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Jean-Michel Coron, Qi Lü

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Jean-Michel Coron* and Qi Lü†

Abstract

This paper is devoted to the study of the rapid exponential stabilization problem for a controlled Korteweg-de Vries equation on a bounded interval with homogeneous Dirichlet boundary conditions and Neumann boundary control at the right endpoint of the interval. For every noncritical length, we build a feedback control law to force the solution of the closed-loop system to decay exponentially to zero with arbitrarily prescribed decay rates, provided that the initial datum is small enough. Our approach relies on the construction of a suitable integral transform and can be applied to many other equations.

2010 Mathematics Subject Classification. 93D15, 35Q53.

Key Words. Korteweg-de Vries equation, stabilization, integral transform.

1 Introduction

Let $L > 0$. Consider the following controlled Korteweg-de Vries (KdV) equation on $(0, L)$

$$\begin{cases} v_t + v_x + v_{xxx} + vv_x = 0 & \text{in } (0, +\infty) \times (0, L), \\ v(t, 0) = v(t, L) = 0 & \text{on } (0, +\infty), \\ v_x(t, L) = f(t) & \text{on } (0, +\infty). \end{cases} \tag{1.1}$$

This is a control system, where, at time $t \in [0, +\infty)$, the state is $v(t, \cdot) \in L^2(0, L)$ and the control is $f(t) \in \mathbb{R}$.

*Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France E-mail: coron@ann.jussieu.fr. JMC was supported by ERC advanced grant 266907 (CPDENL) of the 7th Research Framework Programme (FP7).

†School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 610054, China. E-mail: luqi59@163.com. QL was supported by ERC advanced grant 266907 (CPDENL) of the 7th Research Framework Programme (FP7), the NSF of China under grant 11101070 and the Fundamental Research Funds for the Central Universities in China under grants ZYGX2012J115.

We are concerned with the following stabilization problem for the system (1.1).

Problem **(S)**: Let $\lambda > 0$. Does there exist a linear feedback control $F : L^2(0,L) \to \mathbb{R}$ such that, for some $\delta > 0$, every solution v of (1.1) with $f(t) = F(v(t,\cdot))$ satisfies

$$|v(t,\cdot)|_{L^2(0,L)} \leq Ce^{-\lambda t}|v(0,\cdot)|_{L^2(0,L)}, \tag{1.2}$$

for some $C > 0$, provided that $|v(0,\cdot)|_{L^2(0,L)} \leq \delta$?

Let us point out that, for every continuous linear map $F : L^2(0,L) \to \mathbb{R}$, the Cauchy problem associated with (1.1) and $f(t) = F(v(t,\cdot))$ is locally well-posed in the sense of the following theorem, which is proved in the appendix of this paper.

Theorem 1.1 *Let $F : L^2(0,L) \to \mathbb{R}$ be continuous linear map and let $T_0 \in (0,+\infty)$. Then, for every $v^0 \in L^2(0,L)$, there exists at most one solution $v \in C^0([0,T_0];L^2(0,L)) \cap L^2(0,T_0;H_0^1(0,L))$ of (1.1) with $f(t) = F(v(t,\cdot))$ satisfying the initial condition $v(0,\cdot) = v^0(\cdot)$.*

Moreover, there exists an $r_0 > 0$ such that, for every $v^0 \in L^2(0,L)$ with

$$|v^0|_{L^2(0,L)} \leq r_0, \tag{1.3}$$

there exists one solution $v \in C^0([0,T_0];L^2(0,L)) \cap L^2(0,T_0;H_0^1(0,L))$ of (1.1) with $f(t) = F(v(t,\cdot))$ satisfying the initial condition $v(0,\cdot) = v^0(\cdot)$.

Let $F : L^2(0,L) \to \mathbb{R}$ be continuous linear map and let r_0 be as in Theorem 1.1 for $T_0 \triangleq 1$. Let $v^0 \in L^2(0,L)$ be such that (1.3) holds. From Theorem 1.1, we get that there exists a unique $T \in (0,+\infty]$ such that

(i) There exists $v \in C^0([0,T);L^2(0,L)) \cap L^2_{loc}([0,T);H^1(0,L))$ of (1.1) with $f(t) = F(v(t,\cdot))$ satisfying the initial condition $v(0,\cdot) = v^0(\cdot)$,

(ii) For every $T' \in (T,+\infty)$ there is no $v \in C^0([0,T');L^2(0,L)) \cap L^2_{loc}([0,T');H^1(0,L))$ of (1.1) with $f(t) = F(v(t,\cdot))$ satisfying the initial condition $v(0,\cdot) = v^0(\cdot)$.

From now on, the v as in (i) will be called the solution of (1.1) with $f(t) = F(v(t,\cdot))$ satisfying the initial condition $v(0,\cdot) = v^0$. (In fact (1.3) is not needed: see Remark A.2 below.)

It follows from [21, Lemma 3.5] that Problem **(S)** has a negative answer for the linearized system of (1.1) at 0 (i.e. the control system (1.5) below) for every $\lambda > 0$ and every $L \in \mathcal{N}$ with $\mathcal{N}$ defined by

$$\mathcal{N} \triangleq \left\{ 2\pi\sqrt{\frac{l^2 + lj + j^2}{3}};\, l,j \in \mathbb{Z}^+ \right\}, \tag{1.4}$$

where $\mathbb{Z}^+$ denote the set of positive integers: $\mathbb{Z}^+ \triangleq \{1,2,3,\ldots\}$. (Note, however, that it has been proved in [6] that, for $L = 2\pi \in \mathcal{N}$, $0 \in L^2(0,L)$ is locally asymptotically stable for our KdV equation (1.1) with $f(\cdot) = 0$, even if this property is not true for the linearized system at 0 of our KdV equation (1.1) with $f(\cdot) = 0$.) In this paper, unless otherwise specified, we always keep the assumption that $L \notin \mathcal{N}$.

The stabilization problems for both linear and nonlinear systems were studied extensively in the literature. There are too many related works to list them comprehensively here. As a result, we restrict ourselves to KdV equations. For KdV equations with internal feedback controls, we refer to [14, 18, 19, 20, 23, 24, 25, 26] and the rich references therein, as well as the self-contained survey paper [3]. For KdV equations with boundary feedback controls, we refer to [4, 5, 31] and the numerous references therein.

In this paper, we focus on the boundary feedback stabilization problems. This issue was first studied in [31], where the author proved that the system (1.1) is locally exponentially stable in $H_0^1(0, L)$ for $L = 1$ and for the feedback control $f(t) = \alpha u_x(0, t)$ where $\alpha \in (0, 1)$. In that work, the exponential decay rate depends on α but cannot be arbitrarily large.

There are three natural methods to study Problem **(S)**. The first one is the so called "Gramian approach". It works well for time reversible linear systems (see [13, 27, 29] for example). This method was used in [5] to get the rapid stabilization for the following linearized system of (1.1)

$$\begin{cases} \tilde{v}_t + \tilde{v}_x + \tilde{v}_{xxx} = 0 & \text{in } (0, +\infty) \times (0, L), \\ \tilde{v}(t, 0) = \tilde{v}(t, L) = 0 & \text{on } (0, +\infty), \\ \tilde{v}_x(t, L) = \tilde{f}(t) & \text{on } (0, +\infty). \end{cases} \tag{1.5}$$

For every given λ, by solving a suitable linear quadratic regulator problem, a feedback control law $\tilde{f}(t) = \tilde{F}(\tilde{v}(t, \cdot))$ was constructed in [5] which makes (1.5) exponentially stable with an exponential decay rate at least equal to λ. However, one does not know how to apply this result to treat the nonlinear system (1.1).

The second one is the control Lyapunov function method. It is well known that Lyapunov functions are very useful for the study the asymptotic stability of dynamical systems. For a control system, with the aid of a suitable choice of feedback laws, there are more "chances" for a given function to be a Lyapunov function. Thus, Lyapunov functions are even more useful for the stabilization of control systems than for the stability of dynamical systems without control (see [7, Section 12.1] for example). This method can be used to get rapid stabilization for some partial differential equations (see, for example, [8]). However this method is difficult to use here, as well as for many other control systems, to get rapid stabilization. The reason is that the "natural" control Lyapunov functions do not lead to arbitrarily large exponential decay rate. See, for example, [7, Remark 12.9, p. 318].

The third one is the backstepping method. It is now a standard method for finite dimensional control systems (see, e.g., [15], [28, pages 242-246] and [7, Section 12.5]). The first adaptations of this method to control systems modeled by partial differential equations were given in [10] and [17]. This method has been used on the discretization of partial differential equations in [1]. A key modification of the method by using a Volterra transformation is introduced in [2]. This last article has been the starting point of many works. See, in particular, the references given in [16], where a systematic and clear introduction to this method is given. Note that this method can be useful to handle nonlinearities as it was shown in [4, 12]. In [4], the authors studied the rapid stabilization problem with a left Dirichlet boundary control by the backstepping method (in that case the assumption $L \notin \mathcal{N}$

is no longer required). They consider the following system

$$\begin{cases} \check{v}_t + \check{v}_x + \check{v}_{xxx} + \check{v}\check{v}_x = 0 & \text{in } (0,+\infty)\times(0,L), \\ \check{v}(t,0) = \check{f}(t), \check{v}(t,L) = 0 & \text{on } (0,+\infty), \\ \check{v}_x(t,L) = 0 & \text{on } (0,+\infty), \end{cases} \tag{1.6}$$

and proved that, for every $\lambda > 0$, there exists a continuous linear feedback control $\check{f}(t) = \int_0^L \check{k}(0,y)\check{v}(t,y)dy$, an $r > 0$ and a $C > 0$ such that

$$|\check{v}(t,\cdot)|_{L^2(0,L)} \leq Ce^{-\lambda t}|\check{v}(0,\cdot)|_{L^2(0,L)}$$

for every solution of (1.6) satisfying $|\check{v}(0,\cdot)|_{L^2(0,L)} \leq r$. Here the function

$$\check{k} : \{(x,y) :\, x \in [0,L],\, y \in [x,L]\} \to \mathbb{R}$$

is the solution to

$$\begin{cases} \check{k}_{xxx} + \check{k}_x + \check{k}_{yyy} + \check{k}_y + \lambda\check{k} = 0 & \text{in } \{(x,y) :\, x \in [0,L],\, y \in [x,L]\}, \\ \check{k}(x,L) = 0 & \text{in } [0,L], \\ \check{k}(x,x) = 0 & \text{in } [0,L], \\ \check{k}_x(x,x) = \dfrac{\lambda}{3}(L-x) & \text{in } [0,L], \end{cases} \tag{1.7}$$

as in the backstepping approach.

The main difference between the system (1.1) and the system (1.6) is the position where the control is acting. It is well known that the control properties of KdV equations depend in a crucial way on the location of the controls. For instance, the system is only null controllable if the control acts on the left Dirichlet boundary condition and homogeneous data is considered at the right end point of the interval (see [22]). On the other hand, if the system is controlled from the right boundary condition, then it is exactly controllable (see [21]). In [4], the authors also pointed out the difficulties for employing the backstepping method to solve our rapid stabilization problem for (1.1).

In this paper, in order to stabilize the solutions of (1.1), we use a more general integral transform on the state v than the one allowed by the backstepping method. In fact, no restriction is put on the integral transform which is a priori considered: it takes the following form

$$w(x) \triangleq v(x) - \int_0^L k(x,y)v(y)dy \triangleq v(x) - (Kv)(x). \tag{1.8}$$

Note that no assumption is made on the support of $k(\cdot,\cdot)$. This is in contrast with the backstepping approach where the support of k is assumed to be included in one of the following triangles

$$\{(x,y) \in [0,L]^2;\, x \in [0,L],\, y \in [0,x]\}, \quad \{(x,y) \in [0,L]^2;\, y \in [0,L],\, x \in [0,y]\}.$$

Now let us briefly explain the idea for the linearized KdV equation (1.5).

One can check that (the computations are similar but simpler than the ones for the KdV equation (1.1) given in Section 4, we omit them), if

$$\begin{cases} k_{yyy} + k_y + k_{xxx} + k_x + \lambda k = \lambda\delta(x-y) & \text{in } (0,L)\times(0,L), \\ k(x,0) = k(x,L) = 0 & \text{on } (0,L), \\ k_y(x,0) = k_y(x,L) = 0 & \text{on } (0,L), \\ k(0,y) = k(L,y) = 0 & \text{on } (0,L), \end{cases} \tag{1.9}$$

where $\delta(x-y)$ denotes the Dirac measure on the diagonal of the square $[0,L]\times[0,L]$, has a solution k which is smooth enough, then, for every solution $\tilde{v}$ of (1.5), $\tilde{w}(t,x) \triangleq \tilde{v}(t,x) - \int_0^L k(x,y)\tilde{v}(t,y)dy$ is a solution of

$$\begin{cases} \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + \lambda\tilde{w} = 0 & \text{in } (0,+\infty)\times(0,L), \\ \tilde{w}(t,0) = \tilde{w}(t,L) = 0 & \text{on } (0,+\infty), \\ \tilde{w}_x(t,L) = \tilde{f}(t) - \displaystyle\int_0^L k_x(L,y)\tilde{v}(t,y)dy & \text{on } (0,+\infty). \end{cases} \tag{1.10}$$

If we define the feedback law $F(\cdot)$ by

$$\tilde{F}(\tilde{v}) = \tilde{f}(t) \triangleq \int_0^L k_x(L,y)\tilde{v}(t,y)dy, \tag{1.11}$$

then the last equation of (1.10) becomes $\tilde{w}_x(t,L) = 0$. Multiplying the first equation of (1.10) by $\tilde{w}$ and integrating on $[0,L]$ we get, using integration by parts together with the boundary conditions of (1.10) and $\tilde{w}_x(t,L) = 0$,

$$\frac{d}{dt}\int_0^L \tilde{w}(t,x)^2 dx \leq -\lambda \int_0^L \tilde{w}(t,x)^2 dx, \tag{1.12}$$

which shows the exponential decay of the L^2-norm of $\tilde{w}$. In order to get the same exponential decay of the L^2-norm of $\tilde{v}$, it suffices to prove that

1. $|v|_{L^2(0,L)} \leq C|v - Kv|_{L^2(0,L)}$ for some $C > 0$ independent of $v \in L^2(0,L)$.

Furthermore, in order to show that (1.2) holds, we also need to prove that

2. $k(\cdot,\cdot)$ is smooth enough so that the nonlinearity vv_x will not be a problem provided that $|v(0,\cdot)|_{L^2(0,L)}$ is small enough.

We will check these two points together with the existence of $k(\cdot,\cdot)$ satisfying (1.9) and therefore we prove the following local rapid stabilization result.

Theorem 1.2 *For every $\lambda > 0$, there exist a continuous linear feedback control law $F : L^2(0,L) \to \mathbb{R}$, an $r \in (0,+\infty)$ and a $C > 0$ such that, for every $v^0 \in L^2(0,L)$ satisfying $|v^0|_{L^2(0,L)} \leq r$, the solution v of (1.1) with $f(t) \triangleq F(v(t,\cdot))$ satisfying the initial condition $v(0,\cdot) = v^0(\cdot)$ is defined on $[0,+\infty)$ and satisfies*

$$|v(t,\cdot)|_{L^2(0,L)} \leq Ce^{-\frac{\lambda}{2}t}|v(0,\cdot)|_{L^2(0,L)}, \qquad \textit{for every } t \geq 0. \tag{1.13}$$

To show the existence and uniqueness of $k(\cdot,\cdot)$, we have to utilize the fact that the linear KdV equation is exact controllable by Neumann boundary control on the right end point of the interval. This is different from the backstepping method, which does not use any controllability result.

Although the aim of this paper is to study the rapid stabilization of KdV equations, the method introduced here can be applied to many other partial differential equations on one dimensional bounded domains, such as heat equations, Schrödinger equations, beam equations and Kuramoto-Sivashinsky equations. In particular, we have applied it in [11] to the following Kuramoto-Sivashinsky control system

$$\begin{cases} v_t + v_{xxxx} + \lambda v_{xx} + v v_x = 0 & \text{in } (0,+\infty)\times(0,1), \\ v(t,0) = v(t,1) = 0 & \text{on } (0,+\infty), \\ v_{xx}(t,0) = f(t),\ v_{xx}(t,1) = 0 & \text{on } (0,+\infty), \end{cases} \tag{1.14}$$

where, at time $t \in [0,+\infty)$, the state is $v(t,\cdot) \in L^2(0,1)$ and the control is $f(t) \in \mathbb{R}$. In those cases, the main difference is that the equations satisfied by the kernel k are different from (1.9). However, one can follow the same strategy to show the existence and regularity of the kernel and the invertibility of the transform.

The rest of this paper is organized as follows

- in Section 2, we establish the well-posedness of (1.9) and study the regularity of its solution,
- in Section 3, we prove the invertibility of $I - K$,
- in Section 4, we conclude the proof of Theorem 1.2 by using the results established in Section 2 and Section 3.

2 Well-posedness of (1.9)

This section is devoted to the study of the equation (1.9). We first introduce the definition of the (transposition) solution to (1.9). Let

$$\begin{aligned} \mathcal{E} \triangleq \big\{ &\rho \in C^\infty([0,L]\times[0,L]) :\ \rho(0,y) = \rho(L,y) = \rho(x,0) = \rho(x,L) = 0, \\ &\rho_x(0,y) = \rho_x(0,L) = 0 \big\} \end{aligned} \tag{2.1}$$

and let $\mathcal{G}$ be the set of $k \in H_0^1((0,L)\times(0,L))$ such that

$$\big(x \in (0,L) \mapsto k_x(x,\cdot) \in L^2(0,L)\big) \in C^0([0,L]; L^2(0,L)), \tag{2.2}$$

$$\big(y \in (0,L) \mapsto k_y(\cdot,y) \in L^2(0,L)\big) \in C^0([0,L]; L^2(0,L)), \tag{2.3}$$

$$k_y(\cdot,0) = k_y(\cdot,L) = 0 \text{ in } L^2(0,L). \tag{2.4}$$

We call $k(\cdot,\cdot) \in \mathcal{G}$ a solution to (1.9) if

$$\begin{aligned} &\int_0^L\int_0^L \big[\rho_{yyy}(x,y) + \rho_y(x,y) + \rho_{xxx}(x,y) + \rho_x(x,y) - \lambda\rho(x,y)\big] k(x,y)\,dxdy \\ &+ \int_0^L \lambda\rho(x,x)\,dx = 0, \qquad \text{for every } \rho \in \mathcal{E}. \end{aligned} \tag{2.5}$$

We have the following well-posedness result for (1.9).

Lemma 2.1 *Let $\lambda \neq 0$. Equation* (1.9) *has one and only one solution in $\mathcal{G}$.*

Proof of Lemma 2.1: The proof is divided into two steps:

- Step 1: proof of the uniqueness of the solution to (1.9);
- Step 2: proof of the existence of a solution to (1.9) with the required regularity.

Step 1: proof of the uniqueness of the solution to (1.9)

Assume that $k_1(\cdot,\cdot)$ and $k_2(\cdot,\cdot)$ are two solutions of (1.9). Let $k_3(\cdot,\cdot) \triangleq k_1(\cdot,\cdot) - k_2(\cdot,\cdot)$. Then $k_3(\cdot,\cdot)$ is a (transposition: (2.5) holds without the last integral term) solution of

$$\begin{cases} k_{3,yyy} + k_{3,y} + k_{3,xxx} + k_{3,x} + \lambda k_3 = 0 & \text{in } (0,L)\times(0,L), \\ k_3(x,0) = k_3(x,L) = 0 & \text{on } (0,L), \\ k_{3,y}(x,0) = k_{3,y}(x,L) = 0 & \text{on } (0,L), \\ k_3(0,y) = k_3(L,y) = 0 & \text{on } (0,L). \end{cases} \tag{2.6}$$

Let us define an unbounded linear operator $A : D(A) \subset L^2(0,L) \to L^2(0,L)$ as follows.

$$\begin{cases} D(A) \triangleq \{\varphi :\, \varphi \in H^3(0,L),\, \varphi(0) = \varphi(L) = 0,\, \varphi_x(0) = \varphi_x(L)\}, \\ A\varphi \triangleq -\varphi_{xxx} - \varphi_x. \end{cases} \tag{2.7}$$

The operator A is a skew-adjoint operator with compact resolvent. Furthermore, since $L \notin \mathcal{N}$, we have $L \notin 2\pi\mathbb{Z}^+$, which, as one easily checks, implies that 0 is not an eigenvalue of A. Denote by $\{i\mu_j\}_{j\in\mathbb{Z}\setminus\{0\}}$, $\mu_j \in \mathbb{R}$, the eigenvalues of A, which are organized in the following way:

$$\cdots \leq \mu_{-2} \leq \mu_{-1} < 0 < \mu_1 \leq \mu_2 \leq \cdots. \tag{2.8}$$

Let us point out that all these eigenvalues are simple. Indeed, assume that μ_j is of multiplicity at least 2 and $\varphi_{1,j}$ and $\varphi_{2,j}$ are two linear independent eigenfunctions corresponding to $i\mu_j$. Let

$$\varphi_j \triangleq \begin{cases} \varphi_{2,j} - \dfrac{\varphi'_{2,j}(0)}{\varphi'_{1,j}(0)}\varphi_{1,j} & \text{if } \varphi'_{1,j}(0) \neq 0, \\ \varphi_{1,j} & \text{if } \varphi'_{1,j}(0) = 0. \end{cases} \tag{2.9}$$

Since $\varphi_{1,j}$ and $\varphi_{2,j}$ are linearly independent, we conclude that $\varphi_j \neq 0$. Then φ_j is a nonzero eigenfunction for this eigenvalue $i\mu_j$ such that

$$\varphi_j(0) = \varphi_j(L) = \varphi'_j(0) = \varphi'_j(L) = 0. \tag{2.10}$$

By [21, Lemma 3.5], our assumption $L \notin \mathcal{N}$ implies that, for every a nonzero eigenfunction φ_j of A,

$$\varphi'_j(0) \neq 0, \tag{2.11}$$

which is in contradiction with (2.10). Hence all these eigenvalues are simple and all the inequalities in (2.8) are strict inequalities. Let us write $\{\varphi_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ for the corresponding eigenfunctions with $|\varphi_j|_{L^2(0,L)} = 1$ ($j \in \mathbb{Z}\setminus\{0\}$). It is well known that $\{\varphi_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ constitutes an orthonormal basis of $L^2_{\mathbb{C}}(0, L)$.

In what follows, for simplicity, in $L^2_{\mathbb{C}}(0, L)$, $L^\infty_{\mathbb{C}}(0, L)$ etc., when no confusion is possible, we omit the index $\mathbb{C}$.

Next, we analyze the asymptotic behavior of μ_j as $|j| \to +\infty$. This is done in [5]. However, we present it also here since we need similar information for other sequences of numbers which will be introduced later on. To this end, we first consider the following boundary problem, with μ given in $\mathbb{R}$,

$$\begin{cases} \varphi''' + \varphi' + i\mu\varphi = 0 \quad \text{in } [0, L], \\ \varphi(0) = \varphi(L) = 0, \\ \varphi'(0) = \varphi'(L). \end{cases} \tag{2.12}$$

To solve the equation (2.12), we study the following algebraic equation in $\mathbb{C}$

$$s^3 + s + i\mu = 0. \tag{2.13}$$

Denote by s_1, s_2 and s_3 the three roots of (2.13). These roots are distinct if and only if $4 \neq 27\mu^2$, which we assume from now on. For the special case $4 = 27\mu^2$, the arguments given below can be adapted and we omit this adaptation.

Since the function $x \in \mathbb{R} \mapsto 2x(4x^2 - 1) \in \mathbb{R}$ is a surjective map from $\mathbb{R}$ to $\mathbb{R}$, we get that there always exists at least one $\tau \in \mathbb{R}$ such that

$$\mu = 2\tau(4\tau^2 - 1). \tag{2.14}$$

By this, the three roots of (2.13) are

$$s_1 = \sqrt{3\tau^2 - 1} - i\tau, \; s_2 = -\sqrt{3\tau^2 - 1} - i\tau, \; s_3 = 2i\tau.$$

Since we want to analyze the asymptotic behavior of μ_j as $|j| \to +\infty$, we may assume that $|\tau|$ is large enough. Thus, we only consider the case $3\tau^2 > 1$. Now the eigenfunction is

$$\varphi(x) = e^{-i\tau x}\big[r_1 \cosh(\sqrt{3\tau^2 - 1}x) + r_2 \sinh(\sqrt{3\tau^2 - 1}x)\big] + r_3 e^{2i\tau x}$$

for some suitable complex numbers r_1, r_2 and r_3 which are not all equal to 0 . The boundary conditions of (2.12) are fulfilled if and only if these three complex numbers satisfy that

$$\begin{cases} r_1 + r_3 = 0, \\ e^{-i\tau L}\big[r_1 \cosh(\sqrt{3\tau^2 - 1}L) + r_2 \sinh(\sqrt{3\tau^2 - 1}L)\big] + r_3 e^{2i\tau L} = 0, \\ \quad -i\tau r_1 + 2i\tau r_3 + r_2\sqrt{3\tau^2 - 1} \\ = -i\tau e^{-i\tau L}\big[r_1 \cosh(\sqrt{3\tau^2 - 1}L) + r_2 \sinh(\sqrt{3\tau^2 - 1}L)\big] + 2i\tau r_3 e^{2i\tau L} \\ \quad + e^{-i\tau L}\sqrt{3\tau^2 - 1}\big[r_1 \sinh(\sqrt{3\tau^2 - 1}L) + r_2 \cosh(\sqrt{3\tau^2 - 1}L)\big]. \end{cases}$$

Thus, we have that

$$\sqrt{3\tau^2-1}\cos(2\tau L)-3\tau\sin(\tau L)\sinh(\sqrt{3\tau^2-1}L)-\sqrt{3\tau^2-1}\cos(\tau L)\cosh(\sqrt{3\tau^2-1}L)=0. \tag{2.15}$$

As $|\tau|\to+\infty$,

$$\sinh(\sqrt{3\tau^2-1}L)\sim\cosh(\sqrt{3\tau^2-1}L)\sim\frac{1}{2}e^{\sqrt{3\tau^2-1}L}\sim\frac{1}{2}e^{\sqrt{3}|\tau|L}.$$

Hence, we find that

$$\begin{aligned} e^{\sqrt{3\tau^2-1}L}&=\frac{\cos(2\tau L)}{2\cos(\tau L-\frac{\pi}{3})}+O(1)\ \text{ as } \tau\to+\infty,\\ e^{\sqrt{3\tau^2-1}L}&=\frac{\cos(2\tau L)}{2\cos(\tau L+\frac{\pi}{3})}+O(1)\ \text{ as } \tau\to-\infty. \end{aligned} \tag{2.16}$$

For $j\in\mathbb{Z}$ with $|j|$ large enough, one checks that there exists a unique solution $\tau=\tau_j$ of (2.15) in each interval $[\frac{j\pi}{L},\frac{(j+1)\pi}{L})$, and one has that

$$\begin{cases} \tau_j=\dfrac{j\pi}{L}+\dfrac{5\pi}{6L}+O\Big(\dfrac{1}{j}\Big) & \text{as } j\to+\infty,\\ \tau_j=\dfrac{j\pi}{L}+\dfrac{\pi}{6L}+O\Big(\dfrac{1}{j}\Big) & \text{as } j\to-\infty. \end{cases} \tag{2.17}$$

From this and (2.14) we can see that

$$\begin{cases} \mu_j=\Big(\dfrac{2j\pi}{L}\Big)^3+\dfrac{40\pi^3}{3}j^2+O(j) & \text{as } j\to+\infty,\\ \mu_j=\Big(\dfrac{2j\pi}{L}\Big)^3+\dfrac{8\pi^3}{3}j^2+O(j) & \text{as } j\to-\infty. \end{cases} \tag{2.18}$$

The corresponding eigenfunction φ_j with $|\varphi_j|_{L^2(0,L)}=1$ reads

$$\varphi_j(x)=\alpha_j\Big\{e^{-i\tau_j x}\Big[\cosh\big(\sqrt{3\tau_j^2-1}x\big)+\frac{e^{3i\tau_j L}-\cosh\big(\sqrt{3\tau_j^2-1}L\big)}{\sinh\big(\sqrt{3\tau_j^2-1}L\big)}\sinh\big(\sqrt{3\tau_j^2-1}x\big)\Big]-e^{2i\tau_j x}\Big\}, \tag{2.19}$$

where $\alpha_j\in(0,+\infty)$ is a suitable chosen positive real number such that $|\varphi_j|_{L^2(0,L)}=1$.

Let us analyze the asymptotic behavior of α_j as $|j|\to+\infty$. For this, with some simple computations, we see that

$$\begin{aligned}\varphi_j(x)=\alpha_j\Big\{&\frac{\cosh\big(\sqrt{3\tau_j^2-1}x\big)\sinh\big(\sqrt{3\tau_j^2-1}L\big)-\sinh\big(\sqrt{3\tau_j^2-1}x\big)\cosh\big(\sqrt{3\tau_j^2-1}L\big)}{\sinh\big(\sqrt{3\tau_j^2-1}L\big)}e^{-i\tau_j x}\\ &+\frac{e^{3i\tau_j L}\sinh\big(\sqrt{3\tau_j^2-1}x\big)}{\sinh\big(\sqrt{3\tau_j^2-1}L\big)}e^{-i\tau_j x}-e^{2i\tau_j x}\Big\}.\end{aligned} \tag{2.20}$$

Then, we find that, as $|j| \to +\infty$,

$$\left(\int_0^L \left| \varphi_j(x) - \alpha_j \left[\frac{2e^{3i\tau_j L} \sinh\left(\sqrt{3\tau_j^2 - 1}x\right) + e^{\sqrt{3\tau_j^2-1}(L-x)}}{2\sinh\left(\sqrt{3\tau_j^2-1}L\right)} e^{-i\tau_j x} - e^{2i\tau_j x} \right] \right|^2 dx \right)^{\frac{1}{2}} \tag{2.21}$$
$$= O\left(\alpha_j e^{-\sqrt{3\tau_j^2-1}L}\right),$$

which, using $|\varphi_j|_{L^2(0,L)} = 1$, gives us

$$\alpha_j \to \frac{1}{\sqrt{L}} \text{ as } |j| \to +\infty. \tag{2.22}$$

We now estimate $\varphi_{j,x}$. From (2.19), we obtain that

$$\begin{aligned}\varphi_{j,x}(x) = \alpha_j \Bigg\{ & -i\tau_j e^{-i\tau_j x} \left[\cosh\left(\sqrt{3\tau_j^2-1}x\right) + \frac{e^{3i\tau_j L} - \cosh\left(\sqrt{3\tau_j^2-1}L\right)}{\sinh\left(\sqrt{3\tau_j^2-1}L\right)} \sinh\left(\sqrt{3\tau_j^2-1}x\right) \right] \\ & + e^{-i\tau_j x} \left[\sqrt{3\tau_j^2-1} \sinh\left(\sqrt{3\tau_j^2-1}x\right) + \frac{e^{3i\tau_j L} - \cosh\left(\sqrt{3\tau_j^2-1}L\right)}{\sinh\left(\sqrt{3\tau_j^2-1}L\right)} \sqrt{3\tau_j^2-1} \cosh\left(\sqrt{3\tau_j^2-1}x\right) \right] \\ & -2i\tau_j e^{2i\tau_j x} \Bigg\}.\end{aligned} \tag{2.23}$$

From (2.22) and (2.23), we get that, for $|j|$ large enough,

$$\begin{aligned} & |\varphi_{j,x}|_{L^\infty(0,L)} \\ & \leq \max_{x\in[0,L]} \left| \alpha_j \tau_j \left[\frac{2e^{3i\tau_j L} \sinh\left(\sqrt{3\tau_j^2-1}x\right) + e^{\sqrt{3\tau_j^2-1}L - \sqrt{3\tau_j^2-1}x} - e^{-\sqrt{3\tau_j^2-1}L + \sqrt{3\tau_j^2-1}x}}{2\sinh\left(\sqrt{3\tau_j^2-1}L\right)} \right] \right| \\ & \quad + \max_{x\in[0,L]} \left| \alpha_j \sqrt{3\tau_j^2-1} \left[\frac{2e^{3i\tau_j L} \sinh\left(\sqrt{3\tau_j^2-1}x\right) - e^{\sqrt{3\tau_j^2-1}L - \sqrt{3\tau_j^2-1}x} - e^{-\sqrt{3\tau_j^2-1}L + \sqrt{3\tau_j^2-1}x}}{2\sinh\left(\sqrt{3\tau_j^2-1}L\right)} \right] \right| \\ & \quad + 2|\tau_j| \\ & \leq C|\tau_j|, \end{aligned} \tag{2.24}$$

From (2.17) and (2.24), one gets, for every $j \in \mathbb{Z}^+$,

$$|\varphi_{j,x}|_{L^\infty(0,L)} \leq C|j|. \tag{2.25}$$

From (2.23), as $|j| \to +\infty$,

$$\begin{aligned}\varphi_{j,x}(0) &= \alpha_j \left[\frac{e^{3i\tau_j L} - \cosh\left(\sqrt{3\tau_j^2-1}L\right)}{\sinh\left(\sqrt{3\tau_j^2-1}L\right)} \sqrt{3\tau_j^2-1} - 3i\tau_j \right] \\ &= \alpha_j \left(-3i\tau_j - \sqrt{3\tau_j^2-1} \right) + \alpha_j O\left(e^{-\sqrt{3\tau_j^2-1}L}\right)\end{aligned} \tag{2.26}$$

and

$$
\begin{aligned}
\varphi_{j,x}(L) = \alpha_j \bigg\{ & -3i\tau_j e^{2i\tau_j L} + e^{-i\tau_j L} \bigg[\sqrt{3\tau_j^2 - 1} \sinh\big(\sqrt{3\tau_j^2 - 1}L\big) \\
& + \sqrt{3\tau_j^2 - 1} \frac{e^{3i\tau_j L} - \cosh\big(\sqrt{3\tau_j^2 - 1}L\big)}{\sinh\big(\sqrt{3\tau_j^2 - 1}L\big)} \cosh\big(\sqrt{3\tau_j^2 - 1}L\big) \bigg] \bigg\} \\
= \alpha_j \Big(& -3i\tau_j + \sqrt{3\tau_j^2 - 1} \Big) e^{2i\tau_j L} + \alpha_j O\big(e^{-\sqrt{3\tau_j^2 - 1}L}\big).
\end{aligned} \tag{2.27}
$$

Let us write

$$
k_3(x, y) = \sum_{j \in \mathbb{Z} \setminus \{0\}} \psi_j(x) \varphi_j(y) \tag{2.28}
$$

for the solution to (2.6). Then, we get that ψ_j solves

$$
\begin{cases} \psi_j''' + \psi_j' + \lambda\psi_j - i\mu_j \psi_j = 0 & \text{in } (0, L), \\ \psi_j(0) = \psi_j(L) = 0. \end{cases} \tag{2.29}
$$

Let $c_j \triangleq \psi_{j,x}(L) - \psi_{j,x}(0)$ $(j \in \mathbb{Z} \setminus \{0\})$. We consider the following equation:

$$
\begin{cases} \check{\psi}_j''' + \check{\psi}_j' + \lambda\check{\psi}_j - i\mu_j \check{\psi}_j = 0 & \text{in } (0, L), \\ \check{\psi}_j(0) = \check{\psi}_j(L) = 0, \\ \check{\psi}_j'(L) - \check{\psi}_j'(0) = 1. \end{cases} \tag{2.30}
$$

Since $\lambda - i\mu_j$ is not an eigenvalue of A (recall that $\lambda \neq 0$), the equation (2.30) has one and only one solution. Moreover

$$
\psi_j = c_j \check{\psi}_j \text{ for every } j \in \mathbb{Z} \setminus \{0\}. \tag{2.31}
$$

Denote by $r_j^{(1)}$, $r_j^{(2)}$ and $r_j^{(3)}$ the roots of

$$
r^3 + r = -\lambda + i\mu_j. \tag{2.32}
$$

Let $\sigma_j \in \mathbb{R}$ be such that

$$
2\sigma_j(4\sigma_j^2 - 1) = \begin{cases} -\sqrt{\lambda^2 + \mu_j^2}, & \text{if } \mu_j > 0, \\ \sqrt{\lambda^2 + \mu_j^2}, & \text{if } \mu_j < 0. \end{cases} \tag{2.33}
$$

Then, there exists $C > 0$ such that

$$
|\sigma_{-j} - \tau_j| \le \frac{C}{1 + j^2} \text{ and } |\sigma_{-j}^2 - \tau_j^2| \le \frac{C}{1 + |j|}, \forall\, j \in \mathbb{Z} \setminus \{0\}. \tag{2.34}
$$

Let $\hat{\sigma}_j \in \mathbb{C}$ be such that

$$2\hat{\sigma}_j(4\hat{\sigma}_j^2 - 1) = -i\lambda - \mu_j. \tag{2.35}$$

Roughly speaking, there are three complex numbers which satisfy (2.35). For a good choice of $\hat{\sigma}_j$, we have

$$\lim_{|j|\to+\infty} |\sigma_j - \hat{\sigma}_j| = 0.$$

In this case, it is easy to see that there is a constant $C > 0$ such that

$$|\sigma_j - \hat{\sigma}_j| \leq \frac{C}{1+j^2}, \forall j \in \mathbb{Z} \setminus \{0\}. \tag{2.36}$$

The solutions to (2.32) read

$$r_j^{(1)} = \sqrt{3\hat{\sigma}_j^2 - 1} - i\hat{\sigma}_j,\ r_j^{(2)} = -\sqrt{3\hat{\sigma}_j^2 - 1} - i\hat{\sigma}_j,\ r_j^{(3)} = 2i\hat{\sigma}_j.$$

For every $j \in \mathbb{Z} \setminus \{0\}$, let us define

$$\kappa_j^{(1)} \triangleq \frac{-i\hat{\sigma}_j + \sqrt{3\hat{\sigma}_j^2 - 1} - \sqrt{3\sigma_j^2 - 1}}{-i\sigma_j}, \quad \kappa_j^{(2)} \triangleq \frac{\hat{\sigma}_j}{\sigma_j}, \tag{2.37}$$

$$\theta_j(x) \triangleq e^{(-i\hat{\sigma}_j - \sqrt{3\hat{\sigma}_j^2-1})x} - e^{(-i\hat{\sigma}_j + \sqrt{3\hat{\sigma}_j^2-1} - 2\sqrt{3\sigma_j^2-1})x},\ x \in [0, L]. \tag{2.38}$$

Let $J \in \mathbb{Z}^+$ be such that

$$(j \in \mathbb{Z} \text{ and } |j| \geq J) \Rightarrow \left(3\sigma_j^2 - 1 \geq 1 \text{ and } |3\hat{\sigma}_j^2 - 1| \geq 1\right). \tag{2.39}$$

(The existence of such a J follows from (2.17), (2.34) and (2.36)). From (2.36) and (2.38), one gets the existence of $C_1 > 0$ and $C > 0$ such that, for every $x \in [0, L]$ and every $j \in \mathbb{Z} \setminus \{0\}$ with $|j| \geq J$,

$$\begin{aligned}
|\theta_j(x)| &\leq \left| e^{(-i\hat{\sigma}_j - \sqrt{3\hat{\sigma}_j^2-1} + \sqrt{3\sigma_j^2-1})x} - e^{(-i\hat{\sigma}_j + \sqrt{3\hat{\sigma}_j^2-1} - \sqrt{3\sigma_j^2-1})x} \right| e^{-\sqrt{3\sigma_j^2-1}x} \\
&\leq \left| e^{(-i\hat{\sigma}_j - \sqrt{3\hat{\sigma}_j^2-1} + \sqrt{3\sigma_j^2-1})x} \right| \left| 1 - e^{2(\sqrt{3\hat{\sigma}_j^2-1} - \sqrt{3\sigma_j^2-1})x} \right| e^{-\sqrt{3\sigma_j^2-1}x} \\
&\leq C_1 e^{\frac{C_1}{j^2}} \frac{C_1}{j^2} e^{-\sqrt{3\sigma_j^2-1}x} \leq \frac{C}{j^2} e^{-\sqrt{3\sigma_j^2-1}x}.
\end{aligned} \tag{2.40}$$

Similarly, there exists $C > 0$ such that, for every $x \in [0, L]$ and all integer j with $|j| \geq J$,

$$|\theta_j'(x)| \leq \frac{C\sqrt{3\sigma_j^2 - 1}}{j^2} e^{-\sqrt{3\sigma_j^2-1}x}. \tag{2.41}$$

From (2.40) and (2.41), one gets the existence of $C > 0$ such that, for all integer j with $|j| \geq J$,

$$|\theta_j(L)| \leq Ce^{-\sqrt{3\sigma_j^2-1}L}, \quad |\theta_j'(L)| \leq C\sqrt{3\sigma_j^2 - 1}e^{-\sqrt{3\sigma_j^2-1}L}. \tag{2.42}$$

Similar to the argument for the analysis of μ_j and φ_j, we can get that, if $3\sigma_j^2 \le 1$,

$$\check{\psi}_j(x) = e^{-i\kappa_j^{(1)}\sigma_j x}\Big[\beta_j^{(1)}\cos\big(\sqrt{1-3\sigma_j^2}x\big) + \beta_j^{(2)}\sin\big(\sqrt{1-3\sigma_j^2}x\big)\Big] + \beta_j^{(3)}e^{2i\kappa_j^{(2)}\sigma_j x} + \big(\beta_j^{(1)} - \beta_j^{(2)}\big)\theta_j(x),$$

and, if $3\sigma_j^2 > 1$,

$$\check{\psi}_j(x) = e^{-i\kappa_j^{(1)}\sigma_j x}\Big[\beta_j^{(1)}\cosh\big(\sqrt{3\sigma_j^2-1}x\big) + \beta_j^{(2)}\sinh\big(\sqrt{3\sigma_j^2-1}x\big)\Big] + \beta_j^{(3)}e^{2i\kappa_j^{(2)}\sigma_j x} + \big(\beta_j^{(1)} - \beta_j^{(2)}\big)\theta_j(x).$$

From now on, we assume that $|j|$ is large enough so that $3\sigma_j^2 > 1$. From the boundary conditions satisfied by $\check{\psi}_j$ -see (2.30)-, we get

$$\begin{cases} \beta_j^{(1)} + \beta_j^{(3)} = 0, \\ e^{-i\kappa_j^{(1)}\sigma_j L}\Big[\beta_j^{(1)}\cosh\big(\sqrt{3\sigma_j^2-1}L\big) + \beta_j^{(2)}\sinh\big(\sqrt{3\sigma_j^2-1}L\big)\Big] + \beta_j^{(3)}e^{2i\kappa_j^{(2)}\sigma_j L} \\ +\big(\beta_j^{(1)}-\beta_j^{(2)}\big)\theta_j(L) = 0, \\ \quad -i\kappa_j\sigma_j\beta_j^{(1)} + 2i\kappa_j\sigma_j\beta_j^{(3)} + \beta_j^{(2)}\sqrt{3\sigma_j^2-1} + 1 \\ = -i\kappa_j\sigma_j e^{-i\kappa_j^{(1)}\sigma_j L}\Big[\beta_j^{(1)}\cosh\big(\sqrt{3\sigma_j^2-1}L\big) + \beta_j^{(2)}\sinh\big(\sqrt{3\sigma_j^2-1}L\big)\Big] + 2i\sigma_j\beta_j^{(3)}e^{2i\kappa_j^{(2)}\sigma_j L} \\ \quad + e^{-i\kappa_j^{(1)}\sigma_j L}\sqrt{3\sigma_j^2-1}\Big[\beta_j^{(1)}\sinh\big(\sqrt{3\sigma_j^2-1}L\big) + \beta_j^{(2)}\cosh\big(\sqrt{3\sigma_j^2-1}L\big)\Big] + \big(\beta_j^{(1)}-\beta_j^{(2)}\big)\theta_j'(L). \end{cases} \tag{2.43}$$

Then, we know

$$\check{\psi}_j(x) = \beta_j\Bigg\{e^{-i\kappa_j^{(1)}\sigma_j x}\Bigg[\cosh\big(\sqrt{3\sigma_j^2-1}x\big) + \frac{e^{i(\kappa_j^{(1)}+2\kappa_j^{(2)})\sigma_j L} - \cosh\big(\sqrt{3\sigma_j^2-1}L\big) - e^{i\kappa_j^{(1)}\sigma_j L}\theta_j(L)}{\sinh\big(\sqrt{3\sigma_j^2-1}L\big) - e^{i\kappa_j^{(1)}\sigma_j L}\theta_j(L)} \times\Big(\sinh\big(\sqrt{3\sigma_j^2-1}x\big) - e^{i\kappa_j\sigma_j x}\theta_j(x)\Big)\Bigg] - e^{2i\kappa_j^{(2)}\sigma_j x}\Bigg\}, \tag{2.44}$$

where β_j is a suitable chosen complex number such that the third equality in (2.43) holds.

From (2.44), we have, as $|j| \to +\infty$,

$$
\begin{aligned}
\check{\psi}_{j,x}(0) =& \beta_j\Big[- i\kappa_j^{(1)}\sigma_j - 2i\kappa_j^{(2)}\sigma_j + \frac{e^{i(\kappa_j^{(1)}+2\kappa_j^{(2)})\sigma_j L} - \cosh\big(\sqrt{3\sigma_j^2-1}L\big) - e^{i\kappa_j^{(1)}\sigma_j L}\theta_j(L)}{\sinh\big(\sqrt{3\sigma_j^2-1}L\big) - e^{i\kappa_j^{(1)}\sigma_j L}\theta_j(L)} \\
&\times\Big(2\sqrt{3\hat{\sigma}_j^2-1} - \sqrt{3\sigma_j^2-1}\Big)\Big] \\
=& \beta_j\Big(- i\kappa_j^{(1)}\sigma_j - 2i\kappa_j^{(2)}\sigma_j - 2\sqrt{3\hat{\sigma}_j^2-1} + \sqrt{3\sigma_j^2-1}\Big) + \gamma_j O\Big(e^{-\sqrt{3\sigma_j^2-1}L}\Big)
\end{aligned} \tag{2.45}
$$

and

$$
\begin{aligned}
\check{\psi}_{j,x}(L) =& \beta_j\Big\{ -i\kappa_j^{(1)}\sigma_j e^{2i\kappa_j^{(2)}\sigma_j L} - 2i\kappa_j^{(2)}\sigma_j e^{2i\kappa_j^{(2)}\sigma_j L} + e^{-i\kappa_j^{(1)}\sigma_j L}\Big[\sqrt{3\sigma_j^2-1}\sinh\big(\sqrt{3\sigma_j^2-1}L\big) \\
&+ \frac{e^{i(\kappa_j^{(1)}+2\kappa_j^{(2)})\sigma_j L} - \cosh\big(\sqrt{3\sigma_j^2-1}L\big) - e^{i\kappa_j^{(1)}\sigma_j L}\theta_j(L)}{\sinh\big(\sqrt{3\sigma_j^2-1}L\big) + e^{i\kappa_j^{(1)}\sigma_j L}\theta_j(L)}\Big(\sqrt{3\sigma_j^2-1}\cosh\big(\sqrt{3\sigma_j^2-1}L\big) \\
&-i\kappa_j^{(1)}\sigma_j e^{i\kappa_j^{(1)}\sigma_j L}\theta_j(L) - ie^{i\kappa_j^{(1)}\sigma_j L}\theta_j'(L)\Big)\Big]\Big\} \\
=& \beta_j\Big(- i\kappa_j^{(1)}\sigma_j - 2i\kappa_j^{(2)}\sigma_j + \sqrt{3\sigma_j^2-1}\Big)e^{2i\kappa_j^{(2)}\sigma_j L} + \beta_j O\big(e^{-\sqrt{3\sigma_j^2-1}L}\big).
\end{aligned} \tag{2.46}
$$

Since $\varphi_{j,y}(0) = \varphi_{j,y}(L)$, from (2.26) and (2.27), we get

$$
- 3i\tau_j - \sqrt{3\tau_j^2-1} + O\big(e^{-\sqrt{3\tau_j^2-1}L}\big) = \Big(- 3i\tau_j + \sqrt{3\tau_j^2-1}\Big)e^{2i\tau_j L}. \tag{2.47}
$$

This, together with the choice of σ_j, $\kappa_j^{(1)}$ and $\kappa_j^{(2)}$, implies that

$$
\begin{aligned}
&-i\kappa_j^{(1)}\sigma_j - 2i\kappa_j^{(2)}\sigma_j - 2\sqrt{3\sigma_j^2-1} + \sqrt{3\hat{\sigma}_j^2-1} \\
&= \Big(- i\kappa_j^{(1)}\sigma_j - 2i\kappa_j^{(2)}\sigma_j + \sqrt{3\sigma_j^2-1}\Big)e^{2i\kappa_j^{(2)}\sigma_j L} + O\Big(\frac{1}{j^2}\Big), \quad \text{as } |j| \to +\infty.
\end{aligned} \tag{2.48}
$$

Thus, we see

$$
\beta_j = O(j^2) \ \text{ as } |j| \to +\infty. \tag{2.49}
$$

Let, for $j \in \mathbb{Z} \setminus \{0\}$,

$$
\hat{\psi}_j(x) \triangleq \begin{cases} \dfrac{\alpha_{-j}}{\beta_j}\check{\psi}_j(x), & x \in [0, L] \quad \text{if } 3\sigma_j^2 > 1, \\ \check{\psi}_j(x), & x \in [0, L] \quad \text{if } 3\sigma_j^2 \leq 1. \end{cases} \tag{2.50}
$$

Now, we are going to prove that $\{\hat{\psi}_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ is a Riesz basis of $L^2(0, L)$.

First, we analyze the asymptotic behavior of $\hat{\psi}_j(\cdot)$ as $|j| \to +\infty$. It is clear that

$$\begin{aligned}\hat{\psi}_j(x) = \alpha_{-j}\Bigg\{ & \frac{e^{i(\kappa_j^{(1)}+2\kappa_j^{(2)})\sigma_j L}\sinh\big(\sqrt{3\sigma_j^2-1}x\big)}{\sinh\big(\sqrt{3\sigma_j^2-1}L\big)-e^{i\kappa_j^{(1)}\sigma_j L}\theta_j(L)}e^{-i\kappa_j^{(1)}\sigma_j x} + \frac{\cosh\big(\sqrt{3\sigma_j^2-1}L\big)\theta_j(x)}{\sinh\big(\sqrt{3\sigma_j^2-1}L\big)-e^{i\kappa_j^{(1)}\sigma_j L}\theta_j(L)} \\ & + \frac{\cosh\big(\sqrt{3\sigma_j^2-1}x\big)\sinh\big(\sqrt{3\sigma_j^2-1}L\big)-\sinh\big(\sqrt{3\sigma_j^2-1}x\big)\cosh\big(\sqrt{3\sigma_j^2-1}L\big)}{\sinh\big(\sqrt{3\sigma_j^2-1}L\big)-e^{i\kappa_j^{(1)}\sigma_j L}\theta_j(L)}e^{-i\kappa_j^{(1)}\sigma_j x} \\ & - \frac{\cosh\big(\sqrt{3\sigma_j^2-1}x\big)e^{i\kappa_j^{(1)}\sigma_j(L-x)}\theta_j(L)+e^{i(\kappa_j^{(1)}+2\kappa_j^{(2)})\sigma_j L}\theta_j(x)}{\sinh\big(\sqrt{3\sigma_j^2-1}L\big)-e^{i\kappa_j^{(1)}\sigma_j L}\theta_j(L)} \\ & - \frac{e^{ik_j^{(1)}\sigma_j(L-y)}\theta_j(L)\big(\sinh\big(\sqrt{3\sigma_j^2-1}x\big)-e^{i\kappa_j^{(1)}\sigma_j y}\theta_j(x)\big)}{\sinh\big(\sqrt{3\sigma_j^2-1}L\big)-e^{i\kappa_j^{(1)}\sigma_j L}\theta_j(L)} - e^{2i\kappa_j^{(2)}\sigma_j x}\Bigg\}.\end{aligned} \tag{2.51}$$

Then, using also (2.22), we get that

$$\begin{aligned}&\left(\int_0^L\left|\hat{\psi}_j(x)-\alpha_{-j}\left[\frac{2e^{i(\kappa_j^{(1)}+2\kappa_j^{(2)})\sigma_j L}\sinh\big(\sqrt{3\sigma_j^2-1}x\big)+e^{\sqrt{3\sigma_j^2-1}(L-x)}}{2\sinh\big(\sqrt{3\sigma_j^2-1}L\big)-2e^{i\kappa_j^{(1)}\sigma_j L}\theta_j(L)}e^{-i\kappa_j^{(1)}\sigma_j x}-e^{2i\kappa_j^{(2)}\sigma_j x}\right]\right|^2 dx\right)^{\frac{1}{2}} \\ &= O\Big(\frac{1}{j^3}\Big) \qquad \text{as } |j| \to +\infty.\end{aligned} \tag{2.52}$$

Now, we prove that

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} \big|\hat{\psi}_{-j}-\varphi_j\big|^2_{L^2(0,L)} < \infty. \tag{2.53}$$

Utilizing (2.42) and straightforward computations, one can show that

$$\begin{aligned}&\Bigg|\alpha_j\left[\frac{2e^{3i\tau_j L}\sinh\big(\sqrt{3\tau_j^2-1}x\big)+e^{\sqrt{3\tau_j^2-1}(L-x)}}{2\sinh\big(\sqrt{3\tau_j^2-1}L\big)}e^{-i\tau_j x}-e^{2i\tau_j x}\right] \\ &-\alpha_j\left[\frac{2e^{i(\kappa_j^{(1)}+2\kappa_j^{(2)})\sigma_{-j}L}\sinh\big(\sqrt{3\sigma_{-j}^2-1}x\big)+e^{\sqrt{3\sigma_{-j}^2-1}(L-x)}}{2\sinh\big(\sqrt{3\sigma_{-j}^2-1}L\big)-2e^{i\kappa_j^{(1)}\sigma_{-j}L}\theta_j(L)}e^{-i\kappa_j^{(1)}\sigma_{-j}x}-e^{2i\kappa_j^{(2)}\sigma_{-j}x}\right]\Bigg| \\ &\leq C\alpha_j\Big(\big|1-e^{(\sqrt{3\tau_j^2-1}-\sqrt{3\sigma_{-j}^2-1})x}\big|+\big|1-e^{(\kappa_j^{(1)}\sigma_{-j}-\tau_j)x}\big|+\big|1-e^{(\kappa_j^{(2)}\sigma_{-j}-\tau_j)x}\big|+e^{-\sqrt{3\sigma_{-j}^2-1}L}\Big).\end{aligned} \tag{2.54}$$

Hence, using (2.21), (2.34) and (2.52), we see that there exists $C > 0$ such that for every

$j \in \mathbb{Z} \setminus \{0\}$,

$$\begin{aligned}
&\int_0^L |\hat{\psi}_{-j} - \varphi_j|^2 dx \\
&\leq C\Big(\Big| \sqrt{3\tau_j^2 - 1} - \sqrt{3\sigma_{-j}^2 - 1} \Big|^2 + \big|\tau_j - \kappa_j^{(1)}\sigma_{-j}\big|^2 + \big|\tau_j - \kappa_j^{(2)}\sigma_{-j}\big|^2 + e^{-\sqrt{3\sigma_{-j}^2 - 1}L} + \frac{1}{j^6} \Big) \\
&\leq C\frac{1}{j^4}.
\end{aligned} \tag{2.55}$$

Since

$$\sum_{j=1}^{+\infty} \frac{1}{j^4} < +\infty,$$

we have (2.53).

Let $T > 0$. Consider the following control system:

$$\begin{cases} \vartheta_t + \vartheta_{xxx} + \vartheta_x = 0 & \text{in } (0,T) \times (0,L), \\ \vartheta(t,0) = \vartheta(t,L) = 0 & \text{in } (0,T), \\ \vartheta_x(t,L) - \vartheta_x(t,0) = \eta(t) & \text{in } (0,T), \end{cases} \tag{2.56}$$

where $\eta(\cdot) \in L^2(0,T)$ is the control. Let $\tilde{\vartheta} = A^{-1}\vartheta$ (recall (2.7) for the definition of A). Then, we know $\tilde{\vartheta}$ solves

$$\partial_t \tilde{\vartheta} = A\tilde{\vartheta} - \eta(t)b, \tag{2.57}$$

where $b(\cdot)$ is the solution to

$$\begin{cases} b_{xxx} + b_x = 0 & \text{in } (0,L), \\ b(0) = b(L) = 0, \ b_x(L) - b_x(0) = 1. \end{cases} \tag{2.58}$$

Clearly, $b \in L^2(0,L)$. Let $b = \sum_{j \in \mathbb{Z}\setminus\{0\}} b_j \varphi_j$. Since the system (2.56) is exactly controllable in $L^2(0,L)$ (see [21, Theorem 1.2]), we get that (2.56) is also exactly controllable in $A^{-1}\big(L^2(0,L)\big)$. In particular, $b_j \neq 0$ for every $j \in \mathbb{Z} \setminus \{0\}$.

For $j \in \mathbb{Z} \setminus \{0\}$, let $\tilde{\psi}_j \in D(A)$ be the solution of

$$A\tilde{\psi}_j + (-\lambda + i\mu_j)\tilde{\psi}_j = -\frac{\alpha_{-j}}{\beta_j}(-\lambda + i\mu_j)b. \tag{2.59}$$

Then, for every $j \in \mathbb{Z} \setminus \{0\}$, we have

$$A^{-1}\tilde{\psi}_j = -(-\lambda + i\mu_j)^{-1}\tilde{\psi}_j - \frac{\alpha_{-j}}{\beta_j}A^{-1}b \tag{2.60}$$

and

$$\hat{\psi}_j = \tilde{\psi}_j + \frac{\alpha_{-j}}{\beta_j}b. \tag{2.61}$$

Assume that there exists $\{a_j\}_{j\in\mathbb{Z}\setminus\{0\}} \in \ell^2(\mathbb{Z})$ such that

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j\hat{\psi}_j = 0. \tag{2.62}$$

From (2.61) and (2.62), we obtain that

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j\Big(\tilde{\psi}_j + \frac{\alpha_{-j}}{\beta_j}b\Big) = 0. \tag{2.63}$$

Applying A^{-1} to (2.63), and using (2.60), one gets that

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j(-\lambda+i\mu_j)^{-1}\tilde{\psi}_j = 0, \tag{2.64}$$

which, together with (2.60), implies that

$$\Big[\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j\frac{\alpha_{-j}}{\beta_j}(-\lambda+i\mu_j)^{-1}\Big]b - \sum_{j\in\mathbb{Z}\setminus\{0\}} a_j(-\lambda+i\mu_j)^{-1}\hat{\psi}_j = 0. \tag{2.65}$$

Applying A^{-1} to (2.64) and using (2.60) again, we get

$$\Big[\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j\frac{\alpha_{-j}}{\beta_j}(-\lambda+i\mu_j)^{-1}\Big]A^{-1}b + \sum_{j\in\mathbb{Z}\setminus\{0\}} a_j(-\lambda+i\mu_j)^{-2}\tilde{\psi}_j = 0, \tag{2.66}$$

which indicates that

$$\Big[\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j\frac{\alpha_{-j}}{\beta_j}(-\lambda+i\mu_j)^{-1}\Big]A^{-1}b - \Big[\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j\frac{\alpha_{-j}}{\beta_j}(-\lambda+i\mu_j)^{-2}\Big]b + \sum_{j\in\mathbb{Z}\setminus\{0\}} a_j(-\lambda+i\mu_j)^{-2}\hat{\psi}_j = 0. \tag{2.67}$$

By induction, one gets that, for every positive integer p,

$$\begin{aligned}&\Big(\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j\frac{\alpha_{-j}}{\beta_j}(-\lambda+i\mu_j)^{-1}\Big)A^{-p}b + \sum_{k=2}^{p}\Big(\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j\frac{\alpha_{-j}}{\beta_j}(-\lambda+i\mu_j)^{-k}(-1)^{p+1-k}\Big)A^{p+1-k}b\\ &- \sum_{j\in\mathbb{Z}\setminus\{0\}} a_i\frac{\alpha_{-j}}{\beta_j}(-\lambda+i\mu_j)^{-p-1}(-1)^{p+1}b + \sum_{j\in\mathbb{Z}\setminus\{0\}} a_i(-\lambda+i\mu_j)^{-p-1}(-1)^{p+1}\hat{\psi}_j = 0.\end{aligned} \tag{2.68}$$

If

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j\frac{\alpha_{-j}}{\beta_j}(-\lambda+i\mu_j)^{-1} \neq 0, \tag{2.69}$$

we get from (2.68) that

$$\{A^{-p}b\}_{p\in(\{0\}\cup\mathbb{Z}^+)} \subset \operatorname{span}\{\hat{\psi}_j\}_{j\in\mathbb{Z}\setminus\{0\}}. \tag{2.70}$$

If $\overline{\text{span}\,\{\hat{\psi}_j\}_{j\in\mathbb{Z}\setminus\{0\}}} \neq L^2(0,L)$, then we can find a nonzero function

$$d = \sum_{j\in\mathbb{Z}\setminus\{0\}} d_j\varphi_j \in L^2(0,L) \tag{2.71}$$

such that

$$(h,d)_{L^2(0,L)} = 0, \quad \text{for every } h \in \text{span}\,\{\hat{\psi}_j\}_{j\in\mathbb{Z}\setminus\{0\}}. \tag{2.72}$$

From (2.70) and (2.72), we obtain that $(A^{-p}b,d)_{L^2(0,L)} = 0$ for every $p \in \{0\}\cup\mathbb{Z}^+$. Therefore, we get

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} b_j(i\mu_j)^{-p}d_j = 0 \text{ for all } p \in \{0\}\cup\mathbb{Z}^+. \tag{2.73}$$

Let us define a complex variable function $G(\cdot):\mathbb{C}\to\mathbb{C}$ as

$$G(z) = \sum_{j\in\mathbb{Z}\setminus\{0\}} d_jb_je^{(i\mu_j)^{-1}z}, \quad z \in \mathbb{C}. \tag{2.74}$$

Then, it is clear that $G(\cdot)$ is a holomorphic function. From (2.73), we see that

$$G^{(p)}(0) = 0 \text{ for every } p \in \{0\}\cup\mathbb{Z}^+.$$

Thus, we find that $G(\cdot) = 0$, which educes that $d_jb_j = 0$ for all integer j. Since $b_j \neq 0$ for every $j \in \mathbb{Z}\setminus\{0\}$, we know that $d_j = 0$ for all integer j. Therefore, we get $d = 0$, which leads to a contradiction with that $d \neq 0$. Hence, (2.69) implies that

$$\overline{\text{span}\,\{\hat{\psi}_j\}_{j\in\mathbb{Z}\setminus\{0\}}} = L^2(0,L). \tag{2.75}$$

If

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j\frac{\alpha_{-j}}{\beta_j}(-\lambda+i\mu_j)^{-1} = 0$$

and

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j\frac{\alpha_{-j}}{\beta_j}(-\lambda+i\mu_j)^{-2} \neq 0,$$

then by using (2.68) again, we obtain that

$$\{A^{-p}b\}_{p\in\{0\}\cup\mathbb{Z}^+} \subset \text{span}\,\{\hat{\psi}_j\}_{j\in\mathbb{Z}\setminus\{0\}}.$$

Then, by a similar argument, we find that (2.75) again holds. Similarly, we can get that, if there is a $p \in \mathbb{Z}^+$ such that

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j\frac{\alpha_{-j}}{\beta_j}(-\lambda+i\mu_j)^{-p} \neq 0, \tag{2.76}$$

then (2.75) holds.

On the other hand, if

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j \frac{\alpha_{-j}}{\beta_j}(-\lambda+i\mu_j)^{-p}=0 \text{ for every } p\in\mathbb{Z}^+, \tag{2.77}$$

we define a function

$$\widetilde{G}(z)\triangleq\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j \frac{\alpha_{-j}}{\beta_j}(-\lambda+i\mu_j)^{-1}e^{(-\lambda+i\mu_j)^{-1}z} \text{ for every } z\in\mathbb{C},$$

and it is clear that $\widetilde{G}(\cdot)$ is a holomorphic function and

$$\widetilde{G}^{(p)}(0)=0 \text{ for every } p\in\{0\}\cup\mathbb{Z}^+,$$

which implies that $\widetilde{G}(\cdot)=0$. Therefore, we conclude that $a_j=0$ for every $j\in\mathbb{Z}\setminus\{0\}$.

By the above argument, we know that either $\{\hat{\psi}_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ is ω-independent or is complete in $L^2(0,L)$. Before proceeding our proof, we recall the following two known results.

Lemma 2.2 *[30, Page 45, Theorem 15] Let H be a separable Hilbert space and let $\{e_j\}_{j\in\mathbb{Z}^+}$ be an orthonormal basis for H. If $\{f_j\}_{j\in\mathbb{Z}^+}$ is an ω-independent sequence such that*

$$\sum_{j\in\mathbb{Z}^+}|f_j-e_j|_H^2<+\infty,$$

then $\{f_j\}_{j\in\mathbb{Z}^+}$ is a Riesz basis for H.

Lemma 2.3 *[30, Page 40, Theorem 12] Let $\{e_j\}_{j\in\mathbb{Z}^+}$ be a basis of a Banach space X and let $\{f_j\}_{j\in\mathbb{Z}^+}$ be the associated sequence of coefficient functionals. If $\{b_j\}_{j\in\mathbb{Z}^+}$ is complete in X and if*

$$\sum_{j\in\mathbb{Z}^+}|e_j-b_j|_X|f_j|_{X'}<+\infty,$$

then $\{b_j\}_{j\in\mathbb{Z}^+}$ is a basis for X which is equivalent to $\{e_j\}_{j\in\mathbb{Z}^+}$.

Now we continue the proof. We first deal with the case where $\{\hat{\psi}_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ is ω-independent. Let us take $H=L^2(0,L)$ and put

$$\begin{cases} e_{2j-1}=\varphi_j \text{ for } j\in\mathbb{Z}^+,\\ e_{2j}=\varphi_{-j} \text{ for } j\in\mathbb{Z}^+,\end{cases} \qquad \begin{cases} f_{2j-1}=\hat{\psi}_{-j} \text{ for } j\in\mathbb{Z}^+,\\ f_{2j}=\hat{\psi}_{j} \text{ for } j\in\mathbb{Z}^+.\end{cases}$$

Then, by (2.55), the conditions of Lemma 2.2 are fulfilled. Thus, we get that $\{\hat{\psi}_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ is a Riesz basis of $L^2(0,L)$.

Next, we consider the case that $\{\hat{\psi}_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ is complete in $L^2(0,L)$. Let us set $X=L^2(0,L)$,

$$\begin{cases} e_{2j-1}=\varphi_j \text{ for } j\in\mathbb{Z}^+,\\ e_{2j}=\varphi_{-j} \text{ for } j\in\mathbb{Z}^+,\end{cases} \qquad \begin{cases} f_{2j-1}=\varphi_j \text{ for } j\in\mathbb{Z}^+,\\ f_{2j}=\varphi_{-j} \text{ for } j\in\mathbb{Z}^+,\end{cases} \qquad \begin{cases} b_{2j-1}=\hat{\psi}_{-j} \text{ for } j\in\mathbb{Z}^+,\\ b_{2j}=\hat{\psi}_{j} \text{ for } j\in\mathbb{Z}^+.\end{cases}$$

Then, using (2.55) once more, it is easy to see that the conditions of Lemma 2.3 are fulfilled. Therefore, we get that $\{\hat{\psi}_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ is a Riesz basis of $L^2(0,L)$.

Now we give an estimate of $\{c_j\}_{j\in\mathbb{Z}\setminus\{0\}}$. From (2.17) and (2.24), we get that

$$\frac{1}{C}j^2 \leq \int_0^L |\varphi_{j,y}(y)|^2 dy \leq Cj^2.$$

This, together with the fact that $\{\hat{\psi}_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ is a Riesz basis of $L^2(0,L)$ and $k_3(\cdot,\cdot) \in H_0^1((0,L)\times(0,L))$, implies that

$$\begin{aligned}
+\infty > \int_0^L\int_0^L |k_{3,y}(x,y)|^2 dxdy &\geq \int_0^L\int_0^L \Big|\sum_{j\in\mathbb{Z}\setminus\{0\}} c_j\frac{\beta_j}{\alpha_{-j}}\hat{\psi}_j(x)\varphi_{j,y}(y)\Big|^2 dxdy \\
&\geq C\int_0^L \sum_{j\in\mathbb{Z}\setminus\{0\}} \Big|c_j\frac{\beta_j}{\alpha_{-j}}\varphi_{j,y}(y)\Big|^2 dy \geq C\sum_{j\in\mathbb{Z}\setminus\{0\}}\Big|c_j\frac{\beta_j}{\alpha_{-j}}j\Big|^2.
\end{aligned}$$

Hence, we find that

$$\left\{c_j\frac{\beta_j}{\alpha_{-j}}j\right\}_{j\in\mathbb{Z}\setminus\{0\}} \in \ell^2(\mathbb{Z}\setminus\{0\}). \tag{2.78}$$

From (2.11), (2.17), (2.22) and (2.26), we get the existence of $C>0$ such that

$$C^{-1}|j| \leq |\varphi_{j,y}(0)| \leq C|j|, \quad \forall j\in\mathbb{Z}\setminus\{0\}. \tag{2.79}$$

From (2.78) and (2.79), we find that

$$\left\{c_j\frac{\beta_j}{\alpha_{-j}}\varphi_{j,y}(0)\right\}_{j\in\mathbb{Z}\setminus\{0\}} \in \ell^2(\mathbb{Z}\setminus\{0\}). \tag{2.80}$$

Using (2.28), $k_{3,y}(x,0)=0$ and the second inequality of (2.79), we find that

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} \psi_j(\cdot)\varphi_{j,y}(0) = \sum_{j\in\mathbb{Z}\setminus\{0\}} c_j\frac{\beta_j}{\alpha_{-j}}\varphi_{j,y}(0)\hat{\psi}_j(\cdot) = 0 \quad \text{ in } L^2(0,L).$$

Using that $\{\hat{\psi}_j\}_{j\in\mathbb{Z}}$ is a Riesz basis of $L^2(0,L)$, we get

$$c_j\frac{\beta_j}{\alpha_{-j}}\varphi_{j,y}(0) = 0 \text{ for every } j\in\mathbb{Z}\setminus\{0\}.$$

Then, we see $\varphi_{j,y}(0)=0$ if $c_j\neq 0$. However, by [21, Lemma 3.5], this is impossible since $L\notin\mathcal{N}$. Thus, we get that $c_j=0$ for every $j\in\mathbb{Z}\setminus\{0\}$, which implies that the equation (2.6) admits a unique solution $k_3=0$. Therefore, we obtain that the equation (1.9) admits at most one solution. This concludes Step 1.

Step 2: proof of the existence of a solution to (1.9) with the required regularity

Denote by $D(A)'$ the dual space of $D(A)$ with respect to the pivot space $L^2(0,L)$. Let $h(\cdot)=\sum_{j\in\mathbb{Z}\setminus\{0\}} h_j\varphi_j(\cdot)\in D(A)$, i.e., $|h(\cdot)|^2_{D(A)}=\sum_{j\in\mathbb{Z}\setminus\{0\}}|h_j\mu_j|^2<+\infty$, we have that

$$\sum_{j\in\mathbb{Z}\setminus\{0\}}|h_j\varphi_j'(0)|\le\Big(\sum_{j\in\mathbb{Z}\setminus\{0\}}|h_j\mu_j|^2\Big)^{\frac{1}{2}}\Big(\sum_{j\in\mathbb{Z}\setminus\{0\}}|\varphi_j'(0)\mu_j^{-1}|^2\Big)^{\frac{1}{2}}<+\infty.$$

Hence

$$\sum_{j=-l_1}^{l_2}\varphi_j'(0)\overline{\varphi_j(\cdot)} \text{ is converging in } D(A)' \text{ as } l_1 \text{ and } l_2 \text{ tend to } +\infty, \tag{2.81}$$

which allows us to define

$$\hat{b}(\cdot)\triangleq\sum_{j\in\mathbb{Z}\setminus\{0\}}\varphi_j'(0)\overline{\varphi_j(\cdot)}\in D(A)'. \tag{2.82}$$

Further, it is clear that

$$(h,\hat{b})_{D(A),D(A)'}=\sum_{j\in\mathbb{Z}\setminus\{0\}}h_j\varphi_j'(0)=h'(0)=-\delta_0'(h),$$

which implies that

$$\hat{b}=-\delta_0' \quad \text{in } D(A)'. \tag{2.83}$$

Let

$$a_j\triangleq\frac{\lambda}{\varphi_j'(0)},\quad \phi_j\triangleq\bar{\varphi}_j+(-A-i\mu_j+\lambda)^{-1}(a_j\hat{b}-\lambda\bar{\varphi}_j)\ \text{ for } j\in\mathbb{Z}\setminus\{0\}. \tag{2.84}$$

From (2.18), (2.79) and (2.84), we have that

$$\begin{aligned}
&\int_0^L\Big|\overline{\varphi_j(x)}-\phi_j(x)\Big|^2dx\\
&=\int_0^L\big|(-A-i\mu_j+\lambda)^{-1}(a_j\hat{b}-\lambda\bar{\varphi}_j)\big|^2dx\\
&=\int_0^L\Big|\sum_{k\in\mathbb{Z}\setminus\{0,j\}}(i\mu_k-i\mu_j+\lambda)^{-1}\frac{\lambda\varphi_k'(0)}{\varphi_j'(0)}\bar{\varphi}_k(x)\Big|^2dx\\
&=\sum_{k\in\mathbb{Z}\setminus\{0,j\}}\Big|(i\mu_k-i\mu_j+\lambda)^{-1}\frac{\lambda\varphi_k'(0)}{\varphi_j'(0)}\Big|^2\\
&\le C\lambda^2\sum_{k\in\mathbb{Z}\setminus\{0,j\}}\big|k^3-j^3\big|^{-2}j^{-2}k^2\quad \text{ for all } j\in\mathbb{Z}\setminus\{0\}.
\end{aligned}\tag{2.85}$$

From (2.85), we obtain, for $j>0$,

$$\begin{aligned}
&\int_0^L\Big|\overline{\varphi_j(x)}-\phi_j(x)\Big|^2dx\\
&\le C\Big(\sum_{k>2j}\frac{k^2}{\big|k^3-j^3\big|^2j^2}+\sum_{j<k\le 2j}\frac{k^2}{\big|k^3-j^3\big|^2j^2}+\sum_{-j<k<j,\,k\neq 0}\frac{k^2}{\big|k^3-j^3\big|^2j^2}\\
&\quad+\sum_{k\le -j}\frac{k^2}{\big|k^3-j^3\big|^2j^2}\Big).
\end{aligned}\tag{2.86}$$

Now we estimate the terms in the right hand side of (2.86). First, we have that

$$\sum_{k>2j} \frac{k^2}{\left|k^3-j^3\right|^2 j^2} \leq \sum_{k>2j} \frac{8}{7k^4 j^2} \leq \frac{8}{7j^4} \sum_{k>2j} \frac{1}{k^2} \leq \frac{1}{j^5}. \tag{2.87}$$

Next,

$$\begin{aligned}
&\sum_{j<k\leq 2j} \frac{k^2}{\left|k^3-j^3\right|^2 j^2} \leq 4 \sum_{j<k\leq 2j} \frac{1}{\left|k^3-j^3\right|^2} = 4 \sum_{1\leq l\leq j} \frac{1}{\left|(j+l)^3-j^3\right|^2} \\
&\leq 4 \sum_{1\leq l\leq j} \frac{1}{\left|3j^2 l+3jl^2+l^3\right|^2} \leq \frac{4}{9} \sum_{-j\leq l\leq -1} \frac{1}{j^2} \frac{1}{(j+l)^2 l^2} = \frac{4}{9} \sum_{1\leq l\leq j} \frac{1}{j^2} \Big[\frac{1}{j}\Big(\frac{1}{l}-\frac{1}{j+l}\Big)\Big]^2 \\
&\leq \frac{8}{9} \sum_{1\leq l\leq j} \frac{1}{j^4}\Big(\frac{1}{l^2}+\frac{1}{(j-l)^2}\Big) \leq \frac{8\pi^2}{27}\frac{1}{j^4}.
\end{aligned} \tag{2.88}$$

Similarly, we can get that

$$\sum_{-j<k<j,\, k\neq 0} \frac{k^2}{\left|k^3-j^3\right|^2 j^2} \leq \frac{8\pi^2}{27}\frac{1}{j^4} \tag{2.89}$$

and

$$\sum_{k\leq -j} \frac{k^2}{\left|k^3+j^3\right|^2 j^2} \leq \frac{8}{j^5}. \tag{2.90}$$

From (2.86) to (2.90), we know that there is a constant $C>0$ such that for all positive integer j, it holds that

$$\int_0^L \left|\overline{\varphi_j(x)}-\phi_j(x)\right|^2 dx \leq \frac{C}{j^4}. \tag{2.91}$$

Similarly, we can prove that there is a constant $C>0$ such that for all negative integer j, it holds that

$$\int_0^L \left|\overline{\varphi_j(x)}-\phi_j(x)\right|^2 dx \leq \frac{C}{j^4}. \tag{2.92}$$

Further, by similar arguments, we can obtain that

$$\int_0^L \left|\overline{\varphi_{j,x}(x)}-\phi_{j,x}(x)\right|^2 dx \leq \frac{C}{j^2} \quad \text{for all } j\in\mathbb{Z}\setminus\{0\}. \tag{2.93}$$

With the same strategy to prove that $\{\hat{\psi}_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ is a Riesz basis of $L^2(0,L)$, we also can show that

$$\Big\{-(\lambda-i\mu_j)A^{-1}\phi_j\Big\}_{j\in\mathbb{Z}\setminus\{0\}} \text{ is a Riesz basis of } L^2(0,L). \tag{2.94}$$

From (2.18) and (2.79), we get that

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} \varphi_j'(0)(i\mu_j)^{-1}\bar{\varphi}_j \text{ is converging in } L^2(0,L).$$

From (2.94), there is $\{\hat{c}_j\}_{j\in\mathbb{Z}\setminus\{0\}} \in l^2(\mathbb{Z}\setminus\{0\})$ such that

$$-\sum_{j\in\mathbb{Z}\setminus\{0\}} \hat{c}_j(\lambda - i\mu_j)A^{-1}\phi_j = \sum_{j\in\mathbb{Z}\setminus\{0\}} \varphi_{j,y}(0)(i\mu_j)^{-1}\bar{\varphi}_j \quad \text{in } \ L^2(0,L),$$

which, together with (2.82) and (2.83), implies that

$$-\sum_{j\in\mathbb{Z}\setminus\{0\}} \hat{c}_j(\lambda - i\mu_j)\phi_j = \sum_{j\in\mathbb{Z}\setminus\{0\}} \varphi_{j,y}(0)\bar{\varphi}_j = -\delta_0' \quad \text{in } \ D(A)'. \tag{2.95}$$

Since $\varphi_j \in D(A)$, from (2.95), we find that

$$-\hat{c}_j(\lambda - i\mu_j)\int_0^L \varphi_j(x)\phi_j(x)dx - \sum_{k\in\mathbb{Z}\setminus\{0,j\}} \hat{c}_k(\lambda - i\mu_k)\int_0^L \varphi_j(x)\phi_k(x)dx = \varphi_{j,y}(0). \tag{2.96}$$

For $j \neq k$, we get, using (2.82) and (2.84),

$$\begin{aligned}
&\hat{c}_k(\lambda - i\mu_k)\int_0^L \varphi_j(x)\phi_k(x)dx\\
&= \hat{c}_k(\lambda - i\mu_k)\int_0^L \varphi_j(x)\big\{\overline{\varphi_k(x)} + (-A - i\mu_k + \lambda)^{-1}\big[a_k\hat{b}(x) - \lambda\overline{\varphi_k(x)}\,\big]\big\}dx\\
&= \hat{c}_k(\lambda - i\mu_k)\int_0^L \varphi_j(x)(-A - i\mu_k + \lambda)^{-1}a_k\varphi_j'(0)\overline{\varphi_j(x)}dx\\
&= \hat{c}_k(\lambda - i\mu_k)(i\mu_j - i\mu_k + \lambda)^{-1}a_k\varphi_j'(0)\\
&= \hat{c}_k\lambda\frac{\lambda - i\mu_k}{i\mu_j - i\mu_k + \lambda}\frac{\varphi_j'(0)}{\varphi_k'(0)}.
\end{aligned}$$

This, together with (2.96), implies that

$$-\hat{c}_j(\lambda - i\mu_j) - \sum_{k\in\mathbb{Z}\setminus\{0,j\}} \hat{c}_k\lambda\frac{\lambda - i\mu_k}{i\mu_j - i\mu_k + \lambda}\frac{\varphi_j'(0)}{\varphi_k'(0)} = \varphi_j'(0) \ \text{ for all } j \in \mathbb{Z}\setminus\{0\}. \tag{2.97}$$

For $j \in \mathbb{Z}\setminus\{0\}$ and $k \in \mathbb{Z}\setminus\{0\}$, let

$$c_j = -\frac{\hat{c}_j(\lambda - i\mu_j)}{\varphi_j'(0)},\ a_{jk} = \frac{1}{i\mu_j - i\mu_k + \lambda}. \tag{2.98}$$

From (2.97) and (2.98), we get that

$$c_j + \lambda\sum_{k\in\mathbb{Z}\setminus\{0,j\}} a_{jk}c_k = 1. \tag{2.99}$$

Let us now estimate c_j. From (2.18), (2.79) and (2.99), we have, for every $j \in \mathbb{Z}\setminus\{0\}$,

$$\begin{aligned}
\Big|\sum_{k\in\mathbb{Z}\setminus\{0,j\}} a_{jk}c_k\Big| &= \Big|\sum_{k\in\mathbb{Z}\setminus\{0,j\}} \hat{c}_k\frac{\lambda - i\mu_k}{i\mu_j - i\mu_k + \lambda}\frac{1}{\varphi_k'(0)}\Big|\\
&\leq C\sum_{k\in\mathbb{Z}\setminus\{0,j\}} \Big|\hat{c}_k\frac{\lambda - ik^3}{j^3 - k^3}\frac{1}{k}\Big|\\
&\leq C\Big(\sum_{k\in\mathbb{Z}\setminus\{0,j\}} |\hat{c}_k|^2\Big)^{\frac{1}{2}}\Big(\sum_{k\in\mathbb{Z}\setminus\{0,j\}} \Big|\frac{\lambda - ik^3}{j^3 - k^3}\frac{1}{k}\Big|^2\Big)^{\frac{1}{2}}.
\end{aligned} \tag{2.100}$$

For positive integer j,

$$\begin{aligned}\sum_{k\in\mathbb{Z}\setminus\{0,j\}}\Big|\frac{\lambda-ik^3}{j^3-k^3}\frac{1}{k}\Big|^2 &\le \sum_{k>2j}\Big|\frac{\lambda-ik^3}{j^3-k^3}\frac{1}{k}\Big|^2+\sum_{j<k\le 2j}\Big|\frac{\lambda-ik^3}{j^3-k^3}\frac{1}{k}\Big|^2\\ &\quad+\sum_{-j<k<j,\,k\neq 0}\Big|\frac{\lambda-ik^3}{j^3-k^3}\frac{1}{k}\Big|^2+\sum_{k\le -j}\Big|\frac{\lambda-ik^3}{j^3-k^3}\frac{1}{k}\Big|^2.\end{aligned}\tag{2.101}$$

We now estimate the four terms in the right hand side of (2.101). First, we have that

$$\sum_{k>2j}\Big|\frac{\lambda-ik^3}{j^3-k^3}\frac{1}{k}\Big|^2\le\frac{8}{7}\sum_{k>2j}\frac{1+\lambda^2}{k^2}\le\frac{C}{j}.\tag{2.102}$$

Second,

$$\begin{aligned}&\sum_{j<k\le 2j}\Big|\frac{\lambda-ik^3}{j^3-k^3}\frac{1}{k}\Big|^2\le C+\sum_{j<k\le 2j}\Big|\frac{4j^2}{j^3-k^3}\Big|^2\le C+\sum_{1\le l\le j}\Big|\frac{4j^2}{j^3-(j+l)^3}\Big|^2\\ &\le C+\sum_{1\le l\le j}\Big|\frac{4j^2}{3j^2l+3jl^2+l^3}\Big|^2\le C+\frac{16}{9}\sum_{1\le l\le j}\frac{1}{l^2}\le C.\end{aligned}\tag{2.103}$$

With similar arguments, we can obtain that

$$\sum_{-j<k<j,\,k\neq 0}\Big|\frac{\lambda-ik^3}{j^3-k^3}\frac{1}{k}\Big|^2\le C\tag{2.104}$$

and

$$\sum_{k\le -j}\Big|\frac{\lambda-ik^3}{j^3-k^3}\frac{1}{k}\Big|^2\le C.\tag{2.105}$$

From (2.100) to (2.105), we find that there is a constant $C>0$ such that for all positive integer j,

$$\Big|\sum_{k\in\mathbb{Z}\setminus\{0,j\}}a_{jk}c_k\Big|\le C.\tag{2.106}$$

Similarly, we can prove that for all negative integer j,

$$\Big|\sum_{k\in\mathbb{Z}\setminus\{0,j\}}a_{jk}c_k\Big|\le C.\tag{2.107}$$

Combining (2.99), (2.106) and (2.107), we get that there is a constant $C>0$ such that for all $j\in\mathbb{Z}\setminus\{0\}$,

$$|c_j|\le C.\tag{2.108}$$

We now estimate $|c_j|$ for $|j|$ large. From (2.18) and (2.98), we get that for $j>0$,

$$\begin{aligned}&\sum_{k\in\mathbb{Z}\setminus\{0,j\}}|a_{jk}|=\sum_{k\in\mathbb{Z}\setminus\{0,j\}}\frac{1}{|i\mu_j-i\mu_k+\lambda|}\le\sum_{k\in\mathbb{Z}\setminus\{0,j\}}\frac{1}{|\mu_j-\mu_k|}\le\sum_{k\in\mathbb{Z}\setminus\{0,j\}}\frac{C}{|j^3-k^3|}\\ &\le\sum_{k>2j}\frac{C}{|j^3-k^3|}+\sum_{j<k\le 2j}\frac{C}{|j^3-k^3|}+\sum_{-j<k<j,\,k\neq 0}\frac{C}{|j^3-k^3|}+\sum_{k\le -j}\frac{C}{|j^3-k^3|}.\end{aligned}\tag{2.109}$$

We estimate the terms in the last line of (2.109) one by one. First,

$$\sum_{k>2j}\frac{1}{|j^3-k^3|}\leq\frac{8}{7}\sum_{k>2j}\frac{1}{k^3}\leq\frac{8}{7}\sum_{k>2j}\frac{1}{k}\frac{1}{k(k-1)}\leq\frac{8}{7}\frac{1}{(2j+1)2j}\leq\frac{2}{7j^2}. \tag{2.110}$$

Second,

$$\begin{aligned}\sum_{j<k\leq 2j}\frac{1}{|j^3-k^3|}&\leq\sum_{1<l\leq j}\frac{1}{|j^3-(j+l)^3|}\leq\sum_{1<l\leq j}\frac{1}{3j^2l+3jl^2+l^3}\\&\leq\frac{1}{3}\sum_{1<l\leq j}\frac{1}{j^2l+jl^2}\leq\frac{1}{3j^2}\sum_{1<l\leq j}\Big(\frac{1}{j+l}+\frac{1}{l}\Big)\\&\leq\frac{2\ln|j|}{3j^2}.\end{aligned} \tag{2.111}$$

Similarly, we can obtain that

$$\sum_{-j<k<j,\,k\neq 0}\frac{1}{\left|j^3-k^3\right|}\leq\frac{4\ln|j|}{3j^2} \tag{2.112}$$

and

$$\sum_{k\leq -j}\frac{1}{\left|j^3-k^3\right|}\leq\frac{1}{j^2}. \tag{2.113}$$

From (2.109) to (2.113), we know there is a constant $C>0$ such that for all positive integer j,

$$\sum_{k\in\mathbb{Z}\setminus\{0,j\}}|a_{jk}|\leq\frac{C\ln|j|}{j^2}. \tag{2.114}$$

By similar arguments, we also can show that for all negative integer j,

$$\sum_{k\in\mathbb{Z}\setminus\{0,j\}}|a_{jk}|\leq\frac{C\ln|j|}{j^2}. \tag{2.115}$$

Combining (2.99), (2.108), (2.114) and (2.115), we obtain that

$$1-\frac{C\ln|j|}{j^2}\leq c_j\leq 1+\frac{C\ln|j|}{j^2}. \tag{2.116}$$

We now turn to the definition of $k(\cdot,\cdot)$. From (2.91) and (2.116), one has

$$\begin{aligned}&\sum_{j\in\mathbb{Z}\setminus\{0\}}\int_0^L\left|\overline{\varphi_j(x)}-c_j\phi_j(x)\right|^2dx\\&\leq 2\sum_{j\in\mathbb{Z}\setminus\{0\}}\int_0^L\left|(1-c_j)\overline{\varphi_j(x)}\right|^2dx+2\sum_{j\in\mathbb{Z}\setminus\{0\}}\int_0^L|c_j|^2\left|\overline{\varphi_j(x)}-\phi_j(x)\right|^2dx\\&\leq C\sum_{j\in\mathbb{Z}\setminus\{0\}}\Big(\frac{\ln^2|j|}{j^4}+\frac{1}{j^4}\Big)<+\infty.\end{aligned} \tag{2.117}$$

Inequality (2.117) allows us to define $k(\cdot,\cdot) \in L^2((0,L)\times(0,L))$ by

$$k(x,y) \triangleq \sum_{j\in\mathbb{Z}\setminus\{0\}} [\overline{\varphi_j(x)} - c_j\phi_j(x)]\varphi_j(y), \quad 0 \le x, y \le L. \tag{2.118}$$

Let us prove that $k(\cdot,\cdot) \in H^1((0,L)\times(0,L))$. Thanks to (2.25), (2.93) and (2.116), we find that

$$\begin{aligned}
&\int_0^L\int_0^L \Big|\sum_{j\in\mathbb{Z}\setminus\{0\}} [\overline{\varphi_j'(x)} - c_j\phi_j'(x)]\varphi_j(y)\Big|^2 dxdy \\
&= \sum_{j\in\mathbb{Z}\setminus\{0\}} \int_0^L \big|\overline{\varphi_j'(x)} - c_j\phi_j'(x)\big|^2 dx \\
&\le \sum_{j\in\mathbb{Z}\setminus\{0\}} \int_0^L \big|(1-c_j)\overline{\varphi_j'(x)}\big|^2 dx + \sum_{j\in\mathbb{Z}\setminus\{0\}} \int_0^L |c_j|^2\big|\overline{\varphi_j'(x)} - \phi_j'(x)\big|^2 dx \\
&\le C\sum_{j\in\mathbb{Z}\setminus\{0\}} \Big(\frac{\ln^2|j|}{j^2} + \frac{1}{j^2}\Big) < +\infty,
\end{aligned} \tag{2.119}$$

which shows that

$$k_x(\cdot,\cdot) \in L^2((0,L)\times(0,L)). \tag{2.120}$$

Utilizing (2.18), (2.84) and (2.108), we get

$$\begin{aligned}
&\int_0^L\int_0^L\Big|\sum_{j\in\mathbb{Z}\setminus\{0\}}c_j[\overline{\varphi_j(x)}-\phi_j(x)]\varphi_j'(y)\Big|^2dxdy\\
=&\int_0^L\int_0^L\sum_{j,k\in\mathbb{Z}\setminus\{0\}}c_jc_k\big[\overline{\varphi_j(x)}-\phi_j(x)\big]\big[\varphi_k(x)-\overline{\phi_k(x)}\,\big]\varphi_j'(y)\overline{\varphi_k'(y)}dxdy\\
=&\int_0^L\int_0^L\sum_{j,k\in\mathbb{Z}\setminus\{0\}}c_jc_k(-A-i\mu_j+\lambda)^{-1}(a_j\hat{b}-\lambda\overline{\varphi_j})(x)\overline{(-A-i\mu_k+\lambda)^{-1}(a_k\hat{b}-\lambda\overline{\varphi_k})(x)}\\
&\qquad\times\varphi_j'(y)\overline{\varphi_k'(y)}dxdy\\
=&\int_0^L\int_0^L\sum_{j,k\in\mathbb{Z}\setminus\{0\}}c_jc_k\Big[\sum_{l\in\mathbb{Z}\setminus\{0,j\}}(i\mu_l-i\mu_j+\lambda)^{-1}\frac{\lambda\varphi_l'(0)}{\varphi_j'(0)}\overline{\varphi_l(x)}\Big]\\
&\times\overline{\Big[\sum_{l\in\mathbb{Z}\setminus\{0,k\}}(i\mu_l-i\mu_k+\lambda)^{-1}\frac{\lambda\varphi_l'(0)}{\varphi_k'(0)}\overline{\varphi_l(x)}\Big]}\varphi_j'(y)\overline{\varphi_k'(y)}dxdy\\
\le&\, C\lambda^2\sum_{j,k\in\mathbb{Z}\setminus\{0\}}\sum_{l\in\mathbb{Z}\setminus\{0,j,k\}}\big|(i\mu_l-i\mu_j+\lambda)^{-1}\varphi_l'(0)(i\mu_l-i\mu_k+\lambda)^{-1}\varphi_l'(0)\big|\\
&\times\int_0^L\Big|\varphi_j'(y)\varphi_j'(0)^{-1}\varphi_k'(y)\varphi_k'(0)^{-1}\Big|dy\\
\le&\, C\lambda^2\sum_{l\in\mathbb{Z}\setminus\{0\}}\sum_{j\in\mathbb{Z}\setminus\{0,l\}}\big|(i\mu_l-i\mu_j+\lambda)^{-1}\varphi_l'(0)\big|\sum_{k\in\mathbb{Z}\setminus\{0,l\}}\big|(i\mu_l-i\mu_k+\lambda)^{-1}\varphi_l'(0)\big|.
\end{aligned}\tag{2.121}$$

Similar to the proof of (2.114), we can obtain that

$$\begin{cases}\displaystyle\sum_{j\in\mathbb{Z}\setminus\{0,l\}}\big|(i\mu_l-i\mu_j+\lambda)^{-1}\varphi_l'(0)\big|\le\frac{C\ln|l||\varphi_l'(0)|}{l^2}\le\frac{C\ln|l|}{|l|},\\ \displaystyle\sum_{k\in\mathbb{Z}\setminus\{0,l\}}\big|(i\mu_l-i\mu_k+\lambda)^{-1}\varphi_l'(0)\big|\le\frac{C\ln|l||\varphi_l'(0)|}{l^2}\le\frac{C\ln|l|}{|l|}.\end{cases}\tag{2.122}$$

Combining (2.121) and (2.122), one has that

$$\int_0^L\int_0^L\Big|\sum_{j\in\mathbb{Z}\setminus\{0\}}c_j\big[\,\overline{\varphi_j(x)}-\phi_j(x)\big]\varphi_j'(y)\Big|^2dxdy\le C\lambda^2\sum_{l\in\mathbb{Z}\setminus\{0\}}\frac{\ln^2|l|}{l^2}<+\infty.\tag{2.123}$$

From (2.25), (2.116) and (2.123)

$$
\begin{aligned}
&\int_0^L \int_0^L \Big| \sum_{j\in\mathbb{Z}\setminus\{0\}} \big[\overline{\varphi_j(x)} - c_j\phi_j(x)\big]\varphi_j'(y)\Big|^2 dxdy \\
&\le 2\int_0^L \int_0^L \Big| \sum_{j\in\mathbb{Z}\setminus\{0\}} (1-c_j)\overline{\varphi_j(x)}\varphi_j'(y)\Big|^2 dxdy \\
&\quad +2\int_0^L \int_0^L \Big| \sum_{j\in\mathbb{Z}\setminus\{0\}} c_j\big[\overline{\varphi_j(x)} - \phi_j(x)\big]\varphi_j'(y)\Big|^2 dxdy \\
&\le 2\sum_{j\in\mathbb{Z}\setminus\{0\}} \int_0^L \big|(1-c_j)\varphi_j'(y)\big|^2 dy + 2\int_0^L \int_0^L \Big| \sum_{j\in\mathbb{Z}\setminus\{0\}} c_j\big[\overline{\varphi_j(x)} - \phi_j(x)\big]\varphi_j'(y)\Big|^2 dxdy \\
&\le C\sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{\ln^2|j|}{j^2} < +\infty.
\end{aligned}
\tag{2.124}
$$

From (2.117), (2.119) and (2.124), we see that

$$
k_y(\cdot,\cdot) \in L^2((0,L)\times(0,L)), \tag{2.125}
$$

which, together with (2.120), shows that $k(\cdot,\cdot) \in H^1((0,L)\times(0,L))$. Clearly, $k(\cdot,\cdot) = 0$ on the boundary of $(0,L)\times(0,L)$. Thus, we conclude that $k(\cdot,\cdot) \in H_0^1((0,L)\times(0,L))$.

Let us define

$$
k^{(n)}(x,y) \triangleq \sum_{0<|j|\le n} \big[\overline{\varphi_j(x)} - c_j\phi_j(x)\big]\varphi_j(y) \ \text{ in } (0,L)\times(0,L). \tag{2.126}
$$

Simple estimates show that

$$
\big(x\in(0,L) \mapsto k_x^{(n)}(x,\cdot) \in L^2(0,L)\big) \ \text{ is in } C^0([0,L];L^2(0,L)). \tag{2.127}
$$

For any $m,n\in\mathbb{Z}^+$, $m<n$, we have that

$$
\begin{aligned}
&\int_0^L \Big| \sum_{m<|j|\le n} \big[\overline{\varphi_j'(x)} - c_j\phi_j'(x)\big]\varphi_j(y)\Big|^2 dy \\
&= \sum_{m<|j|\le n} \big|\overline{\varphi_j'(x)} - c_j\phi_j'(x)\big|^2 \\
&\le 2\sum_{m<|j|\le n} \big|(1-c_j)\overline{\varphi_j'(x)}\big|^2 + 2\sum_{m<|j|\le n} |c_j|^2\big|\overline{\varphi_j'(x)} - \phi_j'(x)\big|^2.
\end{aligned}
\tag{2.128}
$$

By means of (2.25) and (2.116), we find that

$$
\max_{x\in[0,L]} \sum_{m<|j|\le n} \big|(1-c_j)\overline{\varphi_j'(x)}\big|^2 \le C\sum_{m<|j|\le n} \frac{\ln^2|j|}{j^2}. \tag{2.129}
$$

From (2.84) and (2.24), similarly to the proof of (2.114), we obtain that

$$\begin{aligned}
&\max_{x\in[0,L]} \sum_{m<|j|\le n} |c_j|^2 \left|\overline{\varphi_j'(x)} - \phi_j'(x)\right|^2 \\
&= \max_{x\in[0,L]} \sum_{m<|j|\le n} |c_j|^2 \Big| \sum_{l\in\mathbb{Z}\setminus\{0,j\}} (i\mu_l - i\mu_j + \lambda)^{-1} \frac{\lambda\varphi_l'(0)}{\varphi_j'(0)} \varphi_l'(x)\Big|^2 \\
&\le C \sum_{m<|j|\le n} \sum_{l\in\mathbb{Z}\setminus\{0,j\}} \Big|\frac{l^2}{(j^3-l^3)j}\Big|^2 \\
&\le C \sum_{m<|j|\le n} \frac{\ln^2|j|}{j^2}.
\end{aligned} \tag{2.130}$$

Combining (2.129) and (2.130), we get that

$$\left\{x \in (0,L) \mapsto k_x^{(n)}(x,\cdot) \in L^2(0,L)\right\}_{n=1}^{+\infty} \text{ is a Cauchy sequence in } C^0([0,L];L^2(0,L)), \tag{2.131}$$

which shows that (2.2) holds. Proceeding as in the proofs of (2.125) and of (2.131), one gets that

$$\left\{y \in (0,L) \mapsto k_y^{(n)}(\cdot,y) \in L^2(0,L)\right\}_{n=1}^{+\infty} \text{ is a Cauchy sequence in } C^0([0,L];L^2(0,L)), \tag{2.132}$$

which also gives (2.3). Moreover (2.95), (2.98), (2.126) and (2.132) imply that

$$k_y(\cdot,0) = \lim_{n\to+\infty} k_y^{(n)}(\cdot,0) = 0 \text{ in } L^2(0,L). \tag{2.133}$$

Similarly, one can show that

$$k_y(\cdot,L) = \lim_{n\to+\infty} k_y^{(n)}(\cdot,L) = 0 \text{ in } L^2(0,L). \tag{2.134}$$

From (2.133) and (2.134), one has (2.4).

Let us finally prove that $k(\cdot,\cdot)$ satisfies (2.5).

First, it is clear that

$$\begin{aligned}
&(\partial_{xxx} + \partial_x + \partial_{yyy} + \partial_y + \lambda)(1-c_j)\overline{\varphi_j(x)}\varphi_j(y) \\
&= (1-c_j)(i\mu_j - i\mu_j + \lambda)\overline{\varphi_j(x)}\varphi_j(y) = (1-c_j)\lambda\overline{\varphi_j(x)}\varphi_j(y).
\end{aligned} \tag{2.135}$$

From (2.82) to (2.84), one has

$$
\begin{aligned}
&(\partial_{xxx}+\partial_x+\partial_{yyy}+\partial_y+\lambda)c_j[\overline{\varphi_j(x)}-\phi_j(x)]\varphi_j(y)\\
&=-(\partial_{xxx}+\partial_x+\partial_{yyy}+\partial_y+\lambda)\sum_{k\in\mathbb{Z}\setminus\{0,j\}}c_j\frac{\lambda\varphi_k'(0)\overline{\varphi_k(x)}}{\varphi_j'(0)(i\mu_k-i\mu_j+\lambda)}\varphi_j(y)\\
&=-c_j\lambda\sum_{k\in\mathbb{Z}\setminus\{0,j\}}\frac{\varphi_k'(0)}{\varphi_j'(0)}\overline{\varphi_k(x)}\varphi_j(y)\\
&=c_j\lambda\overline{\varphi_j(x)}\varphi_j(y)-\frac{c_j\lambda}{\varphi_j'(0)}\sum_{k\in\mathbb{Z}\setminus\{0\}}\varphi_k'(0)\overline{\varphi_k(x)}\varphi_j(y)\\
&=c_j\lambda\overline{\varphi_j(x)}\varphi_j(y)+\frac{c_j\lambda}{\varphi_j'(0)}\delta_{x=0}'\otimes\varphi_j(y)\quad\text{ in } D(A)'\otimes L^2(0,L).
\end{aligned}
\tag{2.136}
$$

From (2.126), (2.135) and (2.136), we get in $D(A)'\otimes L^2(0,L)$,

$$
\begin{aligned}
&k_{xxx}^{(n)}(x,y)+k_x^{(n)}(x,y)+k_{yyy}^{(n)}(x,y)+k_y^{(n)}(x,y)+\lambda k^{(n)}(x,y)-\delta_{x=0}'\otimes\sum_{0<|j|\le n}\frac{c_j\lambda}{\varphi_j'(0)}\varphi_j(y)\\
&=\lambda\sum_{0<|j|\le n}\overline{\varphi_j(x)}\varphi_j(y).
\end{aligned}
$$

Therefore, for any $\rho\in\mathcal{E}\subset D(A)\otimes L^2(0,L)$ (recall (2.1) for the definition of $\mathcal{E}$), we have

$$
\begin{aligned}
0=&\int_0^L\int_0^L\big[k_{xxx}^{(n)}(x,y)+k_x^{(n)}(x,y)+k_{yyy}^{(n)}(x,y)+k_y^{(n)}(x,y)+\lambda k^{(n)}(x,y)\\
&-\lambda\sum_{0<|j|\le n}\overline{\varphi_j(x)}\varphi_j(y)\big]\rho(x,y)dxdy-\int_0^L\sum_{0<|j|\le n}\frac{\lambda}{\varphi_j'(0)}\delta_{x=0}'\big(\rho(x,y)\big)\varphi_j(y)dy\\
=&-\int_0^L\int_0^L\big[\rho_{xxx}(x,y)+\rho_x(x,y)+\rho_{yyy}(x,y)+\rho_y(x,y)+\lambda\rho(x,y)\big]k^{(n)}(x,y)dxdy\\
&-\int_0^L k_y^{(n)}(x,L)\rho(x,L)dx+\int_0^L k_y^{(n)}(x,0)\rho(x,0)dx\\
&-\lambda\int_0^L\int_0^L\rho(x,y)\sum_{0<|j|\le n}\overline{\varphi_j(x)}\varphi_j(y)dxdy.
\end{aligned}
\tag{2.137}
$$

By (2.133), (2.134) and letting $n\to+\infty$ in (2.137), we obtain that

$$
\begin{aligned}
&\int_0^L\int_0^L\big[\rho_{xxx}(x,y)+\rho_x(x,y)+\rho_{yyy}(x,y)+\rho_y(x,y)+\lambda\rho(x,y)\big]k(x,y)dxdy\\
&+\int_0^L\rho(y,y)dy=0.
\end{aligned}
\tag{2.138}
$$

This concludes Step 2 and therefore the proof of Lemma 2.1

Remark 2.1 *Note that $k(\cdot,\cdot)$ is a real valued function since $\bar{k}(\cdot,\cdot)$ is also a solution of* (1.9) *in $\mathcal{G}$ and therefore, by the uniqueness proved in Step 1, we must have $\bar{k}(\cdot,\cdot) = k(\cdot,\cdot)$.*

3 Invertibility of $I - K$

We define a bounded linear operator $K : L^2(0,L) \to L^2(0,L)$ by

$$K(v)(x) \stackrel{\triangle}{=} \int_0^L k(x,y)v(y)dy \quad \text{for every } v \in L^2(0,L).$$

Note that, by Remark 2.1, if v is a real valued function, then Kv is also a real valued function. The goal of this section is to prove the following lemma.

Lemma 3.1 *$I - K$ is an invertible operator.*

Proof of Lemma 3.1: Since $k(\cdot,\cdot) \in L^2((0,L)\times(0,L))$, we get that K is a compact operator. Further, since $k \in H_0^1((0,L)\times(0,L))$, we know that K is a continuous linear map from $L^2(0,L)$ into $H_0^1(0,L)$. Denote by K^* the adjoint operator of K. Then, it is easy to see that

$$K^*(v)(x) = \int_0^L k^*(x,y)v(y)dy \quad \text{for any } v \in L^2(0,L),$$

where k^* is defined by

$$k^*(x,y) \stackrel{\triangle}{=} k(y,x),\ (x,y) \in (0,L)\times(0,L). \tag{3.1}$$

From (1.9) and (3.1), one gets

$$\begin{cases} k^*_y + k^*_{yyy} + k^*_x + k^*_{xxx} + \lambda k^* = \lambda\delta(y-x) & \text{in } (0,L)\times(0,L), \\ k^*(0,y) = k^*(L,y) = 0 & \text{on } (0,L), \\ k^*_x(0,y) = k^*_x(L,y) = 0 & \text{on } (0,L), \\ k^*(x,0) = k^*(x,L) = 0 & \text{on } (0,L). \end{cases} \tag{3.2}$$

Further, from (3.1) again, we have the following regularity for $k^*(\cdot,\cdot)$:

$$y \in (0,L) \mapsto k^*_y(\cdot,y) \in L^2(0,L) \text{ is in } C^0([0,L];L^2(0,L)), \tag{3.3}$$

$$x \in (0,L) \mapsto k^*_x(x,\cdot) \in L^2(0,L) \text{ is in } C^0([0,L];L^2(0,L)). \tag{3.4}$$

Let us point out that, from the (3.2) and (3.4), we know

$$v \in K^*(L^2(0,L)) \Rightarrow \left(v \in C^1([0,L]),\ v(0) = v(L) = v_x(0) = v_x(L) = 0\right). \tag{3.5}$$

We claim that the spectral radius $r(K^*)$ of K^* equals 0. Otherwise, since $K^* : L^2(0,L) \to L^2(0,L)$ is as K a compact linear operator, there is a nonzero eigenvalue α of K^*. Then, there exists a positive integer n_0 such that

$$\mathrm{Ker}(K^* - \alpha I)^{n_0+1} = \mathrm{Ker}(K^* - \alpha I)^{n_0}. \tag{3.6}$$

Let

$$\mathcal{F} \overset{\triangle}{=} \operatorname{Ker}(K^* - \alpha I)^{n_0}.$$

It is a finite dimensional space.

Since $\alpha \neq 0$, $\mathcal{F} \subset K^*(L^2(0,L))$, and, with (3.5),

$$\mathcal{F} \subset C^1([0,L]) \text{ and } v(0) = v_x(0) = v(L) = v_x(L) = 0, \; \forall\, v \in \mathcal{F}. \tag{3.7}$$

Note that, by the fact that $k^* \in H_0^1((0,L)\times(0,L))$,

$$K^* \text{ can be extended to be a continuous linear map from } H^{-1}(0,L) \text{ into } L^2(0,L). \tag{3.8}$$

Let us denote by $\widetilde{K}^*$ this extension and remark that, if $u \in H^{-1}(0,L)$ is such that $\widetilde{K}^* u = \alpha u$, then $u \in L^2(0,L)$. Thus, we see that $\operatorname{Ker}(\widetilde{K}^* - \alpha I) \subset L^2(0,L)$ and $\operatorname{Ker}(\widetilde{K}^* - \alpha I) = \operatorname{Ker}(K^* - \alpha I)$. Similarly, we have

$$\begin{cases} \operatorname{Ker}(\widetilde{K}^* - \alpha I)^{n_0} \subset L^2(0,L), \\ \operatorname{Ker}(\widetilde{K}^* - \alpha I)^{n_0} = \mathcal{F}, \\ \operatorname{Ker}(\widetilde{K}^* - \alpha I)^{n_0} = \operatorname{Ker}(\widetilde{K}^* - \alpha I)^{n_0+1}. \end{cases}$$

By (3.2),

$$K^*(\partial_{xxx} + \partial_x)v = (\partial_{xxx} + \partial_x)K^* v - \lambda K^* v + \lambda v, \; \forall\, v \in C_0^\infty(0,L). \tag{3.9}$$

From (3.5) and (3.9) we also get that K^* can be extended to be a continuous linear map from $H^{-3}(0,L)$ into $H^{-2}(0,L)$. This, together with (3.8) and an interpolation argument, shows that K^* can be extended to be a continuous linear map from $H^{-2}(0,L)$ into $H^{-1}(0,L)$. We denote by $\widehat{K}^*$ this extension. Then, as for $\widetilde{K}^*$, one has

$$\begin{cases} \operatorname{Ker}(\widehat{K}^* - \alpha I)^{n_0} = \operatorname{Ker}(K^* - \alpha I) \in L^2(0,L), \\ \operatorname{Ker}(\widehat{K}^* - \alpha I)^{n_0} = \mathcal{F}, \\ \operatorname{Ker}(\widehat{K}^* - \alpha I)^{n_0} = \operatorname{Ker}(\widehat{K}^* - \alpha I)^{n_0+1}. \end{cases} \tag{3.10}$$

Using (3.9), a density argument and (3.8) we get that

$$\widehat{K}^*(\partial_{xxx} + \partial_x)v = (\partial_{xxx} + \partial_x)\widehat{K}^* v - \lambda \widehat{K}^* v + \lambda v, \; \forall v \in \mathcal{F}. \tag{3.11}$$

From (3.11), (3.7), and induction on n, one gets that

$$(\widehat{K}^*)^n(\partial_{xxx} + \partial_x)v = (\partial_{xxx} + \partial_x)(\widehat{K}^*)^n v - n\lambda(\widehat{K}^*)^n v + n\lambda(\widehat{K}^*)^{n-1} v, \; \forall v \in \mathcal{F}. \tag{3.12}$$

and therefore, for every polynomial P,

$$P(\widehat{K}^*)(\partial_{xxx} + \partial_x)v = (\partial_{xxx} + \partial_x)P(\widehat{K}^*)v - \lambda P'(\widehat{K}^*)\widehat{K}^* v + \lambda P'(\widehat{K}^*)v, \; \forall v \in \mathcal{F}. \tag{3.13}$$

By virtue of (3.10), (3.7) and (3.13) with $P(X) \overset{\triangle}{=} (X-\alpha)^{n_0+1}$, we see that $(\partial_{xxx}+\partial_x)\mathcal{F} \subset \mathcal{F}$. Since $\mathcal{F}$ is finite dimensional, this implies that $(\partial_{xxx} + \partial_x)$ has an eigenfunction in $\mathcal{F}$, that is, there exist $\mu \in \mathbb{C}$ and $\xi \in \mathcal{F} \setminus \{0\}$ such that

$$\begin{cases} (\partial_{xxx} + \partial_x)\xi = \mu\xi & \text{in } (0, L), \\ \xi(0) = \xi(L) = \xi_x(0) = \xi_x(L) = 0. \end{cases}$$

But, by [21, Lemma 3.5] again, this is impossible since $L \notin \mathcal{N}$. Then, we know $r(K^*) = 0$, which implies that the spectral radius $r(K)$ of K is zero. Hence, we know that the real number 1 belongs to the resolvent set of K, which completes the proof of Lemma 3.1.

4 Proof of Theorem 1.2

This section is addressed to a proof of Theorem 1.2.

Proof of Theorem 1.2: Let $T > 0$, which will be given later. Consider the following equation

$$\begin{cases} v_{1,t} + v_{1,x} + v_{1,xxx} + v_1 v_{1,x} = 0 & \text{in } [0, T] \times (0, L), \\ v_1(t, 0) = v_1(t, L) = 0 & \text{on } [0, T], \\ v_{1,x}(t, L) = \displaystyle\int_0^L k_x(L, y) v_1(t, y) dy & \text{on } [0, T], \\ v_1(0) = v^0 & \text{in } (0, L). \end{cases} \tag{4.1}$$

By Theorem 1.1, we know that there is an $r_T > 0$ such that for all $v^0 \in L^2(0, L)$ with $|v^0|_{L^2(0,L)} \le r_T$, the equation (4.1) admits a unique solution $v_1 \in C^0([0, T]; L^2(0, L)) \cap L^2(0, T; H_0^1(0, L))$. Moreover, there is a constant $C_T > 0$ such that

$$|v_1|_{C^0([0,T];L^2(0,L)) \cap L^2(0,T;H_0^1(0,L))} \le C_T |v^0|_{L^2(0,L)}. \tag{4.2}$$

Let $w_1 = (I - K)v_1$. Then, we have that

$$\begin{aligned} w_{1,t}(t, x) &= v_{1,t}(t, x) - \int_0^L k(x, y) v_{1,t}(t, y) dy \\ &= v_{1,t}(t, x) + \int_0^L k(x, y) \big[v_{1,y}(t, y) + v_{1,yyy}(t, y) + v_1(t, y) v_{1,y}(t, y) \big] dy \\ &= v_{1,t}(t, x) + \big[k(x, y) v_1(t, y) \big]\Big|_{y=0}^{y=L} - \int_0^L k_y(x, y) v_1(t, y) dy \\ &\quad + \big[k(x, y) v_{1,yy}(t, y) \big]\Big|_{y=0}^{y=L} - \big[k_y(x, y) v_{1,y}(t, y) \big]\Big|_{y=0}^{y=L} + \big[k_{yy}(x, y) v_1(t, y) \big]\Big|_{y=0}^{y=L} \\ &\quad - \int_0^L k_{yyy}(x, y) v_1(t, y) dy + \frac{1}{2} \big[k(x, y) v_1(t, y)^2 \big]\Big|_{y=0}^{y=L} - \frac{1}{2} \int_0^L k_y(x, y) v_1(t, y)^2 dy \\ &= v_{1,t}(t, x) - \int_0^L \big[k_y(x, y) + k_{yyy}(x, y) \big] v_1(t, y) dy - \frac{1}{2} \int_0^L k_y(x, y) v_1(t, y)^2 dy, \end{aligned} \tag{4.3}$$

$$w_{1,x}(t, x) = v_{1,x}(t, x) - \int_0^L k_x(x, y) v_1(t, y) dy \tag{4.4}$$

and

$$w_{1,xxx}(t,x) = v_{1,xxx}(t,x) - \int_0^L k_{xxx}(x,y)v_1(t,y)dy. \tag{4.5}$$

Therefore, for a given $\lambda \in \mathbb{R}$, utilizing the fact the v_1 solves (4.1), we obtain that

$$\begin{aligned}
& w_{1,t}(t,x) + w_{1,x}(t,x) + w_{1,xxx}(t,x) + \lambda w_1(t,x) + \frac{1}{2}\int_0^L k_y(x,y)v_1(t,y)^2 dy \\
&= \lambda v_1(t,x) - \int_0^L v_1(t,y)\big[k_y(x,y) + k_{yyy}(x,y) + k_x(x,y) + k_{xxx}(x,y) + \lambda k(x,y)\big]dy \\
&= -\int_0^L v_1(t,y)\big[k_y(x,y) + k_{yyy}(x,y) + k_x(x,y) + k_{xxx}(x,y) + \lambda k(x,y) - \lambda\delta(x-y)\big]dy \\
&= -v_1(t,x)v_{1,x}(t,x).
\end{aligned} \tag{4.6}$$

Hence, if we take the feedback control F as

$$F(\varphi) \triangleq \int_0^L k_x(L,y)\varphi(y)dy, \quad \forall \varphi \in L^2(0,L), \tag{4.7}$$

then we get that w_1 solves

$$\begin{cases}
w_{1,t} + w_{1,x} + w_{1,xxx} + w_1 w_{1,x} + \lambda w_1 \\
= -v_1 v_{1,x} - \dfrac{1}{2}\displaystyle\int_0^L k_y(x,y)v_1(t,y)^2 dy & \text{in } [0,T]\times(0,L), \\
w_1(t,0) = w_1(t,L) = 0 & \text{on } [0,T], \\
w_{1,x}(t,L) = 0 & \text{on } [0,T].
\end{cases} \tag{4.8}$$

Thus, we see that

$$\begin{aligned}
& \frac{d}{dt}\int_0^L |w_1(t,x)|^2 dx \\
&= -|w_{1,x}(t,0)|^2 - 2\lambda\int_0^L |w_1(t,x)|^2 dx - \int_0^L w_1(t,x)v_1(t,x)v_{1,x}(t,x)dx \\
&\quad -\frac{1}{2}\int_0^L w_1(t,x)\Big[\int_0^L k_y(x,y)v_1(t,y)^2 dy\Big]dx.
\end{aligned} \tag{4.9}$$

Now we estimate the third and fourth terms in the right hand side of (4.9). First,

$$\begin{aligned}
&\Big|\int_0^L w_1(t,x)v_1(t,x)v_{1,x}(t,x)dx\Big| \\
&=\Big|\int_0^L \Big[v_1(t,x)-\int_0^L k(x,y)v_1(t,y)dy\Big]v_1(t,x)v_{1,x}(t,x)dx\Big| \\
&=\Big|\int_0^L \Big(\int_0^L k(x,y)v_1(t,y)dy\Big)v_1(t,x)v_{1,x}(t,x)dx\Big| \\
&=\frac{1}{2}\Big|\int_0^L \Big(\int_0^L k_x(x,y)v_1(t,y)dy\Big)v_1(t,x)^2dx\Big| \\
&\le \frac{1}{2}\Big|\int_0^L |k_x(\cdot,y)|^2dy\Big|_{L^\infty(0,L)}\Big(\int_0^L v_1(t,x)^2dx\Big)^{\frac{3}{2}} \\
&\le |(I-K)^{-1}|^3_{\mathcal{L}(L^2(0,L))}\frac{1}{2}\Big|\int_0^L |k_x(\cdot,y)|^2dy\Big|_{L^\infty(0,L)}\Big(\int_0^L w_1^2(t,x)dx\Big)^{\frac{3}{2}}.
\end{aligned} \tag{4.10}$$

Next,

$$\begin{aligned}
&\Big|\int_0^L w_1(t,x)\int_0^L k_y(x,y)v_1(t,y)^2dydx\Big| \\
&=\Big|\int_0^L \Big[v_1(t,x)-\int_0^L k(x,y)v_1(t,y)dy\Big]\Big(\int_0^L k_y(x,y)v_1(t,y)^2dy\Big)dx\Big| \\
&\le\Big|\int_0^L v_1(t,x)\Big(\int_0^L k_y(x,y)v_1(t,y)^2dy\Big)dx\Big|+\Big|\int_0^L\Big(\int_0^L k(x,y)v_1(t,y)dy\int_0^L k_y(x,y)v_1(t,y)^2dy\Big)dx\Big|.
\end{aligned} \tag{4.11}$$

The first term in the right hand side of (4.11) satisfies that

$$\begin{aligned}
&\Big|\int_0^L v_1(t,x)\Big(\int_0^L k_y(x,y)v_1(t,y)^2dy\Big)dx\Big| \\
&=\Big|\int_0^L v_1(t,y)^2\Big(\int_0^L k_y(x,y)v_1(t,x)dx\Big)dy\Big| \\
&\le \int_0^L v_1(t,y)^2\Big(\int_0^L |k_y(x,y)|^2dx\Big)^{\frac{1}{2}}dy\Big(\int_0^L v_1(t,x)^2dx\Big)^{\frac{1}{2}} \\
&\le \Big|\Big(\int_0^L |k_y(x,\cdot)|^2dx\Big)^{\frac{1}{2}}\Big|_{L^\infty(0,L)}\Big(\int_0^L v_1(t,x)^2dx\Big)^{\frac{3}{2}} \\
&\le |(I-K)^{-1}|^3_{\mathcal{L}(L^2(0,L))}\Big|\Big(\int_0^L |k_y(x,\cdot)|^2dx\Big)^{\frac{1}{2}}\Big|_{L^\infty(0,L)}\Big(\int_0^L w_1(t,x)^2dx\Big)^{\frac{3}{2}}.
\end{aligned} \tag{4.12}$$

The second term in the right hand side of (4.11) satisfies that

$$\begin{aligned}
&\Big|\int_0^L\Big(\int_0^L k(x,z)v_1(t,z)dz\int_0^L k_y(x,y)v_1(t,y)^2dy\Big)dx\Big|\\
&=\Big|\int_0^L v_1(t,y)^2\int_0^L k_y(x,y)\Big(\int_0^L k(x,z)v_1(t,z)dz\Big)dxdy\Big|\\
&\le\int_0^L v_1(t,y)^2\Big(\int_0^L|k_y(x,y)|^2dx\Big)^{\frac12}dy\Big[\int_0^L\Big(\int_0^L k(x,z)v_1(t,z)dz\Big)^2dx\Big]^{\frac12}\\
&\le\Big|\Big(\int_0^L|k_y(x,\cdot)|^2dx\Big)^{\frac12}\Big|_{L^\infty(0,L)}\Big(\int_0^L|k(x,y)|^2dxdy\Big)^{\frac12}\Big(\int_0^L v_1(t,x)^2dx\Big)^{\frac32}\\
&\le|(I-K)^{-1}|^3_{\mathcal{L}(L^2(0,L))}\Big|\Big(\int_0^L|k_y(x,\cdot)|^2dx\Big)^{\frac12}\Big|_{L^\infty(0,L)}\Big(\int_0^L|k(x,y)|^2dxdy\Big)^{\frac12}\Big(\int_0^L w_1(t,x)^2dx\Big)^{\frac32}.
\end{aligned}\tag{4.13}$$

Let

$$\begin{aligned}
\widehat{C}=&\frac12|(I-K)^{-1}|^3_{\mathcal{L}(L^2(0,L))}\Big|\int_0^L|k_x(\cdot,y)|^2dy\Big|_{L^\infty(0,L)}\\
&+|(I-K)^{-1}|^3_{\mathcal{L}(L^2(0,L))}\Big|\Big(\int_0^L|k_y(x,\cdot)|^2dx\Big)^{\frac12}\Big|_{L^\infty(0,L)}\\
&+|(I-K)^{-1}|^3_{\mathcal{L}(L^2(0,L))}\Big|\Big(\int_0^L|k_y(x,\cdot)|^2dx\Big)^{\frac12}\Big|_{L^\infty(0,L)}\Big(\int_0^L|k(x,y)|^2dxdy\Big)^{\frac12}.
\end{aligned}$$

From (4.9)–(4.13), we get that

$$\frac{d}{dt}\int_0^L|w_1(t,x)|^2dx\le-2\lambda\int_0^L|w_1(t,x)|^2dx+\widehat{C}\Big(\int_0^L w_1(t,x)^2dx\Big)^{\frac32}.\tag{4.14}$$

For a given $\lambda>0$, we know that there is a $\delta_1>0$ such that, if $|w_1(0)|_{L^2(0,L)}\le\delta_1$, then

$$\widehat{C}\Big(\int_0^L w_1(t,x)^2dx\Big)^{\frac12}\le\lambda,\quad\forall\,t\in[0,T].$$

This, together with (4.14), implies that

$$\frac{d}{dt}\int_0^L|w_1(t,x)|^2dx\le-\lambda\int_0^L|w_1(t,x)|^2dx,\quad\forall\,t\in[0,T].$$

In particular

$$|w_1(t,\cdot)|_{L^2(0,L)}\le e^{-\frac{\lambda t}{2}}|w_1(0,\cdot)|_{L^2(0,L)},\quad\forall\,t\in[0,T].\tag{4.15}$$

Then, from Lemma 3.1 and (4.15), we get that if

$$|v_1(0,\cdot)|_{L^2(0,L)}\le\min\{|I-K|^{-1}_{\mathcal{L}(L^2(0,L))}\delta_1,r_T\},$$

then

$$|v_1(t,\cdot)|_{L^2(0,L)}\le|(I-K)^{-1}|_{\mathcal{L}(L^2(0,L))}|I-K|_{\mathcal{L}(L^2(0,L))}e^{-\frac{\lambda t}{2}}|v_1(0,\cdot)|_{L^2(0,L)},\quad\forall\,t\in[0,T].\tag{4.16}$$

Now we choose $T > 0$ such that

$$e^{-\frac{\lambda T}{2}} \leq |(I-K)^{-1}|_{\mathcal{L}(L^2(0,L))}|I-K|_{\mathcal{L}(L^2(0,L))}.$$

From (4.16), we find that $|v_1(T)|_{L^2(0,L)} \leq |v_1(0)|_{L^2(0,L)} \leq r_T$. Thus, by Theorem 1.1, we know that the following equation is well-posed.

$$\begin{cases} v_{2,t} + v_{2,x} + v_{2,xxx} + v_2 v_{2,x} = 0 & \text{in } [0,T] \times (0,L), \\ v_2(t,0) = v_2(t,L) = 0 & \text{on } [0,T], \\ v_{2,x}(t,L) = \displaystyle\int_0^L k_x(L,y) v_2(t,y) dy & \text{on } [0,T], \\ v_2(0) = v_1(T) & \text{in } (0,L). \end{cases} \tag{4.17}$$

Further, by a similar argument, we can find that $w_2 \triangleq (I-K)v_2$ satisfies

$$|w_2(t,\cdot)|_{L^2(0,L)} \leq e^{-\frac{\lambda t}{2}} |w_2(0,\cdot)|_{L^2(0,L)}, \quad \forall\, t \in [0,T]$$

and

$$|v_2(T,\cdot)|_{L^2(0,L)} \leq |v_2(0,\cdot)|_{L^2(0,L)}.$$

Then, we can define v_3 and w_3 in a similar manner. By induction, we can find $v_n \in C^0([0,T]; L^2(0,L)) \cap L^2(0,T; H_0^1(0,L))$ $(n > 1)$, which solves

$$\begin{cases} v_{n,t} + v_{n,x} + v_{n,xxx} + v_n v_{n,x} = 0 & \text{in } [0,T] \times (0,L), \\ v_n(t,0) = v_n(t,L) = 0 & \text{on } [0,T], \\ v_{n,x}(t,L) = \displaystyle\int_0^L k_x(L,y) v_n(t,y) dy & \text{on } [0,T], \\ v_n(0) = v_{n-1}(T) & \text{in } (0,L). \end{cases} \tag{4.18}$$

Further, we have that $w_n = (I-K)v_n$ satisfies

$$|w_n(t,\cdot)|_{L^2(0,L)} \leq e^{-\frac{\lambda t}{2}} |w_n(0,\cdot)|_{L^2(0,L)} = e^{-\frac{\lambda t}{2}} |w_{n-1}(T,\cdot)|_{L^2(0,L)} \tag{4.19}$$

and

$$|v_n(T,\cdot)|_{L^2(0,L)} \leq |v_n(0,\cdot)|_{L^2(0,L)} = |v_{n-1}(T,\cdot)|_{L^2(0,L)}.$$

Now we put

$$v(t+(n-1)T, x) = v_n(t,x), \quad w(t+(n-1)T, x) = w_n(t,x) \quad \text{for } (t,x) \in [0,T) \times [0,L].$$

Then, it is an easy matter to see that v is a solution to (1.1) and $w = (I-K)v$. From (4.19), we get that

$$|w(t,\cdot)|_{L^2(0,L)} \leq e^{-\frac{\lambda t}{2}} |w(0,\cdot)|_{L^2(0,L)}, \quad \forall\, t \geq 0. \tag{4.20}$$

This, together with $w = (I-K)v$, implies that for all $t \geq 0$,

$$|v(t,\cdot)|_{L^2(0,L)} \leq e^{-\frac{\lambda t}{2}} |(I-K)^{-1}|_{\mathcal{L}(L^2(0,L))} |I-K|_{\mathcal{L}(L^2(0,L))} |v(0,\cdot)|_{L^2(0,L)} \leq C e^{-\frac{\lambda t}{2}} |v(0,\cdot)|_{L^2(0,L)}.$$

Let $\delta_0 = \min\{|(I-K)|^{-1}_{\mathcal{L}(L^2(0,L))}\delta_1, r_T\}$. Then, we know that for any $v^0 \in L^2(0,L)$ with $|v^0|_{L^2(0,L)} \le \delta_0$, the equation (1.1) with the initial datum v^0 admits a solution

$$v \in C^0([0,+\infty); L^2(0,L)) \cap L^2_{loc}(0,+\infty; H^1_0(0,L)).$$

Further, we have that

$$|v(t,\cdot)|_{L^2(0,L)} \le Ce^{-\frac{\lambda t}{2}}|v(0,\cdot)|_{L^2(0,L)}, \quad \forall\, t \ge 0. \tag{4.21}$$

A Appendix

This section is devoted to a proof of Theorem 1.1. Before giving the proof, we first recall the following useful results.

Let $T > 0$. We consider the following linearized KdV equation with non-homogeneous boundary condition.

$$\begin{cases} u_t + u_{xxx} + u_x = \tilde{h} & \text{in } [0,T]\times(0,L),\\ u(t,0) = u(t,L) = 0 & \text{on } [0,T],\\ u_x(t,L) = h(t) & \text{on } [0,T],\\ u(0) = u^0 & \text{in } (0,L). \end{cases} \tag{A.1}$$

Here $u^0 \in L^2(0,L)$, $h \in L^2(0,T)$, $\tilde{h} \in L^1(0,T;L^2(0,L))$. In [21] (see also [3]), the author proved the following results:

Lemma A.1 *Let $u^0 \in L^2(0,L)$. There exists a unique solution $u \in C^0([0,T];L^2(0,L)) \cap L^2(0,T;H^1_0(0,L))$ of* (A.1) *such that $u(0,\cdot) = u^0(\cdot)$. Moreover, there is a $C_1 > 0$, independent of $h \in L^2(0,T)$, $\tilde{h} \in L^1(0,T;L^2(0,L))$ and $u^0 \in L^2(0,L)$, such that*

$$|u|^2_{C^0([0,T];L^2(0,L))} + |u|^2_{L^2(0,T;H^1_0(0,L))} \le C_1\big(|u^0|^2_{L^2(0,L)} + |h|^2_{L^2(0,T)} + |\tilde{h}|^2_{L^1(0,T;L^2(0,L))}\big). \tag{A.2}$$

Lemma A.2 *Let $z \in L^2(0,T;H^1(0,L))$. Then $zz_x \in L^1(0,T;L^2(0,L))$ and the map $z \in L^2(0,T;H^1(0,L)) \mapsto zz_x \in L^1(0,T;L^2(0,L))$ is continuous.*

In [9], the following result is proved.

Lemma A.3 *[9, Proposition 15] There exists $C > 0$ such that for every $\tilde{u}_0, \hat{u}_0 \in L^2(0,L)$ and $\tilde{h}, \hat{h} \in L^2(0,T)$ for which there exist solution $\tilde{u}$ and $\hat{u}$ in $C^0([0,T];L^2(0,L)) \cap L^2(0,T;H^1_0(0,L))$ of*

$$\begin{cases} \tilde{u}_t + \tilde{u}_x + \tilde{u}_{xxx} + \tilde{u}\tilde{u}_x = 0 & \textit{in } [0,T]\times(0,L),\\ \tilde{u}(t,0) = \tilde{u}(t,L) = 0 & \textit{on } [0,T],\\ \tilde{u}_x(t,L) = \tilde{h}(t) & \textit{on } [0,T],\\ \tilde{u}(0,x) = \tilde{u}^0(x) & \textit{in } (0,L), \end{cases} \tag{A.3}$$

and of

$$\begin{cases} \hat{u}_t + \hat{u}_x + \hat{u}_{xxx} + \hat{u}\hat{u}_x = 0 & \textit{in } [0,T]\times(0,L), \\ \hat{u}(t,0) = \hat{u}(t,L) = 0 & \textit{on } [0,T], \\ \hat{u}_x(t,L) = \hat{h}(t) & \textit{on } [0,T], \\ \hat{u}(0,x) = \hat{u}^0(x) & \textit{in } (0,L), \end{cases} \tag{A.4}$$

one has the following inequality

$$\begin{aligned} &\int_0^L |\tilde{u}(t,x) - \hat{u}(t,x)|^2 dx \\ &\le e^{C\left(1+|\tilde{u}|^2_{L^2(0,T;H^1(0,L))}+|\hat{u}|_{L^2(0,T;H^1(0,L))}\right)^2}\Big[\int_0^L |\tilde{u}^0(x) - \hat{u}^0(x)|^2 dx + \int_0^t |\tilde{h}(s) - \hat{h}(s)|^2 ds\Big] \end{aligned} \tag{A.5}$$

for all $t \in [0,T]$.

Remark A.1 *In [9], the term* $\int_0^t |\tilde{h}(s)-\hat{h}(s)|^2 ds$ *in the right hand side of* (A.5) *is* $\int_0^T |\tilde{h}(s)-\hat{h}(s)|^2 ds$. *However, one can see that the proof of [9] also gives* (A.5).

Proof of Theorem 1.1:

Uniqueness of the solution

Assume that v_1 and v_2 are two solutions of (1.1), by Lemma A.3, we know that for any $t \in [0,T]$,

$$\begin{aligned} &\int_0^L |v_1(t,x) - v_2(t,x)|^2 dx \\ &\le e^{C\left(1+|v_1|^2_{L^2(0,T;H^1(0,L))}+|v_2|^2_{L^2(0,T;H^1(0,L))}\right)} \int_0^t |F(v_1) - F(v_2)|^2 ds \\ &\le Ce^{C\left(1+|v_1|^2_{L^2(0,T;H^1(0,L))}+|v_2|^2_{L^2(0,T;H^1(0,L))}\right)} \int_0^t \int_0^L |v_1(s,x) - v_2(s,x)|^2 dx ds. \end{aligned} \tag{A.6}$$

This, together with the Gronwall inequality, implies that $v_1 = v_2$ in $[0,T]\times(0,L)$.

Existence of the solution

Let us extend $\tilde{h}$ and h to be a function on $(0,+\infty)\times(0,L)$ and $(0,+\infty)$ by setting them to be equal to zero on $(T,+\infty)\times(0,L)$ and $(T,+\infty)$, respectively. Denote by $||F||$ the norm of the continuous linear map $F : L^2(0,L) \to \mathbb{R}$. Set

$$T_1 \triangleq \min\Big\{\frac{1}{2C_1||F||^2}, T\Big\}. \tag{A.7}$$

Let $u \in C^0([0,T_1]; L^2(0,L))$. We know that

$$h(\cdot) \triangleq F(u(\cdot)) \in L^2(0,T_1).$$

Hence, for v^0 given in $L^2(0,L)$, we can define a map

$$\mathcal{J} : C^0([0,T_1]; L^2(0,L)) \to C^0([0,T_1]; L^2(0,L))$$

as follows: $\mathcal{J}(u) = v$, where $v \in C^0([0,T_1]; L^2(0,L)) \cap L^2(0,T; H_0^1(0,L))$ solves (A.1) with $h(\cdot) = F(u(\cdot))$ and $v(0,\cdot) = v^0(\cdot)$.

For $\tilde{u}, \hat{u} \in C^0([0,T_1]; L^2(0,L))$, one has, using in particular (A.2) and (A.7),

$$
\begin{aligned}
&|\mathcal{J}(\tilde{u}) - \mathcal{J}(\hat{u})|_{C^0([0,T_1];L^2(0,L))} \\
&\leq \sqrt{C_1}\Big(\int_0^{T_1} \big|F(\tilde{u}(t,\cdot)) - F(\hat{u}(t,\cdot))\big|^2 dt\Big)^{\frac{1}{2}} \\
&\leq \sqrt{C_1}||F||\Big(\int_0^{T_1} \int_0^L |\tilde{u}(t,y) - \hat{u}(t,y)|^2 dydt\Big)^{\frac{1}{2}} \\
&\leq \sqrt{T_1}\sqrt{C_1}||F|||\tilde{u} - \hat{u}|_{C^0([0,T_1];L^2(0,L))} \\
&\leq \frac{1}{\sqrt{2}}|\tilde{u} - \hat{u}|_{C^0([0,T_1];L^2(0,L))}.
\end{aligned}
$$

Hence, we get that $\mathcal{J}$ is a contractive map. By the Banach fixed point theorem, we know that $\mathcal{J}$ has a unique fixed point v_1, which is the solution to the following equation

$$
\begin{cases}
v_{1,t} + v_{1,xxx} + v_{1,x} = \tilde{h} & \text{in } [0,T_1] \times (0,L), \\
v_1(t,0) = v_1(t,L) = 0 & \text{on } [0,T_1], \\
v_{1,x}(t,L) = F(v_1(t,\cdot)) & \text{on } [0,T_1], \\
v_1(0,x) = v^0(x) & \text{in } (0,L).
\end{cases}
\tag{A.8}
$$

Using (A.2), (A.7) and (A.8), we find that

$$
\begin{aligned}
&|v_1|^2_{C^0([0,T_1];L^2(0,L))} \\
&\leq C_1\Big(|v^0|^2_{L^2(0,L)} + \int_0^{T_1} \big|F(v(t,\cdot))\big|^2 dt + |\tilde{h}|^2_{L^1(0,T_1;L^2(0,L))}\Big) \\
&\leq C_1\Big(|v^0|^2_{L^2(0,L)} + T_1||F||^2|v_1|^2_{C^0([0,T_1];L^2(0,L))} + |\tilde{h}|^2_{L^1(0,T_1;L^2(0,L))}\Big) \\
&\leq C_1\big(|v^0|^2_{L^2(0,L)} + |\tilde{h}|^2_{L^1(0,T_1;L^2(0,L))}\big) + \frac{1}{2}|v_1|^2_{C^0([0,T_1];L^2(0,L))},
\end{aligned}
$$

which implies that

$$
|v_1|^2_{C^0([0,T_1];L^2(0,L))} \leq 2C_1\big(|v^0|^2_{L^2(0,L)} + |\tilde{h}|^2_{L^1(0,T_1;L^2(0,L))}\big). \tag{A.9}
$$

From (A.2), (A.8) and (A.9), we obtain that

$$
\begin{aligned}
&|v_1|^2_{C^0([0,T_1];L^2(0,L))} + |v_1|^2_{L^2(0,T_1;H_0^1(0,L))} \\
&\leq C_1\Big(|v^0|^2_{L^2(0,L)} + \int_0^{T_1} \big|F(v(t,\cdot))\big|^2 dt + |\tilde{h}|^2_{L^1(0,T_1;L^2(0,L))}\Big) \\
&\leq C_1\big(|v^0|^2_{L^2(0,L)} + |\tilde{h}|^2_{L^1(0,T_1;L^2(0,L))}\big) + C_1T_1||F||^2|v_1|^2_{C^0([0,T_1];L^2(0,L))} \\
&\leq C_1\big(|v^0|^2_{L^2(0,L)} + |\tilde{h}|^2_{L^1(0,T_1;L^2(0,L))}\big) + \frac{1}{2}|v_1|^2_{C^0([0,T_1];L^2(0,L))} \\
&\leq 2C_1\big(|v^0|^2_{L^2(0,L)} + |\tilde{h}|^2_{L^1(0,T_1;L^2(0,L))}\big).
\end{aligned}
\tag{A.10}
$$

By a similar argument, we can prove that the following equation

$$\begin{cases} v_{2,t} + v_{2,xxx} + v_{2,x} = \tilde{h} & \text{in } [T_1, 2T_1] \times (0, L), \\ v_2(t, 0) = v_2(t, L) = 0 & \text{on } [T_1, 2T_1], \\ v_{2,x}(t, L) = F(v_2(t, \cdot)) & \text{on } [T_1, 2T_1], \\ v_2(T_1, x)) = v_1(T_1, x) & \text{in } (0, L). \end{cases} \tag{A.11}$$

admits a unique solution in $C^0([T_1, 2T_1]; L^2(0, L)) \cap L^2(T_1, 2T_1; H_0^1(0, L))$. Furthermore, this solutions satisfies that

$$\begin{aligned} &|v_2|^2_{C^0([T_1,2T_1];L^2(0,L))} + |v_2|^2_{L^2(T_1,2T_1;H_0^1(0,L))} \\ &\leq 2C_1\big(|v_1(T_1)|^2_{L^2(0,L)} + |\tilde{h}|^2_{L^1(T_1,2T_1;L^2(0,L))}\big). \end{aligned} \tag{A.12}$$

By induction, we know that for an integer $n \geq 2$, the following equation admits a unique solution.

$$\begin{cases} v_{n,t} + v_{n,xxx} + v_{n,x} = \tilde{h} & \text{in } [(n-1)T_1, nT_1] \times (0, L), \\ v_n(t, 0) = v_n(t, L) = 0 & \text{on } [(n-1)T_1, nT_1], \\ v_{n,x}(t, L) = F(v_n(t, \cdot)) & \text{on } [(n-1)T_1, nT_1], \\ v_n((n-1)T_1, x) = v_{n-1}((n-1)T_1, x) & \text{in } (0, L). \end{cases} \tag{A.13}$$

Furthermore, one has

$$\begin{aligned} &|v_n|^2_{C^0([(n-1)T_1,nT_1];L^2(0,L))} + |v_n|^2_{L^2((n-1)T_1,nT_1;H_0^1(0,L))} \\ &\leq 2C_1\big(|v_{n-1}((n-1)T_1)|^2_{L^2(0,L)} + |\tilde{h}|^2_{L^1((n-1)T_1,nT_1;L^2(0,L))}\big). \end{aligned} \tag{A.14}$$

Let $n_0 \in \mathbb{Z}^+$ be such that $(n_0 - 1)T_1 \leq T < n_0 T_1$. Let

$$v(t, x) \triangleq v_n(t, x) \text{ for } t \in [(n-1)T_1, nT_1) \cap [0, T], \quad n = 1, 2, \cdots, n_0,\ x \in (0, L).$$

Then v is the solution to

$$\begin{cases} v_t + v_{xxx} + v_x = \tilde{h} & \text{in } [0, T] \times (0, L), \\ v(t, 0) = v(t, L) = 0 & \text{on } [0, T], \\ v_x(t, L) = F(v(t, y)) & \text{on } [0, T], \\ v(0, x) = v^0(x) & \text{in } (0, L). \end{cases} \tag{A.15}$$

Moreover, there is a constant $C = C(T) > 0$ (independent of $v^0 \in L^2(0, L)$ and of $\tilde{h} \in L^1(0, T; L^2(0, L))$) such that

$$|v|^2_{C^0([0,T];L^2(0,L))} + |v|^2_{L^2(0,T;H_0^1(0,L))} \leq C(T)\big(|v^0|^2_{L^2(0,L)} + |\tilde{h}|^2_{L^1(0,T;L^2(0,L))}\big). \tag{A.16}$$

Let $\kappa > 0$, depending only on $L > 0$, such that

$$|w|_{L^\infty} \leq \kappa |w|_{H_0^1(0,L)}, \quad \forall w \in H_0^1(0, L). \tag{A.17}$$

From now on, we assume that $v^0 \in L^2(0,L)$ satisfies

$$|v^0|^2_{L^2(0,L)} \le \frac{5}{36C(T)^2\kappa^2}. \tag{A.18}$$

Let

$$\mathcal{B} \triangleq \Big\{ u \in C^0([0,T];L^2(0,L)) \cap L^2(0,T;H^1_0(0,L)) : \\ |u|^2_{C^0([0,T];L^2(0,L))} + |u|^2_{L^2(0,T;H^1_0(0,L))} \le \frac{1}{6C(T)\kappa^2} \Big\}.$$

Then $\mathcal{B}$ is a nonempty closed subset of the Banach space $C^0([0,T];L^2(0,L)) \cap L^2(0,T;H^1_0(0,L))$. Let us define a map $\mathcal{K}$ from $\mathcal{B}$ to $C^0([0,T];L^2(0,L)) \cap L^2(0,T;H^1_0(0,L))$ as follows:

$$\mathcal{K}(z) = v, \text{ where } v \text{ is the solution to (A.15) with } \tilde{h} = zz_x.$$

From Lemma A.2 and the above well-posedness result for (A.15), we know that $\mathcal{K}$ is well-defined. From (A.16) and (A.18), we have that

$$\begin{aligned}
&|\mathcal{K}(z)|^2_{C^0([0,T];L^2(0,L))} + |\mathcal{K}(z)|^2_{L^2(0,T;H^1_0(0,L))} \\
&\le C(T)\big(|v^0|^2_{L^2(0,L)} + |zz_x|^2_{L^1(0,T;L^2(0,L))}\big) \\
&\le C(T)\big(|v^0|^2_{L^2(0,L)} + |z|^2_{L^2(0,T;L^\infty(0,L))}|z_x|^2_{L^2(0,T;L^2(0,L))}\big) \\
&\le C(T)\big(|v^0|^2_{L^2(0,L)} + \kappa^2|z|^4_{L^2(0,T;H^1_0(0,L))}\big) \\
&\le \frac{1}{6C(T)\kappa^2}
\end{aligned}$$

and

$$\begin{aligned}
&|\mathcal{K}(z_1) - \mathcal{K}(z_2)|^2_{C^0([0,T];L^2(0,L))} + |\mathcal{K}(z_1) - \mathcal{K}(z_2)|^2_{L^2(0,T;H^1_0(0,L))} \\
&\le C(T)|z_1z_{1,x} - z_2z_{2,x}|^2_{L^1(0,T;L^2(0,L))} = C(T)|z_1z_{1,x} - z_1z_{2,x} + z_1z_{2,x} - z_2z_{2,x}|^2_{L^1(0,T;L^2(0,L))} \\
&\le 2C(T)\kappa^2\big(|z_1|^2_{L^2(0,T;H^1_0(0,L))}|z_1 - z_2|^2_{L^2(0,T;H^1_0(0,L))} + |z_1 - z_2|^2_{L^2(0,T;H^1_0(0,L))}|z_2|^2_{L^2(0,T;H^1_0(0,L))}\big) \\
&\le 2C(T)\kappa^2\big(|z_1|^2_{L^2(0,T;H^1_0(0,L))} + |z_2|^2_{L^2(0,T;H^1_0(0,L))}\big)|z_1 - z_2|^2_{L^2(0,T;H^1_0(0,L))} \\
&\le \frac{2}{3}|z_1 - z_2|^2_{L^2(0,T;H^1_0(0,L))}.
\end{aligned}$$

Therefore, we know that $\mathcal{K}$ is from $\mathcal{B}$ to $\mathcal{B}$ and is contractive. Then, by the Banach fixed point theorem, there is a (unique) fixed point v, which is the solution to (1.1) with initial data $v(0,\cdot) = v^0(\cdot)$.

Remark A.2 *Adapting the proof of [19, Section 3.1], one can also prove that, for every* $v^0 \in L^2(0,L)$, *there exist* $T > 0$ *and a solution* $v \in C^0([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$ *of* (1.1) *with* $f(t) = F(v(t,\cdot))$ *satisfying the initial condition* $v(0,\cdot) = v^0(\cdot)$.

Acknowledgement

Jean-Michel Coron was supported by ERC advanced grant 266907 (CPDENL) of the 7th Research Framework Programme (FP7).

Qi Lü was supported by ERC advanced grant 266907 (CPDENL) of the 7th Research Framework Programme (FP7), the NSF of China under grant 11101070 and the Fundamental Research Funds for the Central Universities in China under grants ZYGX2012J115. Qi Lü would like to thank the Laboratoire Jacques-Louis Lions (LJLL) at Université Pierre et Marie Curie for its hospitality. This work was carried out while he was visiting the LJLL.

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