} \preccurlyeq \Delta_{\mathbf{I}}^{p}\mathbf{b}$ since the assumption on $\mathbf{b}$ can be written $\mathbf{b} \preccurlyeq \Delta_{\mathbf{I}}^{p}\mathbf{b}$. Since any atom $\mathbf{t}$ such that $\mathbf{t} \preccurlyeq v(\mathbf{J},\mathbf{I})$ is not in $M_{\mathbf{I}}$ the induction can go on while $i - 1 > 0$.

We get (ii) from the proof of (i): any element $\mathbf{b} \in M(\mathcal{I})$ satisfies the assumption of (i) for $p = 1$ and $\mathbf{I}$ equal to the source of $\mathbf{b}$; whence the result since in the proof of (i) we have seen that $\mathbf{b}$ is a product of elements of the form $v(\mathbf{J},\mathbf{K})$. □

Though in the current paper we need only finite Coxeter groups, we note that the above description of the atoms also extends to the case of Artin monoids which are associated with infinite Coxeter groups —and thus do not have a Garside element. Proposition 4.16 below can be extracted from the proof of Theorem 0.5 in [G].

In the case of an Artin monoid $(B^+, \mathbf{S})$ the Garside family of Corollary 4.10 in $B^+(\mathcal{I})$ is $\mathbf{W} \cap B^+(\mathcal{I}) = \{\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J} \in \operatorname{Conj} B^+(\mathcal{I}) \mid \mathbf{w} \in \mathbf{W} \text{ and } \alpha_{\mathbf{I}}(\mathbf{w}) = 1\}$. For $\mathbf{I} \subset \mathbf{S}$ and $\mathbf{s} \in \mathbf{S}$ we denote by $\mathbf{I}(\mathbf{s})$ the connected component of $\mathbf{s}$ in the Coxeter diagram of $\mathbf{I} \cup \{\mathbf{s}\}$, that is the vertices of the connected component of $\mathbf{s}$ in the graph with vertices $\mathbf{I} \cup \{\mathbf{s}\}$ and an edge between $\mathbf{s}'$ and $\mathbf{s}''$ whenever $\mathbf{s}'$ and $\mathbf{s}''$ do not commute.

When $\mathbf{I}$ is spherical, the subgroup W_I generated by the image I of $\mathbf{I}$ in W is finite even though W is not, in which case we denote by $\mathbf{w}_{\mathbf{I}}$ the lift in $\mathbf{W}$ of the longest element of W_I. With these notations, we have

Proposition 4.16. *The atoms of* $B^+(\mathcal{I})$ *are the morphisms* $\mathbf{I}\xrightarrow{v(\mathbf{s},\mathbf{I})}{}^{v(\mathbf{s},\mathbf{I})}\mathbf{I}$ *where* $\mathbf{I}$ *is in* $\mathcal{I}$ *and* $\mathbf{s}\in\mathbf{S}-\mathbf{I}$ *is such that* $\mathbf{I}(\mathbf{s})$ *is spherical, and where* $v(\mathbf{s},\mathbf{I})=\mathbf{w}_{\mathbf{I}(\mathbf{s})}\mathbf{w}_{\mathbf{I}(\mathbf{s})-\{\mathbf{s}\}}$.

5. Application to Artin groups

We will spell out how the above results can be stated in two particular cases. We try to recall enough notation so this section can be read independently of the previous ones.

Artin monoids with automorphism. We first look at the case of a spherical Artin monoid B^+ attached to a Coxeter system (W,S) with a diagram automorphism ϕ, see 4.1. The category $\mathcal{C}$ we will take is the monoid $B^+\rtimes\langle\phi\rangle$; it has a Garside element $\mathbf{w}_0$ and an attached Garside family $\mathbf{W}$. The Garside automorphism Φ is given by $\mathbf{b}\mapsto\mathbf{b}^{\mathbf{w}_0}$; it is trivial if $\mathbf{W}_0$ is central and has order 2 otherwise. We set $\boldsymbol{\pi}=\mathbf{w}_0^2$, a central element in B^+. An element $\mathbf{b}\phi\in B^+\rtimes\langle\phi\rangle$ is (d,p)-periodic if $(\mathbf{b}\phi)^d=\mathbf{w}_0^p\phi^d$, which can be written $\mathbf{b}^\phi\mathbf{b}^{\phi^2}\mathbf{b}\cdots=\mathbf{w}_0^p$.

Theorem 5.1. *If* $\phi=\mathrm{Id}$*, two periodic elements of* B^+ *of same period are cyclically conjugate.*

Proof. This results from the work of David Bessis on the dual braid monoid. Two periodic elements of same period in B^+ are also periodic and have equal periods in the dual monoid, since the Garside element $\mathbf{w}_0$ of B^+ is a power of the Garside element of the dual monoid. By [B1, 11.21], such elements are conjugate in the dual monoid, so are conjugate in B, hence are conjugate in B^+; indeed if $\mathbf{b}'=\mathbf{h}^{-1}\mathbf{b}\mathbf{h}$ with $\mathbf{b},\mathbf{b}'\in B^+$ and $\mathbf{h}\in B$, then there exists $i>0$ such that $\mathbf{h}\boldsymbol{\pi}^i\in B^+$ and since $\boldsymbol{\pi}$ is central $\mathbf{h}\boldsymbol{\pi}^i$ still conjugates $\mathbf{b}$ to $\mathbf{b}'$. By Proposition 3.9 conjugate periodic elements are cyclically conjugate. □

We conjecture that the same result holds in the case $\phi\neq\mathrm{Id}$.

Taking in account that $\Phi^2=\mathrm{Id}$, statement 3.13 gives:

Proposition 5.2. *Let* $\mathbf{b}'\phi\in B^+\phi$ *be* $(d,2)$*-periodic, that is* $(\mathbf{b}'\phi)^d=\boldsymbol{\pi}\phi^d$*, and let* $e=\lfloor\frac{d}{2}\rfloor$*. Then there exists* $\mathbf{b}\phi\in B^+\phi$ *cyclically conjugate to* $\mathbf{b}'\phi$ *such that* $\mathbf{b}^e\in\mathbf{W}$*, and*

- *If* d *is even then* $(\mathbf{b}\phi)^e=\mathbf{w}_0\phi^e$*. The centralizer* $C_{B^+}(\mathbf{b}\phi)$ *identifies with* $\operatorname{cyc}B^+(\mathbf{b}\phi)$*, and even more specifically to the endomorphisms of* $\mathbf{b}\phi$ *in the category of conjugacy by* $\mathbf{w}_0\phi^e$*-stable divisors.*
- *If* d *is odd there exists* $\mathbf{v}\in\mathbf{W}$ *such that* $(\mathbf{b}\phi)^e\mathbf{v}=\mathbf{w}_0\phi^e$ *and* $\mathbf{b}=\mathbf{v}\phi^{-e}(\mathbf{v}^{\mathbf{w}_0})$*. The centralizer* $C_{B^+}(\mathbf{b}\phi)$ *identifies with the endomorphisms of* $\mathbf{v}\mathbf{w}_0\phi^{-e}$ *in the category of conjugacy by* ϕ^d*-stable divisors.*

Part of the above proposition is already in [BM, 6.8]. The equation $(\mathbf{b}\phi)^d=\boldsymbol{\pi}\phi^d$ for $(d,2)$-periodic elements made the authors of [BM] call such elements d-th ϕ-roots of $\boldsymbol{\pi}$.

Ribbons in Artin monoids. We keep in this subsection a spherical Artin monoid B^+ attached to (W,S) with a diagram automorphism ϕ and consider the ribbon category $B^+\rtimes\langle\phi\rangle(\mathcal{I})$ defined by a conjugacy class $\mathcal{I}$ of subsets of $\mathbf{S}$.

A subset $I\subset S$ and the corresponding subset $\mathbf{I}\subset\mathbf{S}$ determine:

- A *standard parabolic* subgroup W_I generated by I; we denote by w_I its longest element (with this notation $w_0 = w_S$). In every coset $W_I w$ there is a unique shortest element called I-reduced.
- A *parabolic submonoid* $B^+_{\mathbf{I}}$ generated by $\mathbf{I}$; it has the Garside family $\mathbf{W}_{\mathbf{I}} := \mathbf{W} \cap B^+_{\mathbf{I}}$ and the associated Garside element is the lift $\mathbf{w}_{\mathbf{I}}$ of w_I; we set $\boldsymbol{\pi}_{\mathbf{I}} = \mathbf{w}_{\mathbf{I}}^2$. By Lemma 2.13 every element $\mathbf{b} \in B^+$ has a unique longest divisor $\alpha_{\mathbf{I}}(\mathbf{b})$ in $B^+_{\mathbf{I}}$; an element such that $\alpha_{\mathbf{I}}(\mathbf{b}) = 1$ is called $\mathbf{I}$-reduced.

The ribbon category $B^+(\mathcal{I})$ is the category whose objects are the elements of $\mathcal{I}$ and a morphism $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$ is given by an $\mathbf{I}$-reduced element $\mathbf{b} \in B^+$ such that $\mathbf{I}^{\mathbf{b}} = \mathbf{J}$; since $\mathbf{J}$ is determined by $\mathbf{I}$ and $\mathbf{b}$ we denote also by $\mathbf{I} \xrightarrow{\mathbf{b}}$ - this morphism. Proposition 4.8 shows that this definition makes sense, that is if we have a composition $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \xrightarrow{\mathbf{c}} \mathbf{K}$ in $B^+(\mathcal{I})$, then $\alpha_{\mathbf{I}}(\mathbf{bc}) = 1$.

By Corollary 4.10 and Proposition 4.11 $B^+(\mathcal{I})$ has a Garside family $\mathcal{S}$ consisting of the morphisms $\mathbf{I} \xrightarrow{\mathbf{w}}$ - where $\mathbf{w} \in \mathbf{W}$ and a Garside map $\Delta_{\mathcal{I}}(\mathbf{I}) = \mathbf{I} \xrightarrow{\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0} \mathbf{I}^{\mathbf{w}_0}$. These properties include the following:

Lemma 5.3. (i) *$\mathcal{S}$ generates $B^+(\mathcal{I})$; specifically, if $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \in B^+(\mathcal{I})$ and $(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ is the $\mathbf{W}$-strict normal decomposition of $\mathbf{b}$, there exist subsets $\mathbf{I}_i$ with $\mathbf{I}_1 = \mathbf{I}$, $\mathbf{I}_{k+1} = \mathbf{J}$ such that for all i we have $\mathbf{I}_{i+1} = \mathbf{I}_i^{\mathbf{w}_i}$; thus $\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \to \cdots \to \mathbf{I}_k \xrightarrow{\mathbf{w}_k} \mathbf{J}$ is a decomposition of $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$ in $B^+(\mathcal{I})$ as a product of elements of $\mathcal{S}$.*

(ii) *The relations $(\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{J} \xrightarrow{\mathbf{w}_2} \mathbf{K}) = (\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{K})$ when $\mathbf{w} = \mathbf{w}_1\mathbf{w}_2 \in \mathbf{W}$ form a presentation of $B^+(\mathcal{I})$.*

In our case strict normal decompositions are unique. They can be defined as follows: for $\mathbf{b} \in B^+$, let $\alpha(\mathbf{b})$ be the left-gcd of $\mathbf{b}$ and $\mathbf{w}_0$; the restriction of α to $B^+ - \{1\}$ is a $\mathbf{W}$-head function, thus $\mathbf{w}_1 := \alpha(\mathbf{b})$ is the first term of the normal decomposition of $\mathbf{b}$, and the other terms are defined similarly by induction, setting $\mathbf{w}_2 = \alpha(\mathbf{w}_1^{-1}\mathbf{b})$, etc...

For generating the category $B^+ \rtimes \langle\phi\rangle(\mathcal{I})$ we need additionally the invertible morphisms $\mathbf{I} \xrightarrow{\phi} \mathbf{I}^{\phi}$. The family $\mathcal{S}$ is still a Garside family for this category, with the same Garside map $\Delta_{\mathcal{I}}$. When $\mathcal{I} = \{\emptyset\}$, $B^+(\mathcal{I})$ reduces to the Artin-Tits monoid B^+ and $B^+ \rtimes \langle\phi\rangle(\mathcal{I})$ reduces to $B^+ \rtimes \langle\phi\rangle$, thus the results in this subsection generalize those of the previous subsection.

We will be interested in $(d, 2)$-periodic elements in $B^+\phi(\mathcal{I})$. Such an element is an endomorphism of the form $\mathbf{I} \xrightarrow{\mathbf{b}\phi} \mathbf{I}$ or via the correspondence between conjugacy in the semi-direct category and ϕ-conjugacy, a morphism $\mathbf{I} \xrightarrow{\mathbf{b}} {}^{\phi}\mathbf{I}$ in $B^+(\mathcal{I})$ where ${}^{\mathbf{b}\phi}\mathbf{I} = \mathbf{I}$. Since $\Delta_{\mathcal{I}}(\mathbf{I})\Delta_{\mathcal{I}}(\mathbf{I}^{\mathbf{w}_0}) = \mathbf{I} \xrightarrow{\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}} \mathbf{I}$ the condition for this morphism to be $(d, 2)$-periodic is $(\mathbf{b}\phi)^d = \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}\phi^d$.

By the forgetful functor $(\mathbf{I} \xrightarrow{\mathbf{b}\phi}$ -$) \mapsto \mathbf{b}\phi$ the morphisms in $B^+\phi(\mathcal{I})(\mathbf{I}, \text{-})$ identify with the elements $\mathbf{b}\phi \in B^+\phi$ such that ${}^{\mathbf{b}\phi}\mathbf{I} \subset \mathbf{S}$ and $\alpha_{\mathbf{I}}(\mathbf{b}) = 1$. We will thus sometimes write $\mathbf{b}\phi \in B^+\phi(\mathcal{I})(\mathbf{I}, \text{-})$ to mean $\mathbf{I} \xrightarrow{\mathbf{b}\phi}$ - $\in B^+\phi(\mathcal{I})(\mathbf{I}, \text{-})$.

Taking into account the above, and that the Garside automorphism associated to $\Delta_{\mathcal{I}}$ is $\Phi(\mathbf{I} \xrightarrow{\mathbf{v}} \mathbf{I}^{\mathbf{v}}) = \mathbf{I}^{\mathbf{w}_0} \xrightarrow{\mathbf{v}^{\mathbf{w}_0}} \mathbf{I}^{\mathbf{v}\mathbf{w}_0}$, the generalization of Proposition 5.2 is

Proposition 5.4. *Let $\mathbf{b}'\phi \in B^+\phi$ be such that $(\mathbf{b}'\phi)^d = \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}\phi^d$ for some ϕ^d-stable $\mathbf{J} \in \mathcal{I}$, and let $e = \lfloor \frac{d}{2} \rfloor$. Then $\mathbf{b}'\phi$ defines an endomorphism of $\mathbf{J}$ in $B^+\phi(\mathcal{I})$, that is ${}^{\mathbf{b}'\phi}\mathbf{J} = \mathbf{J}$ and $\alpha_{\mathbf{J}}(\mathbf{b}') = 1$. This endomorphism is $(d,2)$-periodic and there exists a ϕ^d-stable $\mathbf{I} \in \mathcal{I}$ and $\mathbf{I} \xrightarrow{\mathbf{b}\phi} \mathbf{I} \in B^+\phi(\mathcal{I})(\mathbf{I})$ cyclically conjugate to $\mathbf{J} \xrightarrow{\mathbf{b}'\phi} \mathbf{J} \in B^+\phi(\mathcal{I})(\mathbf{J})$ such that $(\mathbf{b}\phi)^d = \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}\phi^d$, $(\mathbf{b}\phi)^e \in \mathbf{W}\phi^e$, and*

- *If d is even then $(\mathbf{b}\phi)^e = \mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0\phi^e$. The centralizer* $\operatorname{Conj} B^+(\mathcal{I})(\mathbf{I} \xrightarrow{\mathbf{b}\phi} \mathbf{I})$ *identifies with* $(\operatorname{cyc} B^+(\mathcal{I})(\mathbf{I} \xrightarrow{\mathbf{b}\phi} \mathbf{I}))^{\mathbf{w}_0\phi^e}$.
- *If d is odd there exists $\mathcal{I} \xrightarrow{\mathbf{v}} \mathbf{I}^{\mathbf{w}_0\phi^e} \in \mathbf{W} \cap B^+(\mathcal{I})$ such that $(\mathbf{b}\phi)^e\mathbf{v} = \mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0\phi^e$ and $\mathbf{b} = \mathbf{v}\phi^{-e}(\mathbf{v}^{\mathbf{w}_0})$. The centralizer* $\operatorname{Conj} B^+(\mathcal{I})(\mathbf{I} \xrightarrow{\mathbf{b}\phi} \mathbf{I})$ *identifies with* $(\operatorname{cyc} B^+(\mathcal{I})(\mathbf{I} \xrightarrow{\mathbf{v}\Phi\phi^{-e}} \mathbf{I}))^{\phi^d}$.

Proof. We need to prove that $(\mathbf{b}'\phi)^d = (\boldsymbol{\pi}_{\mathbf{J}})^{-1}\boldsymbol{\pi}\phi^d$ implies $\alpha_{\mathbf{J}}(\mathbf{b}') = 1$ and that ${}^{\mathbf{b}'\phi}\mathbf{J} = \mathbf{J}$. The condition $\alpha_{\mathbf{J}}(\mathbf{b}') = 1$ follows from $\alpha_{\mathbf{J}}(\mathbf{b}') \preccurlyeq \alpha_{\mathbf{J}}((\mathbf{b}'\phi)^d)$ and from the fact that $(\boldsymbol{\pi}_{\mathbf{J}})^{-1}\boldsymbol{\pi}$ defines a morphism in $B^+(\mathcal{I})$ as we have seen above. By Proposition 4.15(i) $\mathbf{b}'\phi$ defines a morphism $\mathbf{J} \xrightarrow{\mathbf{b}'\phi} \mathbf{K}$ in $B^+(\mathcal{I})$. Hence $\mathbf{b}'\phi$ conjugates $\boldsymbol{\pi}_{\mathbf{J}}$ to $\boldsymbol{\pi}_{\mathbf{K}}$ by Lemma 4.13. Since $\mathbf{b}'\phi$ centralizes $\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}\phi^d$ and $\boldsymbol{\pi}$ is central, it thus centralizes $\boldsymbol{\pi}_{\mathbf{J}}\phi^d$, hence it centralizes $\boldsymbol{\pi}_{\mathbf{J}}^\delta$, where δ is the order of ϕ and we get $\boldsymbol{\pi}_{\mathbf{J}}^\delta = \boldsymbol{\pi}_{\mathbf{K}}^\delta$. However the support (see the proof of Lemma 4.5) of $\boldsymbol{\pi}_{\mathbf{J}}^\delta$ is $\mathbf{J}$ and that of $\boldsymbol{\pi}_{\mathbf{K}}^\delta$ is $\mathbf{K}$, thus $\mathbf{J} = \mathbf{K}$ and $\mathbf{b}'\phi$ stabilizes $\mathbf{J}$.

The other assertions of the proposition are straightforward translations of Corollary 3.13. □

We note that any element which conjugates a $(d,2)$-periodic element in $B^+\phi$ to another is ϕ^d-stable. Indeed such an element conjugates some $\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}\phi^d$ to some $\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}\phi^d$; if δ is the order of ϕ since $\boldsymbol{\pi}$ is central it thus conjugates $\boldsymbol{\pi}_{\mathbf{J}}^\delta$ to $\boldsymbol{\pi}_{\mathbf{I}}^\delta$ thus by the same reasoning as the end of the proof above it conjugates $\mathbf{I}$ to $\mathbf{J}$, which finally implies that it commutes with ϕ^d.

We now state 3.14 in the case of ribbons.

Corollary 5.5. *Let $\mathbf{b}'\phi \in B^+\phi$ be such that $(\mathbf{b}'\phi)^d = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}})^k\phi^d$ for some ϕ^d-stable $\mathbf{J} \in \mathcal{I}$. Then $\mathbf{b}'\phi$ defines a $(d,2k)$-periodic endomorphism of $\mathbf{J}$ in $B^+\phi(\mathcal{I})$, and up to cyclic conjugacy in $B^+\phi(\mathcal{I})$, we may assume k prime to d. Then, for any choice of integers d',k' with $dk' = 1 + kd'$ there exists a ϕ^d-stable $\mathbf{I} \in \mathcal{I}$ and $\mathbf{I} \xrightarrow{\mathbf{b}\phi} \mathbf{I} \in B^+\phi(\mathcal{I})(\mathbf{I})$ cyclically conjugate to $\mathbf{J} \xrightarrow{\mathbf{b}'\phi} \mathbf{J}$ such that $(\mathbf{b}\phi)^d = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^k\phi^d$ and $(\mathbf{b}\phi)^{d'} \preccurlyeq (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^{k'}$, and if we define $\mathbf{b}_1 \in B^+(\mathcal{I})$ by $(\mathbf{b}\phi)^{d'}\mathbf{b}_1\phi^{-d'} = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^{k'}$ then $(\mathbf{b}_1\phi^{-d'})^d = \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}\phi^{-dd'}$ and $(\mathbf{b}_1\phi^{-d'})^k = (\mathbf{b}\phi)\phi^{-k'd}$.*

Proof. As in the beginning of the proof 5.4 we deduce from the equality $(\mathbf{b}'\phi)^d = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}})^k\phi^d$ that $\mathbf{b}'\phi$ defines an element of $B^+\phi(\mathcal{I})(\mathbf{J})$. The only other observation needed is that we apply 3.14 for the Garside structure corresponding to the Garside map $\Delta(\mathbf{J}) = \mathbf{J} \xrightarrow{\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}} \mathbf{J}$, the square of the previously introduced Garside map $\Delta_{\mathcal{I}}$ —this is allowed by 2.15. For this Garside map the corresponding functor Φ is the identity, as required by 3.14. □

Corollary 5.6. *As in corollary 5.5 let $\mathbf{b}'\phi \in B^+\phi$ be such that $(\mathbf{b}'\phi)^d = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}})^k\phi^d$ for some ϕ^d-stable $\mathbf{J} \in \mathcal{I}$. Then $\mathbf{I} \xrightarrow{\mathbf{b}'\phi} \mathbf{J}$ is cyclically conjugate in $B^+\phi(\mathcal{I})$ to a $(d,2k)$-periodic endomorphism $\mathbf{I} \xrightarrow{\mathbf{b}\phi} \mathbf{I}$ such that $(\mathbf{b}\phi)^{\lfloor \frac{d}{2k} \rfloor} \in \mathbf{W}\phi^{\lfloor \frac{d}{2k} \rfloor}$.*

Proof. By 5.5 we may first assume that k is prime to d. We then use 5.5 to get $\mathbf{b}_1\phi^{-d'} \in B^+\phi^{-d'}$ satisfying the assumption of 5.4 with ϕ replaced by $\phi^{-d'}$. By 5.4 we may find a cyclic conjugate $\mathbf{b}'_1\phi^{-d'}$ of $\mathbf{b}_1\phi^{-d'}$ such that $(\mathbf{b}'_1\phi^{-d'})^{\lfloor\frac{d}{2}\rfloor} \in \mathbf{W}\phi^{-d'\lfloor\frac{d}{2}\rfloor}$. If this cyclic conjugation conjugates $\mathbf{b}\phi = (\mathbf{b}_1\phi^{-d'})^k\phi^{k'd}$ to $(\mathbf{b}'_1\phi^{-d'})^k\phi^{k'd}$ we are done since $k\lfloor\frac{d}{2k}\rfloor \le \lfloor\frac{d}{2}\rfloor$. Note that the cyclic conjugacy in 5.4 conjugates $\mathbf{J}$ to $\mathbf{I}$ and $\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{J}\phi^d$ to $\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I}\phi^d$, so is ϕ^d-stable ($\phi^{-dd'}$-stable in our application). If we had that any $\phi^{dd'}$-stable element is ϕ^d-stable we would be done since the conjugation would then commute with $\phi^{k'd}$. Thus we finish using Lemma 3.16 which shows that we may choose d' prime to the order of ϕ. □

For $\mathbf{b} \in B^+$, let $\alpha(\mathbf{b}) = \gcd(\mathbf{b}, \mathbf{w}_0)$. It is a $\mathbf{W}$-head function in B^+ thus by Proposition 4.9 and Corollary 4.10 $(\mathbf{I} \xrightarrow{\mathbf{b}} \text{-}) \mapsto (\mathbf{I} \xrightarrow{\alpha(\mathbf{b})} \text{-})$ is a $\mathcal{S}$-head function.

Lemma 5.7. *For $\mathbf{I} \xrightarrow{\mathbf{b}} \text{-} \in B^+(\mathcal{I})$ and $\mathbf{v} \in B^+_\mathbf{I}$ we have $\alpha(\mathbf{vb}) = \alpha(\mathbf{v})\alpha(\mathbf{b})$.*

Proof. Lemma 5.3 implies that $\alpha(\mathbf{b})$ defines an element of $B^+(\mathcal{I})(\mathbf{I},\text{-})$ so that $\mathbf{v}^{\alpha(\mathbf{b})} \in B^+$. We have $\alpha(\mathbf{vb}) = \alpha(\mathbf{v}\alpha(\mathbf{b})) = \alpha(\alpha(\mathbf{b})\mathbf{v}^{\alpha(\mathbf{b})}) = \alpha(\alpha(\mathbf{b})\alpha(\mathbf{v}^{\alpha(\mathbf{b})}))$, the first and last equalities by property $(\mathcal{H})$ of Proposition 2.4. By Lemma 4.13 we have $\alpha(\mathbf{v}^{\alpha(\mathbf{b})}) = \alpha(\mathbf{v})^{\alpha(\mathbf{b})}$, so that $\alpha(\mathbf{vb}) = \alpha(\alpha(\mathbf{b})\alpha(\mathbf{v})^{\alpha(\mathbf{b})}) = \alpha(\alpha(\mathbf{v})\alpha(\mathbf{b}))$. Since $\alpha(\mathbf{b})$ is $\mathbf{I}$-reduced we have $\alpha(\mathbf{v})\alpha(\mathbf{b}) \in \mathbf{W}$, hence $\alpha(\alpha(\mathbf{v})\alpha(\mathbf{b})) = \alpha(\mathbf{v})\alpha(\mathbf{b})$. □

The following proposition shows a compatibility of morphisms in $B^+(\mathcal{I})$ with a "parabolic" situation.

Proposition 5.8. *Fix $\mathbf{I} \in \mathcal{I}$, and for $\mathbf{J} \subset \mathbf{I}$, let $\mathcal{J}$ be the set of $B^+_\mathbf{I}$-conjugates of $\mathbf{J}$. Let $(\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{I}') \in B^+(\mathcal{I})$ and let $(\mathbf{J} \xrightarrow{\mathbf{v}} \mathbf{J}') \in B^+_\mathbf{I}(\mathcal{J})$. Let $(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ be the strict normal decomposition of $\mathbf{vb}$ and let $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k)$ be a normal decomposition of $\mathbf{b}$ (we have added some 1's at the end of the strict normal decomposition so the decompositions have same length); then for each i there exists $\mathbf{v}_i \in B^+$ such that $\mathbf{u}_i = \mathbf{v}_i\mathbf{w}_i$ and $(\mathbf{v}_1, {}^{\mathbf{w}_1}\mathbf{v}_2, {}^{\mathbf{w}_1\mathbf{w}_2}\mathbf{v}_3, \ldots)$ is a normal decomposition of $\mathbf{v}$.*

Proof. We proceed by induction on k. By Lemma 5.7, we have $\mathbf{u}_1 = \alpha(\mathbf{v})\alpha(\mathbf{b}) = \alpha(\mathbf{v})\mathbf{w}_1$. Hence $\mathbf{v}_1 = \alpha(\mathbf{v})$ is a solution. Cancelling $\mathbf{v}_1$ we get $\mathbf{u}_2\cdots\mathbf{u}_k = \omega(\mathbf{v})^{\alpha(\mathbf{b})}\omega(\mathbf{b})$. The induction hypothesis applied to $\omega(\mathbf{v})^{\alpha(\mathbf{b})}$, which defines an element of $B^+_{\mathbf{I}^{\alpha(\mathbf{b})}}(\mathcal{J}^{\alpha(\mathbf{b})})$, and to $\omega(\mathbf{b})$ which defines an element of $B^+(\mathcal{I})$ gives the result. □

The category $\mathcal{D}^\mathcal{I}$. The category $\operatorname{cyc} B^+\phi(\mathcal{I})$ will play an important role in our work: it will be interpreted as a category of morphisms between Deligne-Lusztig varieties. For this reason we will abbreviate its name to $\mathcal{D}^\mathcal{I}$; when $\mathcal{I} = \{\emptyset\}$ it reduces to the category $\mathcal{D}^+$ of [DMR, 5.1].

The objects of $\mathcal{D}^\mathcal{I}$ are endomorphisms $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}$ in $B^+\phi(\mathcal{I})$ and the morphisms are generated by the "simple" morphisms that we will denote by $\operatorname{ad}\mathbf{v}$, defined for $\mathbf{v} \preccurlyeq \mathbf{w}$ such that $\mathbf{I}^\mathbf{v} \subset \mathbf{S}$; such a morphism goes from $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}$ to $\mathbf{J} \xrightarrow{\mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v}} \mathbf{J}$ where $\mathbf{J} = \mathbf{I}^\mathbf{v}$.

By Proposition 3.6 the category $\mathcal{D}^\mathcal{I}$ has a Garside family consisting of the simple morphisms. In particular defining relations for $\mathcal{D}^\mathcal{I}$ are given by the equalities $\operatorname{ad}\mathbf{v}_1\cdots\operatorname{ad}\mathbf{v}_k = \operatorname{ad}\mathbf{v}'_1\cdots\operatorname{ad}\mathbf{v}'_{k'}$ whenever $\operatorname{ad}\mathbf{v}_i$ are simple and $\mathbf{v}_1\cdots\mathbf{v}_k = \mathbf{v}'_1\cdots\mathbf{v}'_{k'}$ in B^+. If $\mathbf{v} = \mathbf{v}_1\cdots\mathbf{v}_k \in B^+$ where the $\operatorname{ad}\mathbf{v}_i$ are simple morphisms of $\mathcal{D}^\mathcal{I}$, we still denote by $\operatorname{ad}\mathbf{v}$ the composed morphism in $\mathcal{D}^\mathcal{I}$.

Note that for $\mathbf{w}\phi \in B^+\phi(\mathcal{I})(\mathbf{I})$, the centralizer $\operatorname{Conj} B^+(\mathcal{I})(\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I})$ identifies via the forgetful functor with the monoid

$$B^+_{\mathbf{w}} := \{\mathbf{b} \in C_{B^+}(\mathbf{w}\phi) \mid \mathbf{I}^{\mathbf{b}} = \mathbf{I} \text{ and } \alpha_{\mathbf{I}}(\mathbf{b}) = 1\}.$$

The following theorem gives a general case where we can describe $\mathcal{D}^{\mathcal{I}}(\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I})$:

Theorem 5.9. *Assume that some power of* $\mathbf{w}\phi$ *is divisible on the left by* $\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0$. *Then* $\mathcal{D}^{\mathcal{I}}(\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}) = \operatorname{Conj} B^+(\mathcal{I})(\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I})$*, thus consists of the morphisms* $\operatorname{ad}\mathbf{b}$ *where* $\mathbf{b} \in B^+_{\mathbf{w}}$.

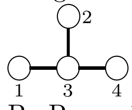
Proof. This is a special case of Proposition 3.9. □

Note that if k is the smallest power such that ${}^{\phi^k}\mathbf{I} = \mathbf{I}$ and ${}^{\phi^k}\mathbf{w} = \mathbf{w}$, then $\mathbf{w}^{(k)} := \mathbf{w}{}^{\phi}\mathbf{w}\cdots{}^{\phi^{k-1}}\mathbf{w}$ is in $B^+_{\mathbf{w}}$. Since $\operatorname{ad}\mathbf{w}$ is equal up to an invertible to the Garside map of $\mathcal{D}^{\mathcal{I}}$ described in Proposition 3.7 and $\operatorname{ad}\mathbf{w}^{(k)}$ is equal up to an invertible to the k-th power of that map, every element of $\mathcal{D}^{\mathcal{I}}(\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I})$ divides a power of $\operatorname{ad}\mathbf{w}^{(k)}$; it follows that under the assumptions of Theorem 5.9 every element of $B^+_{\mathbf{w}}$ divides a power of $\mathbf{w}^{(k)}$. In particular, in the case $\mathbf{I} = \emptyset$, Theorem 5.9 says that $B^+ \cap C_B(\mathbf{w}\phi) = \operatorname{End}_{\mathcal{D}^+}(\mathbf{w})$, with the notations of [DM2, 2.1]. Since $\mathbf{w}_0$ divides a power of $\mathbf{w}\phi$, hence a power of $\mathbf{w}^{(k)}$, any element of the group $C_B(\mathbf{w}\phi)$ multiplied by some power of $\mathbf{w}^{(k)}$ lies in B^+, hence the group $C_B(\mathbf{w}\phi)$ is generated as a monoid by $\operatorname{End}_{\mathcal{D}^+}(\mathbf{w})$ and $(\mathbf{w}^{(k)})^{-1}$. Thus Theorem 5.9 in this particular case gives a positive answer to conjecture [DM2, 2.1].

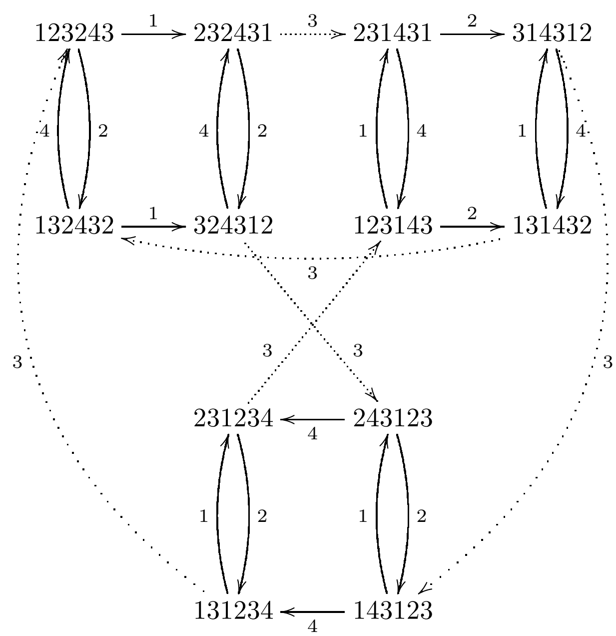
As an example of Theorem 5.9 we get that $\mathcal{D}^{\mathcal{I}}(\mathbf{I} \xrightarrow{\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}\phi} \mathbf{I})$ identifies with $\{\mathbf{b} \in C_{B^+}(\mathbf{I})^{\phi} \mid \alpha_{\mathbf{I}}(\mathbf{b}) = 1\}$ which itself identifies with $B^+(\mathcal{I})(\mathbf{I})^{\phi}$.

Two examples. In two cases we show a picture of the category associated with the centralizer of a periodic element.

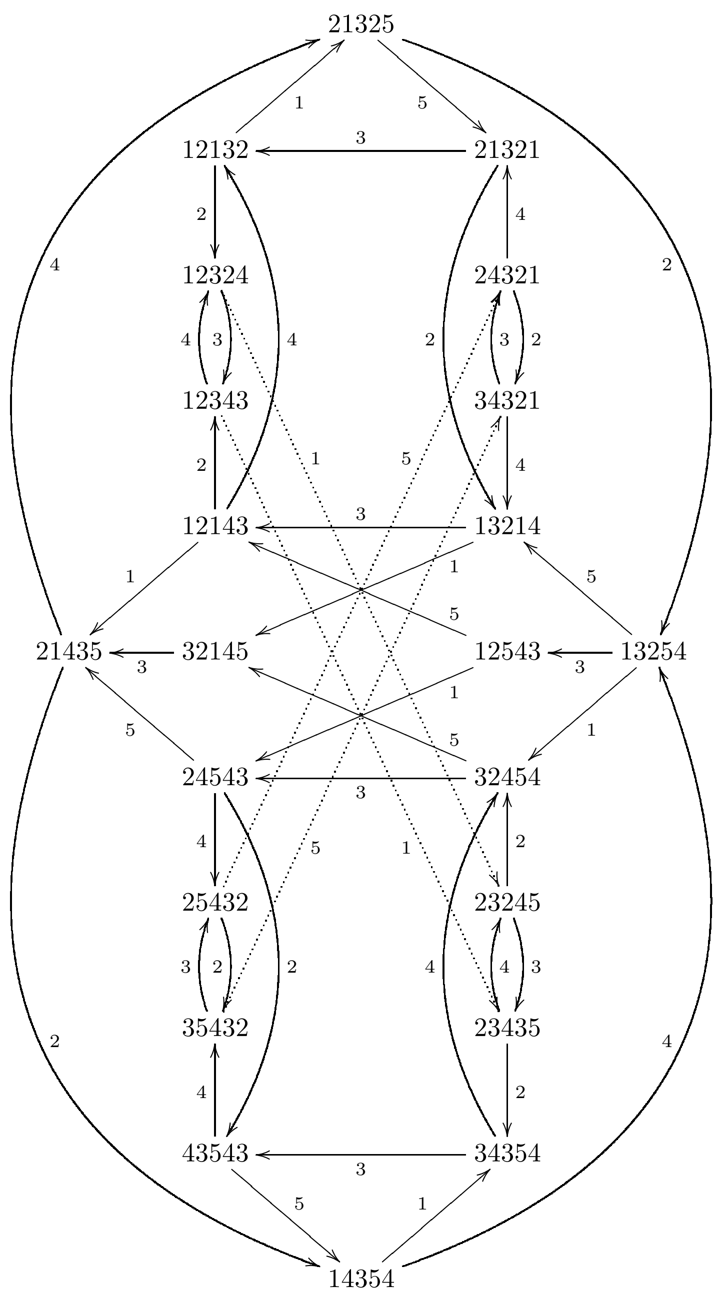
We first look at the case of a $(4,2)$-periodic element $\mathbf{w} \in B^+(W(D_4))$; by Proposition 5.2(i) we may assume $\mathbf{w}^2 = \mathbf{w}_0$; following Proposition 5.2(i) we describe the monoid $(\operatorname{cyc} B^+(\mathbf{w}))^{\mathbf{w}_0}$, in our case equal to $\operatorname{cyc} B^+(\mathbf{w})$ since $\mathbf{w}_0$ is central. As in Theorem 10.11, we choose $\mathbf{w}$ given by the word in the generators 123423 where the labeling of the Coxeter diagram is



By Proposition 5.2(i) the monoid $\operatorname{cyc} B^+(\mathbf{w})$ generates $C_B(\mathbf{w})$; by [B1, 12.5(ii)], $C_B(\mathbf{w})$ is the braid group of $C_W(w) \simeq G(4,2,2)$. This braid group has presentation $\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \mid \mathbf{xyz} = \mathbf{yzx} = \mathbf{zxy}\rangle$. The automorphism $\mathbf{x} \mapsto \mathbf{y} \mapsto \mathbf{z}$ corresponds to the triality in D_4. One of the generators $\mathbf{x}$ corresponds to the morphism 24 in the diagram below. The other generators are the conjugates of the similar morphisms 41 and 21 in the other squares.



We now look at a $(3,2)$-periodic $\mathbf{w} \in B^+(W(A_5))$, that is $\mathbf{w}^3 = \boldsymbol{\pi}$, and following Proposition 5.2(ii) we describe $\operatorname{cyc} B^+(\mathbf{v}\Phi)$ where Φ is the Garside automorphism $\mathbf{b} \mapsto \mathbf{b}^{\mathbf{w}_0}$ and where $\mathbf{w} = \mathbf{v}\Phi(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}^{\mathbf{w}_0}$. By Proposition 5.2(ii) the monoid $\operatorname{cyc} B^+(s\Phi)$ generates $C_B(\mathbf{w})$ and, again by the results of Bessis, $C_B(\mathbf{w})$ is the braid group of $C_W(w) \simeq G(3,1,2)$ (see Theorem 10.4). We choose $\mathbf{w}$ such that $\mathbf{v}$ is given by the word 21325 in the generators. The generator of $C_B(\mathbf{w})$ lifting the generator of order 3 of $G(3,1,2)$ is given by the word 531. The other one is the conjugate of any of the length 2 cycles 23 in the diagram.



6. Representations into bicategories

We give here a theorem on representations of categories with Garside families which generalizes a result of Deligne [D, 1.11] about representations of spherical braid monoids into a category; just as this theorem of Deligne was used to attach a Deligne-Lusztig variety to an element of an Artin monoid, our theorem will be used to attach a Deligne-Lusztig variety to a morphism of a ribbon category. Note that Theorem 6.2 covers the case of non-spherical Artin monoids.

We follow the terminology of [McL, XII.6] for bicategories. By *representation of category $\mathcal{C}$ into bicategory X* we mean a morphism of bicategories between $\mathcal{C}$ viewed as a trivial bicategory into the given bicategory X. This amounts to give a map T from $\mathrm{Obj}(\mathcal{C})$ to the 0-cells of X, and for $f \in \mathcal{C}$ of source x and target y, an element $T(f) \in V(T(x), T(y))$ where $V(T(x), T(y))$ is the category whose objects (resp. morphisms) are the 1-cells of X with domain $T(x)$ and codomain $T(y)$ (resp. the 2-cells between them), together with for each composable pair (f, g) an isomorphism $T(f)T(g) \xrightarrow{\sim} T(fg)$ such that the resulting square

$$\begin{array}{ccc} T(f)T(f')T(f'') & \xrightarrow{\sim} & T(ff')T(f'') \\ \downarrow\sim & & \downarrow\sim \\ T(f)T(f'f'') & \xrightarrow{\sim} & T(ff'f'') \end{array} \tag{6.1}$$

commutes.

We define a representation of the Garside family $\mathcal{S}$ as the same, except that the above square is restricted to the case where f, ff' and $ff'f''$ are in $\mathcal{S}$, (which implies $f', f'', f'f'' \in \mathcal{S}$ since $\mathcal{S}$ is closed under right-divisors). We then have

Theorem 6.2. *Let $\mathcal{C}$ be a right-Noetherian category which admits conditional right-lcms and has a Garside family $\mathcal{S}$. Then any representation of $\mathcal{S}$ into a bicategory extends uniquely to a representation of $\mathcal{C}$ into the same bicategory.*

Proof. The proof goes exactly as in [D], in that what must been proven is a simple connectedness property for the set $E(g)$ of decompositions as a product of elements of $\mathcal{S}$ of an arbitrary morphism $g \in \mathcal{C}$— this generalizes [D, 1.7] and is used in the same way. In his context, Deligne shows more, the contractibility of the set of decompositions; on the other hand our proof, which follows a suggestion by Serge Bouc to use a version of [Bouc, Lemma 6], is simpler and holds in our more general context.

Fix $g \in \mathcal{C}$ with $g \notin \mathcal{C}^\times$. We denote by $E(g)$ the set of decompositions of g into a product of elements of $\mathcal{S} - \mathcal{C}^\times$.

Then $E(g)$ is a poset, the order being defined by

$$(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_n) > (g_1, \ldots, g_{i-1}, a, b, g_{i+1}, \ldots, g_n)$$

if $ab = g_i \in \mathcal{S}$.

We recall the definition of homotopy in a poset E (a translation of the corresponding notion in a simplicial complex isomorphic as a poset to E). A path from x_1 to x_k in E is a sequence $x_1 \cdots x_k$ where each x_i is comparable to x_{i+1}. The composition of paths is defined by concatenation. Homotopy, denoted by $\sim$, is the finest equivalence relation on paths compatible with concatenation and generated by the two following elementary relations: $xyz \sim xz$ if $x \le y \le z$ and both $xyx \sim x$ and $yxy \sim y$ when $x \le y$. Homotopy classes form a groupoid, as the composition of a path with source x and of the inverse path is homotopic to the constant path at x. For $x \in E$ we denote by $\Pi_1(E, x)$ the fundamental group of E with base point x, which is the group of homotopy classes of loops starting from x.

A poset E is said to be *simply connected* if it is connected (there is a path linking any two elements of E) and if the fundamental group with some (or any) base point is trivial.

Note that a poset with a smallest or largest element x is simply connected since any path $xyzt\cdots x$ is homotopic to $xyxzxtx\cdots x$ which is homotopic to the trivial loop.

Proposition 6.3. *The set $E(g)$ is simply connected.*

Proof. First we prove a version of a lemma from [Bouc] on order preserving maps between posets. For a poset E we put $E_{\geq x} = \{x' \in E \mid x' \geq x\}$, which is a simply connected subposet of E since it has a smallest element. If $f : X \to Y$ is an order preserving map it is compatible with homotopy (it corresponds to a continuous map between simplicial complexes), so it induces a homomorphism $f^* : \Pi_1(X, x) \to \Pi_1(Y, f(x))$.

Lemma 6.4 (Bouc)**.** *Let $f : X \to Y$ an order preserving map between two posets. We assume that Y is connected and that for any $y \in Y$ the poset $f^{-1}(Y_{\geq y})$ is connected and non empty. Then f^* is surjective. If moreover $f^{-1}(Y_{\geq y})$ is simply connected for all y then f^* is an isomorphism.*

Proof. Let us first show that X is connected. Let $x, x' \in X$; we choose a path $y_0 \cdots y_n$ in Y from $y_0 = f(x)$ to $y_n = f(x')$. For $i = 0, \ldots, n$, we choose $x_i \in f^{-1}(Y_{\geq y_i})$ with $x_0 = x$ and $x_n = x'$. Then if $y_i \geq y_{i+1}$ we have $f^{-1}(Y_{\geq y_i}) \subset f^{-1}(Y_{\geq y_{i+1}})$ so that there exists a path in $f^{-1}(Y_{\geq y_{i+1}})$ from x_i to x_{i+1}; otherwise $y_i < y_{i+1}$, which implies $f^{-1}(Y_{\geq y_i}) \supset f^{-1}(Y_{\geq y_{i+1}})$ and there exists a path in $f^{-1}(Y_{\geq y_i})$ from x_i to x_{i+1}. Concatenating these paths gives a path connecting x and x'.

We fix now $x_0 \in X$. Let $y_0 = f(x_0)$. We prove that $f^* : \Pi_1(X, x_0) \to \Pi_1(Y, y_0)$ is surjective. Let $y_0 y_1 \cdots y_n$ with $y_n = y_0$ be a loop in Y. We lift arbitrarily this loop into a loop $x_0 \text{—} \cdots \text{—} x_n$ in X as above, (where $x_i \text{—} x_{i+1}$ stands for a path from x_i to x_{i+1} which is either in $f^{-1}(Y_{\geq y_i})$ or in $f^{-1}(Y_{\geq y_{i+1}})$). Then the path $f(x_0 \text{—} x_1 \text{—} \cdots \text{—} x_n)$ is homotopic to $y_0 \cdots y_n$; this can be seen by induction: let us assume that $f(x_0 \text{—} x_1 \cdots \text{—} x_i)$ is homotopic to $y_0 \cdots y_i f(x_i)$; then the same property holds for $i + 1$: indeed $y_i y_{i+1} \sim y_i f(x_i) y_{i+1}$ as they are two paths in a simply connected set which is either $Y_{\geq y_i}$ or $Y_{\geq y_{i+1}}$; similarly we have $f(x_i) y_{i+1} f(x_{i+1}) \sim f(x_i \text{—} x_{i+1})$. Putting things together gives

$$\begin{aligned} y_0 \cdots y_i y_{i+1} f(x_{i+1}) &\sim y_0 y_1 \cdots y_i f(x_i) y_{i+1} f(x_{i+1}) \\ &\sim f(x_0 \text{—} \cdots \text{—} x_i) y_{i+1} f(x_{i+1}) \\ &\sim f(x_0 \text{—} \cdots \text{—} x_i \text{—} x_{i+1}). \end{aligned}$$

We now prove injectivity of f^* when all $f^{-1}(Y_{\geq y})$ are simply connected.

We first prove that if $x_0 \text{—} \cdots \text{—} x_n$ and $x'_0 \text{—} \cdots \text{—} x'_n$ are two loops lifting the same loop $y_0 \cdots y_n$, then they are homotopic. Indeed, we get by induction on i that $x_0 \text{—} \cdots \text{—} x_i \text{—} x'_i$ and $x'_0 \text{—} \cdots \text{—} x'_i$ are homotopic paths, using the fact that x_{i-1}, x_i, x'_{i-1} and x'_i are all in the same simply connected sub-poset, namely either $f^{-1}(Y_{\geq y_{i-1}})$ or $f^{-1}(Y_{\geq y_i})$.

It remains to prove that we can lift homotopies, which amounts to show that if we lift as above two loops which differ by an elementary homotopy, the liftings are homotopic. If $yy'y \sim y$ is an elementary homotopy with $y < y'$ (resp. $y > y'$), then $f^{-1}(Y_{\geq y'}) \subset f^{-1}(Y_{\geq y})$ (resp. $f^{-1}(Y_{\geq y}) \subset f^{-1}(Y_{\geq y'})$) and the lifting of $yy'y$ constructed as above is in $f^{-1}(Y_{\geq y})$ (resp. $f^{-1}(Y_{\geq y'})$) so is homotopic to the trivial path. If $y < y' < y''$, a lifting of $yy'y''$ constructed as above is in $f^{-1}(Y_{\geq y})$ so is homotopic to any path in $f^{-1}(Y_{\geq y})$ with the same endpoints. □

We now prove Proposition 6.3 by contradiction. If it fails we choose $g \in \mathcal{C}$ minimal for proper right-divisibility such that $E(g)$ is not simply connected.

Let L be the set of elements of $\mathcal{S} - \mathcal{C}^\times$ which are left-divisors of g. For any $I \subset L$, since the category admits conditional right-lcms and is right-Noetherian, the elements of I have an lcm. We fix such an lcm Δ_I. Let $E_I(g) = \{(g_1, \ldots, g_n) \in E(g) \mid \Delta_I \preccurlyeq g_1\}$. We claim that $E_I(g)$ is simply connected for $I \neq \emptyset$. This is clear if $g \in \Delta_I\mathcal{C}^\times$, in which case $E_I(g) = \{(g)\}$. Let us assume this is not the case. In the following, if $\Delta_I \preccurlyeq a$, we denote by a^I the element such that $a = \Delta_I a^I$. The set $E(g^I)$ is defined since $g \notin \Delta_I\mathcal{C}^\times$. We apply Lemma 6.4 to the map $f : E_I(g) \to E(g^I)$ defined by

$$(g_1, \ldots, g_n) \mapsto \begin{cases} (g_2, \ldots, g_n) & \text{if } g_1 = \Delta_I \\ (g_1^I, g_2, \ldots g_n) & \text{otherwise} \end{cases}.$$

This map preserves the order and any set $f^{-1}(Y_{\geq(g_1,\ldots,g_n)})$ has a least element, namely $(\Delta_I, g_1, \ldots, g_n)$, so is simply connected. As by minimality of g the set $E(g^I)$ is simply connected Lemma 6.4 implies that $E_I(g)$ is simply connected.

Let Y be the set of non-empty subsets of L. We now apply Lemma 6.4 to the map $f : E(g) \to Y$ defined by $(g_1, \ldots, g_n) \mapsto \{s \in L \mid s \preccurlyeq g_1\}$, where Y is ordered by inclusion. This map is order preserving since $(g_1, \ldots, g_n) < (g_1', \ldots, g_n')$ implies $g_1 \preccurlyeq g_1'$. We have $f^{-1}(Y_{\geq I}) = E_I(g)$, so this set is simply connected. Since Y, having a greatest element, is simply connected, Lemma 6.4 gives that $E(g)$ is simply connected, whence the proposition. □

□

II. Deligne-Lusztig varieties and eigenspaces

In this part, we study the Deligne-Lusztig varieties giving rise to a Lusztig induction functor $R_{\mathbf{L}}^{\mathbf{G}}$ and generalize them to varieties attached to elements of a ribbon category.

In Section 8 we consider the particular ribbons describing varieties which play a role in the Broué conjectures; they are associated with maximal eigenspaces of elements of the Weyl group.

Finally in Section 9 we spell out the geometric form of the Broué conjectures, describing how the action on the ℓ-adic cohomology of the endomorphisms of our varieties coming from the conjugacy category of the ribbon category should factorize through a cyclotomic Hecke algebra.

7. Parabolic Deligne-Lusztig varieties

Let $\mathbf{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_p$, and let F be an isogeny on $\mathbf{G}$ such that some power F^δ is a Frobenius for a split $\mathbb{F}_{q^\delta}$-structure (this defines a positive real number q such that q^δ is an integral power of p).

Let $\mathbf{L}$ be an F-stable Levi subgroup of a (non-necessarily F-stable) parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ and let $\mathbf{P} = \mathbf{L}\mathbf{V}$ be the corresponding Levi decomposition of $\mathbf{P}$. Let

$$\begin{aligned} \mathbf{X}_\mathbf{V} &= \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g\mathbf{V} \cap F(g\mathbf{V}) \neq \emptyset\} = \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g^{-1}\,{}^F g \in \mathbf{V}\,{}^F\mathbf{V}\} \\ &\simeq \{g \in \mathbf{G} \mid g^{-1}\,{}^F g \in {}^F\mathbf{V}\}/(\mathbf{V} \cap {}^F\mathbf{V}). \end{aligned}$$

On this variety $\mathbf{G}^F$ acts by left-multiplication and $\mathbf{L}^F$ acts by right-multiplication.

We choose a prime number $\ell \neq p$. Then the virtual $\mathbf{G}^F$-module-$\mathbf{L}^F$ given by $M = \sum_i (-1)^i H^i_c(\mathbf{X}_\mathbf{V}, \overline{\mathbb{Q}}_\ell)$ defines the Lusztig induction $R_\mathbf{L}^\mathbf{G}$ which by definition maps an $\mathbf{L}^F$-module λ to $M \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{L}^F} \lambda$.

The map $g\mathbf{V} \mapsto g\mathbf{P}$ makes $\mathbf{X}_\mathbf{V}$ an $\mathbf{L}^F$-torsor over

$$\begin{aligned} \mathbf{X}_\mathbf{P} &= \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g\mathbf{P} \cap F(g\mathbf{P}) \neq \emptyset\} = \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}\,{}^F g \in \mathbf{P}\,{}^F\mathbf{P}\} \\ &\simeq \{g \in \mathbf{G} \mid g^{-1}\,{}^F g \in {}^F\mathbf{P}\}/(\mathbf{P} \cap {}^F\mathbf{P}), \end{aligned}$$

a $\mathbf{G}^F$-variety such that $R_\mathbf{L}^\mathbf{G}(\mathrm{Id}) = \sum_i (-1)^i H^i_c(\mathbf{X}_\mathbf{P}, \overline{\mathbb{Q}}_\ell)$. The variety $\mathbf{X}_\mathbf{P}$ is the prototype of the varieties we want to study.

Let $\mathbf{T} \subset \mathbf{B}$ be a pair of an F-stable maximal torus and an F-stable Borel subgroup of $\mathbf{G}$. With this choice is associated a basis Π of the root system Φ of $\mathbf{G}$ with respect to $\mathbf{T}$, and a Coxeter system (W, S) for the Weyl group $W = N_\mathbf{G}(\mathbf{T})/\mathbf{T}$. Let $X_\mathbb{R} = X(\mathbf{T}) \otimes \mathbb{R}$ where $X(\mathbf{T})$ is the group of rational characters of the torus $\mathbf{T}$. On the vector space $X_\mathbb{R}$, the isogeny F acts as $q\phi$ where ϕ is of order δ and stabilizes the positive cone $\mathbb{R}^+\Pi$; we will still denote by ϕ the induced automorphism of (W, S).

To a subset $I \subset \Pi$ corresponds a subgroup $W_I \subset W$, a parabolic subgroup $\mathbf{P}_I = \coprod_{w \in W_I} \mathbf{B}w\mathbf{B}$, and the Levi subgroup $\mathbf{L}_I$ of $\mathbf{P}_I$ which contains $\mathbf{T}$.

Given any $\mathbf{P} = \mathbf{L}\mathbf{V}$ as in the beginning of this section, where $\mathbf{L}$ is F-stable, there exists $I \subset \Pi$ such that $(\mathbf{L}, \mathbf{P})$ is $\mathbf{G}$-conjugate to $(\mathbf{L}_I, \mathbf{P}_I)$; if we choose the conjugating element such that it conjugates a maximally split torus of $\mathbf{L}$ to $\mathbf{T}$ and a rational Borel subgroup of $\mathbf{L}$ containing this torus to $\mathbf{B} \cap \mathbf{L}_I$, then this element conjugates $(\mathbf{L}, \mathbf{P}, F)$ to $(\mathbf{L}_I, \mathbf{P}_I, \dot{w}F)$ where $\dot{w} \in N_\mathbf{G}(\mathbf{T})$ is such that ${}^{w\phi}I = I$, where w is the image of $\dot{w}$ in W.

It will be convenient to consider I as a subset of S instead of a subset of Π; the condition on w must then be stated as "$I^w = {}^\phi I$ and w is I-reduced". Note that w is then also reduced-${}^\phi I$. Via the above conjugation, the variety $\mathbf{X}_\mathbf{P}$ is isomorphic to the variety

$$\mathbf{X}(I, w\phi) = \{g\mathbf{P}_I \in \mathbf{G}/\mathbf{P}_I \mid g^{-1}\,{}^F g \in \mathbf{P}_I w\,{}^F\mathbf{P}_I\}.$$

We will denote by $\mathbf{X}_\mathbf{G}(I, w\phi)$ this variety when there is a possible ambiguity on the group. If we denote by $\mathbf{U}_I$ the unipotent radical of $\mathbf{P}_I$, we have $\dim \mathbf{X}(I, w\phi) = \dim \mathbf{U}_I - \dim(\mathbf{U}_I \cap {}^{wF}\mathbf{U}_I) = l(w)$, the last equality since w is reduced-${}^\phi I$. The ℓ-adic cohomology of the variety $\mathbf{X}(I, w\phi)$ gives rise to the Lusztig induction from $\mathbf{L}_I^{\dot{w}F}$ to $\mathbf{G}^F$ of the trivial representation; to avoid ambiguity on the isogenies involved, we will sometimes denote this Lusztig induction by $R^{\mathbf{G},F}_{\mathbf{L}_I,\dot{w}F}(\mathrm{Id})$.

Definition 7.1. *We say that a pair* $(\mathbf{P},\mathbf{Q})$ *of parabolic subgroups is in relative position* (I,w,J)*, where* $I,J\subset S$ *and* $w\in W$*, if* $(\mathbf{P},\mathbf{Q})$ *is* $\mathbf{G}$*-conjugate to* $(\mathbf{P}_I,{}^w\mathbf{P}_J)$*. We denote this as* $\mathbf{P}\xrightarrow{I,w,J}\mathbf{Q}$.

Since any pair $(\mathbf{P},\mathbf{Q})$ of parabolic subgroups share a common maximal torus, it has a relative position (I,w,J) where I,J is uniquely determined as well as the double coset W_IwW_J.

Let $\mathcal{P}_I$ be the variety of parabolic subgroups conjugate to $\mathbf{P}_I$; this variety is isomorphic to $\mathbf{G}/\mathbf{P}_I$. Via the map $g\mathbf{P}_I\mapsto {}^g\mathbf{P}_I$ we have an isomorphism

$$\mathbf{X}(I,w\phi)\simeq\{\mathbf{P}\in\mathcal{P}_I\mid \mathbf{P}\xrightarrow{I,w,{}^\phi I}{}^F\mathbf{P}\};$$

it is a variety over $\mathcal{P}_I\times\mathcal{P}_{\phi I}$ by the first and second projection.

The varieties $\mathcal{O}$ attached to $B^+(\mathcal{I})$. In order to have a rich enough monoid of endomorphisms (see Definition 7.21), we need to generalize the pairs $(I,w\phi)$ which label our varieties to the larger set of morphisms of the category $B^+(\mathcal{I})$ of Section 5, where $\mathcal{I}$ is the conjugacy class in B^+ of the lift $\mathbf{I}$ of I.

In order to do this, we define in this subsection a representation of $B^+(\mathcal{I})$ into the bicategory $\mathbf{X}$ of varieties over $\mathcal{P}_I\times\mathcal{P}_J$, where I,J vary over $\mathcal{I}$. The bicategory $\mathbf{X}$ has 0-cells which are the elements of $\mathcal{I}$, has 1-cells with domain $\mathbf{I}$ and codomain $\mathbf{J}$ which are the $\mathcal{P}_I\times\mathcal{P}_J$-varieties and has 2-cells which are isomorphisms of $\mathcal{P}_I\times\mathcal{P}_J$-varieties. For $\mathbf{I},\mathbf{J}\in\mathcal{I}$ we denote by $V(\mathbf{I},\mathbf{J})$ the category whose objects (resp. morphisms) are the 1-cells with domain $\mathbf{I}$ and codomain $\mathbf{J}$ (resp. the 2-cells between them); in other words, $V(\mathbf{I},\mathbf{J})$ is the category of $\mathcal{P}_I\times\mathcal{P}_J$-varieties endowed with the isomorphisms of $\mathcal{P}_I\times\mathcal{P}_J$-varieties. The horizontal composition bifunctor $V(\mathbf{I},\mathbf{J})\times V(\mathbf{J},\mathbf{K})\to V(\mathbf{I},\mathbf{K})$ is given by the fibered product over $\mathcal{P}_J$. The vertical composition is given by the composition of isomorphisms.

The representation of $B^+(\mathcal{I})$ in $\mathbf{X}$ we construct will be denoted by T, following the notations of Section 6. For $\mathbf{I}\xrightarrow{\mathbf{b}}\mathbf{J}\in B^+(\mathcal{I})$, we will also write $\mathcal{O}(\mathbf{I},\mathbf{b})$ for $T(\mathbf{I}\xrightarrow{\mathbf{b}}\mathbf{J})$, to lighten the notation. We first define T on the Garside family $\mathcal{S}$ of $B^+(\mathcal{I})$.

Definition 7.2. *For* $(\mathbf{I}\xrightarrow{\mathbf{w}}\mathbf{J})\in\mathcal{S}$ *we define* $\mathcal{O}(\mathbf{I},\mathbf{w})$ *to be the variety* $\{(\mathbf{P},\mathbf{P}')\in\mathcal{P}_I\times\mathcal{P}_J\mid \mathbf{P}\xrightarrow{I,w,J}\mathbf{P}'\}$*, where* I*,* w*,* J *are the images in* W *of* $\mathbf{I}$*,* $\mathbf{w}$*,* $\mathbf{J}$.

The following lemma constructs the isomorphism $T(f)T(g)\xrightarrow{\sim}T(fg)$ when $f,g,fg\in\mathcal{S}$:

Lemma 7.3. *Let* $(\mathbf{I}\xrightarrow{\mathbf{w}_1}\mathbf{I}_2\xrightarrow{\mathbf{w}_2}\mathbf{J})=(\mathbf{I}\xrightarrow{\mathbf{w}}\mathbf{J})$ *where* $\mathbf{w}=\mathbf{w}_1\mathbf{w}_2\in\mathbf{W}$ *be a defining relation of* $B^+(\mathcal{I})$*. Then* $(p',p''):\mathcal{O}(I,\mathbf{w}_1)\times_{\mathcal{P}_{I_2}}\mathcal{O}(I_2,\mathbf{w}_2)\xrightarrow{\sim}\mathcal{O}(I,\mathbf{w}_1\mathbf{w}_2)$ *is an isomorphism, where* p' *and* p'' *are respectively the first and last projections.*

Proof. First notice that for two parabolic subgroups $(\mathbf{P}',\mathbf{P}'')\in\mathcal{P}_I\times\mathcal{P}_J$ we have $\mathbf{P}'\xrightarrow{I,w,J}\mathbf{P}''$ if and only if the pair $(\mathbf{P}',\mathbf{P}'')$ is conjugate to a pair containing termwise the pair $(\mathbf{B},{}^w\mathbf{B})$. This shows that if $\mathbf{P}'\xrightarrow{I,w_1,I_2}\mathbf{P}_1$ and $\mathbf{P}_1\xrightarrow{I_2,w_2,J}\mathbf{P}''$ then $\mathbf{P}'\xrightarrow{I,w_1w_2,J}\mathbf{P}''$, so (p',p'') goes to the claimed variety.

Conversely, we have to show that given $\mathbf{P}'\xrightarrow{I,w,J}\mathbf{P}''$ there is a unique $\mathbf{P}_1$ such that $\mathbf{P}'\xrightarrow{I,w_1,I_2}\mathbf{P}_1\xrightarrow{I_2,w_2,J}\mathbf{P}''$. The image of $(\mathbf{B},{}^w\mathbf{B})$ by the conjugation which sends $(\mathbf{P}_I,{}^w\mathbf{P}_J)$ to $(\mathbf{P}',\mathbf{P}'')$ is a pair of Borel subgroups $(\mathbf{B}'\subset\mathbf{P}',\mathbf{B}''\subset\mathbf{P}'')$ in position w. Since $l(w_1)+l(w_2)=l(w)$, there is a unique Borel subgroup $\mathbf{B}_1$ such that $\mathbf{B}'\xrightarrow{w_1}\mathbf{B}_1\xrightarrow{w_2}\mathbf{B}''$. The unique parabolic subgroup of type I_2 containing $\mathbf{B}_1$ has the desired relative positions, so $\mathbf{P}_1$ exists. And any other parabolic subgroup $\mathbf{P}'_1$ which has the desired relative positions contains a Borel subgroup $\mathbf{B}'_1$ such that $\mathbf{B}'\xrightarrow{w_1}\mathbf{B}'_1\xrightarrow{w_2}\mathbf{B}''$ (take for $\mathbf{B}'_1$ the image of ${}^{w_1}\mathbf{B}$ by the conjugation which maps $(\mathbf{P}_I,{}^{w_1}\mathbf{P}_{I_2})$ to $(\mathbf{P}',\mathbf{P}'_1)$), which implies that $\mathbf{B}'_1=\mathbf{B}_1$ and thus $\mathbf{P}'_1=\mathbf{P}_1$. Thus our map is bijective on points. To show it is an isomorphism, it is sufficient to check that its target is a normal variety, which is given by

Lemma 7.4. *For* $(\mathbf{I}\xrightarrow{\mathbf{w}}\mathbf{J})\in\mathcal{S}$ *the variety* $\mathcal{O}(\mathbf{I},\mathbf{w})$ *is smooth.*

Proof. Consider the locally trivial fibrations with smooth fibers given by $\mathbf{G}\times\mathbf{G}\xrightarrow{p}\mathcal{P}_I\times\mathcal{P}_J:(g_1,g_2)\mapsto({}^{g_1}\mathbf{P}_I,{}^{g_2w}\mathbf{P}_J)$ and $\mathbf{G}\times\mathbf{G}\xrightarrow{q}\mathbf{G}:(g_1,g_2)\mapsto g_1^{-1}g_2$. It is easy to check that $\mathcal{O}(\mathbf{I},\mathbf{w})=p(q^{-1}({}^{w}\mathbf{P}_J))$ thus by for example [DMR, 2.2.3] it is smooth. □

□

From the above lemma we see also that the square 6.1 commutes for elements of $\mathcal{S}$, since the isomorphism "forgetting the middle parabolic" has clearly the corresponding property. We have thus defined a representation T of $\mathcal{S}$ in $\mathbf{X}$.

The extension of T to the whole of $B^+(\mathcal{I})$ associates with a composition $\mathbf{I}\xrightarrow{\mathbf{w}_1}\mathbf{I}_2\to\cdots\to\mathbf{I}_k\xrightarrow{\mathbf{w}_k}\mathbf{J}$ with $\mathbf{w}_i\in\mathbf{W}$ the variety

$$\mathcal{O}(\mathbf{I},\mathbf{w}_1)\times_{\mathcal{P}_{I_2}}\cdots\times_{\mathcal{P}_{I_k}}\mathcal{O}(\mathbf{I}_k,\mathbf{w}_k)=\{(\mathbf{P}_1,\ldots,\mathbf{P}_{k+1})\mid\mathbf{P}_i\xrightarrow{I_i,w_i,I_{i+1}}\mathbf{P}_{i+1}\},$$

where $I_1=I$ and $I_{k+1}=J$. It is a $\mathcal{P}_I\times\mathcal{P}_J$-variety via the first and last projections mapping $(\mathbf{P}_1,\ldots,\mathbf{P}_{k+1})$ respectively to $\mathbf{P}_1$ and $\mathbf{P}_{k+1}$, and Lemma 7.3 shows that up to isomorphism it does not depend on the chosen decomposition of $\mathbf{I}\xrightarrow{\mathbf{w}_1\cdots\mathbf{w}_k}\mathbf{J}$. Theorem 6.2 shows that there is actually a unique isomorphism between the various models attached to different decompositions, so T associates a well-defined variety to any element of $B^+(\mathcal{I})$.

Definition 7.5. *For* $\mathbf{I}\xrightarrow{\mathbf{b}}\mathbf{J}\in B^+(\mathcal{I})$ *we denote by* $\mathcal{O}(\mathbf{I},\mathbf{b})$ *the variety defined by Theorem 6.2. For any decomposition* $(\mathbf{I}\xrightarrow{\mathbf{b}}\mathbf{J})=(\mathbf{I}_1\xrightarrow{\mathbf{w}_1}\mathbf{I}_2\to\cdots\xrightarrow{\mathbf{w}_k}\mathbf{J})$ *into elements of* $\mathcal{S}$ *it has the model* $\{(\mathbf{P}_1,\ldots,\mathbf{P}_{k+1})\mid\mathbf{P}_i\xrightarrow{I_i,w_i,I_{i+1}}\mathbf{P}_{i+1}\}$.

The variety $\mathcal{O}(\mathbf{I},\mathbf{b})$ is endowed with a natural action of $\mathbf{G}$ by simultaneous conjugation of the $\mathbf{P}_i$.

The Deligne-Lusztig varieties attached to $B^+(\mathcal{I})$. The automorphism ϕ lifts naturally to an automorphism of B^+ which stabilizes $\mathbf{S}$, which we will still denote by ϕ, by abuse of notation. For $(\mathbf{I}\xrightarrow{\mathbf{w}}{}^{\phi}\mathbf{I})\in\mathcal{S}$, the variety $\mathbf{X}(I,w\phi)$ is the intersection of $\mathcal{O}(\mathbf{I},\mathbf{w})$ with the graph of F, that is, points whose image under (p',p'') has the form $(\mathbf{P},{}^{F}\mathbf{P})$. Via the correspondance between ϕ-conjugacy and conjugacy in the coset, we interpret $\mathbf{I}\xrightarrow{\mathbf{w}}{}^{\phi}\mathbf{I}$ as the endomorphism $\mathbf{I}\xrightarrow{\mathbf{w}\phi}\mathbf{I}$ in $B^+\phi(\mathcal{I})$. More generally,

Definition 7.6. *Let* $\mathbf{I}\xrightarrow{\mathbf{b}\phi}\mathbf{I}$ *be an endomorphism of* $B^+\phi(\mathcal{I})$; *we define the variety* $\mathbf{X}(\mathbf{I},\mathbf{b}\phi)$ *as the intersection of* $\mathcal{O}(\mathbf{I},\mathbf{b})$ *with the graph of* F.

The action of $\mathbf{G}$ on $\mathcal{O}(\mathbf{I},\mathbf{b})$ restricts to an action of $\mathbf{G}^F$ on $\mathbf{X}(\mathbf{I},\mathbf{b}\phi)$. This last variety may be interpreted as an "ordinary" parabolic Deligne-Lusztig variety in a group which is a restriction of scalars:

Proposition 7.7. *For any decomposition* $(\mathbf{I}\xrightarrow{\mathbf{b}}{}^\phi\mathbf{I})=(\mathbf{I}_1\xrightarrow{\mathbf{w}_1}\mathbf{I}_2\to\cdots\xrightarrow{\mathbf{w}_k}{}^\phi\mathbf{I})$ *in elements of* $\mathcal{S}$ *the variety* $\mathbf{X}(\mathbf{I},\mathbf{b}\phi)$ *has the model* $\{(\mathbf{P}_1,\ldots,\mathbf{P}_{k+1})\mid\mathbf{P}_i\xrightarrow{I_i,w_i,I_{i+1}}\mathbf{P}_{i+1}$ *and* $\mathbf{P}_{k+1}=F(\mathbf{P}_1)\}$. *Let* F_1 *be the isogeny of* $\mathbf{G}^k$ *defined by* $F_1(g_1,\ldots,g_k)=(g_2,\ldots,g_k,F(g_1))$ *and let* ϕ_1 *be the corresponding automorphism of* W^k. *Then the above model is isomorphic to* $\mathbf{X}_{\mathbf{G}^k}(I_1\times\cdots\times I_k,(w_1,\ldots,w_k)\phi_1)$. *By this isomorphism the action of* F^δ *corresponds to that of* $F_1^{k\delta}$ *and the action of* $\mathbf{G}^F$ *corresponds to that of* $(\mathbf{G}^k)^{F_1}$*—the isomorphism* $\mathbf{G}^F\xrightarrow{\sim}(\mathbf{G}^k)^{F_1}$ *is via the diagonal embedding.*

Proof. That $\mathbf{X}(\mathbf{I},\mathbf{b}\phi)$ has the model given above is a consequence of the analogous statement for $\mathcal{O}(\mathbf{I},\mathbf{b})$.

An element $\mathbf{P}_1\times\cdots\times\mathbf{P}_k\in\mathbf{X}_{\mathbf{G}^k}(I_1\times\cdots\times I_k,(w_1,\ldots,w_k)\phi_1)$ by definition satisfies

$$\mathbf{P}_1\times\cdots\times\mathbf{P}_k\xrightarrow{I_1\times\cdots I_k,(w_1,\ldots,w_k),I_2\times\cdots I_k\times{}^\phi I_1}\mathbf{P}_2\times\cdots\times\mathbf{P}_k\times{}^F\mathbf{P}_1$$

thus is equivalently given by a sequence $(\mathbf{P}_1,\ldots,\mathbf{P}_{k+1})$ such that $\mathbf{P}_i\xrightarrow{I_i,w_i,I_{i+1}}\mathbf{P}_{i+1}$ with $\mathbf{P}_{k+1}={}^F\mathbf{P}_1$ and $I_{k+1}={}^\phi I_1$, which is the same as an element

$$(\mathbf{P}_1,\ldots,\mathbf{P}_{k+1})\in\mathcal{O}(\mathbf{I}_1,\mathbf{w}_1)\times_{\mathcal{P}_{I_2}}\mathcal{O}(\mathbf{I}_2,\mathbf{w}_2)\cdots\times_{\mathcal{P}_{I_k}}\mathcal{O}(\mathbf{I}_k,\mathbf{w}_k)$$

such that $\mathbf{P}_{k+1}={}^F\mathbf{P}_1$. But this is a model of $\mathbf{X}_\mathbf{G}(\mathbf{I},\mathbf{b}\phi)$ as explained above.

One checks easily that this sequence of identifications is compatible with the actions of F^δ and $\mathbf{G}^F$ as described by the proposition. □

Proposition 7.8. *The variety* $\mathbf{X}(\mathbf{I},\mathbf{b}\phi)$ *is irreducible if and only if* $\mathbf{I}\cup\operatorname{supp}(\mathbf{b})$ *meets all the orbits of* ϕ *on* $\mathbf{S}$, *where* $\operatorname{supp}(\mathbf{b})$ *is the support of* $\mathbf{b}$ *(see the proof of Lemma 4.5).*

Proof. This is, using Proposition 7.7, an immediate translation in our setting of the result [BR, Theorem 2] of Bonnafé-Rouquier. □

The varieties $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)$. The conjugation which transforms $\mathbf{X}_\mathbf{P}$ into $\mathbf{X}(I,w\phi)$ maps $\mathbf{X}_\mathbf{V}$ to the $\mathbf{G}^F$-variety-$\mathbf{L}_I^{\dot wF}$ given by

$$\tilde{\mathbf{X}}(I,\dot wF)=\{g\mathbf{U}_I\in\mathbf{G}/\mathbf{U}_I\mid g^{-1}\,{}^Fg\in\mathbf{U}_I\dot w\,{}^F\mathbf{U}_I\},\tag{7.9}$$

where $\dot w$ is a representative of w (any representative can be obtained by choosing an appropriate conjugation). The map $g\mathbf{U}_I\mapsto g\mathbf{P}_I$ makes $\tilde{\mathbf{X}}(I,\dot wF)$ a $\mathbf{L}_I^{\dot wF}$-torsor over $\mathbf{X}(I,w\phi)$. We have written $\dot w$ and F together since the variety depends only on the product $\dot wF\in N_\mathbf{G}(\mathbf{T})\rtimes\langle F\rangle$; we will write $\tilde{\mathbf{X}}(I,\dot w\cdot F)$ to separate the Frobenius endomorphism from the representative of the Weyl group element when needed, in the case where the ambient group is a Levi subgroup with Frobenius endomorphism of the form $\dot xF$.

In this section, we define a variety $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)$ which generalizes $\tilde{\mathbf{X}}(I,\dot wF)$ by replacing $\dot w$ by elements of the braid group. Since $\dot w$ represents a choice of a lift of w to $N_\mathbf{G}(\mathbf{T})$, we have to make uniformly such choices for all elements of the braid group, which we do by using a "Tits homomorphism".

First, when $\mathbf{w}\in\mathbf{W}$, we define a variety $\tilde{\mathcal{O}}(I,\dot w)$ "above" $\mathcal{O}(\mathbf{I},\mathbf{w})$ such that $\tilde{\mathbf{X}}(I,\dot wF)$ is the intersection of $\tilde{\mathcal{O}}(I,w)$ with the graph of F, and then we extend this construction to $\mathcal{B}^+(\mathcal{I})$.

Definition 7.10. *Let* $(\mathbf{I}\xrightarrow{\mathbf{w}}\mathbf{J})\in\mathcal{S}$*, and let* $\dot w\in N_{\mathbf{G}}(\mathbf{T})$ *be a representative of* w*. We define* $\tilde{\mathcal{O}}(I,\dot w)=\{(g\mathbf{U}_I,g'\mathbf{U}_J)\in\mathbf{G}/\mathbf{U}_I\times\mathbf{G}/\mathbf{U}_J\mid g^{-1}g'\in\mathbf{U}_I\dot w\mathbf{U}_J\}$.

The variety $\tilde{\mathcal{O}}(I,\dot w)$ has a left action of $\mathbf{G}$ by simultaneous translation and a right action of $\mathbf{L}_I$ by $(g\mathbf{U}_I,g'\mathbf{U}_J)\mapsto(gl\mathbf{U}_I,g'l^{\dot w}\mathbf{U}_J)$.

We can prove an analogue of Lemma 7.3.

Lemma 7.11. *Let* $(\mathbf{I}\xrightarrow{\mathbf{w}_1}\mathbf{I}_2\xrightarrow{\mathbf{w}_2}\mathbf{J})=(\mathbf{I}\xrightarrow{\mathbf{w}_1\mathbf{w}_2}\mathbf{J})$ *where* $\mathbf{w}_1\mathbf{w}_2\in\mathbf{W}$ *be a defining relation of* $B^+(\mathcal{I})$*, and let* $\dot w_1,\dot w_2$ *be representatives of the images of* $\mathbf{w}_1$ *and* $\mathbf{w}_2$ *in* W*. Then* $(p',p''):\tilde{\mathcal{O}}(I,\dot w_1)\times_{\mathbf{G}/\mathbf{U}_{I_2}}\tilde{\mathcal{O}}(I_2,\dot w_2)\xrightarrow{\sim}\tilde{\mathcal{O}}(I,\dot w_1\dot w_2)$ *is an isomorphism where* p' *and* p'' *are the first and last projections.*

Proof. We first note that if $\mathbf{I}\xrightarrow{\mathbf{w}}\mathbf{J}\in B^+(\mathcal{I})$ and $\dot w$ is a representative in $N_{\mathbf{G}}(\mathbf{T})$ of the image of $\mathbf{w}$ in W, then $\mathbf{U}_I\dot w\mathbf{U}_J$ is isomorphic by the product morphism to the direct product of varieties $(\mathbf{U}_I\cap{}^w\mathbf{U}_J^-)\dot w\times\mathbf{U}_J$, where $\mathbf{U}_J^-$ is the unipotent radical of the parabolic subgroup opposed to $\mathbf{P}_J$ containing $\mathbf{T}$. We now use the lemma:

Lemma 7.12. *Under the assumptions of Lemma 7.11, the product gives an isomorphism* $(\mathbf{U}_I\cap{}^{\dot w_1}\mathbf{U}_{I_2}^-)\dot w_1\times(\mathbf{U}_{I_2}\cap{}^{\dot w_2}\mathbf{U}_J^-)\dot w_2\xrightarrow{\sim}(\mathbf{U}_I\cap{}^{\dot w_1\dot w_2}\mathbf{U}_J^-)\dot w_1\dot w_2$.

Proof. Since w is I-reduced and $I^w=J$, we have $\mathbf{U}_I\cap{}^w\mathbf{U}_J^-=\prod_{-\alpha\in{}^wN(w)}\mathbf{U}_\alpha$ as a product of root subgroups, where $N(w)=\{\alpha\in\Phi^+\mid{}^w\alpha\in\Phi^-\}$. The lemma is then a consequence of the equality $N(w_1)^{w_2}\coprod N(w_2)=N(w_1w_2)$ when $l(w_1)+l(w_2)=l(w_1w_2)$. □

The lemma proves in particular that if $g_1^{-1}g_2\in\mathbf{U}_I\dot w_1\mathbf{U}_{I_2}$ and $g_2^{-1}g_3\in\mathbf{U}_{I_2}\dot w_2\mathbf{U}_J$ then $g_1^{-1}g_3\in\mathbf{U}_I\dot w_1\mathbf{U}_{I_2}\dot w_2\mathbf{U}_J=(\mathbf{U}_I\cap{}^{\dot w_1}\mathbf{U}_{I_2}^-)\dot w_1(\mathbf{U}_{I_2}\cap{}^{\dot w_2}\mathbf{U}_J^-)\dot w_2\mathbf{U}_J=(\mathbf{U}_I\cap{}^{\dot w_1\dot w_2}\mathbf{U}_J^-)\dot w_1\dot w_2\mathbf{U}_J=\mathbf{U}_I\dot w_1\dot w_2\mathbf{U}_J$, so the image of the morphism (p',p'') in Lemma 7.11 is indeed in the variety $\tilde{\mathcal{O}}(I,\dot w_1\dot w_2)$.

Conversely, we have to show that given $(g_1\mathbf{U}_I,g_3\mathbf{U}_{\mathbf{J}})\in\tilde{\mathcal{O}}(I,\dot w_1\dot w_2)$, there exists a unique $g_2U_{I_2}$ such that $(g_1\mathbf{U}_I,g_2\mathbf{U}_{I_2})\in\tilde{\mathcal{O}}(I,\dot w_1)$ and $(g_2\mathbf{U}_{I_2},g_3\mathbf{U}_{I_3})\in\tilde{\mathcal{O}}(I_2,\dot w_2)$. The varieties involved being invariant by left-translation by $\mathbf{G}$, it is enough to solve the problem when $g_1=1$. Then we have $g_3\in\mathbf{U}_I\dot w_1\dot w_2\mathbf{U}_J$, and the conditions for $g_2\mathbf{U}_{I_2}$ is that $g_2\mathbf{U}_{I_2}\subset\mathbf{U}_I\dot w_1\mathbf{U}_{I_2}$. Any such coset has then a unique representative in $(\mathbf{U}_I\cap{}^{\dot w_1}\mathbf{U}_{I_2}^-)\dot w_1$ and we will look for such a representative g_2. But we must have $g_2^{-1}g_3\in\mathbf{U}_{I_2}\dot w_2\mathbf{U}_J=(\mathbf{U}_{I_2}\cap{}^{\dot w_2}\mathbf{U}_J^-)\dot w_2\mathbf{U}_J$ and since by the lemma the product gives an isomorphism between $(\mathbf{U}_I\cap{}^{\dot w_1}\mathbf{U}_{I_2}^-)\dot w_1\times(\mathbf{U}_{I_2}\cap{}^{\dot w_2}\mathbf{U}_J^-)\dot w_2\mathbf{U}_J$ and $\mathbf{U}_I\dot w_1\dot w_2\mathbf{U}_J$, the element g_3 can be decomposed in one and only one way in a product $g_2(g_2^{-1}g_3)$ satisfying the conditions. To conclude as in Lemma 7.3 we show that the variety $\tilde{\mathcal{O}}(\mathbf{I},\dot w_1\dot w_2)$ is smooth. An argument similar to the proof of Lemma 7.4, replacing $\mathcal{P}_I$ and $\mathcal{P}_J$ by $\mathbf{G}/\mathbf{U}_I$ and $\mathbf{G}/\mathbf{U}_J$ respectively gives the result. □

The isomorphism of Lemma 7.11 is compatible with the action of $\mathbf{G}$ and of $\mathbf{L}_I$, $\mathbf{L}_{I_2}$ respectively.

We will now use a Tits homomorphism, which is a homomorphism $B\xrightarrow{t}N_{\mathbf{G}}(\mathbf{T})$ which factors the projection $B\to W$ —the existence of such a homomorphism is proved in [T]. Theorem 6.2 implies that, setting $T(\mathbf{I}\xrightarrow{\mathbf{w}}\mathbf{J})=\tilde{\mathcal{O}}(I,t(\mathbf{w}))$ for $(\mathbf{I}\xrightarrow{\mathbf{w}}\mathbf{J})\in\mathcal{S}$ and replacing Lemma 7.3 by Lemma 7.11, we can define a representation of $B^+(\mathcal{I})$ in the bicategory $\tilde{\mathbf{X}}$ of varieties above $\mathbf{G}/\mathbf{U}_I\times\mathbf{G}/\mathbf{U}_J$ for $I,J\in\mathcal{I}$.

Definition 7.13. *The above representation defines for any* $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \in B^+(\mathcal{I})$ *a variety* $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$ *which for any decomposition* $(\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}) = (\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \to \cdots \to \mathbf{I}_k \xrightarrow{\mathbf{w}_k} \mathbf{J})$ *into elements of* $\mathcal{S}$ *has the model* $\tilde{\mathcal{O}}(I, t(\mathbf{w}_1)) \times_{\mathbf{G}/\mathbf{U}_{I_2}} \cdots \times_{\mathbf{G}/\mathbf{U}_{I_k}} \tilde{\mathcal{O}}(I_k, t(\mathbf{w}_k))$.

By the remarks after Lemma 7.11 the variety $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$ affords a natural left action of $\mathbf{G}$ and right action of $\mathbf{L}_I$.

Proposition 7.14. *There exists a Tits homomorphism* t *which is* F*-equivariant, that is such that* $t(\phi(\mathbf{b})) = F(t(\mathbf{b}))$.

Proof. With any simple reflection $s \in S$ is associated a quasi-simple subgroup $\mathbf{G}_s$ of rank 1 of $\mathbf{G}$, generated by the root subgroups $\mathbf{U}_{\alpha_s}$ and $\mathbf{U}_{-\alpha_s}$; the 1-parameter subgroup of $\mathbf{T}$ given by $\mathbf{T} \cap \mathbf{G}_s$ is a maximal torus of $\mathbf{G}_s$. By [T, Theorem 4.4] if for any $s \in S$ we choose a representative $\dot{s}$ of s in $\mathbf{G}_s$, then these representatives satisfy the braid relations, which implies that $\mathbf{s} \mapsto \dot{s}$ induces a well defined Tits homomorphism. We claim that if s is fixed by some power ϕ^d of ϕ then there exists $\dot{s} \in \mathbf{G}_s$ fixed by F^d; we then get an F-equivariant Tits homomorphism by choosing arbitrarily $\dot{s}$ for one s in each orbit of ϕ. If s is fixed by ϕ^d then $\mathbf{G}_s$ is stable by F^d; the group $\mathbf{G}_s$ is isomorphic to either SL_2 or PSL_2 and F^d is a Frobenius endomorphism of this group. In either case the simple reflection s of $\mathbf{G}_s$ has an F^d-stable representative in $N_{\mathbf{G}_s}(\mathbf{T} \cap \mathbf{G}_s)$, whence our claim. □

Notation 7.15. *We assume now that we have chosen, once and for all, an* F*-equivariant Tits homomorphism* t *which is used to define the varieties* $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$.

The equivariance of t allows to extend it to a morphism $B^+ \rtimes \langle\phi\rangle \to N_{\mathbf{G}}(\mathbf{T}) \rtimes \langle F\rangle$—note that here our convention that $\langle\phi\rangle$ is infinite order is useful, since F is of infinite order. This allows to extend t by $t(\phi) = F$ thus we can write indifferently $t(\mathbf{b})F$ or $t(\mathbf{b}\phi)$.

Definition 7.16. *For any endomorphism* $(\mathbf{I} \xrightarrow{\mathbf{b}\phi} \mathbf{I}) \in B^+\phi(\mathcal{I})$ *we define* $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{b}\phi) = \{x \in \tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b}) \mid p''(x) = F(p'(x))\}$.

The action of $\mathbf{L}_I$ on $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$ restricts to an action of $\mathbf{L}_I^{t(\mathbf{b}\phi)}$ on $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{b}\phi)$, compatible with the first projection $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{b}\phi) \to \mathbf{G}/\mathbf{U}_I$.

When $\mathbf{w} \in \mathbf{W}$ we have $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) = \tilde{\mathbf{X}}(I, t(\mathbf{w}\phi))$, the variety defined in 7.9 for $\dot{w}F = t(\mathbf{w}\phi)$. We have the following analogue of Proposition 7.7 for $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{b}\phi)$.

Proposition 7.17. *Let* $\mathbf{I} = \mathbf{I}_1 \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \to \cdots \to \mathbf{I}_k \xrightarrow{\mathbf{w}_k} {}^{\phi}\mathbf{I}$ *be a decomposition into elements of* $\mathcal{S}$ *of* $\mathbf{I} \xrightarrow{\mathbf{b}} {}^{\phi}\mathbf{I} \in B^+(\mathcal{I})$, *let* F_1 *be the isogeny of* $\mathbf{G}^k$ *as in Proposition 7.7.*

Then $\tilde{\mathbf{X}}_{\mathbf{G}}(\mathbf{I}, \mathbf{b}\phi) \simeq \tilde{\mathbf{X}}_{\mathbf{G}^k}(I_1 \times \cdots \times I_k, (t(\mathbf{w}_1), \ldots, t(\mathbf{w}_k))F_1)$. *By this isomorphism the action of* F^δ *corresponds to that of* $F_1^{k\delta}$, *the action of* $\mathbf{G}^F$ *corresponds to that of* $(\mathbf{G}^k)^{F_1}$, *and the action of* $\mathbf{L}_I^{t(\mathbf{b}\phi)}$ *corresponds to that of* $(\mathbf{L}_{I_1} \times \cdots \times \mathbf{L}_{I_k})^{(t(\mathbf{w}_1), \ldots, t(\mathbf{w}_k))F_1}$.

Proof. An element $x_1\mathbf{U}_{I_1} \times \cdots \times x_k\mathbf{U}_{I_k} \in \tilde{\mathbf{X}}_{\mathbf{G}^k}(I_1 \times \cdots \times I_k, (t(\mathbf{w}_1), \ldots, t(\mathbf{w}_k))F_1)$ by definition satisfies $(x_i\mathbf{U}_{I_i}, x_{i+1}\mathbf{U}_{I_{i+1}}) \in \tilde{\mathcal{O}}(I_i, t(\mathbf{w}_i))$ for $i = 1, \ldots, k$, where we have put $I_{k+1} = {}^F I_1$ and $x_{k+1}\mathbf{U}_{I_k+1} = {}^F(x_1\mathbf{U}_{I_1})$. This is the same as an element in the intersection of $\tilde{\mathcal{O}}(\mathbf{I}_1, \mathbf{w}_1) \times_{\mathbf{G}/\mathbf{U}_{I_2}} \tilde{\mathcal{O}}(\mathbf{I}_2, \mathbf{w}_2) \cdots \times_{\mathbf{G}/\mathbf{U}_{I_k}} \tilde{\mathcal{O}}(\mathbf{I}_k, \mathbf{w}_k)$ with the graph of F. Since, by definition, we have

$$\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b}) \simeq \tilde{\mathcal{O}}(\mathbf{I}_1, \mathbf{w}_1) \times_{\mathbf{G}/\mathbf{U}_{I_2}} \tilde{\mathcal{O}}(\mathbf{I}_2, \mathbf{w}_2) \cdots \times_{\mathbf{G}/\mathbf{U}_{I_k}} \tilde{\mathcal{O}}(\mathbf{I}_k, \mathbf{w}_k),$$

via this last isomorphism we get an element of $\tilde{\mathcal{O}}(\mathbf{I},\mathbf{b})$ which is in $\tilde{\mathbf{X}}_{\mathbf{G}}(\mathbf{I},\mathbf{b}\phi)$.

One checks easily that this sequence of identifications is compatible with the actions of F^δ, of $\mathbf{G}^F$ and of $\mathbf{L}_I^{t(\mathbf{b}\phi)}$ as described by the proposition. □

Lemma 7.18. *For any endomorphism $(\mathbf{I}\xrightarrow{\mathbf{b}\phi}\mathbf{I})\in B^+\phi(\mathcal{I})$, there is a natural projection $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{b}\phi)\xrightarrow{\pi}\mathbf{X}(\mathbf{I},\mathbf{b}\phi)$ which makes $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{b}\phi)$ a $\mathbf{L}_I^{t(\mathbf{b}\phi)}$-torsor over $\mathbf{X}(\mathbf{I},\mathbf{b}\phi)$.*

Proof. Let $\mathbf{I}\xrightarrow{\mathbf{w}_1}\mathbf{I}_2\to\cdots\to\mathbf{I}_r\xrightarrow{\mathbf{w}_r}{}^\phi\mathbf{I}$ be a decomposition into elements of $\mathcal{S}$ of $\mathbf{I}\xrightarrow{\mathbf{b}}{}^\phi\mathbf{I}$, so that $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{b}\phi)$ identifies with the set of sequences $(g_1\mathbf{U}_I,g_2\mathbf{U}_{I_2},\ldots,g_r\mathbf{U}_{I_r})$ such that $g_j^{-1}g_{j+1}\in\mathbf{U}_{I_j}t(\mathbf{w}_j)\mathbf{U}_{I_{j+1}}$ for $j<r$ and $g_r^{-1}{}^Fg_1\in\mathbf{U}_{I_r}t(\mathbf{w}_r)\mathbf{U}_{\phi I}$. We define π by $g_j\mathbf{U}_{I_j}\mapsto{}^{g_j}\mathbf{P}_{I_j}$. It is easy to check that the morphism π thus defined commutes with an "elementary morphism" in the bicategories of varieties $\tilde{\mathbf{X}}$ or $\mathbf{X}$ consisting of passing from the decomposition $(\mathbf{w}_1,\ldots,\mathbf{w}_i,\mathbf{w}_{i+1},\ldots,\mathbf{w}_r)$ to $(\mathbf{w}_1,\ldots,\mathbf{w}_i\mathbf{w}_{i+1},\ldots,\mathbf{w}_r)$ when $(\mathbf{I}_i\xrightarrow{\mathbf{w}_i\mathbf{w}_{i+1}}\mathbf{I}_{i+2})\in\mathcal{S}$. Thus by 6.1 the morphism π is well-defined independently of the chosen decomposition of $\mathbf{b}$.

The fact that π makes $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{b}\phi)$ a $\mathbf{L}^{t(\mathbf{b}\phi)}$-torsor over $\mathbf{X}(\mathbf{I},\mathbf{b}\phi)$ results then via Proposition 7.17 from the same statement on the varieties of 7.9. □

We give an isomorphism which reflects the transitivity of Lusztig's induction.

Proposition 7.19. *Let $\mathbf{I}\xrightarrow{\mathbf{w}\phi}\mathbf{I}\in B^+\phi(\mathcal{I})$, and let w be the image of $\mathbf{w}$ in W; the automorphism $w\phi$ of W_I lifts to an automorphism that we will still denote by $w\phi$ of $B^+_{\mathbf{I}}$. For $\mathbf{J}\subset\mathbf{I}$, let $\mathcal{J}$ be the set of $B^+_{\mathbf{I}}$-conjugates of $\mathbf{J}$ and let $\mathbf{J}\xrightarrow{\mathbf{v}w\phi}\mathbf{J}\in B^+_{\mathbf{I}}w\phi(\mathcal{J})$. Then*

(i) *We have an isomorphism $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)\times_{\mathbf{L}_I^{t(\mathbf{w}\phi)}}\tilde{\mathbf{X}}_{\mathbf{L}_I}(\mathbf{J},\mathbf{v}w\phi)\xrightarrow{\sim}\tilde{\mathbf{X}}(\mathbf{J},\mathbf{v}\mathbf{w}\phi)$ of $\mathbf{G}^F$-varieties-$\mathbf{L}_J^{t(\mathbf{v}\mathbf{w}\phi)}$, where the variety $\tilde{\mathbf{X}}_{\mathbf{L}_I}(\mathbf{J},\mathbf{v}w\phi)$ is defined via the (obvious) Tits homomorphism $B^+_{\mathbf{I}}\rtimes\langle w\phi\rangle\to N_{\mathbf{L}_I}(\mathbf{T})\rtimes\langle t(\mathbf{w}\phi)\rangle$. This isomorphism is compatible with the action of F^n for any n such that $\mathbf{I}$, $\mathbf{J}$, $\mathbf{v}$ and $\mathbf{w}$ are ϕ^n-stable.*

(ii) *Through the quotient by $\mathbf{L}_J^{t(\mathbf{v}w\phi)}$ (see Lemma 7.18) we get an isomorphism of $\mathbf{G}^F$-varieties*

$$\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)\times_{\mathbf{L}_I^{t(\mathbf{w}\phi)}}\mathbf{X}_{\mathbf{L}_I}(\mathbf{J},\mathbf{v}w\phi)\xrightarrow{\sim}\mathbf{X}(\mathbf{J},\mathbf{v}\mathbf{w}\phi).$$

Proof. We first look at the case $\mathbf{w},\mathbf{v}\in\mathbf{W}$ (which implies $\mathbf{v}\mathbf{w}\in\mathbf{W}$), in which case we seek an isomorphism

$$\tilde{\mathbf{X}}(I,t(\mathbf{w}\phi))\times_{\mathbf{L}_I^{t(\mathbf{w}\phi)}}\tilde{\mathbf{X}}_{\mathbf{L}_I}(J,t(\mathbf{v}w\phi))\xrightarrow{\sim}\tilde{\mathbf{X}}(J,t(\mathbf{v}\mathbf{w}\phi))$$

where

$$\tilde{\mathbf{X}}(I,t(\mathbf{w}\phi))=\{g\mathbf{U}_I\in\mathbf{G}/\mathbf{U}_I\mid g^{-1}{}^Fg\in\mathbf{U}_It(\mathbf{w})^F\mathbf{U}_I\},$$
$$\tilde{\mathbf{X}}(J,t(\mathbf{v}\mathbf{w}\phi))=\{g\mathbf{U}_J\in\mathbf{G}/\mathbf{U}_J\mid g^{-1}{}^Fg\in\mathbf{U}_Jt(\mathbf{v}\mathbf{w})^F\mathbf{U}_J\}$$
$$\text{and }\tilde{\mathbf{X}}_{\mathbf{L}_I}(J,t(\mathbf{v}w\phi))=\{l\mathbf{V}_J\in\mathbf{L}_I/\mathbf{V}_J\mid l^{-1}{}^{t(w\phi)}l\in\mathbf{V}_It(\mathbf{v})^{t(w\phi)}\mathbf{V}_J\},$$

where $\mathbf{V}_J=\mathbf{L}_I\cap\mathbf{U}_J$.

This is the content of Lusztig's proof of the transitivity of his induction (see [Lu, Lemma 3]), that we recall and detail in our context. We claim that $(g\mathbf{U}_I,l\mathbf{V}_J)\mapsto$

$g\mathbf{U}_I l\mathbf{V}_J = gl\mathbf{U}_J$ induces the isomorphism we want. Using that $\mathbf{U}_J = \mathbf{U}_I\mathbf{V}_J$ and that $\mathbf{V}_J t(\mathbf{v})^{t(w\phi)}\mathbf{V}_J$ is in $\mathbf{L}_I$, thus normalizes $\mathbf{U}_I$, we get

$$\mathbf{U}_J t(\mathbf{vw})^F\mathbf{U}_J = \mathbf{U}_I\mathbf{V}_J t(\mathbf{v})^{t(w\phi)}\mathbf{V}_J t(\mathbf{w})^F\mathbf{U}_I = \mathbf{V}_J t(\mathbf{v})^{t(w\phi)}\mathbf{V}_J\mathbf{U}_I t(\mathbf{w})^F\mathbf{U}_I.$$

Hence if $(g\mathbf{U}_I, l\mathbf{V}_J) \in \tilde{\mathbf{X}}(I, t(\mathbf{w}\phi)) \times \tilde{\mathbf{X}}_{\mathbf{L}_I}(J, t(\mathbf{v}w\phi))$, we have

$$\begin{aligned}(gl)^{-1\,F}(gl) &\in l^{-1}\mathbf{U}_I t(\mathbf{w})^F\mathbf{U}_I{}^F l = l^{-1}\mathbf{U}_I{}^{t(w\phi)}l t(\mathbf{w})^F\mathbf{U}_I\\ &= l^{-1\,t(w\phi)}l\mathbf{U}_I t(\mathbf{w})^F\mathbf{U}_I \subset \mathbf{V}_J t(\mathbf{v})^{t(w\phi)}\mathbf{V}_J\mathbf{U}_I t(\mathbf{w})^F\mathbf{U}_I = \mathbf{U}_J t(\mathbf{vw})^F\mathbf{U}_J.\end{aligned}$$

Hence we have defined a morphism $\tilde{\mathbf{X}}(I, t(\mathbf{w}\phi)) \times \tilde{\mathbf{X}}_{\mathbf{L}_I}(J, t(\mathbf{v}w\phi)) \to \tilde{\mathbf{X}}(J, t(\mathbf{vw}\phi))$ of $\mathbf{G}^F$-varieties-$\mathbf{L}_J^{t(\mathbf{v}w\phi)}$. We show now that it is surjective. The unicity in the decomposition $\mathbf{P}_I \cap {}^{t(\mathbf{w}\phi)}\mathbf{U}_I = \mathbf{L}_I\cdot(\mathbf{U}_I \cap {}^{t(\mathbf{w}\phi)}\mathbf{U}_I)$ implies that the product $\mathbf{L}_I.(\mathbf{U}_I t(\mathbf{w})^F\mathbf{U}_I)$ is direct. Hence an element $x^{-1\,F}x \in \mathbf{U}_J t(\mathbf{vw})^F\mathbf{U}_J$ defines unique elements $l \in \mathbf{V}_J t(\mathbf{v})^{t(w\phi)}\mathbf{V}_J$ and $u \in \mathbf{U}_I t(\mathbf{w})^F\mathbf{U}_I$ such that $x^{-1\,F}x = lu$. If, using Lang's theorem, we write $l = l'^{-1\,t(w\phi)}l'$ with $l' \in \mathbf{L}_I$, the element $g = xl'^{-1}$ satisfies $g^{-1\,F}g = l'x^{-1\,F}x^F l'^{-1} = {}^{t(w\phi)}l'u^F l'^{-1} \in {}^{t(w\phi)}l'\mathbf{U}_I t(\mathbf{w})^F\mathbf{U}_I{}^F l'^{-1} = \mathbf{U}_I t(\mathbf{w})^F\mathbf{U}_I$. Hence $(g\mathbf{U}_I, l'\mathbf{V}_J)$ is a preimage of $x\mathbf{U}_J$ in $\tilde{\mathbf{X}}(I, t(\mathbf{w}\phi)) \times \tilde{\mathbf{X}}_{\mathbf{L}_I}(J, t(\mathbf{v}w\phi))$.

Let us look now at the fibers of the above morphism. If $g'\mathbf{U}_I l'\mathbf{V}_J = g\mathbf{U}_I l\mathbf{V}_J$ then $g'^{-1}g \in \mathbf{P}_I$ so we may choose g' in $g'\mathbf{U}_I$ such that $g' = g\lambda$ with $\lambda \in \mathbf{L}_I$; we have then $\lambda l'\mathbf{U}_J = l\mathbf{U}_J$, so that $l^{-1}\lambda l' \in \mathbf{U}_J \cap \mathbf{L}_I = \mathbf{V}_J$; moreover if $g\lambda\mathbf{U}_I \in \tilde{\mathbf{X}}(I, t(\mathbf{w}\phi))$ with $\lambda \in \mathbf{L}_I$, then $\lambda^{-1}\mathbf{U}_I t(\mathbf{w})^F\mathbf{U}_I{}^F\lambda = \mathbf{U}_I t(\mathbf{w})^F\mathbf{U}_I$ which implies $\lambda \in \mathbf{L}_I^{t(\mathbf{w}\phi)}$. Conversely, the action of $\lambda \in \mathbf{L}_I^{t(\mathbf{w}\phi)}$ given by $(g\mathbf{U}_I, l\mathbf{V}_J) \mapsto (g\lambda\mathbf{U}_I, \lambda^{-1}l\mathbf{V}_J)$ preserves the subvariety $\tilde{\mathbf{X}}(I, t(\mathbf{w}\phi)) \times \tilde{\mathbf{X}}_{\mathbf{L}_I}(J, t(\mathbf{v}w\phi))$, of $\mathbf{G}/\mathbf{U}_I \times \mathbf{L}_I/\mathbf{V}_J$. Hence the fibers are the orbits under this action of $\mathbf{L}_I^{t(\mathbf{w}\phi)}$.

Now the morphism $j : (g\mathbf{U}_I, l\mathbf{V}_J) \mapsto gl\mathbf{U}_J$ is an isomorphism $\mathbf{G}/\mathbf{U}_I \times_{\mathbf{L}_I} \mathbf{L}_I/\mathbf{V}_J \simeq \mathbf{G}/\mathbf{U}_J$ since $g\mathbf{U}_J \mapsto (g\mathbf{U}_I, \mathbf{V}_J)$ is its inverse. By what we have seen above the restriction of j to the closed subvariety $\tilde{\mathbf{X}}(I, t(\mathbf{w}\phi)) \times_{\mathbf{L}_I^{t(\mathbf{w}\phi)}} \tilde{\mathbf{X}}_{\mathbf{L}_I}(J, t(\mathbf{v}w\phi))$ maps this variety surjectively on the closed subvariety $\tilde{\mathbf{X}}(J, t(\mathbf{vw}\phi))$ of $\mathbf{G}/\mathbf{U}_J$, hence we get the isomorphism we want.

We now consider the case of generalized varieties. Let k be the number of terms of the strict normal decomposition of $\mathbf{vw}$ and let $\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \xrightarrow{\mathbf{w}_2} \mathbf{I}_3 \to \cdots \to \mathbf{I}_k \xrightarrow{\mathbf{w}_k} {}^\phi\mathbf{I}$ be a normal decomposition of $\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi\mathbf{I}$ of same length. We have $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \simeq \tilde{\mathbf{X}}(I_1 \times I_2 \times \cdots \times I_k, (t(\mathbf{w}_1), \ldots, t(\mathbf{w}_k))F_1)$, where F_1 is as in Proposition 7.7. Let us write $(\mathbf{v}_1\mathbf{w}_1, \ldots, \mathbf{v}_k\mathbf{w}_k)$ for the normal decomposition of $\mathbf{vw}$, with same notation as in Proposition 5.8. Let $J_1 = J$ and $J_{j+1} = J_j^{v_j w_j} \subset I_{j+1}$ for $j = 1, \ldots, k-1$. We apply the first part of the proof to the group $\mathbf{G}^k$ with isogeny F_1 with I, J, w, v replaced respectively by $I_1 \times \cdots \times I_k$, $J_1 \times \cdots \times J_k$, $(w_1, \ldots, w_k)$ $(v_1, \ldots, v_k)$. Using the isomorphisms from Proposition 7.17;

$$\tilde{\mathbf{X}}_{\mathbf{G}^k}(J_1 \times \cdots \times J_k, (t(\mathbf{v}_1\mathbf{w}_1), \ldots, t(\mathbf{v}_k\mathbf{w}_k))F_1) \simeq \tilde{\mathbf{X}}(\mathbf{J}, \mathbf{vw}\phi)$$

and

$$\tilde{\mathbf{X}}_{\mathbf{L}_{I_1\times\cdots\times I_k}}(J_1 \times \cdots \times J_k, (v_1, \ldots, v_k).(t(\mathbf{w}_1), \ldots, t(\mathbf{w}_k))F_1) \simeq \tilde{\mathbf{X}}_{\mathbf{L}_I}(\mathbf{J}, \mathbf{v}w\phi),$$

we get (i). Now (ii) is immediate from (i) taking the quotient on both sides by $\mathbf{L}_J^{t(\mathbf{v}w\phi)}$. □

In the particular case where $\mathbf{I} = \emptyset$ we write $\mathbf{X}(\mathbf{w}\phi)$ for $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$. Let us recall that in [DMR, 2.3.2] we defined a monoid $\underline{B}^+$ generated by B^+ and symbols $\underline{w}$

where $w \in W$, and attached to any $\mathbf{u} \in \underline{B}^+$ a Deligne-Lusztig variety $\mathbf{X}(\mathbf{u}\phi)$. This variety is denoted by $\mathbf{X}(\mathbf{u})$ in [DMR] and roughly defined by the property that given $\underline{w}$ attached to $w \in W$, we have $\mathbf{X}(\mathbf{u}_1\underline{w}\mathbf{u}_2\phi) = \bigcup_{w' \leq w} \mathbf{X}(\mathbf{u}_1\mathbf{w}'\mathbf{u}_2\phi)$, where $\mathbf{w}'$ is the lift to B^+ of w' and where w' runs over the elements smaller than w for the Bruhat order. Attached to $I \subset S$, we have an analogous monoid $\underline{B}^+_{\mathbf{I}}$ attached to W_I, which has a natural embedding $\underline{B}^+_{\mathbf{I}} \subset \underline{B}^+$.

Corollary 7.20. *With these notations of* [DMR], *for any* $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I} \in B^+\phi(\mathcal{I})$ *and any* $\mathbf{u} \in \underline{B}^+_{\mathbf{I}}$, *we have an isomorphism* $\mathbf{X}(\mathbf{u}\mathbf{w}\phi) \xrightarrow{\sim} \tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \times_{\mathbf{L}_I^{t(\mathbf{w}\phi)}} \mathbf{X}_{\mathbf{L}_I}(\mathbf{u}w\phi)$ *and a surjective morphism* $\mathbf{X}(\mathbf{u}\mathbf{w}\phi) \to \mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$ *whose fibers are isomorphic to* $\mathbf{X}_{\mathbf{L}_I}(\mathbf{u}w\phi)$.

Proof. The variety $\mathbf{X}(\mathbf{u}\mathbf{w}\phi)$ is the union of varieties of the form $\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}w\phi)$ with $\mathbf{v} \in \mathbf{W_I}$. The isomorphisms given for each $\mathbf{v}$ by Proposition 7.19 applied with $\mathbf{J} = \emptyset$ can be glued together to give a global morphism of varieties since they are defined by a formula independent of $\mathbf{v}$. We thus get a bijective morphism $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \times_{\mathbf{L}_I^{t(\mathbf{w}\phi)}} \mathbf{X}_{\mathbf{L}_I}(\mathbf{u}w\phi) \to \mathbf{X}(\mathbf{u}\mathbf{w}\phi)$ which is an isomorphism since $\mathbf{X}(\mathbf{u}\mathbf{w}\phi)$ is normal (see [DMR, 2.3.5]). Composing this isomorphism with the projection of $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \times_{\mathbf{L}_I^{t(\mathbf{w}\phi)}} \mathbf{X}_{\mathbf{L}_I}(\mathbf{u}w\phi)$ onto $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$ (see 7.18), we get the second assertion of the corollary. ∎

Endomorphisms of parabolic Deligne-Lusztig varieties — the category $\mathcal{D}^{\mathcal{I}}$.

Definition 7.21. *Given* $\operatorname{ad}\mathbf{v} \in \mathcal{D}^{\mathcal{I}}(\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}, \mathbf{J} \xrightarrow{\mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v}} \mathbf{J})$ *where* $\mathbf{J} = \mathbf{I}^{\mathbf{v}}$, *we define morphisms of varieties:*

(i) $D_{\mathbf{v}} : \mathbf{X}(\mathbf{I}, \mathbf{w}\phi) \to \mathbf{X}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v})$ *as the restriction of the morphism*

$$(a, b) \mapsto (b, {}^F a) : \mathcal{O}(\mathbf{I}, \mathbf{w}) = \mathcal{O}(\mathbf{I}, \mathbf{v}) \times_{\mathcal{P}_J} \mathcal{O}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}) \to \mathcal{O}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}) \times_{\mathcal{P}_{\phi_I}} \mathcal{O}({}^\phi\mathbf{I}, {}^\phi\mathbf{v}) = \mathcal{O}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}^\phi\mathbf{v}).$$

(ii) $\tilde{D}_{\mathbf{v}} : \tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \to \tilde{\mathbf{X}}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v})$ *as the restriction of the morphism*

$$(a, b) \mapsto (b, {}^F a) : \tilde{\mathcal{O}}(\mathbf{I}, \mathbf{w}) = \tilde{\mathcal{O}}(\mathbf{I}, \mathbf{v}) \times_{\mathbf{G}/\mathbf{U}_J} \tilde{\mathcal{O}}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}) \to \tilde{\mathcal{O}}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}) \times_{\mathbf{G}/\mathbf{U}_{\phi_I}} \tilde{\mathcal{O}}({}^\phi\mathbf{I}, {}^\phi\mathbf{v}) = \tilde{\mathcal{O}}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}^\phi\mathbf{v}).$$

Note that the existence of well-defined decompositions as above of $\mathcal{O}(\mathbf{I}, \mathbf{w})$ and of $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{w})$ are consequences of Theorem 6.2.

Note that when $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v}^{-1}\mathbf{w}^\phi\mathbf{v}$ are in $\mathbf{W}$ the endomorphism $D_{\mathbf{v}}$ maps $g\mathbf{P}_I \in \mathbf{X}(I, w\phi)$ to $g'\mathbf{P}_J \in \mathbf{X}(J, v^{-1}w\phi v)$ such that $g^{-1}g' \in \mathbf{P}_I v \mathbf{P}_J$ and $g'^{-1}\,{}^F g \in \mathbf{P}_J v^{-1} w^F \mathbf{P}_I$ and similarly for $\tilde{D}_{\mathbf{v}}$.

Note also that $D_{\mathbf{v}}$ and $\tilde{D}_{\mathbf{v}}$ are equivalences of étale sites; indeed, the proof of [DMR, 3.1.6] applies without change in our case.

The definition of $\tilde{D}_{\mathbf{v}}$ and $D_{\mathbf{v}}$ shows the following property:

Lemma 7.22. *The following diagram is commutative:*

$$\begin{array}{ccc} \tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi) & \xrightarrow{\tilde{D}_{\mathbf{v}}} & \tilde{\mathbf{X}}(\mathbf{J},\mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v}) \\ \downarrow & & \downarrow \\ \mathbf{X}(\mathbf{I},\mathbf{w}\phi) & \xrightarrow{D_{\mathbf{v}}} & \mathbf{X}(\mathbf{J},\mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v}) \end{array}$$

where the vertical arrows are the respective quotients by $\mathbf{L}_I^{t(\mathbf{w}\phi)}$ and $\mathbf{L}_J^{t(\mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v})}$ (see Lemma 7.18); for $l\in\mathbf{L}_I^{t(\mathbf{w}\phi)}$ we have $\tilde{D}_{\mathbf{v}}\circ l = l^{t(\mathbf{v})}\circ\tilde{D}_{\mathbf{v}}$.

As a further consequence of Theorem 6.2, the map which sends a simple morphism $\operatorname{ad}\mathbf{v}$ to $D_{\mathbf{v}}$ extends to a natural morphism from $\mathcal{D}^{\mathcal{I}}(\mathbf{I}\xrightarrow{\mathbf{w}\phi}\mathbf{I},\mathbf{J}\xrightarrow{\mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v}}\mathbf{J})$ to $\operatorname{Hom}_{\mathbf{G}^F}(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\mathbf{X}(\mathbf{J},\mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v}))$ whose image consists of equivalences of étale sites. We still denote by $D_{\mathbf{v}}$ the image of $\operatorname{ad}\mathbf{v}$ by this morphism.

Lemma 7.23. *Via the isomorphism of 7.17 and with the notations of loc. cit. the morphism $D_{\mathbf{w}_1}$ with source $\tilde{\mathbf{X}}_{\mathbf{G}}(\mathbf{I},\mathbf{b}\phi)$ becomes the morphism $D_{(t(\mathbf{w}_1),1,\ldots,1)}$ with source $\tilde{\mathbf{X}}_{\mathbf{G}^k}(I_1\times\cdots\times I_k,(t(\mathbf{w}_1),\ldots,t(\mathbf{w}_k))F_1)$.*

Proof. The endomorphism $D_{\mathbf{w}_1}$ maps the element $(g_1\mathbf{U}_1,\ldots,g_k\mathbf{U}_k)$ of the model of 7.17 of $\tilde{\mathbf{X}}_{\mathbf{G}}(\mathbf{I},\mathbf{b}\phi)$ to $(g_2\mathbf{U}_2,\ldots,g_k\mathbf{U}_k,{}^Fg_1{}^F\mathbf{U}_1)$. On the other hand the isomorphism of Proposition 7.17 maps $(g_1\mathbf{U}_1,\ldots,g_k\mathbf{U}_k)$ to

$$(g_1,\ldots,g_k)(\mathbf{U}_1,\ldots,\mathbf{U}_k)\in\tilde{X}_{\mathbf{G}^k}(I_1\times\cdots\times I_k,(t(\mathbf{w}_1),\ldots,t(\mathbf{w}_k))F_1)$$

which is sent by $D_{(t(\mathbf{w}_1),1,\ldots,1)}$ to $(g_2,\ldots,g_k,{}^Fg_1)(\mathbf{U}_2,\ldots,\mathbf{U}_k,{}^F\mathbf{U}_1)$ which is the image by the isomorphism of Proposition 7.17 of $(g_2\mathbf{U}_2,\ldots,g_k\mathbf{U}_k,{}^Fg_1{}^F\mathbf{U}_1)$, whence the lemma. □

Proposition 7.24. *For $\mathbf{J}\subset\mathbf{I}$ let $\mathcal{J}$ denote the set of $B^+_{\mathbf{I}}$-conjugates of $\mathbf{J}$. With same assumptions and notation as in Proposition 7.19, let $\mathbf{J}\xrightarrow{\mathbf{x}}\mathbf{J}^{\mathbf{x}}\in B^+_{\mathbf{I}}(\mathcal{J})$ be a left-divisor of $\mathbf{J}\xrightarrow{\mathbf{v}}{}^{\mathbf{w}\phi}\mathbf{J}$. The following diagram is commutative:*

$$\begin{array}{ccc} \tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)\times_{\mathbf{L}_I^{t(\mathbf{w}\phi)}}\tilde{\mathbf{X}}_{\mathbf{L}_I}(\mathbf{J},\mathbf{v}\cdot w\phi) & \xrightarrow{\sim} & \tilde{\mathbf{X}}(\mathbf{J},\mathbf{vw}\phi) \\ \downarrow{\scriptstyle \operatorname{Id}\times\tilde{D}_{\mathbf{x}}} & & \downarrow{\scriptstyle \tilde{D}_{\mathbf{x}}} \\ \tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)\times_{\mathbf{L}_I^{t(\mathbf{w}\phi)}}\tilde{\mathbf{X}}_{\mathbf{L}_I}(\mathbf{J}^{\mathbf{x}},\mathbf{x}^{-1}(\mathbf{v}\cdot w\phi)\mathbf{x}) & \xrightarrow{\sim} & \tilde{\mathbf{X}}(\mathbf{J}^{\mathbf{x}},\mathbf{x}^{-1}\mathbf{vw}\phi\mathbf{x}) \end{array}$$

Proof. Decomposing $\mathbf{x}$ into a product of simples in the category analogous to $\mathcal{D}^{\mathcal{I}}$ where B^+ is replaced by $B^+_{\mathbf{I}}$ and $\mathcal{I}$ by $\mathcal{J}$, the definitions show that it is sufficient to prove the result for $\mathbf{x}\in\mathbf{W}$. We use then Proposition 5.8 and Lemma 7.23 to reduce the proof to the case where $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}^{-1}\mathbf{v}^{w\phi}\mathbf{x}$ are in $\mathbf{W}$ (in which case $\mathbf{vw}$ and $\mathbf{x}^{-1}\mathbf{vw}^{\phi}\mathbf{x}$ are in $\mathbf{W}$ too): we choose compatible decompositions of $\mathbf{v}$ and $\mathbf{w}$ as in 5.8 which we refine if needed so that $\mathbf{x}$ is the first term of that of $\mathbf{v}$ and use Lemma 7.23 once in $\mathbf{G}$ and once in in $\mathbf{L}_I$.

Assume now $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}^{-1}\mathbf{v}^{w\phi}\mathbf{x}$ in $\mathbf{W}$. We start with $(g\mathbf{U}_I,l\mathbf{V}_J)\in\tilde{\mathbf{X}}(I,t(\mathbf{w}\phi))\times\tilde{\mathbf{X}}_{\mathbf{L}_I}(J,vw\phi)$. This element is sent by the top isomorphism of the diagram to $gl\mathbf{U}_J$. On the other hand, we have seen above Lemma 7.22 that it is sent by $\operatorname{Id}\times\tilde{D}_{\mathbf{x}}$ to $(g\mathbf{U}_I,l'\mathbf{V}_{J^x})$ where $l^{-1}l'\in\mathbf{V}_Jx\mathbf{V}_{J^x}$ and $l'^{-1\,t(\mathbf{w}\phi)}l\in\mathbf{V}_{J^x}x^{-1}v^{wF}\mathbf{V}_J$. This element is sent in turn to $gl'\mathbf{U}_{J^x}$ by the bottom isomorphism of the diagram. We have to check that $gl'\mathbf{U}_{J^x} = \tilde{D}_{\mathbf{x}}(gl\mathbf{U}_J)$. But $(gl)^{-1}gl' = l^{-1}l'$ is in $\mathbf{V}_J x \mathbf{V}_{J^x} \subset \mathbf{U}_J x \mathbf{U}_{J^x}$ and

$$(gl')^{-1}\,{}^F(gl) = l'^{-1}g^{-1}\,{}^Fg\,{}^Fl \in l'^{-1}\mathbf{U}_I t(\mathbf{w})\,{}^F\mathbf{U}_I\,{}^Fl = \mathbf{U}_I l'^{-1}\,{}^{t(w\phi)}l t(\mathbf{w})\,{}^F\mathbf{U}_I$$
$$\subset \mathbf{U}_I \mathbf{V}_{J^x} x^{-1} v w\,{}^F\mathbf{V}_J\,{}^F\mathbf{U}_I = \mathbf{U}_{J^x} x^{-1} v w\,{}^F\mathbf{U}_J,$$

so that $(gl'\mathbf{U}_{J^x}) = \tilde{D}_{\mathbf{x}}(gl\mathbf{U}_J)$. □

Using Proposition 7.19(ii), Proposition 7.24 and Lemma 7.22 we get

Corollary 7.25. *The following diagram is commutative:*

$$\begin{array}{ccc}
\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \times_{\mathbf{L}_I^{t(\mathbf{w}\phi)}} \mathbf{X}_{\mathbf{L}_I}(\mathbf{J}, \mathbf{v}\cdot w\phi) & \xrightarrow{\sim} & \mathbf{X}(\mathbf{J}, \mathbf{v}\mathbf{w}\phi) \\
\downarrow{\scriptstyle \mathrm{Id}\times D_{\mathbf{x}}} & & \downarrow{\scriptstyle D_{\mathbf{x}}} \\
\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \times_{\mathbf{L}_I^{t(\mathbf{w}\phi)}} \mathbf{X}_{\mathbf{L}_I}(\mathbf{J}^{\mathbf{x}}, \mathbf{x}^{-1}(\mathbf{v}\cdot w\phi)\mathbf{x}) & \xrightarrow{\sim} & \mathbf{X}(\mathbf{J}^{\mathbf{x}}, \mathbf{x}^{-1}\mathbf{v}\mathbf{w}\phi\mathbf{x})
\end{array}$$

Affineness. Until the end of the text, we will be specially interested in varieties $\mathbf{X}(\mathbf{I}, \mathbf{b}\phi)$ which satisfy the assumption of Theorem 5.9, that is some power of $\mathbf{b}\phi$ is left-divisible by $\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0$. They have many nice properties. We show in this subsection that they are affine, by adapting the proof of Bonnafé and Rouquier [BR2]; we use the existence of the varieties $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$ and $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{b}\phi)$ to replace doing a quotient by $\mathbf{L}_I$ by doing a quotient by $\mathbf{L}_I^{t(\mathbf{w}\phi)}$.

Proposition 7.26. *Assume the morphism* $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \in B^+(\mathcal{I})$ *is left-divisible by* $\Delta_{\mathcal{I}} = \mathbf{I} \xrightarrow{\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0} \mathbf{I}^{\mathbf{w}_0}$. *Then the variety* $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$ *is affine.*

Proof. By assumption there exists a decomposition into elements of $\mathcal{S}$ of $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$ of the form $\mathbf{I} \xrightarrow{\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0} \mathbf{I}_1 \xrightarrow{\mathbf{v}_1} \mathbf{I}_2 \xrightarrow{\mathbf{v}_2} \mathbf{I}_3 \to \cdots \to \mathbf{I}_r \xrightarrow{\mathbf{v}_r} \mathbf{J}$. We show that the map φ defined by:

$$\mathbf{G} \times \prod_{i=1}^{i=r} (\mathbf{U}_{I_i} \cap {}^{t(\mathbf{v}_i)}\mathbf{U}_{I_{i+1}}^-) t(\mathbf{v}_i) \to$$
$$\tilde{\mathcal{O}}(I, t(\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0)) \times_{\mathbf{G}/\mathbf{U}_{I_1}} \tilde{\mathcal{O}}(I_1, t(\mathbf{v}_1)) \cdots \times_{\mathbf{G}/\mathbf{U}_{I_r}} \tilde{\mathcal{O}}(I_r, t(\mathbf{v}_r))$$
$$(g, h_1, \ldots, h_r) \mapsto$$
$$(g\mathbf{U}_I, gt(\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0)\mathbf{U}_{I_1}, gt(\mathbf{w}_I^{-1}\mathbf{w}_0)h_1\mathbf{U}_{I_2}, \ldots, gt(\mathbf{w}_I^{-1}\mathbf{w}_0)h_1 \cdots h_r \mathbf{U}_J)$$

is an isomorphism; since the first variety is a product of affine varieties this will prove our claim.

Since $\mathbf{U}_{I_i} t(\mathbf{v}_i) \mathbf{U}_{I_{i+1}}$ is isomorphic to $(\mathbf{U}_{I_i} \cap {}^{t(\mathbf{v}_i)}\mathbf{U}_{I_{i+1}}^-) t(\mathbf{v}_i) \times \mathbf{U}_{I_{i+1}}$, by composition with the first projection we get a morphism $\eta_i : \mathbf{U}_{I_i} t(\mathbf{v}_i) \mathbf{U}_{I_{i+1}} \to (\mathbf{U}_{I_i} \cap {}^{t(\mathbf{v}_i)}\mathbf{U}_{I_{i+1}}^-) t(\mathbf{v}_i)$ for $i = 1, \ldots, r$, where $I_{r+1} = J$. Similarly we have a morphism $\eta : \mathbf{U}_I t(\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0) \mathbf{U}_{I_1} \to (\mathbf{U}_I \cap {}^{t(\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0)}\mathbf{U}_{I_1}^-) t(\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0)$. For

$$x = (g\mathbf{U}_I, g_1\mathbf{U}_{I_1}, g_2\mathbf{U}_{I_2}, \ldots, g_r\mathbf{U}_{I_r}, g_{r+1}\mathbf{U}_J)$$
$$\in \tilde{\mathcal{O}}(I, t(\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0)) \times_{\mathbf{G}/\mathbf{U}_{I_1}} \tilde{\mathcal{O}}(I_1, t(\mathbf{v}_1)) \cdots \times_{\mathbf{G}/\mathbf{U}_{I_r}} \tilde{\mathcal{O}}(I_r, t(\mathbf{v}_r))$$

let $\psi(x) = g\eta(g^{-1}g_1)$, $\psi_1(x) = \psi(x)t(\mathbf{w}_0)$, $\psi_i(x) = \eta_i((\psi(x)\psi_1(x)\cdots\psi_{i-1}(x))^{-1}g_i)$. We claim that the map ψ (resp. ψ_i) is well defined, that is does not depend on the representative g (resp. g_i) chosen; the morphism $x \mapsto (\psi(x), \psi_1(x), \ldots, \psi_r(x))$ is then clearly inverse to φ. Since $\eta_i(hu) = \eta_i(h)$ for all $h \in \mathbf{U}_{I_i}t(\mathbf{v}_i)\mathbf{U}_{I_{i+1}}$ and all $u \in \mathbf{U}_{I_{i+1}}$, we get that all ψ_i are well-defined. Since moreover $\eta(uh) = u\eta(h)$ for all $h \in \mathbf{U}_I t(\mathbf{w}_\mathbf{I}^{-1}\mathbf{w}_0)\mathbf{U}_{I_1}$ and all $u \in \mathbf{U}_I$, we get that ψ also is well-defined, whence our claim. □

Proposition 7.27. *Assume that we are under the assumptions of Theorem 5.9, that is* $(\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}) \in B^+\phi(\mathcal{I})$ *has some power divisible by* $\Delta_\mathcal{I}$*, or equivalently some power of* $\mathbf{w}\phi$ *is left-divisible by* $\mathbf{w}_\mathbf{I}^{-1}\mathbf{w}_0$*. Then* $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$ *is affine.*

Proof. Let us define k as the smallest integer such that ${}^{\phi^k}\mathbf{I} = \mathbf{I}$, ${}^{\phi^k}\mathbf{w} = \mathbf{w}$ and $\mathbf{w}_\mathbf{I}^{-1}\mathbf{w}_0 \preccurlyeq \mathbf{w}^{(k)}$, where $\mathbf{w}^{(k)} := \mathbf{w}\,{}^{\phi}\mathbf{w}\cdots{}^{\phi^{k-1}}\mathbf{w}$.

We will embed $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$ as a closed subvariety in $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{w}^{(k)})$, which will prove it to be affine.

Let $\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \xrightarrow{\mathbf{w}_2} \mathbf{I}_3 \to \cdots \to \mathbf{I}_r \xrightarrow{\mathbf{w}_r} {}^{\phi}\mathbf{I}$ be a decomposition of $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$ into elements of $\mathcal{S}$, so that $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{w}^{(k)})$ identifies with the set of sequences

$$\begin{aligned}&(g_{1,1}\mathbf{U}_I, g_{1,2}\mathbf{U}_{I_2}, \ldots, g_{1,r}\mathbf{U}_{I_r},\\ &\ g_{2,1}\mathbf{U}_{{}^{\phi}I}, g_{2,2}\mathbf{U}_{{}^{\phi}I_2}, \ldots, g_{2,r}\mathbf{U}_{{}^{\phi}I_r},\\ &\ \ldots,\\ &\ g_{k,1}\mathbf{U}_{{}^{\phi^{k-1}}I}, g_{k,2}\mathbf{U}_{{}^{\phi^{k-1}}I_2}, \ldots, g_{k,r}\mathbf{U}_{{}^{\phi^{k-1}}I_r},\\ &\ g_{k+1,1}\mathbf{U}_I)\end{aligned}$$

such that for $j < r$ we have $g_{i,j}^{-1}g_{i,j+1} \in \mathbf{U}_{{}^{\phi^{i-1}}I_j}t({}^{\phi^{i-1}}\mathbf{w}_j)\mathbf{U}_{{}^{\phi^{i-1}}I_{j+1}}$ and $g_{i,r}^{-1}g_{i+1,1} \in \mathbf{U}_{{}^{\phi^{i-1}}I_r}t({}^{\phi^{i-1}}\mathbf{w}_r)\mathbf{U}_{{}^{\phi^i}I}$.

Similarly $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$ identifies with the set of sequences $(g_1\mathbf{U}_I, g_2\mathbf{U}_{I_2}, \ldots, g_r\mathbf{U}_{I_r})$ such that $g_j^{-1}g_{j+1} \in \mathbf{U}_{I_j}t(\mathbf{w}_j)\mathbf{U}_{I_{j+1}}$ for $j < r$ and $g_r^{-1}\,{}^F g_1 \in \mathbf{U}_{I_r}t(\mathbf{w}_r)\mathbf{U}_{{}^{\phi}I}$. It is thus clear that the map

$$\begin{aligned}(g_1\mathbf{U}_I, g_2\mathbf{U}_{I_2}, \ldots, g_r\mathbf{U}_{I_r}) \mapsto\ &(g_1\mathbf{U}_I, g_2\mathbf{U}_{I_2}, \ldots, g_r\mathbf{U}_{I_r},\\ &\ {}^F g_1\mathbf{U}_{{}^{\phi}I}, {}^F g_2\mathbf{U}_{{}^{\phi}I_2}, \ldots, {}^F g_r\mathbf{U}_{{}^{\phi}I_r},\\ &\ \ldots,\\ &\ {}^{F^{k-1}}g_1\mathbf{U}_{{}^{\phi^{k-1}}I}, \ldots, {}^{F^{k-1}}g_r\mathbf{U}_{{}^{\phi^{k-1}}I_r}, {}^{F^k}g_1\mathbf{U}_I)\end{aligned}$$

identifies $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$ with the closed subvariety of $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{w}^{(k)})$ defined by $g_{i+1,j}\mathbf{U}_{{}^{\phi^i}I_j} = {}^F(g_{i,j}\mathbf{U}_{{}^{\phi^{i-1}}I_j})$ for all i, j. □

Corollary 7.28. *Under the assumptions of Theorem 5.9, that is* $(\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}) \in B^+(\mathcal{I})$ *has some power divisible by* $\Delta_\mathcal{I}$*, or equivalently some power of* $\mathbf{w}\phi$ *is divisible on the left by* $\mathbf{w}_\mathbf{I}^{-1}\mathbf{w}_0$*, the variety* $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$ *is affine.*

Proof. Indeed, by Proposition 7.27 and Lemma 7.18, $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$ is the quotient of an affine variety by a finite group, so it is affine. □

Shintani descent identity. In this subsection we give a formula for the Leftschetz number of a variety $\mathbf{X}(\mathbf{I},\mathbf{w}F)$ which we deduce from a "Shintani descent identity".

Let m be a multiple of δ; if we identify $\mathbf{G}/\mathbf{B}$ with the variety $\mathcal{B}$ of Borel subgroups of $\mathbf{G}$, the $\mathbf{G}^{F^m}$-module $\overline{\mathbb{Q}}_\ell(\mathbf{G}/\mathbf{B})^{F^m}$ identifies with the permutation module of $\mathbf{G}^{F^m}$ on $\mathcal{B}^{F^m}$. Its endomorphism algebra $\mathcal{H}_{q^m}(W):=\operatorname{End}_{\mathbf{G}^{F^m}}(\overline{\mathbb{Q}}_\ell\mathcal{B}^{F^m})$ has a basis consisting of the operators $(T_w)_{w\in W}$ where

$$T_w:\mathbf{B}'\mapsto\sum_{\{\mathbf{B}''\in\mathcal{B}^{F^m}\mid\mathbf{B}''\xrightarrow{w}\mathbf{B}'\}}\mathbf{B}''$$

(see [Bou, Chapitre IV §2, exercice 22]).

Similarly, since I is F^m-stable, the algebra $\mathcal{H}_{q^m}(W,W_I):=\operatorname{End}_{\mathbf{G}^{F^m}}(\overline{\mathbb{Q}}_\ell\mathcal{P}_I^{F^m})$ has a $\overline{\mathbb{Q}}_\ell$-basis consisting of the operators

$$X_w:\mathbf{P}\mapsto\sum_{\{\mathbf{P}'\in\mathcal{P}_I^{F^m}\mid\mathbf{P}'\xrightarrow{I,w,I}\mathbf{P}\}}\mathbf{P}',$$

where w runs over a set of representatives of the double cosets $W_I\backslash W/W_I\simeq\mathbf{P}_I^{F^m}\backslash\mathbf{G}^{F^m}/\mathbf{P}_I^{F^m}$. The map γ which sends $\mathbf{P}\in\mathcal{P}_I^{F^m}$ to the sum of all its F^m-stable Borel subgroups makes $\overline{\mathbb{Q}}_\ell\mathcal{P}_I^{F^m}$ into a direct summand of $\overline{\mathbb{Q}}_\ell\mathcal{B}^{F^m}$. Indeed the image of γ identifies with that of the idempotent $X_1=|(\mathbf{P}_I/\mathbf{B})^{F^m}|^{-1}\sum_{v\in W_I}T_v$, and γ has a left-inverse given up to a scalar by mapping $\mathbf{B}\in\mathcal{B}^{F^m}$ to the unique (thus F^m-stable) parabolic subgroup in $\mathcal{P}_I$ containing it. The operator X_w identifies with the restriction of X_1T_w to the image $\overline{\mathbb{Q}}_\ell\mathcal{P}_I^{F^m}$ of X_1.

We may define a $\overline{\mathbb{Q}}_\ell$-representation of $B^+(\mathcal{I})(\mathbf{I})$ on $\overline{\mathbb{Q}}_\ell\mathcal{P}_I^{F^m}$ by sending $\mathbf{I}\xrightarrow{\mathbf{w}}\mathbf{I}$ to the operator $X_{\mathbf{w}}\in\mathcal{H}(W,W_I)$ defined by

$$X_{\mathbf{w}}(\mathbf{P})=\sum_{\{x\in\mathcal{O}(\mathbf{I},\mathbf{w})^{F^m}\mid p''(x)=\mathbf{P}\}}p'(x).$$

When $\mathbf{w}\in\mathbf{W}$, with image w in W, the operators $X_{\mathbf{w}}$ and X_w coincide. In the particular case where $I=\emptyset$ we get an operator denoted by $T_{\mathbf{w}}$, defined for any $\mathbf{w}$ in B^+. The operator $X_{\mathbf{w}}$ identifies with the restriction of $X_1T_{\mathbf{w}}$ to the image $\overline{\mathbb{Q}}_\ell\mathcal{P}_I^{F^m}$ of X_1.

Similarly, to $\mathbf{I}\xrightarrow{\mathbf{w}\phi}\mathbf{I}\in B^+\phi(\mathcal{I})$ we associate an endomorphism $X_{\mathbf{w}\phi}$ of $\overline{\mathbb{Q}}_\ell\mathcal{P}_I^{F^m}$ by the formula

$$X_{\mathbf{w}\phi}(\mathbf{P})=\sum_{\{x\in\mathcal{O}(\mathbf{I},\mathbf{w})^{F^m}\mid p''(x)=F(\mathbf{P})\}}p'(x).$$

When $\phi(I)=I$ we have $X_{\mathbf{w}\phi}=X_{\mathbf{w}}F$. In general we have $X_{\mathbf{w}\phi}=X_1T_{\mathbf{w}}F$ on $\overline{\mathbb{Q}}_\ell\mathcal{P}_I^{F^m}$ seen as a subspace of $\overline{\mathbb{Q}}_\ell\mathcal{B}^{F^m}$: on this latter module one can separate the action of F; the operator F sends the submodule $\overline{\mathbb{Q}}_\ell\mathcal{P}_I^{F^m}$ to $\overline{\mathbb{Q}}_\ell\mathcal{P}_{\phi(I)}^{F^m}$ which is sent back to $\overline{\mathbb{Q}}_\ell\mathcal{P}_I^{F^m}$ by $X_1T_{\mathbf{w}}$. The endomorphism $X_{\mathbf{w}\phi}$ commutes with $\mathbf{G}^{F^m}$ like F, hence normalizes $\mathcal{H}_{q^m}(W,W_I)$; its action identifies with the conjugation action of $T_{\mathbf{w}}\phi$ on $\mathcal{H}_{q^m}(W,W_I)$ inside $\mathcal{H}_{q^m}(W)\rtimes\langle\phi\rangle$.

Recall that the Shintani descent $\operatorname{Sh}_{F^m/F}$ is the "norm" map which maps the F-class of $g'=h.{}^Fh^{-1}\in\mathbf{G}^{F^m}$ to the class of $g=h^{-1}.{}^{F^m}h\in\mathbf{G}^F$.

Proposition 7.29 (Shintani descent identity)**.** *Let* $\mathbf{I}\xrightarrow{\mathbf{w}\phi}\mathbf{I}\in B^+\phi(\mathcal{I})$*, and let* m *be a multiple of* δ*. We have the following equality of functions on* $\mathbf{G}^F$*:*

$$(g\mapsto|\mathbf{X}(\mathbf{I},\mathbf{w}\phi)^{gF^m}|)=\operatorname{Sh}_{F^m/F}(g'\mapsto\operatorname{Trace}(g'X_{\mathbf{w}\phi}\mid\overline{\mathbb{Q}}_\ell\mathcal{P}_I^{F^m})).$$

Proof. Let $g = h^{-1}.{}^{F^m}h$ and $g' = h.{}^F h^{-1}$, so that the class of g is the image by $\mathrm{Sh}_{F^m/F}$ of the F-class of g'; we have $\mathbf{X}(\mathbf{I},\mathbf{w}\phi)^{gF^m} = \{x \in \mathcal{O}(\mathbf{I},\mathbf{w}) \mid {}^{F^m h}x = {}^h x \text{ and } p''({}^h x) = {}^{g'F}p'({}^h x)\}$. Taking ${}^h x$ as a variable in the last formula we get $|\mathbf{X}(\mathbf{I},\mathbf{w}\phi)^{gF^m}| = |\{x \in \mathcal{O}(\mathbf{I},\mathbf{w})^{F^m} \mid p''(x) = {}^{g'F}p'(x)\}|$. Putting $\mathbf{P} = p'(x)$ this last number becomes $\sum_{\mathbf{P}\in\mathcal{P}_I^{F^m}} |\{x \in \mathcal{O}(\mathbf{I},\mathbf{w})^{F^m} \mid p'(x) = \mathbf{P} \text{ and } p''(x) = {}^{g'F}\mathbf{P}\}|$. On the other hand the trace of $g'X_{\mathbf{w}\phi}$ is the sum over $\mathbf{P} \in \mathcal{P}_I^{F^m}$ of the coefficient of $\mathbf{P}$ in $\sum_{\{x\in\mathcal{O}(\mathbf{I},\mathbf{w})^{F^m} | p''(x)=F(\mathbf{P})\}} g'p'(x)$. This coefficient is equal to $|\{x \in \mathcal{O}(\mathbf{I},\mathbf{w})^{F^m} \mid g'p'(x) = \mathbf{P} \text{ and } p''(x) = {}^F\mathbf{P}\}| = |\{x \in \mathcal{O}(\mathbf{I},\mathbf{w})^{F^m} \mid p'(x) = \mathbf{P} \text{ and } p''(x) = {}^{g'F}\mathbf{P}\}|$, this last equality by changing $g'x$ into x. □

The above computation can be done along different lines, without mentioning $\overline{\mathbb{Q}}_\ell\mathcal{P}_I^{F^m}$; one can use instead Corollary 7.20 for $\mathbf{u} = \underline{w}_I$, which gives a $\mathbf{G}^F$-equivariant morphism $\mathbf{X}(\underline{w}_I\mathbf{w}\phi) \to \mathbf{X}(\mathbf{I},\mathbf{w}\phi)$ whose fibers are isomorphic to the variety of Borel subgroups of $\mathbf{L}_I$; the action of F induces that of $t(w\phi)$ on the fibers. One may then use directly [DMR, 3.3.7] to get $|\mathbf{X}(\underline{w}_I\mathbf{w}\phi)^{gF^m}| = \mathrm{Trace}(g'T_{\underline{w}_I}T_\mathbf{w}\phi \mid \overline{\mathbb{Q}}_\ell\mathcal{B}^{F^m})$, where $T_{\underline{w}_I} = \sum_{v\in W_I} T_v$.

By, for example, [DM1, II, 3.1] the algebras $\mathcal{H}_{q^m}(W)$ and $\mathcal{H}_{q^m}(W) \rtimes \langle\phi\rangle$ split over $\overline{\mathbb{Q}}_\ell[q^{m/2}]$; corresponding to the specialization $q^{m/2} \mapsto 1 : \mathcal{H}_{q^m}(W) \to \overline{\mathbb{Q}}_\ell W$, there is a bijection $\chi \mapsto \chi_{q^m} : \mathrm{Irr}(W) \to \mathrm{Irr}(\mathcal{H}_{q^m}(W))$. Choosing an extension $\tilde\chi$ to $W \rtimes \langle\phi\rangle$ of each character in $\mathrm{Irr}(W)^\phi$, we get a corresponding extension $\tilde\chi_{q^m} \in \mathrm{Irr}(\mathcal{H}_{q^m}(W) \rtimes \langle\phi\rangle)$ which takes its values in $\overline{\mathbb{Q}}_\ell[q^{m/2}]$. If $U_\chi \in \mathrm{Irr}(\mathbf{G}^{F^m})$ is the corresponding character of $\mathbf{G}^{F^m}$, we get a corresponding extension $U_{\tilde\chi}$ of U_χ to $\mathbf{G}^{F^m} \rtimes \langle F\rangle$ (see [DM1, III théorème 1.3]). With these notations, the Shintani descent identity becomes

Proposition 7.30.

$$(g \mapsto |\mathbf{X}(\mathbf{I},\mathbf{w}\phi)^{gF^m}|) = \sum_{\chi\in\mathrm{Irr}(W)^\phi} \tilde\chi_{q^m}(X_1 T_\mathbf{w}\phi)\,\mathrm{Sh}_{F^m/F} U_{\tilde\chi}$$

and the only characters χ in that sum which give a non-zero contribution are those which are a component of $\mathrm{Ind}_{W_I}^W \mathrm{Id}$.

Proof. We have $\mathrm{Trace}(g'X_{\mathbf{w}\phi} \mid \overline{\mathbb{Q}}_\ell\mathcal{P}_I^{F^m}) = \mathrm{Trace}(g'X_1T_\mathbf{w}\phi \mid \overline{\mathbb{Q}}_\ell\mathcal{B}^{F^m})$ since X_1 is the projector onto $\overline{\mathbb{Q}}_\ell\mathcal{P}_I^{F^m}$. Hence $(g \mapsto |\mathbf{X}(\mathbf{I},\mathbf{w}\phi)^{gF^m}|) = \sum_{\chi\in\mathrm{Irr}(W)^\phi} \tilde\chi_{q^m}(X_1T_\mathbf{w}\phi)\,\mathrm{Sh}_{F^m/F} U_{\tilde\chi}$. Since X_1 acts by 0 on the representation of character χ if χ is not a component of $\mathrm{Ind}_{W_I}^W \mathrm{Id}$, we get the second assertion. □

Finally, if λ_ρ is the root of unity attached to $\rho \in \mathcal{E}(\mathbf{G}^F, 1)$ as in [DMR, 3.3.4], the above formula translates, using [DM1, III, 2.3(ii)] as

Corollary 7.31.

$$|\mathbf{X}(\mathbf{I},\mathbf{w}\phi)^{gF^m}| = \sum_{\rho\in\mathcal{E}(\mathbf{G}^F,1)} \lambda_\rho^{m/\delta}\rho(g) \sum_{\chi\in\mathrm{Irr}(W)^\phi} \tilde\chi_{q^m}(X_1T_\mathbf{w}\phi)\langle\rho, R_{\tilde\chi}\rangle_{\mathbf{G}^F},$$

where $R_{\tilde\chi} = |W|^{-1}\sum_{w\in W}\tilde\chi(w\phi)R_{\mathbf{T}_w}^\mathbf{G}(\mathrm{Id})$. *The only characters χ in the above sum which give a non-zero contribution are those which are a component of* $\mathrm{Ind}_{W_I}^W \mathrm{Id}$.

Using the Lefschetz formula and taking the "limit for $m \to 0$" (see for example [DMR, 3.3.8]) we get the equality of virtual characters

Corollary 7.32.

$$\sum_i (-1)^i H^i_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell) = \sum_{\{\chi\in\operatorname{Irr}(W)^\phi \mid \langle \operatorname{Res}^W_{W_I}\chi,\operatorname{Id}\rangle_{W_I}\neq 0\}} \tilde\chi(x_1 w\phi) R_{\tilde\chi},$$

where w is the image of $\mathbf{w}$ in W and $x_1 = |W_I|^{-1}\sum_{v\in W_I} v$.

Cohomology. If π is the projection of Lemma 7.18, the sheaf $\pi_!\overline{\mathbb{Q}}_\ell$ decomposes into a direct sum of sheaves indexed by the irreducible characters of $\mathbf{L}_\mathbf{I}^{t(\mathbf{w}\phi)}$. We will denote by $\boldsymbol{\chi}$ the subsheaf indexed by the character $\chi\in\operatorname{Irr}(\mathbf{L}_\mathbf{I}^{t(\mathbf{w}\phi)})$, and in particular by $\mathbf{St}$ the subsheaf indexed by the Steinberg character $\operatorname{St}\in\operatorname{Irr}(\mathbf{L}_\mathbf{I}^{t(\mathbf{w}\phi)})$. We have the isomorphism of $\mathbf{G}^F\times\mathbf{L}_I^{t(\mathbf{w}\phi)}$-modules

$$H^i_c(\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell) = \oplus_{\chi\in\operatorname{Irr}(\mathbf{L}_I^{t(\mathbf{w}\phi)})} H^i_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\boldsymbol{\chi})\otimes V_\chi$$

where V_χ is an $\mathbf{L}_I^{t(\mathbf{w}\phi)}$-module of character χ and $H^i_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\boldsymbol{\chi})$ is a $\mathbf{G}^F$-module. When χ is F^δ-stable there is an action of F^δ on V_χ such that the inclusion of $H^i_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\boldsymbol{\chi})\otimes V_\chi$ into $H^i_c(\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)$ is an inclusion of $\mathbf{G}^F\times\mathbf{L}_I^{t(\mathbf{w}\phi)}\rtimes\langle F^\delta\rangle$-modules

The following corollary of Proposition 7.19 relates the cohomology of a general variety $\mathbf{X}(\mathbf{I},\mathbf{w}\phi)$ to the case of the varieties $\mathbf{X}(\mathbf{u}\phi)$ considered in [DMR]; its part (ii) is a refinement of Corollary 7.32. In the following corollary, if M is a $\overline{\mathbb{Q}}_\ell$-vector space on wich F acts, we denote by $M(n)$ for $n\in\mathbb{Z}$ the n-th Tate twist of M.

Corollary 7.33. *Let $\mathbf{I}\xrightarrow{\mathbf{w}}{}^\phi\mathbf{I}\in B^+(\mathcal{I})$. Then*

(i) *For any unipotent F^δ-stable character $\chi\in\operatorname{Irr}(\mathbf{L}_I^{t(\mathbf{w}\phi)})$, for any $\mathbf{u}\in\underline{B}^+_\mathbf{I}$ and any i,j we have the inclusion of $\mathbf{G}^F\times\langle F^\delta\rangle$-modules*

$$H^i_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\boldsymbol{\chi})\otimes(H^j_c(\mathbf{X}_{\mathbf{L}_\mathbf{I}}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell)\otimes_{\mathbf{L}_I^{t(\mathbf{w}\phi)}} V_{\overline{\chi}})\subset H^{i+j}_c(\mathbf{X}(\mathbf{u}\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell).$$

(ii) *For all $\mathbf{v}\in B^+_\mathbf{I}$ and all i we have the following inclusions of $\mathbf{G}^F\times\langle F^\delta\rangle$-modules:*

$$H^i_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)\subset H^{i+2l(\mathbf{v})}_c(\mathbf{X}(\mathbf{v}\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)(-l(\mathbf{v}))$$

and

$$H^i_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\mathbf{St})\subset H^{i+l(\mathbf{v})}_c(\mathbf{X}(\mathbf{v}\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)$$

(iii) *For all i we have the following equality of $\mathbf{G}^F\times\langle F^\delta\rangle$-modules:*

$$H^i_c(\mathbf{X}(\underline{w}_I\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell) = \sum_{j+2k=i} H^j_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)\otimes\overline{\mathbb{Q}}_\ell^{\,n_{I,k}}(k)$$

where $n_{I,k} = |\{v\in W_I \mid l(v)=k\}|$.

Note that in (iii) above we have $\mathbf{X}(\underline{w}_I\mathbf{w}\phi)=\bigcup_{\mathbf{v}\in\mathbf{W}_\mathbf{I}}\mathbf{X}(\mathbf{v}\mathbf{w}\phi)$.

Proof. We apply the Künneth formula to the isomorphism of Corollary 7.20 and decompose the equality obtained according to the characters of $\mathbf{L}_I^{t(\mathbf{w}\phi)}$; we get that for any $\mathbf{u}\in\underline{B}^+_\mathbf{I}$, we have

$$\bigoplus_{\substack{0\le j\le 2l(\mathbf{u})\\ \chi\in\operatorname{Irr}(\mathbf{L}_I^{t(\mathbf{w}\phi)})}} H^{i-j}_c(\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)_\chi\otimes_{\mathbf{L}_I^{t(\mathbf{w}\phi)}} H^j_c(\mathbf{X}_{\mathbf{L}_\mathbf{I}}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell)_{\overline{\chi}}\simeq H^i_c(\mathbf{X}(\mathbf{u}\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell),$$

which can be written

$$(7.34)\quad \bigoplus_{\substack{0\le j\le 2l(\mathbf{u})\\ \chi\in\operatorname{Irr}(\mathbf{L}_I^{t(\mathbf{w}\phi)})}} H_c^{i-j}(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\boldsymbol{\chi})\otimes(H_c^j(\mathbf{X}_{\mathbf{L_I}}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell)\otimes_{\mathbf{L}_I^{t(\mathbf{w}\phi)}} V_{\overline{\chi}}) \simeq H_c^i(\mathbf{X}(\mathbf{uw}\phi),\overline{\mathbb{Q}}_\ell).$$

This gives (i). We get also (ii) from equation 7.34 and the facts that for $\mathbf{v}\in B_{\mathbf{I}}^+$

- the only j such that $H_c^j(\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}w\phi),\overline{\mathbb{Q}}_\ell)_{\mathrm{Id}}$ is non-trivial is $j=2l(\mathbf{v})$ and that isotypic component is irreducible and $t(\mathbf{w}\phi)$ acts by $q^{l(\mathbf{v})}$ on it (see [DMR, 3.3.14]) and $t(\mathbf{w}\phi)^{k\delta}$ is equal to $F^{k\delta}$ for some k.
- the only j such that $H_c^j(\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}w\phi),\overline{\mathbb{Q}}_\ell)_{\mathrm{St}}$ is non-trivial is $j=l(\mathbf{v})$ and that isotypic component is irreducible with trivial action of $t(\mathbf{w}\phi)$ (see [DMR, 3.3.15]).

Hence the term $\boldsymbol{\chi}=\overline{\mathbb{Q}}_\ell$ in the LHS of 7.34 for $\mathbf{u}=\mathbf{v}$ and $j=2l(\mathbf{v})$ is $H_c^{i-2l(\mathbf{v})}(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)\otimes\overline{\mathbb{Q}}_\ell(-l(\mathbf{v}))$ and is a submodule of $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{wv}\phi),\overline{\mathbb{Q}}_\ell)$. Similarly the term $\boldsymbol{\chi}=\mathbf{St}$ in the LHS for $\mathbf{u}=\mathbf{v}$ and $j=l(\mathbf{v})$ is $H_c^{i-l(\mathbf{v})}(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\mathbf{St})$ and is a submodule of $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)$.

We now prove (iii). By Corollary 7.20 applied with $\mathbf{u}=\underline{w}_I$ we have an isomorphism $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)\times_{\mathbf{L}_I^{t(\mathbf{w}\phi)}}\mathcal{B}_I\xrightarrow{\sim}\mathbf{X}(\underline{w}_I\mathbf{w}\phi)$ where $\mathcal{B}_I$ is the variety of Borel subgroups of $\mathbf{L}_I$. We get (iii) from the fact that $H_c^k(\mathcal{B}_I,\overline{\mathbb{Q}}_\ell)$ is 0 if k is odd and if $k=2k'$ is a trivial $\mathbf{L}_I^{t(\mathbf{w}\phi)}$-module of dimension $n_{I,k'}$, where F^δ acts by the scalar $q^{\delta k'}$; this results for example from the cellular decomposition into affine spaces given by the Bruhat decomposition and the fact that the action of $\mathbf{L}_I^{t(\mathbf{w}\phi)}$ extends to the connected group $\mathbf{L}_I$ so that it acts trivially on the cohomology. □

Corollary 7.35. *Let* $\mathbf{I}\xrightarrow{\mathbf{w}\phi}\mathbf{I}\in B^+\phi(\mathcal{I})$, *let* $\chi\in\operatorname{Irr}(\mathbf{L}_I^{t(\mathbf{w}\phi)})$ *be unipotent and* F^δ*-stable, and let* $i\in\mathbb{N}$. *Then*

(i) *The* $\mathbf{G}^F$*-module* $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\boldsymbol{\chi})$ *is unipotent. Given* $\rho\in\operatorname{Irr}(\mathbf{G}^F)$ *unipotent, the eigenvalues of* F^δ *on* $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\boldsymbol{\chi})_\rho$ *are in* $q^{\delta\mathbb{N}}\lambda_\rho\omega_\rho$, *where* λ_ρ *is as in Corollary 7.31 and* ω_ρ *is the element of* $\{1,q^{\delta/2}\}$ *attached to* ρ *as in* [DMR, 3.3.4]; λ_ρ *and* ω_ρ *are independent of* i *and* $\mathbf{w}$.

(ii) *The eigenvalues of* F^δ *on* $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\boldsymbol{\chi})$ *have absolute value at most* $q^{\delta i/2}$.

(iii) *We have* $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\boldsymbol{\chi})=0$ *unless* $l(\mathbf{w})\le i\le 2l(\mathbf{w})$.

(iv) *The Steinberg representation does not occur in* $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\boldsymbol{\chi})$ *unless* $\boldsymbol{\chi}=\mathbf{St}$ *and* $i=l(\mathbf{w})$, *in which case it occurs with multiplicity* 1, *associated with the eigenvalue* 1 *of* F^δ.

(v) *The trivial representation does not occur in* $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\boldsymbol{\chi})$ *unless* $\boldsymbol{\chi}=\overline{\mathbb{Q}}_\ell$ *and* $i=2l(\mathbf{w})$, *in which case it occurs with multiplicity* 1, *associated with the eigenvalue* $q^{\delta l(\mathbf{w})}$ *of* F^δ.

Proof. (i) is a straightforward consequence of equation 7.34 applied for any $\mathbf{u}$ such that some term $H_c^j(\mathbf{X}_{\mathbf{L_I}}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell)_{\overline{\chi}}$ is not 0 for some j, since the result is known for $H_c^i(\mathbf{X}(\mathbf{uw}\phi),\overline{\mathbb{Q}}_\ell)$ (see [DMR, 3.3.4] and [DMR, 3.3.10 (i)]).

(ii) and (iii) are a consequence of 7.34 applied for $\mathbf{u}\in B_{\mathbf{I}}^+$ of minimal length such that $\overline{\chi}$ appears in some $H_c^j(\mathbf{X}_{\mathbf{L}_I}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell)$. Then by [DMR, 3.3.21] $\overline{\chi}$ appears in $H_c^{l(\mathbf{u})}(\mathbf{X}_{\mathbf{L}_I}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell)$ and the corresponding eigenvalue of F^δ has module $q^{\delta l(\mathbf{u})/2}$.

It follows then from 7.33(i) applied with $j = l(\mathbf{u})$ that $H^i_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\boldsymbol{\chi})\otimes V \subset H^{i+l(\mathbf{u})}(\mathbf{X}(\mathbf{uw}\phi),\overline{\mathbb{Q}}_\ell)$ where V is an F^δ-module where the eigenvalues of F^δ are of module $q^{\delta l(\mathbf{u})/2}$. The result follows from the facts that $H^{i+l(\mathbf{u})}(\mathbf{X}(\mathbf{uw}\phi),\overline{\mathbb{Q}}_\ell) = 0$ for $i < l(\mathbf{w})$ and that the eigenvalues of F^δ on it have a module at most $q^{\delta(i+l(\mathbf{u}))/2}$.

For (iv), we use

Lemma 7.36. *If* $\chi \in \mathrm{Irr}(\mathbf{L}_I^{t(\mathbf{w}\phi)})$ *is unipotent and* $\chi \neq \mathrm{St}$ *there exists* $\mathbf{u} \in \underline{B}^+_{\mathbf{I}} - B^+_{\mathbf{I}}$ *and* j *such that* $H^j_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell)_\chi \neq 0$.

Proof. First, assume that χ is not in the principal series, and let $\mathbf{v} \in B^+_{\mathbf{I}}$ be of minimal length such that χ appears in some $H^j_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}w\phi),\overline{\mathbb{Q}}_\ell)$. Since χ is not in the principal series we have $l(\mathbf{v}) > 0$ thus there exists $\mathbf{s} \in \mathbf{I}$ and $\mathbf{v}' \in B^+_{\mathbf{I}}$ such that $\mathbf{v} = \mathbf{s}\mathbf{v}'$. Then $H^j_c(\mathbf{X}_{\mathbf{L}_I}(\underline{s}\mathbf{v}'w\phi),\overline{\mathbb{Q}}_\ell)_\chi = H^j_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}w\phi),\overline{\mathbb{Q}}_\ell)_\chi \neq 0$ because of the minimality of $\mathbf{v}$ and the long exact sequence resulting from $\mathbf{X}_{\mathbf{L}_I}(\underline{s}\mathbf{v}'w\phi) = \mathbf{X}_{\mathbf{L}_I}(\mathbf{v}w\phi)\coprod \mathbf{X}_{\mathbf{L}_I}(\mathbf{v}'w\phi)$ where the first (resp. second) term of the RHS is an open (resp. closed) subvariety of the LHS.

When χ is in the principal series, we use that if J is a $w\phi$-stable subset of I and $\mathbf{u} \in \underline{B}^+_J$, then $H^j_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell) = R^{\mathbf{L}_I}_{\mathbf{L}_J} H^j_c(\mathbf{X}_{\mathbf{L}_J}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell)$. It follows that if χ is of the form ρ_ψ for $\psi \in \mathrm{Irr}(W_I^{w\phi})$ (see [DMR, 5.3.1]), and ψ_1 is a component of $\mathrm{Res}^{W_I^{w\phi}}_{W_J^{w\phi}}\psi$ such that $\langle H^j_c(\mathbf{X}_{\mathbf{L}_J}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell),\rho_{\psi_1}\rangle_{\mathbf{L}_J^F} \neq 0$, then $\langle H^j_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell),\rho_\psi\rangle_{\mathbf{L}_I^F} \neq 0$. If J is a $w\phi$-orbit in I, the group $W_J^{w\phi}$ is a Coxeter group of type A_1 and the restriction to $W_J^{w\phi}$ of a character ψ other than the sign character cannot be isotypic of type sign for all orbits J (ψ would then be itself isotypic of type sign). We are thus reduced to the case where I is a single $w\phi$-orbit, so that $L_I^{t(w\phi)}$ has only two unipotent characters, Id and St. For such a group the identity character is a component of $H^2_c(\mathbf{X}_{\mathbf{L}_I}(\underline{s}w\phi)\overline{\mathbb{Q}}_\ell)$ where $W_I^{w\phi} = \langle s\rangle$, so that the lemma is true. ☐

Since for $\mathbf{u}$ as in the lemma we have $H^*_c(\mathbf{X}(\mathbf{uw}\phi),\overline{\mathbb{Q}}_\ell)_{\mathrm{St}} = 0$ (see [DMR, 3.3.15]), by 7.34 we deduce that for $\boldsymbol{\chi} \neq \mathbf{St}$ we have $H^i_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\boldsymbol{\chi})_{\mathrm{St}} = 0$ for all i. Thus, for any $\mathbf{u} \in B^+_{\mathbf{I}}$, using that $H^j_c(\mathbf{X}_{\mathbf{L_I}}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell)\otimes_{\mathbf{L}_I^{t(\mathbf{w}\phi)}} V_{\mathrm{St}} = 0$ when $j \neq l(\mathbf{u})$, the St-part of 7.34 reduces to

$$H^{i-l(\mathbf{u})}_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\mathbf{St})_{\mathrm{St}}\otimes(H^{l(\mathbf{u})}_c(\mathbf{X}_{\mathbf{L_I}}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell)\otimes_{\mathbf{L}_I^{t(\mathbf{w}\phi)}} V_{\mathrm{St}}) \simeq H^i_c(\mathbf{X}(\mathbf{uw}\phi),\overline{\mathbb{Q}}_\ell)_{\mathrm{St}}.$$

We apply this for $\mathbf{u} = \mathbf{v} \in B^+_{\mathbf{I}}$ in which case $H^{l(\mathbf{v})}_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}w\phi),\overline{\mathbb{Q}}_\ell)\otimes_{\mathbf{L}_I^{t(\mathbf{w}\phi)}} V_{\mathrm{St}} = \overline{\mathbb{Q}}_\ell$ with trivial action of F^δ, which gives the isomorphism of $\mathbf{G}^F \times \langle F\rangle^\delta$-modules

$$H^i_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\mathbf{St})_{\mathrm{St}} \simeq H^{i+l(\mathbf{v})}_c(\mathbf{X}(\mathbf{vw}\phi),\overline{\mathbb{Q}}_\ell)_{\mathrm{St}}.$$

using the values of the RHS (known by [DMR, 3.3.15]) $H^{i+l(\mathbf{v})}_c(\mathbf{X}(\mathbf{vw}\phi),\overline{\mathbb{Q}}_\ell)_{\mathrm{St}} = \begin{cases} 0 & \text{if } i \neq l(\mathbf{w}) \\ \overline{\mathbb{Q}}_\ell & \text{with trivial action of } F^\delta \text{ otherwise}\end{cases}$, we get (iv).

For (v), we use

Lemma 7.37. *If* $\chi \in \mathrm{Irr}(\mathbf{L}_I^{t(\mathbf{w}\phi)})$ *is unipotent and* $\chi \neq \mathrm{Id}$ *there exists* $\mathbf{u} \in B^+_{\mathbf{I}}$ *and* $j \neq 2l(\mathbf{u})$ *such that* $H^j_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell)_\chi \neq 0$.

Proof. First, assume that χ is not in the principal series, and let $\mathbf{u} \in B^+_{\mathbf{I}}$ be of minimal length such that χ appears in some $H^j_c(\mathbf{X}_{\mathbf{L}_I}(\mathbf{u}w\phi),\overline{\mathbb{Q}}_\ell)$. Then by [DMR, 3.3.21 (ii)] we have $H_c^{l(\mathbf{u})}(\mathbf{X}_{\mathbf{L}_I}(\mathbf{u}w\phi), \overline{\mathbb{Q}}_\ell)_\chi \neq 0$. Since χ is not in the principal series we have $l(\mathbf{u}) \neq 2l(\mathbf{u})$, whence the lemma in this case.

Now assume χ in the principal series and take $\mathbf{u} = \boldsymbol{\pi}_\mathbf{I}$. It results for example from [DMR, 3.3.8 (i)] that there exists j such that $H_c^j(\mathbf{X}_{\mathbf{L}_I}(\boldsymbol{\pi}_\mathbf{I} w\phi), \overline{\mathbb{Q}}_\ell) \neq 0$. On the other hand, it results for example from Proposition 7.8 that $\mathbf{X}_{\mathbf{L}_I}(\boldsymbol{\pi}_\mathbf{I} w\phi)$ is irreducible, thus $H_c^{2l(\boldsymbol{\pi}_\mathbf{I})}(\mathbf{X}_{\mathbf{L}_I}(\boldsymbol{\pi}_\mathbf{I} w\phi), \overline{\mathbb{Q}}_\ell)$ is a 1-dimensional module affording only the trivial representation of $\mathbf{G}^F$. It follows that $j \neq 2l(\boldsymbol{\pi}_\mathbf{I})$, whence the lemma. □

Applying 7.34 for an $\mathbf{u}$ as in Lemma 7.37 and using that $H_c^i(\mathbf{X}(\mathbf{uw}\phi), \overline{\mathbb{Q}}_\ell)_{\mathrm{Id}} = 0$ for $i \neq 2(l(\mathbf{w}) + l(\mathbf{u}))$, we deduce that for $\boldsymbol{\chi} \neq \mathrm{Id}$ we have $H_c^i(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \boldsymbol{\chi})_{\mathrm{Id}} = 0$ for all i. Taking now $\mathbf{u} = 1$ and using that $H_c^0(\mathbf{X}_{\mathbf{L}_I}(w\phi), \overline{\mathbb{Q}}_\ell) \otimes_{\mathbf{L}_I^{t(\mathbf{w}\phi)}} \mathrm{Id} = \overline{\mathbb{Q}}_\ell$, the Id-part of 7.34 reduces to

$$H_c^i(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \mathrm{Id})_{\mathrm{Id}} \simeq H_c^i(\mathbf{X}(\mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)_{\mathrm{Id}}.$$

whence the result using the value of the RHS given by [DMR, 3.3.14]. □

8. Eigenspaces and roots of $\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I}$

Let $\ell \neq p$ be a prime such that a Sylow ℓ-subgroup S of $\mathbf{G}^F$ is abelian.

Then "generic block theory" (see [BMM]) associates with ℓ a root of unity ζ and some $w\phi \in W\phi$ such that its ζ-eigenspace V in $X := X_\mathbb{R} \otimes \mathbb{C}$ is non-zero and maximal among ζ-eigenspaces of elements of $W\phi$; for any such ζ, there exists a unique minimal subtorus $\mathbf{S}$ of $\mathbf{T}$ such that $V \subset X(\mathbf{S}) \otimes \mathbb{C}$. The space $X(\mathbf{S}) \otimes \mathbb{C}$ is the kernel of $\Phi(w\phi)$, where, if the coset $W\phi$ is rational (that is, ϕ preserves $X(\mathbf{T})$) then Φ is the d-th cyclotomic polynomial, where d is the order of ζ. Otherwise, in the "very twisted" cases ${}^2B_2, {}^2F_4$ (resp. ${}^2G_2$) we have to take for Φ the irreducible cyclotomic polynomial over $\mathbb{Q}(\sqrt{2})$ (resp. $\mathbb{Q}(\sqrt{3})$) of which ζ is a root. The torus $\mathbf{S}$ is wF-stable thus has an F-stable $\mathbf{G}$-conjugate $\mathbf{S}'$ in a maximal torus of type w; the torus $\mathbf{S}'$ is called a Φ-Sylow; we have $|\mathbf{S}'^F| = \Phi(q)^{\dim V}$.

The relationship with ℓ is that S is a subgroup of $\mathbf{S}'^F$, and thus that $|\mathbf{G}^F|/|\mathbf{S}'^F|$ is prime to ℓ; we have $N_{\mathbf{G}^F}(S) = N_{\mathbf{G}^F}(\mathbf{S}') = N_{\mathbf{G}^F}(\mathbf{L})$ where $\mathbf{L} := C_\mathbf{G}(\mathbf{S}')$ is a Levi subgroup of $\mathbf{G}$ whose Weyl group is $C_W(V)$. Conversely, any non-zero maximal ζ-eigenspace determines some primes ℓ giving an abelian Sylow, those which divide $\Phi(q)$ and no other cyclotomic factor of $|\mathbf{G}^F|$.

The classes $C_W(V)w\phi$, where $V = \mathrm{Ker}(w\phi - \zeta)$ is maximal, form a single orbit under W-conjugacy [see eg. [Br2, 5.6(i)]]; the maximality implies that all elements of $C_W(V)w\phi$ have same ζ-eigenspace.

We will see in Theorem 8.1(i) that up to conjugacy we may assume that $C_W(V)$ is a standard parabolic group W_I; then the Broué conjectures predict that for an appropriate choice of coset $C_W(V)w\phi$ in its $N_W(W_I)$-conjugacy class the cohomology complex of the variety $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$ should be a tilting complex realizing a derived equivalence between the unipotent parts of the principal ℓ-blocks of $\mathbf{G}^F$ and of $N_{\mathbf{G}^F}(\mathbf{S}')$. We want to describe explicitly what should be a "good" choice of w (see Conjectures 9.1).

Since it is no more effort to have a result in the context of any finite real reflection group than for a context which includes the Ree and Suzuki groups, we give a more general statement. Our situation generalizes that studied in [BM], which corresponds to the case $\mathbf{I} = \emptyset$, or ζ-regular elements, that is elements of $W\phi$ which have an eigenvector for the eigenvalue ζ outside the reflecting hyperplanes (see [S, above 6.5]); in particular Theorem 8.1 generalizes [BM, 3.11, 6.5] and Theorem 8.3 generalizes [BM, 3.12, 6.6]; in the [BM] case, the "d-good periodic maximal" elements we consider here reduce to "good d-th ϕ-roots of $\boldsymbol{\pi}$". Note that we focus our study on the ℓ-principal block (or $\Phi(q)$-principal block), which corresponds to the maximality condition on eigenspaces and to what we call "non-extendable" periodic elements. Extendable periodic elements would be needed in considering more general blocks.

In what follows we look at real reflection cosets $W\phi$ of finite order, that is W is a finite reflection group acting on the real vector space $X_{\mathbb{R}}$ and ϕ is an element of $N_{\mathrm{GL}(X_{\mathbb{R}})}(W)$, such that $W\phi$ is of finite order δ, that is δ is the smallest integer such that $(W\phi)^\delta = W$ (equivalently ϕ is of finite order). Since W is transitive on the chambers of the real hyperplane arrangement it determines, one can always choose ϕ in its coset so that it preserves a chamber of this arrangement. We will do this; thus ϕ is 1-regular, since it has a fixed point outside the reflecting hyperplanes, thus is of order δ since 1 is the only 1-regular element of W.

Theorem 8.1. *Let $W\phi \subset \mathrm{GL}(X_{\mathbb{R}})$ be a finite order real reflection coset, such that ϕ preserves a chamber of the hyperplane arrangement on $X_{\mathbb{R}}$ determined by W, thus induces an automorphism of the Coxeter system (W,S) determined by this chamber. We call again ϕ the induced automorphism of the braid group B of W, and denote by $\mathbf{S},\mathbf{W}$ the lifts of S,W to B (see Example 4.1).*

Let $\zeta = e^{2i\pi k/d}$, and let V be a subspace of $X := X_{\mathbb{R}}\otimes\mathbb{C}$ on which some element of $W\phi$ acts by ζ. Then we may choose V in its W-orbit such that:

- (i) *$C_W(V) = W_I$ for some $I \subset S$.*
- (ii) *If $W_I w\phi$ is the W_I-coset of elements which act by ζ on V, where w is I-reduced, then $l((w\phi)^i) = (2ik/d)l(w_0 w_I^{-1})$ for $2ik \le d$, where we have extended the length function to $W \rtimes \langle\phi\rangle$ by $l(w\phi^i) = l(w)$.*

Further, we may lift w as in (ii) to $\mathbf{w} \in B^+$ such that ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$ and $(\mathbf{w}\phi)^d = \phi^d(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^k$, where $\mathbf{I} \subset \mathbf{S}$ lifts I. Thus $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}$ is a $(d,2k)$-periodic element in $B^+\phi(\mathcal{I})$, where $\mathcal{I}$ is the set of subsets of $\mathbf{S}$ conjugate to $\mathbf{I}$.

Note that the last part implies that for w as in (ii) we have $(w\phi)^d = \phi^d$. Note also that if $2k \le d$, then (ii) is applicable for $i=1$ and we get $l(\mathbf{w}) = l(w) = (2k/d)l(w_0 w_I^{-1})$ thus $\mathbf{w}$ is the unique lift of w to $\mathbf{W}$.

Since we assume $W\phi$ real, if $e^{2i\pi k/d}$ is an eigenvalue of $w\phi$, then the complex conjugate $e^{2i\pi(d-k)/d}$ is also an eigenvalue, for the complex conjugate eigenspace; thus we may always assume that $2k \le d$, so that $\mathbf{w} \in \mathbf{W}$.

If the coset $W\phi$ preserves a $\mathbb{Q}$-structure on $X_{\mathbb{R}}$ (which is the case for cosets associated with finite reductive groups, except for the "very twisted" cases ${}^2B_2, {}^2G_2$ and ${}^2F_4$), we have more generally that if $e^{2i\pi k/d}$ is an eigenvalue of $w\phi$, with k prime d, the Galois conjugate $e^{2i\pi/d}$ is also an eigenvalue, for a Galois conjugate eigenspace; in these cases we may assume $k=1$.

Recall that by our conventions, even though ϕ is a finite order automorphism of B^+, in the semi-direct product $B^+ \rtimes \langle\phi\rangle$ we take $\langle\phi\rangle$ of infinite order.

Proof of Theorem 8.1. Since $W\langle\phi\rangle$ is finite, we may find a scalar product on $X_{\mathbb{R}}$ (extending to an Hermitian product on X) invariant by W and ϕ. The subspace $X'_{\mathbb{R}}$ of $X_{\mathbb{R}}$ orthogonal to the fixed points of W (the subspace spanned by the root lines of W) identifies with the reflection representation of the Coxeter system (W,S) (see for example [Bou, Chapitre V §3]). We will use the root system Φ on $X'_{\mathbb{R}}$ consisting of the vectors of norm 1 (for the scalar product) along the root lines of W, which is thus preserved by $W\langle\phi\rangle$. By [Bou, Chapitre V §3 Proposition 1] the centralizer of any subspace of X is a parabolic subgroup of W, hence conjugate to a standard parabolic subgroup, whence (i).

To prove (ii) we reprove (i) by changing the order on Φ, which is equivalent to do a conjugation by some element of W. Let v be a regular vector in V, that is $v \in V$ such that $C_W(v) = C_W(V)$. Multiplying v if needed by a complex number of absolute value 1, we may assume that for any $\alpha \in \Phi$ we have $\Re\langle v, \alpha\rangle = 0$ if and only if $\langle v, \alpha\rangle = 0$. Then there exists an order on Φ such that $\Phi^+ \subset \{\alpha \in \Phi \mid \Re(\langle v, \alpha\rangle) \geq 0\}$. Let Π be the corresponding basis; the subset $I = \{\alpha \in \Pi | \Re(\langle v, \alpha\rangle) = 0\}$ is such that $C_W(V) = C_W(v) = W_I$, and $\Phi_I = \{\alpha \in \Phi \mid \langle v, \alpha\rangle = 0\}$ is a root system for W_I.

Note that $(w\phi)^d = \phi^d$. Indeed $(w\phi)^d$ fixes v, thus preserves the sign of any root not in Φ_I; since w is chosen I-reduced we have ${}^{w\phi}I = I$, so that $w\phi$ also preserves the sign of roots in Φ_I. It is thus equal to the only element ϕ^d of $W\phi^d$ which preserves the signs of all roots. We get also that ${}^{\phi^d}I = I$. If we notice that we may lift ϕ to $\phi\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$, this completes the proof in the case $d = 1$.

We now assume that $d \neq 1$ and we first prove the theorem in the case $k = 1$. Since $\langle v, {}^{(w\phi)^m}\alpha\rangle = \langle {}^{(w\phi)^{-m}}v, \alpha\rangle = \zeta^{-m}\langle v, \alpha\rangle$, we get that all orbits of $w\phi$ on $\Phi - \Phi_I$ have cardinality a multiple of d; it is thus possible by partitioning suitably those orbits, to get a partition of $\Phi - \Phi_I$ in subsets $\mathcal{O}$ of the form $\{\alpha, {}^{w\phi}\alpha, \ldots, {}^{(w\phi)^{d-1}}\alpha\}$; and the numbers $\{\langle v, \beta\rangle \mid \beta \in \mathcal{O}\}$ for a given $\mathcal{O}$ form the vertices of a regular d-gon centered at $0 \in \mathbb{C}$; the action of $w\phi$ is the rotation by $-2\pi/d$ of this d-gon. Looking at the real parts of the vertices of this d-gon, we see that for $m \leq d/2$, exactly m positive roots in $\mathcal{O}$ are sent to negative roots by $(w\phi)^m$. Since this holds for all $\mathcal{O}$, we get that for $m \leq d/2$ we have $l((w\phi)^m) = \frac{m|\Phi - \Phi_I|}{d}$; thus if $\mathbf{w}$ is the lift of w to $\mathbf{W}$ we have $(\mathbf{w}\phi)^i \in \mathbf{W}\phi^i$ if $2i \leq d$.

Now we finish the case $k = 1, d \neq 1$ with the following

Lemma 8.2. *Assume that ${}^{w\phi}W_I = W_I$, that w is I-reduced, and that for some $d > 1$ we have $(w\phi)^d = \phi^d$ and $l((w\phi)^i) = (2i/d)l(w_0 w_I^{-1})$ if $2i \leq d$. Then if $\mathbf{w}$ is the lift of w to $\mathbf{W}$ we have ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$ and $(\mathbf{w}\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$.*

Proof. Since w is I-reduced and $w\phi$ normalizes W_I we get that $w\phi$ stabilizes I; these properties imply in the braid monoid the equality ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$.

Assume first d even and let $d = 2d'$ and $x = \phi^{-d'}(w\phi)^{d'}$. Then $l(x) = (1/2)l(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}) = l(w_0) - l(w_I)$ and since x is reduced-I it is equal to the only reduced-I element of that length which is $w_0 w_I^{-1}$. Since the lengths add we can lift the equality $(w\phi)^{d'} = \phi^{d'} w_0 w_I^{-1}$ to the braid monoid as $(\mathbf{w}\phi)^{d'} = \phi^{d'}\mathbf{w}_0\mathbf{w}_I^{-1}$. By a similar reasoning using that $(w\phi)^{d'}\phi^{-d'}$ is the unique I-reduced element of its length, we get also $(\mathbf{w}\phi)^{d'} = \mathbf{w}_I^{-1}\mathbf{w}_0\phi^{d'}$. Thus $(\mathbf{w}\phi)^d = \mathbf{w}_I^{-1}\mathbf{w}_0\phi^{d'}\phi^{d'}\mathbf{w}_0\mathbf{w}_I^{-1} = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$, where the last equality uses that $\phi^d = (w\phi)^d$ preserves $\mathbf{I}$.

Assume now that $d = 2d' + 1$; then $(w\phi)^{d'}\phi^{-d'}$ is I-reduced and $\phi^{-d'}(w\phi)^{d'}$ is reduced-I. Using that any reduced-$\mathbf{I}$ element of $\mathbf{W}$ is a right-divisor of $\mathbf{w}_0\mathbf{w}_I^{-1}$ (resp. any $\mathbf{I}$-reduced element of $\mathbf{W}$ is a left-divisor of $\mathbf{w}_I^{-1}\mathbf{w}_0$), we get that there exists $\mathbf{t}, \mathbf{u} \in \mathbf{W}$ such that $\phi^{d'}\mathbf{w}_I^{-1}\mathbf{w}_0 = \mathbf{t}(\mathbf{w}\phi)^{d'}$ and $\mathbf{w}_0\mathbf{w}_I^{-1}\phi^{d'} = (\mathbf{w}\phi)^{d'}\mathbf{u}$. Thus $\phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}} = \mathbf{w}_0\mathbf{w}_I^{-1}\phi^d\mathbf{w}_I^{-1}\mathbf{w}_0 = (\mathbf{w}\phi)^{d'}\mathbf{u}\phi\mathbf{t}(\mathbf{w}\phi)^{d'}$, the first equality since ${}^{\phi^d}I = I$.

The image in $W\phi^d$ of the left-hand side is ϕ^d, and $(w\phi)^d = \phi^d$. We deduce that the image in $W\phi$ of $\mathbf{u}\phi\mathbf{t}$ is $w\phi$. If $d \neq 1$ then $d' \neq 0$ and we have $l(\mathbf{u}) = l(\mathbf{t}) = l(\mathbf{w})/2$; thus $\mathbf{u}\phi\mathbf{t} = \mathbf{w}\phi$ and $(\mathbf{w}\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$. □

We now consider the case $k \neq 1$, $d \neq 1$. We have seen (before assuming $k = 1$) that (i) holds and that the I-reduced element w of the coset $W_I w\phi$ acting by ζ on V satisfies $(w\phi)^d = \phi^d$.

We first consider the case when k is prime to d. Let d', k' be positive integers such that $kd' = 1 + dk'$, and let $w_1\phi_1 = (w\phi)^{d'}$, where $\phi_1 = \phi^{d'}$. Then $w_1\phi_1$ acts on V by $e^{2i\pi/d}$, so by the case $k = 1$ we have $l((w_1\phi_1)^i) = (2i/d)l(w_0 w_I^{-1})$ for $2i \leq d$. Since $(w_1\phi_1)^{ik} = (w\phi)^{ikd'} = (w\phi)^{i(1+dk')} = (w\phi)^i\phi^{idk'}$, we get (ii).

By Lemma 8.2 the lift $\mathbf{w}_1$ of w_1 to B satisfies ${}^{\mathbf{w}_1\phi_1}\mathbf{I} = \mathbf{I}$ and $(\mathbf{w}_1\phi_1)^d = \phi_1^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$, thus if we define $\mathbf{w}$ by $(\mathbf{w}_1\phi_1)^k = \mathbf{w}\phi^{1+dk'}$, then $\mathbf{w}$ lifts w and satisfies $(\mathbf{w}\phi)^d = \phi^d(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^k$, using ${}^{\phi^d}I = I$.

We finally consider the general case $d = \lambda d_1$, $k = \lambda k_1$ where d_1 is prime to k_1. The theorem holds for d_1, k_1; statement (ii) depends only on k/d thus holds, and we just have to raise the equation $(\mathbf{w}\phi)^{d_1} = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^{k_1}\phi^{d_1}$ to the λ-th power to get the desired equation $(\mathbf{w}\phi)^d = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^k\phi^d$. □

We give now a kind of converse of Theorem 8.1.

Theorem 8.3. *Let (W, S), ϕ, $X_{\mathbb{R}}$, X, $\mathbf{S}, B, B^+$ be as in Theorem 8.1. For $d \in \mathbb{N}$, let $\mathbf{w} \in B^+$ be such that $(\mathbf{w}\phi)^d = \phi^d(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^k$ for some ϕ^d-stable $\mathbf{I} \subset \mathbf{S}$. Then*

(i) *${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$, and $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}$ is a $(d, 2k)$-periodic element in $B^+\phi(\mathcal{I})$, where $\mathcal{I}$ is the set of subsets of $\mathbf{S}$ conjugate to $\mathbf{I}$.*

Denote by w and I the images in W of $\mathbf{w}$ and $\mathbf{I}$, let $\zeta = e^{2i\pi k/d}$, let $V \subset X$ be the ζ-eigenspace of $w\phi$, and let X^{W_I} be the fixed point space of W_I; then

(ii) *$W_I = C_W(X^{W_I} \cap V)$, in particular $C_W(V) \subset W_I$.*

Further, the following two assertions are equivalent:

(iii) *No element of the coset $W_I w\phi$ has a non-zero ζ-eigenvector on the subspace spanned by the root lines of W_I.*

(iv) *$\mathbf{w}\phi$ is "non-extendable", that is, there do not exist a ϕ^d-stable $\mathbf{J} \subsetneq \mathbf{I}$ and $\mathbf{v} \in B^+_{\mathbf{I}}$ such that $(\mathbf{v}\mathbf{w}\phi)^d = \phi^d(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}})^k$.*

Proof. We will deduce the general case from the case $k = 1$.

So we first assume $k = 1$. Then (i) is already in Proposition 5.4 which also states that there exists $\mathbf{I} \xrightarrow{\mathbf{v}} \mathbf{J} \in B^+(\mathcal{I})$ such that if $\mathbf{w}'\phi = (\mathbf{w}\phi)^{\mathbf{v}}$ then $\mathbf{w}'\phi \in B^+\phi$, $(\mathbf{w}'\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}$ and $(\mathbf{w}'\phi)^{\lfloor \frac{d}{2} \rfloor} \in \mathbf{W}\phi^{\lfloor \frac{d}{2} \rfloor}$.

As (ii) and the equivalence of (iii) and (iv) are invariant by a conjugacy in B which sends $\mathbf{w}\phi$ to $B^+\phi$ and $\mathbf{I}$ to another subset of $\mathbf{S}$, we may replace $(\mathbf{w}\phi, \mathbf{I})$ by the conjugate $(\mathbf{w}'\phi, \mathbf{J})$, thus assume that w and I satisfy the assumptions of the next lemma.

Lemma 8.4. *Let $w \in W, I \subset S$ be such that $(w\phi)^d = \phi^d$, ${}^{w\phi}I = I$ and such that $l((w\phi)^i) = \frac{2i}{d}l(w_I^{-1}w_0)$ for any $i \leq d/2$. We have*

(i) *If Φ is a root system for W and Φ^+ is chosen such that $\phi(\Phi^+) = \Phi^+$ (as in the proof of Theorem 8.1), then $\Phi - \Phi_I$ is the disjoint union of sets of the form $\{\alpha, {}^{w\phi}\alpha, \ldots, {}^{(w\phi)^{d-1}}\alpha\}$ with $\alpha, {}^{w\phi}\alpha, \ldots, {}^{(w\phi)^{\lfloor d/2 \rfloor - 1}}\alpha$ of same sign and ${}^{(w\phi)^{\lfloor d/2 \rfloor}}\alpha, \ldots, {}^{(w\phi)^{d-1}}\alpha$ of the opposite sign.*

(ii) *The order of* $w\phi$ *is* $\operatorname{lcm}(d,\delta)$.
(iii) *If* $d>1$, *then* $W_I = C_W(X^{W_I}\cap\ker(w\phi-\zeta))$.

Proof. The statement is empty for $d=1$ so in the following proof we assume $d>1$.

For $x\in W\rtimes\langle\phi\rangle$ let $N(x)=\{\alpha\in\Phi^+ \mid {}^x\alpha\in\Phi^-\}$; for $x\in W$ we have $l(x)=|N(x)|$ (see [Bou, Chapitre VI §1, Corollaire 2]). This still holds for $x=w\phi^i\in W\rtimes\langle\phi\rangle$ since $N(w\phi^i)={}^{\phi^{-i}}N(w)$. It follows that for $x,y\in W\rtimes\langle\phi\rangle$ we have $l(xy)=l(x)+l(y)$ if and only if $N(xy)=N(y)\coprod {}^{y^{-1}}N(x)$. In particular $l((w\phi)^i)=il(w\phi)$ for $i\le d/2$ implies ${}^{(w\phi)^{-i}}N(w\phi)\subset\Phi^+$ for $i\le d/2-1$.

Let us partition each $w\phi$-orbit in $\Phi-\Phi_I$ into "pseudo-orbits" of the form $\{\alpha,{}^{w\phi}\alpha,\ldots,{}^{(w\phi)^{k-1}}\alpha\}$, where k is minimal such that ${}^{(w\phi)^k}\alpha={}^{\phi^k}\alpha$ (then k divides d); a pseudo-orbit is an orbit if $\phi=1$. The action of $w\phi$ defines a cyclic order on each pseudo-orbit. The previous paragraph shows that when there is a sign change in a pseudo-orbit, at least the next $\lfloor d/2\rfloor$ roots for the cyclic order have the same sign. On the other hand, as ϕ^k preserves Φ^+, each pseudo-orbit contains an even number of sign changes. Thus if there is at least one sign change we have $k\ge 2\lfloor d/2\rfloor$. Since k divides d, we must have $k=d$ for pseudo-orbits which have a sign change, and then they have exactly two sign changes. As the total number of sign changes is $2l(w)=2|\Phi-\Phi_I|/d$, there are $|\Phi-\Phi_I|/d$ pseudo-orbits with sign changes; their total cardinality is $|\Phi-\Phi_I|$, thus there are no other pseudo-orbits and up to a cyclic permutation we may assume that each pseudo-orbit consists of $\lfloor d/2\rfloor$ roots of the same sign followed by $d-\lfloor d/2\rfloor$ of the opposite sign. We have proved (i).

Let $d'=\operatorname{lcm}(d,\delta)$. The proof of (i) shows that the order of $w\phi$ is a multiple of d. Since the order of $(w\phi)^d=\phi^d$ is d'/d, we get (ii).

We now prove (iii). Let $V=\ker(w\phi-\zeta)$. Since $W\langle\phi\rangle$ is finite, we may find a scalar product on X invariant by W and ϕ. We have then $X^{W_I}=\Phi_I^\perp$. The map $p=\frac{1}{d'}\sum_{i=0}^{d'-1}\zeta^{-i}(w\phi)^i$ is a $w\phi$-invariant projector on V, thus is the orthogonal projector on V.

We claim that $p(\alpha)\notin<\Phi_I>$ for any $\alpha\in\Phi-\Phi_I$. As $p((w\phi)^i\alpha)=\zeta^ip(\alpha)$ it is enough to assume that α is the first element of a pseudo-orbit; replacing if needed α by $-\alpha$ we may even assume $\alpha\in\Phi^+$. Looking at imaginary parts, we have $\Im(\zeta^i)\ge 0$ for $0\le i<\lfloor d/2\rfloor$, and $\Im(\zeta^i)<0$ for $\lfloor d/2\rfloor\le i<d$. Let λ be a linear form such that λ is 0 on Φ_I and is real strictly positive on $\Phi^+-\Phi_I$; we have $\lambda({}^{(w\phi)^i}\alpha)>0$ for $0\le i<\lfloor d/2\rfloor$, and $\lambda({}^{(w\phi)^i}\alpha)<0$ for $\lfloor d/2\rfloor\le i<d$; it follows that $\Im(\lambda(\zeta^i{}^{(w\phi)^i}\alpha))=\Im(\zeta^i\lambda({}^{(w\phi)^i}\alpha))>0$ for all elements of the pseudo-orbit. If $d'=d$ we have thus $\Im(\lambda(p(\alpha)))>0$, in particular $p(\alpha)\notin<\Phi_I>$. If $d'>d$, since $\phi^d\alpha$ is also a positive root and the first term of the next pseudo-orbit the same computation applies to the other pseudo-orbits and we conclude the same way.

Now $C_W(X^{W_I}\cap V)$ is generated by the reflections whose root is orthogonal to $X^{W_I}\cap V$, that is whose root is in $<\Phi_I>+V^\perp$. If α is such a root we have $p(\alpha)\in<\Phi_I>$, whence $\alpha\in\Phi_I$ by the above claim. This proves that $C_W(X^{W_I}\cap V)\subset W_I$. Since the reverse inclusion is true, we get (iii). □

We return to the proof of the case $k=1$ of Theorem 8.3. Assertion (iii) of Lemma 8.4 gives the second assertion of the theorem. We now show $\neg$(iv)$\Rightarrow\neg$(iii). If $\mathbf{w}\phi$ is extendable, there exists a ϕ^d-stable $\mathbf{J}\subsetneq\mathbf{I}$ and $\mathbf{v}\in B_{\mathbf{I}}^+$ such that $(\mathbf{vw}\phi)^d=\phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}$, which implies ${}^{\mathbf{vw}\phi}\mathbf{J}=\mathbf{J}$. If we denote by ψ the automorphism of $B_{\mathbf{I}}$ induced by the automorphism $\mathbf{w}\phi$ of $\mathbf{I}$, we have ${}^{\mathbf{v}\psi}\mathbf{J}=\mathbf{J}$ and $(\mathbf{v}\psi)^d=\psi^d\boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}}$. Let X_I be the subspace of X spanned by Φ_I. It follows from the first part of the theorem applied with X, ϕ, $\mathbf{w}$, w respectively replaced with X_I, ψ, $\mathbf{v}$, v that $v\psi = vw\phi$ has a non-zero ζ-eigenspace in X_I, since if V' is the ζ-eigenspace of $vw\phi$ we get $C_{W_I}(V') \subset W_J \subsetneq W_I$; this contradicts (iv).

We show finally that $\neg$(iii)$\Rightarrow$ $\neg$(iv). If some element of $W_I\psi$ has a non-zero ζ-eigenvector on X_I, by Theorem 8.1 applied to $W_I\psi$ acting on X_I we get the existence of $\mathbf{J} \subsetneq \mathbf{I}$ and $\mathbf{v} \in B^+_{\mathbf{I}}$ satisfying ${}^{\mathbf{v}\psi}\mathbf{J} = \mathbf{J}$ and $(\mathbf{v}\psi)^d = \psi^d \boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}}$. Using that $(\mathbf{w}\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$, it follows that $(\mathbf{vw}\phi)^d = (\mathbf{w}\phi)^d\boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}} = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}\cdot\boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}} = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}$ so $\mathbf{w}\phi$ is extendable.

We now deal with the general case $k \neq 1$. This time we use 5.5, which gives immediately (i). Let us first consider the case when k is prime to d. Then, by 5.5, up to conjugacy in $B^+(\mathcal{I})$, which we may as well do as observed at the beginning of the proof, we get that with d' and k' as in 5.5 we have $(\mathbf{w}\phi)^{d'} \preccurlyeq (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^{k'}$ and the element $\mathbf{w}_1$ defined by $(\mathbf{w}\phi)^{d'}\mathbf{w}_1\phi^{-d'} = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^{k'}$ satisfies $(\mathbf{w}_1\phi^{-d'})^k = (\mathbf{w}\phi)\phi^{-k'd}$ and $(\mathbf{w}_1\phi^{-d'})^d = \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}\phi^{-dd'}$. Since $\mathbf{I}$ is $\phi^{-dd'}$-stable the last equality shows that we may apply the case $k=1$ to $\mathbf{w}_1\phi^{-d'}$. Since k is prime to d the defining relation for $\mathbf{w}_1$ gives in W that $(w\phi)^{-d'} = w_1\phi^{-d'}$, where w_1 is the image of $\mathbf{w}_1$ in W, which (since d' is prime to d) shows that that the ζ-eigenspace of $w\phi$ is the $e^{2i\pi/d}$-eigenspace of $w_1\phi^{-d'}$. This gives (ii).

Similarly the coset $W_Iw_1\phi^{-d'}$ is the $-d'$-th power of the coset $W_Iw\phi$, so condition (iii) for $w\phi$ and ζ is equivalent to (iii) for $w_1\phi^{-d'}$ and $e^{2i\pi/d}$.

Item (ii) of the following lemma completes the proof of the case $\gcd(d,k)=1$ since by Lemma 3.16 we may choose d' prime to δ;

Lemma 8.5. *Let k, d, k', d' be positive integers satisfying $dk' = 1 + kd'$ with d' prime to the order of ϕ. Let $\mathbf{w}_1\phi^{-d'}$ be $(d,2)$-periodic element. Define $\mathbf{w}\phi$ by $\mathbf{w}\phi = (\mathbf{w}_1\phi^{-d'})^k\phi^{k'd}$. Then*

(i) *$\mathbf{w}\phi$ is $(d,2k)$-periodic.*
(ii) *$\mathbf{w}\phi$ is non-extendable if and only if $\mathbf{w}_1\phi^{-d'}$ is non-extendable.*

Proof. Assertion (i) is an immediate translation of 3.15. Assume $\mathbf{w}_1\phi^{-d'}$ extendable, that is there exists $\mathbf{v}_1$ such that $(\mathbf{v}_1\mathbf{w}_1\phi^{-d'})^d = \phi^{-dd'}\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}$ for some $\mathbf{J} \subsetneq \mathbf{I}$. The k-th power of this equality gives $(\mathbf{v}(\mathbf{w}_1\phi^{-d'})^k)^d = (\mathbf{vw}\phi\cdot\phi^{-k'd})^d = \phi^{-kdd'}(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}})^k$, where $\mathbf{v}$ is defined by $(\mathbf{v}_1\mathbf{w}_1\phi^{-d'})^k = \mathbf{v}(\mathbf{w}_1\phi^{-d'})^k$. Since $\mathbf{w}_1\phi^{-d'}$ is $(d,2)$-periodic, it is $\phi^{dd'}$-stable, and the defining equality for $\mathbf{v}_1$ shows that $\mathbf{v}_1$ also is $\phi^{dd'}$-stable. It follows that $\mathbf{vw}\phi$ is also $\phi^{dd'}$-stable. Since d' is prime to δ any element commuting to $\phi^{dd'}$ commutes to ϕ^d, in particular $(\mathbf{vw}\phi\cdot\phi^{-k'd})^d\phi^{kdd'} = (\mathbf{vw}\phi)^d\phi^{-k'd^2+kdd'} = (\mathbf{vw}\phi)^d\phi^{-d}$, whence the result.

For the converse, if $w\phi$ denotes the automorphism of $B^+_{\mathbf{I}}$ induced by $\mathbf{w}\phi$, using that $(\mathbf{w}\phi)^d = \phi^d(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^k$ and that $\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}} = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})(\boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}})$ we may write the equation $(\mathbf{vw}\phi)^d = \phi^d(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}})^k$ as $(\mathbf{v}w\phi)^d = \phi^d(\boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}})^k$. We now use a relative version of 5.5, where we replace $B^+(\mathcal{I})$ by $B^+_{\mathbf{I}}(\mathcal{J})$ where $\mathcal{J}$ is the set of $B^+_{\mathbf{I}}$-conjugates of $\mathbf{J}$, replace ϕ by $w\phi$ and replace $\mathbf{b}$ by $\mathbf{v}$; we get the existence of $\mathbf{v}_1$ such that $(\mathbf{v}_1(w\phi)^{-d'})^d = \boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}}(w\phi)^{-dd'}$, which can be written $(\mathbf{v}_1(\mathbf{w}\phi)^{-d'})^d(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^{kd'} = \boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}}\phi^{-dd'}$ or $(\mathbf{v}_1(\mathbf{w}\phi)^{-d'}(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^{k'})^d = \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}\phi^{-dd'}$ which using that $(\mathbf{w}\phi)^{d'}\mathbf{w}_1\phi^{-d'} = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^{k'}$ transforms into the equality we seek $(\mathbf{v}_1\mathbf{w}_1\phi^{-d'})^d = \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}\phi^{-dd'}$. ∎

We now consider the case when $\lambda = \gcd(d,k) \neq 1$. We set $d_1 = d/\lambda$ and $k_1 = k/\lambda$. Up to cyclic conjugacy, which we may as well do, we may assume by 5.5 that $(\mathbf{w}\phi)^{d_1} = (\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I})^{k_1}\phi^{d_1}$. Since $e^{2i\pi k_1/d_1} = e^{2i\pi k/d}$ we have (i), (ii) of the theorem as well as the equivalence of (iii) with the "d_1-extendability" of $\mathbf{w}$, that is the existence of $\mathbf{v} \in B^+_\mathbf{I}$ such that $(\mathbf{vw}\phi)^{d_1} = \phi^{d_1}(\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{J})^{k_1}$. The d_1-extendability implies trivially the d-extendability by raising the equation to the λ-th power. Conversely, using as above that the equation $(\mathbf{vw}\phi)^d = \phi^d(\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{J})^k$ is equivalent to $(\mathbf{v}w\phi)^d = \phi^d(\boldsymbol{\pi}_\mathbf{I}/\boldsymbol{\pi}_\mathbf{J})^k$ the relative version of 5.5 as used above shows that up to cyclic conjugacy we have $(\mathbf{v}w\phi)^{d_1} = \phi^{d_1}(\boldsymbol{\pi}_\mathbf{I}/\boldsymbol{\pi}_\mathbf{J})^{k_1}$ which in turn is equivalent to $(\mathbf{vw}\phi)^{d_1} = \phi^{d_1}(\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{J})^{k_1}$. □

The non-extendability condition (iii) or (iv) of Theorem 8.3 is equivalent to the conjunction of two others, thanks to the following lemma which holds for any complex reflection coset and any ζ. For definitions and basic results on complex reflection groups we refer to [Br2]. Recall that a complex reflection group is a finite group generated by pseudo-reflections acting on a finite dimensional complex vector space and that the fixator of a subspace is called a parabolic subgroup. It is still a complex reflection group.

Lemma 8.6. *Let W be finite a reflection group on the complex vector space X and let ϕ be an automorphism of X of finite order which normalizes W. Let V be the ζ-eigenspace of an element $w\phi \in W\phi$. Assume that W' is a parabolic subgroup of W which is $w\phi$-stable and such that $C_W(V) \subset W'$, and let X' denote the subspace of X spanned by the root lines of W'. Then the condition*

(i) $V \cap X' = 0$.

is equivalent to

(ii) $C_W(V) = W'$.

While the stronger condition

(iii) *No element of the coset $W'w\phi$ has a non-zero ζ-eigenvector on X'.*

is equivalent to the conjunction of (ii) and

(iv) *The space V is maximal among the ζ-eigenspaces of elements of $W\phi$.*

Proof. Since $W\langle\phi\rangle$ is finite we may endow X with a $W\langle\phi\rangle$-invariant Hermitian scalar product, which we shall do.

We show (i) ⇔ (ii). Assume (i); since $w\phi$ has no non-zero ζ-eigenvector in X' and X' is $w\phi$-stable, we have $V \perp X'$, so that $W' \subset C_W(V)$, whence (ii) since the reverse inclusion is true by assumption. Conversely, (ii) implies that $V \subset X'^\perp$ thus $V \cap X' = 0$.

We show (iii) ⇒ (iv). There exists an element of $W\phi$ whose ζ-eigenspace V_1 is maximal with $V \subset V_1$. Then $C_W(V_1) \subset C_W(V) \subset W'$ and the $C_W(V_1)$-coset of elements of $W\phi$ which act by ζ on V_1 is a subset of the coset $C_W(V)w\phi$ of elements which act by ζ on V. Thus this coset is of the form $C_W(V_1)vw\phi$ for some $v \in W'$. By (i) ⇒ (ii) applied with $w\phi$ replaced by $vw\phi$ we get $C_W(V_1) = W'$. Since $v \in W'$ this implies that $vw\phi$ and $w\phi$ have same action on V_1 so that $w\phi$ acts by ζ on V_1, thus $V_1 \subset V$.

Conversely, assume that (ii) and (iv) are true. If there exists $v \in W'$ such that $vw\phi$ has a non-zero ζ-eigenvector in X', then since v acts trivially on V by (ii), the element $vw\phi$ acts by ζ on V and on a non-zero vector of X' so has a ζ-eigenspace strictly larger that V, contradicting (iv). □

Let us give now examples which illustrate the need for the conditions in Theorem 8.3 and Lemma 8.6.

We first give an example where $\mathbf{w}\phi$ is a root of $\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$ which is extendable in the sense of Theorem 8.3(iv) and $\ker(w\phi-\zeta)$ is not maximal: let us take $W=W(A_3)$, $\phi=1$, $d=2$, $\zeta=-1$, $\mathbf{I}=\{\mathbf{s}_2\}$ (where the conventions for the generators of W are as in the appendix, see Subsection 10.2), $\mathbf{w}=\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0$. We have $\mathbf{w}^2=\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$ but $\ker(w+1)$ is not maximal: its dimension is 1 and a 2-dimensional -1-eigenspace is obtained for $\mathbf{w}=\mathbf{w}_0$.

In the above example we still have $C_W(V)=W_I$ but even this need not happen; at the same time we illustrate that the maximality of $V=\ker(w\phi-\zeta)$ does not imply the non-extendability of $\mathbf{w}$ if $C_W(V)\subsetneq W_I$; we take $W=W(A_3)$, $\phi=1$, $d=2$, $\zeta=-1$, but this time $I=\{\mathbf{s}_1,\mathbf{s}_3\}$, $\mathbf{w}=\mathbf{w}_I^{-1}\mathbf{w}_0$. We have $\mathbf{w}^2=\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$ and $\ker(w+1)$ is maximal (w is conjugate to w_0, thus -1-regular) but $\mathbf{w}$ is extendable. In this case $C_W(V)=\{1\}$.

The smallest example with a non-extendable $\mathbf{w}$ and non-trivial $\mathbf{I}$ is for $W=W(A_4)$, $\phi=1$, $d=3$, $\mathbf{w}=\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2$ and $\mathbf{I}=\{\mathbf{s}_3\}$. Then $\mathbf{w}^3=\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$; this corresponds to the smallest example with a non-regular eigenvalue (we call regular an eigenvalue of a regular element for which the eigenspace has trivial centralizer): ζ_3 is not regular in A_4.

Finally we give an example which illustrates the necessity of the condition $\phi^d(\mathbf{I})=\mathbf{I}$ in Theorem 8.3. We take $W\phi$ of type ${}^3D_4$, thus ϕ is the triality automorphism $\mathbf{s}_1\mapsto\mathbf{s}_4\mapsto\mathbf{s}_2$. Let $\mathbf{w}=\mathbf{w}_0\mathbf{s}_1^{-1}\mathbf{s}_2^{-1}\mathbf{s}_4$. Then, for $\mathbf{I}=\{\mathbf{s}_1\}$ we have $(\mathbf{w}\phi)^2=\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}\phi^2$, but $\mathbf{I}^{\mathbf{w}\phi}=\{\mathbf{s}_4\}$. The other statements of Theorem 8.3 also fail: if V is the -1-eigenspace of $w\phi$ the group $C_W(V)$ is the parabolic subgroup generated by s_1,s_2 and s_4.

Lemma 8.7. *Let $W\phi$ be a complex reflection coset and let V be the ζ-eigenspace of $w\phi\in W\phi$; then*

(i) $N_W(V)=N_W(C_W(V)w\phi)$.
(ii) *If $W\phi$ is real, and $C_W(V)=W_I$ where (W,S) is a Coxeter system and $I\subset S$, and w is I-reduced, then the subgroup $\{v\in C_W(w\phi)\cap N_W(W_I)\mid v$ is I-reduced$\}$ is a section of $N_W(V)/C_W(V)$ in W.*

Proof. Let W_1 denote the parabolic subgroup $C_W(V)$. All elements of $W_1w\phi$ have the same ζ-eigenspace V, so $N_W(W_1w\phi)$ normalizes V; conversely, an element of $N_W(V)$ normalizes W_1 and conjugates $w\phi$ to an element $w'\phi$ with same ζ-eigenspace, thus w and w' differ by an element of W_1, whence (i).

For the second item, $N_W(W_Iw\phi)/W_I$ admits as a section the set of I-reduced elements, and such an element will conjugate $w\phi$ to the element of the coset $W_Iw\phi$ which is I-reduced, so will centralize $w\phi$. □

We call *essential rank* of a (complex) reflection coset $W\phi\subset\mathrm{GL}(X)$ the dimension of the space generated by its root lines (the dimension of X minus the dimension of the intersection of the reflection hyperplanes of W).

We call *ζ-rank* of an element of $W\phi$ the dimension of its ζ-eigenspace, and ζ-rank of $W\phi$ the maximal ζ-rank of its elements.

Let us say that a $(d,2k)$-periodic element of $B^+\phi(\mathcal{I})$ is non-extendable if it is non-extendable in the sense of Theorem 8.3(iv). Another way to state the non-extendability of a periodic element $\mathbf{I}\xrightarrow{\mathbf{w}\phi}\mathbf{I}\in B^+\phi(\mathcal{I})$ is to require that $|\mathbf{I}|$ be no more than the essential rank of the centralizer of a maximal ζ-eigenspace of $W\phi$, where $\zeta = e^{2ik\pi/d}$: indeed if $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}$ is extendable there exists $\mathbf{J}$ and $\mathbf{v}$ as in Theorem 8.3(iv) and, since condition 8.3(iv) implies Lemma 8.6(iii), the element $vw\phi$ has maximal ζ-rank, and the centralizer of its ζ-eigenspace has essential rank $|\mathbf{J}| < |\mathbf{I}|$. Note that the notion of non-extendable $(d, 2k)$-periodic element makes sense without specifying $\mathcal{I}$, as $\zeta = e^{2ik\pi/d}$ is determined by k/d, and $\mathcal{I}$ in turn is determined as the class of parabolic subgroups which are centralizers of ζ-eigenspaces of elements of $W\phi$ of maximal ζ-rank.

The correspondence between maximal eigenspaces and non-extendable periodic elements, as described by Theorems 8.1 and 8.3, can be summarized as follows:

Corollary 8.8. *Let V' be the ζ-eigenspace of an element of $W\phi$ of maximal ζ-rank, where $\zeta = e^{2i\pi k/d}$. Then there is a W-conjugate V of V' and $I \subset S$ such that $C_W(V) = W_I$ and the corresponding I-reduced $w\phi$ (see Theorem 8.1(ii)) lifts to a non-extendable $(d, 2k)$-periodic element $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}$. Conversely, for a $(d, 2k)$-periodic non-extendable $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}$ the image $w\phi$ in $W\phi$ has maximal ζ-rank.*

We conjecture that Bessis's theorem [B1, 11.21] extends to

Conjecture 8.9. *Two non-extendable $(d, 2k)$-periodic elements of $B^+\phi(\mathcal{I})$ are cyclically conjugate.*

Note that because of Lemma 8.6 the non-extendability condition is necessary in the above.

By 5.6 a $(d, 2k)$-periodic element is cyclically conjugate to an element which satisfies in addition $(\mathbf{w}\phi)^{\lfloor \frac{d}{2k} \rfloor} \in \mathbf{W}\phi^{\lfloor \frac{d}{2k} \rfloor}$. We will call *good* a $(d, 2k)$-periodic element which satisfies this additional condition.

When $k = 1$ we can give conditions purely in terms of W for an element to lift to a good $(d, 2)$-periodic (resp. non-extendable good $(d, 2)$-periodic) element.

Lemma 8.10. *Let $W\phi \subset \mathrm{GL}(X_\mathbb{R})$ be a finite order real reflection coset such that ϕ preserves the chamber of the corresponding hyperplane arrangement determining the Coxeter system (W, S).*

Let $w \in W$ and $I \subset S$ and let $\mathbf{w} \in \mathbf{W}$ and $\mathbf{I} \subset \mathbf{S}$ be their lifts; let $\mathcal{I}$ be the conjugacy orbit of $\mathbf{I}$, then $\mathbf{w}$ induces a morphism $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I} \in B^+(\mathcal{I})$ if and only if:

(i) *${}^{w\phi}I = I$ and w is I-reduced.*

If $\mathbf{w}$ satisfies (i), for $d > 1$ the element $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}$ is good $(d, 2)$-periodic if and only if the following two additional conditions are satisfied.

(ii) *$l((w\phi)^i) = \frac{2i}{d} l(w_I^{-1} w_0)$ for $0 \leq 2i \leq d$.*

(iii) *$(w\phi)^d = \phi^d$.*

If, moreover,

(iv) *$W_I w\phi$ has ζ-rank 0 on the subspace spanned by the root lines of W_I where $\zeta = e^{2i\pi/d}$,*

then $\mathbf{w}$ is non-extendable in the sense of Theorem 8.3(iv).

Proof. By definition w induces a morphism $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}$ if and only if it satisfies (i). By definition again if $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}$ is good $(d, 2)$-periodic then (ii) and (iii) are satisfied. Conversely, Lemma 8.2 shows that the morphism induced by the lift of a w satisfying (i), (ii), (iii) is good $(d, 2)$-periodic.

Property (iv) means that no element $vw\phi$ with $v \in W_I$ has an eigenvalue ζ on the subspace spanned by the root lines of W_I which is exactly the characterization of Theorem 8.3(iv) of a non-extendable element. □

Note that d and I are uniquely determined by $w\phi$ satisfying (i), (ii), (iii) above since d is the smallest power of $w\phi$ which is a power of ϕ and I is determined by $(\mathbf{w}\phi)^d = \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}\phi^d$.

Definition 8.11. *We say that $w\phi \in W\phi$ is d-*good *if it satisfies (i), (ii), (iii) in Lemma 8.10.*

We say $w\phi$ is d-good maximal *if it satisfies in addition (iv) in Lemma 8.10.*

In particular, d-good elements lift to good $(d, 2)$-periodic elements, and d-good maximal elements lift to good non-extendable $(d, 2)$-periodic elements. In the appendix, we will construct a non-extendable $(d, 2k)$-periodic element for each $W\phi$, each d and each k. Actually, we will do this only for $k = 1$ (by constructing d-good maximal elements of $W\phi$), which is sufficient by

Lemma 8.12. (i) *If $\lambda = \gcd(d, k)$ and we set $d_1 = d/\lambda$ and $k_1 = k/\lambda$ and $\mathbf{w}\phi$ is $(d_1, 2k_1)$-periodic (resp. non-extendable $(d_1, 2k_1)$-periodic) then $\mathbf{w}\phi$ is $(d, 2k)$-periodic (resp. non-extendable $(d, 2k)$-periodic).*

(ii) *If k is prime to d there exists integers k' and d' such that $dk' = 1 + kd'$ such that if $\mathbf{w}_1\phi^{-d'}$ is $(d, 2)$-periodic (resp. non-extendable $(d, 2)$-periodic) then the element $\mathbf{w}\phi$ defined by $(\mathbf{w}_1\phi^{-d'})^k = (\mathbf{w}\phi)\phi^{-k'd}$ is $(d, 2k)$-periodic (resp. non-extendable $(d, 2k)$-periodic).*

Proof. (i) is part of what is proved in the last paragraph of the proof of 8.3 and (ii) is Lemma 8.5. □

Any element of $W\phi$ with a maximal ζ-eigenspace is conjugate to an element of $C_W(V)w\phi$ since the maximal eigenspaces are conjugate, see [S, Theorem 3.4(iii) and Theorem 6.2(iii)]. If $w\phi$ is the image of a non-extendable $(d, 2k)$-periodic element, where $\zeta = e^{2ik\pi/d}$, it is 1-regular in this coset by Theorem 8.3 (ii) which implies that it preserves a chamber of the corresponding real arrangement (see remarks above Theorem 8.1). The following lemma shows that the images in $W\phi$ of non-extendable $(d, 2k)$-periodic elements (thus in particular d-good maximal elements) belong to a single conjugacy class under W, characterized by the above property.

Lemma 8.13. *Let $W\phi$ be a finite order real reflection coset. The elements of $W\phi$ which have a ζ-eigenspace V of maximal dimension and among those, have the largest dimension of fixed points, are conjugate.*

Proof. As remarked above, up to W-conjugacy we may fix a ζ-eigenspace V and consider only elements of the coset $C_W(V)w\phi$ where $w\phi$ is some element with ζ-eigenspace V; then W-conjugacy is reduced to $C_W(V)$-conjugacy. Since $C_W(V)$ is a parabolic subgroup of the Coxeter group W and is normalized by $w\phi$, the coset $C_W(V)w\phi$ is a real reflection coset; in this coset there are 1-regular elements, which are those which preserve a chamber of the corresponding real hyperplane arrangement; the 1-regular elements have maximal 1-rank, that is have the largest dimension of fixed points, and they form a single $C_W(V)$-orbit under conjugacy, whence the lemma. □

Lemma 8.14. *Let $w\phi$ be the image in $W\phi$ of a non-extendable $(d, 2k)$-periodic element $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}$, let I be the image of $\mathbf{I}$ and let V_1 be the fixed point subspace of $w\phi$ in the space spanned by the root lines of W_I; then $w\phi$ is regular in the coset $C_W(V_1)w\phi$.*

Proof. Let $W' = C_W(V_1)$; we first note that since $w\phi$ normalizes V_1 it normalizes also W', so $W'w\phi$ is indeed a reflection coset. We have thus only to prove that $C_{W'}(V)$ is trivial, where V is the $e^{2ik\pi/d}$-eigenspace of $w\phi$. This last group is generated by the reflections with respect to roots both orthogonal to V and to V_1, which are the roots of $W_I = C_W(V)$ orthogonal to V_1. Since $w\phi$ preserves a chamber of W_I, the sum v of the positive roots of W_I with respect to the order defined by this chamber is in V_1 and is in the chamber: this is well known for a true root system; here we have taken all the roots to be of length 1 but the usual proof (see [Bou, Chapitre VI §1, Proposition 29]) is still valid. Since no root is orthogonal to a vector v inside a chamber, W_I has no root orthogonal to V_1, hence $C_{W'}(V) = \{1\}$. □

One could hope that the above lemma reduces the classification of d-good maximal elements to that of regular elements; however the map $C_{W'}(w\phi) = N_{W'}(V) \to N_W(V)/C_W(V)$ with the notations of the above proof is injective, but not always surjective: for example, if W of type E_7, and $\phi = \mathrm{Id}$ and $d = 4$, then $N_W(V)/C_W(V)$ is the complex reflection group G_8, while W' is of type D_4 and $N_{W'}(V)/C_{W'}(V)$ is the complex reflection group $G(4, 2, 2)$. However, there are only 3 such cases for irreducible groups W; the group $N_W(V)/C_W(V)$ was determined in appendix 1 in all other cases by the equality $C_{W'}(w\phi) \simeq N_W(V)/C_W(V)$, which is proved by checking that $C_{W'}(w\phi)$ and $N_W(V)/C_W(V)$ have the same reflection degrees, a simple arithmetic check on the reflection degrees of W and W'; indeed, recall that when V is a maximal ζ-eigenspace, the group $N_W(V)/C_W(V)$ is a complex reflection group acting on V, with reflection degrees the reflection degrees of W satisfying the arithmetic condition given for instance in [Br2, 5.6] (when $\phi = \mathrm{Id}$, the reflection degrees divisible by d).

9. Conjectures

The following conjectures extend those of [DM2, §2]. They are a geometric form of Broué conjectures.

Conjectures 9.1. *Let $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I} \in B^+(\mathcal{I})\phi$ be non-extendable $(d, 2k)$-periodic. Then*

(i) *The group $B_{\mathbf{w}}$ generated by the monoid $B^+_{\mathbf{w}}$ of Theorem 5.9 is isomorphic to the braid group of the complex reflection group $W(w\phi) := N_W(W_I w\phi)/W_I$.*

(ii) *The natural morphism $\mathcal{D}^{\mathcal{I}}(\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I}) \to \mathrm{End}_{\mathbf{G}^F}(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$ (see below Lemma 7.22) gives rise to a morphism $B_{\mathbf{w}} \to \mathrm{End}_{\mathbf{G}^F} H^*_c(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$ which factors through a special representation of a ζ-cyclotomic Hecke algebra $\mathcal{H}_{\mathbf{w}}$ for $W(w\phi)$, where $\zeta = e^{2ik\pi/d}$.*

(iii) *The odd and even $H^i_c(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$ are disjoint $\mathbf{G}^F$-modules, and the above morphism extends to a surjective morphism $\overline{\mathbb{Q}}_\ell[B_{\mathbf{w}}] \to \mathrm{End}_{\mathbf{G}^F}(H^*_c(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)))$.*

The group $W(w\phi)$ above is a complex reflection group by the remarks at the end of last section and Lemma 8.7 (i).

The condition that the periodic elements we consider are non-extendable is necessary for assertion (ii) above to hold; in the case of extendable periodic elements the endomorphism algebra should, instead of being a deformation of the group algebra of $W(w\phi)$, be a deformation of an endomorphism algebra of an induced representation from a complex reflection group to another. Whenever a periodic element is extendable, a decomposition as in Theorem 7.19 can be applied. See [Du, 1.3] for such computations.

David Craven has made (iii) above more specific by giving a conjectural formula computing the cohomology degree in which a given unipotent character should occur (see [C]); Craven's formula should be valid for any $(d, 2k)$-periodic element, not only the non-extendable ones. In the current paper we focus on the study of non-extendable periodic elements; this should be a start for the general study of all periodic elements.

Lemma 9.2. *Let* $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I} \in B^+\phi(\mathcal{I})$ *be non-extendable* $(d, 2k)$*-periodic and assume Conjectures 9.1; then for any* $i \neq j$ *the* $\mathbf{G}^F$*-modules* $H^i_c(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$ *and* $H^j_c(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$ *are disjoint.*

Proof. Since the image of the morphism of Conjecture 9.1(ii) consists of equivalences of étale sites, it follows that the action of $\mathcal{H}_\mathbf{w}$ on $H^*_c(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$ preserves individual cohomology groups. The surjectivity of the morphism of (iii) implies that for $\rho \in \operatorname{Irr}(\mathbf{G}^F)$, the ρ-isotypic part of $H^*_c(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$ affords an irreducible $\mathcal{H}_\mathbf{w}$-module; this would not be possible if this ρ-isotypic part was spread over several distinct cohomology groups. □

We will now explore the information given by the Shintani descent identity on the above conjectures

Lemma 9.3. *Let* $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I} \in B^+\phi(\mathcal{I})$ *be* $(d, 2k)$*-periodic With the notations of Proposition 7.30, we have* $\tilde{\chi}_{q^m}(X_1 T_\mathbf{w}\phi) = q^{m\frac{k}{d}(l(\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I}) - a_\chi - A_\chi)}\tilde{\chi}(e_I wF)$ *for* $\chi \in \operatorname{Irr}(W)^\phi$*, where* a_χ *(resp.* A_χ*) is the valuation (resp. the degree) of the generic degree of* χ *and* $e_I = |W_I|^{-1}\sum_{v\in W_I} v$*.*

Proof. We have $(X_1 T_\mathbf{w}\phi)^d = X_1(T_{\boldsymbol{\pi}}/T_{\boldsymbol{\pi}_\mathbf{I}})^k\phi^d = q^{-kl(\boldsymbol{\pi}_\mathbf{I})}X_1 T^k_{\boldsymbol{\pi}}\phi^d$ since X_1 commutes with $T_\mathbf{w}\phi$ and since for any $v \in W_I$ we have $X_1 T_v = q^{l(v)}X_1$. Since $T_{\boldsymbol{\pi}}$ acts on the representation of character χ_{q^m} as the scalar $q^{m(l(\boldsymbol{\pi}) - a_\chi - A_\chi)}$ (see [BM, Corollary 4.20]), it follows that all the non-zero eigenvalues of $X_1 T_\mathbf{w}\phi$ on this representation are equal to $q^{m\frac{k}{d}(l(\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I}) - a_\chi - A_\chi)}$ times a root of unity. To compute the sum of these roots of unity, we may use the specialization $q^{m/2} \mapsto 1$, through which $\tilde{\chi}_{q^m}(X_1 T_\mathbf{w}\phi)$ specializes to $\tilde{\chi}(e_I w\phi)$. □

Proposition 9.4. *Let* $\mathbf{I} \xrightarrow{\mathbf{w}\phi} \mathbf{I} \in B^+\phi(\mathcal{I})$ *be* $(d, 2k)$*-periodic. For any* m *multiple of* δ*, we have*

$$|\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m}| = \sum_{\rho\in\mathcal{E}(\mathbf{G}^F,1)} \lambda_\rho^{m/\delta} q^{m\frac{k}{d}(l(\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I}) - a_\rho - A_\rho)} \langle \rho, R^{\mathbf{G},F}_{\mathbf{L}_I, t(\mathbf{w}\phi)} \operatorname{Id}\rangle_{\mathbf{G}^F}\rho(g),$$

where a_ρ *and* A_ρ *are respectively the valuation and the degree of the generic degree of* ρ*.*

Proof. We start with Corollary 7.31, whose statement reads, using the value of $\tilde\chi_{q^m}(X_1 T_{\mathbf{w}}\phi)$ given by Lemma 9.3:

$$|\mathbf{X}(\mathbf{I},\mathbf{w}\phi)^{gF^m}| = \sum_{\rho\in\mathcal{E}(\mathbf{G}^F,1)} \lambda_\rho^{m/\delta}\rho(g) \sum_{\chi\in\operatorname{Irr}(W)^\phi} q^{m\frac{k}{d}(l(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})-a_\chi-A_\chi)}\tilde\chi(e_I w\phi)\langle\rho, R_{\tilde\chi}\rangle_{\mathbf{G}^F}.$$

Using that for any ρ such that $\langle\rho, R_{\tilde\chi}\rangle_{\mathbf{G}^F}\neq 0$ we have $a_\rho = a_\chi$ and $A_\rho = A_\chi$ (see [BM] around (2.4)) the right-hand side can be rewritten

$$\sum_{\rho\in\mathcal{E}(\mathbf{G}^F,1)} \lambda_\rho^{m/\delta} q^{m\frac{k}{d}(l(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})-a_\rho-A_\rho)}\rho(g)\langle\rho, \sum_{\chi\in\operatorname{Irr}(W)^\phi}\tilde\chi(e_I w\phi)R_{\tilde\chi}\rangle_{\mathbf{G}^F}.$$

The proposition is now just a matter of observing that

$$\sum_{\chi\in\operatorname{Irr}(W)^\phi}\tilde\chi(e_I w\phi)R_{\tilde\chi} = |W_I|^{-1}\sum_{v\in W_I}\sum_{\chi\in\operatorname{Irr}(W)^\phi}\tilde\chi(vw\phi)R_{\tilde\chi} = |W_I|^{-1}\sum_{v\in W_I} R^{\mathbf{G}}_{\mathbf{T}_{vw}}(\mathrm{Id}) = R^{\mathbf{G},F}_{\mathbf{L}_I,t(\mathbf{w}\phi)}(\mathrm{Id}).$$

Where the last equality is obtained by transitivity of $R^{\mathbf{G}}_{\mathbf{L}}$ and the equality $\mathrm{Id}_{\mathbf{L}_I^{t(\mathbf{w}\phi)}} = |W_I|^{-1}\sum_{v\in W_I} R^{\mathbf{L}_I,t(\mathbf{w}\phi)}_{\mathbf{T}_{vw}}(\mathrm{Id})$, a torus $\mathbf{T}$ of $\mathbf{L}_I$ of type v for the isogeny $t(\mathbf{w}\phi)$ being conjugate to $\mathbf{T}_{vw}$ in $\mathbf{G}$. □

Corollary 9.5. *Let $\mathbf{I}\xrightarrow{\mathbf{w}\phi}\mathbf{I}\in B^+\phi(\mathcal{I})$ be non-extendable $(d,2k)$-periodic and assume Conjectures 9.1; then for any $\rho\in\operatorname{Irr}(\mathbf{G}^F)$ such that $\langle\rho, R^{\mathbf{G},F}_{\mathbf{L}_I,t(\mathbf{w}\phi)}(\mathrm{Id})\rangle_{\mathbf{G}^F}\neq 0$ the isogeny F^δ has a single eigenvalue on the ρ-isotypic part of $H^*_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi))$, equal to $\lambda_\rho q^{\delta\frac{k}{d}(l(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})-a_\rho-A_\rho)}$.*

Proof. This follows immediately, in view of Lemma 9.2, from the comparison between Proposition 9.4 and the Lefschetz formula:

$$|\mathbf{X}(\mathbf{I},\mathbf{w}\phi)^{gF^m}| = \sum_i(-1)^i\operatorname{Trace}(gF^m\mid H^i_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)).$$

□

In view of Corollary 7.35(i) it follows that if $\langle\rho, R^{\mathbf{G}}_{\mathbf{L}_I}(\mathrm{Id})\rangle_{\mathbf{G}^F}\neq 0$ then if $\omega_\rho = 1$ then $\frac{k}{d}(l(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})-a_\rho-A_\rho)\in\mathbb{N}$, and if $\omega_\rho=\sqrt{q^\delta}$ then $\frac{k}{d}(l(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})-a_\rho-A_\rho)\in\mathbb{N}+1/2$.

Assuming Conjectures 9.1, we choose once and for all a specialization $q^{1/a}\mapsto\zeta^{1/a}$, where $a\in\mathbb{N}$ is large enough such that $\mathcal{H}_{\mathbf{w}}\otimes\overline{\mathbb{Q}}_\ell[q^{1/a}]$ is split. This gives a bijection $\varphi\mapsto\varphi_q:\operatorname{Irr}(W(w\phi))\to\operatorname{Irr}(\mathcal{H}_{\mathbf{w}})$, and the conjectures give a further bijection $\varphi\mapsto\rho_\varphi$ between $\operatorname{Irr}(W(w\phi))$ and the set $\{\rho\in\operatorname{Irr}(\mathbf{G}^F)\mid\langle\rho, R^{\mathbf{G}}_{\mathbf{L}_I}(\mathrm{Id})\rangle_{\mathbf{G}^F}\neq 0\}$, which is such that $\langle\rho_\varphi, R^{\mathbf{G}}_{\mathbf{L}_I}(\mathrm{Id})\rangle_{\mathbf{G}^F}=\varphi(1)$.

Corollary 9.6. *Under the assumptions of Corollary 9.5, if ω_φ is the central character of φ, then*

$$\lambda_{\rho_\varphi} = \omega_\varphi((w\phi)^\delta)\zeta^{-\delta\frac{k}{d}(l(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})-a_{\rho_\varphi}-A_{\rho_\varphi})}.$$

Proof. We first note that it makes sense to apply ω_φ to $(w\phi)^\delta$, since $(w\phi)^\delta$ is a central element of $W(w\phi)$. Actually $(\mathbf{w}\phi)^\delta$ is a central element of $B_{\mathbf{w}}$ and maps by the morphism of Conjecture 9.1(iii) to F^δ, thus the eigenvalue of F^δ on the ρ_φ-isotypic part of $H^*_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi))$ is equal to $\omega_{\varphi_q}((\mathbf{w}\phi)^\delta)$; thus $\omega_{\varphi_q}((\mathbf{w}\phi)^\delta) = \lambda_{\rho_\varphi} q^{\delta\frac{k}{d}(l(\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I})-a_{\rho_\varphi}-A_{\rho_\varphi})}$. The statement follows by applying the specialization $q^{1/a}\mapsto\zeta^{1/a}$ to this equality. □

10. Appendix: d-good maximal elements in finite Coxeter cosets.

We will describe, in a finite Coxeter coset, for each d, a d-good maximal element.

As explained the introduction of Section 8, when the Coxeter coset is attached to a reductive group $\mathbf{G}$, such an element defines a parabolic Deligne-Lusztig variety whose cohomology should be a tilting complex for the Broué conjectures for an ℓ dividing $\Phi_d(q)$. The properties of this variety do not depend on the isogeny type, thus it is sufficient to study the case when $\mathbf{G}$ is semi-simple and simply connected. Now, a semi-simple and simply connected group is a direct product of restrictions of scalars of simply connected quasi-simple groups. A restriction of scalars is a group of the form $\mathbf{G}^n$, with an isogeny F_1 such that $F_1(x_0,\ldots,x_{n-1}) = (x_1,\ldots,x_{n-1},F(x_0))$. Then $(\mathbf{G}^n)^{F_1}\simeq\mathbf{G}^F$. If F induces ϕ on the Weyl group W of G then $(\mathbf{G}^n,F_1)$ corresponds to the reflection coset $W^n\cdot\sigma$, where $\sigma(x_0,\ldots,x_{n-1}) = (x_1,\ldots,x_{n-1},\phi(x_0))$.

10.1. **Restrictions of scalars.** Restrictions of scalars as above appear in the classification of arbitrary complex reflection cosets. Arbitrary cosets $W\phi$ are direct products of cosets where ϕ is transitive on the irreducible components of W; we call *restriction of scalars* a complex reflection coset with this last property. It is of the form $W^n\cdot\sigma\subset\mathrm{GL}(V^n)$, where V is a complex vector space and $W\phi\subset\mathrm{GL}(V)$ is a complex reflection coset and where $\sigma(x_0,\ldots,x_{n-1}) = (x_1,\ldots,x_{n-1},\phi(x_0))$. We say that $W^n\sigma$ is a *restriction of scalars* of $W\phi$, by analogy with the terminology for reductive groups.

We first look at the invariant theory of a restriction of scalars. Recall (see for example [Br2]) that, if S_W is the coinvariant algebra of W (the quotient of the symmetric algebra of V^* by the ideal generated by the W-invariants of positive degree), for any W-module X the graded vector space $(S_W\otimes X^*)^W$ admits a homogeneous basis formed of eigenvectors of ϕ. The degrees of the elements of this basis are called the X-exponents of W and the corresponding eigenvalues of ϕ the X-factors of $W\phi$. For $X=V$, the V-exponents n_i satisfy $n_i=d_i-1$ where the d_i's are the reflection degrees of W, and the V-factors ε_i are called the *factors* of $W\phi$. For $X=V^*$, the n_i-1 where n_i are the V^*-exponents are called the *codegrees* d_i^* of W and the corresponding V^*-factors ε_i^* are called the *cofactors* of $W\phi$. By Springer [S, 6.4], for a root of unity ζ, the ζ-rank of $W\phi$ is equal to $|\{i\mid\zeta^{d_i}=\varepsilon_i\}|$. By analogy, we define the ζ-corank of $W\phi$ as $|\{i\mid\zeta^{d_i^*}=\varepsilon_i^*\}|$. By for example [Br2, 5.19.2] an eigenvalue is regular if it has same rank and corank.

Proposition 10.1. *Let $W^n\cdot\sigma$ be a restriction of scalars of the complex reflection coset $W\phi$. Then the ζ-rank (resp. corank) of $W^n\cdot\sigma$ is equal to the ζ^n-rank (resp. corank) of $W\phi$.*

In particular, ζ is regular for $W^n\cdot\sigma$ if and only if ζ^n is regular for $W\cdot\phi$.

Proof. It is easy to see from the construction that the pairs of a reflection degree and the corresponding factor of σ for the coset $W^n\cdot\sigma$ are the pairs $(d_i,\eta_{i,j})$, where

$i \in \{1, \ldots, r\}$ and where $\{\eta_{i,j}\}_{j\in\{1\ldots n\}}$ run over the n-th roots of ε_i. Similarly, the pairs of a reflection codegree and the corresponding cofactor are $(d_i^*, \eta_{i,j}^*)$ where $\{\eta_{i,j}^*\}_{j\in\{1\ldots n\}}$ run over the n-th roots of ε_i^*.

In particular the ζ-rank of $W^n \cdot \sigma$ is $|\{(i,j) \mid \zeta^{d_i} = \eta_{i,j}\}|$ and the ζ-corank is $|\{(i,j) \mid \zeta^{d_i^*} = \eta_{i,j}^*\}|$.

Given d, there is at most one j such that $\zeta^d = \eta_{i,j}$, and there is one if and only if $\zeta^{nd} = \varepsilon_i$. Thus $|\{(i,j) \mid \zeta^{d_i} = \eta_{i,j}\}| = |\{i \mid \zeta^{nd_i} = \varepsilon_i\}|$ and similarly for the corank, whence the two assertions of the statement. □

The next lemma can also be used to give a direct proof of the statement on ζ-ranks.

Lemma 10.2. *Let $W^n \cdot \sigma$ be a restriction of scalars of $W\phi$. Then*

(i) *Any element of $W^n\sigma$ is conjugate to an element of the form $(1, \ldots, 1, w)\sigma$.*
(ii) *The vector $(x_0, \ldots, x_{n-1}) \in V^n$ is a ζ-eigenvector of $(1, \ldots, 1, w)\sigma$ if and only if x_0 is a ζ^n-eigenvector of $w\phi$ and $x_i = \zeta^i x_0$ for $i = 1, \ldots, n-1$.*

Proof. The element $(1, w_0, w_0w_1, \ldots, w_0w_1 \ldots w_{n-2})$ conjugates $(w_0, \ldots, w_{n-1})\sigma$ to $(1, \ldots, 1, w_0 \ldots w_{n-1})\sigma$, whence (i). Property (ii) results from an immediate computation. □

In view of Lemma 10.1, the following proposition is enough to determine all possible non-extendable $(d, 2k)$-periodic elements of $W^n\sigma$.

Proposition 10.3. *Let $W^n \cdot \sigma$ be a restriction of scalars of the finite Coxeter coset $W\phi$. Let $(B^+)^n\sigma$ and $B^+\phi$ be the corresponding cosets of braid monoids. Then*

(i) *Any element in $(B^+)^n\sigma$ is conjugate under $(B^+)^n$ to an element of the form $(1, \ldots, 1, \mathbf{w})\sigma$.*
(ii) *The element $(1, \ldots, 1, \mathbf{w})\sigma \in (B^+)^n\sigma$ is $(nd, 2k)$-periodic if and only if $\mathbf{w}\phi$ is $(d, 2k)$ periodic. Moreover the latter is non-extendable if and only if the former is non-extendable.*

Proof. The element $(1, \mathbf{w}_0, \mathbf{w}_0\mathbf{w}_1, \ldots, \mathbf{w}_0\mathbf{w}_1 \ldots \mathbf{w}_{n-2})$ conjugates $(\mathbf{w}_0, \ldots, \mathbf{w}_{n-1})\sigma$ to $(1, \ldots, 1, \mathbf{w}_0 \ldots \mathbf{w}_{n-1})\sigma$, whence (i).

For (ii), we have $((1, \ldots, 1, \mathbf{w})\sigma)^{nd} = ((\mathbf{w}\phi)^d\phi^{-d}, \ldots, (\mathbf{w}\phi)^d\phi^{-d})\sigma^{nd}$, whence the first assertion: $(\mathbf{w}\phi)^d = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^k\phi^d$ is equivalent to $((1, \ldots, 1, \mathbf{w})\sigma)^{nd} = ((\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^k, \ldots, (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^k)\sigma^{nd}$.

If the last equalities hold and $\mathbf{w}\phi$ is extendable, that is there exist $\mathbf{v} \in B_{\mathbf{I}}^+$ and $\mathbf{J} \subsetneq \mathbf{I}$ such that $(\mathbf{v}\mathbf{w})^d = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}})^k\phi^d$, then $((1, \ldots, 1, \mathbf{v})(1, \ldots, 1, \mathbf{w})\sigma)^{nd} = ((\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}})^k, \ldots, (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}})^k)\sigma^{nd}$, so that $(1, \ldots, 1, \mathbf{w})\sigma$ is extendable.

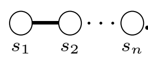
Conversely assume that $(1, \ldots, 1, \mathbf{w})\sigma$ is extendable, that is, there exist $(\mathbf{v}_0 \ldots, \mathbf{v}_{n-1}) \in (B_{\mathbf{I}}^+)^n$ and $\mathbf{J}_0 \times \cdots \times \mathbf{J}_{n-1} \subsetneq \mathbf{I}^n$ such that

$$((\mathbf{v}_0, \ldots, \mathbf{v}_{n-2}, \mathbf{v}_{n-1}\mathbf{w})\sigma)^{nd} = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}_0}, \ldots, \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}_{n-1}})^k\sigma^{nd}.$$

Then since $(\mathbf{v}_0, \ldots, \mathbf{v}_{n-2}, \mathbf{v}_{n-1}\mathbf{w})\sigma$ stabilizes $\mathbf{J}_0 \times \cdots \times \mathbf{J}_{n-1}$, we have $\mathbf{J}_i = {}^{\mathbf{v}_i}\mathbf{J}_{i+1}$ for $i < n-1$ so that $\mathbf{J}_i \subsetneq \mathbf{I}$ for all i. By the same conjugation as in the first line of the proof (by $(1, \mathbf{v}_0, \mathbf{v}_0\mathbf{v}_1, \ldots, \mathbf{v}_0\mathbf{v}_1 \cdots \mathbf{v}_{n-2})$) the above equality conjugates to $((1, \ldots, \mathbf{v}_0 \cdots \mathbf{v}_{n-1}\mathbf{w})\sigma)^{nd} = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}_0}, \ldots, \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}_0})^k\sigma^{nd}$, or equivalently $(\mathbf{v}_0 \cdots \mathbf{v}_{n-1}\mathbf{w}\phi)^d = (\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}_0})^k\phi^d$, thus $\mathbf{w}\phi$ is extendable. □

10.2. **Case of irreducible Coxeter cosets.** We are going to give, for each irreducible finite Coxeter group W, each possible corresponding coset $W\phi$ where ϕ preserves a chamber of the corresponding hyperplane arrangement, and each possible d, a representative $w\phi$ of the d-good maximal elements. Since conjecturally all non-extendable $(d,2)$-periodic elements are conjugate in the ribbon category (see Conjecture 8.9), this should describe also these elements.

We also describe the corresponding ζ_d-eigenspace V where $\zeta_d = e^{2i\pi/d}$, the set I and the relative complex reflection group $W(w\phi) := N_W(V)/C_W(V)$. In the cases where the injection $C_{W'}(w\phi) \to N_W(V)/C_W(V) = W(w\phi)$ of the remark after Lemma 8.14, is surjective, where $W' = C_W(V_1)$ and V_1 is the fixed point subspace of $w\phi$ in the space spanned by the root lines of W_I, we use it to deduce $W(w\phi)$ from $W' = C_W(V_1)$ using the description of centralizers of regular elements in [BM, Annexe 1].

Types A_n **and** ${}^2A_n$ . ${}^2A_n$ is defined by the diagram automorphism ϕ which exchanges s_i and s_{n+1-i}.

For any integer $1 < d \le n+1$, we define

$$v_d = s_1 s_2 \cdots s_{n-\lfloor \frac{d}{2} \rfloor} s_n s_{n-1} \cdots s_{\lfloor \frac{d+1}{2} \rfloor} \text{ and } J_d = \{s_i \mid \lfloor \frac{d+1}{2} \rfloor + 1 \le i \le n - \lfloor \frac{d}{2} \rfloor\}.$$

If d is odd we have $v_d = v'_d {}^\phi v'_d$, where $v'_d = s_1 s_2 \cdots s_{n-\lfloor \frac{d}{2} \rfloor}$.

Now, for $1 < d \le n+1$, let kd be the largest multiple of d less than or equal to $n+1$, so that $\frac{n+1}{2} < kd \le n+1$ and $k = \lfloor \frac{n+1}{d} \rfloor$. We then define $w_d = v_{kd}^k$, $I_d = J_{kd}$ and if d is odd we define w'_d by

$$w'_d\phi = \begin{cases} (v'_{kd}\phi)^k & \text{if } k \text{ is odd,} \\ v_{kd}^{k/2}\phi & \text{if } k \text{ is even,} \end{cases}$$

Theorem 10.4. *For $W = W(A_n)$, d-good maximal elements exist for $1 < d \le n+1$; a representative is w_d, with $I = I_d$ and $W(w_d) = G(d, 1, \lfloor \frac{n+1}{d} \rfloor)$.*

For $W\phi$, d-good maximal elements exist for the following d with representatives as follows:

- $d \equiv 0 \pmod 4$*, $1 < d \le n+1$; a representative is $w_d\phi$ with $I = I_d$ and $W(w_d\phi) = G(d, 1, \lfloor \frac{n+1}{d} \rfloor)$.*
- $d \equiv 2 \pmod 4$*, $1 < d \le 2(n+1)$; a representative is $w'_{d/2}\phi$ with $I = I_{d/2}$ and $W(w'_{d/2}\phi) = G(d/2, 1, \lfloor \frac{2(n+1)}{d} \rfloor)$.*
- *d odd, $1 < d \le \frac{n+1}{2}$. If $d \ne 1$ a representative is $w_{2d}^2\phi$ with $I = I_{2d}$ and $W(w_{2d}^2\phi) = G(2d, 1, \lfloor \frac{n+1}{2d} \rfloor)$.*

Proof. We identify the Weyl group of type A_n as usual with $\mathfrak{S}_{n+1}$ by $s_i \mapsto (i, i+1)$; the automorphism ϕ maps to the exchange of i and $n+2-i$. An easy computation shows that the element v_d maps to the d-cycle $(1, 2, \ldots, \lfloor \frac{d+1}{2} \rfloor, n+1, n, \ldots, n+2-\lfloor \frac{d}{2} \rfloor)$ and that for d odd v'_d maps to the cycle $(1, 2, \ldots, n - \frac{d-3}{2})$.

Lemma 10.5. *If d is even v_d and w_d are ϕ-stable. If d is odd we have $w_d = w'_d . {}^\phi w'_d$.*

Proof. That d is even implies $\lfloor \frac{d+1}{2} \rfloor = \lfloor \frac{d}{2} \rfloor$, thus in the above cycle ϕ exchanges the two sequences $1, 2, \ldots, \lfloor \frac{d+1}{2} \rfloor$ and $n+1, n, \ldots, n+2-\lfloor \frac{d}{2} \rfloor$, thus v_d is ϕ-stable. The same follows for w_d, with $k = \lfloor \frac{n+1}{d} \rfloor$, since kd is even if d is even.

For d odd we have

$$w'_d.{}^{\phi}w'_d = (w'_d\phi)^2 = \begin{cases} (v'_{kd}\phi)^{2k} & \text{if } k \text{ is odd,} \\ v_{kd}^{k/2}.{}^{\phi}(v_{kd}^{k/2}) & \text{if } k \text{ is even.} \end{cases}$$

If k is odd we have $(v'_{kd}\phi)^{2k} = (v'_{kd}{}^{\phi}v'_{kd})^k = v_{kd}^k = w_d$. If k is even then v_{kd} is ϕ-stable thus $v_{kd}^{k/2}.{}^{\phi}(v_{kd}^{k/2}) = v_{kd}^k = w_d$. □

Lemma 10.6. *For $1 < d \leq n+1$,*

- *the element v_d is J_d-reduced and stabilizes J_d.*
- *the element w_d is I_d-reduced and stabilizes I_d.*
- *for d odd, the element v'_d is J_d-reduced and $v'_d\phi$ stabilizes J_d.*
- *for d odd, the element w'_d is I_d-reduced and $w'_d\phi$ stabilizes I_d.*

Proof. The property for w_d (resp. w'_d) follows from that for v_d (resp. v'_d) and the definitions since being I_d-reduced and stabilizing I_d are properties stable by taking a power.

It is clear on the expression of v_d as a cycle that it fixes i and $i+1$ if $s_i \in J_d$ thus it fixes the simple roots corresponding to J_d, whence the lemma for v_d.

For d odd, $1 < d \leq n+1$, an easy computation shows that $v'_d = (1, 2, \ldots, n - \frac{d-3}{2})$, and that $v'_d\phi$ preserves the simple roots corresponding to J_d. □

Lemma 10.7. *For $1 < d \leq n+1$ and for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$, we have*

- *$l(v_d^i) = \frac{2i}{d} l(w_{J_d}^{-1}w_0)$ and $l(w_d^i) = \frac{2i}{d} l(w_{I_d}^{-1}w_0)$*
- *(for d odd) $l((v'_d\phi)^i\phi^{-i}) = \frac{i}{d} l(w_{J_d}^{-1}w_0)$ and $l((w'_d\phi)^i\phi^{-i}) = \frac{i}{d} l(w_{I_d}^{-1}w_0)$.*

Proof. It is straightforward to see that the result for w_d (resp. w'_d) results from the result for v_d (resp. v'_d or v_d) and the definitions.

Note that the group W_{J_d} is of type A_{n-d}, thus $l(w_{J_d}^{-1}w_0) = \frac{n(n+1)}{2} - \frac{(n-d)(n-d+1)}{2} = \frac{(2n-d+1)d}{2}$.

We first prove the result for v_d and v'_d when $i = 1$. For odd d we have by definition $l(v'_d) = n - \frac{d-1}{2} = \frac{2n-d+1}{2}$ which is the formula we want for v'_d. To find the length of v_d one can use that $s_n s_{n-1} \cdots s_{\lfloor \frac{d+1}{2} \rfloor}$ is $\{s_1, s_2, \ldots, s_{n-1}\}$-reduced, thus adds to $s_1 s_2 \cdots s_{n-\lfloor \frac{d}{2} \rfloor}$, which gives $l(v_d) = 2n - d + 1$, the result for v_d.

We now show by direct computation that when d is even $v_d^{d/2} = w_{J_d}^{-1}w_0$. Raising the cycle $(1, 2, \ldots, \frac{d}{2}, n+1, n, \ldots, n+2-\frac{d}{2})$ to the $d/2$-th power we get $(1, n+1)(2, n) \cdots (\frac{d}{2}, n+2-\frac{d}{2})$ which gives the result since $w_{J_d} = (\frac{d}{2}+1, n+1-\frac{d}{2}) \cdots (\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n+1}{2} \rfloor)$. The lemma follows for v_d with d even since its truth for $i = 1$ and $i = \frac{d}{2}$ implies its truth for all i between these values.

We show now similarly that for odd d we have $(v'_d\phi)^d = w_{J_d}^{-1}w_0\phi^d$. Since ϕ acts on W by the inner automorphism given by w_0, this is the same as $(v'_d w_0)^d = w_{J_d}$. We find that $(1, 2, \ldots, n - \frac{d-3}{2})w_0 = (1, n+1, 2, n, 3, n-1, \ldots, n - \frac{d-5}{2}, \frac{d+1}{2})(\frac{d+3}{2}, n - \frac{d-3}{2}) \cdots (\lfloor \frac{n+3}{2} \rfloor, \lfloor \frac{n+4}{2} \rfloor)$ as a product of disjoint cycles, which gives the result since $(1, n+1, 2, n, 3, n-1, \ldots, n - \frac{d-5}{2}, \frac{d+1}{2})$ is a d-cycle and $(\frac{d+3}{2}, n - \frac{d-3}{2}) \cdots (\lfloor \frac{n+3}{2} \rfloor, \lfloor \frac{n+4}{2} \rfloor) = w_{J_d}$. This proves the lemma for w'_d by interpolating the other values of i as above.

It remains the case of v_d for odd d. We then have $v_d = (v'_d\phi)^2$ where the lengths add, and we deduce the result for v_d from the result for v'_d. □

Lemma 10.8. *The following elements are d-good*

- *For $1 < d \leq n+1$, the elements v_d and w_d.*
- *For $d \equiv 0 \pmod 4, d \leq n+1$ the elements $v_d\phi$ and $w_d\phi$.*
- *For $d \equiv 2 \pmod 4, d \leq 2(n+1)$ the elements $v'_{d/2}\phi$ and $w'_{d/2}\phi$.*
- *For d odd, $d \leq \frac{n+1}{2}$ the elements $v_{2d}^2\phi$ and $w_{2d}^2\phi$.*

Proof. In view of the previous lemmas, the only thing left to check is that in each case, the chosen element x in W (resp. $W\phi$) satisfies $x^d = 1$ (resp. $(x\phi)^d = \phi^d$). Once again, it is easy to check that the property for w_d (resp. w'_d) results from that for v_d (resp. v'_d or v_d) and the definitions.

It is clear that $v_d^d = 1$ since then it is a d-cycle, from which it follows that when $d \equiv 2 \pmod 4$ we have $(v'_{d/2}\phi)^d = v_{d/2}^{d/2} = 1$. The other cases are obvious. □

To prove the theorem, it remains to check that:

• The possible d for which the ζ_d-rank of W (resp. $W\phi$) is non-zero are as described in the theorem. In the untwisted case they are the divisors of one of the degrees, which are $2, \ldots, n+1$. In the twisted case the pairs of degrees and factors are $(2,1), \ldots, (i, (-1)^i), \ldots, (n+1, (-1)^{n+1})$ and we get the given list by the formula for the ζ_d-rank recalled above Proposition 10.1.

• The coset $W_I w\phi$ has ζ_d-rank 0 on the subspace spanned by the root lines of W_I. For this we first have to describe the type of the coset, which is a consequence of the analysis we did to show that $w\phi$ stabilizes I. We may assume I non-empty.

Let us look first at the untwisted case. We found that w_d acts trivially on I_d, so the coset is of untwisted type A_{n-kd} where $k = \lfloor \frac{n+1}{d} \rfloor$. Since $1 + n - kd < d$ by construction, this coset has ζ_d-rank 0.

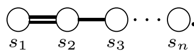
In the twisted case, if $d \equiv 0 \pmod 4$, the coset is $W_{I_d} w_d \phi$, which since w_d acts trivially on I_d and ϕ acts by the non-trivial diagram automorphism, is of type ${}^2A_{n-kd}$ where $k = \lfloor \frac{n+1}{d} \rfloor$. Since $n - kd = n - \lfloor \frac{n+1}{d} \rfloor d < d - 1$, this coset has ζ_d-rank 0.

If d is odd, the coset is $W_{I_{2d}} w_{2d}^2 \phi$, which since w_{2d} acts trivially on I_{2d} and ϕ acts by the non-trivial diagram automorphism, is of type ${}^2A_{n-2kd}$ where $k = \lfloor \frac{n+1}{2d} \rfloor$. Since $n - 2kd = n - \lfloor \frac{n+1}{2d} \rfloor 2d < 2d$, this coset has ζ_d-rank 0.

Finally, if $d \equiv 2$ modulo 4, the coset is $W_{I_{d/2}} w'_{d/2} \phi$. Let $k = \lfloor \frac{2(n+1)}{d} \rfloor$; then $W_{I_{d/2}}$ is of type $A_{n-kd/2}$. If k is even then $w'_{d/2} = w_{kd/2}^{k/2}$ and the coset is of type ${}^2A_{n-kd/2}$. Since $n - kd/2 < d/2 - 1$, this coset has ζ_d-rank 0. Finally if k is odd $w'_{d/2}\phi = (w'_{kd/2}\phi)^k$. Since $kd/2$ is odd, we found that $w'_{kd/2}\phi$ acts trivially on $I_{d/2}$ so the coset is of type $A_{n-kd/2}$, and has also has ζ_d-rank 0.

• Determine the group $W(w\phi)$ (resp. $W(w)$) in each case, We first give V_1 and the coset $C_W(V_1)w\phi$ or $C_W(V_1)w$. In the untwisted case w_d acts trivially on the roots of W_{I_d}, hence V_1 is spanned by these roots and $C_W(V_1)$ is generated by the reflection with respect to the roots orthogonal to those, which gives that $C_W(V_1)$ is of type $A_{d\lfloor \frac{n+1}{d} \rfloor - 1}$ if $d \not| n$ and A_n otherwise. In the twisted case if $d \equiv 0 \pmod 4$ since w_d acts trivially on the roots of W_{I_d} the space V_1 is spanned by the sums of the orbits of the roots under ϕ which is the non-trivial automorphism of that root system. Hence the type of the coset $C_W(V_1)w_d\phi$ is ${}^2A_{d\lfloor \frac{n+1}{d} \rfloor - 1}$ if n is odd and ${}^2A_{d\lfloor \frac{n+1}{d} \rfloor}$ if n is even. If d is odd a similar computation gives that the type of the coset $C_W(V_1)w_{2d}^2\phi$ is ${}^2A_{2d\lfloor \frac{n+1}{2d} \rfloor - 1}$ if n is odd and ${}^2A_{2d\lfloor \frac{n+1}{2d} \rfloor}$ if n is even. If $d \equiv 2 \pmod 4$ $w'_{d/2}\phi$ acts also by the non-trivial automorphism on $W_{I_{d/2}}$ and we get that the coset $C_W(V_1)w'_{d/2}\phi$ is of type ${}^2A_{\frac{d}{2}\lfloor\frac{2(n+1)}{d}\rfloor}$ if n and $\lfloor\frac{2(n+1)}{d}\rfloor$ have the same parity and ${}^2A_{\frac{d}{2}\lfloor\frac{2(n+1)}{d}\rfloor-1}$ otherwise.

Knowing the type of the coset in each case, we deduce the group $W(w\phi)$ (resp. $W(w)$) as in the remark at the beginning of Subsection 10.2. ☐

Type B_n 

. For d even, $2 \leq d \leq 2n$ we define

$$v_d = s_{n+1-d/2}\cdots s_2 s_1 s_2 \cdots s_n \text{ and } J_d = \{s_i \mid 1 \leq i \leq n - d/2\}.$$

Note that v_{2n} is the Coxeter element $s_1 s_2 \cdots s_n$. Now for $1 \leq d \leq 2n$, that we require even if $d > n$, we define w_d as follows: let kd be the largest even multiple of d less than or equal to $2n$ so that $k = \lfloor\frac{2n}{d}\rfloor$ if d is even and $k = 2\lfloor\frac{n}{d}\rfloor$ is d is odd. We define $w_d = v_{kd}^k$ and $I_d = J_{kd}$.

Theorem 10.9. *For $W = W(B_n)$, d-good maximal elements exist for odd d less than or equal to n and even d less than or equal to $2n$. A representative is w_d, with $I = I_d$; we have $W(w_d) = G(d, 1, \lfloor\frac{2n}{d}\rfloor)$ if d is even and $W(w_d) = G(2d, 1, \lfloor\frac{n}{d}\rfloor)$ if d is odd.*

Proof. We identify as usual the Weyl group of type B_n with the group of signed permutations on $\{1, \ldots, n\}$ by $s_i \mapsto (i-1, i)$ for $i \geq 2$ and $s_1 \mapsto (1, -1)$. The element v_d maps to the d-cycle (or signed $d/2$-cycle) given by $(n+1-d/2, n+2-d/2, \ldots, n-1, n, d/2-n-1, d/2-n-2, \ldots, -n)$. This element normalizes J_d and acts trivially on the corresponding roots, so is J_d-reduced. The same is thus true for w_d and I_d, since these properties carry to powers.

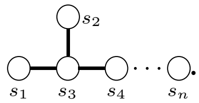
Lemma 10.10. *For $0 \leq i \leq \lfloor\frac{d}{2}\rfloor$ we have $l(v_d^i) = \frac{2i}{d} l(w_{J_d}^{-1} w_0)$ and $l(w_d^i) = \frac{2i}{d} l(w_{I_d}^{-1} w_0)$.*

Proof. As in Lemma 10.7 it is sufficient to prove the lemma for v_d, which we do now. To find the length of v_d we note that $s_1 s_2 \cdots s_n$ is $\{s_2, s_3, \ldots, s_n\}$-reduced so that the lengths of $s_{n+1-d/2}\cdots s_2$ and of $s_1 s_2 \cdots s_n$ add, whence $l(v_d) = 2n - d/2$. Since $l(w_0) = n^2$ and $l(w_{J_d}) = (n - d/2)^2$ we have $l(w_{I_d}^{-1} w_0) = nd - d^2/4$, which gives the result for $i = 1$. Written as permutations w_0 is the product of all sign changes and w_{I_d} is the product of all sign changes on the set $\{1, \ldots, n - d/2\}$; a direct computation shows that $v_d^{d/2}$ is the product of all sign changes on $\{n+1-d/2, \ldots, n\}$, hence $v_d^{d/2} = w_{I_d}^{-1} w_0$. The lemma follows for the other values of d. ☐

Since $v_d^{d/2} = w_{I_d}^{-1} w_0$ we have $v_d^d = 1$, so the same property is true for w_d, thus the above lemma shows that v_d and w_d are d-good elements.

Note also that Theorem 10.9 describes all d such that W has non-zero ζ_d-rank since the degrees of $W(B_n)$ are all the even integers from 2 to $2n$. We prove now the maximality property 8.10(iv) for w_d. If k is as in the definition of w_d, the group W_{I_d} is a Weyl group of type $B_{n-kd/2}$ and w_d acts trivially on I_d. Since $n - kd/2 < d$ the ζ_d-rank of $W_{I_d} w_d$ is zero on the subspace spanned by the roots corresponding to I_d.

It remains to get the type of $W(w_d)$. Since w_d acts trivially on I_d the space V_1 of Lemma 8.14 is spanned by the root lines of W_{I_d} and $C_W(V_1)$ is spanned by the roots orthogonal to those, so is of type $B_{kd/2}$. We then deduce the group $W(w_d)$ as in the remark at the beginning of Subsection 10.2, as the centralizer of a ζ_d-regular element in a group of type $B_{kd/2}$. ☐

Types D_n and ${}^2D_n$ 

. ${}^2D_n$ is defined by the diagram automorphism ϕ which exchanges s_1 and s_2 and fixes s_i for $i > 2$.

For d even, $2 \le d \le 2(n-1)$ we define

$$v_d = s_{n+1-d/2} \cdots s_3 s_2 s_1 s_3 \cdots s_n \text{ and } J_d = \begin{cases} \emptyset \text{ if } d = 2(n-1) \\ \{s_i \mid 1 \le i \le n - d/2\} \text{ otherwise.} \end{cases}$$

Note that $v_{2(n-1)}$ is a Coxeter element. Then for $1 \le d \le 2(n-1)$, that we require even if $d > n$, we let kd be the largest even multiple of d less than $2n$, so that $k = \lfloor \frac{2n-2}{d} \rfloor$ if d is even and $k = 2\lfloor \frac{n-1}{d} \rfloor$ if d is odd, and define $w_d = v_{kd}^k$ and $I_d = J_{kd}$.

Note that v_d, and thus w_d, are ϕ-stable.

Theorem 10.11.
- *For $W = W(D_n)$ there exist d-good maximal elements for odd d less than or equal to n and even d less than or equal to $2(n-1)$. When d does not divide n a representative is w_d, with $I = I_d$; in this case, if d is odd $W(w_d) = G(2d, 1, \lfloor \frac{n-1}{d} \rfloor)$ and if d is even $W(w_d) = G(d, 1, \lfloor \frac{2n-2}{d} \rfloor)$.*

  If $d|n$ a representative is $w_n^{n/d}$ where $w_n = s_1 s_2 s_3 \cdots s_n s_2 s_3 \cdots s_{n-1}$. In this case $I = \emptyset$ and $W(w_n^{n/d}) = G(2d, 2, n/d)$.
- *For $W\phi$ there exist d-good maximal elements for odd d less than n, for even d less than $2(n-1)$ and for $d = 2n$. Except in the case when d divides $2n$ and $2n/d$ is odd a representative is $w_d\phi$, with $I = I_d$ and $W(w_d\phi) = G(2d, 1, \lfloor \frac{n-1}{d} \rfloor)$ if d is odd and $W(w_d\phi) = G(d, 1, \lfloor \frac{2n-2}{d} \rfloor)$ if d is even. In the excluded case a representative is $(w_{2n}\phi)^{2n/d}$ where $w_{2n} = s_1 s_3 s_4 \cdots s_n$. In this case $I = \emptyset$ and $W((w_{2n}\phi)^{2n/d}) = G(d, 2, 2n/d)$.*

Proof. The cases D_n with $d|n$ or ${}^2D_n$ with $d|2n$ and $2n/d$ odd involve regular elements, so are dealt with in [BM]. We thus consider only the other cases.

We identify the Weyl group of type D_n with the group of signed permutations on $\{1, \ldots, n\}$ with an even number of sign changes, by mapping s_i to $(i-1, i)$ for $i \ne 2$ and s_2 to $(1, -2)(-1, 2)$. For d even v_d maps to $(1, -1)(n+1-d/2, n+2-d/2, \ldots, n-1, n, d/2-n-1, \ldots, 1-n, -n)$. This element normalizes J_d: when $J_d \ne \emptyset$, it exchanges the simple roots corresponding to s_1 and s_2 and acts trivially on the other simple roots indexed by J_d, so it is J_d-reduced. It follows that w_d normalizes I_d and is I_d-reduced.

Lemma 10.12. *For $0 \le i \le \lfloor \frac{d}{2} \rfloor$ we have $l(v_d^i) = \frac{2i}{d} l(w_{J_d}^{-1} w_0)$ and $l(w_d^i) = \frac{2i}{d} l(w_{I_d}^{-1} w_0)$.*

Proof. As in Lemma 10.7 it is sufficient to prove the lemma for v_d. To find the length of v_d we note that $s_2 s_1 s_3 s_4 \cdots s_n$ is $\{s_3, \ldots, s_n\}$-reduced so that the lengths of $s_{n+1-d/2} \cdots s_3$ and of $s_2 s_1 s_3 \cdots s_n$ add, whence $l(v_d) = 2n - 1 - d/2$. Since $l(w_0) = n^2 - n$ and $l(w_{J_d}) = (n - d/2)^2 - (n - d/2)$, we have $l(w_{J_d}^{-1} w_0) = d/2(2n - 1 - d/2)$. which gives the result for $i = 1$. Written as permutations $w_0 = (1, -1)^n (2, -2) \cdots (n, -n)$ and $w_{J_d} = (1, -1)^{n-d/2} (2, -2) \cdots (n-d/2, d/2-n)$; a direct computation shows that $v_d^{d/2} = (1, -1)^{d/2} (n+1-d/2, d/2-n-1) \cdots (n, -n)$, hence $v_d^{d/2} = w_{J_d}^{-1} w_0$. The lemma follows for smaller i. ∎

Since $v_d^{d/2} = w_{J_d}^{-1} w_0$ and J_d is w_0 stable we have $v_d^d = 1$, so the same property follows for w_d which shows that v_d and w_d are d-good elements.

We also note that the theorem describes all d such that the ζ_d-rank is not zero, since the degrees of $W(D_n)$ are all the even integers from 2 to $2n-2$ and n, and in the twisted case the factor associated with the degree n is -1 and the other factors are equal to 1.

Since w_d is ϕ-stable the element $w_d\phi$ is also d-good.

We now check Lemma 8.10(iv), that is that the ζ_d-rank of $W_{I_d} w_d$ in the untwisted case, resp. $W_{I_d} w_d \phi$ in the twisted case is 0 on the subspace spanned by the roots corresponding to I_d. This property is clear if $I_d = \emptyset$. Otherwise:

• In the untwisted case the type of the coset is $D_{n-kd/2}$ if k is even and ${}^2D_{n-kd/2}$ if k is odd, where k is as in the definition of w_d. In both cases the set of values i such that the ζ_i-rank is not 0 consists of the even i less than $2n-kd$, the odd i less than $n-kd/2$ and in the twisted case (k odd) $i = 2n-kd$. Since if d is even we have $2n-kd \le d$ and if d is odd we have $n-kd/2 \le d$, the only case where d could be in this set is k odd and $d = 2n-kd$, which means that $\frac{k+1}{2}d = n$. But d is assumed not to divide n, so this case does not happen.

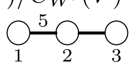
• In the twisted case the type of the coset is $D_{n-kd/2}$ if k is odd and ${}^2D_{n-kd/2}$ if k is even. In both cases the set of values i such that the ζ_i-rank is not 0 consists of the even i less than $2n-kd$, the odd i less than $n-kd/2$ and in the twisted case (k even) $i = 2n-kd$. Since if d is even we have $2n-kd \le d$ and if d is odd we have $n-kd/2 \le d$, the only case where d could be in this set is k even and $d = 2n-kd$, which means that $(k+1)d = 2n$. But this is precisely the excluded case.

We now give $C_W(V_1)$, where V_1 is as in Lemma 8.14, in each case where I is not empty. In the untwisted case, if d is odd the group $C_W(V_1)$ is of type $D_{d\lfloor\frac{n-1}{d}\rfloor}$; if d is even the group $C_W(V_1)$ is of type $D_{\frac{d}{2}\lfloor\frac{2n-2}{d}\rfloor+1}$ if $\lfloor\frac{2n-2}{d}\rfloor$ is odd and $D_{\frac{d}{2}\lfloor\frac{2n-2}{d}\rfloor}$ if $\lfloor\frac{2n-2}{d}\rfloor$ is even. In the twisted case, if d is odd the coset $C_W(V_1)w\phi$ is of type ${}^2D_{d\lfloor\frac{n-1}{d}\rfloor+1}$ and if d is even the coset is of type ${}^2D_{\frac{d}{2}\lfloor\frac{2n-2}{d}\rfloor+1}$ if $\lfloor\frac{2n-2}{d}\rfloor$ is even and $D_{\frac{d}{2}\lfloor\frac{2n-2}{d}\rfloor}$ if $\lfloor\frac{2n-2}{d}\rfloor$ is odd. In all cases except if d is even and $\lfloor\frac{2n-2}{d}\rfloor$ is even (resp. odd) in the untwisted case (resp. twisted case) we then deduce the group $W(w\phi)$ (resp. $W(w)$) as in the remarks at the beginning of Subsection 10.2 and after Lemma 8.14, since in these cases the centralizer of the regular element $w\phi$ (resp. w) in the parabolic subgroup $W' = C_W(V_1)$ has the (known) reflection degrees of $W(w\phi)$ (resp. $W(w)$). In the excluded cases the group $C_{W'}(w\phi)$ or $C_{W'}(w)$ is isomorphic to $G(d,2,\lfloor\frac{2n-2}{d}\rfloor)$ which does not have the reflection degrees of $W(w\phi)$, resp. $W(w)$. This means that the morphism of the remark after Lemma 8.14 is not surjective. We can prove in this case that $W(w\phi)$ or $W(w)$ is $G(d,1,\lfloor\frac{2n-2}{d}\rfloor)$ since it is an irreducible complex reflection group by [Br2, 5.6.6] and it is the only one which has the right reflection degrees apart from the exceptions in low rank given by $G_5, G_{10}, G_{15}, G_{18}, G_{26}$; we can exclude these since they do not have $G(d,2,\lfloor\frac{2n-2}{d}\rfloor)$ as a reflection subgroup. □

Types $I_2(n)$ and ${}^2I_2(n)$. All eigenvalues ζ such that the ζ-rank is non-zero are regular, so this case can be found in [BM].

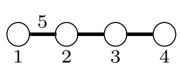
Exceptional types. Below are tables for exceptional finite Coxeter groups giving information on d-good maximal elements for each d. They were obtained with the `GAP` package `Chevie` (see [Chevie]): first, the conjugacy class of good ζ_d-maximal elements as described in Lemma 8.13 was determined; then we determined I for an element of that class, which gave $l(w_I)$. The next step was to determine the elements of the right length $2(l(w_0) - l(w_I))/d$ in that conjugacy class; this required care in large groups like E_8. The best algorithm is to start from an element of minimal length in the class (known by [GP]) and conjugate by Coxeter generators until all elements of the right length are reached.

In the following tables, we give for each possible d and each possible I for that d a representative good $w\phi$, and give the number of possible $w\phi$. We then describe the coset $W_I w\phi$ by giving, if $I \neq \emptyset$, in the column I the permutation induced by $w\phi$ of the nodes of the Coxeter diagram indexed by I. Then we describe the isomorphism type of the complex reflection group $N_W(W_I w\phi)/W_I = N_W(V)/C_W(V)$, where V is the ζ_d-eigenspace of $w\phi$. Finally, in the cases where $I \neq \emptyset$, we give the isomorphism type of $W' = C_W(V_1)$, where V_1 is the 1-eigenspace of $w\phi$ on the subspace spanned by the root lines of I. We note that there are 3 cases where $N_{W'}(V)/C_{W'}(V) \lneq N_W(V)/C_W(V)$: for $d = 4$ or 5 in E_7 and for $d = 9$ in E_8.

H_3: 

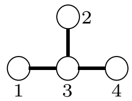
The reflection degrees are $2, 6, 10$.

d	representative w	#good w	$C_W(w)$
10	$w_{10} = 123$	4	Z_{10}
6	$w_6 = 32121$	6	Z_6
5	w_{10}^2	4	Z_{10}
3	w_6^2	6	Z_6
2	w_0	1	H_3
1	$\cdot$	1	H_3

H_4: 

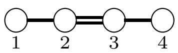
The reflection degrees are $2, 12, 20, 30$.

d	representative w	#good w	$C_W(w)$
30	$w_{30} = 1234$	8	Z_{30}
20	$w_{20} = 432121$	12	Z_{20}
15	w_{30}^2	8	Z_{30}
12	$w_{12} = 2121432123$	22	Z_{12}
10	w_{30}^3 or w_{20}^2	24	G_{16}
6	w_{30}^5 or w_{12}^2	40	G_{20}
5	w_{30}^6 or w_{20}^4	24	G_{16}
4	w_{20}^5 or w_{12}^3	60	G_{22}
3	w_{30}^{10} or w_{12}^4	40	G_{20}
2	w_0	1	H_4
1	$\cdot$	1	H_4

$^3D_4$: 

ϕ does the permutation $(1,2,4)$. The reflection degrees are $2,4,4,6$ with corresponding factors $1,\zeta_3,\zeta_3^2,1$.

d	representative $w\phi$	#good $w\phi$	$C_W(w\phi)$
12	$w_{12}\phi = 13\phi$	6	Z_4
6	$w_6\phi = 1243\phi$	8	G_4
3	$w_6^2\phi$	8	G_4
2	$w_0\phi$	1	G_2
1	ϕ	1	G_2

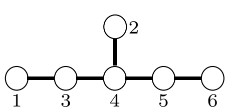
F_4: 

The reflection degrees are $2,6,8,12$.

d	representative w	#good w	$C_W(w)$
12	$w_{12} = 1234$	8	Z_{12}
8	$w_8 = 214323$	14	Z_8
6	w_{12}^2	16	G_5
4	w_{12}^3 or w_8^2	12	G_8
3	w_{12}^4	16	G_5
2	w_0	1	F_4
1	$\cdot$	1	F_4

$^2F_4$: ϕ does the permutation $(1,4)(2,3)$. The factors, in increasing order of the degrees, are $1,-1,1,-1$.

d	representative $w\phi$	#good $w\phi$	$C_W(w\phi)$
24	$w_{24}\phi = 12\phi$	6	Z_{12}
12	$w_{12}\phi = 3231\phi$	10	Z_6
8	$(w_{24}\phi)^3$	12	G_8
4	$(w_{12}\phi)^3$	24	G_{12}
2	$w_0\phi$	1	$I_2(8)$
1	ϕ	1	$I_2(8)$

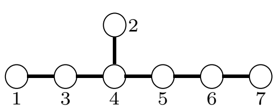
E_6: 

The reflection degrees are $2,5,6,8,9,12$.

d	representative w	#good w	I	$N_W(W_I w)/W_I$	$C_W(V_1)$
12	$w_{12} = 123654$	8		Z_{12}	
9	$w_9 = 12342654$	24		Z_9	
8	$w_8 = 123436543$	14		Z_8	
6	w_{12}^2	16		G_5	
5	24231454234565	8	(3)	Z_5	A_5
	12435423456543	8	(4)		
	12314235423654	8	(5)		
4	w_8^2 or w_{12}^3	12		G_8	
3	w_{12}^4 or w_9^3	80		G_{25}	
2	w_0	1		F_4	
1	$\cdot$	1		E_6	

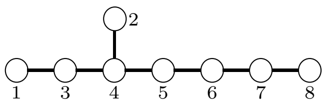
${}^2E_6$: ϕ does the permutation $(1,6)(3,5)$. The factors, in increasing order of the degrees, are $1,-1,1,1,-1,1$.

d	representative $w\phi$	#good$w\phi$	I	$N_W(W_I w\phi)/W_I$	$C_W(V_1)w\phi$
18	$w_{18}\phi = 1234\phi$	24		Z_9	
12	$w_{12}\phi = 123654\phi$	8		Z_{12}	
10	2431543ϕ	8	(3)	Z_5	${}^2A_5$
	5423145ϕ	8	(4)		
	3143542ϕ	8	(5)		
8	$w_8\phi = 123436543\phi$	14		Z_8	
6	$(w_{18}\phi)^3$	80		G_{25}	
4	$(w_{12}\phi)^3$	12		G_8	
3	$w_{12}^4\phi$	16		G_5	
2	$w_0\phi$	1		E_6	
1	ϕ	1		F_4	

E_7: 

The reflection degrees are $2,6,8,10,12,14,18$.

d	representative w	#good w	I	$N_W(W_I w)/W_I$	$C_W(V_1)$
18	$w_{18} = 1234567$	64		Z_{18}	
14	$w_{14} = 123425467$	160		Z_{14}	
12	$w_{12} = 1342546576$	8	$(2,5,7)$	Z_{12}	E_6
10	$w_{10a} = 134254234567$	8	$(2,4)$	Z_{10}	D_6
	$w_{10b} = 243154234567$	8	$(3,4)$		
	$w_{10c} = 124354265437$	8	$(4,5)$		
9	w_{18}^2	64		Z_{18}	
8	134234542346576	14	$(2)(5,7)$	Z_8	D_5
7	w_{14}^2	160		Z_{14}	
6	w_{18}^3 or w_{12}^2	800		G_{26}	
5	w_{10a}^2	8	$(2)(4)$	Z_{10}	A_5
	w_{10b}^2	8	$(3)(4)$		
	w_{10c}^2	8	$(4)(5)$		
4	w_8^2 or w_{12}^3	12	$(2)(5)(7)$	G_8	D_4
3	w_{18}^6 or w_{12}^4	800		G_{26}	
2	w_0	1		E_7	
1	$\cdot$	1		E_7	

E_8: 

The reflection degrees are $2, 8, 12, 14, 18, 20, 24, 30$.

d	representative w	#good w	I	$N_W(W_I w)/W_I$	$C_W(V_1)$
30	$w_{30} = 12345678$	128		Z_{30}	
24	$w_{24} = 1234254678$	320		Z_{24}	
20	$w_{20} = 123425465478$	624		Z_{20}	
18	$w_{18a} = 1342542345678$	16	$(2,4)$	Z_{18}	E_7
	$w_{18b} = 2431542345678$	16	$(3,4)$		
	$w_{18c} = 1243542654378$	16	$(4,5)$		
15	w_{30}^2	128		Z_{30}	
14	$w_{14a} = 13423454234565768$	128	(2)	Z_{14}	E_7
	$w_{14b} = 24231454234565768$	88	(3)		
	$w_{14c} = 12435423456543768$	108	(4)		
	$w_{14d} = 12342543654276548$	68	(5)		
12	w_{24}^2	2696		G_{10}	
10	w_{30}^3 or w_{20}^2	3370		G_{16}	
9	w_{18a}^2	16	$(2)(4)$	Z_{18}	E_6
	w_{18b}^2	16	$(3)(4)$		
	w_{18c}^2	16	$(4)(5)$		
8	w_{24}^3	7748		G_9	
7	w_{14a}^2	128	(2)	Z_{14}	E_7
	w_{14b}^2	88	(3)		
	w_{14c}^2	108	(4)		
	w_{14d}^2	68	(5)		
6	w_{30}^5 or w_{24}^4	4480		G_{32}	
5	w_{30}^6 or w_{20}^4	3370		G_{16}	
4	w_{24}^6 or w_{20}^5	15120		G_{31}	
3	w_{30}^{10} or w_{24}^8	4480		G_{32}	
2	w_0	1		E_8	
1	$\cdot$	1		E_8	

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