Maximum likelihood estimator consistency for recurrent random walk in a parametric random environment with finite support

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Abstract

We consider a one-dimensional recurrent random walk in random environment (RWRE) when the environment is i.i.d. with a parametric, finitely supported distribution. Based on a single observation of the path, we provide a maximum likelihood estimation procedure of the parameters of the environment. Unlike most of the classical maximum likelihood approach, the limit of the criterion function is in general a nondegenerate random variable and convergence does not hold in probability. Not only the leading term but also the second order asymptotics is needed to fully identify the unknown parameter. We present different frameworks to illustrate these facts. We also explore the numerical performance of our estimation procedure.

Key words : Recurrent regime, maximum likelihood estimation, random walk in random environment. *MSC 2010* : Primary 62M05, 62F12; secondary 62F12.

1 Introduction

Since the pioneer works of Chernov (1967) and Temkin (1972), random walks in random environments (RWRE) have attracted many probabilists and physicists, and the related literature in these fields has become richer and source of fine probabilistic results that the reader may find in surveys including Hughes (1996)

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and Zeitouni (2004). The literature dealing with the statistical analysis of RWRE is far from being as rich and we aim at making a fundamental contribution to the inference of parameters of the environment distribution for a one-dimensional nearest neighbour path.

Let $\omega = (\omega_x)_{x\in\mathbb{Z}}$ be an independent and identically distributed (i.i.d.) collection of $(0,1)$-valued random variables with a parametric distribution η_θ of the form

$$\eta_\theta = \sum_{i=1}^{d} p_i \delta_{a_i}, \tag{1}$$

with d an integer, $\mathbf{p} = (p_i)_{1\le i\le d}$ a probability vector and $\mathbf{a} = (a_i)_{1\le i\le d}$ the ordered support. We further assume that $d \ge 2$ is known, and the unknown parameter is $\theta = (\mathbf{a}, \mathbf{p})$.

Denote by $\mathbb{P}^\theta = \eta_\theta^{\otimes\mathbb{Z}}$ the law on $(0,1)^{\mathbb{Z}}$ of the environment ω and by $\mathbb{E}^\theta$ the expectation under this law. The process ω represents a random environment in which the random walk evolves. For fixed environment ω, let $X = (X_t)_{t\in\mathbb{Z}_+}$ be the Markov chain on $\mathbb{Z}_+$ starting at $X_0 = 0$ and with transition probabilities $P_\omega(X_{t+1} = 1|X_t = 0) = 1$, and for $x > 0$

$$P_\omega(X_{t+1} = y|X_t = x) = \begin{cases} \omega_x & \text{if } y = x+1, \\ 1-\omega_x & \text{if } y = x-1, \\ 0 & \text{otherwise.} \end{cases}$$

For simplicity, we stick to the RWRE on the positive integers reflected at 0, but our results apply for the RWRE on the integer axis as well. The symbol P_ω denotes the measure on the path space of X given ω, usually called *quenched* law. The (unconditional) law of X is given by

$$\mathbf{P}^\theta(\cdot) = \int P_\omega(\cdot)\mathrm{d}\mathbb{P}^\theta(\omega),$$

this is the so-called *annealed* law. We write E_ω and $\mathbf{E}^\theta$ for the corresponding quenched and annealed expectations, respectively.

Random environment is a classical paradigm for inhomogeneous media which possess some regularity at large scale. Introduced by Chernov (1967) as a model for DNA replication, RWRE was recently used by Baldazzi et al. (2006, 2007) and Andreoletti and Diel (2012) to analyse experiments on DNA unzipping, pointing the need of sound statistical procedures for these models. In (1) we restrict the model to environments with finite support, a framework which already covers many interesting applications and also reveals the main features of the estimation problem. Considering a general setup would increase the technical complexity without any further appeal.

The behaviour of the process X is related to the ratio sequence

$$\rho_x = \frac{1-\omega_x}{\omega_x}, \quad x \in \mathbb{Z}_+, \tag{2}$$

and we refer to Solomon (1975) for the classification of X between transient or recurrent cases according to whether $\mathbb{E}^{\theta}(\log \rho_0)$ is different or not from zero. The transient case may be further split into two sub-cases, called *ballistic* and *sub-ballistic* that correspond to a linear and a sub-linear displacement for the walk, respectively.

Comets et al. (2014) provided a maximum likelihood estimator (MLE) of the parameter θ in the transient *ballistic* case. The estimator maximizes the annealed log-likelihood function, it only depends on the sequence of the number of left steps performed by the random walk. In the *ballistic* transient case this normalized criterion function converges in probability to a *finite* deterministic limit function, which identifies the true value of the parameter. Comets et al. (2014) establishes the consistency of MLE while asymptotic normality together with asymptotic efficiency (namely, that it asymptotically achieves the Cramér-Rao bound) is investigated in Falconnet et al. (2013). Falconnet et al. proved that a slight modification of the above criterion function also provides a well-designed limiting function in the *sub-ballistic* case. In all these works, the results rely on the branching structure of the sequence of the number of left steps performed by the walk, which was originally observed by Kesten et al. (1975).

The salient probabilistic feature of a recurrent RWRE is the strong localization revealed by Sinaĭ (1982). Inhomogeneities in the medium create deep traps the walker falls into. Typically, on a time interval $[0, n]$, the walker sneaks most of the time around the very bottom of the main “valley” which is at random distance of order $\log^2 n$. Visiting repeatedly the medium at the bottom of the main "valley", it collects precise information there. Unfortunately the medium at the bottom is *not typical* from the unknown distribution η_θ, but on the contrary, it is strongly biased by the implicit information that there is a deep trap right there. However, the localization mechanism can be completely analysed using the analogy between nearest neighbour walks and electrical networks, as explained in the book of Doyle and Snell (1984). From Gantert et al. (2010), we know that the empirical distributions of the RWRE converge weakly to a certain limit law which describes the stationary distribution of a random walk in an *infinite* valley whose construction goes back to Golosov (1984). This allows to unbias our observation of the medium, and to prove consistency of MLE. We recall at this point the moment estimator introduced in Adelman and Enriquez (2004), which relies on the observation that the step when leaving a site x for the first time yields an unbiased estimator of the environment.

In the course of the proof, expanding the log-likelihood for a large observation time n, we prove convergence of the first order term at scale n and of the second order term at scale $\log^2 n$. Though the first order term is random in general, it allows to identify the support $\mathbf{a}$ of the environment, while the second one with a correct estimate of $\mathbf{a}$, allows to identify the probability vector $\mathbf{p}$. This discrepancy reflects that the amount of information gathered on the unknown parameter $\mathbf{a}$

is proportional to the duration of the observation, whereas the one for $\mathbf{p}$ is only proportional to the number of distinct visited sites. As emphasized above, the expansion of the likelihood, Lemma 2.2 below, is the key. Therefore it is natural to introduce a maximum pseudo-likelihood estimator (MPLE), defined by the above first two terms of the expansion, as a intermediate step to study MLE.

The rest of the article is organized as follows. In Section 2, we present the construction of our M-estimators (*i.e.* an estimator maximising some criterion function), state the assumptions required on the model as well as our consistency results. Sections 3 and Section 4 are respectively devoted to the proofs of the consistency result for the estimators of $\mathbf{a}$, and $\mathbf{p}$ respectively. Section 5 presents some examples of environment distributions for which we provide an explicit expression of the limit of the criterion function, and check whether it is a non-degenerate or a constant random variable. Finally, numerical experiments are presented in Section 6, focusing on the three examples that were developed in Section 5.

2 Statistical problem, M-estimators and results

We address the following statistical problem: we assume that the process X is generated under the true parameter value $\theta^\star = (\mathbf{a}^\star, \mathbf{p}^\star)$, an interior point of the parameter space Θ, and we want to estimate the unknown $\theta^\star$ from a single observation $(X_t)_{0\le t\le n}$ of the RWRE path with time length n. Let $d \ge 2$ an integer. We always assume that $\Theta \subset (0,1)^{2d}$ is compact and satisfies Assumptions I and II below, which ensure that the environment is recurrent and has d atoms.

Assumption I (Recurrent environment)**.** *For any* $\theta = (\mathbf{a}, \mathbf{p})$ *in* Θ,

$$p_i > 0, \textit{ for any } i \le d, \quad \textit{and} \quad \sum_{i=1}^{d} p_i \log \frac{1-a_i}{a_i} = 0. \tag{3}$$

Assumption II (Identifiability)**.** *For all* θ *in* Θ,

$$0 < a_1 < a_2 < \ldots < a_d < 1. \tag{4}$$

Note that, since Θ is compact, there exists ε_0 in $(0, 1/2d)$ such that

$$a_1 \ge \varepsilon_0, \quad 1 - a_d \ge \varepsilon_0, \quad a_{i+1} - a_i \ge \varepsilon_0, \quad p_i \ge \varepsilon_0, \quad \text{for any } i \ge 1. \tag{5}$$

We shorten to $\mathbf{P}^\star$ and $\mathbf{E}^\star$ (resp. $\mathbb{P}^\star$ and $\mathbb{E}^\star$) the annealed probability $\mathbf{P}^{\theta^\star}$ and its corresponding expectation $\mathbf{E}^{\theta^\star}$ (resp. the law of the environment $\mathbb{P}^{\theta^\star}$ and its corresponding expectation $\mathbb{E}^{\theta^\star}$) under parameter value $\theta^\star$.

Define for all $x \in \mathbb{Z}$,

$$\xi(n,x) := \sum_{t=0}^{n} \mathbb{1}\{X_t = x\}, \tag{6}$$

$$\xi^-(n,x) := \sum_{t=0}^{n-1} \mathbb{1}\{X_t = x;\ X_{t+1} = x-1\}, \tag{7}$$

$$\xi^+(n,x) := \sum_{t=0}^{n-1} \mathbb{1}\{X_t = x;\ X_{t+1} = x+1\}, \tag{8}$$

which are respectively the local time of the RWRE in x and the number of left steps (resp. right steps) from site x at time n. Note that

$$\xi(n-1,x) = \xi^+(n,x) + \xi^-(n,x) \quad \text{and} \quad |\xi^-(n,x+1) - \xi^+(n,x)| \le 1.$$

Denote by R_n the range of the walk,

$$R_n = |\mathscr{R}_n|, \qquad \mathscr{R}_n = \big\{x > 0 : \xi(n-1,x) \ge 1\big\}, \tag{9}$$

with $|E|$ the cardinality of the set E. It is straightforward to compute the quenched and annealed likelihood of a nearest neighbour path $X_{[0,n]}$ of length n starting from 0

$$P_\omega(X_{[0,n]}) = \prod_{x\in\mathscr{R}_n} \omega_x^{\xi^+(n,x)}(1-\omega_x)^{\xi^-(n,x)},$$

and

$$\mathbf{P}^\theta(X_{[0,n]}) = \prod_{x\in\mathscr{R}_n} \int_0^1 a^{\xi^+(n,x)}(1-a)^{\xi^-(n,x)} \mathrm{d}\eta_\theta(a).$$

Then, the annealed log-likelihood function $\theta \mapsto \ell_n(\theta)$ is defined for all $\theta = (\mathbf{a},\mathbf{p}) \in \Theta$ as

$$\ell_n(\theta) = \log \mathbf{P}^\theta(X_{[0,n]}) = \sum_{x\in\mathscr{R}_n} \log\Big[\sum_{i=1}^{d} a_i^{\xi^+(n,x)}(1-a_i)^{\xi^-(n,x)} p_i\Big]. \tag{10}$$

Definition 2.1. *A Maximum Likelihood Estimator (MLE) $\widehat{\theta}_n$ of $\theta^\star$ is defined as a measurable choice*

$$\widehat{\theta}_n \in \operatorname*{Argmax}_{\theta\in\Theta} \ell_n(\theta). \tag{11}$$

We denote $\widehat{\mathbf{a}}_n$ and $\widehat{\mathbf{p}}_n$ the first and second projection of $\widehat{\theta}_n$.

Due to an analysis of the log-likelihood function provided by Lemma 2.2 below, we are lead to Definition 2.3 of a Maximum Pseudo-Likelihood Estimator (MPLE). To do so, some additional notations are required. Let

$$\nu(n,x) = \frac{\xi(n-1,x)}{n}, \quad \nu^+(n,x) = \frac{\xi^+(n,x)}{n}, \quad \text{and} \quad \nu^-(n,x) = \frac{\xi^-(n,x)}{n}.$$

Let $\Theta_{\mathbf{a}} = \{\mathbf{a} : \exists \mathbf{p}, (\mathbf{a},\mathbf{p}) \in \Theta\}$ be the first projection of the parameter space Θ. Introduce for $\mathbf{a}$ in $\Theta_{\mathbf{a}}$ and $\pi^+, \pi^- : \mathbb{Z} \to \mathbb{R}_+$ with $\sum_x [\pi^+(x) + \pi^-(x)] = 1$

$$L(\mathbf{a}, \pi^+, \pi^-) = \sum_{x\in\mathbb{Z}} \max_i \{\pi^+(x) \log a_i + \pi^-(x) \log(1-a_i)\}, \tag{12}$$

and

$$L_n(\mathbf{a}) = L\big(\mathbf{a}, \nu^+(n,\cdot), \nu^-(n,\cdot)\big), \tag{13}$$

where we set $\nu^+(n,x) = \nu^-(n,x) = 0$ for any non-positive integer x. Define the increasing sequence $\boldsymbol{\beta} = (\beta_i)_{0\le i\le d}$ depending on $\mathbf{a}$ as

$$\beta_0 = -\infty, \quad \beta_i = \log\Big(\frac{1-a_i}{1-a_{i+1}}\Big) \Big/ \log\Big(\frac{a_{i+1}}{a_i}\Big), \text{ for any } i \le d-1, \quad \text{and} \quad \beta_d = \infty. \tag{14}$$

For any x in $\mathscr{R}_n$, define the random integer $\hat{\imath}(\mathbf{a}, n, x)$ as

$$\hat{\imath}(\mathbf{a}, n, x) = i, \qquad \text{if} \quad \xi^+(n,x)/\xi^-(n,x) \in (\beta_{i-1}, \beta_i], \tag{15}$$

which is designed to satisfy

$$\hat{\imath}(\mathbf{a}, n, x) \in \operatorname{Argmax}\Big\{\xi^+(n,x) \log a_i + \xi^-(n,x) \log(1-a_i) : i \le d\Big\}, \tag{16}$$

when x is in $\mathscr{R}_n$. Finally, introduce $K_n(\theta)$ as

$$K_n(\theta) = \frac{1}{R_n} \sum_{x\in\mathscr{R}_n} \log p_{\hat{\imath}(\mathbf{a},n,x)} = \sum_{i=1}^{d} \frac{R_n(\mathbf{a}, i)}{R_n} \log p_i, \tag{17}$$

where $R_n(\mathbf{a}, i)$ is the random integer defined by

$$R_n(\mathbf{a}, i) = \sum_{x\in\mathscr{R}_n} \mathbb{1}\{\hat{\imath}(\mathbf{a}, n, x) = i\} = \sum_{x\in\mathscr{R}_n} \mathbb{1}\Big\{\beta_{i-1} < \frac{\xi^+(n,x)}{\xi^-(n,x)} \le \beta_i\Big\}. \tag{18}$$

Lemma 2.2. *We have*

$$\ell_n(\theta) = n \cdot L_n(\mathbf{a}) + R_n \cdot K_n(\theta) + r_n(\theta), \tag{19}$$

where

$$r_n(\theta) = \sum_{x\in\mathscr{R}_n} \log\left(1 + \sum_{i\neq\hat{\imath}(\mathbf{a},n,x)} \frac{p_i}{p_{\hat{\imath}(\mathbf{a},n,x)}} U_i(\mathbf{a}, n, x)^{\xi(n-1,x)}\right), \tag{20}$$

with

$$U_i(\mathbf{a}, n, x) = \Big(\frac{a_i}{a_{\hat{\imath}(\mathbf{a},n,x)}}\Big)^{\xi^+(n,x)/\xi(n-1,x)} \Big(\frac{1-a_i}{1-a_{\hat{\imath}(\mathbf{a},n,x)}}\Big)^{\xi^-(n,x)/\xi(n-1,x)}. \tag{21}$$

Furthermore, for any $\theta \in \Theta$,

$$\frac{1}{n}\Big(R_n \cdot K_n(\theta) + r_n(\theta)\Big) \xrightarrow[n\to\infty]{} 0, \quad \mathbf{P}^\star\text{-a.s.}, \tag{22}$$

as well as

$$\frac{r_n(\theta)}{\log^2 n} \xrightarrow[n\to\infty]{} 0, \quad \text{in } \mathbf{P}^\star\text{-probability.} \tag{23}$$

Hence, the first term in the RHS of (19) is of order n and depends only on $\mathbf{a}$, whereas the second term is of order $\log^2 n$ and depends on $\mathbf{a}$ and $\mathbf{p}$. The proofs of (22) and (23) are respectively provided in Sections 3.2 and 4.2. As claimed above, in view of Lemma 2.2, the following definition appears natural for an estimator of $\theta^\star$.

Definition 2.3. *A Maximum Pseudo-Likelihood Estimator (MPLE)* $\overline{\theta_n} = (\overline{\mathbf{a}_n}, \overline{\mathbf{p}_n})$ *of* θ *is a measurable choice of*

$$\begin{cases} \overline{\mathbf{a}_n} & \in \quad \operatorname*{Argmax}_{\mathbf{a}\in\Theta_{\mathbf{a}}} L_n(\mathbf{a}), \\ \overline{\mathbf{p}_n} & \in \quad \operatorname*{Argmax}_{\mathbf{p}} K_n(\overline{\mathbf{a}_n}, \mathbf{p}). \end{cases} \tag{24}$$

In the beginning of Section 4, we prove that

$$\overline{\mathbf{p}_n} = \left(\frac{R_n(\overline{\mathbf{a}_n}, i)}{R_n} : i = 1, \dots, d \right). \tag{25}$$

Now, we can state our consistency results.

Theorem 2.4. *Let Assumptions I and II hold. Both the ML estimator* $\widehat{\mathbf{a}}_n$ *and the MPL estimator* $\overline{\mathbf{a}_n}$ *converge in* $\mathbf{P}^\star$*-probability to the true parameter value* $\mathbf{a}^\star$.

Theorem 2.5. *Let Assumptions I and II hold. Both the ML estimator* $\widehat{\mathbf{p}}_n$ *and the MPL estimator* $\overline{\mathbf{p}_n}$ *converges in* $\mathbf{P}^\star$*-probability to the true parameter value* $\mathbf{p}^\star$.

We expect the speed of convergence for estimating $\mathbf{p}$ is much slower than the one for estimating $\mathbf{a}$. This is supported by our simulation experiment provided in Section 6 but we leave this question for further research. Section 3 is devoted to the proof of the consistency of $\overline{\mathbf{a}_n}$ and $\widehat{\mathbf{a}}_n$ whereas Section 4 is devoted to the proof of the consistency of $\widehat{\mathbf{p}}_n$ and $\overline{\mathbf{p}_n}$.

Concluding remarks. Let us start to describe a naive estimation based on recurrence. For all x in $\mathcal{R}_n$, we can estimate the environment at this point by

$$\hat{\omega}_x^{(n)} = \frac{\xi^+(n,x)}{\xi(n-1,x)}.$$

By recurrence, $\hat{\omega}_x^{(n)}$ converges to ω_x, $\mathbf{P}^\star$-a.s. With some extra work, it can be shown that,

$$\frac{1}{R_n} \sum_{x\in\mathcal{R}_n} \delta_{\hat{\omega}_x^{(n)}} \xrightarrow[n\to\infty]{} \eta_{\theta^\star} \quad \mathbf{P}^\star\text{-a.s.},$$

where $\mathcal{R}_n$ is defined by (9), and this leads to estimators of the parameters. This empirical estimator is then consistent. However, it gives equal weight to all visited sites x in $\mathcal{R}_n$, without any notice of the number of visits there, which is certainly far from optimal. This is essentially how Adelman and Enriquez (2004) devise their estimators, and the simulations in Section 6 indicate they perform poorly at some extend. On the other hand, Andreoletti (2011) proposes estimators based on local time of the walk, which indeed take care of this flaw, but which are adhoc in essence and difficult to use in an optimal manner. In contrast, our estimate relies on first principles - maximum likelihood - and uses the full information gathered by the walk all through.

3 Proof of consistency for the MLE and MPLE of the ordered support

In the present section, we first recall the weak convergence result established by Gantert et al. for the empirical distributions of the RWRE. Then, we identify the limit of our criterion as a functional of **a** only, and provide some information on it. Using its regularity properties, we can adapt the proof of consistency for M-estimators to our context.

3.1 Potential and infinite valley

The environment ω in which the random walk evolves is visualized as a potential landscape V where $V = \{V(x) : x \in \mathbb{Z}\}$ is defined by

$$V(x) = \begin{cases} \sum_{y=1}^{x} \log \rho_y & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\sum_{y=x+1}^{0} \log \rho_y & \text{if } x < 0, \end{cases} \tag{26}$$

with $\rho_y = (1-\omega_y)/\omega_y$. An example of a realization of V can be seen on Figure 1.

Setting $C(x, x+1) = \exp[-V(x)]$, for any integer x, the Markov chain X is an electric network in the sense of Doyle and Snell (1984) or Levin et al. (2009), where $C(x, x+1)$ is the conductance of the (unoriented) bond $(x, x+1)$. In particular, the measure μ defined as

$$\mu(x) = \exp[-V(x-1)] + \exp[-V(x)], \quad x \in \mathbb{Z}, \tag{27}$$

is a reversible and invariant measure for the Markov chain X.

Now, define the right border c_n of the "valley" with depth $\log n + (\log n)^{1/2}$ as

$$c_n = \min\left\{x \geq 0 : V(x) - \min_{0\leq y\leq x} V(y) \geq \log n + (\log n)^{1/2}\right\}, \tag{28}$$

and the bottom b_n of the "valley" as

$$b_n = \min\left\{x \geq 0 : V(x) = \min_{0\leq y\leq c_n} V(y)\right\}.$$

On Figure 1, one can see a representation of b_n and c_n. We are interested in the shape of the "valley" $(0, b_n, c_n)$ when n tends to infinity and we recall the concept of infinite valley introduced by Golosov (1984).

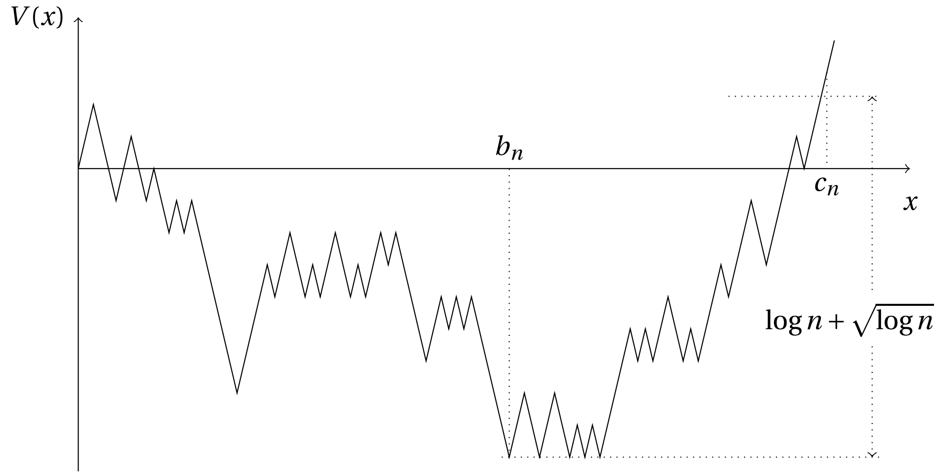

Figure 1: Example of potential derived from a random environment distributed as in Example I with parameter $a = 0.3$. Simulation with $n = 1000$.

Let $\widetilde{V} = \{\widetilde{V}(x) : x \in \mathbb{Z}\}$ be a collection of random variables distributed as V conditioned to stay positive for any negative x, and non-negative for any non negative x. For simplicity, we assume that without loss of generality, $\widetilde{V}$ is also realised under $\mathbb{P}^\star$. For each realization of $\widetilde{V}$, consider the corresponding Markov chain on $\mathbb{Z}$, which is an electrical network with conductances $\widetilde{C}(x, x+1) = \exp[-\widetilde{V}(x)]$. As usual, the measure $\widetilde{\mu}$ defined as $\widetilde{\mu}(x) = \widetilde{C}(x-1, x) + \widetilde{C}(x, x+1)$, $x \in \mathbb{Z}$, is a (reversible) and invariant measure for the Markov chain X. Furthermore, the measure $\widetilde{\mu}$ can be normalized to get a reversible probability measure ν, defined by

$$\nu(x) = \frac{\exp[-\widetilde{V}(x-1)] + \exp[-\widetilde{V}(x)]}{2\sum_{z\in\mathbb{Z}} \exp[-\widetilde{V}(z)]}. \tag{29}$$

Note that $\nu(x)$ can be written as the sum of $\nu^+(x)$ and $\nu^-(x)$, where for any $x \in \mathbb{Z}$

$$\nu^+(x) = \frac{\exp[-\widetilde{V}(x)]}{2\sum_{z\in\mathbb{Z}} \exp[-\widetilde{V}(z)]} \quad \text{and} \quad \nu^-(x) = \frac{\exp[-\widetilde{V}(x-1)]}{2\sum_{z\in\mathbb{Z}} \exp[-\widetilde{V}(z)]}. \tag{30}$$

We have $\nu^+(x) = \nu^-(x+1)$. Define $\widetilde{\omega}(x) \in (0,1)$, for any $x \in \mathbb{Z}$, as

$$\widetilde{\omega}(x) = \frac{\nu^+(x)}{\nu(x)} = \frac{1}{1 + \exp[\widetilde{V}(x) - \widetilde{V}(x-1)]}. \tag{31}$$

Remark 3.1. *Noting that the possible values of* $\widetilde{V}(x) - \tilde{V}(x-1)$ *are those of* $V(x) - V(x-1)$ *for any integer* x*, we deduce that under* $\mathbb{P}^\star$, $\widetilde{\omega}(x)$ *is equal to one of the coordinates of* $\mathbf{a}^\star$.

Gantert et al. (2010) showed that the empirical distribution of the RWRE, suitably centered at b_n converges to the stationary distribution of a random walk in an infinite valley. More precisely, let

$$\nu_n(x) = \nu(n, x + b_n), \quad \nu_n^+(x) = \nu^+(n, x + b_n), \quad \text{and} \quad \nu_n^-(x) = \nu^-(n, x + b_n).$$

Theorem 3.2 (Gantert et al. (2010))**.** *Let Assumptions I and II hold. The distributions of* $\{\nu_n(x) : x \in \mathbb{Z}\}$ *converge weakly to the distribution of* ν *(as probability measures on* ℓ^1 *equipped with the* ℓ_1*-norm). As a consequence, for each strongly continuous functional* $f : \ell^1 \to \mathbb{R}$ *which is shift invariant, we have*

$$f\big(\{\nu(n, x) : x \in \mathbb{Z}\}\big) \xrightarrow[n\to\infty]{law} f\big(\{\nu(x) : x \in \mathbb{Z}\}\big).$$

In Gantert et al. (2010) is mentioned that the result still holds with obvious extensions for the non-reflected case, i.e., of a RWRE on $\mathbb{Z}$. An inspection of the proof of Gantert et al. immediatly yields:

Corollary 3.3. *Let Assumptions I and II hold. The distributions of*

$$\{(\nu_n^+(x), \nu_n^-(x)) : x \in \mathbb{Z}\}$$

converge weakly to the distribution of $\{(\nu^+(x), \nu^-(x)) : x \in \mathbb{Z}\}$*. As a consequence, for each strongly continuous functional* $f : \ell^1 \times \ell^1 \to \mathbb{R}$ *which is shift invariant, we have*

$$f\big(\{(\nu^+(n, x), \nu^-(n, x)) : x \in \mathbb{Z}\}\big) \xrightarrow[n\to\infty]{law} f\big(\{(\nu^+(x), \nu^-(x)) : x \in \mathbb{Z}\}\big).$$

3.2 Identification and properties of the criterion limit

First, we start with

Proof of (22) *in Lemma 2.2.* Clearly,

$$\log \varepsilon_0 \times \max_{0\le t\le n} X_t \le \ell_n(\theta) - \sum_{x\in\mathcal{R}_n} \max_{i\le d} \Big\{\xi^+(n, x)\log a_i + \xi^-(n, x)\log(1 - a_i)\Big\} \le 0, \quad (32)$$

with ε_0 from (5), and from (12) and (13)

$$\sum_{x\in\mathcal{R}_n} \max_{i\le d} \Big\{\xi^+(n, x)\log a_i + \xi^-(n, x)\log(1 - a_i)\Big\} = n \cdot L_n(\mathbf{a}).$$

Since $\mathbf{E}^\star(\rho_0) \ge \exp[\mathbf{E}^\star \log(\rho_0)] = 1$, case (c') in Theorem 2.1.9 in Zeitouni (2004) applies, and X_n/n converges to 0, $\mathbf{P}^\star$-a.s. Hence, $\frac{1}{n}\max_{0\le t\le n} X_t$ converges to 0 $\mathbf{P}^\star$-a.s and the claim is proved. ☐

For any $\mathbf{a} \in \Theta_{\mathbf{a}}$, denote $L_\infty(\mathbf{a})$

$$L_\infty(\mathbf{a}) = L(\mathbf{a}, \nu^+, \nu^-) = \sum_{x\in\mathbb{Z}} \max_i \big\{\nu^+(x)\log a_i + \nu^-(x)\log(1 - a_i)\big\}, \quad (33)$$

where ν^+ and ν^- are defined by (30). Anticipating Lemma 3.5 below, Corollary 3.3 and (22) immediatly yields

$$L_n(\mathbf{a}) \xrightarrow[n\to\infty]{\text{law}} L_\infty(\mathbf{a}).$$

Therefore, we provide some useful information on the functional $L_\infty(\cdot)$. To do so, some additional notations are required.

Definition 3.4 (Boltzmann entropy function and Kullback-Leibler distance)**.** *Define the Boltzmann entropy function* $H(\cdot)$ *on* $(0,1)$ *as*

$$H(q) = -[q\log q + (1-q)\log(1-q)] \geq 0,$$

the Kullback-Leibler distance $d_{\mathrm{KL}}(\cdot|\cdot)$ *on* $(0,1)\times(0,1)$ *as*

$$d_{\mathrm{KL}}(q|q') = q\log\frac{q}{q'} + (1-q)\log\frac{1-q}{1-q'} \geq 0,$$

and their multinomial extensions on probability vectors $\mathbf{q}$ *and* $\mathbf{q}'$

$$H(\mathbf{q}) = -\sum_i q_i \log q_i \geq 0, \quad d_{\mathrm{KL}}(\mathbf{q}|\mathbf{q}') = \sum_i q_i \log\frac{q_i}{q_i'} \geq 0.$$

With ν and $\widetilde{\omega}$ defined by (29) and (31), we have for all $\mathbf{a}$ in Θ_a the identity

$$L_\infty(\mathbf{a}) = -\sum_{x\in\mathbb{Z}} \nu(x) H\big[\widetilde{\omega}(x)\big] - \sum_{x\in\mathbb{Z}} \nu(x) \min_i \big(d_{\mathrm{KL}}\big[\widetilde{\omega}(x)|a_i\big]\big). \tag{34}$$

From Remark 3.1, we have $\min_i\big(d_{\mathrm{KL}}\big[\widetilde{\omega}(x)|a_i^\star\big]\big) = 0$, and using the fact that $d_{\mathrm{KL}}(q|q') > 0$ for $q \neq q'$, we deduce that

$$L_\infty(\mathbf{a}) < L_\infty(\mathbf{a}^\star) = -\sum_{x\in\mathbb{Z}} \nu(x) H\big[\widetilde{\omega}(x)\big], \qquad \mathbf{a} \neq \mathbf{a}^\star. \tag{35}$$

More precisely, a useful bound is

$$L_\infty(\mathbf{a}) \leq L_\infty(\mathbf{a}^\star) - \min_k \big(\nu\{x\in\mathbb{Z} : \widetilde{\omega}(x) = a_k^\star\}\big) \times \sum_{j=1}^d \min_i \big(d_{\mathrm{KL}}\big[a_j^\star|a_i\big]\big), \tag{36}$$

where the sum is deterministic and positive for $\mathbf{a} \neq \mathbf{a}^\star$, though its factor ($\min_j(\cdot)$) is a.s. positive and does not depend on $\mathbf{a}$.

Lemma 3.5. *The function L defined by* (12) *is Lipschitz continuous:*

$$|L(\mathbf{a},\pi^+,\pi^-) - L(\mathbf{a},\mu^+,\mu^-)| \leq |\log\varepsilon_0| \cdot \big(\|\pi^+ - \mu^+\|_1 + \|\pi^- - \mu^-\|_1\big), \tag{37}$$

$$|L(\mathbf{a},\pi^+,\pi^-) - L(\mathbf{a}',\pi^+,\pi^-)| \leq 2\varepsilon_0^{-1}\|\mathbf{a}-\mathbf{a}'\|_2. \tag{38}$$

Proof of Lemma 3.5. Recall that $\mathbb{R}^d \ni u \mapsto \max_i u_i \in \mathbb{R}$ is 1-lipschitz continuous for the norm $\|\cdot\|_\infty$. Moreover, the mapping

$$f_i : (\mathbf{a}, v^+, v^-) \mapsto v^+ \log a_i + v^- \log(1-a_i)$$

is $|\log \varepsilon_0|$-lipschitz continuous in v^+, resp. in v^-, with ε_0 from (5). By composition, $(v^+, v^-) \mapsto \max_i f_i(\mathbf{a}, v^+, v^-)$ is $|\log \varepsilon_0|$-lipschitz continuous in the $\|\cdot\|_1$ norm, and (37) follows by summing over x. By Cauchy-Schwarz,

$$\begin{aligned}|f_i(\mathbf{a}, v^+, v^-) - f_i(\mathbf{a}', v^+, v^-)| &= \left| \int_0^1 \partial_a f_i(\mathbf{a}' + t(\mathbf{a}-\mathbf{a}'), v^+, v^-) \cdot (\mathbf{a}-\mathbf{a}') \mathrm{d}t \right| \\ &\le \|\mathbf{a}-\mathbf{a}'\|_2 \times \sup_{\mathbf{a}"} \|\partial_a f_i(\mathbf{a}", v^+, v^-)\|_2,\end{aligned}$$

where derivative can be bounded using

$$\|\partial_a f_i(\mathbf{a}, v^+, v^-)\|_2 = \left(a_i^{-2}(v^+)^2 + (1-a_i)^{-2}(v^-)^2\right)^{1/2} \le 2\varepsilon_0^{-1}(v^+ + v^-).$$

Taking the maximum over i and summing over x, this yields (38). □

3.3 Proof of Theorem 2.4

Recall that $\overline{\mathbf{a}_n}$ is defined by (24). Fix some $\varepsilon > 0$. We prove that for all $\varepsilon_1 > 0$, there exists an integer n_1 such that for all $n \ge n_1$,

$$\mathbf{P}^\star\left(\overline{\mathbf{a}_n} \in \mathscr{B}(\mathbf{a}^\star, \varepsilon)\right) \ge 1 - 2\varepsilon_1, \tag{39}$$

where $\mathscr{B}(\mathbf{a}^\star, \varepsilon)$ is the open ball of radius ε centered at $\mathbf{a}^\star$.

Under $\mathbb{P}^\star$, the sequence $(g[\widetilde{\omega}(x)] : x \in \mathbb{Z})$ with $g(u) = \log(u^{-1}-1)$ is the sequence of the jumps of a random walk V conditioned to stay positive. The jump of unconditionned V takes the value $g(a_j^\star)$ with positive probability $p_j^\star$. Hence it is not difficult to check that for all j in $\{1, \dots, d\}$, we can find $\mathbb{P}^\star$-a.s. some random x such that $\widetilde{\omega}(x) = a_j^\star$. Since ν is $\mathbb{P}^\star$-a.s. supported by the whole $\mathbb{Z}$, we have

$$\mathbb{P}^\star\left(\min_j \left\{\nu\{x : \widetilde{\omega}(x) = a_j^\star\}\right\} > 0\right) = 1.$$

By continuity, for arbitrary $\varepsilon_1 > 0$ we can fix $\delta_1 > 0$ such that

$$\mathbb{P}^\star\left(\min_j \left\{\nu\{x : \widetilde{\omega}(x) = a_j^\star\}\right\} \ge \delta_1\right] \ge 1 - \varepsilon_1. \tag{40}$$

By compactness of $K = \Theta_{\mathbf{a}} \setminus \mathscr{B}(\mathbf{a}^\star, \varepsilon)$,

$$\kappa = \inf\left\{\sum_{j=1}^d \min_i \left\{d_{\mathrm{KL}}\left[a_j^\star | a_i\right]\right\} : \mathbf{a} \in K\right\} > 0.$$

By (38),

$$\mathbf{a}' \in \mathscr{B}\big(\mathbf{a}, \kappa\delta_1\varepsilon_0/6\big) \Rightarrow |L_\infty(\mathbf{a}) - L_\infty(\mathbf{a}')| \le \kappa\delta_1/3. \tag{41}$$

Since K is totally bounded, we can select a finite covering $\cup_{k=1}^{k_0}\mathscr{B}(\mathbf{a}^k, \kappa\delta_1\varepsilon_0/6)$ of K with balls of that radius. From (37) we can apply Corollary 3.3, and we have

$$\big(L_n(\mathbf{a}^k) : 0 \le k \le k_0\big) \xrightarrow[n\to\infty]{\text{law}} \big(L_\infty(\mathbf{a}^k) : 0 \le k \le k_0\big),$$

where we have set $\mathbf{a}^0 = \mathbf{a}^\star$. Also,

$$\big(L_n(\mathbf{a}^\star) - L_n(\mathbf{a}^k)\big)_{k\le k_0} \xrightarrow[n\to\infty]{\text{law}} \big(L_\infty(\mathbf{a}^\star) - L_\infty(\mathbf{a}^k)\big)_{k\le k_0},$$

where the limits are simultaneously larger than $\kappa\delta_1$ on a set of large probability (larger than $1-\varepsilon_1$) by (36) and (40). Then, we find n_1 such that for $n \ge n_1$,

$$\mathbf{P}^\star\big(L_n(\mathbf{a}^\star) - L_n(\mathbf{a}^k) \ge 2\kappa\delta_1/3,\, 1 \le k \le k_0\big) \ge 1 - 2\varepsilon_1.$$

Taking (41) into account, we obtain that, for all $n \ge n_1$,

$$\mathbf{P}^\star\big(L_n(\mathbf{a}^\star) - L_n(\mathbf{a}) \ge \kappa\delta_1/3, \mathbf{a} \in K\big) \ge 1 - 2\varepsilon_1.$$

This implies (39) for all $n \ge n_1$. Hence $\overline{\mathbf{a}_n} \to \mathbf{a}^\star$ in $\mathbf{P}^\star$-probability. Now, we turn to the convergence of $\widehat{\mathbf{a}}_n$ to $\mathbf{a}^\star$ in $\mathbf{P}^\star$-probability. According to (32), we have

$$L_n(\mathbf{a}) - u_n \le \frac{\ell_n(\theta)}{n} \le L_n(\mathbf{a}),$$

where u_n is non-negative and converges $\mathbf{P}^\star$-a.s. to 0 independently from θ. Choosing n_2 such that for all $n \ge n_2$

$$\mathbf{P}^\star\big(u_n \ge \kappa\delta_1/3\big) \le \varepsilon_1,$$

achieves the proof of the convergence $\widehat{\mathbf{a}}_n \to \mathbf{a}^\star$.

4 Proof of consistency of estimates for the probability vector

First, note that $K_n(\theta)$ can be rewritten

$$K_n(\theta) = -H\Big(\frac{R_n(\mathbf{a},\cdot)}{R_n}\Big) - d_{\mathrm{KL}}\Big(\frac{R_n(\mathbf{a},\cdot)}{R_n}\Big|\mathbf{p}\Big). \tag{42}$$

Therefore,

$$K_n(\overline{\mathbf{a}_n}, \mathbf{p}) \le -H\Big(\frac{R_n(\overline{\mathbf{a}_n},\cdot)}{R_n}\Big),$$

with equality if and only if $\mathbf{p} = \Big(\frac{R_n(\overline{\mathbf{a}_n},i)}{R_n} : i = 1,\dots,d\Big)$. Hence, we can use the alternative definition of $\overline{\mathbf{p}_n}$ given by (25) to prove Theorem 2.5.

4.1 Proof of Theorem 2.5: convergence of $\overline{\mathbf{p}_n}$

For $0 < \delta < 1$ let

$$\mathscr{G}_n^\delta = \{x \le b_n : \max_{z \in [x, b_n]} V(z) - V(x) \ge \delta \log n\}, \tag{43}$$

$$\mathscr{D}_n^\delta = \{x > b_n : \max_{z \in [b_n, x]} V(z) - V(b_n) \le (1-\delta) \log n\}, \tag{44}$$

$$\mathscr{R}_n^\delta = \mathscr{G}_n^\delta \cup \mathscr{D}_n^\delta. \tag{45}$$

Note that $\mathscr{D}_n^\delta$ is an interval, $\mathscr{D}_n^\delta =]b_n, c_n^\delta]$ with $c_n^\delta \le c_n$.

Denote $\boldsymbol{\beta}^\star$ and $\overline{\boldsymbol{\beta}}$ the sequences defined by (14) replacing $\mathbf{a}$ by $\mathbf{a}^\star$ and $\overline{\mathbf{a}_n}$ respectively. Noting that for any i in $\{1, \dots, d-1\}$, $d_{\mathrm{KL}}(a_i^\star | a_{i+1}^\star) > 0$, we can choose $\varepsilon' > 0$ small enough such that for any $i \in \{1, \dots, d\}$

$$\beta_{i-1}^\star \le \frac{a_i^\star}{1 - a_i^\star} - 2\varepsilon' < \frac{a_i^\star}{1 - a_i^\star} + 2\varepsilon' \le \beta_i^\star. \tag{46}$$

Let $\varepsilon > 0$. Define the events $A_n^\delta(\varepsilon')$, $B_n(\varepsilon')$ and $C_n^\delta(\varepsilon, i)$ by

$$A_n^\delta(\varepsilon') = \left\{ \forall x \in \mathscr{R}_n^\delta : \left| \frac{\xi + (n,x)}{\xi^-(n,x)} - \frac{1}{\rho_x} \right| \le \varepsilon' \right\}, \tag{47}$$

$$B_n(\varepsilon') = \left\{ \forall i \in \{1, \dots, d\} : |\overline{\beta_i} - \beta_i^\star| \le \varepsilon' \right\}, \tag{48}$$

$$C_n^\delta(\varepsilon, i) = \left\{ \left| \frac{1}{|\mathscr{R}_n \cap \mathscr{R}_n^\delta|} \sum_{x \in \mathscr{R}_n \cap \mathscr{R}_n^\delta} \mathbb{1}\{\omega_x = a_i^\star\} - p_i^\star \right| > \frac{\varepsilon}{4} \right\}, \tag{49}$$

Denote $^\complement A_n^\delta(\varepsilon')$ and $^\complement B_n(\varepsilon')$ the respective complementary events of $A_n^\delta(\varepsilon')$ and $B_n(\varepsilon')$, and define the quantities $\phi_n^\delta(\varepsilon')$, $\psi_n^\delta(\varepsilon)$ and $\mu_n^\delta(\varepsilon)$ by

$$\phi_n^\delta(\varepsilon') = \mathbf{P}^\star\left({}^\complement A_n^\delta(\varepsilon')\right) + \mathbf{P}^\star\left({}^\complement B_n(\varepsilon')\right)$$

$$\psi_n^\delta(\varepsilon) = \mathbf{P}^\star\left(\frac{|\mathscr{R}_n \setminus \mathscr{R}_n^\delta|}{R_n} > \frac{\varepsilon}{2}\right),$$

$$\mu_n^\delta(\varepsilon, i) = \mathbf{P}^\star\left(C_n^\delta(\varepsilon, i)\right).$$

Then, we have

$$\mathbf{P}^\star\left(|\overline{\mathbf{p}_n}(i) - p_i^\star| > \varepsilon\right) \le \phi_n^\delta(\varepsilon') + \psi_n^\delta(\varepsilon) + \psi_n^\delta(\varepsilon/2) + \mu_n^\delta(\varepsilon, i). \tag{50}$$

Indeed, it is clear that

$$\mathbf{P}^\star\left(|\overline{\mathbf{p}_n}(i) - p_i^\star| > \varepsilon\right) \le \mathbf{P}^\star\left(|\overline{\mathbf{p}_n}(i) - p_i^\star| > \varepsilon, A_n^\delta(\varepsilon'), B_n(\varepsilon')\right) + \phi_n^\delta(\varepsilon'), \tag{51}$$

and writing the set $\mathscr{R}_n$ as the union of the two disjoint sets $\left(\mathscr{R}_n \cap \mathscr{R}_n^\delta\right)$ and $\left(\mathscr{R}_n \setminus \mathscr{R}_n^\delta\right)$ in (18) and using (25) yields

$$\mathbf{P}^\star\left(|\overline{\mathbf{p}_n}(i) - p_i^\star| > \varepsilon, A_n^\delta(\varepsilon'), B_n(\varepsilon')\right) \le \mathbf{P}^\star\left(A_n^\delta(\varepsilon'), B_n(\varepsilon'), D_n^\delta(\varepsilon, i)\right) + \psi_n^\delta(\varepsilon), \tag{52}$$

with

$$D_n^\delta(\varepsilon,i)=\left\{\Big|\frac{1}{R_n}\sum_{x\in\mathscr{R}_n\cap\mathscr{R}_n^\delta}\mathbb{1}\left\{\overline{\beta_i}<\frac{\xi^+(n,x)}{\xi(n,x)}\le\overline{\beta_{i+1}}\right\}-p_i^\star\Big|>\frac{\varepsilon}{2}\right\}.$$

Using our choice of ε' to satisfy (46), we have for all i in $\{1,\ldots,d\}$

$$\left\{\overline{\beta_i}<\frac{\xi^+(n,x)}{\xi(n,x)}\le\overline{\beta_{i+1}},A_n^\delta(\varepsilon'),B_n(\varepsilon')\right\}=\left\{\omega_x=a_i^\star,A_n^\delta(\varepsilon'),B_n(\varepsilon')\right\}.$$

Hence,

$$\mathbf{P}^\star\Big(A_n^\delta(\varepsilon'),B_n(\varepsilon'),D_n^\delta(\varepsilon,i)\Big)\le\mathbf{P}^\star\Big(\Big|\frac{1}{R_n}\sum_{x\in\mathscr{R}_n\cap\mathscr{R}_n^\delta}\mathbb{1}\{\omega_x=a_i^\star\}-p_i^\star\Big|>\frac{\varepsilon}{2}\Big),\qquad(53)$$

and clearly,

$$\mathbf{P}^\star\Big(\Big|\frac{1}{R_n}\sum_{x\in\mathscr{R}_n\cap\mathscr{R}_n^\delta}\mathbb{1}\{\omega_x=a_i^\star\}-p_i^\star\Big|>\frac{\varepsilon}{2}\Big)\le\psi_n^\delta(\varepsilon/2)+\mu_n^\delta(\varepsilon,i).\qquad(54)$$

Combining (51), (52), (53) and (54) yields (50).

Anticipating some lemmas which are proved independently in the next section, we conclude the proof. From Lemma 4.1, we have

$$\lim_{\delta\to0}\limsup_{n\to\infty}\psi_n^\delta(\varepsilon)=0.$$

By the law of large numbers,

$$\lim_{n\to\infty}\mu_n^\delta(\varepsilon)=0.$$

In view of (50), to conclude the proof of Theorem 2.5 it suffices to prove that, for all $\delta>0$, $\phi_n^\delta(\varepsilon)$ vanishes as $n\to\infty$. On the one hand, $\overline{\mathbf{a}_n}$ converges to $\mathbf{a}^\star$ in $\mathbf{P}^\star$-probability, and thus

$$\mathbf{P}^\star\big(\complement B_n(\varepsilon')\big)\longrightarrow0.$$

On the other hand, by Lemma 4.2,

$$\mathbf{P}^\star\big(\complement A_n^\delta(\varepsilon')\big)\longrightarrow0.$$

This concludes the proof of the convergence in $\mathbf{P}^\star$-probability of $\overline{\mathbf{p}_n}$ to $\mathbf{p}^\star$. □

4.2 Proof of (23) in Lemma 2.2

For all $x\in\mathscr{R}_n$, one can write

$$\log U_i(\mathbf{a},n,x)=\Big[\frac{\xi^+(n,x)}{\xi(n-1,x)}-a_{\hat\imath(\mathbf{a},n,x)}\Big]\Big[\log\frac{a_i}{a_{\hat\imath(\mathbf{a},n,x)}}-\log\frac{1-a_i}{1-a_{\hat\imath(\mathbf{a},n,x)}}\Big]-d_{\mathrm{KL}}\big(a_{\hat\imath(\mathbf{a},n,x)}|a_i\big).\qquad(55)$$

Let c_0 be the quantity defined as

$$c_0 = \inf_{\mathbf{a}\in\Theta_{\mathbf{a}}} \inf_{i\neq j} d_{\mathrm{KL}}(a_i|a_j) > 0. \tag{56}$$

where the strict inequality comes from (5) and the continuity of $d_{\mathrm{KL}}(\cdot|\cdot)$. Recall ε_0 in (5) and ε' in (46), and let ε'' be the quantity defined as

$$\varepsilon'' = \min\left\{\varepsilon', \frac{c_0}{4}\left(\log\frac{1-\varepsilon_0}{\varepsilon_0}\right)^{-1}\right\}. \tag{57}$$

Let $\mathcal{V}(\varepsilon'') \subset \Theta_{\mathbf{a}}$ be the neighborhood of $\mathbf{a}^\star$ defined as

$$\mathcal{V}(\varepsilon'') = \left\{\mathbf{a}\in\Theta_{\mathbf{a}} : \max\left\{\|\mathbf{a}-\mathbf{a}^\star\|, \|\boldsymbol{\beta}-\boldsymbol{\beta}^\star\|\right\} \leq \varepsilon''\right\}. \tag{58}$$

Fix $0<\delta<1$. Let $\mathbf{a}$ be in $\mathcal{V}(\varepsilon'')$ and assume that $A_n^\delta(\varepsilon'')$ defined by (47) occurs, then for all $x\in\mathcal{R}_n^\delta$ and any $i\neq\hat{\imath}(\mathbf{a},n,x)$

$$\log U_i(\mathbf{a},n,x) \leq -\frac{c_0}{2}. \tag{59}$$

Indeed, using (58) and the fact that $A_n^\delta(\varepsilon'')$ occurs with $\varepsilon''\leq\varepsilon'$, we have for all $x\in\mathcal{R}_n^\delta$

$$\hat{\imath}(\mathbf{a},n,x) = \hat{\imath}(\mathbf{a}^\star,n,x) \quad\text{and}\quad \omega_x = a^\star_{\hat{\imath}(\mathbf{a},n,x)} = a^\star_{\hat{\imath}(\mathbf{a}^\star,n,x)}. \tag{60}$$

Then, if $i\neq\hat{\imath}(\mathbf{a},n,x)$, using (55), (56), (60), and the fact that

$$\left|\log\frac{a_i}{a_{\hat{\imath}(\mathbf{a},n,x)}} - \log\frac{1-a_i}{1-a_{\hat{\imath}(\mathbf{a},n,x)}}\right| \leq 2\log\frac{1-\varepsilon_0}{\varepsilon_0},$$

yield

$$\log U_i(\mathbf{a},n,x) \leq 2\log\frac{1-\varepsilon_0}{\varepsilon_0}\left(\left|h\Big(\frac{\xi^+(n,x)}{\xi^-(n,x)}\Big) - h\Big(\frac{1}{\rho_x}\Big)\right| + \|\mathbf{a}-\mathbf{a}^\star\|\right) - c_0$$

with

$$h(u) = \frac{u}{1+u}.$$

Using the fact that $0\leq h'(u)\leq 1$, for any $u\geq 0$, and the fact that $A_n^\delta(\varepsilon'')$ occurs, we have

$$\log U_i(\mathbf{a},n,x) \leq 2\log\frac{1-\varepsilon_0}{\varepsilon_0}\left(\varepsilon'' + \|\mathbf{a}-\mathbf{a}^\star\|\right) - c_0.$$

Using our choice of $\mathbf{a}$ and ε'' achieves the proof of (59). Now, from (59), we deduce that if $A_n^\delta(\varepsilon'')$ occurs,

$$0 \leq r_n(\theta) \leq \left|\mathcal{R}_n\setminus\mathcal{R}_n^\delta\right| \cdot \log\left(1+\frac{d-1}{\varepsilon_0}\right) + \sum_{x\in\mathcal{R}_n^\delta}\log\left(1+\frac{d-1}{\varepsilon_0}\mathrm{e}^{-\xi(n-1,x)c_0/2}\right). \tag{61}$$

Assume that the event E_n^δ defined by

$$E_n^\delta = \left\{\forall x\in\mathcal{R}_n^\delta : \xi(n,x)\geq n^{\delta/2}\right\}, \tag{62}$$

occurs. Then

$$0 \leq r_n(\theta) \leq \frac{d-1}{\varepsilon_0}\left(\left|\mathcal{R}_n\setminus\mathcal{R}_n^\delta\right| + |\mathcal{R}_n^\delta|\mathrm{e}^{-n^{\delta/2}c_0/2}\right). \tag{63}$$

From Lemma 4.1, 4.2 and 4.3, we conclude that (23) occurs. □

4.3 Convergence of $\widehat{\mathbf{p}}_n$

From the definitions of $\widehat{\theta}_n$ and $\overline{\mathbf{a}_n}$ respectively given by (11) and (24), we have

$$\ell_n(\widehat{\theta}_n) - \ell_n(\overline{\theta_n}) \geq 0 \quad \text{and} \quad L_n(\widehat{\mathbf{a}}_n) - L_n(\overline{\mathbf{a}_n}) \leq 0. \tag{64}$$

Recalling (19), (64) implies

$$K_n(\widehat{\theta}_n) - K_n(\overline{\theta_n}) + \frac{1}{R_n}[r_n(\widehat{\theta}_n) - r_n(\overline{\theta_n})] \geq 0. \tag{65}$$

From Theorem 2.4, when n is large enough, both $\widehat{\mathbf{a}}_n$ and $\overline{\mathbf{a}_n}$ belong to $\mathcal{V}(\varepsilon'')$ introduced in (58) with large probability. Assuming that the event $A_n^\delta(\varepsilon'')$ defined by (47) occurs, we have

$$\hat{\imath}(\overline{\mathbf{a}_n}, n, x) = \hat{\imath}(\widehat{\mathbf{a}}_n, n, x) = \hat{\imath}(\mathbf{a}^\star, n, x), \quad \text{for all } x \in \mathcal{R}_n^\delta,$$

and

$$K_n(\widehat{\theta}_n) \leq \log\Big(\frac{1-\varepsilon_0}{\varepsilon_0}\Big)\frac{|\mathcal{R}_n \setminus \mathcal{R}_n^\delta|}{R_n} + \frac{1}{R_n}\sum_{x\in\mathcal{R}_n} \log \widehat{p}_{\hat{\imath}(\mathbf{a}^\star,n,x)},$$

as well as

$$K_n(\overline{\theta_n}) \geq \log\Big(\frac{\varepsilon_0}{1-\varepsilon_0}\Big)\frac{|\mathcal{R}_n \setminus \mathcal{R}_n^\delta|}{R_n} + \frac{1}{R_n}\sum_{x\in\mathcal{R}_n} \log \bar{p}_{\hat{\imath}(\mathbf{a}^\star,n,x)},$$

which holds for large n when $\overline{\mathbf{p}_n}$ is in $[\varepsilon_0, 1-\varepsilon_0]$, and finally

$$K_n(\widehat{\theta}_n) - K_n(\overline{\theta_n}) \leq 2\log\Big(\frac{1-\varepsilon_0}{\varepsilon_0}\Big)\frac{|\mathcal{R}_n \setminus \mathcal{R}_n^\delta|}{R_n} - d_{\mathrm{KL}}(\overline{\mathbf{p}_n}|\widehat{\mathbf{p}}_n). \tag{66}$$

Assuming furthermore that E_n^δ defined by (62) occurs, then (63) occurs. All in all, combining (63), (65) and (66) yields the existence of a positive constant C, depending on ε_0 and d only, such that

$$d_{\mathrm{KL}}(\overline{\mathbf{p}_n}|\widehat{\mathbf{p}}_n) \leq C\Big(\frac{|\mathcal{R}_n \setminus \mathcal{R}_n^\delta|}{R_n} + \frac{|\mathcal{R}_n^\delta|}{R_n}\mathrm{e}^{-n^{\delta/2}c_0/2}\Big),$$

with large probability. From the fact that $\overline{\mathbf{p}_n}$ converges in $\mathbf{P}^\star$-probability to $\mathbf{p}^\star$, the continuity of $d_{\mathrm{KL}}(\cdot|\cdot)$, Lemma 4.1, 4.2 and 4.3, we conclude that $\widehat{\mathbf{p}}_n$ converges to $\mathbf{p}^\star$ in $\mathbf{P}^\star$-probability. □

4.4 Intermediate lemmas

Lemma 4.1. *There exists a random variable $Z(\delta) \geq 0$ such that*

$$\frac{|\mathcal{R}_n \setminus \mathcal{R}_n^\delta|}{\log^2 n} \xrightarrow[n\to\infty]{\text{law}} Z(\delta), \quad \text{with} \tag{67}$$

$$Z(\delta) \xrightarrow[\delta\to 0]{} 0 \quad \text{in } \mathbf{P}^\star\text{-probability}. \tag{68}$$

Proof. By the invariance principle,

$$\left(\frac{V([u\log^2 n])}{\log n}; u\geq 0\right) \xrightarrow[n\to\infty]{\text{law}} (W(u); u\geq 0), \tag{69}$$

with W a standard Brownian motion. Recall the definition of c_n^δ, $\mathscr{D}_n^\delta =]b_n, c_n^\delta]$ with $\mathscr{D}_n^\delta$ from (44), and define

$$c_\infty = \inf\{u>0 : W(u) - \min_{v\in[0,u]} W(v) \geq 1\}, \qquad b_\infty = \operatorname{Argmin}_{v\in[0,c_\infty]} W(v),$$
$$c_\infty^\delta = \min\{u > b_\infty : W(u) - W(b_\infty) \geq 1-\delta\}.$$

By (69) and from well-known results on RWRE (e.g., (Zeitouni, 2004, Sect. 2.5)), we have the joint convergence of

$$\frac{b_n}{\log^2 n} \xrightarrow[n\to\infty]{\text{law}} b_\infty, \qquad \frac{c_n^\delta}{\log^2 n} \xrightarrow[n\to\infty]{\text{law}} c_\infty^\delta, \qquad \frac{\max \mathscr{R}_n}{\log^2 n} \xrightarrow[n\to\infty]{\text{law}} c_\infty.$$

Then, the convergence in (67) follows, with the limit given by

$$\begin{aligned} Z(\delta) &= \text{Leb}\Big(\{u<b_\infty : \max_{v\in[u,b_\infty]} W(v) - W(u) \leq \delta\} \cup [c_\infty^\delta, c_\infty]\Big) \\ &= Z_1(\delta) + Z_2(\delta) \end{aligned}$$

with Leb(A) the Lebesgue measure of a Borel set A and $Z_2(\delta) = c_\infty - c_\infty^\delta$. It is not difficult to see that $c_\infty^\delta \nearrow c_\infty$ a.s. as $\delta \searrow 0$, and then $Z_2(\delta)$ vanishes. Let us prove in details that $Z_1(\delta)$ vanishes. Letting

$$A_\delta = \{u < b_\infty : \max_{v\in[u,b_\infty]} W(v) - W(u) \leq \delta\},$$

we have, as $\delta \searrow 0$,

$$A_\delta \searrow A_{0^+} \subset A' = \{u<b_\infty : W(u) \geq W(v), v\in[u,b_\infty]\}.$$

By Fubini, we compute

$$\mathbf{E}^\star \text{Leb}(A') = \int_0^\infty \mathbf{P}^\star(u<b_\infty, u\in A')\,du,$$

and we show that the integrand is zero. For all $u\geq 0$,

$$\begin{aligned} \mathbf{P}^\star(u<b_\infty, u\in A') &= \lim_{\alpha\searrow 0} \mathbf{P}^\star(u<b_\infty, u\in A', u\leq b_\infty+\alpha) \\ &\leq \limsup_{\alpha\searrow 0} \mathbf{P}^\star(W(u)\geq W(v), v\in[u,u+\alpha]), \end{aligned}$$

and the last probability is zero by Iterated Logarithm law (Karatzas and Shreve, 1991, Th. 9.12)). Finally, $\lim_{\delta\to 0} \text{Leb}(A_\delta) = 0$ a.s., ending the proof of (68). □

Lemma 4.2. *Let Assumptions I and II hold, let ε' and ε'' be such that* (46) *and* (57) *are satisfied, and let $A_n^\delta(\varepsilon'')$ and $A_n^\delta(\varepsilon'')$ be the events defined by* (47)*. Then the following convergence holds*

$$\mathbf{P}^\star\left(\complement A_n^\delta(\varepsilon')\right),\ \mathbf{P}^\star\left(\complement A_n^\delta(\varepsilon'')\right) \xrightarrow[n\to\infty]{} 0, \quad \textit{for any } \delta \textit{ in } (0,1). \tag{70}$$

Proof. Under $P_\omega(\cdot|\xi_(n-1,x)=m)$, the pair $(\xi^+(n,x),\xi^-(n,x))$ is distributed as $(\mathrm{B}(m,\omega_x), m-\mathrm{B}(m,\omega_x))$, where $\mathrm{B}(m,q)$ is a binomial random variable with sample size m and probability of success q. Using Hoeffding's concentration inequality yields

$$P_\omega\left(\left|\frac{\xi^+(n,x)}{m}-\omega_x\right|>\alpha\,\middle|\,\xi(n-1,x)=m\right)\le 2\exp[-2m\alpha^2], \quad \forall\alpha>0. \tag{71}$$

Assume that the event $\{\xi(n-1,x)=m\}$ occurs. Then, we have

$$\left|\frac{\xi^+(n,x)}{\xi^-(n,x)}-\frac{1}{\rho_x}\right| = \left|f\left(\frac{\xi^+(n,x)}{m}\right)-f(\omega_x)\right| \le \sup_I f'\cdot\left|\frac{\xi^+(n,x)}{m}-\omega_x\right|,$$

with

$$f(u)=\frac{u}{1-u} \quad\text{and}\quad I=\left[\frac{\xi^+(n,x)}{m}\wedge\omega_x, \frac{\xi^+(n,x)}{m}\vee\omega_x\right].$$

Under Assumption II, uniform ellipticity occurs and (71) implies that we can find a constant $K>0$ depending on ε'' only such that

$$P_\omega\left(\left|\frac{\xi^+(n,x)}{\xi^-(n,x)}-\frac{1}{\rho_x}\right|>\varepsilon''\,\middle|\,\xi(n-1,x)=m\right)\le K^{-1}\exp[-Km]. \tag{72}$$

Recall the event E_n^δ defined by (62) and denote its complement by $\complement E_n^\delta$. We have

$$P_\omega\left(\complement A_n^\delta(\varepsilon'')\right)\le P_\omega\left(\complement A_n^\delta(\varepsilon''),E_n^\delta\right)+P_\omega\left(\complement E_n^\delta\right).$$

Now,

$$P_\omega\left(\complement A_n^\delta(\varepsilon''),E_n^\delta\right)\le\sum_{x\in\mathscr{R}_n^\delta}P_\omega\left(\left|\frac{\xi^+(n,x)}{\xi^-(n,x)}-\frac{1}{\rho_x}\right|>\varepsilon'',\xi(n-1,x)\ge n^{\delta/2}\right),$$

and writing $\left\{\xi(n-1,x)\ge n^{\delta/2}\right\}=\bigcup_{m\ge n^{\delta/2}}\{\xi(n-1,x)=m\}$ and using (72) yield

$$P_\omega\left(\complement A_n^\delta(\varepsilon''),E_n^\delta\right)\le\sum_{x\in\mathscr{R}_n^\delta}\sum_{m\ge n^{\delta/2}}K^{-1}\exp[-Km]\cdot P_\omega(\xi(n-1,x)=m).$$

Using the fact that

$$\exp[-Km]\le\exp[-Kn^{\delta/2}], \quad\text{for any } m\ge n^{\delta/2},$$

the fact that

$$\sum_{m\ge n^{\delta/2}}P_\omega(\xi(n-1,x)=m)\le 1,$$

and the fact that $|\mathscr{R}_n^\delta| \le c_n$, yield

$$P_\omega\left(\complement A_n^\delta(\varepsilon''), E_n^\delta\right) \le K^{-1} c_n \exp[-Kn^{\delta/2}]. \tag{73}$$

Then, combining (73) and

$$\mathbf{P}^\star\left(\complement A_n^\delta(\varepsilon''), E_n^\delta\right) = \mathbb{E}^\star\left(P_\omega\left(\complement A_n^\delta(\varepsilon''), E_n^\delta\right)\right)$$

yields

$$\mathbf{P}^\star\left(\complement A_n^\delta(\varepsilon''), E_n^\delta\right) \le K^{-1}\exp[-Kn^{\delta/2}]\log^3 n + \mathbb{P}^\star(c_n > \log^3 n),$$

and from the fact that $c_n/\log^2 n$ converges in $\mathbb{P}^\star$-distribution to c_∞, we deduce that

$$\mathbf{P}^\star\left(\complement A_n^\delta(\varepsilon''), E_n^\delta\right) \xrightarrow[n\to\infty]{} 0, \quad \text{for any } \delta \text{ in } (0,1).$$

From Lemma 4.3 below, we have

$$\mathbf{P}^\star\left(\complement E_n^\delta\right) = \mathbb{E}^\star\left(P_\omega\left(\complement E_n^\delta\right)\right) \xrightarrow[n\to\infty]{} 0, \quad \text{for any } \delta \text{ in } (0,1),$$

and this achieves the proof of (70) since $A_n^\delta(\varepsilon'') \subset A_n^\delta(\varepsilon')$. □

Lemma 4.3. *Let Assumptions I and II hold and let E_n^δ be the event defined by* (62). *Then the following convergence holds*

$$\mathbf{P}^\star\left(\complement E_n^\delta\right) \xrightarrow[n\to\infty]{} 0, \quad \textit{for any } \delta \textit{ in } (0,1). \tag{74}$$

Proof. Note that

$$\mathbf{P}^\star\left(\complement E_n^\delta\right) \le \mathbf{P}^\star\left(\exists x \in \mathscr{G}_n^\delta : \xi(n,x) < n^{\delta/2}\right) + \mathbf{P}^\star\left(\exists x \in \mathscr{D}_n^\delta : \xi(n,x) < n^{\delta/2}\right),$$

where $\mathscr{G}_n^\delta$ and $\mathscr{D}_n^\delta$ are respectively defined by (43) and (44). We first show that

$$\mathbf{P}^\star\left(\exists x \in \mathscr{D}_n^\delta : \xi(n,x) < n^{\delta/2}\right) \xrightarrow[n\to\infty]{} 0, \quad \text{for any } \delta \text{ in } (0,1). \tag{75}$$

Let x be in $\mathscr{D}_n^\delta$. Define $T(y) = \inf\{t \ge 1 : X_t = y\}$ the first hitting time of y. The probability of visiting $x > b_n$ during a given excursion from b_n to b_n is (e.g., (Zeitouni, 2004, formula (2.1.4)))

$$P_\omega(T(x) < T(b_n) \mid X_0 = b_n) = \frac{\omega_{b_n}}{\sum_{j=b_n}^x \exp[V(j) - V(b_n)]} > \frac{\varepsilon_0}{c_n} \cdot n^{\delta-1}, \tag{76}$$

where the lower bound follows from (4) and the fact that x belongs to $\mathscr{D}_n^\delta$.

Fix $\varepsilon > 0$ and let k_n denote the number of excursions of X from b_n to b_n before time n. From (2.14) in Gantert et al. (2010), we have

$$P_\omega\left(\left|\frac{k_n}{n} - \frac{1}{\gamma_n}\right| \ge \varepsilon\right) \xrightarrow[n\to\infty]{} 0, \quad \text{for} \quad \mathbb{P}^\star\text{-a.a. } \omega,$$

where γ_n is the average length of an excursion from b_n to b_n. Let F_n denote the event $F_n = \{k_n \geq n \cdot \gamma_n^{-1}/2\}$. We have

$$P_\omega\left(\exists x \in \mathscr{D}_n^\delta : \xi(n,x) < n^{\delta/2}, F_n\right) \leq \sum_{x \in \mathscr{D}_n^\delta} P_\omega\left(\xi(n,x) < n^{\delta/2}, F_n\right).$$

Using the independence of excursions from b_n to b_n yields

$$P_\omega\left(\xi(n,x) < n^{\delta/2}, F_n\right) \leq \mathrm{Prob}\big(\mathrm{B}\left(n \cdot \gamma_n^{-1}/2, \varepsilon_0 c_n^{-1} n^{\delta-1}\right) \leq n^{\delta/2}\big),$$

where $\mathrm{B}(m,q)$ is a binomial random variable with sample size m and probability of success q. Combining the Chernov's bound

$$\mathrm{Prob}\big(\mathrm{B}(m,q) \leq m(q-\alpha)\big) \leq \exp\left[-m d_{\mathrm{KL}}(q-\alpha|q)\right].$$

with $m = n \cdot \gamma_n^{-1}/2$, $q = \varepsilon_0 c_n^{-1} n^{\delta-1}$ and $m(q-\alpha) = n^{\delta/2}$, the fact that $|\mathscr{D}_n^\delta| \leq c_n$, that

$$\gamma_n = \sum_{x=0}^{c_n} \frac{\mu(x)}{\mu(b_n)} \leq 2(1+c_n),$$

and that

$$\frac{c_n}{\log^2 n} \xrightarrow[n\to\infty]{\text{law}} c_\infty,$$

yield (75).

Now, we turn to the case where x is in $\mathscr{G}_n^\delta$, and a different argument is needed. From Lemma 1 in Golosov (1984),

$$\mathbf{P}^\star(T(b_n) > n) \xrightarrow[n\to\infty]{} 0,$$

hence,

$$\mathbf{P}^\star(\xi(n,b_n) = 0) \xrightarrow[n\to\infty]{} 0.$$

Note that

$$\left\{\exists x \in \mathscr{G}_n^\delta : \xi(n,x) = 0\right\} \subset \left\{\xi(n,b_n) = 0\right\}.$$

The probability of visiting b_n during a given excursion from x to x is

$$P_\omega(T(b_n) < T(x) \,|\, X_0 = x) = \frac{\omega_x}{\sum_{j=x}^{b_n} \exp[V(j) - V(x)]} < n^{-\delta}, \tag{77}$$

where the larger bound follows the fact that x belongs to $\mathscr{G}_n^\delta$. Let $h_n(x)$ denote the number of returns to x before reaching b_n. From (77), this is a geometric variable with success probability less than $n^{-\delta}$. Now,

$$\begin{aligned} P_\omega\left(\exists x \in \mathscr{G}_n^\delta : \xi(n,x) \leq n^{\delta/2}\right) \leq\; & P_\omega\left(\exists x \in \mathscr{G}_n^\delta : 1 \leq \xi(n,x) \leq n^{\delta/2}\right) \\ & + P_\omega\left(\exists x \in \mathscr{G}_n^\delta : \xi(n,x) = 0\right) \end{aligned}$$

Hence,

$$P_\omega\left(\exists x \in \mathscr{G}_n^\delta : 1 \le \xi(n,x) \le n^{\delta/2}\right) \le \sum_{x\in\mathscr{G}_n^\delta} P_\omega\left(1 \le \xi(n,x) \le n^{\delta/2}\right)$$
$$\le \sum_{x\in\mathscr{G}_n^\delta} P_\omega\left(1 \le h_n(x) \le n^{\delta/2}\right)$$
$$\le |\mathscr{G}_n^\delta| \cdot [1-(1-n^{-\delta})^{n^{\delta/2}}]$$
$$\le c_n \cdot [1-(1-n^{-\delta})^{n^{\delta/2}}].$$

Hence,

$$\mathbf{P}^\star\left(\exists x \in \mathscr{G}_n^\delta : 1 \le \xi(n,x) \le n^{\delta/2}\right) \le [1-(1-n^{-\delta})^{n^{\delta/2}}]\log^3 n + \mathbb{P}^\star(c_n > \log^3 n),$$

which combined with the fact that $c_n/\log^2 n$ converges in distribution to c_∞ proves that

$$\mathbf{P}^\star\left(\exists x \in \mathscr{G}_n^\delta : \xi(n,x) < n^{\delta/2}\right).$$ □

5 Examples

5.1 Particular Case: recurrent Temkin model

Example I. *Let $\eta = \frac{1}{2}\delta_a + \frac{1}{2}\delta_{1-a}$. Here, the unknown parameter is the support $a \in \Theta \subset (0,1/2)$ (namely $\theta = a$).*

Proposition 5.1. *In the framework of Example I, the limiting function L_∞ defined by* (33) *is deterministic and given by*

$$L_\infty(a) = -\big[H(a^\star) + d_{\mathrm{KL}}(a^\star|a)\big].$$

Proof. Note that

$$\{\widetilde{\omega}(x), 1-\widetilde{\omega}(x)\} = \{a^\star, 1-a^\star\}, \quad \text{for any } x \in \mathbb{Z}. \tag{78}$$

Recalling that $\nu^+(x) = \nu(x)\cdot\widetilde{\omega}(x)$ and $\nu^-(x) = \nu(x)\cdot[1-\widetilde{\omega}(x)]$ and noting that

$$\max_{u\in\{a,1-a\}}\left\{a^\star \log u + (1-a^\star)\log(1-u)\right\} = a^\star \log a + (1-a^\star)\log(1-a),$$

implies that (33) might be rewritten as

$$L_\infty(a) = \left[a^\star \log a + (1-a^\star)\log(1-a)\right]\sum_x \nu(x).$$

The fact that $\sum_{x\in\mathbb{Z}}\nu(x) = 1$ achieves the proof. □

5.2 Case of two-point support

Example II. *Let* $\eta = p_1\delta_{a_1} + p_2\delta_{a_2}$ *with* $a_1 < 1/2 < a_2$*, and*

$$p_1 = \frac{\log\frac{1-a_1}{a_1}}{\log\frac{a_2(1-a_1)}{a_1(1-a_2)}}, \quad p_2 = \frac{\log\frac{a_2}{1-a_2}}{\log\frac{a_2(1-a_1)}{a_1(1-a_2)}}.$$

Here, the unknown parameter is the support $\mathbf{a} = (a_1, a_2)$.

The following result extends Proposition 5.1.

Proposition 5.2. *In the framework of Example II, the limiting function* L_∞ *defined by* (33) *is deterministic and given by*

$$L_\infty(\mathbf{a}) = -\frac{1}{a_2^\star - a_1^\star}\sum_{j=1}^{2}\left|a_j^\star - \frac{1}{2}\right|\left[H(a_j^\star) + \min_{i=1,2} d_{\mathrm{KL}}(a_j^\star|a_i)\right].$$

Proof. By (34),

$$L_\infty(\mathbf{a}) = -\sum_{j=1}^{2}\nu^{(j)}\cdot\left[H(a_j^\star) + \min_{i=1,2} d_{\mathrm{KL}}(a_j^\star|a_i)\right], \qquad \nu^{(j)} = \sum_{x:\widetilde{\omega}(x)=a_j^\star}\nu(x).$$

So, the formula for $L_\infty(\mathbf{a})$ follows from the values of the coefficients $\nu^{(j)}$: these are determined by

$$\sum_{j=1}^{2}\nu^{(j)} = 1, \qquad \sum_{j=1}^{2}\nu^{(j)}\cdot(2a_j^\star - 1) = 0,$$

where the second equality means that the mean drift is zero in the recurrent case. Indeed, the drift per time unit on time interval $[0,n]$, $n^{-1}\sum_{t=0}^{n-1}(2\omega_{X_t} - 1) = \sum_x \nu_n(x)(2\widetilde{\omega}(x) - 1)$, vanishes as $n \to \infty$ by the law of large numbers for martingales. For a direct derivation, we can write, from (30) and (31),

$$\sum_x \nu(x)\widetilde{\omega}(x) = \sum_x \nu^+(x) = \sum_x \nu^-(x) = \sum_x \nu(x)(1-\widetilde{\omega}(x)),$$

and then $\sum_x \nu(x)(2\widetilde{\omega}(x) - 1) = 0$, which, after reorganizing the terms, gives the second equality above. □

5.3 Particular Case: recurrent lazy Temkin model

We present a simple example for which $L_\infty(\cdot)$ is a random variable.

Example III. *Let* $\eta = \frac{1-r}{2}\delta_a + r\delta_{1/2} + \frac{1-r}{2}\delta_{1-a}$ *with* $a < 1/2, r \in (0,1)$*. Here, we have* $\mathbf{a} = (a, \frac{1}{2}, 1-a)$ *and* $\mathbf{p} = (\frac{1-r}{2}, r, \frac{1-r}{2})$*, and the unknown parameter is* (a, r).

Proposition 5.3. *In the framework of Example III, there exists $a' \in (0, a^\star)$, depending only on $a^\star$, such that the limiting function $L_\infty(\mathbf{a})$ is deterministic when $a \in (0, a']$ and random when $a \in (a', 1/2)$:*

$$\mathbb{V}ar^\star(L_\infty(\mathbf{a})) > 0.$$

Proof. Set $\underline{\mathbf{a}} = \{a, 1/2, 1-a\}$. By (34),

$$\begin{aligned} L_\infty(\mathbf{a}) = &- \nu^{(a^\star)} \cdot \big[H(a^\star) + \min_{\underline{\mathbf{a}}} d_{\mathrm{KL}}(a^\star|\cdot)\big] \\ &- \nu^{(1-a^\star)} \cdot \big[H(1-a^\star) + \min_{\underline{\mathbf{a}}} d_{\mathrm{KL}}(1-a^\star|\cdot)\big] - \nu^{(1/2)} \cdot H(1/2), \end{aligned}$$

with

$$\nu^{(u)} = \sum_{x:\widetilde{\omega}(x)=u} \nu(x), \quad u \in \underline{\mathbf{a}}.$$

So, the formula for $L_\infty(\mathbf{a})$ follows from the values of the coefficients $\nu^{(u)}$: these satisfy

$$\sum_{u\in\underline{\mathbf{a}}} \nu^{(u)} = 1, \qquad \sum_{u\in\underline{\mathbf{a}}} \nu^{(u)}(2u-1) = 0,$$

where the second equality is equivalent to $\nu^{(a^\star)} = \nu^{(1-a^\star)}$. Hence, we have $2\nu^{(a^\star)} + \nu^{(1/2)} = 1$. Furthermore, noting that

$$H(a^\star) = H(1-a^\star) \quad \text{and} \quad \min_{\underline{\mathbf{a}}} d_{\mathrm{KL}}(a^\star|\cdot) = \min_{\underline{\mathbf{a}}} d_{\mathrm{KL}}(1-a^\star|\cdot)$$

yields

$$L_\infty(\mathbf{a}) = \nu^{(1/2)}\big[H(a^\star) + \min_{\underline{\mathbf{a}}} d_{\mathrm{KL}}(a^\star|\cdot) - H(1/2)\big] - \big[H(a^\star) + \min_{\underline{\mathbf{a}}} d_{\mathrm{KL}}(a^\star|\cdot)\big]. \quad (79)$$

One can see that in (79), the quantities

$$\big[H(a^\star) + \min_{\underline{\mathbf{a}}} d_{\mathrm{KL}}(a^\star|\cdot) - H(1/2)\big] \quad \text{and} \quad \min_{\underline{\mathbf{a}}} d_{\mathrm{KL}}(a^\star|\cdot)\big]$$

are deterministic. Hence, $L_\infty(\mathbf{a})$ is random if and only if $\nu^{(1/2)}$ is random and $\big[H(a^\star) + \min_{\underline{\mathbf{a}}} d_{\mathrm{KL}}(a^\star|\cdot) - H(1/2)\big]$ is non zero.

The facts that $d_{\mathrm{KL}}(a^\star|\cdot)$ is continuous, decreasing on $(0, a^\star)$, increasing on $(a^\star, 1)$, null at $a^\star$ and that

$$d_{\mathrm{KL}}(a^\star|1/2) = H(1/2) - H(a^\star) \quad \text{and} \quad d_{\mathrm{KL}}(a^\star|a) \xrightarrow[a\to 0]{} +\infty,$$

yield the existence of a unique $a' \in (0, a^\star)$ such that

$$d_{\mathrm{KL}}(a^\star|a') = H(1/2) - H(a^\star),$$

and as a consequence that for any $a \in (0, a']$,

$$\min_{\underline{\mathbf{a}}} d_{\mathrm{KL}}(a^\star|\cdot) = d_{\mathrm{KL}}(a^\star|a') = H(1/2) - H(a^\star).$$

Hence, the limiting function $L_\infty(\mathbf{a})$ is deterministic and equal to $-H(1/2) = -\log 2$, for any $a \in (0, a']$.

Now, let a be in $(a', 1/2)$. Recall that

$$\nu^{(1/2)} = \sum_{x:\widetilde{\omega}(x)=1/2} \nu(x), \quad \text{with} \quad \nu(x) = \frac{\exp[-\widetilde{V}(x-1)] + \exp[-\widetilde{V}(x)]}{2\sum_{z\in\mathbb{Z}} \exp[-\widetilde{V}(z)]},$$

and that $\widetilde{\omega}(x) = 1/2$ if and only if $\widetilde{V}(x) = \widetilde{V}(x-1)$. Now, we focus on the potential of the infinite valley. Recall that $\widetilde{V}(0) = 0$, $\{\widetilde{V}(x)\}_{x\geq 0}$ and $\{\widetilde{V}(-x)\}_{x\geq 0}$ are two independent Markov chains where $\widetilde{V}(x)$ is non-negative for any non-negative x and $\widetilde{V}(x)$ is positive for any negative x, see Golosov (1984). Especially, we have

$$\mathbb{P}^\star\big(\widetilde{V}(x+1) = 0 \,|\, \widetilde{V}(x) = 0\big) = 1 - \mathbb{P}^\star\big(\widetilde{V}(x+1) = \log \rho^\star \,|\, \widetilde{V}(x) = 0\big) = r,$$

where we set $\rho^\star = \frac{1-a^\star}{a^\star}$. Define the random variable $T \geq 1$ as $T = \inf\{x \in \mathbb{N} : \widetilde{V}(x) = \log \rho^\star\}$. The random variable T is distributed according to a geometric distribution with parameter of success $1-r$, and for any $0 \leq x \leq T-1$, the potential $\widetilde{V}(x)$ is equal to 0. We have

$$\nu^{(1/2)} = \frac{\sum_{x<0} \mathrm{e}^{-\widetilde{V}(x)} \mathbb{1}_{\widetilde{V}(x-1)=\widetilde{V}(x)} + (T-1) + \sum_{x=T}^\infty \mathrm{e}^{-\widetilde{V}(x)} \mathbb{1}_{\widetilde{V}(x-1)=\widetilde{V}(x)}}{\sum_{x\leq 0} \mathrm{e}^{-\widetilde{V}(x)} + (T-1) + \sum_{x=T}^\infty \mathrm{e}^{-\widetilde{V}(x)}}$$

Using the definition of $(\widetilde{V}(x) : x \in \mathbb{Z})$ and the strong Markov property, T is independent of $\mathscr{F} := \mathfrak{S}(\widetilde{V}(x) : x \leq 0 \text{ or } x \geq T)$, hence

$$\mathbb{V}\mathrm{ar}^\star[\nu^{(1/2)}|\mathscr{F}] = \mathbb{V}\mathrm{ar}^\star\left(\frac{T-1+\Upsilon'}{T-1+\Upsilon}\right),$$

with

$$\Upsilon' = \sum_{x<0; x\geq T} \mathrm{e}^{-\widetilde{V}(x)} \mathbb{1}_{\widetilde{V}(x-1)=\widetilde{V}(x)} \quad \text{and} \quad \Upsilon = \sum_{x\leq 0; x\geq T} \mathrm{e}^{-\widetilde{V}(x)}.$$

Since $\Upsilon > \Upsilon'$, $\mathbb{V}\mathrm{ar}^\star[\nu^{(1/2)}|\mathscr{F}]$ is a non constant strictly positive random variable, hence

$$\mathbb{V}\mathrm{ar}^\star(\nu^{(1/2)}) = \mathbb{E}^\star\big(\mathbb{V}\mathrm{ar}^\star[\nu^{(1/2)}|\mathscr{F}]\big) + \mathbb{V}\mathrm{ar}^\star(\mathbb{E}^\star[\nu^{(1/2)}|\mathscr{F}]) > 0. \qquad \square$$

6 Numerical performance

In this section, we explore the numerical performance of our estimation procedure in the frameworks of Examples I to III. We compare our performance with the performance of the estimator proposed by Adelman and Enriquez (2004) in the framework of Example I. An explicit description of the form of Adelman and Enriquez's estimator in the particular case of the one-dimensional nearest neighbour path is provided in Section 5.1 of Comets et al. (2014). Therefore, one can estimate $\theta^\star$ by the solution of an appropriate system of equations, as illustrated below.

Example I (continued) In this case the parameter θ equals a and we have

$$v = \mathbb{E}^\star[\omega_0] = \frac{1}{2}a^\star + \frac{1}{2}(1-a^\star) = \frac{1}{2},$$
$$w = \frac{\mathbb{E}^\star[\omega_0^2]}{\mathbb{E}^\star[\omega_0]} = \{\frac{1}{2}[a^\star]^2 + \frac{1}{2}(1-a^\star)^2\}\cdot v^{-1} = [a^\star]^2 + (1-a^\star)^2.$$

Hence, among the visited sites, the proportion of those from which the first move is to the right (or to the left) gives no information on the parameter $a^\star$. We need to push to the sites visited at least twice to get some information. Among those from which the first move is to the right, the proportion of those from which the second move is also to the right gives an estimator for $[a^\star]^2 + (1-a^\star)^2$. Using this observation, we can estimate $a^\star$.

We now present the three simulation experiments corresponding respectively to Examples I to III. The comparison with Adelman and Enriquez's procedure is given only for Example I. For each of the three simulations, we *a priori* fix a parameter value $\theta^\star$ as given in Table 1 and repeat 1,000 times the procedure described below. We first generate a random environment according to $\eta_{\theta^\star}$ on the

Simulation	Estimated parameter
Example I	$a^\star = 0.3$
Example II	$(a_1^\star, a_2^\star) = (0.4, 0.7)$
Example III	$(a^\star, r^\star) = (0.3, 0.2)$

Table 1: Parameter values for each experiment.

set of sites $\{1,\dots,10^5\}$. In fact, we do not use the environment values for all the sites, since only few of these sites are visited by the walk. However the computation cost is very low comparing to the rest of the estimation procedure. Then, we run a random walk in this environment and stop it successively at n, with $n \in \{10^4 \cdot k : 1 \le k \le 10\}$. For each stop, we estimate $\theta^\star$ according to our MPLE procedure (and Adelman and Enriquez's one for the first simulation). In the cases of Examples I and III, the likelihood optimization procedure for the support $a^\star$ was performed as a combination of golden section search and successive parabolic interpolation. In the case of Example II, the likelihood optimization procedure for $(a_1^\star, a_2^\star)$ was performed according to the "L-BFGS-B" method of Byrd et al. (1995) which uses a limited-memory modification of the “BFGS” quasi-Newton method published simultaneously in 1970 by Broyden, Fletcher, Goldfarb and Shanno. In all three cases, the parameters in Table 1 are chosen such that the RWRE is recurrent.

Figure 2 shows the boxplots of our estimator and Adelman and Enriquez's estimator obtained from 1,000 iterations of the procedures in Example I. First, we shall notify that in order to simplify the visualisation of the results, we removed in the boxplots corresponding to Example I about 13.5% of outliers values (outside 1.5 times the interquartile range above the upper quartile and below the lower quartile) from our estimator. Second, we shall also notify that about 16% of Adelman and Enriquez's estimation procedures could not be achieved for the simple reason that the equation involving $\mathbf{a}^\star$ had no solution. We observe that our procedure is far more accurate than Adelman and Enriquez's, and that the accuracies of the procedure increase with the value of n.

Figure 3 shows the boxplots of our estimator obtained from 1,000 iterations of the procedure in Example II. Once again, we shall notify that in order to simplify the visualisation of the results, we removed in the boxplots corresponding to Example II about 13% of outliers values from both estimators of $a_1^\star$ and $a_2^\star$. We observe that the accuracy of the procedure is high, and that in this case, it does increase with the value of n.

Figure 4 shows the boxplots of our estimator obtained from 1,000 iterations of the procedure in Example III. We shall notify that in order to simplify the visualisation of the results, we removed in the boxplots corresponding to Example III about 12.3% of outliers values from our estimator of $a^\star$ and 2.8% of outliers values from our estimator of $r^\star$. We observe that the accuracy of the procedure is high for the parameter of the support and low for the probability parameter, even if in both cases, the accuracy increases with the value of n.

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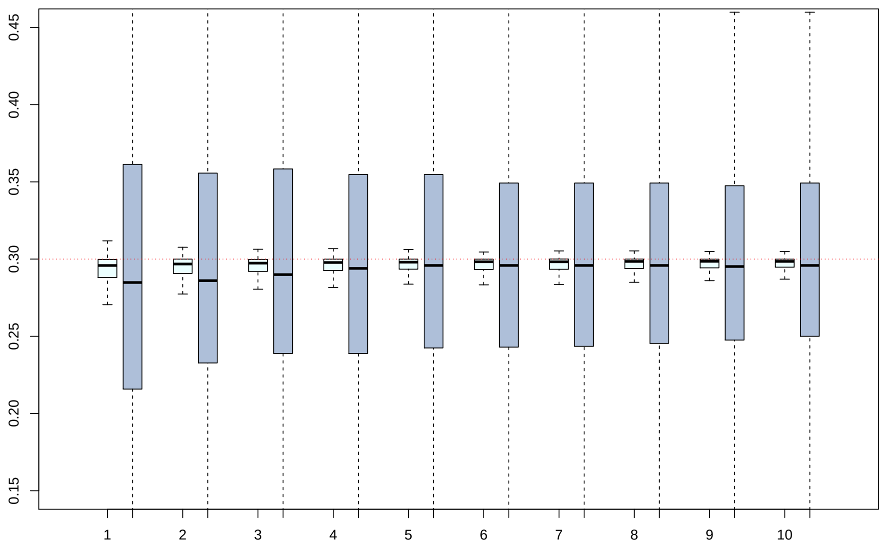

Figure 2: Boxplots of our estimator (left and white) and Adelman and Enriquez's estimator (right and grey) obtained from 1,000 iterations and for values n ranging in $\{10^4 \cdot k : 1 \leq k \leq 10\}$ (x-axis indicates the value k). The panel displays estimation of $a^\star$ in Example I. The true value is indicated by an horizontal line.

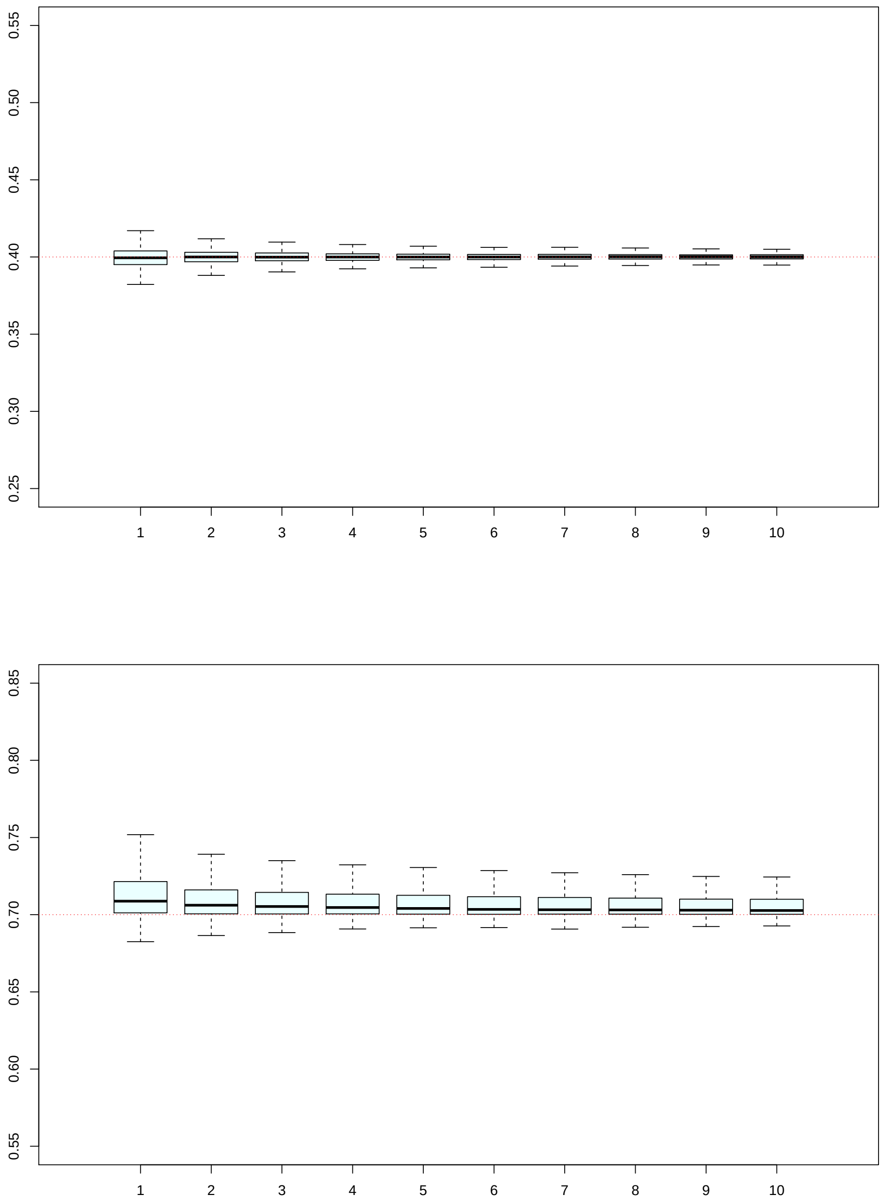

Figure 3: Boxplots of our estimator obtained from 1,000 iterations in Example II and for values n ranging in $\{10^4 \cdot k : 1 \leq k \leq 10\}$ (x-axis indicates the value k). Estimation of $a_1^\star$ (top panel) and $a_2^\star$ (bottom panel). The true values are indicated by horizontal lines.

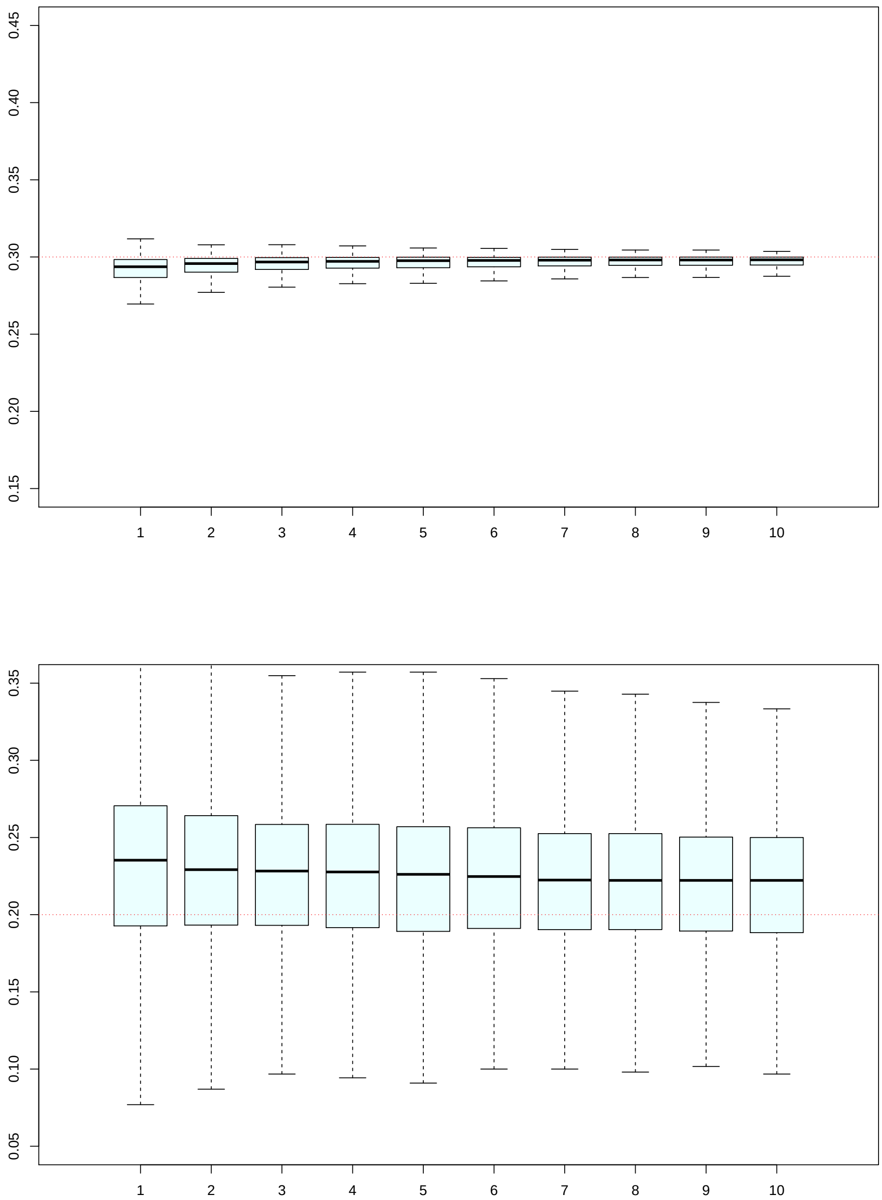

Figure 4: Boxplots of our estimator obtained from 1,000 iterations in Example III and for values n ranging in $\{10^4 \cdot k : 1 \le k \le 10\}$ (x-axis indicates the value k). Estimation of $a^\star$ (top panel) and $r^\star$ (bottom panel). The true values are indicated by horizontal lines.