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David Gérard-Varet, Matthieu Hillairet

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Abstract

We study in this paper the movement of a rigid solid inside an incompressible Navier-Stokes flow, within a bounded domain. We consider the case where slip is allowed at the fluid/solid interface, through a Navier condition. Taking into account slip at the interface is very natural within this model, as classical no-slip conditions lead to unrealistic collisional behavior between the solid and the domain boundary. We prove for this model existence of weak solutions of Leray type, up to collision, in three dimensions. The key point is that, due to the slip condition, the velocity field is discontinuous across the fluid/solid interface. This prevents from obtaining global H^1 bounds on the velocity, which makes many aspects of the theory of weak solutions for Dirichlet conditions inappropriate.

1 Introduction

The general concern of this paper is the dynamics of solid bodies in a fluid flow. This dynamics is relevant to many natural and industrial processes, like blood flows, sprays, or design of micro swimmers.

A main problem to understand this dynamics is to compute the drag exerted by the flow on the bodies. From the mathematical point of view, a natural approach to this problem is to use the Euler or Navier-Stokes equations to model the flow. However, this generates serious difficulties. A famous one is *D'Alembert's paradox*, related to the Euler equation: in an incompressible and inviscid potential flow, a solid body undergoes no drag [21].

But the Navier-Stokes equations also raise modeling issues. Let us consider for instance a single solid moving in a viscous fluid. We denote by $S(t) \subset \mathbb{R}^3,\ F(t) \subset \mathbb{R}^3$ the solid and fluid domains at time t, and $\Omega := \overline{S(t)} \cup F(t)$ the total domain. We assume that the fluid is governed by the Navier-Stokes equations. We denote u_F and p_F its velocity and internal pressure, ρ_F its density, μ_F its viscosity. Thus:

$$\left\{ \begin{aligned} \rho_F \left(\partial_t u_F + u_F \cdot \nabla u_F\right) - \mu_F \Delta u_F \; &= \; -\nabla p_F - \rho_F g, \quad t > 0,\ x \in F(t), \\ \operatorname{div} u_F \; &= \; 0, \qquad\quad t > 0,\ x \in F(t), \end{aligned} \right. \tag{1.1}$$

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with $-\rho_F g$ the gravitational force. In parallel to the fluid modeling, we write the conservation of linear and angular momentum for the body. Denoting $x_S(t) \in \mathbb{R}^3$ the position of the center of mass, $U_S(t) \in \mathbb{R}^3$ its velocity, and $\omega_S(t) \in \mathbb{R}^3$ the angular velocity at time t, these conservation laws read

$$\left\{ \begin{aligned} m_S \frac{d}{dt} U_S(t) &= -\int_{\partial S(t)} \Sigma_F \nu \, d\sigma \, - \, m_S \, g, \\ \frac{d}{dt} \left(J_S \, \omega_S(t) \right) &= -\int_{\partial S(t)} (x - x_S(t)) \times (\Sigma_F \, \nu) \, d\sigma \, + \, \rho_S \int_{S(t)} (x - x_S(t)) \times (-g). \end{aligned} \right. \tag{1.2}$$

Following standard notations, ρ_S and $m_S := \rho_S |S(0)|$ are the density and mass of the solid (independent of t and x), $\Sigma_F(t,x) \in M_3(\mathbb{R})$ is the newtonian tensor of the fluid:

$$\Sigma_F = (2\mu_F D(u_F) - p_F \, I_d),$$

and $J_S(t) \in M_3(\mathbb{R})$ is the inertia matrix of the solid:

$$J_S(t) := \rho_S \int_{S(t)} \left(|x - x_S(t)|^2 \, I_d - (x - x_S(t)) \otimes (x - x_S(t)) \right) dx.$$

The vector $\nu = \nu(t, \cdot)$ is the unit normal vector pointing inside the solid $S(t)$. Note that the velocity $u_S(t,x)$ at each point x of the solid reads

$$u_S(t,x) := U_S(t) + \omega_S(t) \times (x - x_S(t)).$$

To close the system, one usually imposes no-slip conditions, both at the fluid-solid interface and the cavity boundary:

$$\left\{ \begin{aligned} & u_F|_{\partial S(t)} = u_S|_{\partial S(t)} \\ & u_F|_{\partial\Omega} = 0, \end{aligned} \right. \tag{1.3}$$

and one specifies the initial data: the initial position of the solid S_0,

$$u_{F,0} := u_F|_{t=0} \text{ and } u_{S,0} := U_{S,0} + \omega_S(0) \times (x - x_{S_0}).$$

One could believe that system (1.1)-(1.2)-(1.3) is a good model for the interaction between a solid and a viscous fluid. Far from it: in the case of a sphere falling over a flat wall

$$S(0) := e_3 + B(0, 1/2), \quad \Omega := \{x_3 > 0\},$$

it predicts that no collision is possible between the solid and the wall ! This no-collision paradox has been known from specialists since the 1960's, after articles by Cox and Brenner [4] and Cooley and O'Neill [5] in the context of Stokes equations. Since then, the no-collision paradox has been confirmed at the level of the Navier-Stokes equations (see [16, 17], and the preliminary result in [23]).

Of course, such a result is unrealistic, as it goes against Archimedes' principle. It suggests that the Navier-Stokes equations are not relevant to collisional and post-collisional descriptions. Hence, many physicists have tried to find an explanation for the paradox. *We shall focus here on one possible explanation, namely the no-slip condition.* The idea is that, when the distance between the solids gets very small (below the micrometer), the no-slip condition is no longer accurate, and must be replaced by a Navier condition:

$$\left\{ \begin{aligned} (u_F - u_S) \cdot \nu|_{\partial S(t)} &= 0, \qquad (u_F - u_S) \times \nu|_{\partial S(t)} = -2\beta_S (\Sigma_F \nu) \times \nu|_{\partial S(t)}, \\ u_F \cdot \nu|_{\partial\Omega} &= 0, \qquad u_F \times \nu|_{\partial\Omega} = -2\beta_\Omega \, (\Sigma_F \nu) \times \nu|_{\partial\Omega}. \end{aligned} \right. \tag{1.4}$$

In other words, only the normal component of the relative velocity of the fluid is zero, to ensure impermeability. The tangential ones are non-zero, and proportional to the stress constraint, with constant slip lengths $\beta_S, \beta_\Omega > 0$. For a recent discussion of the Navier condition, notably in the context of microfluidics, we refer to [19]. See also the seminal paper [18]. Let us point out that the Navier-condition is sometimes used as a wall law, to describe the averaged effect of rough hydrophobic surfaces [2].

The effect of slip conditions (1.4) on collision was investigated recently by the authors in article [13]. More precisely, this article is devoted to a simplified model, in which

- The Navier-Stokes equations are replaced by the steady Stokes ones (quasi-static regime).
- The domain Ω is a half-space, the solid S is a sphere.

In this context, denoting $h(t)$ the distance between $S(t)$ and the plane $\partial\Omega$, it is shown that the dynamics obeys the reduced ODE

$$\ddot{h} \,=\, \dot{h}\,\mathcal{D}(h) \,+\, \frac{(\rho_F - \rho_S)}{\rho_S}\,g$$

where the drag term $\mathcal{D}(h)$ satisfies $\mathcal{D}(h) = O(|\ln h|)$ as $h \to 0$. This is in sharp contrast with the case of no-slip conditions (1.3), for which $\mathcal{D}(h) \sim \frac{C}{h}$. In particular, it allows for collisions in finite time. We refer to [13] for all details and other results in the context of rough boundaries.

Hence, paper [13] provides a resolution of the paradox: one can *a priori* keep the Navier-Stokes equations, up to considering the Navier boundary conditions (1.4). Nevertheless, the analysis in [13] is limited to simple configurations and to Stokes flows. In the context of the full 3D Navier-Stokes system (1.1), more complicated behaviors may occur. For instance, smooth solutions may exhibit singularities prior to any collision. To describe the qualitative features of the collision, *one needs to consider weak solutions*. The theory of weak solutions is well understood in the case of no-slip conditions and many references will be given in the next section. *However, the existence of weak solutions with Navier conditions has been so far an open question, due to serious additional mathematical difficulties. To address this question is the purpose of the present paper*. Broadly, we shall build weak solutions for system (1.1)-(1.2)-(1.4), up to collision between the solid and the cavity Ω.

The paper is organized as follows. Section 2 contains the statement of our main result: we give a definition of weak solutions, and state the existence of such solutions as long as no contact occurs. We explain the main difficulties in proving their existence, in comparison to the results available for no-slip conditions. We conclude Section 2 by an outline of our proof, to be carried out in sections 3 to 5. More precisely:

- Section 3 is devoted to an auxiliary nonlinear transport equation, which is crucial to our approximation procedure.
- Section 4 is dedicated to the construction of solutions for well-chosen approximations of the Navier-Stokes / solid dynamics.
- Section 5 describes the limit procedure, from the approximate to the exact system.

2 Main result and ideas

2.1 Weak solutions of Navier-Stokes with slip conditions

The aim of this paragraph is to define a weak formulation and weak solutions for system (1.1)-(1.2)-(1.4), that is in the case of slip conditions of Navier type. We remind that in the case of no-slip conditions, the theory of weak solutions has been successfully achieved over the last ten years, first up to collision (see [8]) and then globally in time (see [23] in the 2D case, [10] in the 3D case). Let us also mention the alternative approach in [3], and the recent result [14] on the uniqueness of 2D weak solutions up to collision.

As usual, in order to derive a weak formulation, the starting point is formal multiplication by appropriate test functions. These test functions must of course look like the solution itself. Notably, they must be rigid vector fields in the solid domain S. This forces the space of test functions to depend on the solution itself: it is a classical difficulty, already recognized in the no-slip case. *A key feature of the slip conditions is that these test functions, and also the solution, will be moreover discontinuous across the fluid/solid interface.* Indeed, the first line of (1.4) ensures the continuity of the normal component, but the tangential ones may have a jump. It is a strong difference with regards to boundary conditions (1.3), and it will generate many difficulties throughout the paper.

We first introduce some notation for the classical spaces of solenoidal vector fields. Let $\mathcal{O}$ be a Lipschitz domain. We set

$$
\begin{aligned}
&\mathcal{D}_\sigma(\mathcal{O}) := \{\varphi \in \mathcal{D}(\mathcal{O}), \quad \operatorname{div}\varphi = 0\}, \quad \mathcal{D}_\sigma(\overline{\mathcal{O}}) := \left\{\varphi|_{\mathcal{O}},\ \varphi \in \mathcal{D}_\sigma(\mathbb{R}^3)\right\},\\
&L^2_\sigma(\mathcal{O}) := \text{ the closure of } \mathcal{D}_\sigma(\mathcal{O}) \text{ in } L^2(\mathcal{O}), \quad H^1_\sigma(\mathcal{O}) := H^1(\mathcal{O}) \cap L^2_\sigma(\mathcal{O}),\\
&H^1_\sigma(\overline{\mathcal{O}}) := \text{ the closure of } \mathcal{D}_\sigma(\overline{\mathcal{O}}) \text{ in } H^1(\mathcal{O})
\end{aligned}
$$

We remind that elements u of $L^2_\sigma(\mathcal{O})$ satisfy $u \cdot \nu = 0$ in $H^{-1/2}(\partial\mathcal{O})$.

We also introduce the finite dimensional space of rigid vector fields in $\mathbb{R}^3$:

$$
\mathcal{R} := \{\varphi_s, \quad \varphi_s(x) = V + \omega \times x, \quad \text{for some } V \in \mathbb{R}^3,\ \omega \in \mathbb{R}^3\}.
$$

Finally, we define for any $T > 0$ the space $\mathcal{T}_T$ of test functions over $[0, T)$:

$$
\begin{aligned}
\mathcal{T}_T := \Big\{ &\varphi \in C([0,T]; L^2_\sigma(\Omega)), \text{ there exists } \varphi_F \in \mathcal{D}([0,T); \mathcal{D}_\sigma(\overline{\Omega})),\ \varphi_S \in \mathcal{D}([0,T); \mathcal{R})\\
&\text{such that } \varphi(t,\cdot) = \varphi_F(t,\cdot) \text{ on } F(t),\ \varphi(t,\cdot) = \varphi_S(t,\cdot) \text{ on } S(t), \text{ for all } t \in [0,T] \Big\}.
\end{aligned}
$$

Let us point out once again that this space of test functions depends on the solution itself through the domains $S(t)$ and $F(t)$. Let us also notice that the constraint $\varphi(t,\cdot) \in L^2_\sigma(\Omega)$ encodes in a weak form the continuity of the normal component at $\partial S(t)$:

$$
\varphi_F(t,\cdot) \cdot \nu = \varphi_S(t,\cdot) \cdot \nu \text{ at } \partial S(t), \quad \forall t \in [0,T).
$$

Multiplying (1.1) by $\varphi \in \mathcal{T}_T$, integrating over $F(t)$, and integrating by parts, we obtain (formally)

$$
\begin{aligned}
\frac{d}{dt}\int_{F(t)} \rho_F\, u_F \cdot \varphi_F - \int_{F(t)} \rho_F\, u_F \cdot \partial_t \varphi_F - \int_{F(t)} \rho_F\, u_F \otimes u_F : \nabla\varphi_F + \int_{F(t)} 2\mu_F D(u_F) : D(\varphi_F)\\
= \int_{\partial\Omega} (\Sigma_F \nu) \cdot \varphi_F + \int_{\partial S(t)} (\Sigma_F \nu) \cdot \varphi_F + \int_{F(t)} \rho_F(-g) \cdot \varphi_F,
\end{aligned}
$$

where the normal vectors ν, in the integrals at the right-hand side, point resp. outside Ω and inside $S(t)$. Using (1.4):

$$\int_{\partial\Omega} (\Sigma_F \nu)\cdot\varphi_F = -\frac{1}{2\beta_\Omega}\int_{\partial\Omega} (u_F\times\nu)\cdot(\varphi_F\times\nu),$$
$$\int_{\partial S(t)} (\Sigma_F \nu)\cdot\varphi_F = -\frac{1}{2\beta_S}\int_{\partial S(t)} ((u_F-u_S)\times\nu)\cdot((\varphi_F-\varphi_S)\times\nu) \;+\; \int_{\partial S(t)} (\Sigma_F\nu)\cdot\varphi_S$$

Eventually, one can use (1.2) to write differently the last integral: tedious but straightforward manipulations yield

$$\int_{\partial S(t)} (\Sigma_F\nu)\cdot\varphi_S = -\frac{d}{dt}\int_{S(t)} \rho_S\, u_S\cdot\varphi_S + \int_{S(t)} \rho_S\, u_S\cdot\partial_t\varphi_S + \int_{S(t)} \rho_S(-g)\cdot\varphi_S.$$

Combining the previous identities and integrating from 0 to T entails

$$\begin{aligned}
&-\int_0^T\int_{F(t)} \rho_F\, u_F\cdot\partial_t\varphi_F - \int_0^T\int_{S(t)} \rho_S\, u_S\cdot\partial_t\varphi_S + \int_0^T\int_{F(t)} \rho_F\, u_F\otimes u_F : \nabla\varphi_F \\
&+ \int_0^T\int_{F(t)} 2\mu_F D(u_F) : D(\varphi_F) + \frac{1}{2\beta_\Omega}\int_0^T\int_{\partial\Omega} (u_F\times\nu)\cdot(\varphi_F\times\nu) \\
&+ \frac{1}{2\beta_S}\int_0^T\int_{\partial S(t)} ((u_F-u_S)\times\nu)\cdot((\varphi_F-\varphi_S)\times\nu) \\
&= \int_0^T\int_{F(t)} \rho_F(-g)\cdot\varphi_F + \int_0^T\int_{S(t)} \rho_S(-g)\cdot\varphi_S \\
&+ \int_{F(0)} \rho_F u_{F,0}\cdot\varphi_F|_{t=0} + \int_{S(0)} \rho_S u_{S,0}\cdot\varphi_S|_{t=0}
\end{aligned} \tag{2.1}$$

Equation (2.1) is a global weak formulation of the momentum equations (1.1) and (1.2), taking the slip conditions (1.4) into account. Setting $\varphi = u$ in the above formal computations yields that, for all $t\in[0,T]$:

$$\begin{aligned}
&\int_{F(t)} \frac{1}{2}\rho_F|u_F(t,\cdot)|^2 + \int_{S(t)} \frac{1}{2}\rho_S|u_S(t,\cdot)|^2 \;+\; \int_0^t\int_{F(s)} 2\mu_F|D(u_F)|^2 ds \\
&+\; \frac{1}{2\beta_\Omega}\int_0^t\int_{\partial\Omega} |u_F\times\nu|^2 \;+\; \frac{1}{2\beta_S}\int_0^t\int_{\partial S(t)} |(u_F-u_S)\times\nu|^2 \\
&\le \int_0^t\int_{F(t)} \rho_F(-g)\cdot u_F \;+\; \int_0^t\int_{S(t)} \rho_S(-g)\cdot u_S \;+\; \int_{\Omega\setminus S_0} \rho_F\,|u_{F,0}|^2 \;+\; \int_{S_0} \rho_S\,|u_{S,0}|^2.
\end{aligned} \tag{2.2}$$

This goes together with the conservation of mass, that amounts to the transport of S by the rigid vector field u_S. It reads

$$\partial_t\chi_S + \mathrm{div}\,(u_S\chi_S) \;=\; 0 \;\text{ in }\Omega, \quad \chi_S(t,x) \;:=\; 1_{S(t)}(x),$$

or in a weak form: for all $\Psi\in\mathcal{D}([0,T),\mathcal{D}(\overline{\Omega}))$,

$$-\int_0^T\int_{S(t)} \partial_t\Psi \;-\; \int_0^T\int_{S(t)} u_S\cdot\nabla\Psi \;=\; \int_{S_0} \Psi|_{t=0}. \tag{2.3}$$

Pondering on these formal manipulations, we can now introduce our definition of a weak solution on $[0,T)$. *We fix once for all the positive constants* $\rho_S, \rho_F, \mu_F, \beta_S, \beta_\Omega$.

Definition 1 *Let Ω and $S_0 \subset \Omega$ two Lipschitz bounded domains of $\mathbb{R}^3$. Let $u_{F,0} \in L^2_\sigma(\Omega)$, $u_{S,0} \in \mathcal{R}$ such that $u_{F,0} \cdot \nu = u_{S,0} \cdot \nu$ on ∂S_0.*
A weak solution of (1.1)-(1.2)-(1.4) on $[0,T)$ (associated to the initial data S_0, $u_{F,0}$ and $u_{S,0}$) is a couple (S,u) satisfying

- *$S(t) \subset \Omega$ is a bounded domain of $\mathbb{R}^3$ for all $t \in [0,T)$, such that*

$$\chi_S(t,x) := 1_{S(t)}(x) \in L^\infty((0,T)\times\Omega).$$

- *u belongs to the space*

$$\mathcal{S}_T := \Big\{ u \in L^\infty(0,T;L^2_\sigma(\Omega)),\ \text{there exists}\ u_F \in L^2(0,T;H^1_\sigma(\Omega)),\ u_S \in L^\infty(0,T;\mathcal{R})$$
$$\text{such that}\ u(t,\cdot) = u_F(t,\cdot)\ \text{on}\ F(t),\ u(t,\cdot) = u_S(t,\cdot)\ \text{on}\ S(t),\ \text{for a. e.}\ t \in [0,T] \Big\},$$

where $F(t) := \Omega \setminus \overline{S(t)}$ for all $t \in [0,T)$.

- *Equation (2.1) is satisfied for all $\varphi \in \mathcal{T}_T$.*
- *Equation (2.3) is satisfied for all $\psi \in \mathcal{D}([0,T);\mathcal{D}(\overline{\Omega}))$.*
- *Equation (2.2) is satisfied for almost every $t \in (0,T)$.*

Let us conclude this paragraph by a few comments on this definition of weak solutions:

1. As $\chi_S \in L^\infty((0,T)\times\Omega)$, the integrals over $S(t)$ in (2.3) are integrable with respect to time: namely,

$$t \mapsto \int_{S(t)} \partial_t \Psi = \int_\Omega \chi_S \partial_t \Psi \quad \text{and} \quad t \mapsto \int_{S(t)} u_S \cdot \nabla\Psi = \int_\Omega \chi_S u_S \cdot \nabla \Psi$$

belong to $L^1(0,T)$. Actually, by the method of characteristics, as $u_S \in L^\infty(0,T;\mathcal{R})$ (rigid velocity field), it is easily seen that

$$S(t) = \phi_{t,0}(S_0)$$

for an isometric propagator $\phi_{t,s}$ which is Lipschitz continuous in time, smooth in space. It follows that all integrals in equation (2.1) make sense. For instance, as $\partial S(t)$ is Lipschitz for all t and fields $u_F, u_S, \varphi_F, \varphi_S$ have at least L^2H^1 regularity, the surface integral

$$\int_{\partial S(t)} ((u_F - u_S)\times n)\cdot((\varphi_F - \varphi_S)\times n)$$

can be defined for almost every t in the trace sense. Moreover, it defines an element of $L^1(0,T)$. This can be seen through the change of variable $x = \phi_{t,0}(y)$: the surface integral turns into

$$\int_{\partial S_0} \jmath\,(t, \phi_{t,0}(y))\ \mathrm{Jac}_\tau(y)\, dy,$$

where

$$\jmath(t,x) := ((u_F(t,x) - u_S(t,x))\times\nu)\cdot((\varphi_F(t,x) - \varphi_S(x))\times\nu)$$

and where

$$\mathrm{Jac}_\tau(y) = \|[\nabla\phi_{t,0}(y)]^{-1}\,\nu(y)\|_2 \det(\nabla\phi_{t,0}(y))\ (=1)$$

is the tangential jacobian (see [15, Lemme 5.4.1] for details). This clearly defines an element of $L^1(0,T)$.

2. Equations (2.1) and (2.3) involve fields $u_F, u_S, \varphi_F, \varphi_S$ defined over Ω and such that

$$u = (1-\chi_S)u_F + \chi_S u_S, \quad \varphi = (1-\chi_S)\varphi_F + \chi_S\varphi_S,$$

However, a closer look at equations (2.1) and (2.3) shows that they only involve

$$\chi_S u_S,\ \chi_F(1,\nabla)u_F, \quad \text{as well as } \chi_S(1,\partial_t)\varphi_S \text{ and } \chi_F(1,\partial_t,\nabla)\varphi_F.$$

In particular, they only depend on u and φ, not on the choice of the extended fields u_F, u_S and φ_F, φ_S.

3. The condition $u \in L^\infty(0,T;L^2_\sigma(\Omega))$ implies that

$$u_F \cdot \nu|_{\partial S(t)} = u_S \cdot \nu|_{\partial S(t)} \text{ for a.e. } t$$

all terms being again understood in the trace sense.

4. It is easy to see that equation (2.3), that is the transport equation

$$\partial_t \chi_S + \text{div}\,(\chi_S u_S) = 0 \text{ in } \mathcal{D}'([0,T)\times\overline{\Omega})$$

can be written

$$\partial_t \chi_S + \text{div}\,(\chi_S u) = 0 \text{ in } \mathcal{D}'([0,T)\times\overline{\Omega}) \tag{2.4}$$

and implies

$$\partial_t \chi_F + \text{div}\,(\chi_F u) = 0 \text{ in } \mathcal{D}'([0,T)\times\overline{\Omega}), \quad \chi_F(t,x) = \chi_{F(t)}(x) \tag{2.5}$$

(remind that $F(t) = \Omega\setminus\overline{S(t)}$). More generally, one can replace u by any $v \in L^\infty(0,T;L^2_\sigma(\Omega))$ satisfying

$$v(t,\cdot)\cdot\nu|_{\partial S(t)} = u\cdot\nu|_{\partial S(t)} = u_S\cdot\nu|_{\partial S(t)} \quad \text{for a.e. } t$$

where the last equality holds in the space $H^{-1/2}(\partial S(t))$ (see [11, Theorem 3.2.2]). Note that equations (2.4) and (2.5) should be replaced by

$$\partial_t\rho_s + \text{div}\,(\rho_s u) = 0, \quad \partial_t\rho_f + \text{div}\,(\rho_f u) = 0$$

in the case of inhomogeneous solid and fluid, with variable density functions ρ_s and ρ_f. See [10] in the case of no-slip conditions. Extension of the present work (on a single rigid and homogeneous solid in a homogeneous fluid) to more general configurations will be the matter of a forthcoming paper.

5. Noticing that

$$D(u(t,\cdot)) = D(u_S(t,\cdot)) = 0 \text{ in } S(t), \quad D(\varphi(t,\cdot)) = D(\varphi_S(t,\cdot)) = 0 \text{ in } S(t)$$

it is tempting to write (2.1) under the condensed form

$$-\int_0^T\int_\Omega \rho u\cdot\partial_t\varphi + \int_0^T\int_\Omega \rho u\otimes u : D(\varphi) + \int_0^T\int_\Omega 2\mu_F D(u):D(\varphi) = \text{"boundary terms"} \tag{2.6}$$

where $\rho := \rho_F \chi_F + \rho_S \chi_S$, coupled with the global transport equation

$$\partial_t \rho + \text{div } (\rho u) = 0 \text{ in } \Omega. \tag{2.7}$$

This kind of global formulation, reminiscent of the inhomogeneous Navier-Stokes equations, is used in the construction of weak solutions with Dirichlet boundary conditions: *cf* [23]. *However, it is not valid here*: due to the discontinuity of the tangential components of u and φ, neither $\partial_t \varphi$ nor $D(u)$ and $D(\varphi)$ belong to $L^2(\Omega)$. For instance,

$$\partial_t \varphi = \chi_F \partial_t \varphi_F \ + \ \chi_S \partial_t \varphi_S \ + \ u_S \cdot \nu \, (\varphi_F - \varphi_S) \, \delta_{\partial S}$$

where $\delta_{\partial S}$ is the Dirac mass along the solid boundary ∂S. This is why we keep the formulation (2.1), distinguishing between the solid and the fluid part.

6. The definition of a weak solution that we consider can not be satisfactory after collision. Indeed, we do not specify any rebound law. Moreover, in the case of Dirichlet conditions at the fluid-solid interface, explicit examples show that the analogue of our weak solution is not unique [26].

2.2 Main result

Our result is the following

Theorem 1 (Existence of weak solutions up to collision)
Let Ω and $S_0 \Subset \Omega$ two $C^{1,1}$ bounded domains of $\mathbb{R}^3$. Let $u_{F,0} \in L^2_\sigma(\Omega)$, $u_{S,0} \in \mathcal{R}$ such that $u_{F,0} \cdot \nu = u_{S,0} \cdot \nu$ on ∂S_0.
There exists $T > 0$ and a weak solution of (1.1)-(1.2)-(1.4) on $[0, T)$ (associated to the initial data S_0, $u_{F,0}$ and $u_{S,0}$). Moreover, such weak solution exists up to collision, that is

$$S(t) \Subset \Omega \quad \textit{for all } \ t \in [0, T), \quad \textit{and} \ \lim_{t \to T^-} \textit{dist}(S(t), \partial\Omega) = 0.$$

The rest of the paper will be devoted to the proof of the theorem. Briefly, there are two main difficulties compared to the case of Dirichlet conditions:

- The lack of a unified formulation such as (2.6).
- The lack of a uniform H^1 bound on solutions u.

These difficulties appear both in the construction of approximate solutions, and in the convergence process.

Indeed, the approximation of fluid-solid systems is usually adressed by relaxing the solid constraint, through a penalization term. In this way, one is left with approximate systems that are close to density dependent Navier-Stokes equations. Roughly, they read

$$\begin{cases} \partial_t(\rho_n u_n) \ + \ \text{div } (\rho_n u_n \otimes u_n) \ + \ \ldots \ = \ \text{penalization} \\ \partial_t \rho_n + \text{div } (\rho_n u_n) = 0. \end{cases} \tag{2.8}$$

In the case of no slip conditions, in which a global formulation of type (2.6)-(2.7) already holds, to build such approximation is quite natural. But in the case of Navier conditions, this is not easy.

Once an approximate sequence of solutions (ρ_n, u_n) has been derived, Dirichlet conditions allow for uniform H^1 bounds on u_n. This simplifies a lot of convergence arguments, notably with regards to the transport equation

$$\partial_t \rho_n + \text{div } (\rho_n u_n) = 0$$

to which the classical DiPerna-Lions theory applies straightforwardly [9]. Also, it helps to provide strong convergence of u_n in $L^2((0,T) \times \Omega)$. In short, the lack of bound on $\partial_t(\rho_n u_n)$ (due to the penalization term) can be overcome by considering the fields $P_\delta(t) u_n$, where $P_\delta(t)$ is the orthogonal projection in $H^1_\sigma(\Omega)$ over the fields that are rigid in a δ-neighborhood of $S(t)$. One can show that $P_\delta(t) u_n$ has good equicontinuity properties uniformly in δ and n.

In the case of Navier boundary conditions, no uniform bound is available in H^1. This forces us to use more the structure of the solution u, in particular the H^1 bounds on the fluid and solid domains separately. This is also a source of trouble for the construction of approximate solutions, as one must find an approximation scheme in which such structure is not too much broken.

2.3 Strategy of proof

Let us describe here briefly the main lines of our proof. Let $S_0, u_{F,0}, u_{S,0}$ as in Theorem 1, and

$$\rho_0 \;:=\; \rho_F(1 - 1_{S_0}) \;+\; \rho_S 1_{S_0}, \quad u_0 \;:=\; (1 - 1_{S_0}) u_{F,0} \;+\; 1_{S_0} u_{S,0}\,.$$

The keypoint is to consider approximate problems of the following type: *find* (S^n, u^n) *such that*

a) *$S^n(t) \subset \Omega$ is a bounded Lipschitz domain for all $t \in [0,T]$, such that*

$$\chi^n_S(t,x) \;:=\; 1_{S^n(t)}(x) \in L^\infty((0,T) \times \Omega) \cap C([0,T]; L^p(\Omega)), \; \forall\, p < +\infty$$

b) $u^n \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_\sigma(\Omega))$.

c) *For all $\varphi \in H^1(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_\sigma(\Omega))$ s.t. $\varphi|_{t=T} = 0$:*

$$\begin{aligned}
&- \int_0^T \int_\Omega \rho^n \, (u^n \partial_t \varphi + v^n \otimes u^n : \nabla \varphi) \;+\; \int_0^T \int_\Omega 2\mu^n D(u^n) : D(\varphi) \\
&+ \frac{1}{2\beta_\Omega} \int_0^T \int_{\partial\Omega} (u^n \times \nu) \cdot (\varphi \times \nu) \;+\; \frac{1}{2\beta_S} \int_0^T \int_{\partial S^n(t)} ((u^n - P^n_S u^n) \times \nu) \cdot ((\varphi - P^n_S \varphi) \times \nu) \\
&+ n \int_0^T \int_\Omega \chi^n_S (u^n - P^n_S u^n) \cdot (\varphi - P^n_S \varphi) \;=\; \int_0^T \int_\Omega \rho^n (-g) \cdot \varphi \;+\; \int_\Omega \rho_0 \, u_0 \cdot \varphi|_{t=0}
\end{aligned}$$

d) $\partial_t \chi^n_S \;+\; P^n_S u^n \cdot \nabla \chi^n_S = 0, \quad \chi^n_S|_{t=0} = 1_{S_0}$.

In above lines,

- $\rho^n \;:=\; \rho_F(1 - \chi^n_S) \;+\; \rho_S \chi^n_S$ is the total density function.
- $\mu^n \;:=\; \mu_F(1 - \chi^n_S) \;+\; \frac{1}{n^2} \chi^n_S$ is an inhomogeneous viscosity coefficient.
- $P^n_S = P^n_S(t)$ is the orthogonal projection in $L^2(S^n(t))$ over rigid fields. This means that:

$$\forall\, 0 \le t < T, \quad \forall u_S \in \mathcal{R}, \quad \forall u \in L^2_\sigma(\Omega), \quad P^n_S(t)u \in \mathcal{R} \quad \text{and} \quad \int_\Omega \chi^n_S(t,\cdot)(u - P^n_S(t)u)\cdot u_S = 0.$$

- Eventually,

$$v^n \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_\sigma(\Omega))$$

  is a field that satisfies

$$\begin{aligned} v^n(t,\cdot) &= P^n_S(t)\, u^n(t,\cdot) \text{ in } S^n(t), \\ v^n(t,\cdot) &= u^n(t,\cdot) \text{ outside a } \delta \text{ neighborhood of } S^n(t),\ t \in [0,T), \end{aligned}$$

  for some δ fixed and arbitrary in $(0, \mathrm{dist}(S_0, \partial\Omega)/2)$. Moreover, v^n will be chosen so that it is close to u^n outside S^n (in L^p topology). In this way, it will asymptotically coincide with the limit u of u^n. Further details on the definition of v^n will be provided in due course.

Let us make a few comments on such approximate problems:

1. They rely on the use of the fields $P^n_S u^n$, that were already introduced in [3] in the context of Dirichlet conditions. These fields appear both:

   i) in the transport equation for χ^n_S. They will allow for a good control of the trajectories of the approximate solid bodies S^n.

   ii) in the penalization term $n \int_0^T \int_\Omega \chi^n_S (u^n - P^n_S u^n)\cdot(\varphi - P^n_S \varphi)$. Formally, as n goes to infinity, this term will allow to recover the rigid constraint inside the solid.

2. Note that *a contrario* to the large penalization term, the viscosity term μ^n vanishes asymptotically in the solid part. Hence, there will be no uniform bound in $H^1_\sigma(\Omega)$ for u^n, as expected (see the discussion in paragraph 2.2).

3. A specificity of these approximate problems is that the transport equation **d)** is nonlinear in χ^n_S for a given u^n. Indeed, P^n_S depends on χ^n_S (*cf* the formula in section 3). The whole section 3 is dedicated to this auxiliary nonlinear transport equation, which is a keystone of the approximation procedure.

4. Once the solution χ^n_S of **d)** is found and seen as a functional of u^n, equation **c)** can be written as $\mathcal{F}(u^n) = 0$ for some functional $\mathcal{F}$ from $L^\infty L^2_\sigma \cap L^2 H^1_\sigma$ into itself. In short, we shall solve this equation by a Galerkin procedure: we shall look for an approximate solution $u^{n,N}(t,x) = \sum_{k=0}^N \alpha_k(t) e_k(x)$ where (e_k) is an orthonormal basis of $L^2_\sigma(\Omega)$. We shall solve approximate equations $\mathcal{F}^{n,N}(u^{n,N}) = 0$ by Schauder's theorem and pass to the limit with respect to N. This process is explained in section 4.

5. Note that the field v^n satisfies

$$v^n(t,\cdot)\cdot\nu|_{\partial S^n(t)} \;=\; P^n_S(t) u^n(t,\cdot)\cdot \nu|_{\partial S^n(t)}.$$

   In particular, one can write

$$\partial_t \chi^n_S \;+\; v^n \cdot \nabla \chi^n_S = 0, \quad \text{and} \quad \partial_t \rho^n \;+\; v^n\cdot\nabla\rho^n = 0.$$

This will allow to obtain energy estimates in a standard way, in the spirit of the approximate systems (2.8) used for Dirichlet conditions. The price to pay is the necessary control of $u^n - v^n$, which will exhibit strong gradients near ∂S^n as $n \to +\infty$. Moreover, a similar "boundary layer behaviour" will be involved in the approximation of discontinuous test functions $\varphi \in \mathcal{T}_T$ by continuous test functions φ^n (involved in **c)**). The whole convergence process will be analyzed in section 5.

3 A nonlinear transport equation

Let $T > 0$, $u \in L^\infty(0, T; L^2_\sigma(\Omega))$. This section is devoted to the equation

$$\partial_t \chi_S \; + \; P_S u \cdot \nabla \chi_S, \quad \chi_S|_{t=0} = 1_{S_0},$$

where $P_S u$ is defined by the following formula

$$P_S u \; := \; \frac{1}{M} \int_\Omega \rho_S \, \chi_S \, u \; + \; \left(J^{-1} \int_\Omega \rho_S \, \chi_S \, \big((x' - x_S) \times u \big) \, dx' \right) \times (x - x_S) \tag{3.1}$$

where the center of mass, total mass and inertia tensor of the solid are defined by

$$x_S \; := \; \int_{\mathbb{R}^3} \rho_S \chi_S, \quad M \; := \; \int_{\mathbb{R}^3} \rho_S \chi_S, \tag{3.2}$$

and

$$J \; := \; \int_{\mathbb{R}^3} \rho_S \chi_S \left(|x - x_S|^2 I_d - (x - x_S) \otimes (x - x_S) \right) dx. \tag{3.3}$$

If $\chi_S(t, x) = 1_{S(t)}(x)$ with $S(t)$ a subdomain of Ω , $P_S(t)$ is the orthogonal projection in $L^2(S(t))$ over rigid vector fields, see [3].

We start with the regular case, that is when $u \in C([0, T]; \mathcal{D}_\sigma(\overline{\Omega}))$. This case will be useful for Galerkin approximations of **a)-d)**.

Proposition 2 (Well-posedness)
Let $u \in C([0, T]; \mathcal{D}_\sigma(\overline{\Omega}))$.
i) There is a unique solution $\chi_S \in L^\infty((0, T) \times \mathbb{R}^3) \cap C([0, T]; L^p(\mathbb{R}^3))$ $(p < \infty)$ of

$$\partial_t \chi_S + div \; (\chi_S \, P_S u) = 0 \;\; in \; \mathbb{R}^3, \quad \chi_S|_{t=0} \; = \; 1_{S_0}. \tag{3.4}$$

ii) Moreover $\chi_S(t, \cdot) = 1_{S(t)}$ for all t, with $S(t)$ a Lipschitz bounded domain. More precisely,

$$S(t) = \phi_{t,0}(S_0)$$

for the isometric propagator $\phi_{t,s}$ associated to $P_S u$: $(t, s) \mapsto \phi_{t,s} \in C^1([0, T]^2; C^\infty_{loc}(\mathbb{R}^3))$.

Proof. We can suppose $u \in C([0, T]; \mathcal{D}_\sigma(\mathbb{R}^3))$ with no loss of generality.

Assume for a moment that we have found a solution χ_S of (3.4). Then, we can see (3.4) as a linear transport equation, with given transport $P_S u \in C([0, T]; \; \mathcal{R})$. By the method of characteristics, we get easily

$$\chi_S(t, \phi_{t,0}(y)) = 1_{S_0}(y), \tag{3.5}$$

where $\phi_{t,s}$ is the isometric propagator defined by

$$\begin{cases} \phi_{s,s}(y) = y\,, \quad \forall\, y \in \mathbb{R}^3, \\ \partial_t \phi_{t,s}(y) = P_S u(t, \phi_{t,s}(y)) \quad \forall\,(s,t,x) \in (0,T)^2 \times \mathbb{R}^3. \end{cases} \tag{3.6}$$

Now, we use (3.5) in the expression (3.1) for $P_S u$. We obtain:

$$\begin{aligned} P_S u(t,x) = &\frac{1}{M} \int_{S_0} \rho_S\, 1_\Omega(\phi_{t,0}(y))\, u(t, \phi_{t,0}(y)) dy \\ &+ \left(J^{-1}(t) \int_{S_0} \rho_S\, 1_\Omega(\phi_{t,0}(y))\, (\phi_{t,0}(y) - x_S(t)) \times u(t, \phi_{t,0}(y)) dy \right) \times (x - x_S(t)) \end{aligned} \tag{3.7}$$

where $M := |S_0|\, \rho_S, \quad x_S(t) := \displaystyle\int_{S_0} \rho_S \phi_{t,0}(y) dy,$ and

$$J(t) := \int_{S_0} \rho_S \Big(|\phi_{t,0}(y) - x_S(t)|^2 I_d - (\phi_{t,0}(y) - x_S(t)) \otimes (\phi_{t,0}(y) - x_S(t)) \Big) dy.$$

In particular, denoting $\mathrm{Isom}(\mathbb{R}^3) \approx \mathbb{R}^3 \times O_3(\mathbb{R})$ the finite dimensional manifold of affine isometries, we deduce from (3.6) and (3.7) that $t \mapsto \phi_{t,0},\ [0,T] \mapsto \mathrm{Isom}(\mathbb{R}^3)$ satisfies an ordinary differential equation, of the type

$$\frac{d}{dt} \phi_{t,0} = U_S(t, \phi_{t,0}), \quad \phi_{0,0} = I_d, \tag{3.8}$$

for a time-dependent vector field U_S over $\mathrm{Isom}(\mathbb{R}^3)$. Namely, $U_S(t,\phi) \in T_\phi(\mathrm{Isom}(\mathbb{R}^3)) \approx \mathcal{R}$ is defined by the same formula as in (3.7), replacing everywhere $\phi_{t,0}$ by ϕ.

Conversely, if we manage to show existence of and uniqueness of a C^1 solution of (3.8) over $[0,T]$, then formula (3.5) will define the unique solution χ_S of the nonlinear equation (3.4), proving Theorem 2.

Hence, it only remains to study the well-posedness of (3.8). We can identify $\mathrm{Isom}(\mathbb{R}^3)$ with $\mathbb{R}^3 \times O_3(\mathbb{R}) \subset \mathbb{R}^3 \times \mathbb{R}^9$, and identify all tangent spaces with $\mathcal{R} \subset \mathbb{R}^3 \times \mathbb{R}^9$. By the Cauchy-Lipschitz theorem, there is existence and uniqueness of a C^1 maximal solution if U_S is continuous in t, ϕ, locally Lipschitz in ϕ. Considering the expression of U_S, see (3.7), this follows from

Lemma 3 *Let $v \in C([0,T]; C^\infty_{loc}(\mathbb{R}^3))$. Then, the function*

$$\mathcal{M} : [0,T] \times \mathrm{Isom}(\mathbb{R}^3) \mapsto \mathbb{R}, \quad \mathcal{M}(t,\phi) = \int_{S^0} 1_\Omega(\phi(y)) v(t, \phi(y)) dy$$

is continuous in (t,ϕ), and uniformly Lipschitz in ϕ over $[0,T]$.

Proof of the lemma. The continuity is obvious. Then, for two affine isometries ϕ and ϕ', we write

$$\begin{aligned} \mathcal{M}(t,\phi) - \mathcal{M}(t,\phi') = &\int_{S_0} 1_\Omega(\phi(y)) \, \big(v(t,\phi) - v(t,\phi'(y))\big) \; + \; \int_{S_0} \big(1_\Omega(\phi(y)) - 1_\Omega(\phi'(y))\big)\; v(t,\phi'(y))\, dy \\ := &\; M_1(t) + M_2(t). \end{aligned}$$

Clearly,

$$|M_1(t)| \;\le \sup_{\substack{t\in[0,T], \\ |x| \le \|(\phi,\phi')\|_\infty}} |\partial_x v(t,x)| \int_{S_0} |\phi(y) - \phi'(y)| dy \;\le\; C_{\phi,\phi'} ||\phi - \phi'||_\infty.$$

As regards M_2, we write

$$M_2(t) \;\le\; \sup_{\substack{t\in[0,T],\\ |x|\le\|\phi'\|_\infty}} |v(t,x)| \int_{\mathbb{R}^3} \left|1_\Omega(\phi(y)) - 1_\Omega(\phi'(y))\right| \, dy \;\le\; C_{\phi'} \int_{\mathbb{R}^3} \left|1_\Omega(\phi(y)) - 1_\Omega(\phi'(y))\right| \, dy$$

For each y, the integrand is non-zero if and only if $\phi(y) \in \Omega$ and $\phi'(y) \in \Omega^c$ or vice-versa. As $|\phi(y) - \phi'(y)| \le ||\phi - \phi'||_\infty$, this is only possible if $\phi(y)$ and $\phi'(y)$ are in a $||\phi - \phi'||_\infty$-neighborhood (say V) of $\partial\Omega$. Hence,

$$|M_2(t)| \;\le\; C_{\phi,\phi'} \left(\int_{\phi^{-1}(V)} dy + \int_{\phi'^{-1}(V)} dy \right) \;\le\; 2C_{\phi,\phi'}|V| \;\le\; C'_{\phi,\phi'}||\phi - \phi'||_\infty.$$

This concludes the proof of the lemma.

Last step is to prove that the maximal solution is defined over the whole interval $[0, T]$. From (3.6)-(3.7), one can write

$$\phi_{t,0}(y) = x_S(t) + Q_S(t)y$$

where $x_S(t)$ is defined in (3.6) and $Q_S(t)$ is an orthogonal matrix. In particular, the only way that the maximal solution is not global on $[0, T]$ is through a blow-up of x_S. But, again, from (3.7),

$$|\frac{d}{dt}x_S(t)| \;=\; \frac{1}{M} \left| \int_{S_0} \rho_S \, 1_\Omega(\phi_{t,0}(y)) \, u(t, \phi_{t,0}(y)) dy \right| \;\le\; C \, \|u\|_{L^\infty((0,T)\times\Omega)}$$

which prevents any blow-up. This ends the proof of the theorem.

Proposition 4 (Strong sequential continuity) *Assume that*

$$u^n \to u \quad \textit{in } C([0,T]; \mathcal{D}_\sigma(\overline{\Omega})).$$

Then with obvious notations, one has

$$\chi^n_S \to \chi_S \quad \textit{weakly * in } L^\infty((0,T)\times\mathbb{R}^3), \quad \textit{strongly in } C([0,T]; L^p_{loc}(\mathbb{R}^3)) \; (p < \infty),$$

as well as

$$P^n_S u^n \to P_S u \quad \textit{strongly in } C([0,T]; C^\infty_{loc}(\mathbb{R}^3)), \quad \phi^n \to \phi \quad \textit{strongly in } C^1([0,T]^2; C^\infty_{loc}(\mathbb{R}^3)).$$

Proof of the proposition. As u^n converges in $C([0,T]; \mathcal{D}_\sigma(\overline{\Omega}))$, we have that $P^n_S u^n$ is bounded in $L^\infty(0,T; L^2(\Omega))$. Similarly χ^n_S is bounded in $L^\infty((0,T) \times \Omega)$. Furthermore, up to a subsequence that we do not relabel, $P^n_S u^n$ converges weakly-* in $L^\infty(0,T; H^1_{loc}(\mathbb{R}^3))$ to some $\bar{u}_S$ and $\chi^n_S(0)(= 1_{S_0},$ for all $n \in \mathbb{N})$ converges strongly in $L^1(\Omega)$. Applying Di Perna-Lions theory, we obtain that χ^n_S converges weakly-* in $L^\infty((0,T)\times\Omega)$ and strongly in $C([0,T]; L^p_{loc}(\Omega))$ for all finite p. Its limit $\bar{\chi}_S$, satisfies :

$$\partial_t \bar{\chi}_S + \mathrm{div}(\bar{\chi}_S \bar{u}_S) = 0\,.$$

Using the convergence of both χ^n_S and u^n in equation (3.1), we obtain that $\bar{u}_S = \bar{P}_S u$, where $\bar{P}_S$ is defined similarly to $P_S u$, replacing χ_S by $\bar{\chi}_S$. Moreover, the convergence of $P^n_S u$ holds in $C([0,T]; C^\infty_{loc}(\mathbb{R}^3))$. Consequently, $(\bar{\chi}_S, \bar{P}_S u)$ is the unique solution of (3.4) so that $\bar{\chi}_S = \chi_S$ and $\bar{P}_S u = P_S u$, and all the sequence converges.

To derive the convergence of the propagators ϕ^n from the convergence of the vector fields $P^n_S u^n$ is then standard, and we omit it for brevity.

Proposition 5 **(Weak sequential continuity)**
Let (u^n, χ_S^n) *be a bounded sequence in* $L^\infty(0,T; L^2_\sigma(\Omega)) \times L^\infty((0,T)\times\Omega)$*, satisfying*

$$\partial_t \chi_S^n + \mathit{div}\,(P_S^n u^n\, \chi_S^n) = 0 \;\; \mathit{in}\; \mathbb{R}^3, \quad \chi_S^n|_{t=0} \;=\; 1_{S_0}.$$

Then, up to a subsequence, one has

$$u^n \to u \quad \mathit{weakly * in}\;\; L^\infty(0,T; L^2_\sigma(\Omega))$$

$$\chi_S^n \to \chi_S \quad \mathit{weakly * in}\;\; L^\infty((0,T)\times\mathbb{R}^3), \quad \mathit{strongly\ in}\; C([0,T]; L^p_{loc}(\mathbb{R}^3))\; (p < \infty),$$

with (u_S, χ_S) *a solution of*

$$\partial_t \chi_S + \mathit{div}\,(P_S u\, \chi_S) = 0 \;\; \mathit{in}\; \mathbb{R}^3, \quad \chi_S|_{t=0} \;=\; 1_{S_0}.$$

Moreover, χ_S *satisfies condition ii) of Proposition 2, and the following additional convergences hold:*

$$\begin{aligned} P_S^n u^n \to P_S u \qquad & \mathit{weakly * in} \qquad L^\infty(0,T; C^\infty_{loc}(\mathbb{R}^3)), \\ \phi^n \to \phi \qquad & \mathit{weakly * in} \qquad W^{1,\infty}((0,T)^2; C^\infty_{loc}(\mathbb{R}^3)) \quad \mathit{strongly\ in}\; C([0,T]^2; C^\infty_{loc}(\mathbb{R}^3))\,. \end{aligned}$$

Proof. The proof follows the same scheme as the previous one. We only sketch the arguments. First, up to the extraction of a subsequence, we obtain that

$$u^n \to u \quad \text{weakly * in}\;\; L^\infty(0,T; L^2_\sigma(\Omega))$$

Then, as before, we obtain that $P_S^n u^n$ is bounded in $L^\infty(0,T; C^\infty_{loc}(\mathbb{R}^3))$. This yields that

$$P_S^n u^n \to \bar{P} u \text{ weakly * in }\; L^\infty(0,T; H^1_{loc}(\mathbb{R}^3))$$

(still up to a subsequence). We then deduce applying Di Perna-Lions theory that, up to the extraction of a subsequence, χ_S^n converges strongly in $C([0,T]; L^p_{loc}(\mathbb{R}^3))$ to some χ_S, which in turn implies that $\bar{u}_S = P_S u$ and that $(\chi_S, P_S u)$ is a solution to our tranport equation. Eventually, uniform bounds on ϕ^n and $\partial_t \phi^n$ (which imply weak-* convergence of a subsequence in $W^{1,\infty}$) follow easily.

4 Approximation

This section is devoted to the resolution of approximate fluid-solid systems. These approximate systems were introduced in paragraph 2.3, *cf* **a)-d)**. The previous section has focused on the transport equation **d)**. It remains to examine **c)**. At first, we explain a little how the field v^n connecting $P_S^n u^n$ to u_S^n is defined. The detailed definition of v^n will be achieved in section 5.

4.1 Connecting velocity

We first remind a classical result on the equation div $u = f$, taken from [11, Exercise III.3.5]:

Proposition 6 *Let* $\mathcal{O}$ *be a bounded Lipschitz domain. Let* $f \in L^2(\mathcal{O})$ *and* $\varphi \in H^{1/2}(\partial\mathcal{O})$ *satisfying the compatibility condition*

$$\int_{\mathcal{O}} f = \int_{\partial\mathcal{O}} \varphi \cdot \nu.$$

Then there exists a solution $u \in H^1(\mathcal{O})$ *of*

$$div\, u = f \quad in\ \mathcal{O}, \quad u = \varphi \quad at\ \partial\mathcal{O}$$

with

$$\|u\|_{H^1(\mathcal{O})} \,\le\, C_{\mathcal{O}} \left(\|\varphi\|_{H^{1/2}(\partial\mathcal{O})} \,+\, \|f\|_{L^2(\mathcal{O})} \right).$$

The previous proposition yields easily

Corollary 7 (Extension of solenoidal vector fields)
There exists a continuous linear operator $E_\Omega \,:\, H^1_\sigma(\overline{\Omega}) \,\mapsto\, H^1_\sigma(\mathbb{R}^3)$ *satisfying* $E_\Omega\, u \,=\, u$ *on* Ω*. Moreover, for all open subset* $\omega \Subset \Omega$*,*

$$\|E_\Omega\, u\|_{H^1(\mathbb{R}^3 \setminus \overline{\omega})} \,\le\, C_\omega \|u\|_{H^1(\Omega\setminus\overline{\omega})}, \quad \forall\, u \in H^1_\sigma(\overline{\Omega}).$$

Corollary 8 (Connection of solenoidal vector fields)
For all $\delta > 0$*, there exists a continuous linear operator*

$$V^\delta \,:\, H^1_\sigma\left(\mathbb{R}^3 \setminus S_0\right) \times H^1_\sigma(\overline{S_0}) \,\mapsto\, H^1_\sigma(\mathbb{R}^3), \quad (U, U_S) \,\mapsto\, V^\delta[U, U_S]$$

such that

$$\begin{aligned} V^\delta[U, U_S] \,&=\, U_S \quad in\ S_0, \\ V^\delta[U, U_S] \,&=\, U \quad outside\ a\ \delta\ neighborhood\ of\ S_0. \end{aligned}$$

From there, we have the following

Proposition 9 *For all* $\delta > 0$*, there exists a continuous mapping*

$$v^\delta \,:\, L^2(0, T; H^1_\sigma(\mathbb{R}^3)) \times L^\infty(0, T; \mathcal{R}) \,\mapsto\, L^2(0, T; H^1_\sigma(\mathbb{R}^3)), \quad (u, u_S) \mapsto v^\delta[u, u_S]$$

such that

$$\begin{aligned} v^\delta[u, u_S](t, \cdot) \,&=\, u_S(t, \cdot) \quad in\ S(t), \\ v^\delta[u, u_S](t, \cdot) \,&=\, u(t, \cdot) \quad outside\ a\ \delta\ neighborhood\ of\ S(t),\ t \in [0, T), \end{aligned}$$

where, as usual, $S(t) \,:=\, \phi_{t,0}(S_0)$ *and* $\phi \,=\, \phi_{t,s}$ *is the isometric propagator associated to* u_S*. Moreover,* v^δ *can be chosen so that*

$$\|v^\delta[u, u_S]\|^2_{L^2(0,T;H^1(\mathbb{R}^3))} \,\le\, \mathcal{C} \int_0^T \left(\|u(t, \cdot)\|^2_{H^1(\mathbb{R}^3 \setminus \overline{S(t)})} \,+\, \|u_S(t, \cdot)\|^2_{L^2(S(t))} \right) dt,$$

where $\mathcal{C}$ *depends on* δ *and* T*.*

Proof of the proposition. The proposition can be deduced from Corollary 8 using Lagrangian coordinates. Namely, we introduce U and U_S through the relations

$$u\left(t, \phi_{t,0}(y)\right) \,=\, d\phi_{t,0}|_y \left(U(t, y)\right), \quad u_S\left(t, \phi_{t,0}(y)\right) \,=\, d\phi_{t,0}|_y \left(U_S(t, y)\right).$$

Clearly, for all t, $U(t,\cdot)$ and $U_S(t,\cdot)$ define elements of $H^1_\sigma(\mathbb{R}^3 \setminus S_0)$ and $H^1_\sigma(\overline{S_0})$ respectively. Using Corollary 8, we define $v^\delta[u, u_S]$ through the relation

$$v^\delta[u, u_S]\,(t, \phi_{t,0}(y)) \;=\; d\phi_{t,0}|_y \left(V^\delta[U(t,\cdot), U_S(t,\cdot)](y) \right).$$

It fulfills all requirements, which ends the proof.

Back to system **c)**, the idea is to define

$$v^n \;:=\; v^\delta[E_\Omega u^n, P^n_S u^n].$$

Clearly, for any time T^n such that

$$\mathrm{dist}(S^n(t), \partial\Omega) \geq 2\delta, \quad t \in [0, T^n],$$

$v^n|_\Omega$ will belong to $L^2(0, T^n; H^1_\sigma(\Omega))$ and will satisfy

$$\begin{aligned} v^n(t,\cdot) \;&=\; P^n_S(t)\, u^n(t,\cdot) \quad \text{in } S^n(t), \\ v^n(t,\cdot) \;&=\; u^n(t,\cdot) \quad \text{outside a } \delta \text{ neighborhood of } S^n(t),\ t \in [0, T^n). \end{aligned}$$

Let us stress that there is still some latitude left in the construction of v^n, through the choice of the operator V^δ in Corollary 8. As will be shown in section 5, this operator can be chosen depending on n ($V^\delta = V^{\delta,n}$) so that v^n is close to u^n outside S^n (in L^p topology). However, this additional property will not be needed until section 5.

Last remark: the resolution of **a)-d)**, and the whole construction of weak solutions, will be first performed on a small time interval $[0, T]$, for a time T that is uniform in n. Existence of weak solutions up to collision will follow from a continuation argument, to be explained at the end of section 5.

4.2 Galerkin approximation

As pointed out in paragraph 2.3, the resolution of **a)-d)** is carried out through a Galerkin scheme. Let $(e_k)_{k\geq 1}$ being both an orthonormal basis of $L^2_\sigma(\Omega)$ and a basis of $H^1_\sigma(\Omega)$, with elements in $\mathcal{D}_\sigma(\overline{\Omega})$. The aim of this paragraph is to find for all N, n and some $T > 0$ a couple (S^N, u^N) satisfying

a') $S^N(t) \subset \Omega$ is a bounded Lipschitz domain for all $t \in [0, T]$, such that

$$\chi^N_S(t, x) \;:=\; 1_{S^N(t)}(x) \in L^\infty((0,T) \times \Omega) \cap C([0,T]; L^p(\Omega))\ (p < \infty)$$

b') $u^N(t,\cdot) \;=\; \displaystyle\sum_{i=1}^N \alpha_k(t) e_k, \quad$ with $\alpha = (\alpha_1, \ldots, \alpha_N) \in C([0,T])^N$.

c') For all $\varphi \in \mathcal{D}([0,T);\ \mathrm{span}(e_1, \ldots, e_N))$

$$\begin{aligned} &-\int_0^T \int_\Omega \rho^N \left(u^N \cdot \partial_t \varphi + v^N \otimes u^N : \nabla\varphi\right) + \int_0^T \int_\Omega 2\mu^N D(u^N) : D(\varphi) \\ &+ \frac{1}{2\beta_\Omega} \int_0^T \int_{\partial\Omega} (u^N \times \nu) \cdot (\varphi \times \nu) + \frac{1}{2\beta_S} \int_0^T \int_{\partial S^N(t)} ((u^N - P^N_S u^N) \times \nu) \cdot ((\varphi - P^N_S \varphi) \times \nu) \\ &+ n \int_0^T \int_\Omega \chi^N_S (u^N - P^N_S u^N) \cdot (\varphi - P^N_S \varphi) \;=\; \int_0^T \int_\Omega \rho^N (-g) \cdot \varphi + \int_\Omega \rho_0\, u_0 \cdot \varphi|_{t=0} \end{aligned}$$

d') $\partial_t \chi_S^N \,+\, P_S^N u^N \cdot \nabla \chi_S^N = 0$ in Ω, $\quad \chi_S^N|_{t=0} = 1_{S_0}$.

In above lines, similarly to the original problem:

- $\rho^N \,:=\, \rho_F(1-\chi_S^N) \,+\, \rho_S \chi_S^N$ is the total density function.
- $\mu^N \,:=\, \mu_F(1-\chi_S^N) \,+\, \frac{1}{n^2}\chi_S^N$ is an inhomogeneous viscosity coefficient.
- $P_S^N = P_S^N(t)$ is defined by (3.1), adding the upperscript N everywhere.
- Eventually, $v^N = v^\delta[u^N, P_S^N u^N]$, see paragraph 4.1.

Note that all quantities above depend on n, notably through the penalization term and the viscosity coefficient. But we omit n from the notations to lighten writings. Also, note that u^N can be seen as an element of $L^2(0,T; H^1_\sigma(\mathbb{R}^3))$, as the e_k are defined globally. In particular, $v^N = v^\delta[u^N, P_S^N u^N]$ is well-defined.

The main result of this paragraph is

Theorem 10 *There is $T > 0$, $R > 0$, such that for all n, N,* **a')-d')** *has at least one solution such that $\|u^N\|_{L^\infty(0,T;L^2(\Omega))} \,\le\, R$.*

To prove Theorem 10, we shall express our Galerkin problem as a fixed point problem, and will apply Schauder's theorem to it. Thus, we want to identify u^N as the fixed point of an application

$$\mathcal{F}^N : u \mapsto \tilde{u},$$

defined on $B_{R,T} \,:=\, \big\{u \in C([0,T];\, \text{span}(e_1,\dots,e_N)), \quad \|u\|_{L^\infty(0,T;L^2(\Omega))} \le R\big\}$. We proceed as follows. Let $u \in B_{R,T}$.

- *Step 1.* Let χ_S be the solution of

$$\partial_t \chi_S \,+\, P_S u \cdot \nabla \chi_S = 0, \quad \chi_S|_{t=0} = 1_{S^0},$$

  given by Proposition 2. We know that $\chi_S(t,x) \,=\, 1_{S(t)}(x)$ with $S(t)$ a bounded Lipschitz domain, $t \in [0,T]$. We define accordingly:

$$\rho \,:=\, \rho_F(1-\chi_S) \,+\, \rho_S\chi_S, \quad \mu \,:=\, \mu_F(1-\chi_S) \,+\, \frac{1}{n^2}\chi_S, \quad v(t,x) \,:=\, v^\delta[u, P_S u].$$

- *Step 2.* We consider the following ODE, with unknown $\tilde{u} : [0,T] \mapsto \text{span}(e_1,\dots,e_N)$:

$$A(t)\frac{d}{dt}\tilde{u}(t) + B(t)\tilde{u}(t) = f(t), \quad \tilde{u}(0) \,=\, u_0^N \,:=\, \sum_{k=1}^N \left(\int_\Omega u_0 \cdot e_k\right) e_k, \tag{4.1}$$

  in which $A(t) \,:=\, (a_{i,j}(t))_{1\le i,j\le N}$, $B(t) \,:=\, (b_{i,j}(t))_{1\le i,j\le N}$ and $f(t) \,:=\, (f_i(t))_{1\le i\le N}$ are defined by

$$\begin{aligned}
a_{i,j} \,&:=\, \int_\Omega \rho e_i \cdot e_j, \\
b_{i,j} \,&:=\, \int_\Omega \rho (v\cdot\nabla e_j)\cdot e_i \,+\, \int_\Omega 2\mu D(e_i) : D(e_j) \,+\, \frac{1}{2\beta_\Omega}\int_{\partial\Omega} (e_i \times \nu)\cdot(e_j\times\nu) \\
&\quad +\, \frac{1}{2\beta_S}\int_{\partial S(t)} ((e_i - P_S e_i)\times\nu)\cdot((e_j - P_S e_j)\times\nu) \,+\, n\int_\Omega \chi_S (e_i - P_S e_i)\cdot(e_j - P_S e_j) \\
f_i \,&:=\, \int_\Omega \rho(-g)\cdot e_i.
\end{aligned}$$

We have identified here the function $\tilde{u}$ with its coefficients in the basis $e_1, \dots, e_N$. Note that the function ρ defined in step 1 has a positive lower bound, so that $A(t) \geq \min(\rho_S, \rho_F) I_N$ in the sense of symmetric matrices, whatever the value of χ_S. Also, the continuity of A and B over $[0, T]$ is easy and will be proved below. In particular, equation (4.1) has a unique solution

$$\tilde{u} \in C^1([0,T];\ \text{span}(e_1, \dots, e_N)).$$

In this way, we can associate to each $u \in B_{R,T}$ some field

$$\tilde{u} = \mathcal{F}^N(u) \in C([0,T];\ \text{span}(e_1, \dots, e_N)).$$

The whole point is to prove

Proposition 11 *There exists $T > 0$, $R > 0$, uniform in n and N, such that $\mathcal{F}^N$ is a well-defined mapping from $B_{R,T}$ to itself, continuous and compact.*

Before proving this proposition, let us show how it implies Theorem 10. By Schauder's theorem, it yields the existence of a fixed point $u^N \in B_{R,T}$ of $\mathcal{F}^N$. Let $\chi_S^N = 1_{S^N}$ be the corresponding solution of the transport equation on $[0,T] \times \mathbb{R}^3$. As will be clear from the proof, the time T of the proposition satisfies

$$\text{dist}(S^N(t), \partial\Omega) \ \geq\ 2\delta, \quad \forall\, t \in [0,T],$$

for some δ fixed and arbitrary in $(0, \text{dist}(S_0, \partial\Omega)/2)$. Hence, **a')** is satisfied, and $v^N := v^\delta[u^N, P_S^N u^N]$ satisfies $v^N \cdot \nu|_{\partial\Omega} = 0$, as well as

$$\partial_t \rho^N + v^N \cdot \nabla \rho^N = 0 \ \text{ in } \Omega$$

(see remark 4 after the definition of weak solutions, and remark 5, paragraph 2.3). Finally, we notice that ODE (4.1) is equivalent to: for all $\varphi \in \mathcal{D}([0,T);\ \text{span}(e_1, \dots, e_N))$

$$\begin{aligned}
&\int_0^T \int_\Omega \rho^N \partial_t u^N \cdot \varphi + \int_0^T \int_\Omega \rho^N v^N \cdot \nabla u^N \cdot \varphi + \int_0^T \int_\Omega 2\mu^N D(u^N) : D(\varphi) \\
&+ \frac{1}{2\beta_\Omega} \int_0^T \int_{\partial\Omega} (u^N \times \nu) \cdot (\varphi \times \nu) + \frac{1}{2\beta_S} \int_0^T \int_{\partial S^N(t)} ((u^N - P_S^N u^N) \times \nu) \cdot ((\varphi - P_S^N \varphi) \times \nu) \\
&+ n \int_0^T \int_\Omega \chi_S^N (u^N - P_S^N u^N) \cdot (\varphi - P_S^N \varphi) \ =\ \int_0^T \int_\Omega \rho^N (-g) \cdot \varphi - \int_\Omega \rho_0\, u_0^N \varphi|_{t=0}\, .
\end{aligned} \tag{4.2}$$

Combining this equation with the previous one on ρ^N leads to **c')**. Note that condition $v^N \cdot \nu|_{\partial\Omega} = 0$ is needed for the convective term to vanish through integration by parts.

Proof of the proposition.

Step 1: Definition of $\mathcal{F}^N$.

We first prove that $\mathcal{F}^N$ is well-defined from $B_{R,T}$ to $C([0,T];\ \text{span}(e_1, \dots, e_N))$ for any T and $R > 0$. The only thing to check is the continuity of matrices A and B in (4.1) with respect to time, which will guarantee the existence of a solution to the linear ODE (4.1). As χ_S belongs to $C([0,T]; L^p(\Omega))$ for all finite p, so does ρ, and A is clearly continuous. As regards B, the only difficult terms are

$$I(t) \ :=\ \int_\Omega \rho(v(t,\cdot) \cdot \nabla e_j) \cdot e_i, \quad J(t) \ :=\ \frac{1}{2\beta_S} \int_{\partial S(t)} ((e_i - P_S(t) e_i) \times \nu) \cdot ((e_j - P_S(t) e_j) \times \nu).$$

We remind that the propagator $\phi = \phi_{t,s}$ associated to $P_S u$ satisfies

$$\phi \in C^1\left([0,T]^2;\, C^\infty_{loc}(\mathbb{R}^3)\right)$$

Hence, a look at the construction of v^δ, *cf* Corollary 8 and Proposition 9 (see also Lemma 16 in the appendix A), yields

$$v \in C([0,T]; H^1_\sigma(\mathbb{R}^3)).$$

It implies that $t \mapsto I(t)$ is continuous.

As regards $J(t)$, we change variables to go back to a fixed domain. We set $x = \phi_{t,0}(y)$ to obtain

$$J(t) \;:=\; \frac{1}{2\beta_S} \int_{\partial S_0} \jmath\,(t, \phi_{t,0}(y))\; \mathrm{Jac}_\tau(y)\, dy,$$

where

$$\jmath(t,x) \;:=\; ((e_i(x) - P_S(t)e_i(x)) \times \nu) \cdot ((e_j(x) - P_S(t)e_j(x)) \times \nu)$$

and where

$$\mathrm{Jac}_\tau(y) \;=\; \|d\phi_{t,0}|_y^{-1}\, \nu(y)\|_2 \det(d\phi_{t,0}|_y)(=1)$$

is the tangential jacobian. See [15, Lemme 5.4.1] for details. As $\jmath$ is continuous in t and smooth in x, we obtain that $t \mapsto J(t)$ is continuous.

Step 2: $\mathcal{F}^N$ sends $B_{R,T}$ to itself.

Here, we need to restrict to small T. More precisely, we fix $0 < \delta < \frac{1}{2}\mathrm{dist}(S_0, \partial\Omega)$, and consider a time T such that

$$\inf_{u \in B_{R,T}} \mathrm{dist}(S(t), \partial\Omega) \geq 2\delta > 0 \tag{4.3}$$

Let us prove that such time T does exist and can be chosen uniformly with respect to N and n. For all $u \in B_{R,T}$, we write

$$S(t) \;=\; \phi_{t,0}(S_0)$$

with ϕ the propagator associated to the rigid field $P_S u = \dot{x}_S + \omega_S \times (x - x_S)$ defined in (3.1). It is enough that

$$\sup_{t\in[0,T]} |\partial_t \phi_{t,0}(t,y)| < \frac{\mathrm{dist}(S_0, \partial\Omega) - 2\delta}{T}, \quad t \in [0,T], \quad y \in S_0.$$

We find

$$|\partial_t \phi_{t,0}(t,y)| \;<\; |u_S(t, \phi_{t,0}(t,y))| < |\dot{x}_S(t)| \;+\; |\omega_S(t)|\,|y - x_{S_0}|$$

using that the propagator is isometric. Moreover, classical calculations yield

$$|\dot{x}_S(t)|^2 + J(t)\,\omega_S(t)\cdot\omega_S(t) = \int_{S(t)} \rho_S |P_S u(t,\cdot)|^2 \leq \int_{S(t)} \rho_S |u(t,\cdot)|^2 \;\leq\; \rho_S\, R^2.$$

We can then use the inequality

$$\begin{aligned}
|\dot{x}_S(t)| \;+\; |\omega_S(t)|\,|y - x_{S_0}| \;&\leq\; \sqrt{2}\,\max(1, |y - x_{S_0}|)\, \left(|\dot{x}_S(t)|^2 + |\omega_S(t)|^2\right)^{1/2} \\
&\leq\; C_0\, \left(|\dot{x}_S(t)|^2 + J(t)\omega_S(t)\cdot\omega_S(t)\right)^{1/2}
\end{aligned}$$

where for instance

$$C_0 \;:=\; \sqrt{2}\,\frac{\max\left(1, \sup_{y\in S_0}|y - x_{S_0}|\right)}{\min(1,\lambda_0)^{1/2}}, \quad \lambda_0 \;:\; \text{ smallest eigenvalue of } J(0).$$

Eventually, any $T \;<\; \dfrac{\text{dist}(S_0, \partial\Omega) - 2\delta}{C_0(\rho_S)^{1/2}R}$ will satisfy (4.3).

Let now u be arbitrary in $B_{R,T}$. Thanks to (4.3), we have that $v = v^\delta[u, P_S u]$ satisfies $v\cdot\nu|_{\partial\Omega} = 0$, and

$$\partial_t\rho + v\cdot\nabla\rho = 0 \text{ in } \Omega.$$

Mutiplying (4.1) by $\tilde{u}$, integrating in time, and combining with the last transport equation, we obtain the energy estimate

$$\begin{aligned} &\|\sqrt{\rho}\tilde{u}(t,\cdot)\|^2_{L^2} \;+\; \int_0^t\int_\Omega 2\mu|D(\tilde{u})|^2 \\ &+\; \frac{1}{2\beta_\Omega}\int_0^t\int_{\partial\Omega}|\tilde{u}\times\nu|^2 \;+\; \frac{1}{2\beta_S}\int_0^t\int_{\partial S(t)}|(\tilde{u} - P_S\tilde{u})\times\nu|^2 + n\int_0^t\int_\Omega \chi_S|\tilde{u} - P_S\tilde{u}|^2 \\ &\le\; \int_0^t\int_\Omega \rho(-g)\cdot\tilde{u} \;+\; \int_\Omega \rho_0\,|u_0^N|^2 \end{aligned} \tag{4.4}$$

As $\min(\rho_F,\rho_S) \le \rho \le \max(\rho_F,\rho_S)$, we deduce easily that

$$\|\tilde{u}\|_{L^\infty(0,T;L^2(\Omega))} \;\le\; R$$

for $R = R(T, u_0)$ large enough. Hence, $\mathcal{F}$ sends $B_{R,T}$ to itself.

Step 3. Compactness of $\mathcal{F}^N$.

For any $u = \sum_{k=1}^N \alpha_k e_k$, we get from equation (4.1):

$$|\frac{d}{dt}\tilde{\alpha}(t)| \;\le\; |A^{-1}(t)|\,|B(t)|\,|\alpha(t)| \;+\; |f(t)| \;\le\; R\,|A^{-1}(t)|\,|B(t)| \;+\; |f(t)|.$$

Integrating with respect to time, we obtain

$$\sup_{t\in[0,T]}\left(|\tilde{\alpha}(t)| + |\frac{d}{dt}\tilde{\alpha}(t)|\right) \;\le\; C'$$

(where the constant at the r.h.s. may depend on N or n). In other words,

$$\sup_{u\in B_{R,T}} \|\mathcal{F}^N(u)\|_{C^1([0,T];\,\text{span}(e_1,\dots,e_N))} \;\le\; C''$$

which provides compactness in $B_{R,T}$ by Ascoli's theorem.

Step 4. Continuity of $\mathcal{F}^N$.

Let (u^k) a sequence in $B_{R,T}$, such that $u^k \to u$ in $B_{R,T}$ (that is uniformly over $[0,T]$). We want to show that $\mathcal{F}^N(u^k) \to \mathcal{F}^N(u)$ in $B_{R,T}$. First, we note that, as $span(e_1,\dots,e_N)$ is a finite-dimensional subspace of $\mathcal{D}_\sigma(\bar{\Omega})$ we have that u^k converges to u in $C([0,T];\mathcal{D}_\sigma(\bar{\Omega}))$. Then, we use Proposition 4. With obvious notations,

$$\chi_S^k \to \chi_S \quad \text{weakly * in } L^\infty((0,T)\times\mathbb{R}^3), \quad \text{strongly in } C([0,T];L^p_{loc}(\mathbb{R}^3))\ (p<\infty),$$

as well as

$$P_S^k u^k \to P_S u \quad \text{strongly in } L^\infty(0,T; C^\infty_{loc}(\mathbb{R}^3)), \quad \phi^k \to \phi \quad \text{strongly in } W^{1,\infty}((0,T)^2; C^\infty_{loc}(\mathbb{R}^3)).$$

From there, and the construction of v^δ (Corollary 8, Proposition 9, Lemma 17 in appendix A), it is easy to see that

$$v^k \to v \text{ strongly in } C([0,T]; H^1_\sigma(\mathbb{R}^3)).$$

By slightly adapting the arguments of Step 1, one can then show that the matrices in (4.1) satisfy

$$B^k \to B, \quad A^k \to A \quad \text{strongly in } C([0,T]).$$

From classical results for ODE's, it follows that

$$\tilde{u}^k = \mathcal{F}(u^k) \to \tilde{u} = \mathcal{F}(u) \text{ strongly in } C([0,T]; \text{ span}(e_1, \dots, e_N)).$$

For the sake of brevity, we leave the details to the reader.

4.3 Convergence of the Galerkin scheme

In the previous paragraph, we have built for each n, N a solution $u^{n,N}$ (denoted u^N for brevity) of **a')-d')**. It is defined on $[0,T]$ for some time T uniform in n, N, satisfying (4.3). The next step is to let N go to infinity, to recover a solution u^n of **a)-d)**. We remind the uniform energy estimate (see (4.2))

$$\begin{aligned} &\|\sqrt{\rho^N} u^N(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega 2\mu^N |D(u^N)|^2 + \frac{1}{2\beta_\Omega} \int_0^t \int_{\partial\Omega} |u^N \times \nu|^2 \\ &+ \frac{1}{2\beta_S} \int_0^t \int_{\partial S^N(t)} |(u^N - P_S^N u^N) \times \nu|^2 + n \int_0^t \int_\Omega \chi_S^N |u^N - P_S^N u^N|^2 \\ &\le \int_0^t \int_\Omega \rho(-g) \cdot u^N + \int_\Omega \rho_0 |u_0^N|^2 \end{aligned} \tag{4.5}$$

It yields that

$$(u^N)_{N \in \mathbb{N}} \quad \text{is bounded uniformly with respect to } N \text{ in } L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_\sigma(\Omega))$$

The bound in H^1 follows from the L^2 bound on $D(u^N)$ and Korn's inequality, see [22]. From there, we will be able to show strong convergence both in the transport equation **d')** and in the momentum equation **c')**. As regards the transport equation, we rely on Proposition 5. Up to a subsequence, one has

$$u^N \to u \quad \text{weakly * in } L^\infty(0,T; L^2_\sigma(\Omega)) \text{ and weakly in } L^2(0,T; H^1_\sigma(\Omega))$$

for some $u(= u^n)$, and it follows from this proposition that

$$\chi_S^N \to \chi_S \quad \text{weakly * in } L^\infty((0,T) \times \mathbb{R}^3), \quad \text{strongly in } C([0,T]; L^p_{loc}(\mathbb{R}^3)) \ (p < \infty),$$

as well as

$$\begin{aligned} P_S^N u^N \to P_S u \qquad & \text{weakly * in} \quad L^\infty(0,T; C^\infty_{loc}(\mathbb{R}^3)), \\ \phi^N \to \phi \qquad & \text{weakly * in} \quad W^{1,\infty}((0,T)^2; C^\infty_{loc}(\mathbb{R}^3)), \quad \text{strongly in } C([0,T]; C^\infty_{loc}(\mathbb{R}^3)). \end{aligned}$$

up to another extraction. We stress again that all limits depend on n.

It remains to study the convergence of equation **c')**. Therefore, we fix the test function: we take

$$\varphi(t,x) := \chi(t)\, e_j, \quad \chi \in \mathcal{D}([0,T)).$$

for some fixed j. The point is to obtain as $N \to +\infty$ the limit equation **c)**, still with $\varphi(t,x) = \chi(t) e_j(x)$. But as j is arbitrary, and as $(e_k)_{k\geq 1}$ is a basis of $H^1_\sigma(\Omega)$, standard density arguments will allow to extend the formulation to general test functions.

At first, we need to prove that,

$$v^N|_\Omega \;\to\; v = v^\delta[E_\Omega u, P_S u]|_\Omega \quad \text{in } \; L^2(0,T; H^1_\sigma(\Omega)).$$

It is enough to prove that

$$\hat{v}^N \;:=\; v^\delta[E_\Omega u^N, P^N_S u^N] \;\to\; \hat{v} \;:=\; v^\delta[E_\Omega u, P_S u]$$

weakly in $L^2(0,T;\, H^1_{loc}(\mathbb{R}^3))$. In view of Corollary 8 and Proposition 9, it is an easy consequence of Lemma 17.

We are now ready to handle the asymptotics of **c')** (with $\varphi(t,x) = \chi(t)\, e_j(x)$). As before, for the sake of brevity, we focus on the two most difficult terms, those which involve

$$\begin{aligned} I^N(t) \quad &:= \quad \int_\Omega \rho^N (v^N \otimes u^N) : \nabla e_j, \\ J^N(t) \quad &:= \quad \frac{1}{2\beta_S} \int_{\partial S^N(t)} ((u^N - P^N_S(t) u^N) \times \nu) \cdot ((e_j - P^N_S(t) e_j) \times \nu). \end{aligned}$$

As regards $J^N(t)$, once again we change variables to go back to a fixed domain. We obtain

$$J^N(t) \;:=\; \frac{1}{2\beta_S} \int_{\partial S_0} \jmath^N \left(t, \phi^N_{t,0}(y)\right) \, \mathrm{Jac}^N_\tau(y)\, dy,$$

where

$$\jmath^N(t,x) \;:=\; \big((u^N(t,x) - P^N_S(t) u^N(t,x)) \times \nu\big) \cdot \big((e_j(x) - P^N_S(t) e_j(x)) \times \nu\big)$$

and where

$$\mathrm{Jac}^N_\tau(y) \;=\; \|d\phi^N_{t,0}|_y^{-1}\, \nu(y)\|_2 \det(d\phi^N_{t,0}|_y) = 1.$$

Let $r^N \;:=\; E_\Omega u^N - P^N_S u^N$, resp. $\eta^N_j \;:= e_j - P^N_S e_j$ to which we associate R^N, resp. H^N_j through the change of coordinates:

$$r^N(t, \phi^N(t,y)) := d\phi^N_t|_y R^N(t,y)\,, \qquad \eta^N_j(t, \phi^N(t,y)) := d\phi^N_t|_y H^N_j(t,y)\,.$$

From the weak convergence of u^N, we deduce that r^N converges weakly in $L^2(0,T; H^1_{loc}(\mathbb{R}^3))$. Given the strong convergence of χ^N in $C([0,T]; L^p(\Omega))$ we also have that η^N_j converges strongly to $\eta_j \;:= e_j - P_S e_j$ in $L^2(0,T; H^1_{loc}(\mathbb{R}^3))$. Furthermore, as $d\phi^N_{t,0}|_y$ is an isometric mapping for all N, we get that :

$$\jmath^N(t, \phi_{t,0}(y)) = (R^N \times \nu) \cdot (H^N_j \times \nu)\,, \quad \forall\, N \in \mathbb{N}$$

where, because of lemma 17 :

$$R^N \to R \text{ weakly in } L^2(0,T;H^1(\Omega)), \quad H_j^N \to H_j \text{ strongly in } L^2(0,T;H^1(\Omega)).$$

with obvious notations. This yields corresponding weak and strong convergences of the traces of these functions on ∂S_0. Having in mind that $\mathrm{Jac}_\tau^N \equiv 1$ for all N, and going back to the moving domain, we obtain easily that J^N converges weakly in $L^1(0,T)$ to :

$$J(t) := \frac{1}{2\beta_S}\int_{\partial S(t)} ((u - P_S(t)u)\times\nu)\cdot((e_j - P_S(t)e_j)\times\nu).$$

We finally turn to the convergence of I^N, for which we will need some compactness on $(\rho^N u^N)$. Therefore, we introduce some notations: we denote by P the orthogonal projection from $L^2(\Omega)$ onto $L^2_\sigma(\Omega)$, respectively P_k the orthogonal projection from $L^2(\Omega)$ onto $\mathrm{span}(e_1,\dots,e_k)$. We also remind that our strong, resp. weak, convergence results on ρ^N, resp. u^N imply that.

$$\rho^N u^N \to \rho u \text{ weakly-* in } L^\infty(0,T;L^2(\Omega)).$$

In particular, we have for any fixed k:

$$P_k(\rho^N u^N) \rightharpoonup P_k(\rho u) \quad \text{weakly-* in } L^\infty(0,T;L^2_\sigma(\Omega)) \text{ as } N\to\infty. \tag{4.6}$$

Moreover, equation **c')** can be written: for all $1 \le k \le N$,

$$\partial_t P_k(\rho^N u^N) + P_k F^N = 0 \quad \text{in} \quad \mathcal{D}'\left(0,T;[H^1_\sigma(\Omega)]^*\right)$$

where $F^N \in \mathcal{D}'\left(0,T;[H^1_\sigma(\Omega)]^*\right)$ is defined by the duality relation:

$$\begin{aligned}
\langle F^N,\varphi\rangle &= \int_0^T\int_\Omega \rho^N v^N\otimes u^N : \nabla\varphi - \int_0^T\int_\Omega 2\mu^N D(u^N) : D(\varphi) \\
&+ \frac{1}{2\beta_S}\int_0^T\int_{\partial S^N(t)} ((u^N - P_S^N u^N)\times\nu)\cdot((\varphi - P_S^N\varphi)\times\nu) \\
&+ n\int_0^T\int_\Omega \chi_S^N(u^N - P_S^N u^N)\cdot(\varphi - P_S^N\varphi) + \frac{1}{2\beta_\Omega}\int_0^T\int_{\partial\Omega}(u^N\times\nu)\cdot(\varphi\times\nu) \\
&- \int_0^T\int_\Omega \rho^N(-g)\cdot\varphi, \qquad \text{for all } \varphi\in\mathcal{D}\left(0,T;H^1_\sigma(\Omega)\right).
\end{aligned}$$

We remind that for $f \in [H^1_\sigma(\Omega)]^*$, P_k is defined by duality: $< P_k f,\varphi > := < f, P_k\varphi >$. From the above expression for F^N and the various bounds already obtained, it is easily seen that for any fixed k, $(P_k F^N)$ is bounded (in N) in $L^2(0,T;[H^1_\sigma(\Omega)]^*)$. Hence, the same conclusion applies to $(\partial_t P_k(\rho^N u^N))$. Combining with (4.6), it follows that for any fixed k,

$$P_k(\rho^N u^N) \to P_k(\rho u) \quad \text{strongly in } L^\infty(0,T;[H^1_\sigma(\Omega)]^*) \text{ as } N\to\infty. \tag{4.7}$$

Now, we note that, for arbitrary k and N, and a.a. $t\in(0,T)$ there holds

$$\begin{aligned}
\|P(\rho^N u^N)(t) - P_k(\rho^N u^N)(t)\|_{[H^1_\sigma(\Omega)]^*} &= \sup_{\|\varphi\|_{[H^1_\sigma(\Omega)]}=1}\int_\Omega \left(P(\rho^N u^N(t)) - P_k(\rho^N u^N)(t)]\right)\varphi \\
&= \sup_{\|\varphi\|_{[H^1_\sigma(\Omega)]}=1}\int_\Omega \rho^N u^N(t)(\varphi - P_k\varphi) \\
&\le \left[\sup_{\|\varphi\|_{[H^1_\sigma(\Omega)]}=1}\|\varphi - P_k\varphi\|_{L^2(\Omega)}\right]\|\rho^N u^N\|_{L^\infty L^2(\Omega)},
\end{aligned}$$

By a standard argument based on Rellich Lemma, one shows that

$$\sup_{\|\varphi\|_{H^1_\sigma(\Omega)}=1} \|\varphi - P_k\varphi\|_{L^2(\Omega)} \to 0$$

as $k \to \infty$. With the uniform bound on $\rho^N u^N$ in $L^\infty(0,T;L^2(\Omega))$, we can conclude that

$$P_k(\rho^N u^N) - P(\rho^N u^N) \ \to 0 \quad \text{strongly in } \ L^\infty(0,T;[H^1_\sigma(\Omega)]^*), \quad \text{as } k \to +\infty, \text{uniformly in } N. \tag{4.8}$$

Of course, with a similar but simpler estimate, we also have

$$P_k(\rho u) - P(\rho u) \ \to 0 \quad \text{strongly in } \ L^\infty(0,T;[H^1_\sigma(\Omega)]^*), \quad \text{as } k \to +\infty. \tag{4.9}$$

Combining (4.7), (4.8) and (4.9), we obtain finally that : $P(\rho^N u^N)$ converges to $P(\rho u)$ strongly in $L^2(0,T;[H^1_\sigma(\Omega)]^*)$. Combining this strong convergence with the weak convergence of (u^N) in $L^2(0,T;H^1_\sigma(\Omega))$, we might apply the method of P.L. Lions [20, p.47] with the duality bracket $[H^1_\sigma(\Omega)]^* - H^1_\sigma(\Omega)$ to prove that $\sqrt{\rho^N}u^N$ converges to $\sqrt{\rho}u$ strongly in $L^2((0,T)\times\Omega)$. Finally, we rewrite :

$$I_N(t) = \int_\Omega \sqrt{\rho^N}u^N \otimes \sqrt{\rho^N}v^N : \nabla e_j\,,$$

where :

- $\sqrt{\rho^N}u^N$ converges to $\sqrt{\rho}u$ strongly in $L^2((0,T)\times\Omega)$
- $\sqrt{\rho^N}$ converges to $\sqrt{\rho}$ strongly in $L^\infty(0,T;L^3(\Omega))$
- v^N converges to v weakly in $L^2(0,T;L^6(\Omega))$ (thanks to the imbedding $H^1(\Omega)\subset L^6(\Omega)$).

Combining these statements, we get that I_N converges to I (with obvious notations) weakly in $L^1(0,T)$.

Such convergences result yield that (ρ^n, u^n) satisfy **c')** for test functions φ of the form $\chi(t)\psi$ with $\chi \in \mathcal{D}([0,T))$ and $\psi \in \mathrm{span}(\{e_k, k\in\mathbb{N}\})$. Via a classical density argument, the convergence extends to all $\varphi \in H^1(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;H^1_\sigma(\Omega))$ such that $\varphi|_{t=T}=0$.

4.4 Energy inequality

We end this section by proving that the approximate solution (ρ^n, u^n) satisfies the further estimate :

$$\begin{aligned} &\|\sqrt{\rho^n}u^n(t,\cdot)\|^2_{L^2(\Omega)} \ + \int_0^t\int_\Omega 2\mu^n|D(u^n)|^2 \ + \ \frac{1}{2\beta_\Omega}\int_0^t\int_{\partial\Omega}|u^n\times\nu|^2 \\ &+ \ \frac{1}{2\beta_S}\int_0^t\int_{\partial S^N(t)}|(u^n - P^n_S u^n)\times\nu|^2 + n\int_0^t\int_\Omega \chi^n_S|u^n - P^n_S u^n|^2 \\ &\le \int_0^t\int_\Omega \rho(-g)\cdot u^n \ + \int_\Omega \rho_0\,|u_0|^2 \end{aligned} \tag{4.10}$$

for almost all $t\in[0,T]$. For simplicity we drop exponent n in what follows.

First, we note that the solutions (ρ^N, u^N) of the Galerkin scheme satisfy (4.5) uniformly in N and that, up to the extraction of a subsequence $\sqrt{\rho^N}u^N$ converges to $\sqrt{\rho}u$ in $L^2((0,T)\times\Omega)$. Hence, we may pass to the limit in (4.5) for almost all $t\in[0,T]$. On the other hand, there holds:

- By construction of the Galerkin scheme, $u_0^N \to u_0$ in $L^2(\Omega)$ so that :

$$\lim_{N\to\infty} \int_\Omega \rho_0 |u_0^N|^2 = \int_\Omega \rho_0 |u_0|^2 \,.$$

- Given the strong convergence of (u^N) in $L^2((0,T)\times\Omega)$:

$$\lim_{N\to\infty} \int_0^T \int_\Omega \rho(-g)\cdot u^N = \int_0^T \int_\Omega \rho(-g)\cdot u^n \,.$$

- Given the weak convergence of u^N in $L^2((0,T);H^1(\Omega))$ and the strong convergence of χ_S^N in $C([0,T];L^p(\Omega))$ we get that $\sqrt{\mu^N}D(u^N)$ converges weakly to $\sqrt{\mu}D(u)$ in $L^{2-\varepsilon}((0,T)\times\Omega)$. In particular, in the weak limit, there holds :

$$\int_0^T \int_\Omega \mu |D(u)|^2 \leq \liminf \int_0^T \int_\Omega \mu^N |D(u^N)|^2 \,.$$

  With similar arguments, we obtain :

$$\int_0^T \int_\Omega \chi_S |u - P_S u|^2 \leq \liminf \int_0^T \int_\Omega \chi_S^N |u^N - P_S^N u^N|^2 \,.$$

- Finally, we pass to the limit in the boundary terms. First, we introduce U^N and U_S^N associated to the extension $E_\Omega[u^N]$ and the rigid vector field $P_S^N u^N$ respectively, computed through the change of variable $\phi_{t,0}^N$. As previously, we have:

$$\int_0^T \int_{\partial S^N(t)} |(u^N - P_S^N u^N)\times\nu|^2 = \int_0^T \int_{\partial S^N(t)} |(u^N - u_S^N)\times\nu|^2 = \int_0^T \int_{\partial S_0} |(U^N - U_S^N)\times\nu|^2.$$

  Because of the weak convergence of u^N and u_S^N in $L^2(0,T;H_\sigma^1(\Omega))$, we have that U^N and U_S^N converge also weakly in $L^2(0,T;H_{loc}^1(\mathbb{R}^3))$ (see Lemma 17). Hence, we have also weak convergence of the traces on S_0. The lower semi-continuity of the L^2-norm on ∂S_0 yields :

$$\begin{aligned}\int_0^T \int_{\partial S(t)} |(u - P_S u)\times\nu|^2 &= \int_0^T \int_{\partial S_0} |(U - U_S)\times\nu|^2 \\ &\leq \liminf \int_0^T \int_{\partial S_0} |(U^N - U_S^N)\times\nu|^2 \\ &\leq \liminf \int_0^T \int_{\partial S^N(t)} |(u^N - P_S^N u^N)\times\nu|^2\end{aligned}$$

  Similar weak-convergence and semi-continuity arguments yield also :

$$\int_0^T \int_{\partial\Omega} |u\times\nu|^2 \leq \liminf \int_0^T \int_{\partial\Omega} |u^N\times\nu|^2 \,.$$

This ends the proof of (4.10).

5 Convergence

In the previous section, we have obtained the existence of solutions u^n of approximate fluid-solid systems, namely satisfying **a)-d)**. These solutions u^n are defined on some (uniform in n) time interval $(0,T)$ such that

$$\mathrm{dist}(S^n(t), \partial\Omega) \geq 2\delta, \quad \text{for } t \in [0,T), \quad \text{for some fixed } \delta > 0. \tag{5.1}$$

We must now study the asymptotics of u^n as n goes to infinity, and recover a weak solution at the limit.

In what follows, we will often make use of the notation

$$(\mathcal{O})_\eta := \{x \in \mathbb{R}^3, \ \mathrm{dist}(x, \mathcal{O}) < \eta\}$$

for $\mathcal{O}$ an open set and $\eta > 0$.

5.1 A priori bounds on u^n. Convergence in the transport equation

The density ρ^n clearly satisfies the uniform bound

$$\min(\rho_F, \rho_S) \leq \rho^n \leq \max(\rho_F, \rho_S). \tag{5.2}$$

Combining (4.10) and (5.2) yields that

$$\|u^n\|^2_{L^\infty(0,T;L^2(\Omega))} + n\|\sqrt{\chi^n_S}(u^n - P^n_S u^n)\|^2_{L^2((0,T)\times\Omega)} + \|\sqrt{\mu^n} D(u^n)\|^2_{L^2((0,T)\times\Omega)} \leq C. \tag{5.3}$$

for some constant C depending only on ρ_F, ρ_S, u_0 and T.

In particular, up to a subsequence, the first inequality gives

$$u^n \to u \quad \text{weakly* in } L^\infty(0,T; L^2_\sigma(\Omega)).$$

We can then pass to the limit of the transport equation **d)**, using Proposition 5. The following convergence holds up to a subsequence:

$$\chi^n_S \to \chi_S \quad \text{weakly * in } L^\infty((0,T)\times\mathbb{R}^3), \quad \text{strongly in } C([0,T]; L^p_{loc}(\mathbb{R}^3)) \ (p < \infty),$$

with

$$\chi_S(t,\cdot) = 1_{S(t)}, \quad S(t) = \phi_{t,0}(S_0)$$

for an isometric propagator $\phi = \phi_{t,s} \in W^{1,\infty}((0,T)^2; C^\infty_{loc}(\mathbb{R}^3))$. Moreover, one has

$$P^n_S u^n \to P_S u \quad \text{weakly * in } L^\infty(0,T; C^\infty_{loc}(\mathbb{R}^3)), \quad \phi^n \to \phi \quad \text{weakly * in } W^{1,\infty}((0,T)^2; C^\infty_{loc}(\mathbb{R}^3)).$$

In particular, one recovers the transport equation (2.3) setting $u_S := P_S u$.

Now, we can combine the second inequality in (5.3), that yields

$$\chi^n_S(u^n - P^n_S u^n) \to 0 \text{ strongly in } L^2,$$

with the strong (resp. weak) convergence of χ^n_S (resp. u^n and $P^n_S u^n$). As n goes to infinity, we derive easily:

$$\chi_S(u - u_S) = 0. \tag{5.4}$$

Finally, the last inequality in (5.3) and Korn's inequality imply that

$$\int_0^T \|u^n\|^2_{H^1(F^n(t))} dt \le C \int_0^T \left(\|D(u^n)\|^2_{L^2(F^n(t))} \; + \; \|u^n\|^2_{L^2(F^n(t))} \right) dt \le C, \quad F^n(t) := \Omega \setminus \overline{S^n(t)}.$$

We then introduce continuous extension operators

$$E_n(t) : \; \{u \in H^1(F^n(t)), \text{ div } u = 0 \text{ in } F^n(t), \; u \cdot \nu|_{\partial\Omega} = 0\} \; \mapsto \; H^1_\sigma(\Omega),$$

in the spirit of Corollary 7. As long as the $S^n(t)$ are 2δ away from $\partial\Omega$, it is standard to construct these extension operators in such a way that

$$\|E_n(t)\|_{\mathcal{L}(H^1)} \le C_\delta, \quad \forall \, t \in [0, T].$$

Hence, if we set $u^n_F(t, \cdot) \; := E_n(t) u^n(t, \cdot)$, we have that

$$(u^n_F) \quad \text{is bounded in } \; L^2(0, T; \; H^1_\sigma(\Omega)), \quad (1 - \chi^n_S)(u^n_F - u^n) = 0, \, \forall \, n.$$

From the $L^2(0, T; H^1(\Omega))$ bound, we can assume up to another extraction that

$$u^n_F \to u_F \quad \text{weakly in } \; L^2(0, T; H^1_\sigma(\Omega)).$$

From above equality and from the strong convergence of χ^n_S, we then get:

$$(1 - \chi_S)(u_F - u) \; = \; 0. \tag{5.5}$$

Eventually, considering relations (5.4) and (5.5), we get that the limit u of u^n belongs to $\mathcal{S}_T$. Hence, back to the definition of a weak solution, it only remains to show that the momentum equation (2.1) is satisfied by $S(\cdot), u_S, u_F$.

5.2 A priori bounds on v^n.

Prior to the analysis of the momentum equation, we must establish some refined estimates on the *connecting velocity* v^n. We remind that v^n was defined in Lagrangian like coordinates, see paragraph 4.1. More precisely,

$$v^n \left(t, \phi^n_{t,0}(y) \right) \; := \; d\phi^n_{t,0}|_y \left(V^\delta[U^n(t, \cdot), U^n_S(t, \cdot)] \right),$$

where

$$E_\Omega u^n \left(t, \phi^n_{t,0}(y) \right) \; = \; d\phi^n_{t,0}|_y \left(U^n(t, y) \right), \quad P^n_S u^n \left(t, \phi^n_{t,0}(y) \right) \; = \; d\phi^n_{t,0}|_y \left(U^n_S(t, y) \right),$$

and $V^\delta = V^\delta[U, U_S]$ is some linear operator connecting $U \in H^1_\sigma(\mathbb{R}^3 \setminus S_0)$ to $U_S \in H^1_\sigma(\overline{S_0})$ over a band of width δ outside S_0: see Corollary 8. We shall here specify our choice for the operator V^δ. Actually, we shall make it depend on n ($V^\delta = V^{\delta,n}$), in order for the following additional assumption to be satisfied:

$$\|V^{\delta,n}[U, U_S] - U\|_{L^p((S_0)_\delta \setminus S_0)} \le C_{\delta,p} \left(\|(U - U_S) \cdot \nu\|_{L^p(\partial S_0)} + n^{1/6 - 1/p} \|(U, U_S)\|_{H^1((S_0)_\delta \setminus S_0)} \right), \quad \forall \, 2 \le p \le 6. \tag{5.6}$$

We postpone the construction of such operator $V^{\delta,n}$ to the end of the paragraph.

Back to v^n, the additional assumption (5.6) implies easily that for all $2 \le p \le 6$,

$$\|(1-\chi_S^n)(v^n - u^n)\|_{L^2(0,T;L^p(\Omega))} \le C_{\delta,p}\left(\int_0^T \|(u^n - P_S^n u^n)(t,\cdot)\cdot\nu\|^2_{L^p(\partial S^n(t))}\,dt \,+\, n^{1/6-1/p}\right). \quad (5.7)$$

But we know that

$$\int_0^T \|(u^n - P_S^n u^n)(t,\cdot)\cdot\nu\|^2_{H^{-1/2}(\partial S^n(t))}\,dt \,\le\, C\int_0^T \|(u^n - P_S^n u^n)(t,\cdot)\cdot\nu\|^2_{L^2(S^n(t))}\,dt \le \frac{C}{n} \quad (5.8)$$

where the last bound comes from the second inequality in (5.3). We emphasize here that, as the $S^n(t)$'s are all isometric to one another, the constant C does not depend on n, t. Interpolation with the similar other bound

$$\begin{aligned}\int_0^T \|(u^n - P_S^n u^n)(t,\cdot)\cdot\nu\|^2_{H^{1/2}(\partial S^n(t))} &\le C\int_0^T \left(\|u^n\|^2_{H^1((S^n(t))_\delta\cap F^n(t))} + \|u^n\|^2_{L^2(S^n(t))}\right)\mathrm{d}t \\ &\le C\int_0^T \left(\|u^n\|^2_{H^1(F^n(t))} + \|u^n\|^2_{L^2(S^n(t))}\right)dt \,\le\, C\end{aligned} \quad (5.9)$$

yields that

$$\int_0^T \|(u^n - P_S^n u^n)(t,\cdot)\cdot\nu\|^2_{L^p(\partial S^n(t))}\,dt \,\to\, 0, \quad \forall p < 4. \quad (5.10)$$

Eventually, we get that

$$\|(1-\chi_S^n)(v^n - u^n)\|_{L^2(0,T;L^p(\Omega))} \to 0, \quad \forall p < 4. \quad (5.11)$$

This will be much important in the treatment of the nonlinear terms.

We conclude this paragraph with the construction of the operator $V^{\delta,n}$ satisfying (5.6). We take U, U_S in $H^1_\sigma(\mathbb{R}^3 \setminus S_0) \times H^1_\sigma(\overline{S_0})$. Up to an extension of U_S, there is no loss of generality assuming that $U_S \in H^1_\sigma(\mathbb{R}^3)$.

Step 1. We shall construct a field V such that div V = 0,

$$V|_{\partial S_0} = U_S + (U - U_S)\cdot\nu\,\nu, \quad \text{and } V|_{\partial (S_0)_\delta} = U. \quad (5.12)$$

Therefore, we introduce a system of orthogonal curvilinear coordinates (s_1, s_2, z) in a tubular neighborhood of ∂S_0: s_1, s_2 are coordinates along the surface ∂S_0, whereas z denotes a transverse coordinate. In particular, $\partial S_0 = \{z = 0\}$. We set

$$e_1 \,:=\, \frac{1}{h_1}\frac{\partial}{\partial s_1}, \quad e_2 \,:=\, \frac{1}{h_2}\frac{\partial}{\partial s_2}, \quad e_z := \nu \,=\, \frac{1}{h_z}\frac{\partial}{\partial z}$$

the associated orthonormal vectors, with scale factors $h_1, h_2, h_z \geq 0$. We remind that

$$\nabla f = \frac{1}{h_1}\partial_{s_1} f e_1 \; + \; \frac{1}{h_2}\partial_{s_2} f e_2 \; + \; \frac{1}{h_z}\partial_z f e_z \tag{5.13}$$

for a scalar function f, whereas

$$\text{div } f = \frac{1}{h_1 h_2 h_z}\left(\partial_{s_1}(h_2 h_z f_1) \; + \; \partial_{s_2}(h_1 h_z f_2) \; + \; \partial_z(h_1 h_2 f_z)\right) \tag{5.14}$$

for any field $f = f_1 e_1 + f_2 e_2 + f_z e_z$. We then set

$$V_1 \; := \; (1 - \chi(nz))\, U + \chi(nz)(U_S + [(U - U_S)\cdot e_z]\, e_z)$$

for a smooth truncation function $\chi : \mathbb{R} \to [0,1]$ equal to 1 in a neighborhood of 0. Clearly, for all $p \leq 6$, and $\frac{1}{q} + \frac{1}{6} = \frac{1}{p}$,

$$\begin{aligned} \|V_1 - U\|_{L^p((S_0)_\delta \setminus S_0)} \; &\leq \; C_{p,\delta}\, n^{-1/q}\, \|(U, U_S)\|_{L^6((S_0)_\delta \setminus S_0)} \\ &\leq \; C'_{p,\delta}\, n^{-1/q}\, \|(U, U_S)\|_{H^1((S_0)_\delta \setminus S_0)}. \end{aligned} \tag{5.15}$$

Also, V_1 satisfies (5.12). But it is not divergence-free: formula (5.14) yields

$$\text{div } V_1 \; = \; \chi(nz)\, \text{div }([(U - U_S)\cdot e_z] e_z)$$

so that for all $p \leq 2$,

$$\|\text{div } V_1\|_{L^p((S_0)_\delta \setminus S_0)} \; \leq \; C\, n^{1/2 - 1/p} \|U - U_S\|_{H^1((S_0)_\delta \setminus S_0)}.$$

To obtain a divergence-free field, we note that both U and U_S have zero flux through ∂S_0 and [11, Theorem 3.1]: there exists a field V_2 such that

$$\text{div } V_2 = -\text{div } V_1 \quad \text{in} \quad (S_0)_\delta \setminus S_0, \quad V_2|_{\partial S_0} = V_2|_{\partial (S_0)_\delta} = 0,$$

and for all $p \in]1, 2]$,

$$\|V_2\|_{W^{1,p}((S_0)_\delta \setminus S_0)} \; \leq \; C_\delta\, n^{1/2 - 1/p}\, \|U - U_S\|_{H^1((S_0)_\delta \setminus S_0)}.$$

In particular, by Sobolev imbedding, one has for all $p_* \leq 6$

$$\|V_2\|_{L^{p_*}((S_0)_\delta \setminus S_0)} \; \leq \; C_\delta\, n^{1/6 - 1/p_*}\, \|(U, U_S)\|_{H^1((S_0)_\delta \setminus S_0)}. \tag{5.16}$$

Finally, the field $V \; := \; V_1 + V_2$ fulfills our requirements.

Step 2. We construct a field W such that div $W = 0$,

$$W|_{\partial S_0} = [(U - U_S)\cdot \nu]\, \nu, \quad \text{and } \; W|_{\partial (S_0)_\delta} = 0. \tag{5.17}$$

In the same spirit as in the first step, we take

$$W_1 \; := \; \chi(\frac{2z}{\delta})[(U - U_S)\cdot \nu|_{z=0}]\, e_z$$

where χ is again a truncation function: $\chi = 1$ near 0, and $\chi = 0$ outside $[-1, 1]$. A rapid computation shows that

$$\|W_1\|_{L^p((S_0)_\delta \setminus S_0)} \; \leq \; C_\delta \|(U - U_S)\cdot \nu\|_{L^p(\partial S_0)}, \quad \forall\, p \tag{5.18}$$

By Proposition 6, there exists a field W_2 such that

$$\text{div } W_2 = -\text{div } W_1 \quad \text{in} \quad (S_0)_\delta \setminus S_0, \quad W_2|_{\partial S_0} = W_2|_{\partial (S_0)_\delta} = 0,$$

and

$$\|W_2\|_{H^1((S_0)_\delta \setminus S_0)} \le C_\delta \, \|(U - U_S) \cdot \nu\|_{L^2(\partial S_0)}.$$

In particular, by Sobolev imbedding, one has for all $p \le 6$

$$\|W_2\|_{L^p((S_0)_\delta \setminus S_0)} \le C_\delta \, \|(U - U_S) \cdot \nu\|_{L^2(\partial S_0)}. \tag{5.19}$$

Finally, the field $W := W_1 + W_2$ fulfills our requirements.

Eventually, we set

$$\begin{cases} V^\delta[U, U_S] = U \text{ outside } (S_0)_\delta, \\ V^\delta[U, U_S] = V - W \text{ in } (S_0)_\delta \setminus S_0, \\ V^\delta[U, U_S] = U_S \text{ in } S_0. \end{cases}$$

Combining (5.15), (5.16), (5.18) and (5.19) leads to (5.6).

5.3 Approximation of the test functions

The weak formulation of the momentum equation involves discontinuous test functions $\varphi \in \mathcal{T}_T$:

$$\varphi = (1 - \chi_S)\varphi_F + \chi_S \varphi_S, \quad \varphi_F \in \mathcal{D}([0,T); \mathcal{D}_\sigma(\overline{\Omega})), \quad \varphi_S \in \mathcal{D}([0,T); \mathcal{R}),$$

with

$$\varphi_F \cdot \nu|_{\partial\Omega} = 0, \quad \varphi_F \cdot \nu|_{\partial S(t)} = \varphi_S \cdot \nu|_{\partial S(t)} \quad \forall\, t.$$

On the contrary, the approximate momentum equation **c)** involves continuous (or at least H^1) test functions. Hence, we will have to approach φ by a sequence (φ^n) in $L^2(0,T; H^1_\sigma(\Omega))$. Due to the discontinuity of the limit, the $\varphi^n(t,\cdot)$'s will exhibit strong gradients near $\partial S^n(t)$. Precise estimates are needed, that are the purpose of

Proposition 12 *Let $\alpha > 0$. There exists a sequence (φ^n) in $W^{1,\infty}(0,T; L^2_\sigma(\Omega)) \cap L^\infty(0,T; H^1_\sigma(\Omega))$, of the form*

$$\varphi^n = (1 - \chi^n_S)\varphi_F + \chi^n_S \varphi^n_S,$$

that satisfies

- $\|\sqrt{\chi^n_S}(\varphi^n_S - \varphi_S)\|_{C([0,T]; L^p(\Omega))} = O(n^{-\alpha/p})$ *for all* $p \in [2,6]$.
- $\varphi^n \to \varphi$ *strongly in* $C([0,T]; L^6(\Omega))$.
- $\|\varphi^n\|_{C([0,T]; H^1(\Omega))} = O(n^{\alpha/2})$.
- $\|\chi^n_S(\partial_t + P^n_S u^n \cdot \nabla)(\varphi^n - \varphi_S)\|_{L^\infty(0,T; L^6(\Omega))} = O(n^{-\alpha/6})$.
- $(\partial_t + P^n_S u^n \cdot \nabla)\varphi^n \to (\partial_t + P_S u \cdot \nabla)\varphi$ *weakly * in* $L^\infty(0,T; L^6(\Omega))$.

Proof of the proposition.

The point is to build a good approximation φ_S^n of φ_S over the solid domain. Broadly, we want

$$\varphi_S^n(t,\cdot)|_{\partial S^n(t)} = \varphi_F(t,\cdot)|_{\partial S^n(t)} \quad \forall\, t,$$

and

$$\varphi_S^n(t,\cdot) \approx \varphi_S(t,\cdot) \quad \text{ in } S^n(t) \text{ away from a } n^{-\alpha} \text{ neighborhood of } \partial S^n(t) \quad \forall\, t.$$

Therefore, we proceed as for v^n, by using lagrangian coordinates: we define Φ_S and Φ_F through the formulas

$$\varphi_S\left(t, \phi^n_{t,0}(y)\right) \;=\; d\phi^n_{t,0}|_y\left(\Phi_S(t,y)\right), \quad \varphi_F\left(t, \phi^n_{t,0}(y)\right) \;=\; d\phi^n_{t,0}|_y\left(\Phi_F(t,y)\right),$$

and the goal is to define properly some Φ_S^n, related to φ_S^n by the formula

$$\varphi_S^n\left(t, \phi^n_{t,0}(y)\right) \;=\; d\phi^n_{t,0}|_y\left(\Phi_S^n(t,y)\right).$$

Note that Φ_S and Φ_F depend on n through the propagator ϕ^n, but we omit it from our notations. The only thing we have to keep in mind is that the bounds on ϕ^n guarantee that Φ_S and Φ_F are uniformly bounded in $W^{1,\infty}(0,T; H^k_{loc}(\mathbb{R}^3))$ for all k.

Thanks to the change of coordinates, the problem is now in the fixed domain S_0. Roughly, we want to build Φ_S^n in such a way that

$$\Phi_S^n(t,\cdot)|_{\partial S_0} = \Phi_F(t,\cdot)|_{\partial S_0} \quad \forall\, t,$$

and

$$\Phi_S^n(t,\cdot) \approx \Phi_S(t,\cdot) \quad \text{ in } S_0 \text{ away from a } n^{-\alpha} \text{ neighborhood of } \partial S_0 \quad \forall\, t.$$

Note that time is only a parameter in the system. The construction of Φ_S^n follows the one of V, performed in the previous paragraph, Step 1. We take Φ_S^n under the form

$$\Phi_S^n \;=\; \Phi_{S,1}^n \;+\; \Phi_{S,2}^n.$$

The first term has the explicit form

$$\Phi_{S,1}^n \;=\; \Phi_S \;+\; \chi(n^\alpha\, z)\left((\Phi_F - \Phi_S) - \left[(\Phi_S - \Phi_F)\cdot e_z\right] e_z\right).$$

Again, χ is a smooth truncation function near 0, and z is a coordinate transverse to the boundary: $\partial S_0 = \{z = 0\}$. It is easily seen that $\Phi_{S,1}^n$ satisfies the right boundary condition at ∂S_0. Moreover,

$$\|\Phi_{S,1}^n - \Phi_S\|_{W^{1,\infty}(0,T;L^p(S_0))} \;\le\; C\, n^{-\alpha/p} \quad \forall\, p < \infty, \quad \|\Phi_{S,1}^n - \Phi_S\|_{W^{1,\infty}(0,T;H^1(S_0))} \;\le\; C\, n^{\alpha/2}. \tag{5.20}$$

But it is not divergence-free. By applying formula (5.14), we get

$$\text{div } \Phi_{S,1}^n \;=\; \chi(n^\alpha\, z) \jmath^n, \quad \jmath^n \;:=\; \text{div }\left(x \mapsto \left((\Phi_F - \Phi_S) - \left[(\Phi_S - \Phi_F)\cdot e_z\right] e_z\right)\right).$$

In particular, $\jmath^n$ is uniformly bounded in $W^{1,\infty}(0,T; L^2(S_0))$

By Proposition 6, there exists some field $\Phi_{S,2}^n$ satisfying

$$\text{div } \Phi_{S,2}^n = -\text{div } \Phi_{S,1}^n \quad \text{ in } S_0, \quad \Phi_{S,2}^n|_{\partial S_0} = 0,$$

and

$$\|\Phi^n_{S,2}\|_{W^{1,\infty}(0,T;H^1(S_0))} \le C\,\|\chi(n^\alpha z)\,\jmath^n\|_{W^{1,\infty}(0,T;L^2(S_0))} \le C\,n^{-\alpha/2}. \tag{5.21}$$

In particular,

$$\|\Phi^n_{S,2}\|_{W^{1,\infty}(0,T;L^6(S_0))} \le C\,n^{-\alpha/2}. \tag{5.22}$$

Back to the moving domain (in variable x), one can combine the estimates (5.20)-(5.21)-(5.22) with the uniform bound on ϕ^n in $W^{1,\infty}(0,T;\,C^\infty(\Omega))$. From there, one can deduce the estimates of the proposition. For the sake of brevity, we only treat the two last items. Namely, we write

$$\begin{aligned}\|\chi^n_S(\partial_t + P^n_S u^n\cdot\nabla)\,(\varphi^n - \varphi_S)\,\|_{L^\infty((0,T);L^6(\Omega))} &\le C\,\|\frac{\partial}{\partial t}\,d\phi^n_{t,0}|_y\,(\Phi^n_S - \Phi_S)\,\|_{L^\infty((0,T);L^6(S_0))}\\ &\le C\,n^{-\alpha/6}\,,\end{aligned}$$

where the last inequality involves (5.20) and (5.22). This bound implies in turn that

$$(\partial_t + P^n_S u^n\cdot\nabla)\varphi^n = (1-\chi^n_S)(\partial_t + P^n_S u^n\cdot\nabla)\varphi_F + \chi^n_S(\partial_t + P^n_S u^n\cdot\nabla)\varphi_S + O(n^{-\alpha/6}) \quad \text{in } L^6(\Omega)$$

The products at the r.h.s. are then easily handled using the strong convergence of χ^n_S (and the weak convergence of $P^n_S u^n$). We obtain

$$(\partial_t + P^n_S u^n\cdot\nabla)\varphi^n \to (\partial_t + P_S u\cdot\nabla)\varphi \quad \text{weakly *in } L^\infty(0,T;L^6(\Omega))$$

as expected. This concludes the proof of the proposition.

5.4 Convergence in the momentum equation: linear terms

We now have all the elements to study the asymptotics of the approximate momentum equation **c)**. Given an arbitrary $\varphi \in \mathcal{T}_T$, we consider an approximate sequence (φ^n) as in Proposition 12. We shall take φ^n as a test function in **c)**, and let n tend to infinity, so as to recover (2.1). We shall rely on the fields u^n_F and u_F introduced in paragraph 5.1. We remind that

$$(1-\chi^n_S)\,u^n_F = (1-\chi^n_S)\,u^n, \quad u^n_F \to u_F \quad \text{weakly in } L^2(0,T;H^1_\sigma(\Omega)). \tag{5.23}$$

To lighten notations, we shall write $u^n_S := P^n_S u^n$, $u_S := P_S u$. We remind that these rigid fields satisfy

$$u^n_S \to u_S \quad \text{weakly * in } L^\infty(0,T;W^{k}_{loc}(\mathbb{R}^3)) \quad \forall k. \tag{5.24}$$

In this paragraph, we consider the asymptotics of all terms but the convection one.

- We write the diffusion term as:

$$\begin{aligned}\int_0^T\int_\Omega 2\mu^n D(u^n) : D(\varphi^n) &= \int_0^T\int_\Omega \left(2\mu_F(1-\chi^n_S)D(u^n_F) + \frac{1}{n^2}\chi^n_S D(u^n)\right) : D(\varphi^n)\\ &= \int_0^T\int_\Omega 2\mu_F(1-\chi^n_S)D(u^n_F) : D(\varphi_F) + \frac{1}{n^2}\int_0^T\int_\Omega \chi^n_S\,D(u^n_S) : D(\varphi^n) := I^n_1 + I^n_2.\end{aligned}$$

  From the strong convergence of χ^n_S to χ_S in $C([0,T];L^p(\Omega))$, and the weak convergence of u^n_F to u_F in $L^2(0,T;H^1(\Omega))$, we deduce

$$I^n_1 \to \int_0^T\int_\Omega 2\mu_F(1-\chi_S)D(u_F) : D(\varphi_F).$$

As regards I_2^n, we use the bounds

$$\|\sqrt{\mu^n} D(u^n)\|^2_{L^2((0,T)\times\Omega)} = O(1), \quad \|\varphi^n\|_{L^\infty(0,T;H^1(\Omega))} = O(n^{\alpha/2})$$

established in the previous paragraphs. They imply

$$|I_2^n| \leq \frac{C}{n^2} \|\chi_S D(u^n)\|_{L^2((0,T)\times\Omega)} \|D(\varphi^n)\|_{L^2((0,T)\times\Omega)} \leq \frac{C}{n^{1-\alpha/2}}$$

If we choose $\alpha < 2$, then I_2^n goes to zero as n goes to infinity, and finally

$$\int_0^T \int_\Omega 2\mu^n D(u^n) : D(\varphi^n) \to \int_0^T \int_{F(t)} 2\mu_F \, D(u_F) : D(\varphi_F).$$

- The boundary term at $\partial\Omega$ reads

$$\frac{1}{2\beta_\Omega} \int_0^T \int_{\partial\Omega} (u^n \times \nu) \cdot (\varphi^n \times \nu) = \frac{1}{2\beta_\Omega} \int_0^T \int_{\partial\Omega} (u_F^n \times \nu) \cdot (\varphi_F \times \nu)$$
$$\to \frac{1}{2\beta_\Omega} \int_0^T \int_{\partial\Omega} (u_F \times \nu) \cdot (\varphi_F \times \nu)$$

by the weak convergence of u_F^n in $L^2(0,T;H^1(\Omega))$.

- We deal with the boundary term at ∂S^n as in the Galerkin approximation. We introduce $u_S^n := P_S^n u^n$, $r_S^n := P_S^n \varphi^n = P_S^n \varphi_S^n$ and capital letters to denote velocity fields when seen through the change of variable. We then have, as in the computation for the Galerkin method :

$$\frac{1}{2\beta_S} \int_0^T \int_{\partial S^n(t)} ((u_F^n - u_S^n) \times \nu) \cdot ((\varphi_F^n - r_S^n) \times \nu)$$
$$= \frac{1}{2\beta_S} \int_0^T \int_{\partial S_0} ((U_F^n - U_S^n) \times \nu) \cdot ((\Phi_F - R_S^n) \times \nu),$$

where we used that $\varphi_F^n = \varphi_F$. We note here that φ^n converges to φ in $C([0,T];L^6(\Omega))$ so that combining with the strong convergence of χ_S^n it yields that r_S^n converges to $r_S := P_S\varphi$ in $L^2(0,T;H^1_{loc}(\mathbb{R}^3))$. Through the change of variable, Lemma 17 yields that :

$$R_S^n \to R_S \text{ strongly in } L^2(0,T;H^{1/2}(\partial S_0)).$$

Then, we combine the respective convergences of u_F^n, u_S^n with Lemma 17 yielding, with obvious notations :

$$U_F^n \to U_F \text{ weakly in } L^2(0,T;H^1(\Omega)), \quad U_S^n \to U_S \text{ weakly in } L^2(0,T;H^1(\Omega)).$$

We apply these convergences together with the continuity of traces on $\partial S_0 \subset\subset \Omega$, and go back to the moving geometry, to obtain finally :

$$\frac{1}{2\beta_S} \int_0^T \int_{\partial S^n(t)} ((u^n - u_S^n) \times \nu) \cdot ((\varphi^n - \varphi_S^n) \times \nu)$$
$$\to \frac{1}{2\beta_S} \int_0^T \int_{\partial S(t)} ((u_F - u_S) \times \nu) \cdot ((\varphi_F - \varphi_S) \times \nu).$$

- to treat the penalization term we use the bounds

$$n\|\sqrt{\chi^n_S}(u^n - P^n_S u^n)\|^2_{L^2((0,T)\times\Omega)} = O(1), \quad \|\sqrt{\chi^n_S}(\varphi^n_S - \varphi_S)\|_{C([0,T];L^6(\Omega))} = O(n^{-\alpha/2})$$

established in the previous paragraph. We also remind that φ_S, as a rigid vector field, satisfies $\varphi_S = P^n_S \varphi_S$. From there,

$$\begin{aligned} & \left| n \int_0^T \int_\Omega \chi^n_S (u^n - P^n_S u^n) \cdot (\varphi^n - P^n_S \varphi^n) \right| \\ = & \left| n \int_0^T \int_\Omega \chi^n_S (u^n - P^n_S u^n) \cdot ((\varphi^n_S - \varphi_S) - P^n_S(\varphi^n - \varphi_S)) \right| \\ = & \left| n \int_0^T \int_\Omega \chi^n_S (u^n - P^n_S u^n) \cdot (\varphi^n_S - \varphi_S) \right| \le Cn^{1/2-\alpha/2} \end{aligned}$$

If we choose $\alpha > 1$ (which is compatible with the former constraint $\alpha < 2$), the penalization term vanishes as $n \to +\infty$.

5.5 Strong convergence of (u^n)

To show that (S, u) is a weak solution over $(0, T)$, we still have to pass to the limit in the convection term

$$\text{conv}^n := \int_0^T \int_\Omega \rho^n \left(u^n \cdot \partial_t \varphi^n + v^n \otimes u^n : \nabla \varphi^n\right).$$

To compute this limit, we first prove

Proposition 13 *Up to the extraction of a subsequence, (u^n) converges to u in $L^2((0,T)\times\Omega)$.*

This result is obtained applying the method introduced in the reference [23] (see also [10] for the 3D case). We first introduce some notations. Given $0 \le s \le 1$ and S a bounded connected subset $\Subset \Omega$, we denote

$$\mathcal{R}^s[S] = \text{ the closure of } \{v \in H^1_\sigma(\Omega) \text{ such that } v_{|S} \in \mathcal{R}\} \text{ in } H^s(\Omega).$$

As $\mathcal{R}^s[S]$ is a closed subspace of $H^s(\Omega)$ we denote $P^s[S]$ the orthogonal projector from $H^s(\Omega)$ onto this subspace. Given $s' > s$, we recall that $\mathcal{R}^{s'}[S]$ is a dense subspace of $\mathcal{R}^s[S]$, and that the imbbeding $\mathcal{R}^{s'}[S] \subset \mathcal{R}^s[S]$ is compact. If $s = 0$, we shall drop exponent s. We emphasize that in the case $s = 0$ the projector $P[S]$ does not coincide with the P_S introduced in (3.1).

Our first step is the following approximation lemma :

Lemma 14 *Let $s < \frac{1}{3}$.*

i) *The sequence (u_n) is uniformly bounded in $L^2(0,T;H^s(\Omega))$. Moreover, there is $\varepsilon = \varepsilon(s) > 0$ such that for all $h < \delta/2$,*

$$\int_0^T \|u^n(t,\cdot) - P^s[(S^n(t))_h]\, u^n(t,\cdot)\|^2_{H^s(\Omega)} \le C\left(h^\varepsilon + n^{-\varepsilon}\right). \tag{5.25}$$

ii) *One has $u \in L^2(0,T;H^s(\Omega))$. Moreover, there exists $\varepsilon = \varepsilon(s)$ such that for all $h < \delta/2$,*

$$\int_0^T \|u(t,\cdot) - P^s[(S(t))_h]\, u(t,\cdot)\|^2_{H^s(\Omega)} dt \le Ch^\varepsilon, \tag{5.26}$$

where, in both cases, the constant C depends only on initial data.

Proof of the lemma. We only prove the first item of the lemma, the second one being simpler. It relies on the construction of a suitable approximation v_h^n of u^n, rigid in a h-neighborhood of S^n. This approximation will satisfy the following properties :

- (v_h^n) is bounded in $L^2(0,T;H^s(\Omega))$ for s small enough
- $v_h^n(t,\cdot) = P_{S^n}u^n(t,\cdot)$ in $(S^n(t))_h$ and $v_h^n(t,\cdot) = u^n(t,\cdot)$ outside $(S(t))_\delta$ for a.a. $t \in (0,T)$.

  Note that it implies $v_h^n(t,\cdot) \in \mathcal{R}^s[(S^n(t))_h]$ for a.a. $t \in (0,T)$.
- for h sufficiently small and for a.a. $t \in (0,T)$ there holds

$$
\begin{aligned}
\|u^n(t,\cdot) - v_h^n(t,\cdot)\|_{L^2(\Omega\setminus(S^n(t))_h)} &\leq C\, h^{\frac{1}{3}} \left(\|P_S^n\, u^n(t,\cdot)\|_{L^2(\Omega)} + \|u^n(t,\cdot)\|_{H^1(F^n(t))}\right) \\
&\quad + C\,\|(u^n - P_S^n u^n)\cdot\nu\|_{L^2(\partial S^n(t))}\,, \\
\|\, v_h^n(t,\cdot)\|_{H^s(\Omega)} &\leq C(1 + h^{\frac{1}{3}-s})\left(\|P_S^n\, u^n(t,\cdot)\|_{L^2(\Omega)} + \|u^n(t,\cdot)\|_{H^1(F^n(t))}\right) \\
&\quad + C\,\|(u^n - P_S^n u^n)\cdot\nu\|_{L^2(\partial S^n(t))}.
\end{aligned}
\tag{5.27}
$$

Before giving further details on the construction of v_h^n we explain how the previous properties imply Lemma 5.25. By interpolation of (5.8) and (5.9), we obtain

$$
\int_0^T \|(u^n - P_S^n u^n)(t,\cdot)\cdot\nu\|^2_{L^2(\partial S^n(t))}\,dt \;\leq\; \frac{C}{\sqrt{n}}. \tag{5.28}
$$

We square the inequalities in (5.27) and integrate from 0 to T. Using (5.28) with the uniform bounds (5.3), we end up with

$$
\left(\int_0^T \|u^n - v_h^n\|^2_{L^2(\Omega\setminus(S^n(t))_h)}\right)^{1/2} \leq C\left(h^{1/3} + \frac{1}{\sqrt{n}}\right), \quad \|v_h^n\|_{L^2(0,T;H^s(\Omega))} \leq C, \quad \forall s \leq 1/3. \tag{5.29}
$$

Moreover,

$$
\begin{aligned}
&\left(\int_0^T \|u^n - v_h^n\|^2_{L^2((S^n(t))_h)}\right)^{1/2} = \left(\int_0^T \|u^n - P_S^n u^n\|^2_{L^2((S^n(t))_h)}\right)^{1/2} \\
&\leq \left(\int_0^T \|u^n - P_S^n u^n\|^2_{L^2(S^n(t))}\right)^{1/2} + \left(\int_0^T \|u^n - P_S^n u^n\|^2_{L^2((S^n(t))_h\setminus S^n(t))}\right)^{1/2}
\end{aligned}
$$

Using (5.3), we get

$$
\begin{aligned}
\left(\int_0^T \|u^n - v_h^n\|^2_{L^2((S^n(t))_h)}\right)^{1/2} &\leq \frac{C}{\sqrt{n}} + \left(\int_0^T \|u^n - P_S^n u^n\|^2_{L^2((S^n(t))_h\setminus S^n(t))}\right)^{1/2} \\
&\leq \frac{C}{\sqrt{n}} + C\sqrt{h}\left(\int_0^T \|u^n - P_S^n u^n\|^2_{L^4((S^n(t))_h\setminus S^n(t))}\right)^{1/2} \\
&\leq \frac{C}{\sqrt{n}} + C\sqrt{h}\left(\int_0^T \|u^n - P_S^n u^n\|^2_{H^1(F^n(t))}\right)^{1/2} \leq \frac{C}{\sqrt{n}} + C\sqrt{h}
\end{aligned}
$$

Combining this last inequality with the first inequality in (5.29) yields

$$\|u^n - v_h^n\|_{L^2((0,T)\times\Omega)} \;\le\; C\left(h^{1/3} + \frac{1}{\sqrt{n}}\right) \tag{5.30}$$

As regards the H^s norm, $s \le 1/3$, we use the second inequality in (5.29) to write

$$\begin{aligned}\|u^n - v_h^n\|_{L^2(0,T;H^s(\Omega))} \;\le\; & \left(\int_0^T \|u^n - P_S^n u^n\|^2_{H^s(S^n(t))} dt\right)^{1/2} + \left(\int_0^T \|u^n\|^2_{H^s(F^n(t))} dt\right)^{1/2} \\ & + \left(\int_0^T \|v_h^n\|^2_{H^s(F^n(t))} dt\right)^{1/2} \;\le\; \|u^n - P_S^n u^n\|_{L^2(0,T;H^s(S^n))} \;+\; O(1)\end{aligned}$$

Finally, we have

$$\begin{aligned}\|u^n - P_S^n u^n\|_{L^2(0,T;H^s(S^n))} \;&\le\; C\,\|u^n - P_S^n u^n\|^{1-s}_{L^2((0,T)\times S^n)}\,\|u^n - P_S^n u^n\|^s_{L^2(0,T;H^1(S^n))} \\ &\le\; C\left(\frac{1}{\sqrt{n}}\right)^{1-s} n^s \;\le\; C \quad \text{as soon as } s \le \frac{1}{3}.\end{aligned}$$

We end up with

$$\|u^n - v_h^n\|_{L^2(0,T;H^s(\Omega))} \;\le\; C \quad \text{as soon as } s \le \frac{1}{3}. \tag{5.31}$$

One last interpolation between (5.30) and (5.31) shows that for all $s < 1/3$ and $\varepsilon = \varepsilon(s) > 0$,

$$\|u^n - v_h^n\|_{L^2(0,T;H^s(\Omega))} \;\le\; C\left(h^\varepsilon + n^{-\varepsilon}\right).$$

As $v_h^n(t,\cdot)$ belongs to $\mathcal{R}^s[(S^n(t))_h]$ for all t, by definition of the projection, the same inequality holds replacing v_h^n by $P[(S^n(t))_h]$, as expected.

We still have to achieve the construction of v_h^n. It follows the construction of v^n, *cf* paragraph 5.2. It is actually simpler, because we only look for a v_h^n with H^s regularity for small s. In particular, jump on the tangential part at $\partial(S^n)_h$ and $\partial(S^n)_\delta$ will be allowed.

As before, we go back to Lagrangian coordinates : we look for a v_h^n under the form

$$v_h^n(t, \phi_{t,0}(y)) = d\phi_{t,0}|_y V_h^n(t,y).$$

Also, we define U^n and U_S^n through

$$E_\Omega u^n\left(t, \phi_{t,0}^n(y)\right) \;=\; d\phi_{t,0}^n|_y\, U^n(t,y), \quad P_S^n u^n\left(t, \phi_{t,0}^n(y)\right) \;=\; d\phi_{t,0}^n|_y\, U_S^n(t,y).$$

In this way, we are back to a static problem. *For brevity, we shall omit temporarily the time dependence in our notations.* The point is to build a field V_h^n satisfying

$$V_h^n = U_S^n \quad \text{in } (S_0)_h, \quad V_h^n = U^n \quad \text{outside } (S_0)_\delta,$$

and suitable estimates.

Therefore, we follow paragraph 5.2. We parametrize $(S_0)_\delta \backslash S_0$ by curvilinear coordinates (s_1, s_2, z), z being the distance at ∂S_0. Hence, $\partial(S_0)_h = \{z = h\}$. Then, we introduce

$$V_{h,1}^n \;:=\; \left(1 - \chi\left(\frac{z-h}{h}\right)\right) U^n + \chi\left(\frac{z-h}{h}\right)\left(U_S^n + \left[(U^n - U_S^n)\cdot e_z\right] e_z\right)$$

and the solution $V^n_{h,2}$ of

$$\left\{\begin{array}{rcll} \mathrm{div} V^n_{h,2} & = & -\mathrm{div} V^n_{h,1}\,, & \text{in } (S_0)_\delta \setminus (S_0)_h\,, \\ V^n_{h,2} & = & 0 & \text{on } \partial(S_0)_\delta \text{ and } \partial(S_0)_h \end{array}\right.$$

Computations similar to those of paragraph 5.2 yield :

$$\|V^n_{h,1} - U^n\|_{L^2((S_0)_\delta\setminus(S_0)_h)} \leq h^{\frac{1}{3}}\|(U^n, U^n_S)\|_{H^1((S_0)_\delta\setminus(S_0)_h)\times\mathcal{R}}, \tag{5.32}$$
$$\|V^n_{h,1}\|_{H^1((S_0)_\delta\setminus(S_0)_h)} \leq h^{-\frac{2}{3}}\|(U^n, U^n_S)\|_{H^1((S_0)_\delta\setminus(S_0)_h)\times\mathcal{R}}, \tag{5.33}$$
$$\|V^n_{h,2}\|_{H^1((S_0)_\delta\setminus(S_0)_h)} \leq C\,\|(U^n, U^n_S)\|_{H^1((S_0)_\delta\setminus(S_0)_h)\times\mathcal{R}}. \tag{5.34}$$

Let us emphasize that the constant C in the last inequality can be chosen uniformly in h, see [11, Theorem III.3.1]. It follows by interpolation that

$$\|V^n_{h,1} + V^n_{h,2}\|_{H^s((S_0)_\delta\setminus(S_0)_h)} \leq C\,h^{\frac{1}{3}-s}\,\|(U^n, U^n_S)\|_{H^1((S_0)_\delta\setminus(S_0)_h)\times\mathcal{R}}. \tag{5.35}$$

Finally, we build some $W^n_h = \nabla Y^n_h$ where Y^n_h is the unique solution of :

$$\left\{\begin{array}{rcll} \Delta Y^n_h & = & 0 & \text{in } (S_0)_\delta \setminus (S_0)_h\,, \\ \partial_z Y^n_h & = & (U^n_S - U^n)\cdot e_z\,, & \text{on } \partial(S_0)_h\,, \\ \partial_z Y^n & = & 0\,, & \text{on } \partial(S_0)_\delta\,, \end{array}\right. \qquad \text{such that } \int_{(S_0)_\delta\setminus(S_0)_h} Y^n_h = 0\,.$$

we recall that $\nu = e_z$ on $\partial(S_0)_h$. By standard elliptic regularity results, there exists a constant C independent of h such that :

$$\|W^n_h\|_{L^2((S_0)_\delta\setminus(S_0)_h)} \leq \|Y^n_h\|_{H^1((S_0)_\delta\setminus(S_0)_h)} \leq C\|(U^n_S - U^n)\cdot e_z\|_{H^{-1/2}(\partial(S_0)_h)}\,,$$
$$\|W^n_h\|_{H^1((S_0)_\delta\setminus(S_0)_h)} \leq \|Y^n_h\|_{H^2((S_0)_\delta\setminus(S_0)_h)} \leq C\|(U^n_S - U^n)\cdot e_z\|_{H^{1/2}(\partial(S_0)_h)}\,.$$

By interpolation, we get

$$\|W^n_h\|_{H^{1/2}((S_0)_\delta\setminus(S_0)_h)} \leq C\|(U^n_S - U^n)\cdot e_z\|_{L^2(\partial(S_0)_h)} \tag{5.36}$$

Now, we write

$$\|(U^n_S - U^n)\cdot e_z\|_{L^2(\partial(S_0)_h)} \leq C\left(h^{\frac{1}{2}}\|\nabla(U^n_S - U^n)\|_{L^2((S_0)_\delta\setminus(S_0)_h)} + \|(U^n_S - U^n)\cdot e_z\|_{L^2(\partial S_0)}\right). \tag{5.37}$$

Eventually, we set $V^n_h := V^n_{h,1} + V^n_{h,2} - W^n_h$. We stress that the normal component of V^n_h is continuous across $\partial(S_0)_h$ and $\partial(S_0)_\delta$. Hence, for any $s < \frac{1}{2}$, the H^s norm of V^n_h over the whole domain is controlled by the sum of the H^s norms over $(S_0)_h$, $(S_0)_\delta \setminus (S_0)_h$ and $\Omega^n \setminus (S_0)_\delta$, where Ω^n is a shorthand for $\phi^n_{0,t}(\Omega)$. It follows from this remark and the previous inequalities that: for all $s < 1/2$

$$\|V^n_h\|_{H^s(\Omega^n)} \leq C\left((1 + h^{\frac{2-5s}{6}})\left(\|U^n\|_{H^1(F^n)} + \|U^n_S\|_{\mathcal{R}}\right) + \|(U^n_S - U^n)\cdot e_z\|_{L^2(\partial S_0)}\right). \tag{5.38}$$

Also, one has

$$\|V^n_h - U^n\|_{L^2(\Omega^n\setminus(S_0)_h)} \leq C\left(h^{1/3}\left(\|U^n\|_{H^1(F^n)} + \|U^n_S\|_{\mathcal{R}}\right) + \|(U^n_S - U^n)\cdot e_z\|_{L^2(\partial S_0)}\right). \tag{5.39}$$

Back to the moving domain, and accouting for time dependence, we obtain (5.27).

The second step in the treatment of the nonlinear terms is a control of the Hausdorff distance between $S^n(t)$ and $S(t')$ for close times $t, t' \in [0,T]$. This is the purpose of

Lemma 15 *Let $h > 0$.*

i) *There exists $n_0 \geq 0$ such that for all $n \geq n_0$,*

$$S^n(t) \subset (S(t))_{h/4} \subset (S^n(t))_{h/2} \quad \forall t \in [0,T].$$

ii) *There exists $\eta > 0$ such that for all $t_0 \in [0,T]$, for all $t \in [t_0 - \eta, t_0 + \eta] \cap [0,T]$*

$$(S(t))_{h/2} \subset (S(t_0))_h \subset (S(t))_{2h}.$$

Note that condition (5.1) and point i) of the lemma (applied with $h = \delta$) imply that

$$\mathrm{dist}(S(t), \partial\Omega) \geq \frac{3}{2}\delta, \quad \text{for } t \in [0,T], \quad \text{for some fixed } \delta > 0. \tag{5.40}$$

Proof of the lemma. We first treat **i)**, focusing on the first inclusion (the other one is proved in the same way). To this end, we recall that the associated sequence of characteristic functions χ_S^n converges to χ_S in $C([0,T]; L^1(\Omega))$. This implies that

$$\sup_{t\in[0,T]} |S^n(t) \triangle S(t)| = \sup_{t\in[0,T]} \|\chi_S^n(t,\cdot) - \chi_S(t,\cdot)\|_{L^1(\Omega)} \to 0 \text{ when } n \to \infty\,,$$

where we denoted $\triangle$ the symmetric difference of subsets of $\mathbb{R}^3$. Let us now take $h > 0$ and assume *a contrario* that there exists a sequence of times $t_k \in [0,T]$ and of integers n_k going to infinity such that

$$S^{n_k}(t_k) \setminus (S(t_k))_{h/4} \neq \emptyset\,.$$

As $S^{n_k}(t_k)$ is isometric to S_0, which satisfies:

$\exists\, r > 0$ s.t. for all $x \in S_0$ there exists a euclidean ball B with radius r satisfying $x \in B \subset S_0$

there exists for all k a ball B'_k with radius $r' = \min(r, h/16)$ such that

$$B'_k \subset S^{n_k}(t_k) \setminus S(t_k)\,,$$

so that

$$\sup_{t\in[0,T]} |S^{n_k}(t) \triangle S(t)| \geq \frac{4\pi|r'|^3}{3}\,,$$

which yields a contradiction. Consequently, there exists n_0 such that, for all $n \geq n_0$,

$$S^n(t) \subset (S(t))_{h/4}\,, \quad \forall t \in [0,T].$$

The second item **ii)** is obtained in the same way. Let $h > 0$ and assume for instance that the first inclusion does not hold. Arguing as previously, we are able to construct two sequences (t_0^k) and (t^k) converging both to $t_0 \in [0,T]$ and such that $S(t^k) \setminus S(t_0^k)$ contains a ball of fixed radius. Once again, this contradicts the continuity in $L^1(\Omega)$ of χ_S at t_0.

Thanks to the previous lemmas, we can conclude the proof of Proposition 13, following very closely [23]. At first, very minor adaptation of the proof of [23, Proposition 7.1] yields: *for $s \in (0,1/3)$, there exists h_0, such that, for all $h \in (0,h_0)$:*

$$\lim_{n\to\infty} \int_0^T \int_\Omega \rho^n u^n \cdot P^s[(S(t))_h]u^n = \int_0^T \int_\Omega \rho u \cdot P^s[(S(t))_h]u. \tag{5.41}$$

We remind that the main idea behind this limit is the following: thanks to Lemma 15, for any field ξ, the projected field $P^s[(S(t))_h](\xi)$ is rigid in a neighborhood of $S^n(t)$ for n large enough. Hence, if one uses $P^s[(S(t))_h](\xi)$ as a test function in the momentum equation, the boundary term at $\partial S^n(t)$ and the penalization term vanish: roughly, one recovers a uniform bound on $\partial_t P^s[(S(t))_h](\rho^n u^n)$ in a Sobolev space of negative index, and from there compactness. For all details, see [23, Proposition 7.1].

Then, one establishes that

$$\lim_{n\to\infty} \int_0^T \int_\Omega \rho^n |u^n|^2 = \int_0^T \int_\Omega \rho |u|^2, \tag{5.42}$$

$\left(\rho^n = \rho_F(1-\chi_S^n) + \rho_S \chi_S^n, \quad \rho = \rho_F(1-\chi_S) + \rho_S\chi_S\right)$. The idea is to write

$$\begin{aligned}
&\int_0^T \int_\Omega \rho^n |u^n|^2 - \int_0^T \int_\Omega \rho |u|^2 \\
&= \left(\int_0^T \int_\Omega \rho^n u^n \cdot P^s[(S(t))_h](u^n) - \int_0^T \int_\Omega \rho u \cdot P^s[(S(t))_h](u) \right) \\
&\quad + \int_0^T \int_\Omega \rho^n u^n \cdot (u^n - P^s[S(t)_h]u^n) dt \; + \; \int_0^T \int_\Omega \rho u \cdot (P^s[S(t)_h]u - u) dt
\end{aligned}$$

The first term at the r.h.s. is controlled using (5.41), whereas the last two are treated thanks to Lemma 13: note that $(S(t))_h \subset (S^n(t))_{2h}$ for n large enough by Lemma 15, so that

$$\begin{aligned}
\int_0^T \|u^n(t,\cdot) - P^s[(S(t))_h]\, u^n(t,\cdot)\|^2_{H^s(\Omega)} \; &\le \; \int_0^T \|u^n(t,\cdot) - P^s[(S^n(t))_{2h}]\, u^n(t,\cdot)\|^2_{H^s(\Omega)} \\
&\le \; C\left(h^\varepsilon \, + \, n^{-\varepsilon}\right).
\end{aligned}$$

The final step of the proof consists in showing that

$$\int_0^T \int_\Omega \rho |u^n|^2 \; \to \; \int_0^T \int_\Omega \rho |u|^2$$

which yields the strong compactness of u^n (ρ having positive lower and upper bounds). The idea here is to write

$$\left| \int_0^T \int_\Omega \rho(|u^n|^2 - |u|^2) \right| \; \le \; \left| \int_0^T \int_\Omega \left(\rho^n |u^n|^2 - \rho |u|^2\right) \right| \; + \; \left| \int_0^T \int_\Omega (\rho^n - \rho)|u^n|^2 \right|.$$

The first term at the r.h.s. goes to zero by (5.42). For the second one, we use that $\rho^n \to \rho$ strongly in $C([9,T]; L^p(\Omega))$ for all finite p and that $|u^n|^2$ is uniformly bounded in $L^{p'}$ for some $p' > 1$, thanks to the uniform H^s bound on u^n. Again, we refer to [23] for all details.

5.6 Convergence in the momentum equation: nonlinear terms

Thanks to the strong convergence of Proposition 13, we are now able to split conv^n in a suitable way. Let us first remind that v^n is identically equal to u_S^n inside S^n, whereas φ^n is identically equal to φ_F outside S^n. This allows us to decompose the convection term as follows:

$$\mathrm{conv}^n = \int_0^T \int_\Omega \rho_F (1-\chi_S^n)\, u_F^n \cdot \partial_t \varphi_F$$
$$+ \int_0^T \int_\Omega \rho_F (1-\chi_S^n) v^n \otimes u_F^n : \nabla \varphi_F + \int_0^T \rho_S \chi_S^n (\partial_t + u_S^n \cdot \nabla) \varphi^n \cdot u^n \; := \; I_1^n + I_2^n + I_3^n$$

The convergence of I_1^n is clear:

$$I_1^n \to \int_0^T \int_\Omega \rho_F (1-\chi_S)\, u_F \cdot \partial_t \varphi_F.$$

The convergence of I_3^n follows from the fourth item in Proposition 12, which clearly implies that

$$I_3^n = \int_0^T \int_\Omega \rho_S \chi_S^n (\partial_t + u_S^n \cdot \nabla) \varphi_S \cdot u^n + o(1).$$

Using the strong convergence of $\chi_S^n u^n$ to $\chi_S u_S$ in $L^2((0,T)\times\Omega)$, it is then easily shown that

$$I_3^n = \int_0^T \int_\Omega \rho_S \chi_S \partial_t \varphi_S \cdot u_S + \int_0^T \int_\Omega \rho_S \chi_S^n u_S^n \cdot \nabla \varphi_S \cdot u_S^n \; + \; o(1).$$

Now, we write the second term at the r.h.s. as

$$\int_0^T \int_\Omega \rho_S \chi_S^n u_S^n \cdot \nabla \varphi_S \cdot u_S^n \; = \; \int_0^T \int_\Omega \rho_S \chi_S^n u_S^n \otimes u_S^n : \nabla \varphi_S$$
$$= \int_0^T \int_\Omega \rho_S \chi_S^n u_S^n \otimes u_S^n : D(\varphi_S) \; = \; 0$$

as φ_S is a rigid vector field.

It remains to study I_2^n. We know from paragraph 5.2 that

$$(1-\chi_S^n)(v^n - u^n) = (1-\chi_S^n)(v^n - u_F^n) \to 0 \text{ in } \; L^2(0,T;L^p(\Omega)), \quad \forall p \le 6.$$

It follows that

$$I_2^n \;=\; \int_0^T \int_\Omega \rho_F (1-\chi_S^n) u_F^n \otimes u_F^n : \nabla \varphi_F + o(1).$$
$$= \int_0^T \int_\Omega \rho_F (1-\chi_S^n) u^n \otimes u^n : \nabla \varphi_F + o(1).$$

In this last identity we collect the strong convergences of u^n to u in $L^2((0,T)\times\Omega)$ and of χ_S^n to χ in $\mathcal{C}([0,T];L^{15}(\Omega))$, together with the uniform regularity of (u^n, u) in $L^2(0,T;H^{1/5}(\Omega))$ (see Lemma 14), which yields that (u_n, u) are uniformly bounded in $L^2(0,T;L^{30/13}(\Omega))$. We obtain then :

$$\lim_{n\to\infty} \int_0^T \int_\Omega \rho_F (1-\chi_S^n) u^n \otimes u^n : \nabla \varphi_F \;=\; \int_0^T \int_\Omega \rho_F (1-\chi_S) u \otimes u : \nabla \varphi_F$$
$$= \int_0^T \int_\Omega \rho_F (1-\chi_S) u_F \otimes u_F : \nabla \varphi_F$$

This concludes our proof.

5.7 Energy inequality and extension to collision time.

We pass to the weak limit in (4.10) and prove that the solution (ρ, u) satisfies the further energy estimate (2.2). First, we note that (4.10) implies

$$\|\sqrt{\rho^n}u^n(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega 2\mu^n|D(u^n)|^2 + \frac{1}{2\beta_\Omega}\int_0^t\int_{\partial\Omega}|u^n\times\nu|^2$$
$$+ \frac{1}{2\beta_S}\int_0^t\int_{\partial S^N(t)}|(u^n - P^n_S u^n)\times\nu|^2 \leq \int_0^t\int_\Omega \rho(-g)\cdot u^n + \int_\Omega \rho_0\,|u_0|^2$$

for all n. As we have convergence of $\sqrt{\rho^n}u^n$ in $L^{2-\varepsilon}((0,T)\times\Omega)$ we can pass to the weak limit in this inequality for almost all $t\in[0,T]$. As $S(t)$ remains far from $\partial\Omega$ we treat boundary terms in a similar way as in paragraph 4.4. The only term which requires a new treatment is :

$$\int_0^t\int_\Omega 2\mu^n|D(u^n)|^2.$$

For this term, we note that, because of Lemma 15, there holds for arbitrary $h>0$ and n sufficiently large :

$$\int_0^t\int_{\Omega\setminus(S(t))_h} 2\mu_F|D(u^n)|^2 \leq \int_0^t\int_\Omega 2\mu^n|D(u^n)|^2.$$

If we let n go to infinity, and then h go to 0, we obtain :

$$\int_0^t\int_{F(t)} 2\mu_F|D(u_F)|^2 \leq \liminf\int_0^t\int_\Omega 2\mu^n|D(u^n)|^2,$$

for almost all $t\in[0,T]$. Hence, passing to the limit in (4.10) yields (2.2).

Our solutions are limited in time to avoid collision. Namely, the only shortcoming of our construction is that it requires the distance between $S(t)$ and $\partial\Omega$ to be larger than a fixed positive distance δ through time. However, as long as we are given an initial data $u_0\in L^2(\Omega)$ and an initial position S_0 such that $S_0\Subset\Omega$, we are able to construct a small time T depending only on the inital position of S_0 in Ω and the L^2 norm of u_0 such that the solution exists and satisfies this property on $[0,T]$. As our solutions satisfy also energy estimate (2.2) we might reproduce the arguments of [10, Lemma 2.2] to concatenate solutions in time and prove existence of at least one weak solution until collision time.

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A Weak/Strong convergence and isometries

In this appendix, we study the influence of isometric transformations on weak and strong convergence of sequences. First, we prove :

Lemma 16 *Let $\phi \in C([0,T]; Isom(\mathbb{R}^3))$. Given $w : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ we define :*

$$w(t,\phi(t,y)) := d\phi_t|_y W(t,y)\,, \quad \forall\,(t,y) \in (0,T) \times \mathbb{R}^3. \tag{A.1}$$

Then

- *If $w \in L^2(0,T;H^1(\mathbb{R}^3))$ then $W \in L^2(0,T;H^1(\mathbb{R}^3))$.*
- *If $w \in C([0,T];H^1(\mathbb{R}^3))$ then $W \in C([0,T];H^1(\mathbb{R}^3))$.*
- *The same assertions hold true replacing $H^1(\mathbb{R}^3)$ by $H^1_{loc}(\mathbb{R}^3)$.*

The proof of this lemma is based on the fact that formula (A.1) for fixed t defines a unitary transformation of $H^1(\mathbb{R}^3)$. The details are left to the reader. Second, we obtain :

Lemma 17 *Let $\phi^N : [0,T] \times \mathbb{R}^3$ such that $\phi^N(t,\cdot) \in Isom(\mathbb{R}^3)$ for all $t \in [0,T]$. We assume that ϕ^N converges to ϕ in $C([0,T]; C^\infty_{loc}(\mathbb{R}^3))$. Given a sequence $(w^N) : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ we define :*

$$w^N(t,\phi^N(t,y)) := d\phi^N_t|_y W^N(t,y).$$

Then, with obvious notations:

- *If* (w^N) *converges to* w *strongly (resp. weakly) in* $L^2(0,T;H^1(\mathbb{R}^3))$ *then* (W^N) *converges to* W *strongly (resp. weakly) in* $L^2(0,T;H^1(\mathbb{R}^3))$.
- *If* (w^N) *converges to* w *in* $C([0,T];H^1(\mathbb{R}^3))$ *then* (W^N) *converges to* W *in* $C([0,T];H^1(\mathbb{R}^3))$.
- *The same assertions hold true replacing* $H^1(\mathbb{R}^3)$ *by* $H^1_{loc}(\mathbb{R}^3)$.

Remark. We point out that w^N and W^N satisfy symmetric relations:

$$w^N\left(t,\phi_t^N(y)\right) \;=\; d\phi_t^N|_y\, W^N(t,y) \quad \Leftrightarrow \quad W^N\left(t,[\phi_t^N]^{-1}(x)\right) \;=\; d[\phi_t^N]^{-1}|_x\, w^N(t,x),$$

so that fields W^n and w^n, resp. W and w can be switched in this lemma.

Proof of Lemma 17. We first remind that ϕ_t^N is an affine isometry, so that (for all N,t)

$$\left|d\phi_t^N|_y\, x\right| = |x|\,, \quad \left|[d\phi_t^N|_y]^{-1}\, M\, d\phi_t^N|_y\right| = |M|, \quad \forall (x,y)\in\mathbb{R}^3\times\mathbb{R}^3, \quad \forall M\in M_3(\mathbb{R}). \qquad \text{(A.2)}$$

The same relations hold for ϕ instead of ϕ^N.

Strong convergence. We focus on convergence in $C([0,T];H^1(\mathbb{R}^3))$, the strong convergence in L^2H^1 being treated in the same way. First, we note that our previous lemma yields:

$$W^N\in C([0,T];H^1(\mathbb{R}^3)) \text{ for any } N, \quad W\in C([0,T];H^1(\mathbb{R}^3)).$$

Then, we write

$$\|W^N - W\|_{C([0,T];L^2(\mathbb{R}^3))} \;\le\; \sup_t I_1^N(t) \;+\sup_t\; I_2^N(t) \;+\sup_t\; I_3^N(t)$$

where

$$|I_1^N(t)|^2 := \int_{\mathbb{R}^3} \left|W^N(t,y) - d\phi_t^N\circ[d\phi_t]^{-1}|_y\, W(t,[\phi_t]^{-1}\circ\phi_t^N(y))\right|^2 dy,$$

$$|I_2^N(t)|^2 := \int_{\mathbb{R}^3} \left|[d\phi_t^N]^{-1}\circ d\phi_t|_y\, W(t,[\phi_t]^{-1}\circ\phi_t^N(y)) - W(t,[\phi_t]^{-1}\circ\phi_t^N(y))\right|^2 dy,$$

and

$$|I_3^N(t)|^2 := \int_{\mathbb{R}^3} \left|W(t,[\phi_t]^{-1}\circ\phi_t^N(y)) - W(t,y)\right|^2 dy.$$

Using (A.2), we have easily

$$\begin{aligned} |I_1^N(t)|^2 \;&=\; \int_{\mathbb{R}^3} \left|w^N(t,\phi_t^N(y)) - w(t,\phi_t^N(y))\right|^2 dydt \\ &=\; \int_{\mathbb{R}^3} \left|w^N(t,x) - w(t,x)\right|^2 dx, \end{aligned}$$

which tends uniformly to 0 when $N\to\infty$ by assumption. We then get

$$\begin{aligned} |I_2^N(t)|^2 \;&\le\; \sup_{t,y}\left|d\phi_{0,t}^N\circ d\phi_{t,0} - I_d\right|^2 \int_{\mathbb{R}^3}\left|W(t,[\phi_t]^{-1}\circ\phi_t^N(y))\right|^2 dy \\ &\le\; \sup_{t,y}\left|d\phi_{0,t}^N\circ d\phi_{t,0} - I_d\right|^2 \|W\|_{C([0,T];H^1(\mathbb{R}^3))} \to 0. \end{aligned}$$

Finally, the continuity of W with values in $L^2(\mathbb{R}^3)$ implies that:

$$\int_{|y|\geq A} |W(t,y)|^2 dy$$

can be made arbitrary small uniformly in time, taking A sufficiently large. So, we apply the local convergence of ϕ^N to ϕ to obtain that, for N sufficiently large, there holds :

$$\begin{aligned}
|I_3^N(t)| \;\leq\; & \left(\int_{|y|\geq A} |W(t,[\phi_t]^{-1}\circ\phi_t^N(y))|^2 dy\right)^{1/2} + \left(\int_{|y|\geq A} |W(t,y)|^2 dy\right)^{1/2} \\
& + \left(\int_{|y|<A} \left|W(t,[\phi_t]^{-1}\circ\phi_t^N(y)) - W(t,y)\right|^2 dy\right)^{1/2} \\
\leq\; & 2\left(\int_0^T \int_{|y|\geq A/2} |W(t,(y))|^2\right)^{1/2} + \left(\int_{|y|<A} \left|W(t,[\phi_t]^{-1}\circ\phi_t^N(y)) - W(t,y)\right|^2 dy\right)^{1/2}
\end{aligned}$$

The first term at the r.h.s. is independent of N and goes to zero as A goes to infinity. Moreover, for fixed A, $[\phi_t]^{-1}\circ\phi_t^N(y)$ converges to y uniformly in $[0,T]\times\{|y|\leq A\}$. Hence, for fixed A, the second term at the r.h.s. converges to zero as N goes to infinity (continuity of translations in L^2) uniformly in t. We conclude that I_3^N goes to zero, so that W^N converges to W in $C([0,T];L^2(\mathbb{R}^3))$. The convergence of ∇W^N to ∇W follows the same lines, which yields the result.

Weak convergence. Again, we only prove the convergence on $\mathbb{R}^3$. The convergence in $H^1_{loc}(\mathbb{R}^3)$ is similar. We assume here that (w^N) converges to w in $L^2(0,T;H^1(\mathbb{R}^3))$ weak. Given $\chi \in C_c^\infty((0,T)\times\mathbb{R}^3)$ there holds :

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^3} W^N(t,y)\cdot\chi(t,y)\,dt\,dy &= \int_0^T\int_{\mathbb{R}^3} w^N(t,\phi_t^N(y))\cdot\, d\phi_{t,0}^N|_y\,\chi(t,y)\,dt\,dy \\
&= \int_0^T\int_{\mathbb{R}^3} w^N(t,x)\cdot\left(d[\phi_t^N]^{-1}|_x\right)^{-1}\chi(t,[\phi_t^N]^{-1}(x))\,dt\,dy\,,
\end{aligned}$$

where we applied again that $d\phi_t^N|_y$ is a linear isometry. Because of the strong convergence of ϕ^N in $C([0,T];C^1_{loc}(\mathbb{R}^3))$ there holds :

$$\left(d[\phi_t^N]^{-1}|_x\right)^{-1}\chi(t,[\phi_t^N]^{-1}(x)) \to \left(d[\phi_t]^{-1}|_x\right)^{-1}\chi(t,[\phi_t]^{-1}(x)) \text{ strongly in } L^2((0,T)\times\mathbb{R}^3)$$

so that with the weak convergence of w^N we obtain :

$$\int_0^T\int_{\mathbb{R}^3} W^N(t,y)\cdot\chi(t,y)\,dt\,dy \to \int_0^T\int_{\mathbb{R}^3} W(t,y)\cdot\chi(t,y)\,dt\,dy\,.$$

Similar arguments yield also that:

$$\int_0^T\int_{\mathbb{R}^3} \nabla W^N(t,y) : \nabla\chi(t,y)\,dt\,dy \to \int_0^T\int_{\mathbb{R}^3} \nabla W(t,y) : \nabla\chi(t,y)\,dt\,dy\,.$$

which ends the proof.