Some existence results for the modified binormal curvature flow equation

Haidar Mohamad

Some existence results for the modified binormal curvature flow equation

Haidar Mohamad [1]

February 18, 2014

Abstract

We establish some existence results for the modified binormal curvature flow equation from ($\mathbb{R}$ or $\mathbb{T}^l$) to $\mathbb{R}^3$ where the velocity of the curve depends not only on the binormal vector but the parametrization of the curve, the time and the position of the point in the space. We achieve our objective via the Schrödinger map equation. A Local well-posedness result is proved for the Schrödinger map equation in the space $L^\infty(0, T_1, H^3_{loc}(\mathbb{R}))$.

AMS: 76B99

[1]Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Boîte Courrier 187, 75252 Paris Cedex 05, France. E-mail: `tartousi@ann.jussieu.fr`

1 Introduction

The modified binormal curvature flow equation for $\gamma : [0,T[\times\mathbb{R} \to \mathbb{R}^3$ is

$$\partial_t \gamma = g\left(\partial_x \gamma \wedge \partial_x^2 \gamma\right), \tag{1.1}$$

where $T \in \mathbb{R}_+^* \cup \{+\infty\}$, x is the arc-length parameter of the curve $\gamma(t,.)$ for all $t \in [0,T[$ and g is a real function.
The first goal of this article will be to consider the case where $g = g(t,x)$ and to prove the existence of solution $\gamma \in L^\infty([0,T[, H^2_{loc}(\mathbb{R}))$. Then, we prove a well-posedness result in more regular space ($\gamma \in L^\infty([0,T[, H^4_{loc}(\mathbb{R}))$) via the Schrödinger map equation

$$\partial_t u = \partial_x \left(u \wedge g \partial_x u\right) = u \wedge \Delta_g(u), \tag{1.2}$$

where $\Delta_g(u) \equiv \partial_x\left(g(x)\partial_x u\right)$ and $u \equiv \partial_x \gamma$.
Finally, we consider the case where $g = g(t,x,\gamma)$ and we prove a local existence result of solution $\gamma \in L^\infty([0,T_1[, H^3_{loc}(\mathbb{R}))$, with $T_1 > 0$ depending on $\gamma_0 \equiv \gamma(0,.)$ and g. The transition from results for (1.2) to results for (1.1) occurs by Lemma 1.7.

Theorem 1.1 *Let $u_0 : \mathbb{R} \to S^2$ be such that $\frac{du_0}{dx} \in L^2(\mathbb{R})$, $T > 0$ and let $g \in W^{1,\infty}(\mathbb{R}^+, L^\infty(\mathbb{R}))$ be such there exists $\alpha > 0$ with $g \geq \alpha$. Then the equation (1.2) has a solution $u \in L^\infty(0,T,H^1_{loc}(\mathbb{R},S^2))$ with $u(0,.) = u_0$. Moreover, if $g = g(x)$, then $u \in L^\infty(\mathbb{R}^+, H^1_{loc}(\mathbb{R},S^2))$*

Theorem 1.2 *Let $l > 0$ and $T > 0$. We denote $\mathbb{T}^l \simeq \mathbb{R}/l\mathbb{Z}$. Let $u_0 : \mathbb{T}^l \to S^2$, and let $g \in W^{1,\infty}(\mathbb{R}^+, L^\infty(\mathbb{T}^l))$ such that there exists $\alpha > 0$ with $g \geq \alpha$. Then the equation (1.2) has a solution $u \in L^\infty(0,T,H^1(\mathbb{T}^l,S^2))$ with $u(0,.) = u_0$. Moreover, if $g = g(x)$, then $u \in L^\infty(\mathbb{R}^+, H^1_{loc}(\mathbb{T}^l,S^2))$.*

Theorem 1.3 *Let $u_0 : \mathbb{R} \to S^2$ be such that $\frac{du_0}{dx}$ belongs to $H^2(\mathbb{R})$, and let $g \in W^{1,\infty}(\mathbb{R}^+, W^{3,\infty}(\mathbb{R}))$. Assume that there exists $\alpha > 0$ with $g \geq \alpha$. Then there exists $T_1 = T_1(g,u_0) > 0$ such that equation (1.2) has a unique solution $u \in L^\infty(0,T_1,H^3_{loc}(\mathbb{R}))$ with $u(0,.) = u_0$.*

The uniqueness is deduced from the following quantitative theorem

Theorem 1.4 *Let $T > 0$ and $g : \mathbb{R} \to \mathbb{R}$ be a function verifying the conditions of Theorem 1.3. Let u and $\tilde{u}$ be two solutions for (1.2) with initial datum $u_0, \tilde{u}_0 : \mathbb{R} \to S^2$ respectively. Assume that $\partial_x u, \partial_x \tilde{u}$ belong to $L^\infty(0,T,H^2(\mathbb{R}))$. There exists two positive constants C_1, C_2 depending on g, T and the H^2 norm of $\frac{\partial u_0}{\partial x}$ and $\frac{\partial \tilde{u}_0}{\partial x}$ with*

$$\|u(t,.) - \tilde{u}(t,.)\|_{H^1(\mathbb{R})} \leq C_1 \|u_0 - \tilde{u}_0\|_{H^1(\mathbb{R})},$$

$$\|u(t,.) - \tilde{u}(t,.)\|_{H^2(\mathbb{R})} \leq C_2 \|u_0 - \tilde{u}_0\|_{H^2(\mathbb{R})},$$

for almost every $t \in]0,T[$.

In what concerns the case $g = g(t,x,\gamma)$, we have

Theorem 1.5 *Assume that $g = g(t,x,\gamma)$ and let $g \in W^{1,\infty}(\mathbb{R}^+, W^{2,\infty}(\mathbb{R}^3 \times \mathbb{R})$. We further assume that there exists $\alpha > 0$ with $g \geq \alpha$. Let $\gamma_0 : \mathbb{R} \to \mathbb{R}^3$, be such that $\frac{d^2\gamma_0}{dx^2} \in H^1(\mathbb{R})$. There exists $T_1 = T_1(g,\gamma_0)$ such that equation (1.1) has a solution $\gamma \in L^\infty(0,T_1,H^3_{loc}(\mathbb{R}))$ with $\gamma(0,.) = \gamma_0$.*

Equation (1.1) (with $g \equiv 1$) forms a model of the motion of a very thin vortex with radius ϵ and arc-length parameter x in an incompressible fluid by its own induction. The original equation for this model is given by

$$\partial_t \gamma = G\kappa B, \tag{1.3}$$

where κ is the curvature of γ, B is the binormal vector of the Frenet-Serret formula

$$\partial_x \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{1.4}$$

and

$$G = \frac{\Gamma}{4\pi}\left(\log\left(\frac{1}{\epsilon}\right) + O(1)\right),$$

is the coefficient of local induction which is proportional to the circulation Γ of the vortex and may be regarded as constant if we neglect the slow variation of the logarithm with respect to ϵ^{-1}. In this approximation, the local motion is approximated by that of a very thin circular ring with the same curvature and the tangential motion due to stretching is neglected. This model is called Localized Induction Approximation (LIA). It was developed in 1965 by Arms and Hama [1]. More analysis concerning the limitation of this model was realized in [3, 6].

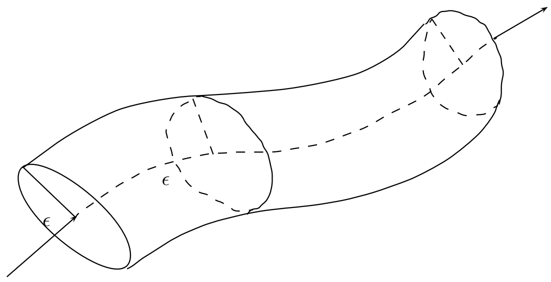

Figure 1: Approximation (LIA).

Our aim in this paper is to prove some existence results for Cauchy problem associated to some generalization of (1.1). Namely, in the formula (1.1) the velocity is proportional to the curvature with identical coefficient in every point of the curve. In our case, we assume that this coefficient can be depending on the time t, the arc-length parameter x and eventually on the position of the point in the space $\gamma(t, x)$:

$$\partial_t \gamma = g\kappa B, \tag{1.5}$$

with $(g = g(t, x, \gamma(t, x)))$. Since we have $\partial_x \gamma = T$ and $B = N \wedge T$, (1.5) becomes

$$\partial_t \gamma = g\partial_x \gamma \wedge \partial_x^2 \gamma. \tag{1.6}$$

Equation (1.6) (with $g \equiv 1$) was presented in 1906 by Da Rios [5]. We denote $u = \partial_x \gamma$, then by deriving (1.6) with respect to x, we obtain at least formally

$$\partial_t u = u \wedge \partial_x (g\partial_x u). \tag{1.7}$$

When $g = g(t, x)$ does not depend on γ, we use the last formula together with Lemme 1.7 in the next part to study the Cauchy problem of (1.6). The case $g \equiv 1$ belongs to the Schrödinger map equation

$$\partial_t u = u \wedge \partial_x^2 u, \tag{1.8}$$

whose Cauchy problem was first studied by Zhou and Guo [4] in 1984 when $u(t, .)$ is defined on an interval $I \subset \mathbb{R}$ into $S^2 = \{v \in \mathbb{R}^3 \quad \text{s. t.} \quad |v| = 1\}$, and by Sulem, Sulem and Bardos [2] in 1986 when $u(t, .)$ is defined on $\mathbb{R}^N$ $(N \geq 1)$ into S^2. They proved that (1.8) has a weak solution in $L^\infty(H^1_{loc})$. Namely,

Theorem 1.6 *Let $u_0 : \mathbb{R}^N \to S^2$ to be such that $\nabla u_0 \in (L^2(\mathbb{R}^N))^N$. Then there exists a weak solution $u : \mathbb{R}^+ \times \mathbb{R}^N \to S^2$ for (1.8) such that $\nabla_x u \in L^\infty(\mathbb{R}^+, \mathbb{R}^N)$ with $u(0, .) = u_0$.*

1.1 Reconstruction of flow γ

Let $I \subset \mathbb{R}^+$ be an interval containing 0, and let $u \in L^\infty(I, H^1_{loc}(\mathbb{R}))$ be a solution for (1.2). We define the function $\Gamma_u \in L^\infty(I, H^2_{loc}(\mathbb{R}))$ by

$$\Gamma_u(t,x) = \int_0^x u(t,z)dz. \tag{1.9}$$

We have, In the sense of distributions on $I \times \mathbb{R}$,

$$\partial_x \left(\partial_t \Gamma_u - g\partial_x \Gamma_u \wedge \partial_x^2 \Gamma_u\right) = 0. \tag{1.10}$$

By construction, the curves $\Gamma_u(t,.)$ all have the same base point $\Gamma_u(t,0)$ fixed at the origin. If they were smooth, equation (1.10) would directly imply the existence of a function $c_u = c_u(t)$ such that the function

$$\gamma_u(t,x) = \Gamma_u(t,x) + c_u(t)$$

is a solution for (1.1) (with $g = g(t,x)$). In this case, we have

$$\begin{aligned} c_u(t) &= \gamma_u(t,x) - \Gamma_u(t,x) \\ &= \gamma_u(0,x) + \int_0^t g(\tau,x)u(\tau,x) \wedge \partial_x u(\tau,x)d\tau - \int_0^x u(t,z)dz \\ &= \gamma_u(0,0) + \int_0^x (u(0,z) - u(t,z))\, dz + \int_0^t g(\tau,x)u(\tau,x) \wedge \partial_x u(\tau,x)d\tau. \end{aligned}$$

In fact, the function c_u represents the evolution in time of the actual base point of the curves.

The relation between the modified binormal curvature flow equation and the Schrödinger map equation is specified in the following lemma.

Lemma 1.7 *Let $\omega \in L^\infty(I, H^1_{loc}(\mathbb{R}, S^1))$ be a solution for (1.2) such that $\partial_x \omega \in L^\infty(I, L^2(\mathbb{R}, S^1))$. Let Γ_ω be defined by (1.9). Then there exists a unique continuous function $c_\omega : I \to \mathbb{R}^3$ satisfying $c_\omega(0) = 0$ such that the function $\gamma_\omega \in L^\infty(I, H^2_{loc}(\mathbb{R}, \mathbb{R}^3))$ defined by*

$$\gamma_\omega(t,x) = \Gamma_\omega(t,x) + c_\omega(t)$$

is a solution for equation (1.1) on $I \times \mathbb{R}$.

Proof. We define $a \in \mathcal{D}'(I \times \mathbb{R}, \mathbb{R}^3)$ by

$$a(t,x) = \int_0^x (\omega(0,z) - \omega(t,z))\, dz + \int_0^t g(\tau,x)\omega(\tau,x) \wedge \partial_x \omega(\tau,x)d\tau.$$

Let $\chi \in \mathcal{D}(\mathbb{R}, \mathbb{R})$ be such that $\int_{\mathbb{R}} \chi(z)dz = 1$. We set

$$c_\omega(t) = \int_{\mathbb{R}} \chi(z)a(t,z)dz.$$

By construction, we have $c_\omega(0) = 0$, and since $\omega \in W^{1,\infty}(I, H^{-1}(\mathbb{R}))$, we have $c_\omega \in \mathcal{C}(I, \mathbb{R}^3)$. On the other hand, we have

$$\begin{aligned} \partial_x(\partial_t a) &= \partial_t \partial_x \Gamma_\omega - \partial_x (g\omega \wedge \partial_x \omega) \\ &= \partial_t \omega - \omega \wedge \Delta_g \omega \\ &= 0, \end{aligned} \tag{1.11}$$

since ω is a solution to (1.2). Since $\partial_t a(t,x)$ does not depend on x, we have for all $\varphi \in \mathcal{D}(I,\mathbb{R}^3)$

$$\begin{aligned}\int_I c_\omega(t)\cdot\varphi'(t)dt &= \int_I\int_{\mathbb{R}} \chi(z)a(t,z)\cdot\varphi'(t)dtdz \\ &= -\int_{\mathbb{R}}\chi(z)\int_I \partial_t a(t,z)\cdot\varphi(t)dtdz \\ &= -\int_I \partial_t a(t,z)\cdot\varphi(t)dt, \end{aligned} \tag{1.12}$$

Relation (1.12) means that

$$c'_\omega = \partial_t a = -\partial_t\Gamma + g\omega\wedge\partial_x\omega \quad \text{in} \quad \mathcal{D}'(I,\mathbb{R}^3). \tag{1.13}$$

We show now that the function γ_ω, defined on $I\times\mathbb{R}$ by

$$\gamma_\omega(t,x) = \Gamma_\omega(t,x) + c_\omega(t),$$

is a solution to (1.1) on $I\times\mathbb{R}$. For this aim, assume that $\psi\in\mathcal{D}(I\times\mathbb{R},\mathbb{R}^3)$ and

$$\varphi(t) = \int_{\mathbb{R}}\psi(t,z)dz \in \mathcal{D}(I,\mathbb{R}^3).$$

Using (1.13), we finally find that

$$\begin{aligned}\langle\partial_t\gamma_\omega - g\partial_x\gamma_\omega\wedge\partial_x^2\gamma_\omega,\psi\rangle_{I\times\mathbb{R}} &= \langle\partial_t\Gamma_\omega - g\omega\wedge\partial_x\omega,\psi\rangle_{I\times\mathbb{R}} + \langle c_\omega,\psi\rangle_{I\times\mathbb{R}} \\ &= -\langle\partial_t a,\varphi\rangle_I + \langle c'_\omega,\varphi\rangle_I \\ &= 0,\end{aligned}$$

where $\langle,\rangle_{I\times\mathbb{R}}$ is the duality pairing between $\mathcal{D}'(I\times\mathbb{R},\mathbb{R}^3)$ and $\mathcal{D}(I\times\mathbb{R},\mathbb{R}^3)$, and $\langle,\rangle_I$ is that between $\mathcal{D}'(I,\mathbb{R}^3)$ and $\mathcal{D}(I,\mathbb{R}^3)$. This proves the existence of c_ω. Since c_ω is required to be continuous with $c_\omega(0)=0$ and since its distributional derivative $c'_\omega = \partial_t a$, its uniqueness follows. ■

1.2 Approximation by discretization of the Schrödinger map equation

We present here the strategy of proof of theorems 1.1, 1.2 and 1.3. We discretise, in space, the continuous system

$$\begin{cases}\partial_t u = \partial_x(u\wedge g\partial_x u) = u\wedge\partial_x(g\partial_x u), & t\geq 0, \quad x\in\mathbb{R}, \\ u(0,.) = u_0.\end{cases} \tag{1.14}$$

in the following sense:
For some $h>0$, we consider the sequence $u_h \equiv \{u_h(t,x_i)\}_{i\in\mathbb{Z}}$ satisfying the semi-discrete system

$$\begin{cases}\frac{du_h}{dt} = D^+(u_h\wedge g_h D^- u_h) = u_h\wedge D^+(g_h D^- u_h), & t\geq 0, \\ u_h(0,x_i) = u_h^0(x_i), \quad i\in\mathbb{Z}\end{cases} \tag{1.15}$$

where $\{x_i\}_{i\in\mathbb{Z}}$, is a uniform subdivision of $\mathbb{R}$ with step h, $g_h\equiv\{g(t,x_i)\}_{i\in\mathbb{Z}}$, and D^+, D^- are two operators approximating the derivative operator ∂_x. The sequence $\{u_h^0(x_i)\}_{i\in\mathbb{Z}}$ is constructed such that it converges to u_0 in certain sense (for example: since $u_0\in H^1_{loc}(\mathbb{R})$, we can choose $u_h^0(x_i) = u_0(x_i) \quad \forall i\in\mathbb{Z}$). We solve the problem (1.15) in some space discretising the space $L^\infty(\mathbb{R}^+, H^1_{loc}(\mathbb{R}))$ where our research for solving the continuous problem (1.14) takes a place. Then, we prove the boundedness properties for discrete derivatives (D^+u_h in the case of Theorems 1.1 and 1.2; and $D^-D^+u_h$, $D^+D^-D^+u_h$ in the case of Theorem 1.3) which allows us, using the compactness properties in spaces $L^2(\mathbb{R})$ and $H^1_{loc}(\mathbb{R})$, to extract a subsequence $\{u_h\}_h$ [1] converging to a solution of (1.14). The proof of Theorem 1.4 is standard. It consists of considering two solutions u and $\tilde{u}$ with initial datum u_0 and $\tilde{u}_0$ respectively and then proving Grönwall-type inequalities for $\|u-\tilde{u}\|_{H^1}$ and $\|u-\tilde{u}\|_{H^2}$. For Theorem 1.5, we follow the same strategy followed in the proof of Theorem 1.3.

In what follows, we define the elements of the discrete problem (1.15). Then, we prove some convergence properties before we skip to the proofs of previous theorems.

Definition 1.8 *Let $h>0$. Let*

$$\mathbb{Z}_h = \{x_i \in \mathbb{R}, \quad x_{i+1} - x_i = h \quad \forall i \in \mathbb{Z}\}.$$

We define the two spaces L^2_h and L^∞_h by

$$L^2_h = \{v_h = \{v_h(x_i)\}_i \in (\mathbb{R}^3)^{\mathbb{Z}_h}, \quad \sum_i |v_h(x_i)|^2 < +\infty\},$$

$$L^\infty_h = \{v_h = \{v_h(x_i)\}_i \in (\mathbb{R}^3)^{\mathbb{Z}_h}, \quad \sup_i |v_h(x_i)| < +\infty\}.$$

We define the scalar product $(,)_h$ on L^2_h by

$$(u_h, v_h)_h = h \sum_i v_h(x_i) \cdot u_h(x_i), \quad u_h, v_h \in L^2_h.$$

Its associated norm $|.|_h$ is defined by

$$|v_h|_h^2 = h \sum_i |v_h(x_i)|^2.$$

Let $l>0$, $N \in \mathbb{N}$ and $h = \frac{l}{N}$. We define the space of N-periodic sequences

$$P_{l,N} = \{v_h \in (\mathbb{R}^3)^{\mathbb{Z}_h}, \quad v_h(x_i) = v_h(x_{i+N}), \quad i \in \mathbb{Z}\}.$$

We define the scalar product $(,)_{l,N}$ by

$$(u_h, v_h)_{l,N} = h \sum_{i=1}^{N} v_h(x_i) \cdot u_h(x_i).$$

Its associated norm $|.|_{l,N}$ is defined by

$$|v_h|_{l,N}^2 = h \sum_{i=1}^{i=N} |v_h(x_i)|^2.$$

Let $v_h \in (\mathbb{R}^3)^{\mathbb{Z}_h}$. We define the left and the right approximations of the derivatives in x_i by the form

$$\begin{cases} D^- v_h(x_i) = \frac{v_h(x_i) - v_h(x_{i-1})}{h}, \\ D^+ v_h(x_i) = \frac{v_h(x_{i+1}) - v_h(x_i)}{h}. \end{cases}$$

It is clear that for two sequences $u_h = \{u_h(x_i)\}_i$ and $v_h = \{v_h(x_i)\}_i$ we have

$$D^\pm(u_h v_h) = \tau^\pm u_h D^\pm v_h + D^\pm u_h v_h,$$

with

$$\tau^\pm u_h(x_i) = u_h(x_{i\pm 1}).$$

[1] To give sense to the notation $\{u_h\}_h$, we can consider $h : \mathbb{N} \to \mathbb{R}^+$ to be a strictly decreasing function which goes to zero when $n \to +\infty$. We have made this choice for its simplicity.

The two spaces L_h^2 and $P_{l,N}$ verify the following property

Lemma 1.9 *1) If $v_h \in L_h^2$, then we have $D^+ v_h \in L_h^2$, and*

$$|D^+ v_h|_h \le \frac{2}{h} |v_h|_h.$$

2) If $v_h \in P_{l,N}$, then we have also

$$|D^+ v_h|_{l,N} \le \frac{2}{h} |v_h|_{l,N}, \quad h = \frac{l}{N}.$$

Proof. It follows directly from the inequality

$$|D_h^+ v_h(x_i)|^2 \le \frac{2}{h^2} (|v_h(x_i)|^2 + |v_h(x_{i+1})|^2).$$

■

Definition 1.10 *We define the norm*

$$|v_h|_{H_h^1}^2 = |v_h|_h^2 + |D^+ v_h|_h^2, \quad v_h \in L_h^2,$$

and the space

$$H_h^{-1} = \left\{ v_h \in (\mathbb{R}^3)^{\mathbb{Z}_h}, \quad \sup_{u_h \in L_h^2} \frac{\langle v_h, u_h \rangle_h}{|u_h|_{H_h^1}} < +\infty \right\}.$$

Its clear that $L_h^2 \subset H_h^{-1}$ and the function $v_h \mapsto |v_h|_{H_h^{-1}} \equiv \sup_{u_h \in L_h^2} \frac{\langle v_h, u_h \rangle_h}{|u_h|_{H_h^1}}$ define a norm on H_h^{-1}. Similarly, we define the norms

$$|v_h|_{H_{l,N}^1}^2 = |v_h|_{l,N}^2 + |D^+ v_h|_{l,N}^2,$$

$$|v_h|_{H_{l,N}^{-1}} = \sup_{u_h \in P_{l,N}} \frac{\langle v_h, u_h \rangle_{l,N}}{|u_h|_{H_{l,N}^1}}, \quad v_h \in P_{l,N}.$$

The two norms $|.|_{H_h^{-1}}$ and $|.|_{H_{l,N}^{-1}}$ are the dual norms of $|.|_{H_h^1}$ and $|.|_{H_{l,N}^1}$ with respect to scalar product $\langle , \rangle_h$ et $\langle , \rangle_{l,N}$ respectively.

Lemma 1.11 *For each $(v_h, u_h) \in L_h^\infty \times \in L_h^2$, we have (discrete integration by parts formula)*

$$\sum_i v_h(x_i) \cdot D^+ u_h(x_i) = - \sum_i u_h(x_i) \cdot D^- v_h(x_i). \tag{1.16}$$

Similarly, for all $v_h, u_h \in P_{l,N}$, we have

$$\sum_{i=1}^N v_h(x_i) \cdot D^+ u_h(x_i) = - \sum_{i=1}^N u_h(x_i) \cdot D^- v_h(x_i). \tag{1.17}$$

Proof. Let $v_h \in L_h^\infty$, $u_h \in L_h^2$ and $K \in \mathbb{N}$. We develop the sum $\sum_{i=-K}^K v_h(x_i) \cdot D^+ u_h(x_i)$ and we make a change in index, then (1.16) holds by using the property $\lim_{|i| \to +\infty} |u_h(x_i)| = 0$ and the assemption ($v_h \in L_h^\infty$). In the second case, we simply develop the sum $\sum_{i=1}^N v_h(x_i) \cdot D^+ u_h(x_i)$ and make a change in index, then we use the periodicity of v_h and u_h. ■

Definition 1.12 *Let $h > 0$. We set $C_i = [x_i, x_{i+1}[, i \in \mathbb{Z}$. Let P_h and Q_h be the two interpolation operators defined, for all $v_h = \{v_h(x_i)\}_i \in (\mathbb{R}^3)^{\mathbb{Z}_h}$, by the functions*

$$Q_h v_h : \quad \mathbb{R} \to \mathbb{R}^3, \quad x \mapsto Q_h v_h(x) = v_h(x_i), \quad \forall x \in C_i, \forall i \in \mathbb{Z},$$

$$P_h v_h : \quad \mathbb{R} \to \mathbb{R}^3, \quad x \mapsto P_h v_h(x) = v_h(x_i) + D^+ v_h(x_i)(x - x_i), \quad \forall x \in C_i, \forall i \in \mathbb{Z}.$$

In all that follows we keep the notation of this definition. We have the following important lemma

Lemma 1.13 *1) Let $\{v_h\}_h$ be a sequence satisfying*

$$\begin{cases} v_h \in H_h^{-1}, \quad \forall h > 0, \\ \exists C > 0, \quad |v_h|_{H_h^{-1}} < C \end{cases}$$

Then the sequence $\{P_h v_h\}_h$ is bounded in $H^{-1}(\mathbb{R})$.
2) Let $l > 0$ and $\{v_h\}_h$ be a sequence satisfying

$$\begin{cases} h = \frac{l}{N}, \\ v_h \in P_{l,N}, \quad \forall N \in \mathbb{N}, \\ \exists C > 0, \quad |v_h|_{H_{l,N}^{-1}} < C, \quad \forall N \in \mathbb{N}. \end{cases}$$

Then the sequence $\{P_h v_h\}_h$ is bounded in $H^{-1}(\mathbb{T}^l)$.

Proof. 1) We have

$$\begin{aligned} \|P_h v_h\|_{H^{-1}(\mathbb{R})} &= \sup_{\varphi \in \mathcal{D}(\mathbb{R})} \frac{\langle P_h v_h, \varphi \rangle_{L^2(\mathbb{R})}}{\|\varphi\|_{H^1(\mathbb{R})}} \\ &\leq \sup_{\varphi \in \mathcal{D}(\mathbb{R})} \frac{\langle P_h v_h, P_h \varphi_h \rangle_{L^2(\mathbb{R})}}{\|\varphi\|_{H^1(\mathbb{R})}} + \sup_{\varphi \in \mathcal{D}(\mathbb{R})} \frac{\langle P_h v_h, \varphi - P_h \varphi_h \rangle_{L^2(\mathbb{R})}}{\|\varphi\|_{H^1(\mathbb{R})}}, \end{aligned} \tag{1.18}$$

with $\varphi_h = \{\varphi(x_i)\}_i$. Since

$$\|\varphi - P_h \varphi_h\|_{L^2(\mathbb{R})} \leq h \|(\varphi - P_h \varphi_h)'\|_{L^2(\mathbb{R})} \quad \text{(Poincaré)},$$

we have

$$\begin{aligned} \|\varphi\|^2_{H^1(\mathbb{R})} &= \|P_h \varphi_h\|^2_{H^1(\mathbb{R})} + \|\varphi - P_h \varphi_h\|^2_{H^1(\mathbb{R})} + 2 \int_{\mathbb{R}} (P_h \varphi_h).(\varphi - P_h \varphi_h) dx + 2 \int_{\mathbb{R}} (P_h \varphi_h)'.(\varphi - P_h \varphi_h)' dx \\ &\geq \|P_h \varphi_h\|^2_{H^1(\mathbb{R})} + \|\varphi - P_h \varphi_h\|^2_{H^1(\mathbb{R})} - 2h \|P_h \varphi_h\|_{L^2(\mathbb{R})} \|(\varphi - P_h \varphi_h)'\|_{L^2(\mathbb{R})}. \end{aligned}$$

Then there exists $h_0 > 0$ such that for all $h < h_0$, we have

$$\begin{aligned} \|\varphi\|^2_{H^1(\mathbb{R})} &\geq \frac{1}{2} \left(\|P_h \varphi_h\|^2_{H^1(\mathbb{R})} + \|\varphi - P_h \varphi_h\|^2_{H^1(\mathbb{R})} \right) \\ &\geq \frac{1}{2} \max \left(\|P_h \varphi_h\|^2_{H^1(\mathbb{R})}, \|\varphi - P_h \varphi_h\|^2_{H^1(\mathbb{R})} \right). \end{aligned}$$

We obtain by substituting in (1.18)

$$\|P_h v_h\|_{H^{-1}(\mathbb{R})} \leq \sup_{\varphi \in \mathcal{D}(\mathbb{R})} \frac{\langle P_h v_h, P_h \varphi_h \rangle_{L^2(\mathbb{R})}}{\frac{1}{\sqrt{2}} \|P_h \varphi_h\|_{H^1(\mathbb{R})}} + \sqrt{2} h \|P_h v_h\|_{L^2(\mathbb{R})}. \tag{1.19}$$

Next, we have

$$
\begin{aligned}
\|P_h\varphi_h\|^2_{H^1(\mathbb{R})} &= \sum_i \int_{x_i}^{x_{i+1}} \left| \frac{x_i - x}{h}\varphi(x_i) + \frac{x - x_i}{h}\varphi(x_{i+1}) \right|^2 dx + \sum_i h \left| \frac{\varphi(x_i) - \varphi(x_{i+1})}{h} \right|^2 dx \\
&= \sum_i \frac{h}{3} \left(|\varphi(x_i)|^2 + |\varphi(x_{i+1})|^2 + \varphi(x_{i+1})\varphi(x_i) \right) + |D^+\varphi_h|_h^2 \\
&\geq \sum_i \frac{h}{6} \left(|\varphi(x_i)|^2 + |\varphi(x_{i+1})|^2 \right) + |D^+\varphi_h|_h^2,
\end{aligned}
$$

from which we can write

$$
\|P_h\varphi_h\|^2_{H^1(\mathbb{R})} \geq \frac{1}{3}|\varphi_h|_h^2 + |D^+\varphi_h|_h^2 \geq \frac{1}{3}|\varphi_h|^2_{H^1_h}. \tag{1.20}
$$

We have on the one hand

$$
\begin{aligned}
\langle P_h v_h, P_h\varphi_h\rangle_{L^2(\mathbb{R})} &= \sum_i \int_{x_i}^{x_{i+1}} (v_h(x_i) + D^+v_h(x_i)(x - x_i)).(\varphi(x_i) + D^+\varphi_h(x_i)(x - x_i))dx \\
&= (v_h, \varphi_h)_h + \frac{h}{2}(v_h, D^+\varphi_h)_h + \frac{h}{2}(D^+v_h, \varphi_h)_h + \frac{h^2}{3}(D^+v_h, D^+\varphi_h)_h \\
&= (v_h, \varphi_h)_h + \frac{h}{2}(v_h, D^+\varphi_h)_h - \frac{h}{2}(v_h, D^-\varphi_h)_h + \frac{h^2}{3}(D^+v_h, D^+\varphi_h)_h \\
&\leq (v_h, \varphi_h)_h + h|v_h|_h|D^+\varphi_h|_h + \frac{h^2}{3}|D^+v_h|_h|D^+\varphi_h|_h, \qquad (1.21)
\end{aligned}
$$

and on the other hand

$$
\begin{aligned}
\|P_h v_h\|^2_{L^2(\mathbb{R})} &= \sum_i \int_{x_i}^{x_{i+1}} |v_h(x_i) + D^+v_h(x_i)(x - x_i)|^2 dx \\
&\leq 2\sum_i \int_{x_i}^{x_{i+1}} (|v_h(x_i)|^2 + |D^+v_h(x_i)|^2(x - x_i)^2)dx \\
&= 2|v_h|_h^2 + \frac{2h^2}{3}|D^+v_h|_h^2. \qquad (1.22)
\end{aligned}
$$

Then by combining (1.19), (1.20), (1.21) and (1.22) we get

$$
\begin{aligned}
\|P_h v_h\|_{H^{-1}(\mathbb{R})} &\leq \sup_{\varphi\in\mathcal{D}(\mathbb{R})} \frac{(v_h, \varphi_h)_h + h|v_h|_h|D^+\varphi_h|_h + \frac{h^2}{3}|D^+v_h|_h|D^+\varphi_h|_h}{\frac{1}{\sqrt{6}}|\varphi_h|_{H^1_h}} + 2h|v_h|_h + \frac{2h^2}{\sqrt{3}}|D^+v_h|_h \\
&\leq \sqrt{6}(|v_h|_{H_h^{-1}} + h|v_h|_h + \frac{h^2}{3}|D^+v_h|_h) + 2h|v_h|_h + \frac{2h^2}{\sqrt{3}}|D^+v_h|_h \\
&\leq \sqrt{6}|v_h|_{H_h^{-1}} + (\sqrt{6} + 2)h|v_h|_h + 2(\frac{\sqrt{6}}{3} + \frac{2}{\sqrt{3}})h|v_h|_h{}^2 \\
&\leq \sqrt{6}|v_h|_{H_h^{-1}} + (\sqrt{6} + 2 + 2\frac{2 + \sqrt{2}}{\sqrt{3}})h|v_h|_h \\
&\leq C|v_h|_{H_h^{-1}},
\end{aligned}
$$

since

$$\begin{aligned} |v_h|_{H_h^{-1}} &= \sup_{u_h} \frac{(v_h,u_h)_h}{|u_h|_{H_h^1}} \\ &\geq \sup_{u_h} \frac{(v_h,u_h)_h}{[|u_h|_h^2 + \frac{4}{h^2}|u_h|_h^2]^{\frac{1}{2}}} \\ &= \frac{h}{\sqrt{h^2+4}} \sup_{u_h} \frac{(v_h,u_h)_h}{|u_h|_h} \\ &\geq \frac{h}{\sqrt{h_0^2+4}} |v_h|_h, \quad \forall h \geq h_0. \end{aligned}$$

The proof of 2) is similar to that of 1). ■

The following lemma shows that the space L_h^2, equipped with the norm $|.|_{H_h^1}$, is continuously embedded in L_h^∞.

Lemma 1.14 *There exist two constants $C_1, C_2 > 0$ such that for all $h > 0$ and $v_h \in L_h^2$ we have*

$$C_2|v_h|_h \leq \|P_h v_h\|_{L^2(\mathbb{R})} \leq C_1|v_h|_h.$$

Proof. Since

$$\begin{aligned} \int_{x_i}^{x_{i+1}} |u_h(x_i) + D^+u_h(x_i)(x-x_i)|^2 dx &= h|u_h(x_i)|^2 + \frac{1}{2}h^2 u_h(x_i) D^+u_h(x_i) + \frac{1}{3}h^3|D^+u_h(x_i)|^2 \\ &= \frac{5}{6}h|u_h(x_i)|^2 - \frac{1}{6}h u_h(x_i) D^+u_h(x_i) + \frac{1}{3}h|u_h(x_{i+1})|^2, \end{aligned}$$

and

$$\frac{3}{4}|u_h(x_i)|^2 + \frac{1}{4}|u_h(x_{i+1})|^2 \leq \frac{5}{6}|u_h(x_i)|^2 - \frac{1}{6}u_h(x_i)D^+u_h(x_i) + \frac{1}{3}|u_h(x_{i+1})|^2 \leq \frac{11}{12}|u_h(x_i)|^2 + \frac{5}{12}|u_h(x_{i+1})|^2,$$

we have

$$|v_h|_h^2 \leq \|P_h v_h\|_{L^2(\mathbb{R})}^2 \leq \frac{4}{3}|v_h|_h^2.$$

■

Corollary 1.15 *If $v_h \in L_h^2 \subset L_h^\infty$, then $P_h v_h \in H^1(\mathbb{R})$ and there exists $C > 0$ (which does not depend on h) such that*

$$|v_h|_{L_h^\infty} \leq C|v_h|_{H_h^1}.$$

Proof. Since $\frac{dP_h v_h}{dx} = Q_h D^+ v_h \in L^2(\mathbb{R})$, we have $P_h v_h \in H^1(\mathbb{R})$ (Lemma 1.9). On the other hand, we have

$$\begin{aligned} \|P_h v_h\|_{L^\infty(\mathbb{R})} &= \sup_{i\in\mathbb{Z}} \sup_{x\in[x_i,x_{i+1}[} |u_h(x_i) + D^+u_h(x_i)(x-x_i)| \\ &= \sup_{i\in\mathbb{Z}} \max(|u_h(x_i)|, |u_h(x_{i+1})|) \\ &= |v_h|_{L_h^\infty}. \end{aligned}$$

The space $L^\infty(\mathbb{R})$ is continuously embedded in the space $H^1(\mathbb{R})$ (Sobolev) and there exists $\tilde{C} > 0$ such that

$$\|v|_{L^\infty(\mathbb{R})} \leq \tilde{C}\|v|_{H^1(\mathbb{R})}, \quad \forall v \in H^1(\mathbb{R}).$$

Consequently,

$$\begin{aligned}|v_h|^2_{L^\infty_h} &= \|P_h v_h\|^2_{L^\infty(\mathbb{R})}\\ &\le \tilde{C}^2\|P_h v_h\|^2_{H^1(\mathbb{R})}\\ &\le \tilde{C}^2(C_1^2|v_h|_h^2 + \|Q_h D^+ v_h\|^2_{L^2(\mathbb{R})})\\ &\le C^2|v_h|^2_{H^1_h}.\end{aligned}$$

■

2 Proofs of principal theorems

Let us first show some important properties.

2.1 Convergence properties

Lemma 2.1 *1) Let $\{v_h\}_h$ be a sequence satisfying*

$$v_h \in L^2_h, \quad \forall h,$$

and

$$\exists C > 0, \quad |v_h|_h \le C. \tag{2.1}$$

Then the sequence $\{P_h v_h - Q_h v_h\}_h$ converges weakly to zero in $L^2(\mathbb{R})$.
2) Let $l > 0$ and $\{v_h\}_h$ be a sequence satisfying

$$\begin{cases} h = \frac{l}{N}, \\ v_h \in P_{l,N}, \quad \forall N \in \mathbb{N}, \end{cases}$$

and

$$\exists C > 0, \quad |v_h|_{l,N} \le C. \tag{2.2}$$

Then $\{P_h v_h - Q_h v_h\}_h$ converges weakly to zero in $L^2(\mathbb{T}^l)$. Moreover, if $\{Q_h v_h\}_h$ converges to v in L^2 ($L^2(\mathbb{R})$ or $L^2(\mathbb{T}^l)$), then $\{P_h v_h\}_h$ converges to the same limit in L^2.

Proof. 1) We write

$$\begin{aligned}\|P_h v_h - Q_h v_h\|^2_{L^2(\mathbb{R})} &= \sum_i \int_{x_i}^{x_{i+1}} |D^+ v_h(x_i)|^2 (x - x_i)^2 dx\\ &\le \frac{1}{3}h^3 \sum_i |D^+ v_h(x_i)|^2\\ &= \frac{1}{3}h^2 |D^+ v_h|_h^2\\ &\le \frac{4}{3}|v_h|_h^2\\ &\le \frac{4}{3}C^2. \end{aligned}\tag{2.3}$$

Furthermore, for all $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned}|\langle P_h v_h - Q_h v_h, \varphi\rangle_{L^2(\mathbb{R})}| \le\ & |\langle P_h v_h - Q_h v_h, Q_h \varphi\rangle_{L^2(\mathbb{R})}|\\ &+ \|P_h v_h - Q_h v_h\|_{L^2(\mathbb{R})}\|\varphi - Q_h \varphi_h\|_{L^2(\mathbb{R})},\end{aligned}\tag{2.4}$$

where $\varphi_h = \{\varphi_(x_i)\}_i$. We have on the one hand

$$\begin{aligned}
\|\varphi - Q_h\varphi_h\|^2_{L^2(\mathbb{R})} &= \sum_i \int_{x_i}^{x_{i+1}} |\varphi(x) - \varphi(x_i)|^2 dx \\
&= \sum_i \int_{x_i}^{x_{i+1}} |\int_{x_i}^{x} \varphi'(s) ds|^2 dx \\
&\leq \sum_i \int_{x_i}^{x_{i+1}} (\int_{x_i}^{x} |\varphi'(s)|^2 ds)(x - x_i) dx \\
&\leq \frac{h^2}{2} \sum_i \int_{x_i}^{x} |\varphi'(s)|^2 ds \\
&= \frac{h^2}{2} \|\varphi'\|^2_{L^2(\mathbb{R})}. \qquad (2.5)
\end{aligned}$$

On the other hand, we can write

$$\begin{aligned}
|\langle P_h v_h - Q_h v_h, Q_h\varphi\rangle_{L^2(\mathbb{R})}| &= \frac{h}{2}|\langle D^+ v_h, \varphi_h\rangle_h| \\
&= \frac{h}{2}|\langle v_h, D^-\varphi_h\rangle_h| \\
&\leq \frac{h}{2}|v_h|_h |D^-\varphi_h|_h \\
&\leq \frac{1}{2}C \left[h \sum_i |\int_{x_{i-1}}^{x_i} \varphi'(s) ds|^2 \right]^{\frac{1}{2}} \\
&\leq \frac{1}{2}Ch\|\varphi'\|_{L^2(\mathbb{R})}. \qquad (2.6)
\end{aligned}$$

Then combining (2.3), (2.4), (2.5) and (2.6), we obtain

$$|\langle P_h v_h - Q_h v_h, \varphi\rangle_{L^2(\mathbb{R})}| \leq (\frac{2}{\sqrt{6}} + \frac{1}{2})C\|\varphi'\|_{L^2(\mathbb{R})} h.$$

Thus the proof of 1) is completed. The proof of 2) is similar to that of 1). To prove the strong convergence property, let $v \in L^2$, then it suffices to note that

$$\begin{aligned}
\|P_h v_h - Q_h v_h\|^2_{L^2} &= \sum_i \int_{x_i}^{x_{i+1}} |D^+ v_h(x_i)|^2 (x - x_i)^2 dx \\
&= \frac{1}{3} h^3 \sum_i |D^+ v_h(x_i)|^2 \\
&= \frac{1}{3} \|\tau_{-h} Q_h v_h - Q_h v_h\|^2_{L^2},
\end{aligned}$$

with $\tau_h w = w(\cdot - h)$, and

$$\begin{aligned}
\|\tau_{-h} Q_h v_h - Q_h v_h\|_{L^2} &\leq \|\tau_{-h} Q_h v_h - v\|_{L^2} + \|Q_h v_h - v\|_{L^2} \\
&\leq \|\tau_h v - v\|_{L^2} + 2\|Q_h v_h - v\|_{L^2}.
\end{aligned}$$

Thus the convergence $\lim_{h\to 0} \|\tau_h v - v\|_{L^2} = 0$ completes the proof. ■

Lemma 2.2 *1) Let $v \in H^{-1}(\mathbb{R})$, and $\{v_h\}_h$ be a sequence such that the sequence $\{Q_h v_h\}_h$ converges to v in $H^{-1}(\mathbb{R})$ weak star. Then the sequence $\{P_h v_h\}_h$ converges to v in $H^{-1}(\mathbb{R})$ weak star.*
2) Let $l > 0$, $v^l \in H^{-1}(\mathbb{T}^l)$ and $\{v_h\}_h$ be a sequence satisfying

$$\begin{cases} h = \frac{l}{N}, \\ v_h \in l_N, \quad \forall N \in \mathbb{N}, \\ Q_h v_h \to v^l \quad in \quad H^{-1}(\mathbb{T}^l) \quad weak\ star. \end{cases}$$

Then $\{P_h^l v_h\}_h$ converges to v^l in $H^{-1}(\mathbb{T}^l)$ weak star.

Proof. 1) First, we prove that $P_h v_h \in H^{-1}(\mathbb{R}), \forall h$. To this end, we first write

$$P_h v_h = Q_h v_h + (P_h - Q_h) v_h.$$

Then it suffices to prove that $(P_h - Q_h) v_h \in H^{-1}(\mathbb{R}), \forall h$. Let $\varphi \in \mathcal{D}(\mathbb{R})$, and $\varphi_h = \{\varphi(x_i)\}_i$. We have

$$\begin{aligned}
|\langle P_h v_h - Q_h v_h, \varphi\rangle_{L^2(\mathbb{R})}| &\leq |\langle P_h v_h - Q_h v_h, Q_h\varphi\rangle_{L^2(\mathbb{R})}| + |\langle P_h v_h - Q_h v_h, \varphi - Q_h\varphi\rangle_{L^2(\mathbb{R})}| \\
&\leq \frac{h}{2}|(D^+ v_h, \varphi_h)_h| + |\sum_i \int_{x_i}^{x_{i+1}} (D^+ v_h(x_i). \int_{x_i}^{x} \varphi'(s)ds)(x - x_i)dx| \\
&\leq \frac{h}{2}|(v_h, D^- \varphi_h)_h| + |h^2\sqrt{h}\sum_i |D^+ v_h(x_i)|. \int_{x_i}^{x_{i+1}} |\varphi'(x)|^2 dx| \\
&\leq \frac{h}{2}|v_h|_h |D^-\varphi_h|_h + h^2 |D^+ v_h|_h \|\varphi'\|_{L^2(\mathbb{R})} \\
&\leq \frac{h}{2}|v_h|_h \|\varphi'\|_{L^2(\mathbb{R})} + 2h|v_h|_h \|\varphi'\|_{L^2(\mathbb{R})} \\
&\leq \frac{5}{2}h|v_h|_h \|\varphi'\|_{L^2(\mathbb{R})},
\end{aligned}$$

where the sequence $\{h|v_h|_h\}_h$ is bounded. Indeed, the sequence $\{Q_h v_h\}_h$ converges to v in $H^{-1}(\mathbb{R})$ weak star. Then there exists $C > 0$ such that $\|Q_h v_h\|_{H^{-1}(\mathbb{R})} \leq C$ for all h, hence we have

$$\frac{\langle Q_h v_h, R_h^N v_h\rangle_{L^2(\mathbb{R})}}{\|R_h^N v_h\|_{H^1(\mathbb{R})}} \leq C, \quad \forall h, \forall N \in \mathbb{N}, \tag{2.7}$$

where $R_h^N v_h$ is a piecwise function with compact support (hence $R_h^N v_h \in H^1(\mathbb{R})$) such that

$$\begin{cases} \langle Q_h v_h, R_h^N v_h\rangle_{L^2(\mathbb{R})} = h\sum_{-N}^{N} |v_i|^2, \\ \|R_h^N v_h\|_{H^1(\mathbb{R})}^2 \leq h^{-1}\sum_{-N}^{N} |v_i|^2. \end{cases} \tag{2.8}$$

For example, we can take

$$R_h^N v_h = Q_h \tilde{v}_h + \sum_i D^+ \tilde{v}_h \chi(x - x_i),$$

where $\tilde{v}_h = \{\tilde{v}_h(x_i)\}_i$ with $\tilde{v}_h(x_i) = \begin{cases} v_h(x_i), & |i| \leq N \\ 0, & |i| > N, \end{cases}$ and χ is given by

$$\chi(x) = \begin{cases} 0, \quad x < 0 \quad \text{or} \quad x > h \\ -\frac{3}{2}x, \quad x \in [0, \frac{h}{3}[\\ \frac{3}{2}(x - \frac{2h}{3}), \quad x \in [\frac{h}{3}, \frac{2h}{3}[\\ 3(x - \frac{2h}{3}), \quad x \in [\frac{2h}{3}, h[. \end{cases}$$

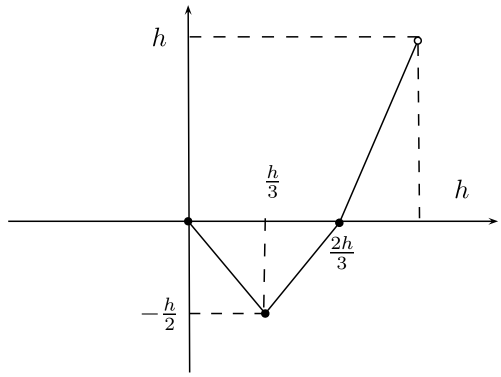

Figure 2: The function χ.

Since

$$\frac{h\sum_N^N |v_i|^2}{\left[h^{-1}\sum_{-N}^N |v_i|^2\right]^{\frac{1}{2}}} \leq C, \quad \forall h, \forall N \in \mathbb{N},$$

we get $h|v_h|_h \leq C, \forall h$. Finally, we have $\|(P_h - Q_h)v_h\|_{H^{-1}(\mathbb{R})} \leq C, \forall h$, then $\|P_h v_h\|_{H^{-1}(\mathbb{R})} \leq C, \forall h$.

To show that $\{P_h v_h\}_h$ converges to v in $H^{-1}(\mathbb{R})$ weak star, we need to prove that

$$P_h v_h \rightharpoonup v, \quad \text{in} \quad \mathcal{D}'(\mathbb{R}).$$

To this end, let $\varphi \in \mathcal{D}(\mathbb{R})$. We denote $\tau_h\varphi = \frac{1}{2}(\varphi + \varphi(. - h))$. Then we have

$$\begin{aligned}
\langle P_h v_h, \varphi\rangle_{L^2(\mathbb{R})} &= \sum_i \int_{x_i}^{x_{i+1}} (v_h(x_i) + D^+v_h(x_i)(x - x_i)) \cdot \varphi(x)dx \\
&= \sum_i \int_{x_i}^{x_{i+1}} \left(\frac{v_h(x_i) + v_h(x_{i+1})}{2} + D^+v_h(x_i)(x - x_i - \frac{h}{2})\right) \cdot \varphi(x)dx \\
&= \langle Q_h v_v, \tau_h\varphi\rangle_{L^2(\mathbb{R})} + \sum_i \int_{x_i}^{x_{i+1}} \left(D^+v_h(x_i)(x - x_i - \frac{h}{2}) \cdot \int_{x_i}^{x} \varphi(t)dt\right) dx \\
&= \langle Q_h v_v, \tau_h\varphi\rangle_{L^2(\mathbb{R})} + \int_0^h \int_0^s (s - \frac{h}{2}) \left(\sum_i D^+v_h(x_i) \cdot \varphi'(x_i + \rho)\right) d\rho ds \\
&= \langle Q_h v_v, \tau_h\varphi\rangle_{L^2(\mathbb{R})} + \frac{1}{h}\int_0^h \int_0^s (s - \frac{h}{2}) \left(\sum_i v_h(x_i) \cdot (\varphi'(x_{i-1} + \rho) - \varphi'(x_i + \rho))\right) d\rho ds \\
&= \langle Q_h v_v, \tau_h\varphi\rangle_{L^2(\mathbb{R})} + \frac{1}{h}\int_0^h \int_0^s (s - \frac{h}{2}) \left(\sum_i v_h(x_i) \cdot \int_{x_i}^{x_{i+1}} \varphi''(x + \rho)dx\right) d\rho ds,
\end{aligned}$$

where

$$\left\{\begin{array}{l} Q_h v_h \to v, \quad \text{dans} \quad H^{-1}(\mathbb{R}) \quad \text{weak star}, \\ \tau_h\varphi \to \varphi, \quad \text{in} \quad H^1(\mathbb{R}); \end{array}\right. \tag{2.9}$$

hence $\langle Q_h v_v, \tau_h\varphi\rangle_{L^2(\mathbb{R})} \to \langle v, \varphi\rangle_{L^2(\mathbb{R})}$. On the other hand, we have

$$\sum_i v_i \cdot \int_{x_i}^{x_{i+1}} \varphi''(x + \rho)dx \leq |v_h|_h \|\varphi''\|_{L^2(\mathbb{R})}.$$

It follows that

$$\left|\frac{1}{h}\int_0^h\int_0^s (s-\frac{h}{2})\left(\sum_i v_h(x_i)\cdot\int_{x_i}^{x_{i+1}}\varphi''(x+\rho)dx\right)d\rho ds\right| \leq h^2|v_h|_h\|\varphi''\|_{L^2(\mathbb{R})}$$
$$\leq Ch\|\varphi''\|_{L^2(\mathbb{R})},$$

and thus the proof of 1) is completed. The proof of 2) is similar to that of 1). ■

We establish now a compactness result which will be useful in the proofs of principal theorems.

Lemma 2.3 *Let $T>0$ and $\{u_h\}_h$ be a sequence whose elements belong to the space $L^\infty(0,T,H^1_{loc}(\mathbb{R}))$. Assume that $\{u_h\}_h$ is bounded in $L^\infty(0,T,H^1_{loc}(\mathbb{R}))$ and further the sequence $\{\partial_t u_h\}_h$ is bounded in $L^\infty(0,T,H^{-1}(\mathbb{R}))$. Then we can extract from $\{u_h\}_h$ a subsequence converging in $\mathcal{C}(0,T,L^2_{loc}(\mathbb{R}))$.*

Proof. The proof is a consequence of the following proposition

Proposition 2.4 *([7]) Let X,B and Y be three Banach spaces such that $X\subset B\subset Y$. Assume that the embedding $X\subset B$ is compact. Let F be some bounded subset in $L^\infty(0,T,X)$ such that the subset $G=\{\partial_t f,\quad f\in F\}$ is bounded in $L^r(0,T,Y)$, with $1<r\leq\infty$. Then F is relatively compact in $\mathcal{C}(0,T,B)$.*

We denote by $I_k=]-k,k[$ with $k\in\mathbb{N}$. We consider the three spaces $X=H^1(I_k)$, $B=L^2(I_k)$ and $Y=H^{-1}(I_k)$. The embedding $H^1(I_k)\subset L^2(I_k)$ is compact, hence using previous proposition, we can extract from $\{u_h\}_h$ a subsequence (depending on k) which converges in $\mathcal{C}(0,T,L^2(I_k))$. Thus the diagonal subsequence of Cantor converges in $C(0,T,L^2(I_k))$ for all $k\in\mathbb{N}$. ■

2.2 Proof of Theorem 1.1

We construct a weak solution for the system

$$\begin{cases} \partial_t u = \partial_x\left(u\wedge g(x)\partial_x u\right) = u\wedge\partial_x\left(g\partial_x u\right), \quad t\geq 0, \quad x\in\mathbb{R}, \\ u(0,x)=u_0(x), \end{cases} \tag{2.10}$$

as a limit, when $h\to 0$, of a sequence $\{u_h\}_h$ of solutions for the semi-discrete system

$$\begin{cases} \frac{du_h}{dt} = D^+\left(u_h\wedge g_h D^- u_h\right) = u_h\wedge D^+\left(g_h D^- u_h\right), \quad t\geq 0, \\ u_h(0)=u_h^0, \end{cases} \tag{2.11}$$

where $u_h^0=\{u_h^0(x_i)\}_i\in(\mathbb{R}^3)^{\mathbb{Z}_h}$ with $|u_h^0(x_i)|=1$ and $g_h=\{g(t,x_i)\}_i$.

Proposition 2.5 *Let $u_h^0=\{u_h^0(x_i)\}_i\in(\mathbb{R}^3)^{\mathbb{Z}_h}$ be such that $|u_h^0(x_i)|=1$, and $D^+u_h^0\in L^2_h$. Let $g\in W^{1,\infty}(\mathbb{R}^+,L^\infty(\mathbb{R}))$ such that there exists $\alpha>0$ with $g\geq\alpha$. Then equation (2.11) has a global solution $u_h=\{u_h(x_i)\}_i\in\mathcal{C}^1(\mathbb{R}^+,(\mathbb{R}^3)^{\mathbb{Z}_h})$ with $|u_h(t,x_i)|=1$ and $D^+u_h\in\mathcal{C}^1(\mathbb{R}^+,L^2_h)$.*

Proof. Let $h>0$. We endow the space

$$E_h=\{v_h\in(\mathbb{R}^3)^{\mathbb{Z}_h},\quad v_h\in L^\infty_h\quad\text{and}\quad D^+v_h\in L^2_h\},$$

with the norm

$$\|v_h\|_h=|v_h|_{L^\infty_h}+|D^+v_h|_h,\quad\forall v_h\in E_h,$$

for which the space $(E_h,\|.\|_h)$ is a Banach space. Let $R>0$ and $\Omega=B_{E_h}(u_h^0,R)$. We define the function

$$\begin{cases} F:\Omega\to E_h:\quad v_h\mapsto F(v_h), \\ (F(v_h))(x_i)=D^+(v_h\wedge(g_hD^-v_h))(x_i)=\frac{1}{h^2}\left(g(x_i)v_h(x_i)\wedge v_h(x_{i-1})-g_h(i+1)v_h(x_{i+1})\wedge v_h(x_i)\right). \end{cases}$$

In what follows we denote $\beta = \|g\|_{L^\infty(\mathbb{R})}$. Let $u_h, v_h \in \Omega$. We have on the one hand

$$\begin{aligned} F(u_h)(x_i) - F(v_h)(x_i) &= \frac{g_h(x_i)}{h^2}\left[u_h(x_i) \wedge (u_h(x_{i-1}) - v_h(x_{i-1})) + (u_h(x_i) - v_h(x_i)) \wedge v_h(x_i)\right] \\ &+ \frac{g_h(x_{i+1})}{h^2}\left[(v_h(x_{i+1}) - u_h(x_{i+1})) \wedge v_h(x_i) + u_h(x_{i+1})(v_h(x_i) - u_h(x_i))\right], \end{aligned}$$

then

$$|F(v_h) - F(u_h)|_{L_h^\infty} \leq \frac{4\beta}{h^2}(R + \|u_h^0\|_h)|v_h - u_h|_{L_h^\infty}, \tag{2.12}$$

On the other hand, using Lemma 1.9 we get

$$\begin{aligned} |D^+(F(v_h) - F(u_h))|_h &= |D^+[D^+(g_h(v_h \wedge D^- v_h - u_h \wedge D^- u_h))]|_h \\ &\leq \frac{4\beta}{h^2}|v_h \wedge D^- v_h - u_h \wedge D^- u_h|_h \\ &\leq \frac{4\beta}{h^2}(|v_h|_{L_h^\infty}|D^-(v_h - u_h)|_h + |D^- u_h|_h|D^-(v_h - u_h)|_h \\ &\leq \frac{4\beta}{h^2}(R + \|u_h^0\|_h)(|D^-(v_h - u_h)|_h + |D^-(v_h - u_h)|_h. \end{aligned}$$

It follows that

$$|F(v_h) - F(u_h)|_h \leq \frac{4\beta}{h^2}(R + \|u_h^0\|)\|v - u\|_h, \tag{2.13}$$

where, combining (2.12) et (2.13), we deduce that

$$\|F(v_h) - F(u_h)\|_h \leq \frac{8\beta}{h^2}(R + \|u_h^0\|_h)\|v_h - u_h\|_h.$$

Thus F is locally Lipschitz-continuous and Cauchy-Lipschitz theorem holds. Hence there exists $T^* \in \mathbb{R}_*^+ \cup \{+\infty\}$ and $u_h : [0, T^*[\to (E_h, \|\|_h)$ satisfying (2.11). Taking the usual $\mathbb{R}^3-$scalar product in (2.11) with u_h, we find that $\frac{d}{dt}|u_h(t, x_i)| = 0$, hence $|u_h(t, x_i)| = |u_h^0(x_i)| = 1$ on $[0, T^*[$. Then we have $\|u_h\|_h = 1 + |D^+ u_h|_h$ which gives T^* the following characterisation

$$\limsup_{t \to T^*} |D^+ u_h(t)|_h = +\infty \quad \text{if} \quad T^* < +\infty.$$

Taking the L_h^2-scalar product in (2.11) with $D^+(g_h D^- u_h)$, we get

$$\frac{d}{dt}\sum_i g_h|D^- u_h(x_i)|^2(t, x_i) = \sum_i \partial_t g(t, x_i)|D^- u_h(x_i)|^2(t, x_i),$$

from which and by using the Grönwall lemma, we obtain

$$|D^+ u_h(t)|_h = |D^- u_h(t)|_h \leq \sqrt{\frac{\beta}{\alpha}}|D^+ u_h^0|_h \exp\left(\frac{\beta_1 t}{2\alpha}\right) \quad \forall t \in [0, T^*[.$$

This means that $\lim_{t \to T^*} \|u_h\|_h \neq +\infty$, hence we finally get $T^* = +\infty$. ■

In what follows, we consider $T > 0$ fixed. For each sequence $\{v_h\}_h$ of elements in L_h^2, we have $\left(\frac{du_h}{dt}, v_h\right)_h = -\left(u_h \wedge g_h D^- u_h, D^- v_h\right)_h$, hence

$$\left|\frac{du_h}{dt}\right|_{H_h^{-1}} \leq \beta\sqrt{\frac{\beta}{\alpha}}|D^+ u_h^0| \exp\left(\frac{\beta_1 t}{2\alpha}\right). \tag{2.14}$$

Let $\{u_h^0\}_h$ be a sequence satisfying

$$\left\{\begin{array}{lll} Q_h u_h^0 \to u_0 & \text{in} & L_{loc}^2(\mathbb{R}), \\ Q_h D^+ u_h^0 \to \frac{du_0}{dx} & \text{in} & L^2(\mathbb{R}). \end{array}\right. \tag{2.15}$$

Then we have

Lemma 2.6 *The sequence of solutions $\{u_h\}_h$ satisfying (2.11), with initial data $\{u_h^0\}_h$ satisfying (2.15), has the properties*
i) $\{\partial_t P_h u_h\}_h$ is bounded in $L^\infty(0,T,H^{-1}(\mathbb{R}))$.
ii) $\{P_h u_h\}_h$ is bounded in $L^\infty(0,T,H^1_{loc}(\mathbb{R}))$.

Proof. Property i) is an immediate result of (2.14) and Lemma 1.13.
ii) Let $I=[a,b]\subset\mathbb{R}$. Then we have

$$\begin{aligned}
\|P_h u_h\|^2_{H^1(I)} &= \sum_i \int_{x_i}^{x_{i+1}} \left|\frac{x_i-x}{h}u_h(x_i)+\frac{x-x_i}{h}u_h(x_{i+1})\right|^2 dx + \sum_i h\left|\frac{u_h(x_i)-u_h(x_{i+1})}{h}\right|^2 dx \\
&\leq \sum_i \frac{h}{3}\big(|u_h(x_i)|^2+|u_h(x_{i+1})|^2+u_h(x_i)u_h(x_{i+1})\big)+|D^+u_h|_h^2 \\
&\leq b-a+2h+|D^+u_h^0|_h^2,
\end{aligned}$$

where the sequence $\{|D^+u_h^0|_h\}_h$ is bounded, since $Q_hD^+u_h^0\to\frac{du_0}{dx}$ in $L^2(\mathbb{R})$. ■

Since $\{P_hu_h\}_h$ and $\{\partial_tP_hu_h\}_h$ are bounded in $L^\infty(0,T,H^1_{loc}(\mathbb{R}))$ and $L^\infty(0,T,H^{-1}(\mathbb{R}))$ respectively and in view of Lemma 2.3, there exists a subsequence $\{u_h\}_h$ and u such that $\{P_hu_h\}_h$ converges to u in $L^2(0,T,L^2_{loc}(\mathbb{R}))$ and almost everywhere. Moreover, $\{\partial_tP_hu_h\}_h$ converges to ∂_tu in $L^\infty(0,T,H^{-1}(\mathbb{R}))$ weak star. The sequence $\{Q_hu_h\}$ converges also to u almost everywhere. To show that the second member $\{P_hD^+(u_h\wedge g_hD^-u_h)\}_h$ converges to $\partial_x(u\wedge g(x)\partial_xu)$, we note first that by Lemma 2.1, the two sequences $\{P_h(u_h\wedge g_hD^-u_h)\}_h$ and $\{Q_h(u_h\wedge g_hD^-u_h)\}_h$ converge to the same limit in $L^\infty(0,T,L^2(\mathbb{R}))$ weak star. Since

$$Q_h(u_h\wedge g_hD^-u_h)=Q_hu_h\wedge(Q_hg_hQ_hD^-u_h),$$

and

$$\begin{cases} Q_hg_h\to g \quad \text{almost everywhere,}\\ Q_hu_h\to u \quad \text{almost everywhere,}\\ Q_hD^-u_h\to\partial_xu \quad \text{in} \quad L^\infty(0,T,L^2(\mathbb{R})) \quad \text{weak star,}\end{cases} \tag{2.16}$$

we have

$$Q_h(u_h\wedge g_hD^-u_h)\to u\wedge(g\partial_xu) \quad \text{in} \quad L^\infty(0,T,L^2(\mathbb{R})) \quad \text{weak star,}$$

and

$$\begin{cases} P_h(u_h\wedge g_hD^-u_h)\to u\wedge(g\partial_xu) \quad \text{in} \quad L^\infty(0,T,L^2(\mathbb{R})) \quad \text{weak star,}\\ \partial_xP_h(u_h\wedge g_hD^-u_h)\to\partial_x\left(u\wedge(g\partial_xu)\right) \quad \text{in} \quad L^\infty(0,T,H^{-1}(\mathbb{R})) \quad \text{weak star.}\end{cases} \tag{2.17}$$

It is clear that

$$Q_hD^+\big(u_h\wedge g_hD^-u_h\big)=\partial_xP_h(u_h\wedge g_hD^-u_h),$$

then using lemma 2.2, the sequence $\{P_hD^+(u_h\wedge g_hD^-u_h)\}_h$ converges to $\partial_x(u\wedge(g\partial_xu))$ in $L^\infty(0,T,H^{-1}(\mathbb{R}))$ weak star.

When $g=g(x)$ does not depend on time, we have

$$\frac{d}{dt}\int_{\mathbb{R}} g(x)|\partial_xu(t,x)|^2dx=0,$$

then

$$\|\partial_xu(t)\|^2_{L^2(\mathbb{R})}\leq\frac{\|g\|_{L^\infty(\mathbb{R})}}{\alpha}\left\|\frac{du_0}{dx}\right\|_{L^2(\mathbb{R})},$$

and $u\in L^\infty(\mathbb{R}^+,H^1_{loc}(\mathbb{R}))$. Thus the proof of Theorem 1.1 is completed.

2.3 Proof of Theorem 1.2

In this proof we use, without details, the same techniques of previous proof. Let $l > 0$. We construct a solution $u \in L^\infty(\mathbb{R}^+, H^1(\mathbb{T}^l, S^2))$ for the system

$$\begin{cases} \partial_t u = \partial_x (u \wedge g\partial_x u) = u \wedge \partial_x (g\partial_x u), \quad t \geq 0, \quad x \in \mathbb{T}^l, \\ u(0,x) = u_0(x). \end{cases} \tag{2.18}$$

as a limit, when $h \to 0$, of a sequence $\{u_h = \{u_h(x_i)\}_i \in P_{l,N}\}_h$ (with $h = \frac{l}{N}$) of solutions for the semi-discrete system

$$\begin{cases} \frac{du_h}{dt} = D^+ (u_h \wedge g_h D^- u_h) = u_h \wedge D^+ (g_h D^- u_h), \quad t > 0, \\ u_h(0) = u_h^0, \\ u_h(t, x_0) = u_h(t, x_N), \quad t \geq 0, \end{cases} \tag{2.19}$$

with $|u_h(x_i)^0| = 1$, and $g_h = \{g(x_i)\}_i$ such that $g(t, x_0) = g(t, x_N)$.

Proposition 2.7 *Let $u_h^0 \in P_{l,N}$ (with $h = \frac{l}{N}$) be such that $|u_h^0(x_i)| = 1$, and $g \in W^{1,\infty}(\mathbb{R}^+, L^\infty(\mathbb{T}^l))$ be such that there exists $\alpha > 0$ with $g \geq \alpha$. Then there exists a solution $u_h = \{u_h(x_i)\}_i \in \mathcal{C}^1(\mathbb{R}^+, P_{l,N})$ for (2.19) with $|u_h(t, x_i)| = 1$ for every i.*

Proof. Let $l > 0$ and $N \in \mathbb{N}$. We denote $h = \frac{l}{N}$. We endow the space $P_{l,N}$ by the norm

$$|v_h|_{L_h^\infty} = \sup_{i \in \mathbb{Z}} |v_h(x_i)|, \quad \forall v_h \in P_{l,N},$$

which makes $(P_{l,N}, |.|_{L_h^\infty})$ a Banach space. Let $R > 0$ and $\Omega = B_{P_{l,N}}(u_h^0, R)$. We define the function $F : \Omega \to P_{l,N}$ by

$$\begin{aligned} (F(v_h))(x_i) &= D^+(v_h \wedge (g_h D^- v_h))(x_i) \\ &= \frac{1}{h^2} (g_h(x_i)v_h(x_i) \wedge v_h(x_{i-1}) - g_h(x_{i+1})v_h(x_{i+1}) \wedge v_h(x_i)). \end{aligned}$$

Then we follow the same steps followed to demonstrate Proposition 2.5. ■

The rest of proof is similar to that of Theorem 1.1 and requires property (1.17) and results of Lemmas 1.13, 2.1 and 2.2.

2.4 Proof of Theorem 1.3

We denote

$$\Delta_{g_h} v_h = D^+(g_h D^- v_h) = D^-(\tau^+ g_h D^+ v_h), \quad D^2 = D^+ D^- = D^+ D^-, D^3 = D^+ D^- D^+,$$

and $g_h^t = \{\partial_t g(t, x_i)\}_i$. Since g is given in $W^{1,\infty}(\mathbb{R}^+, W^{3,\infty}(\mathbb{R}, \mathbb{R}))$, then there exist $\beta, \beta_1, \beta', \beta_1', \beta'', \beta_2''$ and β''' such that

$$\begin{cases} |g_h|_{L_h^\infty} \leq \beta, \quad |g_h^t|_{L_h^\infty} \leq \beta_1 \\ |D^+ g_h|_{L_h^\infty} = |D^- g_h|_{L_h^\infty} \leq \beta', \quad |D^+ g_h^t|_{L_h^\infty} = |D^- g_h^t|_{L_h^\infty} \leq \beta_1' \\ |D^2 g_h|_{L_h^\infty} \leq \beta'', \quad |D^2 g_h^t|_{L_h^\infty} \leq \beta_2'' \\ |D^3 g_h|_{L_h^\infty} \leq \beta'''. \end{cases}$$

Our proof consists of several steps

2.4.1 Step 1

In this step, we establish two a priori estimates in $\frac{du_h}{dt}$, $D^-\frac{du_h}{dt}$, $\Delta_{g_h}u_h$ and $D^-\Delta_{g_h}u_h$.
We start by proving that

$$\frac{d}{dt}\left(\left|\frac{du_h}{dt}\right|_h^2+|\Delta_{g_h}u_h|_h^2\right)\leq C_1\left(\left|\frac{du_h}{dt}\right|_h^2+|\Delta_{g_h}u_h|_h^2\right)^2+C_2, \tag{2.20}$$

where C_1 and C_2 are two positive constants independent of h. For any two sequences $u_h=\{u_h(x_i)\}_i$ and $v_h=\{u_h(x_i)\}_i$, we have

$$\begin{aligned}\Delta_{g_h}(u_hv_h) &= D^+(g_h\tau^-v_hD^-uh+g_hu_hD^-v_h)\\ &= \tau^+\tau^-v_h\Delta_{g_h}u_h+g_hD^+(\tau^-v_h)D^-u_h+\tau^+(g_hD^-v_h)D^+u_h+u_h\Delta_{g_h}v_h\\ &= v_h\Delta_{g_h}uh+g_hD^-v_hD^-u_h+\tau^+g_hD^+v_hD^+u_h+u_h\Delta_{g_h}v_h.\end{aligned} \tag{2.21}$$

We derive (2.11) with respect to t

$$\frac{d^2u_h}{dt^2}=(u_h\wedge\Delta_{g_h}u_h)\wedge\Delta_{g_h}u_h+u_h\wedge\Delta_{g_h}(u_h\wedge\Delta_{g_h}u_h)+u_h\wedge\Delta_{g_h^t}u_h. \tag{2.22}$$

Using (2.21) and $|u_h(t,x_i)|=1$, we deduce from equation (2.22) that

$$\begin{aligned}\frac{d^2u_h}{dt^2} &= (u_h\cdot\Delta_{g_h}u_h)\Delta_{g_h}u_h-|\Delta_{g_h}u_h|^2u_h\\ &\quad +u_h\wedge(g_hD^-u_h\wedge D^-\Delta_{g_h}u_h+\tau^+g_hD^+u_h\wedge D^+\Delta_{g_h}u_h+u_h\wedge\Delta_{g_h}^2u_h)\\ &= u_h\wedge\Delta_{g_h^t}u_h+(u_h\cdot\Delta_{g_h}u_h)\Delta_{g_h}u_h-|\Delta_{g_h}u_h|^2u_h+(u_h\cdot\Delta_{g_h}^2u_h)u_h-\Delta_{g_h}^2u_h\\ &\quad +E,\end{aligned} \tag{2.23}$$

where

$$\begin{aligned}E &= g_hu_h\wedge(D^-u_h\wedge D^-\Delta_{g_h}u_h)+\tau^+g_hu_h\wedge(D^+u_h\wedge D^+\Delta_{g_h}u_h)\\ &= g_h(u_h\cdot D^-\Delta_{g_h}u_h)D^-u_h+\tau^+g_h(u_h\cdot D^+\Delta_{g_h}u_h)D^+u_h\\ &\quad -g_h(u_h\cdot D^-u_h)D^-\Delta_{g_h}u_h-\tau^+g_h(u_h\cdot D^+u_h)D^+\Delta_{g_h}u_h.\end{aligned}$$

Furthermore, we have

$$u_h\cdot D^\pm u_h=\mp\frac{h}{2}(D^\pm u_h)^2,$$

hence

$$\begin{aligned}\tau^+g_h(u_h\cdot D^+u_h)D^+\Delta_{g_h}u_h &= -\frac{h}{2}\tau^+g_h(D^+u_h)^2D^+\Delta_{g_h}u_h\\ &= -\frac{h}{2}\left\{D^-[(D^+u_h)^2\tau^+(g_h\Delta_{g_h}u_h)]-D^-(\tau^+g_h(D^+u_h)^2)\Delta_{g_h}u_h\right\}\\ &= -\frac{h}{2}\left\{D^+[g_h(D^-u_h)^2\Delta_{g_h}u_h]-D^+(g_h(D^-u_h)^2)\Delta_{g_h}u_h\right\},\end{aligned}$$

and

$$\begin{aligned}g_h(u_h\cdot D^-u_h)D^-\Delta_{g_h}u_h &= \frac{h}{2}g_h(D^-u_h)^2D^-\Delta_{g_h}u_h\\ &= \frac{h}{2}\left\{D^+[g_h(D^-u_h)^2\tau^-\Delta_{g_h}u_h]-D^+(g_h(D^-u_h)^2)\Delta_{g_h}u_h\right\},\end{aligned}$$

which together give

$$-\tau^+ g_h(u_h \cdot D^+u_h)D^+\Delta_{g_h}u_h - g_h(u_h \cdot D^-u_h)D^-\Delta_{g_h}u_h = \frac{h^2}{2}D^+[g_h(D^-u_h)^2D^-\Delta_{g_h}u_h]. \qquad (2.24)$$

On the other hand, we have

$$\begin{aligned} u_h \cdot \Delta_{g_h}u_h &= u_h \cdot (D^+g_hD^-u_h + \tau^+g_hD^+D^-u_h) \\ &= \frac{h}{2}D^+g_h(D^-u_h)^2 - \frac{1}{2}\tau^+g_h((D^-u_h)^2 + (D^+u_h)^2) \\ &= -\frac{1}{2}(g_h(D^-u_h)^2 + \tau^+g_h(D^+u_h)^2), \end{aligned}$$

hence

$$\begin{aligned} u_h \cdot D^\pm\Delta_{g_h}u_h &= D^\pm(u_h \cdot \Delta_{g_h}u_h) - D^\pm u_h \cdot \tau^\pm(\Delta_{g_h}u_h) \\ &= -\frac{1}{2}D^\pm\left(g_h(D^-u_h)^2 + \tau^+g_h(D^+u_h)^2\right) - D^\pm u_h \cdot \tau^\pm(\Delta_{g_h}u_h). \qquad (2.25) \end{aligned}$$

Combining (2.24) and (2.25) we find that

$$\begin{aligned} E = {} & \frac{h^2}{2}D^+[g_h(D^-u_h)^2D^-\Delta_{g_h}u_h] \\ & -\frac{1}{2}g_hD^-\left(g_h(D^-u_h)^2 + \tau^+g_h(D^+u_h)^2\right)D^-u_h - g_h(D^-u_h \cdot \tau^-\Delta_{g_h}u_h)D^-u_h \\ & -\frac{1}{2}\tau^+g_hD^+\left(g_h(D^-u_h)^2 + \tau^+g_h(D^+u_h)^2\right)D^+u_h - \tau^+g_h(D^+u_h \cdot \tau^+\Delta_{g_h}u_h)D^+u_h. \end{aligned}$$

Taking the L_h^2-scalar product in (2.23) with $\frac{du_h}{dt}$ and using $u_h \cdot \frac{du_h}{dt} = 0$, $\Delta_{g_h}u_h \cdot \frac{du_h}{dt} = 0$ and

$$\Delta_{g_h}(\frac{du_h}{dt}) = \frac{d}{dt}\Delta_{g_h}(u_h) - \Delta_{g_h^t}u_h,$$

we obtain by integration by parts

$$\frac{1}{2}\frac{d}{dt}\left(\left|\frac{du_h}{dt}\right|_h^2 + |\Delta_{g_h}u_h|_h^2\right) = J_1 + J_2 + I_1 + I_2^+ + I_2^- + I_3^+ + I_3^-,$$

where

$$J_1 = (\Delta_{g_h^t}u_h, \Delta_{g_h}u_h)_h,$$

$$J_2 = (u_h \wedge \Delta_{g_h^t}u_h, u_h \wedge \Delta_{g_h}u_h)_h,$$

$$I_1 = \frac{h^2}{2}\left(D^+[g_h(D^-u_h)^2D^-\Delta_{g_h}u_h], \frac{du_h}{dt}\right)_h,$$

$$I_2^+ = -\frac{1}{2}\left(\tau^+g_hD^+\left(g_h(D^-u_h)^2 + \tau^+g_h(D^+u_h)^2\right)D^+u_h, \frac{du_h}{dt}\right)_h,$$

$$I_2^- = -\frac{1}{2}\left(g_hD^-\left(g_h(D^-u_h)^2 + \tau^+g_h(D^+u_h)^2\right)D^-u_h, \frac{du_h}{dt}\right)_h,$$

$$I_3^+ = -\frac{1}{2}\left(\tau^+g_h(D^+u_h \cdot \tau^+\Delta_{g_h}u_h)D^+u_h, \frac{du_h}{dt}\right)_h,$$

$$I_3^- = -\frac{1}{2}\left(g_h(D^-u_h \cdot \tau^-\Delta_{g_h}u_h)D^-u_h, \frac{du_h}{dt}\right)_h.$$

To bound from above these terms we apply essentially the Hölder inequality and Lemmas 1.9 and 1.15. We start by

$$\begin{aligned} J_1 + J_2 &\leq 2|\Delta_{g_h^t} u_h|_h |\Delta_{g_h} u_h|_h \\ &\leq 2(\beta_1' |D^+ u_h|_h + \beta_1 |D^2 u_h|_h) |\Delta_{g_h} u_h|_h. \end{aligned} \tag{2.26}$$

Then, we have on the one hand

$$\begin{aligned} I_1 &\leq \frac{h^2}{2} |D^+ g_h (D^- u_h)^2 D^- \Delta_{g_h} u_h|_h \left| \frac{du_h}{dt} \right|_h \\ &\leq h |g_h (D^- u_h)^2 D^- \Delta_{g_h} u_h|_h \left| \frac{du_h}{dt} \right|_h \\ &\leq h\beta |D^- u_h|^2_{L_h^\infty} |D^- \Delta_{g_h} u_h|_h \left| \frac{du_h}{dt} \right|_h \\ &\leq 2C\beta |D^- u_h|^2_{H_h^1} |\Delta_{g_h} u_h|_h \left| \frac{du_h}{dt} \right|_h. \end{aligned} \tag{2.27}$$

and on the other hand $I_2^+ = I_{21}^+ + I_{22}^+$, with

$$I_{21}^+ = -\frac{1}{2} \left(\tau^+ g_h D^+ (g_h (D^- u_h)^2) D^+ u_h, \frac{du_h}{dt} \right)_h, \quad I_{22}^+ = -\frac{1}{2} \left(\tau^+ g_h D^+ (\tau^+ g_h (D^+ u_h)^2) D^+ u_h, \frac{du_h}{dt} \right)_h.$$

Moreover,

$$I_{21}^+ = -\frac{1}{2} \left(\tau^+ g_h \left(D^+ g_h (D^- u_h)^2 + \tau^+ g_h (D^- + \tau^+ D^-) u_h \cdot D^+ D^- u_h \right) D^+ u_h, \frac{du_h}{dt} \right)_h,$$

hence

$$\begin{aligned} I_{21}^+ &\leq \frac{1}{2} \beta \left(\beta' |D^- u_h|_h + 2\beta |D^+ D^- u_h|_h \right) |D^- u_h|^2_{L_h^\infty} \left| \frac{du_h}{dt} \right|_h \\ &\leq \frac{1}{2} C\beta \left(\beta' |D^- u_h|_h + 2\frac{\beta}{\alpha} |\Delta_{g_h} u_h|_h \right) |D^- u_h|^2_{H_h^1} \left| \frac{du_h}{dt} \right|_h. \end{aligned}$$

Similarly, we find that

$$I_{22}^+ \leq \frac{1}{2} C\beta \left(\beta' |D^- uh|_h + 2\frac{\beta}{\alpha} |\Delta_{g_h} u_h|_h \right) |D^- u_h|^2_{H_h^1} \left| \frac{du_h}{dt} \right|_h,$$

then

$$I_2^+ \leq C\beta \left(\beta' |D^- u_h|_h + 2\frac{\beta}{\alpha} |\Delta_{g_h} u_h|_h \right) |D^- u_h|^2_{H_h^1} \left| \frac{du_h}{dt} \right|_h. \tag{2.28}$$

For I_3^+ we easily note that

$$I_3^+ \leq \frac{1}{2} C\beta |D^- u_h|^2_{H_h^1} |\Delta_{g_h} u_h|_h \left| \frac{du_h}{dt} \right|_h. \tag{2.29}$$

The two terms I_3^- and I_2^- can be treated in the same way followed to bound I_3^+ and I_2^+. Since

$$\begin{aligned} |D^- u_h|^2_{H_h^1} &= |D^- u_h|_h^2 + |D^+ D^- u_h|_h^2 \\ &\leq |D^- u_h|_h^2 + \frac{1}{\alpha^2} |\Delta_{g_h} u_h|_h^2, \end{aligned} \tag{2.30}$$

we get by combining (2.26), (2.27), (2.28), (2.29), (2.30) and (2.14)

$$\frac{d}{dt}\left(\left|\frac{du_h}{dt}\right|_h^2 + |\Delta_{g_h} u_h|_h^2\right) \leq C_1\left(\left|\frac{du_h}{dt}\right|_h^2 + |\Delta_{g_h} u_h|_h^2\right)^2 + C_2, \tag{2.31}$$

where $C_1, C_2 > 0$ are two constants depending on $\alpha, \beta, \beta_1, \beta', \beta_1'$ and $|D^+ u_h^0|_h$. Then we establish an a priori estimate in $D^-\frac{du_h}{dt}$ and $D^-\Delta_{g_h} u_h$. Let

$$A_{g_h} u_h = \frac{1}{2}(g_h(D^- u_h)^2 + \tau^+ g(D^+ u_h)^2).$$

We have found that

$$\frac{d^2 u_h}{dt^2} + \Delta_{g_h}^2 u_h = (u_h \cdot \Delta_{g_h} u_h)\Delta_{g_h} u_h - |\Delta_{g_h} u_h|^2 u_h + (u_h \cdot \Delta_{g_h}^2 u_h)u_h + u_h \wedge \Delta_{g_h^t} u_h + E, \tag{2.32}$$

where

$$\begin{aligned} E \;\; = \;\; & \frac{h^2}{2} D^+[g_h(D^- u_h)^2 D^- \Delta_{g_h} u_h] \\ & -g_h D^-(A_{g_h} u_h) D^- u_h - g_h(D^- u_h \cdot \tau^- \Delta_{g_h} u_h) D^- u_h \\ & -\tau^+ g_h D^+(A_{g_h} u_h) D^+ u_h - \tau^+ g_h(D^+ u_h \cdot \tau^+ \Delta_{g_h} u_h) D^+ u_h. \end{aligned} \tag{2.33}$$

Moreover, we deduce from (2.21) that

$$\begin{aligned} u_h \cdot \Delta_{g_h}^2(u_h) \;\; = \;\; & \Delta_{g_h}(u_h \cdot \Delta_{g_h} u_h) - |\Delta_{g_h} u_h|^2 - g_h D^- \Delta_{g_h} u \cdot D^- u_h - \tau^+ g_h D^+ \Delta_{g_h} u \cdot D^+ u_h \\ = \;\; & -\Delta_{g_h}(A_{g_h} u_h) - |\Delta_{g_h} u_h|^2 - g_h D^- \Delta_{g_h} u \cdot D^- u_h \\ & -\tau^+ g_h D^+ \Delta_{g_h} u \cdot D^+ u_h. \end{aligned} \tag{2.34}$$

Thus Combining (2.32), (2.33) and (2.34), we get

$$\begin{aligned} \frac{d^2 u_h}{dt^2} + \Delta_{g_h}^2 u_h \;\; = \;\; & -\Delta_{g_h}(A_{g_h} u_h) u_h - A_{g_h} u_h \Delta_{g_h} u_h - \tau^+ g_h D^+(A_{g_h} u_h) D^+ u_h - g_h D^-(A_{g_h} u_h) D^- u_h \\ & -g_h(D^- u_h \cdot \tau^- \Delta_{g_h} u_h) D^- u - g_h D^- \Delta_{g_h} u \cdot D^- u_h - |\Delta_{g_h} u_h|^2 \\ & -\tau^+ g_h(D^+ u_h \cdot \tau^+ \Delta_{g_h} u_h) D^+ u_h - \tau^+ g_h D^+ \Delta_{g_h} u \cdot D^+ u_h - |\Delta_{g_h} u_h|^2 \\ & +\frac{h^2}{2} D^+[g_h(D^- u_h)^2 D^- \Delta_{g_h} u_h] + u_h \wedge \Delta_{g_h^t} u_h, \end{aligned} \tag{2.35}$$

where

$$-g_h(D^- u_h \cdot \tau^- \Delta_{g_h} u_h) D^- u - g_h D^- \Delta_{g_h} u \cdot D^- u_h - |\Delta_{g_h} u_h|^2 = -D^-(\tau^+ g_h(D^+ u_h \cdot \Delta_{g_h} u_h) u_h),$$

$$-\tau^+ g_h(D^+ u_h \cdot \tau^+ \Delta_{g_h} u_h) D^+ u_h - \tau^+ g_h D^+ \Delta_{g_h} u \cdot D^+ u_h - |\Delta_{g_h} u_h|^2 = -D^+(g_h(D^- u_h \cdot \Delta_{g_h} u_h) u_h).$$

We have

$$\begin{cases} D^+ u_h \cdot \Delta_{g_h} u_h = D^+ g_h |D^+ u_h|^2 + \frac{1}{2} g_h D^+(|D^- u_h|^2) + \frac{h}{2} g_h |D^+ D^- u_h|^2, \\ D^- u_h \cdot \Delta_{g_h} u_h = D^- g_h |D^- u_h|^2 + \frac{1}{2}\tau^+ g_h D^-(|D^+ u_h|^2) + \frac{h}{2}\tau^+ g_h |D^+ D^- u_h|^2, \end{cases}$$

and

$$\begin{cases} g_h D^+(|D^- u_h|^2) u_h = D^+(g_h |D^- u_h|^2 u_h) - \tau^+(|D^- u_h|^2) D^+(g_h u_h), \\ \tau^+ g_h D^-(|D^+ u_h|^2) u_h = D^-(\tau^+ g_h |D^+ u_h|^2 u_h) - \tau^-(|D^+ u_h|^2) D^-(g_h u_h), \end{cases}$$

then

$$\begin{aligned} D^-(\tau^+ g_h(D^+ u_h \cdot \Delta_{g_h} u_h) u_h) \;\; = \;\; & \frac{1}{2}\Delta_{g_h}(g_h |D^- u_h|^2 u_h) + \frac{1}{2} D^-(\tau^+ g_h |D^- u_h|^2 [D^+ g_h u_h - \tau^+ g_h D^+ u_h]) \\ & +\frac{h}{2} D^-(\tau^+ g_h g_h |D^+ D^- u_h|^2), \end{aligned}$$

and

$$D^+(g_h(D^-u_h\cdot\Delta_{g_h}u_h)u_h) = \frac{1}{2}\Delta_{g_h}(\tau^+g_h|D^+u_h|^2u_h)+\frac{1}{2}D^+(g_h|D^+u_h|^2[D^+g_hu_h-g_hD^-u_h]) +\frac{h}{2}D^+(\tau^+g_hg_h|D^+D^-u_h|^2).$$

Thus equation (2.35) can be rewritten as

$$\begin{aligned}\frac{d^2u_h}{dt^2}+\Delta^2_{g_h}u_h &= -2\Delta_{g_h}((A_{g_h}u_h)u_h)+u_h\wedge\Delta_{g_h^t}u_h\\ &+\frac{1}{2}D^+\left(g_h|D^+u_h|^2[2g_hD^-u_h-D^+g_hu_h-D^-g_h\tau^-u_h]\right)\\ &-\frac{h}{2}(D^++D^-)(\tau^+g_hg_h|D^+D^-u_h|^2)+\frac{h^2}{2}D^+\left(g_h(D^-u_h)^2D^-\Delta_{g_h}u_h\right)\end{aligned}\tag{2.36}$$

Applying operator D^- on (2.36) and taking the L^2_h-scalar product with $g_hD^-\frac{du_h}{dt}$, we get, after integration by parts,

$$\frac{h}{2}\frac{d}{dt}\sum_i g_h(x_i)\left(\left|D^-\frac{du_h}{dt}(x_i)\right|^2+|D^-\Delta_{g_h}u_h(x_i)|^2\right)=I_1+I_2+I_3+I_4+J_1+J_2+J_3,$$

with

$$I_1=-2\left(D^-\Delta_{g_h}((A_{g_h}u_h)u_h),g_hD^-\frac{du_h}{dt}\right)_h,$$

$$I_2=\frac{1}{2}\left(D^-D^+(g_h|D^+u_h|^2[2g_hD^-u_h-D^+g_hu_h-D^-g_h\tau^-u_h]),g_hD^-\frac{du_h}{dt}\right)_h,$$

$$I_3=-\frac{1}{2}\left(hD^-(D^++D^-)(\tau^+g_hg_h|D^+D^-u_h|^2),g_hD^-\frac{du_h}{dt}\right)_h,$$

$$I_4=\frac{1}{2}\left(h^2D^-D^+[g_h(D^-u_h)^2D^-\Delta_{g_h}u_h],g_hD^-\frac{du_h}{dt}\right)_h,$$

$$J_1=\left(D^-(u_h\wedge\Delta_{g_h^t}u_h,g_hD^-\frac{du_h}{dt})\right)_h,$$

$$J_2=(g_hD^-\Delta_{g_h}u_h,D^-\Delta_{g_h^t}u_h)_h,$$

$$J_3=\frac{h}{2}\frac{d}{dt}\sum_i g_h^t(x_i)\left(\left|D^-\frac{du_h}{dt}(x_i)\right|^2+|D^-\Delta_{g_h}u_h(x_i)|^2\right).$$

We start by bounding J_1, J_2 and J_3. We have

$$\begin{aligned}|J_1| &\le \beta|D^-u_h|_{L^\infty_h}(\beta_1|D^2u_h|_h+\beta_1'|D^+u_h|_h)\left|D^-\frac{du_h}{dt}\right|_h\\ &+\beta(2\beta_1'|D^2u_h|_h+\beta_1''|D^+u_h|_h+\beta|D^3u_h|_h)\left|D^-\frac{du_h}{dt}\right|_h,\end{aligned}\tag{2.37}$$

$$|J_2|\le\beta(2\beta_1'|D^2u_h|_h+\beta_1''|D^+u_h|_h+\beta|D^3u_h|_h)\left|D^-\Delta_{g_h}u_h\right|_h,\tag{2.38}$$

$$|J_3|\le\frac{1}{2}\beta_1\left(\left|D^-\Delta_{g_h}u_h\right|_h^2+\left|D^-\frac{du_h}{dt}\right|^2_h\right).\tag{2.39}$$

For the term I_2, we have

$$\begin{aligned} |I_2| \quad \le \quad & \frac{1}{2}\beta\{2|D^2(g_h^2|D^+u_h|^2D^-u_h)|_h + |D^2(g_hD^+g_h|D^+u_h|^2u_h)|_h \\ & + |D^2(g_hD^-g_h|D^+u_h|^2\tau^-u_h)|_h\}\left|D^-\frac{du_h}{dt}\right|_h, \end{aligned} \tag{2.40}$$

and

$$\begin{aligned} |D^2(g_h^2|D^+u_h|^2D^-u_h)|_h \quad \le \quad & C\{((\beta'^2+\beta\beta'')|D^+u_h|^2_{L^\infty_h} + \beta^2|D^2u_h|^2_{L^\infty_h})|D^+u_h|_h \\ & + \beta\beta'|D^+u_h|^2_{L^\infty_h}|D^2u_h|_h + \beta^2|D^+u_h|^2_{L^\infty_h}|D^3u_h|_h\}. \end{aligned} \tag{2.41}$$

We also have

$$\begin{aligned} |D^2(g_hD^+g_h|D^+u_h|^2u_h)|_h \quad \le \quad & C\{((\beta\beta'''+2\beta'\beta'')|D^+u_h|_{L^\infty_h} + (\beta'^2+\beta\beta'')|D^+u_h|^2_{L^\infty_h})|D^+u_h|_h \\ & + (\beta\beta'|D^+u_h|^2_{L^\infty_h} + (\beta'^2+\beta\beta'')|D^+u_h|_{L^\infty_h})|D^2u_h|_h \\ & + \beta\beta'|D^+u_h|_{L^\infty_h}|D^3u_h|_h\}. \end{aligned} \tag{2.42}$$

The term $|D^2(g_hD^-g_h|D^+u_h|^2\tau^-u_h)|_h$ can be bounded from above by the same term of the right-hand side of (2.42). To find a suitable bound for I_1, we write first

$$\begin{aligned} D^-\Delta_{g_h}(g_h|D^-u_h|^2u_h) \quad &= \quad D^2(g_hD^-(g_h|D^-u_h|^2u_h)) \\ &= \quad D^2(g_h^2\tau^-|D^-u_h|^2D^-u_h + g_hD^-g_h\tau^-|D^-u_h|^2\tau^-u_h + g_h^2D^-(|D^-u_h|^2)u_h). \end{aligned}$$

Thus the two terms $|D^2(g_h^2\tau^-|D^-u_h|^2D^-u_h)|_h$ and $|D^2(g_hD^-g_h\tau^-|D^-u_h|^2\tau^-u_h)|_h$ can be bounded from above by the members of right-hand side of (2.41) and (2.42) respectively. For the term $D^2(g_h^2D^-(|D^-u_h|^2)u_h)$, we have

$$\left(D^2(g_h^2D^-(|D^-u_h|^2)u_h), g_hD^-\frac{du_h}{dt}\right)_h = I_{21} + \left(D^3(|D^-u_h|^2)u_h, g_h^3D^-\frac{du_h}{dt}\right)_h, \tag{2.43}$$

with

$$\begin{aligned} I_{21} \quad \le \quad & C\beta\{\beta^2|D^2u_h|^2_{L^\infty_h}|D^+u_h|h + ((\beta\beta''+\beta'^2)|D^+u_h|_{L^\infty_h} + \beta\beta'|D^2u_h|_{L^\infty_h} + \beta\beta'|D^+u_h|^2_{L^\infty_h})|D^2u_h|_h \\ & + (\beta\beta'|D^+u_h|_{L^\infty_h} + \beta^2|D^2u_h|_{L^\infty_h})|D^3u_h|_h\}|D^-\frac{du_h}{dt}|_h. \end{aligned} \tag{2.44}$$

Integrating by parts the second term of the right-hand side member of (2.43), we obtain

$$\begin{aligned} \left(D^3(|D^-u_h|^2)u_h, g_h^3D^-\frac{du_h}{dt}\right)_h \quad = \quad & -\left(D^2(|D^-u_h|^2)u_h, D^+g_h^3D^-\frac{du_h}{dt}\right)_h \\ & -h\sum_i g^3(x_i)D^2(|D^-u_h|^2)(x_i)D^+(u_h\cdot D^-\frac{du_h}{dt})(x_i). \end{aligned}$$

Moreover, since $u_h\cdot\frac{du_h}{dt}=0$, we have

$$\begin{aligned} D^+(u_h\cdot D^-\frac{du_h}{dt}) \quad &= \quad D^+u_h\cdot D^+\frac{du_h}{dt} + u_h\cdot D^2\frac{du_h}{dt} - D^2(u_h\cdot\frac{du_h}{dt}) \\ &= \quad -D^2u_h\cdot\frac{du_h}{dt} - D^-u_h\cdot D^-\frac{du_h}{dt}. \end{aligned}$$

Consequently, we get

$$\begin{aligned}\left(D^3(|D^-u_h|^2)u_h, g_h^3 D^-\frac{du_h}{dt}\right)_h \quad \leq \quad & C\{\beta^3|D^2u_h|^2_{L_h^\infty}|D^+u_h|_h + \beta'\beta^2|D^2u_h|_{L_h^\infty}|D^2u_h|_h \\ & +(\beta'\beta^2|D^+u_h|_{L_h^\infty} + \beta^3|D^+u_h|^2_{L_h^\infty})|D^3u_h|_h\}\left|D^-\frac{du_h}{dt}\right|_h \\ & +\beta^3\{|D^+u_h|_{L_h^\infty}|D^2u_h|_{L_h^\infty}|D^3u_h|_h \\ & +|D^2u_h|^2_{L_h^\infty}|D^2u_h|_h\}\left|\frac{du_h}{dt}\right|_h. \end{aligned} \tag{2.45}$$

According to the definition of I_3 and I_4, we have

$$|I_3| \leq h\beta|D^2(g_h\tau^+g_h|D^2u_h|^2)|_h\left|D^-\frac{du_h}{dt}\right|_h,$$

and

$$|I_4| \leq \frac{1}{2}h^2\beta|D^2(g_h|D^-u_h|^2D^-\Delta_{g_h}u_h)|_h\left|D^-\frac{du_h}{dt}\right|_h;$$

where, applying Lemma 1.9, we get

$$h|D^2(g_h\tau^+g_h|D^2u_h|^2)|_h \leq 2|D^+(g_h\tau^+g_h|D^2u_h|^2)|_h,$$

and

$$h^2|D^2(g_h|D^2u_h|^2D^-\Delta_{g_h}u_h)|_h \leq 4|g_h|D^-u_h|^2D^-\Delta_{g_h}u_h|_h,$$

which gives together with previous estimates of I_3 and I_4

$$|I_3| \leq C\beta^2|D^2u_h|_{L_h^\infty}(\beta'|D^2u_h|_h + \beta|D^3u_h|_h)|D^-\frac{du_h}{dt}|_h, \tag{2.46}$$

and

$$|I_4| \leq 2\beta^2|D^-u_h|^2_{L_h^\infty}|D^-\Delta_{g_h}u_h|_h|D^-\frac{du_h}{dt}|_h.$$

Since

$$D^-\Delta_{g_h}u_h = D^2g_hD^-u_h + g_hD^3u_h + D^+g_hD^2u_h + D^-g_hD^-D^-u_h, \tag{2.47}$$

we obtain

$$|I_4| \leq 2\beta^2|D^-u_h|^2_{L_h^\infty}(\beta''|D^-u_h|_h + 2\beta'|D^2u_h|_h + \beta|D^3u_h|_h)\left|D^-\frac{du_h}{dt}\right|_h. \tag{2.48}$$

Combining (2.37 - 2.46) and (2.48), we finally get

$$\frac{1}{2}\frac{d}{dt}h\sum_i g_h(x_i)\left(\left|D^-\frac{du_h}{dt}(x_i)\right|^2 + |D^-\Delta_{g_h}u_h(x_i)|^2\right) \leq CA_1A_2, \tag{2.49}$$

with $A_1 = |D^+u_h|_{L_h^\infty} + |D^+u_h|^2_{L_h^\infty} + |D^2u_h|_{L_h^\infty} + |D^2u_h|^2_{L_h^\infty}$, $A_2 = |\frac{du_h}{dt}|^2_{H_h^1} + |D^2u_h|^2_{H_h^1} + |D^+u_h|^2_h$ and $C > 0$ is some constant depending on $\beta, \beta_1\beta', \beta_1', \beta'', \beta_1''$ and β'''.

2.4.2 Step 2

We construct the sequence $\{u_h^0\}_h$ such that

$$\begin{cases} Q_hu_h^0 \to u_0 & \text{in} \quad L^2_{loc}(\mathbb{R}), \\ Q_hD^+u_h^0 \to \frac{du_0}{dx} & \text{in} \quad L^2(\mathbb{R}), \\ Q_hD^2u_h^0 \to \frac{d^2u_0}{dx^2} & \text{in} \quad L^2(\mathbb{R}), \\ Q_hD^3u_h^0 \to \frac{d^3u_0}{dx^3} & \text{in} \quad L^2(\mathbb{R}), \end{cases} \tag{2.50}$$

then

Lemma 2.8 *There exists $T_1 > 0$ such that the sequences $\{\partial_t P_h u_h\}_h$, $\{\partial_t P_h D^- u_h\}_h$, $\{P_h D^2 u_h\}_h$ and $\{P_h D^3 u_h\}_h$ are bounded in $L^\infty(0, T_1, L^2(\mathbb{R}))$.*

Proof. Let $T > \frac{1}{\sqrt{C_1 C_2}}$. For $t \in [0, T]$ we denote

$$G(t) = C_2 T + \left|\frac{du_h}{dt}(0)\right|_h^2 + |\Delta_{g_h} u_h(0)|_h^2 + C_1 \int_0^t \left(\left|\frac{du_h}{dt}(\tau)\right|_h^2 + |\Delta_{g_h} u_h(\tau)|_h^2 \right)^2 d\tau,$$

where C_1 and C_2 are the constants of inequality (2.31), hence $\frac{1}{G} \in W^{1,\infty}(0, T)$ and in view of (2.31) we have

$$\left(\frac{1}{G(t)}\right)' \leq C_1, \quad \text{for almost everywhere on} \quad]0, T[.$$

then we have

$$C_1 t + \frac{1}{G(t)} \geq \frac{1}{G(0)}, \quad \forall t \in [0, T[,$$

and

$$G(t) \leq \frac{G(0)}{1 - C_1 G(0) t}, \quad \forall t \in [0, (C_1 G(0))^{-1}[.$$

Since

$$\begin{aligned} G(0) = C_2 T + \left|\frac{du_h}{dt}(0)\right|_h^2 + |\Delta_{g_h} u_h(0)|_h^2 & \\ &\leq 2|\Delta_{g_h} u_h(0)|_h^2 + C_2 T \\ &\leq 4\beta'^2 |D^+ u_h^0|_h^2 + 4\beta^2 |D^+ D^- u_h^0|_h^2 + C_2 T, \end{aligned}$$

the sequences $\{|D^+ u_h^0|_h\}_h$ and $\{|D^+ D^- u_h^0|_h\}_h$ are bounded. Thus there exists $M > 0$ such that

$$4\beta'^2 |D^+ u_h^0|_h^2 + 4\beta^2 |D^+ D^- u_h^0|_h^2 + C_2 T \leq M,$$

then

$$G(0)^{-1} \geq M^{-1} > 0.$$

Let $\tilde{T} = \frac{1}{2}(C_1 M)^{-1}$. Then, for all $t \in [0, \tilde{T}]$, we have

$$\left|\frac{du_h}{dt}\right|_h^2 + |\Delta_{g_h} u_h|_h^2 \leq G(t) \leq \frac{M}{1 - \frac{1}{2} M^{-1} G(0)} \leq 2M. \tag{2.51}$$

According to Corollary 1.15, there exists $C > 0$ such that

$$|D^+ u_h|_{L_h^\infty} \leq C |D^+ u_h|_{H_h^1}, \quad |D^2 u_h|_{L_h^\infty} \leq C |D^2 u_h|_{H_h^1}.$$

Thus combining (2.49) and (2.51), we have for all $t \in [0, \tilde{T}]$

$$\frac{1}{2}\frac{d}{dt} h \sum_i g_h(x_i) \left(\left|D^- \frac{du_h}{dt}(x_i)\right|^2 + |D^- \Delta_{g_h} u_h(x_i)|^2 \right) \leq C_1 (|D^- \frac{du_h}{dt}|_h^2 + |D^- \Delta_{g_h} u_h|_h^2)^2 + C_2, \tag{2.52}$$

where $C_1, C_2 > 0$ depend on β, β_1, β', β_1', β'', β_1'', β''', α, and M. Following the same argument in the previous part of this step, we find that there exists $K > 0$ and $0 < T_1 \leq \tilde{T}$ such that, for all $t \in [0, T_1]$, we have

$$\left|D^- \frac{du_h}{dt}\right|_h + |D^- \Delta_{g_h} u_h|_h \leq K. \tag{2.53}$$

Since

$$\Delta_{g_h} u_h = D^+ g_h D^+ u_h + g_h D^2 u_h,$$
$$D^- \Delta_{g_h} u_h = D^2 g_h D^- u_h + g_h D^3 u_h + D^+ g_h D^2 u_h + D^- g_h D^- D^- u_h,$$

we deduce from (2.51) and (2.53) that sequences $\{\left|D^- \frac{du_h}{dt}\right|_h\}_h$, $\{\left|\frac{du_h}{dt}\right|_h\}_h$, $\{|D^2 u_h|_h\}_h$, and $\{|D^3 u_h|_h\}_h$ are bounded in $L^\infty(0, T_1)$. The result then yields from Lemma 1.14. ∎

2.4.3 Étape 3

We already proved, by Lemma (2.6), that there exists $u \in L^\infty(0,T,H^1_{loc}(\mathbb{R}))$ and a subsequence $\{u_h\}_h$ such that

$$P_hD^-u_h \to \partial_x u \quad \text{in} \quad L^\infty(0,T,L^2(\mathbb{R})) \quad \text{weak star},$$

for all $T>0$. According to lemma 2.8, there exist $v,w \in L^\infty(0,T_1,L^2(\mathbb{R}))$ and a subsequence $\{u_h\}_h$ such that

$$\begin{cases} P_hD^2u_h \to v \quad \text{in} \quad L^\infty(0,T_1,L^2(\mathbb{R})) \quad \text{weak star}, \\ P_hD^3u_h \to w \quad \text{in} \quad L^\infty(0,T_1,L^2(\mathbb{R})) \quad \text{weak star}. \end{cases} \tag{2.54}$$

Consequently, the sequence $\{\partial_x P_hD^-u_h\}_h$ converges to $\partial_x^2 u$ in the sense of distributions. On the other hand, $\partial_x P_hD^-u_h = Q_hD^2u_h$, and the two sequences $\{Q_hD^2u_h\}_h$ and $\{P_hD^2u_h\}_h$ converge to the same limit in $L^\infty(0,T_1,L^2(\mathbb{R}))$ weak star (Lemma 2.1). It follows that $\partial_x^2 u = v \in L^\infty(0,T_1,L^2(\mathbb{R}))$, hence $\{P_hD^2u_h\}_h$ converges to $\partial_x^2 u$ in $L^\infty(0,T_1,L^2(\mathbb{R}))$ weak star. A similar argument shows that $\partial_x^3 u \in L^\infty(0,T_1,L^2(\mathbb{R}))$ and thus the proof is completed.

2.5 Proof of Theorem 1.4

First, we establishing the following two lemmas

Lemma 2.9 *Let $g \in W^{1,\infty}(\mathbb{R}^+,\mathbb{R})$ be such that there exists $\alpha>0$ with $g \geq \alpha$. Let $T>0$ and $u:[0,T]\times\mathbb{R} \to S^2$ be some solution for (1.2) such that $\partial_x u \in L^\infty(0,T,H^1(\mathbb{R}))$. Then there exist $C_1, C_2 > 0$ depending on g and $\|\partial_x u(0,.)\|_{H^1(\mathbb{R})}$ such that for almost every $t\in[0,T]$ we have*

$$\|\partial_t u\|^2_{L^2(\mathbb{R})} + \|\Delta_g u\|^2_{L^2(\mathbb{R})} \leq C_1 + C_2 \int_0^t \left(\|\partial_t u(\tau)\|^2_{L^2(\mathbb{R})} + \|\Delta_g u(\tau)\|^2_{L^2(\mathbb{R})} \right) d\tau. \tag{2.55}$$

Proof. Taking the L^2-scalar product in (1.2) with $\Delta_g u$ and integrating by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} g(x)|\partial_x u|^2 dx = \int_{\mathbb{R}} \partial_t g(x)|\partial_x u|^2 dx,$$

which gives

$$\|\partial_x u(t,.)\|_{L^2(\mathbb{R})} \leq \sqrt{\frac{\|g\|_{L^\infty}}{\alpha}} \|\partial_x u(0,.)\|_{L^2(\mathbb{R})} \exp\left(\frac{\|\partial_t g\|_{L^\infty}}{2\alpha}\right), \forall t \in [0,T]. \tag{2.56}$$

Since $2u\cdot\partial_x u = \partial_x |u|^2 = 0$, and by deriving (1.2) with respect to t, we obtain

$$\begin{aligned} \partial_t^2 u &= (u\wedge\Delta_g u)\wedge\Delta_g u + u\wedge\Delta_g(u\wedge\Delta_g u) + u\wedge\Delta_{\partial_t g}u \\ &= (u\cdot\Delta_g u)\Delta_g u - |\Delta_g u|^2 u + u\wedge(\Delta_g u\wedge\Delta_g u + 2g\partial_x u\wedge\partial_x\Delta_g u + u\wedge\Delta_g^2 u) + u\wedge\Delta_{\partial_t g} \\ &= (u\cdot\Delta_g u)\Delta_g u - |\Delta_g u|^2 u + 2g(u\cdot\partial_x\Delta_g u)\partial_x u + (u\cdot\Delta_g^2 u)u - \Delta_g^2 u + u\wedge\Delta_{\partial_t g}u. \end{aligned} \tag{2.57}$$

It is clear that $\partial_t u\cdot u = 0$, then we get by taking the L^2-scalar product in (2.57) with $\partial_t u$

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}}(|\partial_t u|^2 + |\Delta_g u|^2)dx &= 4\int_{\mathbb{R}} g(u\cdot\partial_x\Delta_g u)(\partial_x u\cdot\partial_t u)dx \\ &\quad +2\int_{\mathbb{R}}(u\wedge\Delta_{\partial_t g}u)\cdot(u\wedge\Delta_g u)dx \\ &\quad +2\int_{\mathbb{R}}\Delta_{\partial_t g}u\cdot\Delta_g u dx. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} u\cdot\partial_x\Delta_g u &= \partial_x(u\cdot\Delta_g u)-\partial_x u\cdot\Delta_g u \\ &= -\frac{3}{2}\partial_x(g|\partial_x u|^2)-\frac{1}{2}\partial_x g|\partial_x u|^2, \end{aligned} \tag{2.58}$$

and

$$\begin{aligned} (u\wedge\Delta_{\partial_t g}u)\cdot(u\wedge\Delta_g u) &= \Delta_{\partial_t g}u\cdot\Delta_g u-(u\cdot\Delta_{\partial_t g}u)(u\cdot\Delta_g u) \\ &= \Delta_{\partial_t g}u\cdot\Delta_g u-\frac{1}{2}\partial_t g^2|\partial_x u|^4. \end{aligned} \tag{2.59}$$

Then, integrating by parts, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(|\partial_t u|^2+|\Delta_g|^2)dx &= \frac{3}{4}\frac{d}{dt}\int_{\mathbb{R}}g^2|\partial_x u|^4dx-\int_{\mathbb{R}}g\partial_x g|\partial_x u|^2(\partial_x u\cdot\partial_t u)dx \\ &\quad -\frac{5}{4}\int_{\mathbb{R}}\partial_t g^2|\partial_x u|^4dx+2\int_{\mathbb{R}}\Delta_{\partial_t g}u\cdot\Delta_g udx \end{aligned} \tag{2.60}$$

Let

$$I(u)=\|\partial_t u\|^2_{L^2(\mathbb{R})}+\|\Delta_g u\|^2_{L^2(\mathbb{R})}-\frac{3}{2}\int_{\mathbb{R}}g^2|\partial_x u|^4dx,$$

$$J(u)=-\int_{\mathbb{R}}g\partial_x g|\partial_x u|^2(\partial_x u\cdot\partial_t u)dx-\frac{5}{4}\int_{\mathbb{R}}\partial_t g^2|\partial_x u|^4dx+2\int_{\mathbb{R}}\Delta_{\partial_t g}u\cdot\Delta_g udx.$$

Relation (2.60) can be rewritten as

$$\|\partial_t u\|^2_{L^2(\mathbb{R})}+\|\Delta_g u\|^2_{L^2(\mathbb{R})}=I(u(0,.))+\frac{3}{2}\int_{\mathbb{R}}g^2|\partial_x u|^4dx+2\int_0^t J(u(\tau))d\tau. \tag{2.61}$$

Then applying Gagliardo-Nirenberg inequalities on $\partial_x u$, we get

$$\begin{cases} \|\partial_x u\|_{L^6(\mathbb{R})}\le K_6\|\partial_x u\|^{\frac{2}{3}}_{L^2(\mathbb{R})}\|\partial_x^2 u\|^{\frac{1}{3}}_{L^2(\mathbb{R})}, \\ \|\partial_x u\|_{L^4(\mathbb{R})}\le K_4\|\partial_x u\|^{\frac{3}{4}}_{L^2(\mathbb{R})}\|\partial_x^2 u\|^{\frac{1}{4}}_{L^2(\mathbb{R})}, \end{cases} \tag{2.62}$$

with $K_6, K_4>0$. On the other hand, we have

$$\|g\partial_x^2 u\|^2_{L^2(\mathbb{R})}\le 2\|\Delta_g u\|^2_{L^2(\mathbb{R})}+2\|\partial_x g\partial_x u\|^2_{L^2(\mathbb{R})}. \tag{2.63}$$

To find a suitable uper bound for $I(u(0,.))$, we use the relation

$$|\partial_t u|^2=|u\wedge\Delta_g u|^2=|\Delta_g u|^2-g^2|\partial_x u|^4,$$

which implies that

$$\begin{aligned} I(u(0,.)) &= 2\|\Delta_g u(0,.)\|^2_{L^2(\mathbb{R})}-\frac{5}{2}\int_{\mathbb{R}}g^2|\partial_x u(0,.)|^4dx \\ &\le 2\|\Delta_g u(0,.)\|^2_{L^2(\mathbb{R})}+\frac{5}{2}K_4^4\|\partial_x u(0,.)\|^3_{L^2(\mathbb{R})}\|\partial_x^2 u(0,.)\|_{L^2(\mathbb{R})}. \end{aligned} \tag{2.64}$$

Thus, inequalities (2.56), (2.62), (2.63) and (2.64) together with $g\in W^{1,\infty}(\mathbb{R}^+,\mathbb{R})$ allow, by using Hölder inequality, to upper-bound the second member of (2.61) by

$$C_1+C_2\int_0^t\left(\|\partial_t u(\tau)\|^2_{L^2(\mathbb{R})}+\|\Delta_g u(\tau)\|^2_{L^2(\mathbb{R})}\right)d\tau,$$

where the two constants above depend on g and $\|\partial_x u(0,.)\|_{H^1(\mathbb{R})}$.

■

Corollary 2.10 *Under the assumptions of lemma 2.9, we have for all $t \in]0,T[$*

$$\|\partial_x^2 u(t,.)\|^2_{L^2(\mathbb{R})} \leq D_1 e^{D_2 t},$$

where D_1 and D_2 are two positive constants depending on g and $\|\partial_x u(0,.)\|_{H^1(\mathbb{R})}$.

Proof. Let

$$\psi(t) = \|\partial_t u(t)\|^2_{L^2(\mathbb{R})} + \|\partial_x^2 u(t)\|^2_{L^2(\mathbb{R})}.$$

Inequality (2.55) implies that

$$\psi(t) \leq C_1 + C_2 \int_0^t \psi(\tau) d\tau,$$

then conclusion follows from Grönwall lemma. ■

Lemma 2.11 *Let $g \in W^{1,\infty}(\mathbb{R}^+, W^{3,\infty}(\mathbb{R}))$ be such that there exists $\alpha > 0$ with $g \geq \alpha$. Let $T > 0$ and $u : [0,T] \times \mathbb{R} \to S^2$ be a solution for (1.2) such that $\partial_x u \in L^\infty(0,T,H^2(\mathbb{R}))$. Then there exist $C_1, C_2 > 0$ depending on g and $\|\partial_x u(0,.)\|_{H^2(\mathbb{R})}$ such that for almost every $t \in]0,T[$ we have*

$$\|\partial_t\partial_x u\|^2_{L^2(\mathbb{R})} + \|\partial_x^3 u\|^2_{L^2(\mathbb{R})} \leq C_1 + C_2 \int_0^t \left(\|\partial_t\partial_x u(\tau)\|^2_{L^2(\mathbb{R})} + \|\partial_x^3 u(\tau)\|^2_{L^2(\mathbb{R})}\right) d\tau.$$

Proof. Since

$$u \cdot \Delta_g^2 u = \Delta_g(u \cdot \Delta_g u) - 2g\partial_x u \cdot \partial_x \Delta_g u - |\Delta_g u|^2, \tag{2.65}$$

we get by combining (2.57), (2.58) and (2.65)

$$\begin{aligned}
\partial_t^2 u + \Delta_g^2 u &= u \wedge \Delta_{\partial_t g} u - \Delta_g(g|\partial_x u|^2) - g|\partial_x u|^2 \Delta_g u - 2\partial_x(g|\partial_x u|^2)\partial_x u \\
&\quad -2g(\partial_x u \cdot \Delta_g u)\partial_x u - 2g(\partial_x u \cdot \partial_x \Delta_g u)u - 2|\Delta_g u|^2 \\
&= u \wedge \Delta_{\partial_t g} - \Delta_g(g|\partial_x u|^2 u) - 2\partial_x(g(\partial_x u \cdot \Delta_g u)u) \\
&= u \wedge \Delta_{\partial_t g} - 2\Delta_g\left(|\partial_x u|^2 u\right) + \partial_x\left(|\partial_x u|^2(g\partial_x u - \partial_x g u)\right). \qquad (2.66)
\end{aligned}$$

Deriving (2.66) with respect to x and taking the L^2-scalar product with $g\partial_t\partial_x u$, we get by integrating by parts

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} g\left(|\partial_t\partial_x u|^2 + |\partial_x\Delta_g u|^2\right) dx &= -2\int_{\mathbb{R}} g\partial_x\Delta_g\left(|\partial_x u|^2 u\right) \cdot \partial_t\partial_x u dx \\
&\quad + \int_{\mathbb{R}} g\partial_x^2\left(|\partial_x u|^2(g\partial_x u - g'u)\right) \cdot \partial_t\partial_x u dx \\
&\quad + \int_{\mathbb{R}} g\partial_x\left(u \wedge \Delta_{\partial_t g} u\right) \cdot \partial_t\partial_x u dx \\
&\quad + \int_{\mathbb{R}} g\partial_x\Delta_{\partial_t g} u \cdot \partial_x\partial_t\Delta_g u dx + \int_{\mathbb{R}} \partial_t g|\partial_x\Delta_g u|^2 dx. \quad (2.67)
\end{aligned}$$

We upper-bound the L^2 norm of the right-hand side member of (2.66) by applying the chain rule on operators $\partial_x\Delta_g$ and ∂_x^2. All the terms of the right hand side member of (2.67) except for

$$J_1 = -2\int_{\mathbb{R}} g^3\partial_x^3(|\partial_x u|^2)u \cdot \partial_t\partial_x u dx,$$

can be upper-bounded by $C\left(\|\partial_t\partial_x u(\tau)\|^2_{L^2(\mathbb{R})} + \|\partial_x^3 u(\tau)\|^2_{L^2(\mathbb{R})}\right)$. To upper-bound J_1, we integrate by parts hence we get

$$J_1 = 2\int_{\mathbb{R}} \partial_x^2(|\partial_x u|^2)\partial_x(g^3 u \cdot \partial_t\partial_x u)dx,$$

then we develop

$$\partial_x(u\cdot\partial_t\partial_x u) = \partial_x u\cdot\partial_t\partial_x u + u\cdot\partial_t\partial_x^2 u = \partial_x u\cdot\partial_t\partial_x u + u\cdot\partial_t\partial_x^2 u - \partial_x^2(u\cdot\partial_t u) = -\partial_x u\cdot\partial_t\partial_x u - \partial_x^2 u\cdot\partial_t u.$$

Thus we get

$$J_1 = 6\int_{\mathbb{R}} g' g^2 \partial_x^2(|\partial_x u|^2)u\cdot\partial_t\partial_x u dx - 2\int_{\mathbb{R}} g^3\partial_x^2(|\partial_x u|^2)(\partial_x u\cdot\partial_t\partial_x u + \partial_x^2 u\cdot\partial_t u)dx,$$

and the conclusion holds from Hölder inequality and Sobolev embedding. ■

Corollary 2.12 *Under the assumptions of Lemma 2.11, we have for all* $t\in]0,T[$

$$\|\partial_x^3 u(t,.)\|^2_{L^2(\mathbb{R})} \le D_1 e^{D_2 t},$$

where D_1 *and* D_2 *are two positive constants depending on* g *and* $\|\partial_x u(0,.)\|_{H^2(\mathbb{R})}$.

Proof. The proof is an immediate result of Grönwall lemma. ■

2.5.1 Proof of Theorem 1.4

Let u and $\tilde{u}$ be two regular solutions for (1.2) with initial data u_0 and $\tilde{u}_0$ respectively such that $\frac{d\tilde{u}_0}{dx}, \frac{du_0}{dx} \in H^2(\mathbb{R})$. We denote $\omega = u - \tilde{u}$ and $\omega_0 = u_0 - \tilde{u}_0$. In what follows, we prove that there exist $C_k > 0,\quad k = 1,..,5$, depending on g and the H^2 norm of $\frac{d\tilde{u}_0}{dx}$ and $\frac{du_0}{dx}$, such that for almost every $t\in]0,T_1[$ we have

$$\|\omega\|^2_{H^1(\mathbb{R})} \le C_1\|\omega_0\|^2_{H^1(\mathbb{R})} + C_2\int_0^t \|\omega(\tau)\|^2_{H^1(\mathbb{R})}d\tau, \tag{2.68}$$

and

$$\begin{aligned}\|\partial_t\omega\|^2_{L^2(\mathbb{R})} + \|\partial_x^2\omega\|^2_{L^2(\mathbb{R})} \quad &\le \quad C_3\|\omega_0\|^2_{H^1(\mathbb{R})} + C_4\left(\|\partial_t\omega|_{t=0}\|^2_{L^2(\mathbb{R})} + \|\partial_x^2\omega_0\|^2_{L^2(\mathbb{R})}\right)\\ &\quad + C_5\int_0^t\left(\|\partial_t\omega(\tau)\|^2_{L^2(\mathbb{R})} + \|\partial_x^2\omega(\tau)\|^2_{L^2(\mathbb{R})}\right)d\tau.\end{aligned} \tag{2.69}$$

Applying (2.10) and (2.66) on u and $\tilde{u}$ and subtracting, we get

$$\partial_t\omega = z\wedge\Delta_g\omega + \omega\wedge\Delta_g z, \tag{2.70}$$

and

$$\begin{aligned}\partial_t^2\omega + \Delta_g^2\omega \quad &= \quad z\wedge\Delta_{\partial_t g}\omega + \omega\wedge\Delta_{\partial_t g}z - 2\Delta_g(gQ\omega) + \partial_x\left(Q(g\partial_x\omega - \partial_x g\omega)\right)\\ &\quad -4\Delta_g\left(g(\partial_x z\cdot\partial_x\omega)z\right) + 2\partial_x\left((\partial_x z\cdot\partial_x\omega)(g\partial_x z - \partial_x g z)\right),\end{aligned} \tag{2.71}$$

with $z = \frac{1}{2}(u+\tilde{u})$ and $Q = \frac{1}{2}\left(|\partial_x u|^2 + |\partial_x\tilde{u}|^2\right)$. Multiplying (2.70) by ω, we find that $|\omega|^2 = 2(z\wedge\Delta_g\omega)\cdot\omega$, which means that $\omega\in L^2(\mathbb{R})$. Then, integrating by parts and using Hölder's inequality, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}|\omega|^2 \quad &= \quad -2\int_{\mathbb{R}} g(\omega\wedge\partial_x z)\cdot\partial_x\omega\\ &\le \quad 2\|g\partial_x z\|_{L^\infty(\mathbb{R})}\|\omega\|_{L^2(\mathbb{R})}\|\partial_x\omega\|_{L^2(\mathbb{R})}\\ &\le \quad \|g\partial_x z\|_{L^\infty(\mathbb{R})}\|\omega\|^2_{H^1(\mathbb{R})}.\end{aligned} \tag{2.72}$$

Next, we take the L^2-scalar product in (2.70) with $\Delta_g\omega$. Integrating by parts and using Hölder's inequality, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} g|\partial_x\omega|^2 &= \int_{\mathbb{R}} \partial_t|\partial_x\omega|^2 - 2\int_{\mathbb{R}} g\partial_x(\omega\wedge\Delta_g z)\cdot\partial_x\omega \\ &= \int_{\mathbb{R}} \partial_t|\partial_x\omega|^2 - 2\int_{\mathbb{R}} g(\omega\wedge\partial_x\Delta_g z)\cdot\partial_x\omega \\ &\leq (\|\partial_t g\|_{L^\infty(\mathbb{R})} + \|g\partial_x\Delta_g z\|_{L^\infty(\mathbb{R})})\|\omega\|^2_{H^1(\mathbb{R})}. \end{aligned} \tag{2.73}$$

Thus, (2.68) holds from Corollaries 2.10 and 2.12 and from Sobolev's embedding [3] after summing (2.72) and (2.73).

Finally, taking the L^2-scalar product in (2.71) with $\partial_t\omega$ and integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(|\partial_t\omega|^2 + |\Delta_g\omega|^2) = I_1 + I_2 + I_3 - 2E_1 - 4E_2 + E_3 + 2E_4,$$

with

$$I_1 = \int_{\mathbb{R}} \Delta_{\partial_t g}\omega\cdot\Delta_g\omega,$$

$$I_2 = \int_{\mathbb{R}} z\wedge\Delta_{\partial_t g}\omega\cdot\partial_t\omega, \quad I_3 = \int_R \omega\wedge\Delta_{\partial_t g}z\cdot\partial_t\omega,$$

$$E_1 = \int_{\mathbb{R}}\Delta_g(gQ\omega)\cdot\partial_t\omega, \quad E_3 = \int_{\mathbb{R}}\partial_x\left(Q(g\partial_x\omega - g'\omega)\right)\cdot\partial_t\omega,$$

$$E_2 = \int_{\mathbb{R}}\Delta_g\left(g(\partial_x z\cdot\partial_x\omega)z\right)\cdot\partial_t\omega, \quad E_4 = \int_{\mathbb{R}}\partial_x\left((\partial_x z\cdot\partial_x\omega)(g\partial_x z - g'z)\right)\cdot\partial_t\omega.$$

The terms I_1, I_2, I_3, E_1, E_3 and E_4 can be treated by applying Hölder's inequality and Sobolev's embedding $H^1(\mathbb{R})\subset L^\infty(\mathbb{R})$. Applying the chain rule on Δ_g, the term E_2 can be written

$$E_2 = \int_{\mathbb{R}} g^2(\partial_x z\cdot\partial_x^3\omega)(z\cdot\partial_t\omega) + E_{21},$$

where E_{21} can be treated by Hölder's inequality and Sobolev's embedding. Finally, we have $z\cdot\partial_t\omega = -\omega\cdot\partial_t z$ (since $|u|^2 - |\tilde{u}|^2 = 0$) and

$$\int_{\mathbb{R}} g^2(\partial_x z\cdot\partial_x^3\omega)(z\cdot\partial_t\omega) = -2\int_{\mathbb{R}} g'g(\partial_x z\cdot\partial_x^2\omega)(z\cdot\partial_t\omega) + \int_{\mathbb{R}} g^2\partial_x^2\omega\cdot\partial_x\left((\omega\cdot\partial_t z)\partial_x z\right),$$

which is now in a suitable form to be upper-bounded as above. This yields the desired claim at the H^2 level.

2.6 Proof of Theorem 1.5

We construct a solution $\gamma\in L^\infty(0,T_1,H^3_{loc}(\mathbb{R}))$ for the system

$$\begin{cases}\partial_t\gamma = g(t,x,\gamma)\partial_x\gamma\wedge\partial_x^2\gamma, \\ \gamma(0,.) = \gamma_0.\end{cases} \tag{2.74}$$

[3] There exists $C>0$ such that
$$\|u\|_{L^\infty(\mathbb{R})} \leq C\|u\|_{H^1(\mathbb{R})}, \forall u\in H^1(\mathbb{R}).$$

as a limit, when $h \to 0$, of a sequence $\{\gamma_h\}_h$ of solutions for the semi-discrete system

$$\begin{cases} \frac{d\gamma_h}{dt} = g_h D^+\gamma_h \wedge D^2\gamma_h, \quad t>0, \\ \gamma_h(0) = \gamma_h^0, \end{cases} \tag{2.75}$$

where $\gamma_h^0 = \{\gamma_h^0(x_i)\}_i \in (\mathbb{R}^3)^{\mathbb{Z}_h}$ is such that $|D^+\gamma_h^0(x_i)| = 1$, and $g_h = \{g(t, x_i, \gamma_h^0(x_i))\}_i$. We denote $u_h = D^+\gamma_h, g_h^t = \partial_t g(t, x_i, \gamma(x_i))$ and $\Delta_{g_h} u_h = D^+(g_h D^- u_h)$. Then, applying D^+ on (2.75), we get

$$\frac{du_h}{dt} = u_h \wedge \Delta_{g_h} u_h. \tag{2.76}$$

We have

$$\begin{aligned} \frac{d}{dt}\sum_i (g\gamma_h|D^-u_h|^2)(x_i) &= \sum_i \frac{d\gamma_h(x_i)}{dt}\cdot\nabla_\gamma g(t,x_i,\gamma_h(x_i))|D^-u_h(x_i)|^2 \\ &\quad + \sum_i g_h^t(x_i)|D^-u_h(x_i)|^2 + \sum_i \left(g_h D^-u_h \cdot D^-\frac{du_h}{dt}\right)(x_i). \end{aligned}$$

Then, using Lemma 1.16, we obtain

$$h\sum_i (g_h D^-u_h \cdot D^-\frac{du_h}{dt})(x_i) = -\left(\Delta_{g_h} u_h, \frac{du_h}{dt}\right)_h = 0.$$

Thus, using $\frac{d\gamma_h}{dt} = g_h u_h \wedge D^- u_h$, we can write

$$\begin{aligned} \frac{d}{dt}\sum_i (g_h|D^-u_h|^2)(x_i) &\leq \|\nabla_\gamma g\|_{L^\infty}|D^-u_h|_{L_h^\infty}\sum_i (g_h|D^-u_h|^2)(x_i) \\ &\quad + \|\partial_t g\|_{L^\infty}\sum_i |D^-u_h(x_i)|^2. \end{aligned} \tag{2.77}$$

To get another estimate in $|\Delta_{g_h}|_h$, we derive (2.76) with respect to t. This yields

$$\begin{aligned} \frac{d^2u_h}{dt^2} &= \frac{du_h}{dt}\wedge\Delta_{g_h} uh + u_h \wedge \frac{d}{dt}\Delta_{g_h} u_h \\ &= (u_h\wedge\Delta_{g_h}u_h)\wedge\Delta_{g_h}u_h \\ &\quad + u_h \wedge \left(D^+\left(\frac{d\gamma_h}{dt}\cdot\nabla g(\gamma_h)D^-u_h\right) + \Delta_{g_h}\left(\frac{du_h}{dt}\right) + \Delta_{g_h^t}u_h\right). \end{aligned} \tag{2.78}$$

Next, we denote

$$\tilde{\Delta}_{g_h} u_h = D^+\left(g_h(u_h \wedge D^-u_h\cdot\nabla g(\gamma_h))D^-u_h\right),$$

then (2.78) becomes

$$\frac{d^2u_h}{dt^2} = (u_h\wedge\Delta_{g_h}u_h)\wedge\Delta_{g_h}u_h + u_h\wedge\Delta_{g_h}(u_h\wedge\Delta_{g_h}u_h) + u_h\wedge(\tilde{\Delta}_{g_h}u_h + \Delta_{g_h^t}u_h). \tag{2.79}$$

Repeating the same calculus as in (2.32), we get

$$\frac{d^2u_h}{dt^2} + \Delta_{g_h}^2 u_h = (u_h\cdot\Delta_{g_h}u_h)\Delta_{g_h}u_h - |\Delta_{g_h}u_h|^2u_h + (u_h\cdot\Delta_{g_h}^2u_h)u_h + u_h\wedge(\tilde{\Delta}_{g_h}u_h + \Delta_{g_h^t}u_h) + E, \tag{2.80}$$

where

$$\begin{aligned} E &= \frac{h^2}{2}D^+[g_h(D^-u_h)^2D^-\Delta_{g_h}u_h] \\ &\quad - g_hD^-(A_{g_h}u_h)D^-u_h - g_h(D^-u_h\cdot\tau^-\Delta_{g_h}u_h)D^-u_h \\ &\quad - \tau^+g_hD^+(A_{g_h}u_h)D^+u_h - \tau^+g_h(D^+u_h\cdot\tau^+\Delta_{g_h}u_h)D^+u_h. \end{aligned}$$

Taking the L_h^2-scalar product in (2.80) with $\frac{du_h}{dt}$ and using both $u_h \cdot \frac{du_h}{dt} = 0$ and $\Delta_{g_h} u_h \cdot \frac{du_h}{dt} = 0$, we get by integration by parts

$$\frac{1}{2}\frac{d}{dt}\left|\frac{du_h}{dt}\right|_h^2 + \left(\Delta_{g_h} u_h, \Delta_{g_h}\left(\frac{du_h}{dt}\right)\right)_h = I + \left(u_h \wedge \tilde{\Delta}_{g_h} u_h, \frac{du_h}{dt}\right)_h .$$

where $I = \left(E, \frac{du_h}{dt}\right)_h$. We have

$$\Delta_{g_h}\left(\frac{du_h}{dt}\right) = \frac{d}{dt}\Delta_{g_h} u_h - \tilde{\Delta}_{g_h} u_h - \Delta_{g_h^t} u_h.$$

Consequently,

$$\frac{1}{2}\frac{d}{dt}\left(\left|\frac{du_h}{dt}\right|_h^2 + |\Delta_{g_h} u_h|_h\right) = I + \left(\tilde{\Delta}_{g_h} u_h + \Delta_{g_h^t} u_h, \Delta_{g_h} u_h\right)_h + \left(u_h \wedge (\tilde{\Delta}_{g_h} u_h + \Delta_{g_h^t} u_h), u_h \wedge \Delta_{g_h} u_h\right)_h .$$

We know that g_h and $D^+ g_h$ are upper-bounded in norm $L^\infty(0,T,L_h^\infty)$ by $\beta = \|g\|_{L^\infty(0,T,L^\infty)}$ and $\beta' = \|\partial_x g\|_{L^\infty(0,T,L^\infty)} + \|\nabla_\gamma g\|_{L^\infty(0,T,L^\infty)}$ respectively. Thus by following the same calculus in the proof Theorem 1.3, we find that there exists $C_1 = C_1(\alpha,\beta,\beta') > 0$ such that

$$I \le C_1 |D^+ u_h|_{L_h^\infty}^2 (|\Delta_{g_h} u_h|_h^2 + |D^+ u_h|_h^2 + \left|\frac{du_h}{dt}\right|_h^2). \tag{2.81}$$

To find a suitable upper bound for the term $\left(\tilde{\Delta}_{g_h} u_h, \Delta_{g_h} u_h\right)_h$, we first rewrite

$$\begin{aligned}
\tilde{\Delta}_{g_h} u_h &= D^+\left(g_h (u_h \wedge D^- u_h \cdot \nabla_\gamma g(\gamma_h)) D^- u_h\right) \\
&= (u_h \wedge D^- u_h . \nabla_\gamma g(\gamma_h)) \Delta_{g_h} u_h + \tau^+ g_h D^+(u_h \wedge D^- u_h . \nabla_\gamma g(\gamma_h)) D^+ u_h \\
&= (u_h \wedge D^- u_h . \nabla_\gamma g(\gamma_h)) \Delta_{g_h} u_h + \tau^+ g_h (u_h \wedge D^2 u_h . \nabla_\gamma g(\gamma_h)) D^+ u_h \\
&\quad + \tau^+ g_h \left(u_h \wedge D^- u_h . D^+(\nabla_\gamma g(\gamma_h))\right) D^+ u_h .
\end{aligned}$$

The term $D^+(\nabla_\gamma g(\gamma_h))$ is upper-bounded in norm $L^\infty(0,T,L_h^\infty)$ by $\beta'' = \|\partial_x \nabla_\gamma g\|_{L^\infty(0,T,L^\infty)} + \|\nabla_\gamma^2 g\|_{L^\infty(0,T,L^\infty)}$. It follows that

$$\begin{aligned}
\left(\tilde{\Delta}_{g_h} u_h, \Delta_{g_h} u_h\right)_h &\le \beta' |D^- u_h|_{L_h^\infty} |\Delta_{g_h} u_h|_h^2 + \beta' |D^+ u_h|_{L_h^\infty} |\tau^+ D^2 u_h|_h |\Delta_{g_h} u_h|_h \\
&\quad + \beta\beta'' |D^+ u_h|_{L_h^\infty} |D^+ u_h|_h .
\end{aligned} \tag{2.82}$$

Furthermore, we have $\Delta_{g_h^t} u_h = D^+ g_h^t D^- u_h + \tau^+ g_h^t D^2 u_h$, then the two terms $\tau^+ g_h^t$ and $D^+ g_h^t$ are upper-bounded in norm $L^\infty(0,T,L_h^\infty)$ by $\beta_1 = \|\partial_t g\|_{L^\infty(0,T,L^\infty)}$ and $\beta_1' = \|\partial_t \partial_x g\|_{L^\infty(0,T,L^\infty)} + \|\partial_t \nabla_\gamma g\|_{L^\infty(0,T,L^\infty)}$ respectively. Then we have

$$\left(\Delta_{g_h^t} u_h, \Delta_{g_h} u_h\right)_h \le (\beta_1' |D^- u_h|_h + \beta_1 |D^2 u_h|_h) |\Delta_{g_h} u_h|_h . \tag{2.83}$$

Using inequality $|\tau^+ D^2 u_h|_h \le |\Delta_{g_h} u_h|_h + \beta' |D^+ u_h|_h$ together with (2.77), (2.81), (2.83) and (2.82), we find that there exists $C = C(\alpha,\beta,\beta_1,\beta',\beta_1',\beta'')$ such that

$$\begin{aligned}
\frac{d}{dt}\left(\left|\frac{du_h}{dt}\right|_h^2 + |\Delta_{g_h} u_h|_h + h\sum_i [g_h |D^+ u_h|^2](x_i)\right) &\le C\left(|D^+ u_h|_{L_h^\infty}^2 + |D^+ u_h|_{L_h^\infty}\right) \\
&\quad \times \left(|\Delta_{g_h} u_h|_h^2 + |D^+ u_h|_h^2 + |D^+ u_h|_h + \left|\frac{du_h}{dt}\right|_h^2\right).
\end{aligned}$$

In view of Lemma 1.15, there exist $\tilde{C} > 0$ and $C = C(\alpha, \beta') > 0$ such that

$$|D^+ u_h|^2_{L^\infty_h} \leq C|D^+ u_h|^2_{H^1_h} \leq C\tilde{C}(|\Delta_{g_h} u_h|^2_h + |D^+ u_h|^2_h).$$

This implies the existence of two constants $C_1, C_2 > 0$ depending on α, β, β_1, β' β'_1, and β'' such that

$$\frac{d}{dt}\left(\left|\frac{du_h}{dt}\right|^2_h + |\Delta_{g_h} u_h|_h + h\sum_i [g_h|D^+ u_h|^2](x_i)\right) \leq C_1\left(|\Delta_{g_h} u_h|^2_h + |D^+ u_h|^2_h + \left|\frac{du_h}{dt}\right|^2_h\right)^2 + C_2. \tag{2.84}$$

We construct now the sequence $\{\gamma^0_h\}$ such that

$$\begin{cases} Q_h \gamma^0_h \to \gamma_0 & \text{in} \quad L^2_{loc}(\mathbb{R}), \\ Q_h D^+ \gamma^0_h \to \frac{d\gamma_0}{dx} & \text{in} \quad L^2_{loc}(\mathbb{R}), \\ Q_h D^2 \gamma^0_h \to \frac{d^2\gamma_0}{dx^2} & \text{in} \quad L^2(\mathbb{R}), \\ Q_h D^3 \gamma^0_h \to \frac{d^3\gamma_0}{dx^3} & \text{in} \quad L^2(\mathbb{R}). \end{cases} \tag{2.85}$$

Thus we have

Lemma 2.13 *There exists $T_1 > 0$ such that*
i) The two sequences $\{P_h\gamma_h\}_h$ and $\{P_h u_h\}_h$ are upper-bounded in $L^\infty(0, T_1, H^1_{loc}(\mathbb{R}))$.
ii) The sequences $\{\partial_t P_h u_h\}_h$, $\{\partial_t P_h \gamma_h\}_h$, $\{P_h D^+ u_h\}_h$ and $\{P_h D^2 u_h\}_h$ are upper-bounded in $L^\infty(0, T_1, L^2(\mathbb{R}))$.

Proof. Following the same steps in the proof of Lemma 2.8, we find that there exists $T_1 > 0$ and $M > 0$ such that for almost every $t \in]0, T_1[$, we have

$$|D^+ u_h|^2_h + |D^2 u_h|^2_h + \left|\frac{du_h}{dt}\right|^2_h \leq M. \tag{2.86}$$

To prove i), let $L > 0$. For some $1 > h > 0$, we denote $N = E(\frac{L}{h}) + 1$. Since

$$\|P_h\gamma_h(t)\|_{H^1(-L,L)} \leq \sqrt{2L}|P_h\gamma_h(t,0)| + (2L+1)\|\partial_x P_h\gamma_h(t)\|_{L^2(-L,L)}, \tag{2.87}$$

and

$$\begin{aligned} |P_h\gamma_h(t,0)| &= |\gamma_h(t,0)| \\ &\leq |\gamma_h(0,0)| + T_1\left\|\frac{d}{dt}\gamma_h(.,0)\right\|_{L^\infty(0,T_1)} \\ &\leq |\gamma_h(0,0)| + T_1\beta\|D^- u_h(.,0)\|_{L^\infty(0,T_1)} \\ &\leq |\gamma_h(0,0)| + T_1\beta \sup_{\tau\in[0,T]} |D^- u_h(\tau,.)|_{L^\infty_h}, \end{aligned}$$

inequality (2.86) together with Lemma 1.15 imply the existence of a constant $C > 0$ such that

$$|P_h\gamma_h(t,0)| \leq |\gamma^0_h(0)| + CT_1\beta\sqrt{M}, \tag{2.88}$$

for almost every $t \in]0, T_1[$. To treat the second term of the right-hand side of (2.87), we write

$$\|\partial_x P_h\gamma_h\|^2_{L^2(-L,L)} = \sum_{i=-N}^{N-1}\int_{x_i}^{x_{i+1}} |D^+\gamma_h(x_i)|^2 dx \leq 2L + h, \tag{2.89}$$

hence we find that for almost every $t \in]0, T_1[$, [4]

$$\|P_h\gamma_h(t)\|_{H^1(-L,L)} \le \sqrt{2L}(|\gamma_0(0)| + CT_1\beta\sqrt{M}) + (2L+1)^2.$$

On the other hand, we have

$$\begin{aligned} \|P_h u_h\|^2_{H^1(-L,L)} &= \sum_{i=-N}^{N-1} \int_{x_i}^{x_{i+1}} \left| \frac{x_i - x}{h} u_h(x_i) + \frac{x - x_i}{h} u_h(x_{i+1}) \right|^2 dx + \sum_i h \left| \frac{u_h(x_i) - u_h(x_{i+1})}{h} \right|^2 dx \\ &\le \sum_{i=-N}^{N-1} \frac{h}{3} \left(|u_h(x_i)|^2 + |u_h(x_{i+1})|^2 + u_h(x_i)u_h(x_{i+1}) \right) + |D^+u_h|_h^2 \\ &\le 2L + 1 + M. \end{aligned}$$

This completes the proof of i).

Property ii) is an immediate result of (2.88) and Lemma 1.14. ∎

Lemma 2.13 together with Lemma 2.3 imply the existence of $u, \gamma \in L^\infty(0, T_1, L^2_{loc}(\mathbb{R}))$, $\omega, v \in L^\infty(0, T_1, L^2(\mathbb{R}))$, and two subsequences $\{\gamma_h\}_h$ and $\{u_h\}_h$ such that

$$\begin{cases} P_h\gamma_h \to \gamma \quad \text{in} \quad L^2(0, T_1, L^2_{loc}(\mathbb{R})) \quad \text{and almost everywhere}, \\ \partial_t P_h\gamma_h \to \partial_t\gamma \quad \text{in} \quad L^\infty(0, T_1, L^2(\mathbb{R})) \quad \text{weak star}, \\ P_h u_h \to u \quad \text{in} \quad L^2(0, T_1, L^2_{loc}(\mathbb{R})) \quad \text{and almost everywhere}, \\ P_h D^- u_h \to v \quad \text{in} \quad L^\infty(0, T_1, L^2(\mathbb{R})) \quad \text{weak star}, \\ P_h D^2 u_h \to w \quad \text{in} \quad L^\infty(0, T_1, L^2(\mathbb{R})) \quad \text{weak star}. \end{cases} \tag{2.90}$$

It follows that $\{\partial_x P_h u_h\}_h$ converges to $\partial_x u$ in the sense of distributions and, since $\partial_x P_h u_h = Q_h D^+ u_h$, we also have $\partial_x u = v \in L^\infty(0, T_1, L^2(\mathbb{R}))$.

We now prove that $\{P_h(g_h u_h \wedge D^- u_h)\}_h$ converges to $g(\gamma)u \wedge \partial_x u$ in $L^\infty(0, T_1, L^2(\mathbb{R}))$ weak star. We first note that

$$Q_h(g_h u_h \wedge D^- u_h) = g(Q_h\gamma_h) Q_h u_h \wedge Q_h D^- u_h.$$

This implies that the sequence $\{Q_h(g_h u_h \wedge D^- u_h)\}_h$ converges to $g(\gamma)u \wedge \partial_x u$ in $L^\infty(0, T_1, L^2(\mathbb{R}))$ weak star. In view of Lemma 2.1, the two sequences $\{Q_h(g_h u_h \wedge D^- u_h)\}_h$ and $\{P_h(g_h u_h \wedge D^- u_h)\}_h$ converge to the same limit. Since $\{\partial_t P_h \gamma_h\}_h$ converges to $\partial_t\gamma$ in $L^\infty(0, T_1, L^2(\mathbb{R}))$ weak star, we finally get

$$\partial_t\gamma = g(\gamma)u \wedge \partial_x u. \tag{2.91}$$

Thus to complete this proof, it suffices to show that $\partial_x\gamma = u$ and that $\partial_x^2 u \in L^\infty(0, T_1, L^2(\mathbb{R}))$. The sequence $\{\partial_x P_h\gamma_h\}_h$ converges to $\partial_x\gamma$ in the sense of distributions. On the other hand, we have $\partial_x P_h\gamma_h = Q_h D^+\gamma_h = Q_h u_h$, and the sequence $\{Q_h u_h\}_h$ converges to u in $L^\infty(0, T_1, L^2_{loc}(\mathbb{R}))$. Indeed, for $L > 0$ and $N = E(\frac{L}{h}) + 1$, we have

$$\begin{aligned} \|Q_h u_h - P_h u_h\|^2_{L^2(-L,L)} &\le \sum_{i=-N}^{N-1} \int_{x_i}^{x_{i+1}} |D^+u_h(x_i)|^2 (x - x_i)^2 dx \\ &\le \frac{2}{3} N |D^+u_h|^2_{L^\infty_h} h^3 \\ &\le \frac{2}{3} C(L+h) |D^+u_h|^2_{H^1_h} h^2 \\ &\le \frac{2}{3} CM(L+h)h^2, \end{aligned}$$

[4] It is possible to define $\{\gamma_h^0\}_h$ by

$$\gamma_h^0(x_i) = \gamma_0(x_i), \forall i \in \mathbb{Z},$$

hence $\gamma_h^0(0) = \gamma_0(0)$.

hence

$$\partial_x \gamma = u.$$

The sequence $\{\partial_x P_h D^- u_h\}_h$ converges to $\partial_x^2 u$ in the sense of distributions. We have $\partial_x P_h D^- u_h = Q_h D^2 u_h$, and in view of Lemma 2.1, the two sequences $\{Q_h D^2 u_h\}_h$ and $\{P_h D^2 u_h\}_h$ converge to the same limit in $L^\infty(0, T_1, L^2(\mathbb{R}))$ weak star. Thus $\partial_x^2 u = w \in L^\infty(0, T_1, L^2(\mathbb{R}))$.

References

[1] R. J. Arms and F. R. Hama. Localized-induction concept on a curved vortex and motion of an elliptic vortex ring. *Phys. Fluids*, 8:553–559, 1965.

[2] C. Bardos, P. Sulem, and C. Sulem. On the continuous limits for a system of classical spins. *Comm. Math. Phys.*, 107:5782–5804, 1985.

[3] G. K. Batchelor. *An introduction to the fluid dynamics*, volume Cambridge Math. Library. Cambridge University Press, Cambridge, 1967.

[4] B.L. Guo and Y.L. Zhou. Existence of weak solution for boundary problems of systems of ferromagnetic chain. *Sinica Ser. A*, 27(8):799–811, 1984.

[5] L. S. Da Rios. Sul moto di un filetto vorticoso di forma qualunque. *Rend. del Circolo Mat. di Palermo*, 22:117–135, 1906.

[6] P. G. Saffman. *Vortex dynamics*, volume Cambridge Monographs on Mechanics and App. Maths. Cambridge University Press, New York, 1992.

[7] J. Simon. Compact sets in the space $L^p(0, T, B)$. *Annali di matematica pura ed applicata*, CXLVI(4):65–96, 1987.