Quantitative Types for Intuitionistic Calculi

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Abstract. We define quantitative type systems for two intuitionistic term languages. While the first language in natural deduction style is already known in the literature, the second one is one of the contributions of the paper, and turns out to be a natural computational interpretation of sequent calculus style by means of a non-idempotent type discipline. The type systems are able to characterize linear-head, weak and strong normalization sets of terms. All such characterizations are given by means of combinatorial arguments, *i.e.* there is a measure based on type derivations which is decreasing with respect to the different reduction relations considered in the paper.

It is quite difficult to reason about programs in general by only taking into account their syntax, so that many different semantic approaches were proposed to analyze them in a more abstract way. One typical tool to analyze relevant aspects of programs is the use of *type systems*. In particular, *intersection types* allow to characterize *head/weakly/strongly* normalizing terms, *i.e.* a term t is typable in an intersection type system iff t is head/weakly/strongly normalizing; *quantitative* information about the behaviour of programs can also be obtained if they enjoy *non-idempotence*.

Intersection Types (IT): *Simply typed* terms are *strongly normalizing* (*cf.* [6]) but the converse does not hold, *e.g.* the term $t := \lambda x.xx$. *Intersection Types* [15] extend the simply typed discipline with a finitary notion of polymorphism, listing type usage, that exactly captures the set of strongly normalizing terms. This is done by introducing a new construct $\sigma \wedge \tau$ in the syntax of types together with a corresponding set of typing rules. Thus for example, the previous term t is typable with $((\sigma \to \sigma) \wedge \sigma) \to \sigma$ so that the first occurrence of the variable x is typed with $\sigma \to \sigma$ while the second one is just typed with σ. Typically, intersection types are *idempotent*, *i.e.* $\sigma \wedge \sigma = \sigma$. Moreover, the intersection constructor is usually *commutative* and *associative*. Intersection types in their full generality provide a characterization of various properties of terms: models of λ-calculus [7], characterization of head [17] as well as weakly [13, 17] and strongly normalizing terms [33].

Non-Idempotent Intersection Types: The use of non-idempotent types [11] gives rise to resource aware semantics, which is relevant for computational complexity because it gives tools to extract *quantitative* information about reduction sequences. Indeed, the inequality $\sigma \wedge \sigma \neq \sigma$ can be read as the fact that two different uses of the variable x are not isomorphic to a single use. The relationship with Linear Logic [25] and Relevant Logic [18], gives an insight on the information refinement aspects of non-idempotent intersection types. In the framework of the λ-calculus, the relationship between the size of a non-idempotent intersection typing derivation and the *head/weak*-normalization execution time of λ-terms by means of abstract machines was established by D. de Carvalho [21]. Non-idempotent types for different calculi based on linear logic are explored in [35]. They are also used in [8, 9, 20] to reason about the longest derivations of *strongly* β-normalizing terms in the λ-calculus by means of combinatorial arguments.

Calculi with Explicit Substitutions (ES) and Intersection Types: Calculi with ES refine the λ-calculus by decomposing β-reduction into small steps in order to specify different evaluation strategies by means of abstract machines. In traditional calculi with ES [27, 1], the operational semantics specifies the propagation of ES through the term's structure until they reach variable occurrences, on which they finally substitute or get garbage collected. But calculi with ES can also be interpreted in *Linear Logic* [22, 30, 28, 5] by implementing another kind of operational semantics: their dynamics is defined using contexts (*i.e.* terms with holes) that allows the ES to act directly *at a distance* on single variable occurrences, with no need to commute with any other constructor in between. In other words, the propagation of substitutions is not performed by structural induction on terms, since they are only consumed according to the multiplicity of the variables.

Idempotent intersection type systems were used to characterize [34, 29] strongly normalizing terms of calculi with ES in natural deduction style. Similar results [24, 23] exist for intuitionistic sequent calculus style. Non-idempotence is used in [10] to prove the exact relationship between typing derivations and the number of steps of the longest reduction sequences of strongly-normalizing terms in the λ`s`-calculus [28] and in the λ`lxr`-calculus [30]. These systems are not syntax-directed (*i.e.* they need generation lemmas) and only deal with strong normalization.

This paper focuses on *functional programs* specified – via the Curry-Howard isomorphism – by *intuitionistic logic*, in both *natural deduction* and *sequent calculus* style. The operational semantics for both languages/styles implements resource control by means of reduction rules describing the behaviour of *explicit* operators for *erasure* and *duplication*. The term language in natural deduction style is the *linear substitution calculus* [3], called here `M`-*calculus*, and obtained from Milner's calculus [37] and the structural λ-calculus [5]. For the computational interpretation of intuitionistic sequent calculus style we propose a new term language called `J`-*calculus*, whose syntax comes from Herbelin [27] and whose semantics is inspired by the linear substitution calculus. In fact, the operational rules of [27] allow the useless duplication of empty resources, which is inefficient and fails to be interpreted by a quantitative approach. We then propose an alternative semantics for Herbelin's syntax, which, in particular, only allows duplication of resources that are useful, *i.e.* those that are non empty. This is achieved by the *partial substitution operation* of the linear substitution calculus.

Partial substitution naturally allows to express *linear-head reduction* [19, 36], a notion of evaluation of proof nets that is strongly related to significant aspects of computer science [33, 2, 4]. Linear-head reduction cannot be expressed as a simple strategy of the λ-calculus, where substitution acts on all free occurrences of a variable at once; this is probably one of the reasons why there are so few works investigating it. In this paper we use logical systems to reason about different notions of normalization of terms, including those obtained with linear-head reduction.

More precisely, the quantitative semantics of programs used in this paper is given by two kind of non-idempotent intersection type system. The first one, based on [21], allows a characterization of linear-head and weakly normalizing terms. While full logical characterizations of head/weakly β-normalizing λ-terms were already given in the literature, the use of a logical/type system to directly characterize linear-head normalization in calculi with ES is new. The second kind of system, another main contributions of this paper, gives a characterization of strongly normalizing terms.

Main contributions: they can be summarized as follows.

- We define two type systems for the linear substitution calculus. The first system characterizes linear-head and weak normalization while the second one characterizes strong normalization. No previous logical characterization of linear-head normalization for ES was known in the literature.
- We propose a new term language with resources in sequent calculus style which naturally admits a quantitative type semantics. As for the linear substitution calculus, we characterize linear-head, weak and strong normalization sets of terms.
- The type systems use multiset notation and are syntax directed so that no generation lemmas are needed. Moreover, the type systems for strong normalization make use of a special notion of *witness* derivation for the arguments of applications and explicit substitutions which makes them particular natural and simple.

- Similar proof schemes can be applied to both calculi, thus obtaining an homogenous technical development for natural deduction and sequent calculus.
- All the characterizations are given by means of simple combinatorial arguments, *i.e.* there is a measure that can be associated to each typing derivation which is decreasing with respect to the different reduction relations considered in the paper.

Structure of the paper: We recall some general notions of rewriting in Section 1 and we conclude in Section 8. The rest of the paper is organized as follows:

	Syntax and Semantics	Linear-Head and Weak-Normalization	Strong-Normalization
M-calculus	Section 2	Section 3	Section 4
J-calculus	Section 5	Section 6	Section 7

All the detailed proofs of our results are contained in the Appendix.

1 Some General Notions of Rewriting

We use the following general notions of rewriting.

Let $\rightarrow_{\mathcal{R}}$ and $\rightarrow_{\mathcal{S}}$ be two reduction relations on a set $\mathcal{O}$. The **concatenation** (resp. **union**) of $\rightarrow_{\mathcal{R}}$ and $\rightarrow_{\mathcal{S}}$ is written $\rightarrow_{\mathcal{R}\cdot\mathcal{S}}$ or $\rightarrow_{\mathcal{R}} \cdot \rightarrow_{\mathcal{S}}$ (resp. $\rightarrow_{\mathcal{R}\cup\mathcal{S}}$). The reflexive-transitive (resp. transitive) closure of $\rightarrow_{\mathcal{R}}$ is written $\rightarrow^*_{\mathcal{R}}$ (resp. $\rightarrow^+_{\mathcal{R}}$); they both denote **finite $\mathcal{R}$-reduction sequences**. Moreover, $\rightarrow^n_{\mathcal{R}}$ denotes a reduction sequence of length n ($n \geq 0$).

Given $o \in \mathcal{O}$, o is **in $\mathcal{R}$-normal form**, written $o \in \mathcal{R}$-nf, if there is no o' such that $o \rightarrow_{\mathcal{R}} o'$, and o **has an $\mathcal{R}$-normal** form iff there exists $o' \in \mathcal{R}$-nf such that $o \rightarrow^*_{\mathcal{R}} o'$. Moreover, o is **weakly $\mathcal{R}$-normalizing**, written $o \in \mathcal{WN}(\mathcal{R})$, iff o has an $\mathcal{R}$-normal form, o is **strongly $\mathcal{R}$-normalizing** or $\mathcal{R}$-terminating, written $o \in \mathcal{SN}(\mathcal{R})$, if there is no infinite $\mathcal{R}$-reduction sequence starting at o, and o is **$\mathcal{R}$-finitely branching** if the set $\{o' \mid o \rightarrow_{\mathcal{R}} o'\}$ is finite. If $o \in \mathcal{O}$ is $\mathcal{R}$-strongly normalizing and $\mathcal{R}$-finitely branching then the **depth of** o, written $\eta_{\mathcal{R}}(o)$, is the maximal length of $\mathcal{R}$-reduction sequences starting at o.

2 An Intuitionistic Term Calculus in Natural Deduction Style

We first describe the syntax and the operational semantics of the M-calculus, including some particular notions of rewriting such as linear-head reduction. We then introduce a notion of type together with two different type systems that play a central role in the first part the paper.

Syntax: Given a countable infinite set of symbols $x, y, z, \ldots$, three different syntactic categories for terms ($\mathcal{T}_{\mathtt{M}}$) and contexts ($\mathcal{C}_{\mathtt{M}}$) are defined by the following grammars:

$$
\begin{array}{rll}
\textbf{(terms)} & t,u,v & ::= x \mid tt \mid \lambda x.t \mid t[x/t] \\
\textbf{(term contexts)} & \mathtt{C} & ::= \Box \mid \lambda x.\mathtt{C} \mid \mathtt{C}\ t \mid t\ \mathtt{C} \mid \mathtt{C}[x/t] \mid t[x/\mathtt{C}] \\
\textbf{(list contexts)} & \mathtt{L} & ::= \Box \mid \mathtt{L}[x/t]
\end{array}
$$

A term x is called a **variable**, tu an **application**, $\lambda x.t$ an **abstraction** and $t[x/u]$ a **closure** where $[x/u]$ is an **explicit substitution (ES)**. We write $tt_1 \ldots t_n$ for $(\ldots(tt_1)\ldots t_n)$ and $|t|$ for the **size** of t. The notions of **free** and **bound** variables are defined as usual, in particular, $\mathtt{fv}(t[x/u]) := \mathtt{fv}(t) \setminus \{x\} \cup \mathtt{fv}(u)$, $\mathtt{fv}(\lambda x.t) := \mathtt{fv}(t) \setminus \{x\}$, $\mathtt{bv}(t[x/u]) := \mathtt{bv}(t) \cup \{x\} \cup \mathtt{bv}(u)$ and $\mathtt{bv}(\lambda x.t) := \mathtt{bv}(t) \cup \{x\}$. We work with the standard notion of α-conversion *i.e.* renaming of bound variables for abstractions and substitutions. We write $\mathtt{C}[t]$ (resp. $\mathtt{L}[t]$) for the term obtained by replacing the hole of $\mathtt{C}$ (resp. $\mathtt{L}$) by the term t. We write $\mathtt{C}[\![u]\!]$ or $\mathtt{L}[\![u]\!]$ when the free variables of u are not captured by the context, *i.e.* there are no abstractions or explicit substitutions in the context that bind the free variables of u. The set of **positions** of t, written $\texttt{pos}(t)$, is the finite language over $\{0,1\}$ inductively defined as follows: $\epsilon \in \texttt{pos}(t)$ for every t; $0p \in \texttt{pos}(\lambda x.t)$ if $p \in \texttt{pos}(t)$; $0p \in \texttt{pos}(tu)$ (resp. $\texttt{pos}(t[x/u])$) if $p \in \texttt{pos}(t)$; $1p \in \texttt{pos}(tu)$ (resp. $\texttt{pos}(t[x/u])$) if $p \in \texttt{pos}(u)$. The **subterm of t at position** p is written $t|_p$ and defined as expected. The term u **has an occurrence** in t iff there is $p \in \texttt{pos}(t)$ such that $t|_p = u$. We write $|t|_x$ to denote the **number of free occurrences** of the variable x in the term t. All these notions are extended to contexts as expected.

Operational Semantics: The $\texttt{M}$**-calculus** is given by the set of terms $\mathcal{T}_\texttt{M}$ and the **reduction relation** $\rightarrow_\texttt{M}$ on $\mathcal{T}_\texttt{M}$ defined as the union of $\rightarrow_\texttt{dB}$, $\rightarrow_\texttt{c}$, and $\rightarrow_\texttt{w}$, which are, respectively, the closure by term contexts $\texttt{C}$ of the following rewriting rules:

$$\begin{array}{llll} \texttt{L}[\lambda x.t]u & \mapsto_\texttt{dB} & \texttt{L}[t[x/u]] & \\ \texttt{C}[\![x]\!][x/u] & \mapsto_\texttt{c} & \texttt{C}[\![u]\!][x/u] & \\ t[x/u] & \mapsto_\texttt{w} & t & \text{if } |t|_x = 0 \end{array}$$

The names $\texttt{dB}$, $\texttt{c}$ and $\texttt{w}$ stand for $\texttt{d}$*istant* $\texttt{B}$*eta*, $\texttt{c}$*ontraction* and $\texttt{w}$*eakening*, respectively. Rule $\mapsto_\texttt{dB}$ (resp. $\mapsto_\texttt{c}$) comes from the structural λ-calculus [5] (resp. Milner's calculus [37]), while $\mapsto_\texttt{w}$ belongs to both calculi. Rule $\texttt{dB}$ could also be written as $(\lambda x.t)[y_1/v_1]\dots[y_n/v_n]u \mapsto_\texttt{dB} t[x/u][y_1/v_1]\dots[y_n/v_n]$. Notice that the use of a list (resp. term) context $\texttt{L}$ (resp. $\texttt{C}$) in rule $\texttt{dB}$ (resp. $\texttt{c}$) makes the reduction *at a distance*. By α-conversion we can assume in the rule $\texttt{dB}$ that x may only be free in t and no variable in the domain of $\texttt{L}$, defined as expected, has free occurrences in the term u. The *pushed out* list context $\texttt{L}$ in rule $\texttt{dB}$ corresponds to Regnier's σ-equivalence [40]: $\texttt{L}[\lambda x.t]u \sim_\sigma \texttt{L}[(\lambda x.t)u] \rightarrow_\texttt{dB} \texttt{L}[t[x/u]]$. We will come back on this equivalence in Section 4.

The notion of *redex occurrence* in this calculus is more subtle than the one in standard rewriting because one unique term may give rise to different reduction steps at the root, as the following example shows: $(xu)[x/u] \; {}_\texttt{c}\!\leftarrow (xx)[x/u] \rightarrow_\texttt{c} (ux)[x/u]$. Thus, a position $p \in \texttt{pos}(t)$ is said to be a $\texttt{dB}$ (resp. $\texttt{w}$ and $\texttt{c}$) **redex occurrence** of t if $t|_p = \texttt{L}[\lambda x.t]u$ (resp. $t|_p = v[x/u]$ with $|v|_x = 0$, and $p = p_1p_2$ with $t|_{p_1} = \texttt{C}[\![x]\!][x/u]$ and $\texttt{C}|_{p_2} = \Box$). For example 000 and 001 are both $\texttt{c}$-redex occurrences of the term $\lambda z.(xx)[x/u]$.

The $\texttt{M}$-calculus enjoys good properties required for calculi with ES (including simulation of β-reduction, preservation of strong normalization, confluence on terms and metaterms and full composition) [31]. It was recently used in investigations related to cost models [4], π-calculus [2], and axiomatic standardization [3].

The reduction relation $\rightarrow_\texttt{M}$ can be refined in different ways. The **non-erasing** reduction relation $\rightarrow_{\texttt{M}\setminus\texttt{w}}$ is given by $\rightarrow_{\texttt{dB}\cup\texttt{c}}$, and plays a key role in the characterization of strongly normalizing terms in Section 4. Another key subrelation studied in this paper is *linear-head reduction* [19, 36], a strategy related to abstract machines [19] and linear logic [25]. To introduce this notion, we first define the set of **linear-head contexts** that are generated by the following grammar: $\texttt{L}_\texttt{H} ::= \Box \mid \lambda x.\texttt{L}_\texttt{H} \mid \texttt{L}_\texttt{H}t \mid \texttt{L}_\texttt{H}[x/t]$. **Linear-head** $\texttt{M}$**-reduction**, written $\rightarrow_{\texttt{L}_\texttt{H}\texttt{M}}$, is the closure under *linear-head contexts* of the rewriting rules $\{\mapsto_\texttt{dB}, \mapsto_{\texttt{c}|_{\texttt{L}_\texttt{H}}}\}$, where $\mapsto_{\texttt{c}|_{\texttt{L}_\texttt{H}}}$ is the following variation of the rewriting rule $\mapsto_\texttt{c}$:

$$\texttt{L}_\texttt{H}[\![x]\!][x/u] \mapsto_{\texttt{c}|_{\texttt{L}_\texttt{H}}} \texttt{L}_\texttt{H}[\![u]\!][x/u]$$

Indeed, the leftmost (*i.e. head*) occurrence of the variable x in $\texttt{L}_\texttt{H}[\![x]\!]$ is substituted by u and this partial (*i.e. linear*) substitution is only performed on that head occurrence. The notion of $\texttt{c}|_{\texttt{L}_\texttt{H}}$-**redex occurrence** is defined as for the $\texttt{c}$-rule. A term t is **linear-head** $\texttt{M}$**-normalizing**, written $t \in \mathcal{L_H N}(\texttt{M})$, iff t has an $\texttt{L}_\texttt{H}\texttt{M}$-nf. For example, if $t_0 := \lambda x.xy$ and $t_1 := x[y/z](\texttt{II})$, where $\texttt{I} := \lambda w.w$,

$$\frac{}{x{:}[\tau] \vdash x{:}\tau}\ (\mathtt{ax}) \qquad \frac{x{:}[\sigma_i]_{i\in I}; \Gamma \vdash t{:}\tau \qquad (\Delta_i \vdash u{:}\sigma_i)_{i\in I}}{\Gamma +_{i\in I} \Delta_i \vdash t[x/u]{:}\tau}\ (\mathtt{cut}_{\mathcal{HW}})$$

$$\frac{\Gamma \vdash t{:}\tau}{\Gamma \setminus\!\!\setminus x \vdash \lambda x.t{:}\Gamma(x) \to \tau}\ (\to \mathtt{i}) \qquad \frac{\Gamma \vdash t{:}[\sigma_i]_{i\in I} \to \tau \qquad (\Delta_i \vdash u{:}\sigma_i)_{i\in I}}{\Gamma +_{i\in I} \Delta_i \vdash tu{:}\tau}\ (\to \mathtt{e}_{\mathcal{HW}})$$

Fig. 1. The Type System $\mathcal{HW}$ for the M-Calculus

then $t_0 \in$ M-nf, and so also $t_0 \in \mathtt{L_H}$M-nf, while $t_1 \notin$ M-nf but $t_1 \in \mathtt{L_H}$M-nf.

Types: We denote finite multisets by brackets, so that $[\,]$ denotes the empty multiset; $[a, a, b]$ denotes a multiset having two occurrences of the element a and one occurrence of b. We use $+$ for multiset union. Given a countable infinite set of *base types* $\alpha, \beta, \gamma, \ldots$ we consider **types** and **multiset types** defined by the following grammars:

(**types**) $\tau, \sigma, \rho ::= \alpha \mid \mathcal{M} \to \tau$ (**multiset types**) $\mathcal{M} ::= [\tau_i]_{i\in I}$ where I is a finite set

Observe that types are *strict*, *i.e.* the type on the right hand side of a functional type is never a multiset [16]. They make use of usual notations for non-idempotent intersection types via multisets, as in [21]. Thus for instance, an intersection type $[\sigma, \sigma, \tau]$ must be understood as $\sigma \wedge \sigma \wedge \tau$, where the intersection symbol $\wedge$ is defined to enjoy commutative and associative laws. When $\wedge$ verifies the axiom $\sigma \wedge \sigma = \sigma$, the underlying type system is called **idempotent**, otherwise, like in this paper, the type system is called **non-idempotent**.

Type assignments, denoted by $\Gamma, \Delta, \ldots$ are (possibly empty) finite sets of assignments of the form $x{:}\mathcal{M}$, where x is a variable and $\mathcal{M}$ is a *non-empty* multiset type. The **domain** of a type assignment Γ is given by $\mathtt{dom}(\Gamma) := \{x \mid x{:}\mathcal{M} \in \Gamma\}$. The **type of a variable** x **at the assignment** Γ is given by $\Gamma(x) := \mathcal{M}$ if $x{:}\mathcal{M} \in \Gamma$, and $\Gamma(x) := [\,]$ otherwise; so that $\Gamma(x) = [\,]$ iff $x \notin \mathtt{dom}(\Gamma)$. The **intersection of type assignments**, written $\Gamma + \Delta$, is defined by $(\Gamma + \Delta)(x) := \Gamma(x) + \Delta(x)$, where the symbol $+$ denotes multiset union. As a consequence $\mathtt{dom}(\Gamma + \Delta) = \mathtt{dom}(\Gamma) \cup \mathtt{dom}(\Delta)$. We write $\Gamma +_{i\in I} \Delta_i$ as an abbreviation of $\Gamma + \Delta_1 + \ldots + \Delta_n$, where $|I| = n$. The **disjoint union** of an assignment Γ and $x \notin \mathtt{dom}(\Gamma)$ is defined by $x{:}[\,]; \Gamma := \Gamma$ and $x{:}\mathcal{M}; \Gamma := \{x{:}\mathcal{M}\} \cup \Gamma$ if $\mathcal{M} \neq [\,]$. The assignment Γ **deprived of** x is defined by $\Gamma \setminus\!\!\setminus x := \Gamma \setminus \{x{:}\Gamma(x)\}$. The extension $\Gamma \setminus\!\!\setminus (x_1, \ldots, x_n)$ is defined as expected.

The Type Systems: Type judgments are triples of the form $\Gamma \vdash t{:}\tau$, where Γ is a type assignment, t is a term and τ is a type. The two type systems $\mathcal{HW}$ and $\mathcal{S}$ for the M-calculus are given respectively in Figure 1 and 2. A **(typing) derivation** in system X is a tree obtained by applying the typing rules of system X. The notation $\Gamma \vdash_X t{:}\tau$ is used if there is a derivation of the judgment $\Gamma \vdash t{:}\tau$ in system X. The term t is **typable** in the type system X, or X**-typable**, iff there is an assignment Γ and a type τ such that $\Gamma \vdash_X t{:}\tau$. We use the capital Greek letters $\Phi, \Psi, \ldots$ to name type derivations, *e.g.* we write $\Phi \triangleright \Gamma \vdash_X t{:}\tau$ or $\Phi_t \triangleright \Gamma \vdash_X t{:}\tau$. The **size** of a type derivation Φ is a positive natural number written $\mathtt{sz}(\Phi)$ and is defined as expected.

The rules $(\mathtt{ax})$, $(\to \mathtt{i})$ and $(\to \mathtt{e}_{\mathcal{HW}})$ in the type system $\mathcal{HW}$ come from a relational semantics for linear logic in [21]. Remark in particular the absence of weakened axioms and the use of *multiplicative* rules for application and substitution. A particular case of rule $(\to \mathtt{e}_{\mathcal{HW}})$ is when $I = \emptyset$: the subterm u occuring in the *typed* term tu turns out to be *untyped*. Thus for example, from the derivation $x{:}[\sigma] \vdash_{\mathcal{HW}} \lambda y.x{:}[\,] \to \sigma$ we can construct $x{:}[\sigma] \vdash_{\mathcal{HW}} (\lambda y.x)\Omega{:}\sigma$, where Ω is the non-terminating term $(\lambda z.zz)(\lambda z.zz)$. This is precisely the reason why rules $(\to \mathtt{e}_{\mathcal{S}})$ and $(\mathtt{cut}_{\mathcal{S}})$ in

Typing Rules $\{(\mathtt{ax}), (\rightarrow \mathtt{i})\}$ plus

$$\frac{x{:}[\sigma_i]_{i\in I}; \Gamma \vdash t{:}\tau \qquad (\Delta_i \vdash u{:}\sigma_i)_{i\in I\cup\{w\}}}{\Gamma +_{i\in I\cup\{w\}} \Delta_j \vdash t[x/u]{:}\tau}\ (\mathtt{cut}_{\mathcal{S}}) \qquad \frac{\Gamma \vdash t{:}[\sigma_i]_{i\in I} \rightarrow \tau \qquad (\Delta_i \vdash u{:}\sigma_i)_{i\in I\cup\{w\}}}{\Gamma +_{i\in I\cup\{w\}} \Delta_i \vdash tu{:}\tau}\ (\rightarrow \mathtt{e}_{\mathcal{S}})$$

Fig. 2. The Type System $\mathcal{S}$ for the M-Calculus

Figure 2, the system which characterizes strongly-normalizing terms, always asks a **witness** typing derivation for the arguments of applications and substitutions. Indeed, if $I = \emptyset$, then the argument u will be typed with the witness derivation $\Delta_w \vdash u{:}\sigma_w$, whatever the type σ_w is. This witness derivation for u is essential to guarantee strong-normalization of u (and thus of tu and $t[x/u]$). When $I \neq \emptyset$ the rules $(\rightarrow \mathtt{e}_{\mathcal{S}})$ and $(\mathtt{cut}_{\mathcal{S}})$ also require a witness derivation for u, whose use is necessary in order to deal with the c-rule when $|\mathtt{C}[\![x]\!]|_x = 1$ (see discussion after Lemma 4).

Given a derivation $\Phi \triangleright \Gamma \vdash_{\mathcal{HW}} t{:}\sigma$, not every free variable of t necessarily appears in the domain of Γ, this is for example the case in $x{:}[\sigma] \vdash_{\mathcal{HW}} (\lambda y.x)z{:}\sigma$. More precisely, the systems enjoy the following (weak/strong) relevance properties, that can be easily shown by induction on derivations.

Lemma 1. *If $\Phi \triangleright \Gamma \vdash_{\mathcal{HW}} t{:}\sigma$ then $\mathtt{dom}(\Gamma) \subseteq \mathtt{fv}(t)$. If $\Phi \triangleright \Gamma \vdash_{\mathcal{S}} t{:}\sigma$, then $\mathtt{dom}(\Gamma) = \mathtt{fv}(t)$.*

In contrast to many intersection type systems for explicit substitutions in the literature, the typing rules of systems $\mathcal{HW}$ and $\mathcal{S}$ are syntax oriented, so that generation lemmas are not needed to distinguish particular syntactical forms of derivations when proving the previous lemma.

3 Characterization of Linear-Head and Weak M-Normalization

In this section we show two main results. The first one (Section 3.1) characterizes linear-head M-normalizing terms with ES by means of $\mathcal{HW}$-typability. This result generalizes to calculi with ES the well-known characterization of *head* β-normalizing terms in the λ-calculus [17, 21], by giving in particular a logical argument instead of the operational one in [4]. The $\mathcal{HW}$-type system is known to type also some *non* weakly M-normalizing terms: for instance, if Ω is a non-terminating term, then $x{:}[\,] \rightarrow \sigma \vdash_{\mathcal{HW}} x\Omega{:}\sigma$ but $x\Omega$ is not weakly M-normalizing. We are then going to characterize the set of weakly M-normalizing terms, our second result in Section 3.2, by restricting the $\mathcal{HW}$-typing derivations to some particular ones. But first, we develop some key technical tools.

In order to understand which are the redex occurrences actually constrained by the type system, *i.e.* the reducible subterms that verify some particular type specification, let us consider a derivation $\Phi \triangleright \Gamma \vdash_{\mathcal{HW}} t{:}\tau$. A position $p \in \mathtt{pos}(t)$ is a **typed occurrence** of Φ if either $p = \epsilon$, or $p = ip'$ $(i = 0, 1)$ and $p' \in \mathtt{pos}(t|_i)$ is a typed occurrence of *some* of their corresponding subderivations of Φ. A redex occurrence of t which is also a typed occurrence of Φ is a **redex typed occurrence** or **redex T-occurrence** of t in Φ. Thus for example, given the following derivations Φ and Φ', we have that ϵ, 0, 1 and 10 are T-occurrences in Φ and Φ', while 11 is a T-occurrence in Φ but not in Φ'.

$$\Phi \triangleright \frac{\dfrac{}{x{:}[[\tau,\tau]\rightarrow\tau] \vdash x{:}[\tau,\tau]\rightarrow\tau} \quad \dfrac{y{:}[[\,]\rightarrow\tau] \vdash y{:}[\,]\rightarrow\tau}{y{:}[[\,]\rightarrow\tau] \vdash yz{:}\tau} \quad \dfrac{y{:}[[\tau]\rightarrow\tau] \vdash y{:}[\tau]\rightarrow\tau \qquad z{:}[\tau] \vdash z{:}\tau}{y{:}[[\tau]\rightarrow\tau], z{:}[\tau] \vdash yz{:}\tau}}{x{:}[[\tau,\tau]\rightarrow\tau], y{:}[[\,]\rightarrow\tau, [\tau]\rightarrow\tau], z{:}[\tau] \vdash x(yz){:}\tau}$$

$$\Phi' \triangleright \frac{\dfrac{}{x{:}[[\tau,\tau]\rightarrow\tau] \vdash x{:}[\tau,\tau]\rightarrow\tau} \quad \dfrac{y{:}[[\,]\rightarrow\tau] \vdash y{:}[\,]\rightarrow\tau}{y{:}[[\,]\rightarrow\tau] \vdash yz{:}\tau} \quad \dfrac{y{:}[[\,]\rightarrow\tau] \vdash y{:}[\,]\rightarrow\tau}{y{:}[[\,]\rightarrow\tau] \vdash yz{:}\tau}}{x{:}[[\tau,\tau]\rightarrow\tau], y{:}[[\,]\rightarrow\tau, [\,]\rightarrow\tau] \vdash x(yz){:}\tau}$$

The notion of T-occurrence plays a key role in the Subject Reduction (SR) lemma, which is based on a subtle *partial* substitution lemma, a refinement of the standard substitution lemmas used in the λ-calculus. See Appendix A for details.

Lemma 2 (SR I). *Let $\Phi \triangleright \Gamma \vdash_{\mathcal{HW}} t{:}\tau$. If $t \to_{\mathtt{M}} t'$ reduces a $(\mathtt{dB}, \mathtt{c}, \mathtt{w})$-redex T-occurrence of t in Φ then $\Phi' \triangleright \Gamma \vdash_{\mathcal{HW}} t'{:}\tau$ and $\mathtt{sz}(\Phi) > \mathtt{sz}(\Phi')$.*

Indeed, consider a derivation $\Phi'' \triangleright y{:}[[\,] \to [\,] \to \tau] \vdash_{\mathcal{HW}} (xxx)[x/y]{:}\tau$. Then the (typed) reduction step $(xxx)[x/y] \to_{\mathtt{c}} (yxx)[x/y]$ decreases the measure of Φ'' but thereafter $(yxx)[x/y] \to_{\mathtt{c}} (yyx)[x/y] \to_{\mathtt{c}} (yyy)[x/y]$ are not decreasing reduction steps since they act on untyped occurrences.

As a corollary, termination holds for *any* strategy reducing only redexes in T-occurrences, this is an important key point that will be used in Sections 3.1 an 3.2.

Corollary 1. *If $\Phi \triangleright \Gamma \vdash_{\mathcal{HW}} t{:}\tau$, then any $\mathtt{M}$-reduction sequence contracting only $(\mathtt{dB}, \mathtt{c}, \mathtt{w})$-redex T-occurrences is finite.*

Types of terms can also be recovered by means of a Subject Expansion (SE) lemma, a property which will be particular useful in Sections 3.1 and 3.2, and which reads as follows (see Appendix A for details):

Lemma 3 (SE I). *If $\Gamma \vdash_{\mathcal{HW}} t'{:}\tau$ and $t \to_{\mathtt{M}} t'$ then $\Gamma \vdash_{\mathcal{HW}} t{:}\tau$.*

3.1 Linear-Head M-Normalization

Linear-head reduction [19, 36] comes from a fine notion of evaluation for proof nets [26]. It is a particular reduction strategy of the M-calculus although it is not a strategy of β-reduction. In contrast to the standard deterministic notion of *head*-reduction, $\to_{\mathtt{L_H M}}$ is non-deterministic, and in particular, it can occur under λ-abstractions, *e.g.* $y[y/w][x/z] {}_{\mathtt{L_H M}}\!\leftarrow (\lambda x.y[y/w])z \to_{\mathtt{L_H M}} (\lambda x.w[y/w])z$. This behaviour is however safe since $\to_{\mathtt{L_H M}}$ has the diamond property [6].

Another remarkable property of linear-head reduction is that the hole of the head contexts $\mathtt{L_H}$ cannot be duplicated nor erased. This is related to a recent result [3] stating that linear-head reduction is *standard* for the M-calculus, as well as *left-to-right* reduction is standard for the λ-calculus. Other applications concern Abstract Machines [19], π-calculus [2] and cost models [4].

We now refine a known result in the λ-calculus which characterizes *head*-normalizing terms by means of intersection types, either idempotent [17, 7][3] or non-idempotent [21]. Indeed, the set of linear-head M-normalizing terms coincides with the set of $\mathcal{HW}$-typable terms.

Lemma 4. *If $\Phi \triangleright \Gamma \vdash_{\mathcal{HW}} t{:}\tau$ and t has no $(\mathtt{dB}, \mathtt{c}|_{\mathtt{L_H}})$-redexes in T-occurrences in Φ, then $t \in \mathtt{L_H M}$-nf.*

It is worth noticing that Lemma 4 does not hold for *head*-nfs [4]. Indeed, the term $(yxx)[x/y]$ in the example just after Lemma 2 does not have any redex in a T-occurrence (the only two c-redexes occurrences are untyped), and is not a *head*-nf. This emphasizes the fact that linear-head reduction is more pertinent for calculi with ES than head reduction. We can conclude this section by

Theorem 1. *Let $t \in \mathcal{T}_{\mathtt{M}}$. Then $t \in \mathcal{L_H N}(\mathtt{M})$ iff t is $\mathcal{HW}$-typable.*

Proof. Let $t \in \mathcal{L_H N}(\mathtt{M})$. We proceed by induction on the length of the linear-head M-normalizing reduction using Lemma 3 (see Lemma 25 in the Appendix A for details).

Let t be $\mathcal{HW}$-typable. By Corollary 1 the strategy consisting in the contraction of $(\mathtt{dB}, \mathtt{c}|_{\mathtt{L_H}})$-redex T-occurrences terminates in a term t' without such redexes. The term t' is typable by Lemma 2 and then t' turns out to be a $\mathtt{L_H M}$-nf by Lemma 4. Thus, $t \in \mathcal{L_H N}(\mathtt{M})$.

[3] Although idempotency was not explicity mentioned in [17], a remark on pp. 55 points out the meaninglessness of duplication of types in a sequence.

[4] A head-nf is a λ-term having the shape $\lambda x_1 \ldots \lambda x_n.yt_1 \ldots t_m$, with $n, m \geq 0$

3.2 Weak $\mathtt{M}$-Normalization

In this section we use the type system $\mathcal{HW}$ to characterize weakly $\mathtt{M}$-normalizing terms, a result that extends the well-known characterization [17] of weakly β-normalizing in the λ-calculus. As in [17, 13], $\mathcal{HW}$-typability alone does not suffice to characterize weak $\mathtt{M}$-normalizing terms (see an example at the beginning of Section 3). The type $[\,]$ plays a similar rôle the universal ω type in [17, 13], although it is restricted to only occur in the type of the domain of a function that accepts any kind of argument. We then restrict the allowed typing derivations in order to recover such a characterization. Indeed, the set of **positive** (resp. **negative**) subtypes of a type is the smallest set satisfying the following conditions (*cf.*[13]).

- $A \in \mathcal{P}(A)$.
- $A \in \mathcal{P}([\sigma_i]_{i\in I})$ if $\exists i\; A \in \mathcal{P}(\sigma_i)$; $A \in \mathcal{N}([\sigma_i]_{i\in I})$ if $\exists i\; A \in \mathcal{N}(\sigma_i)$.
- $A \in \mathcal{P}(\mathcal{M} \to \tau)$ if $A \in \mathcal{N}(\mathcal{M})$ or $A \in \mathcal{P}(\tau)$; $A \in \mathcal{N}(\mathcal{M} \to \tau)$ if $A \in \mathcal{P}(\mathcal{M})$ or $A \in \mathcal{N}(\tau)$.
- $A \in \mathcal{P}(\Gamma)$ if $\exists\; y \in \mathtt{dom}(\Gamma)$ s.t. $A \in \mathcal{N}(\Gamma(y))$; $A \in \mathcal{N}(\Gamma)$ if $\exists\; y \in \mathtt{dom}(\Gamma)$ s.t. $A \in \mathcal{P}(\Gamma(y))$.
- $A \in \mathcal{P}(\langle \Gamma, \tau\rangle)$ if $A \in \mathcal{P}(\Gamma)$ or $A \in \mathcal{P}(\tau)$; $A \in \mathcal{N}(\langle \Gamma, \tau\rangle)$ if $A \in \mathcal{N}(\Gamma)$ or $A \in \mathcal{N}(\tau)$.

As an example, $[\,] \in \mathcal{P}([\,])$, so that $[\,] \in \mathcal{N}([\,] \to \sigma)$, $[\,] \in \mathcal{P}(x{:}[[\,] \to \sigma])$ and $[\,] \in \mathcal{P}(\langle x{:}[[\,] \to \sigma], \sigma\rangle)$.

Lemma 5. *Let $\Phi \triangleright \Gamma \vdash_{\mathcal{HW}} t{:}\tau$ s.t. $[\,] \notin \mathcal{P}(\langle \Gamma, \tau\rangle)$. If t has no $(\mathtt{dB}, \mathtt{c}, \mathtt{w})$-redex T-occurrences in Φ, then $t \in \mathtt{M}$-nf.*

Theorem 2. *Let $t \in \mathcal{T}_{\mathtt{M}}$. Then, $t \in \mathcal{WN}(\mathtt{M})$ iff $\Gamma \vdash_{\mathcal{HW}} t{:}\tau$ and $[\,] \notin \mathcal{P}(\langle \Gamma, \tau\rangle)$.*

Proof. If $t \in \mathcal{WN}(\mathtt{M})$, we proceed by induction on the length of the $\mathtt{M}$-normalizing reduction sequence using Lemma 3 (see Lemma 28 in the Appendix A for details).

Suppose $\Gamma \vdash_{\mathcal{HW}} t{:}\tau$ and $[\,] \notin \mathcal{P}(\langle \Gamma, \tau\rangle)$. By Corollary 1 the strategy of contracting only redex T-occurrences terminates in a term t' without such redexes. The term t' is typable by Lemma 2 and then t' turns out to be a $\mathtt{M}$-nf by Lemma 5. Thus, $t \in \mathcal{WN}(\mathtt{M})$.

4 Characterization of Strong $\mathtt{M}$-Normalization

In this section we show the third main result concerning the $\mathtt{M}$-calculus that is a characterization of the set of strongly $\mathtt{M}$-normalizing terms by means of $\mathcal{S}$-typability. The proof is done in several steps. The first key point is the characterization of the set of strongly $\mathtt{M}\backslash\mathtt{w}$-normalizing terms (instead of $\mathtt{M}$-normalizing terms). For that, SR and SE lemmas for the $\mathcal{S}$-type system are needed, and an inductive characterization of the set $\mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$, based on Regnier's σ-equivalence [40], turns out to be helpful to obtain them. The second key point is the equivalence between strongly $\mathtt{M}$ and $\mathtt{M}\backslash\mathtt{w}$-normalizing terms. While the inclusion $\mathcal{SN}(\mathtt{M}) \subseteq \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ is straightforward, the fact that every $\mathtt{w}$-reduction step can be *postponed* w.r.t. any $\mathtt{M}\backslash\mathtt{w}$-step (Lemma 10) turns out to be crucial to show $\mathcal{SN}(\mathtt{M}\backslash\mathtt{w}) \subseteq \mathcal{SN}(\mathtt{M})$.

We first introduce the **head graphical equivalence** $\sim$ on $\mathtt{M}$-terms, given by the contextual, transitive, symmetric and reflexive closure of the axiom $(tv)[x/u] \approx t[x/u]v$, where $x \notin \mathtt{fv}(v)$. This equivalence, which comes from Regnier's σ-equivalence [40] on λ-terms (resp. σ-equivalence on terms with ES [5]), preserves types, a property used to perform some *safe* transformations of terms in order to inductively define the set $\mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ (*cf.* clause (E)). We also need lemmas for SE and SR (their proofs can be found in Appendix B).

Lemma 6 (Invariance for $\sim$). *Let $t, t' \in \mathcal{T}_{\mathtt{M}}$ such that $t \sim t'$. Then, 1) $\eta_{\mathtt{M}\backslash\mathtt{w}}(t) = \eta_{\mathtt{M}\backslash\mathtt{w}}(t')$. 2) $\Phi \triangleright \Gamma \vdash_{\mathcal{S}} t{:}\tau$ iff $\Phi' \triangleright \Gamma \vdash_{\mathcal{S}} t'{:}\tau$. Moreover, $\mathtt{sz}(\Phi) = \mathtt{sz}(\Phi')$.*

In contrast to system $\mathcal{HW}$, whose typing measure $\mathtt{sz}()$ is only decreasing w.r.t. reduction of redex *typed* occurrences, the system $\mathcal{S}$ enjoys a stronger subject reduction property which guarantees that *every* reduction decreases the measure $\mathtt{sz}()$ of terms (whose redexes are now all typed).

Lemma 7 (SR II). *Let $\Phi \triangleright \Gamma \vdash_{\mathcal{S}} t{:}\tau$. If $t \to_{\mathtt{M}\backslash\mathtt{w}} t'$ then $\Phi' \triangleright \Gamma \vdash_{\mathcal{S}} t'{:}\tau$ and $\mathtt{sz}(\Phi) > \mathtt{sz}(\Phi')$.*

The "witnesses derivation everywhere" in the typing rules ($\to \mathtt{e}_{\mathcal{S}}$) and ($\mathtt{cut}_{\mathcal{S}}$) is justified by the proofs of the Lemma above. Indeed, let $\Phi \triangleright \Gamma \vdash x[x/u]{:}\tau$ and $x[x/u] \to_{\mathtt{c}} u[x/u]$ thus $\Phi' \triangleright \Gamma \vdash u[x/u]{:}\tau$. In a type system without "witness everywhere", one would have $\mathtt{sz}(\Phi) = \mathtt{sz}(\Phi_u) + 2 \leq 2\mathtt{sz}(\Phi_u) + 1 = \mathtt{sz}(\Phi')$. Therefore, in order to decrease the measure $\mathtt{sz}(\_)$, a witness derivation of u must also be required in the typing derivation of $x[x/u]$ so that one gets $\mathtt{sz}(\Phi) = 2 \cdot \mathtt{sz}(\Phi_u) + 2 > 2 \cdot \mathtt{sz}(\Phi_u) + 1 = \mathtt{sz}(\Phi')$. An alternative approach would be to change the operational semantics of the calculus, spliting the $\mathtt{c}$-rule in two cases: $|t|_x = 1$ would be handled by the dereliction rule $t[x/u] \to_{\mathtt{d}} t\{x/u\}$ and $|t|_x > 1$ by our $\mathtt{c}$-rule. The witness derivation would then be required only when $I = \emptyset$.

Notice that $\to_{\mathtt{w}}$-reduction also decreases the measure $\mathtt{sz}(\_)$ but the type assignment Γ may change, *i.e.* decrease.

Lemma 8 (SE II). *Let $\Gamma \vdash_{\mathcal{S}} t'{:}\tau$. If $t \to_{\mathtt{M}\backslash\mathtt{w}} t'$ then $\Gamma \vdash_{\mathcal{S}} t{:}\tau$.*

Notice that expansion does not hold for $\to_{\mathtt{w}}$-reduction. Thus for example $x : [\sigma] \vdash_{\mathcal{S}} x{:}\sigma$ and $x[y/\Omega] \to_{\mathtt{w}} x$, but $x : [\sigma] \vdash_{\mathcal{S}} x[y/\Omega]{:}\sigma$ does not hold.

These technical tools are now used to prove that $\mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ coincides exactly with the set of $\mathcal{S}$-typable terms. To close the picture, *i.e.* to show that also $\mathcal{SN}(\mathtt{M})$ coincides with the set of $\mathcal{S}$-typable terms, we establish an equivalence between $\mathcal{SN}(\mathtt{M})$ and $\mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$. This is done constructively thanks to the use of an inductive definition for $\mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$. Indeed, the **inductive set of $\mathtt{M}\backslash\mathtt{w}$-strongly-normalizing terms**, written $\mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, is the smallest subset of $\mathcal{T}_{\mathtt{M}}$ that satisfies the following properties:

- (V) If $t_1, \ldots, t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, then $xt_1 \ldots t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$.
- (L) If $t \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, then $\lambda x.t \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$.
- (W) If $t, s \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ and $|t|_x = 0$, then $t[x/s] \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$.
- (B) If $u[x/v]t_1, \ldots, t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, then $(\lambda x.u)vt_1, \ldots, t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$.
- (C) If $\mathtt{C}[\![u]\!][x/u] \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, then $\mathtt{C}[\![x]\!][x/u] \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$.
- (E) If $(tu)[x/s] \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ and $|u|_x = 0$, then $t[x/s]u \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$.

Notice that use of the $\sim$-equivalence in the last clause. It is not surprising that $\mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ turns out to be equivalent to $\mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ (see Lemma 32 in Appendix B for details). We then have:

Lemma 9. *Let $t \in \mathcal{T}_{\mathtt{M}}$. If $t \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ then t is $\mathcal{S}$-typable.*

Proof. We use the equality $\mathcal{SN}(\mathtt{M}\backslash\mathtt{w}) = \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ to reason by induction on $t \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$. The proof also uses Lemma 8. See Appendix B for details.

In order to infer $\mathcal{SN}(\mathtt{M}\backslash\mathtt{w}) \subseteq \mathcal{SN}(\mathtt{M})$, the following postponement property is crucial.

Lemma 10 (Postponement). *Let $v \in \mathcal{T}_{\mathtt{M}}$. If $v \to^+_{\mathtt{w}} \to_{\mathtt{M}\backslash\mathtt{w}} v'$ then $v \to_{\mathtt{M}\backslash\mathtt{w}} \to^+_{\mathtt{w}} v'$.*

Proof. We first show by cases $v \to_{\mathtt{w}} \to_{\mathtt{M}\backslash\mathtt{w}} v'$ implies $v \to_{\mathtt{M}\backslash\mathtt{w}} \to^+_{\mathtt{w}} v'$. Then, the statement holds by induction on the number of $\mathtt{w}$-steps from v.

Lemma 11 (From $\mathtt{M}\backslash\mathtt{w}$ to $\mathtt{M}$). *Let $t \in \mathcal{T}_{\mathtt{M}}$. If $t \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$, then $t \in \mathcal{SN}(\mathtt{M})$.*

Proof. We show that any reduction sequence $\rho : t \to_{\mathtt{M}} \ldots$ is finite by induction on the pair $\langle t, n\rangle$, where n is the maximal number such that ρ can be decomposed as $\rho : t \to^n_{\mathtt{w}} t' \to_{\mathtt{M}\setminus\mathtt{w}} t'' \to \ldots$ (this is well-defined since $\to_{\mathtt{w}}$ is trivially terminating). We compare the pair $\langle t, n\rangle$ using $\to_{\mathtt{M}\setminus\mathtt{w}}$ for the first component (this is well-founded since $t \in \mathcal{SN}(\mathtt{M}\setminus\mathtt{w})$ by hypothesis) and the standard order on natural numbers for the second one. When the reduction sequence starts with at least one $\mathtt{w}$-step we conclude by Lemma 10. All the other cases are straightforward.

We conclude this section with the third main theorem for $\mathtt{M}$-calculus:

Theorem 3. *Let $t \in \mathcal{T}_{\mathtt{M}}$. Then t is $\mathcal{S}$-typable iff $t \in \mathcal{SN}(\mathtt{M})$.*

Proof. Let $\Phi \triangleright \Gamma \vdash_{\mathcal{S}} t{:}\tau$. Assume $t \notin \mathcal{SN}(\mathtt{M}\setminus\mathtt{w})$ so that $\exists\infty$ sequence $t = t_0 \to_{\mathtt{M}\setminus\mathtt{w}} t_1 \to_{\mathtt{M}\setminus\mathtt{w}} t_2 \to_{\mathtt{M}\setminus\mathtt{w}} \cdots$. By Lemma 7 $\Phi_i \triangleright \Gamma \vdash t_i{:}\tau$ for very i, and $\exists\infty$ sequence $\mathtt{sz}(\Phi_0) > \mathtt{sz}(\Phi_1) > \mathtt{sz}(\Phi_2) > \ldots$, which leads to a contradiction. Therefore, $t \in \mathcal{SN}(\mathtt{M}\setminus\mathtt{w}) \subseteq_{\text{Lemma 11}} \mathcal{SN}(\mathtt{M})$.

For the converse, $t \in \mathcal{SN}(\mathtt{M}) \subseteq \mathcal{SN}(\mathtt{M}\setminus\mathtt{w})$ because $\to_{\mathtt{M}\setminus\mathtt{w}} \subseteq \to_{\mathtt{M}}$. We conclude by Lemma 9.

5 An Intuitionistic Term Calculus in Gentzen Sequent Style

In this second part of the paper we develop linear-head, weak and strong normalization characterizations for a term calculus based on the intuitionistic sequent calculus. We first give the syntax and the operational semantics of the $\mathtt{J}$-calculus (including the notion of linear-head reduction). We then introduce the two type systems, called again respectively $\mathcal{HW}$ and $\mathcal{S}$.

Syntax: We use Herbelin's syntax [27]. We consider a countable infinite set of symbols $x, y, z, \ldots$. Three different syntactic categories for objects ($\mathcal{O}_{\mathtt{J}}$), terms ($\mathcal{T}_{\mathtt{J}}$) and lists ($\mathcal{L}_{\mathtt{J}}$) are defined by the following grammars:

$$(\textbf{objects})\ o, p ::= t \mid l \qquad (\textbf{terms})\ t, u ::= xl \mid tl \mid \lambda x.t \mid t[x/t] \qquad (\textbf{lists})\ l, m ::= \mathtt{nil} \mid t; l$$

The term xl is a **headed list**, tl is an **application**, $\lambda x.t$ an **abstraction**, $t[x/u]$ a **closure** affected by an **explicit substitution** $[x/u]$, $\mathtt{nil}$ the **empty list** and $t; l$ a **nonempty list**. We write $tl_1 \ldots l_n$ for $(\ldots(tl_1)\ldots l_n)$ and $x_{\mathtt{nil}}$ for $x\ \mathtt{nil}$. The **size** of the object o is denoted by $|o|$. Remark that the symbol x alone is not an object of the calculus. In contrast to [27] we do not consider a concatenation symbol for lists inside the syntax, but we define a meta-operation of concatenation of lists by induction as follows: $\mathtt{nil}@l := l$ and $(u; m)@l := u; (m@l)$. Moreover, explicit substitutions do not apply to lists, but only to terms, *i.e.* $l[x/u]$ is not in the grammar.

Free and **bound** symbols of objects are defined as expected, written resp. $\mathtt{fs}(o)$ and $\mathtt{bs}(o)$, and $|o|_x$ denotes the **number of free occurrences of the symbol** x **in** o. As before, we work with the standard notion of α-conversion. Positions, subterms and term occurrences are defined as expected, in particular, $0 \in \mathtt{pos}(xl)$ and $(xl)|_0 = l$ since the symbol x is not a subterm of xl (it is not even a term). We also consider two notions of contexts given by the following grammars:

$$(\textbf{list contexts})\ \mathtt{L} ::= \Box \mid \mathtt{L}[x/t] \qquad (\textbf{object contexts})\ \begin{array}{l} \mathtt{O}, \mathtt{P} ::= \mathtt{C} \mid \mathtt{V} \\ \mathtt{C}, \mathtt{D} ::= \Box \mid x\mathtt{V} \mid \mathtt{C}l \mid \lambda y.\mathtt{C} \mid \mathtt{C}[y/u] \mid t[y/\mathtt{C}] \mid t\mathtt{V} \\ \mathtt{V}, \mathtt{U} ::= \mathtt{C}; l \mid t; \mathtt{V} \end{array}$$

Operational Semantics: The **$\mathtt{J}$-calculus** is given by the set of objects $\mathcal{O}_{\mathtt{J}}$ and the **reduction relation** $\to_{\mathtt{J}}$ on $\mathcal{O}_{\mathtt{J}}$ defined as the closure by contexts $\mathtt{O}$ of the following rewriting rules:

$$\begin{array}{lll} \mathtt{L}[\lambda x.t]\mathtt{nil} & \mapsto_{\mathtt{dB_{nil}}} & \mathtt{L}[\lambda x.t] \\ \mathtt{L}[\lambda x.t](u; l) & \mapsto_{\mathtt{dB_{cons}}} & \mathtt{L}[t[x/u]l] \\ \mathtt{C}[\![x\, l]\!][x/u] & \mapsto_{\mathtt{c}} & \mathtt{C}[\![u\, l]\!][x/u] \end{array} \qquad \begin{array}{llll} t[x/u] & \mapsto_{\mathtt{w}} & t & \text{if } |t|_x = 0 \\ \mathtt{L}[xl]m & \mapsto_{@_{\mathtt{var}}} & \mathtt{L}[x(l@m)] & \\ \mathtt{L}[tl]m & \mapsto_{@_{\mathtt{app}}} & \mathtt{L}[t(l@m)] & \end{array}$$

The reader will notice some differences from the reduction rules in [27]. First of all, the use of the meta-operation for concatenating lists in the rules $\mapsto_{@_{\mathtt{var}}}$ and $\mapsto_{@_{\mathtt{app}}}$ replaces the explicit concatenation rules in [27]. This is particularly convenient since we only reduce objects that are terms (even if these terms occur inside lists) and so the proofs are simpler/shorter because there are less rules and only of one kind. Another difference from [27] is the use of rules *at a distance*, specified by means of list contexts $\mathtt{L}$, as we did in the $\mathtt{M}$-calculus. Moreover, the rule $\mapsto_{\mathtt{c}}$ is not exactly the same as that of $\mathtt{M}$-calculus (even if denoted by the same name); the reason being that $\mathtt{C}[\![x]\!]$ is meaningless in the $\mathtt{J}$-calculus because the symbol x does not belong to the set $\mathcal{O}_{\mathtt{J}}$. Last, but not least, the operational semantics of the $\mathtt{J}$-calculus prevents the useless duplication of empty resources which does happens in [27], *e.g.* the reduction step $(tl)[x/u] \to t[x/u]l[x/u]$, where $|tl|_x = 0$.

The notion of **redex occurrence** follows the same idea used for the $\mathtt{M}$-calculus. Thus, we define a position $p \in \mathtt{pos}(t)$ to be a X**-redex** (for $X \in \{\mathtt{dB}_{\mathtt{nil}}, \mathtt{dB}_{\mathtt{cons}}, \mathtt{W}, @_{\mathtt{var}}, @_{\mathtt{app}}\}$) if $t|_p$ has the form of the left-hand side of the rule $\mapsto_X$, and $p \in \mathtt{pos}(t)$ is a $\mathtt{c}$**-redex** if $p = p_1 p_2$, where $t|_{p_1} = \mathtt{C}[\![xl]\!][x/u]$ and $\mathtt{C}|_{p_2} = \Box$. For example 00 and 01 are both $\mathtt{c}$-redex occurrences of the term $x_{\mathtt{nil}}[y/x_{\mathtt{nil}}][x/z_{\mathtt{nil}}]$.

The reduction relation $\to_{\mathtt{J}}$ can also be refined. We write $\to_X$ for the closure by contexts $\mathtt{O}$ of the rewriting rule $\mapsto_X$ for every X. We define $\mathtt{B@} := \{\mathtt{dB}_{\mathtt{nil}} \cup \mathtt{dB}_{\mathtt{cons}} \cup @_{\mathtt{var}} \cup @_{\mathtt{app}}\}$ and $\to_{\mathtt{B@}} := \bigcup_{X \in \mathtt{B@}} \to_X$. The **non-erasing** reduction relation $\to_{\mathtt{J}\setminus\mathtt{W}}$ is given by $\to_{\mathtt{B@}\cup\mathtt{c}}$, *i.e.* $\to_{\mathtt{J}\setminus\mathtt{W}} = \to_{\mathtt{J}} \setminus \to_{\mathtt{W}}$, and plays a key role in the characterization of strongly normalizing terms in Section 7.

To define linear-head $\mathtt{J}$-reduction we first introduce the set of **linear-head contexts** that are generated by the grammar: $\mathtt{L_H} ::= \Box \mid \lambda x.\mathtt{L_H} \mid \mathtt{L_H} l \mid \mathtt{L_H}[x/t]$, obtained by adapting the one for the $\mathtt{M}$-calculus given in Section 2. **Linear-head J-reduction**, written $\to_{\mathtt{L_H J}}$, is the closure under *linear-head contexts* of the relation generated by the rules $\mathtt{B@} \cup \{\mathtt{c}|_{\mathtt{L_H}}\}$, where $\mapsto_{\mathtt{c}|_{\mathtt{L_H}}}$ is given by:

$$\mathtt{L_H}[\![xl]\!][x/u] \mapsto_{\mathtt{c}|_{\mathtt{L_H}}} \mathtt{L_H}[\![ul]\!][x/u]$$

As for the rule $\mapsto_{\mathtt{c}}$ introduced in Section 2, the hole of the context $\mathtt{L_H}$ contains the term xl instead of the symbol x, which is not an object of this calculus. An object o is **linear-head J-normalizing**, written $t \in \mathcal{L_H N}(\mathtt{J})$, iff t has an $\mathtt{L_H J}$-normal form.

As expected the postponement property also holds in this calculus:

Lemma 12 (Postponement). *Let* $o \in \mathcal{O}_{\mathtt{J}}$. *If* $o \to_{\mathtt{W}}^{+} \to_{\mathtt{J}\setminus\mathtt{W}} o'$ *then* $o \to_{\mathtt{J}\setminus\mathtt{W}} \to_{\mathtt{W}}^{+} o'$.

The Type Systems: The set of **types** is the same that we considered in Section 2. The symbol $\_$ is called the **empty stoup**. A **stoup** Σ is either a type σ or the empty stoup. **Type environments** are pairs of the form $\Gamma \mid \Sigma$, where Γ is a type assignment and Σ is a stoup. **Type judgments** are triples of the form $\Gamma \mid \Sigma \vdash o{:}\tau$, where o is an object, $\Gamma \mid \Sigma$ a type environment and τ a type. The two type systems for the $\mathtt{J}$-calculus, called again $\mathcal{HW}$ and $\mathcal{S}$, and given respectively in Figure 3 and 4, are used to derive type judgments of the form $\Gamma \mid \_ \vdash t{:}\tau$ and $\Gamma \mid \sigma \vdash l{:}\tau$, where t is a term and l is a list. Remark the absence of weakened axioms and the multiplicative presentation of the rules, resulting in a logical system which can be shown to be equivalent to the original one [27]. As before $\Gamma \mid \Sigma \vdash_{\mathcal{HW}} o{:}\tau$ (resp. $\Gamma \mid \Sigma \vdash_{\mathcal{S}} o{:}\tau$) denotes derivability in system $\mathcal{HW}$ (resp. $\mathcal{S}$). The **hlist-size** of the type derivation Φ is a positive natural number written $\mathtt{sz2}(\Phi)$ which denotes the size of Φ where every node $\mathtt{hlist}$ counts 2 and the other ones count 1.

The two systems are syntax oriented so we do not need generation lemmas. The following (weak/strong) relevance properties can easily be shown by induction on derivations.

Lemma 13. *If* $\Gamma \mid \Sigma \vdash_{\mathcal{HW}} o{:}\tau$, *then* $\mathtt{dom}(\Gamma) \subseteq \mathtt{fs}(o)$. *If* $\Gamma \mid \Sigma \vdash_{\mathcal{S}} o{:}\tau$, *then* $\mathtt{dom}(\Gamma) = \mathtt{fs}(o)$.

$$\frac{}{\emptyset \mid \tau \vdash \mathtt{nil}{:}\tau}\ (\mathtt{ax}) \qquad \frac{\Gamma \mid \_ \vdash t{:}\tau}{\Gamma \setminus\!\setminus x \mid \_ \vdash \lambda x.t{:}\Gamma(x) \to \tau}\ (\to \mathtt{r}) \qquad \frac{(\Delta_i \mid \_ \vdash u{:}\sigma_i)_{i\in I} \qquad x{:}[\sigma_i]_{i\in I}; \Gamma \mid \_ \vdash t{:}\tau}{\Gamma +_{i\in I} \Delta_i \mid \_ \vdash t[x/u]{:}\tau}\ (\mathtt{es}_{\mathcal{HW}})$$

$$\frac{\Gamma \mid \sigma \vdash l{:}\tau}{\Gamma + \{x{:}[\sigma]\} \mid \_ \vdash xl{:}\tau}\ (\mathtt{hlist}) \quad \frac{\Gamma \mid \_ \vdash t{:}\sigma \qquad \Delta \mid \sigma \vdash l{:}\tau}{\Gamma + \Delta \mid \_ \vdash tl{:}\tau}\ (\mathtt{app}) \quad \frac{(\Delta_i \mid \_ \vdash t{:}\sigma_i)_{i\in I} \qquad \Gamma \mid \sigma \vdash l{:}\tau}{\Gamma +_{i\in I} \Delta_i \mid [\sigma_i]_{i\in I} \to \sigma \vdash t; l{:}\tau}\ (\to \mathtt{l}_{\mathcal{HW}})$$

Fig. 3. The Type System $\mathcal{HW}$ for the J-Calculus

Typing Rules $\{(\mathtt{ax}), (\mathtt{hlist}), (\mathtt{app}), (\to \mathtt{r})\}$ plus

$$\frac{(\Delta_i \mid \_ \vdash t{:}\sigma_i)_{i\in I\cup\{w\}} \qquad \Gamma \mid \sigma \vdash l{:}\tau}{\Gamma +_{i\in I\cup\{w\}} \Delta_i \mid [\sigma_i]_{i\in I} \to \sigma \vdash t; l{:}\tau}\ (\to \mathtt{l}_{\mathcal{S}}) \quad \frac{(\Delta_i \mid \_ \vdash u{:}\sigma_i)_{i\in I\cup\{w\}} \qquad x{:}[\sigma_i]_{i\in I}; \Gamma \mid \_ \vdash t{:}\tau}{\Gamma +_{i\in I\cup\{w\}} \Delta_i \mid \_ \vdash t[x/u]{:}\tau}\ (\mathtt{es}_{\mathcal{S}})$$

Fig. 4. The Type System $\mathcal{S}$ for the J-Calculus

6 Characterization of Linear-Head and Weak J-Normalization

In this section we show the characterizations of linear-head and weakly normalizing terms by means of $\mathcal{HW}$-typability. We use the same notion of (redex) T-occurrence introduced in Section 3 for the M-calculus. The proof of the SR lemma can be found in Appendix C, and makes use of a subtle partial substitution lemma as well as the measure $\mathtt{sz2}()$ introduced in Section 5.

Lemma 14 (SR III). *Let $\Phi \triangleright \Gamma \mid \Sigma \vdash_{\mathcal{HW}} o{:}\tau$. If $o \to_{\mathtt{J}} o'$ reduces a $(\mathtt{B@}, \mathtt{c}, \mathtt{w})$-redex T-occurrence of o in Φ then $\Phi' \triangleright \Gamma \vdash_{\mathcal{HW}} o'{:}\tau$ and $\mathtt{sz2}(\Phi) > \mathtt{sz2}(\Phi')$.*

To illustrate the need of the measure $\mathtt{sz2}()$ instead of the size $\mathtt{sz}()$ used in Section 2 consider the derivation $\Phi \triangleright y{:}\sigma \mid \_ \vdash_{\mathcal{HW}} x_{\mathtt{nil}}[x/y_{\mathtt{nil}}]{:}\sigma$ and the reduction step $t = x_{\mathtt{nil}}[x/y_{\mathtt{nil}}] \to_{\mathtt{c}} y_{\mathtt{nil}}\mathtt{nil}[x/y_{\mathtt{nil}}] = t'$. Let Φ' be the typing derivation obtained from Φ for the term t'. Then it is not difficult to see that the size of the derivation Φ is not strictly bigger than that of Φ' if we only count 1 for the $(\mathtt{hlist})$ rules (indeed, the size is 5 for both terms). However, $7 = \mathtt{sz2}(\Phi) > \mathtt{sz2}(\Phi') = 6$.

Corollary 2. *If $\Phi \triangleright \Gamma \mid \Sigma \vdash_{\mathcal{HW}} o{:}\tau$, then any J-reduction sequence contracting only $(\mathtt{B@}, \mathtt{c}, \mathtt{w})$-redex T-occurrences is finite.*

As expected, subject expansion also holds in this framework:

Lemma 15 (SE III). *If $\Gamma \mid \Sigma \vdash_{\mathcal{HW}} o'{:}\tau$ and $o \to_{\mathtt{J}} o'$ then $\Gamma \mid \Sigma \vdash_{\mathcal{HW}} o{:}\tau$.*

We first use $\mathcal{HW}$-typability to characterize linear-head normalization for the J-calculus, which can be seen as a particular reduction strategy of the relation $\to_{\mathtt{J}}$, even if it induces non-deterministic behaviours. This is however safe since $\to_{\mathtt{L_H J}}$ has the diamond property. The characterization of linear-head normalizing terms follows from the lemma below.

Lemma 16. *If $\Phi \triangleright \Gamma \mid \_ \vdash_{\mathcal{HW}} u{:}\tau$ and u has no $(\mathtt{B@}, \mathtt{c}|_{\mathtt{L_H}})$-redex T-occurrences in Φ then $u \in \mathtt{L_H J}$-nf.*

Theorem 4. *Let $u \in \mathcal{T}_{\mathtt{J}}$. Then $u \in \mathcal{L_H N}(\mathtt{J})$ iff u is $\mathcal{HW}$-typable.*

Proof. Exactly as the proof of Theorem 1, but using Corollary 2, Lemmas 14, 15 and 16 and Lemma 37 in Appendix C in place of Corollary 1, Lemmas 2, 3 and 4, and Lemma 25 in the Appendix A respectively.

Typing derivations in the $\mathcal{HW}$-system must be restricted in order to characterize weak J-normalizing terms, as we did for the M-calculus, so that positive/negative types need to be extended to type environments. Therefore, the set of **positive** (resp. **negative**) subtypes of a type is the smallest set satisfying the conditions in Section 3.2 extended with the following cases.

- $A \in \mathcal{P}(\Gamma|\Sigma)$ if $A \in \mathcal{P}(\Gamma)$ or $\Sigma = \sigma$ & $A \in \mathcal{N}(\sigma)$; $A \in \mathcal{N}(\Gamma|\Sigma)$ if $A \in \mathcal{N}(\Gamma)$ or $\Sigma = \sigma$ & $A \in \mathcal{P}(\sigma)$.
- $A \in \mathcal{P}(\langle \Gamma \mid \Sigma, \tau \rangle)$ if $A \in \mathcal{P}(\Gamma|\Sigma)$ or $A \in \mathcal{P}(\tau)$; $A \in \mathcal{N}(\langle \Gamma|\Sigma, \tau\rangle)$ if $A \in \mathcal{N}(\Gamma|\Sigma)$ or $A \in \mathcal{N}(\tau)$.

Lemma 17. *Let $\Phi \triangleright \Gamma \mid \Sigma \vdash_{\mathcal{HW}} o{:}\tau$ s.t. $[\,] \notin \mathcal{P}(\langle \Gamma \mid \Sigma, \tau\rangle)$. If o has no $(\mathtt{B@}, \mathtt{c}, \mathtt{w})$-redex T-occurrences in Φ, then $o \in$ J-nf.*

Proof. By induction on Φ (see Appendix C for details).

Theorem 5. *Let $o \in \mathcal{O}_{\mathtt{J}}$. Then, $o \in \mathcal{WN}(\mathtt{J})$ iff $\Gamma \mid \Sigma \vdash_{\mathcal{HW}} o{:}\tau$ and $[\,] \notin \mathcal{P}(\langle \Gamma \mid \Sigma, \tau\rangle)$.*

Proof. If $o \in \mathcal{WN}(\mathtt{J})$, we proceed by induction on the length of the J-normalizing sequence using Lemma 15 (see Lemma 39 in the Appendix C for details).

Suppose $\Gamma \mid \Sigma \vdash_{\mathcal{HW}} o{:}\tau$ and $[\,] \notin \mathcal{P}(\langle \Gamma \mid \Sigma, \tau\rangle)$. By Corollary 2 the strategy of contracting only redex T-occurrences terminates in a object o' without such redexes. The object o' is typable by Lemma 14 and then o' turns out to be a J-nf by Lemma 17. Thus, o is weakly J-normalizing.

7 Characterization of Strong J-Normalization

This section is devoted to the characterization of J-strong normalization. Since the techniques already presented in Section 4 were developed to be applied to both calculi M and J, the schemes of the proofs in this section are the same we used for the M-calculus.

The **head graphical equivalence** $\sim$ on J-terms is given by the contextual, transitive, symmetric and reflexive closure of the axiom $(tl)[x/u] \approx t[x/u]l$, where $|l|_x = 0$. Notice that $(xl)[x/u]$ cannot be $\sim$-converted into $x[x/u]l$ when $x \notin \mathtt{fv}(l)$, since x alone is not a term of the calculus.

The main properties of system $\mathcal{S}$ follow (see Appendix D for details).

Lemma 18 (Invariance for $\sim$). *Let $o, o' \in \mathcal{O}_{\mathtt{J}}$ s.t. $o \sim o'$. Then 1) $\eta_{\mathtt{J}\backslash\mathtt{w}}(o) = \eta_{\mathtt{J}\backslash\mathtt{w}}(o')$. 2) $\Phi \triangleright \Gamma \vdash_{\mathcal{S}} o{:}\tau$ iff $\Phi' \triangleright \Gamma \vdash_{\mathcal{S}} o'{:}\tau$. Moreover, $\mathtt{sz2}(\Phi) = \mathtt{sz2}(\Phi')$.*

Lemma 19 (SR IV). *Let $\Phi \triangleright \Gamma \mid \Sigma \vdash_{\mathcal{S}} o{:}\tau$. If $o \to_{\mathtt{J}\backslash\mathtt{w}} o'$, then $\Phi' \triangleright \Gamma \mid \Sigma \vdash_{\mathcal{S}} o'{:}\tau$ and $\mathtt{sz2}(\Phi) > \mathtt{sz2}(\Phi')$.*

Lemma 20 (SE IV). *Let $\Gamma \mid \Sigma \vdash_{\mathcal{S}} o'{:}\tau$. If $o \to_{\mathtt{J}\backslash\mathtt{w}} o'$, then $\Gamma \mid \Sigma \vdash_{\mathcal{S}} o{:}\tau$.*

We now use the previous technical tools to characterize strongly J-terms by means of the type system $\mathcal{S}$. We start by giving an inductive definition for the set of strongly-normalizing terms w.r.t. the non-erasing reduction relation $\mathtt{J}\backslash\mathtt{w}$ such that the resulting set $\mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$ coincides with $\mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$. Indeed, the **inductive set of $\mathtt{J}\backslash\mathtt{w}$-strongly-normalizing objects** is the smallest subset of $\mathcal{O}_{\mathtt{J}}$ that satisfies the following properties:

(EL) $\mathtt{nil} \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$.
(NEL) If $t, l \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$, then $t;l \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$,
(L) If $t \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$, then $\lambda x.t \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$.
(HL) If $l \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$, then $xl \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$.
(W) If $t, s \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$ and $|t|_x = 0$, then $t[x/s] \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$.

($\mathtt{dB_{nil}}$) If $(\lambda x.t)l_1 \dots l_n$ $(n \geq 0) \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$, then $(\lambda x.t)\mathtt{nil}l_1 \dots l_n \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$.

($\mathtt{dB_{cons}}$) If $t[x/u]ml_1 \dots l_n$ $(n \geq 0) \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$, then $(\lambda x.t)(u;m)l_1 \dots l_n \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$.

($@_{\mathtt{var}}$) If $x(n@m)l_1 \dots l_n$ $(n \geq 0) \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$, then $(xn)ml_1 \dots l_n \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$.

($@_{\mathtt{app}}$) If $t(n@m)l_1 \dots l_n$ $(n \geq 0) \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$, then $(tn)ml_1 \dots l_n \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$.

(C) If $\mathtt{C}[\![u\,l]\!][x/u] \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$, then $\mathtt{C}[\![x\,l]\!][x/u] \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$.

(E) If $(tl)[x/s] \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$ and $|l|_x = 0$, then $t[x/s]l \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$.

The sets $\mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$ and $\mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$ coincide (see Lemma 47 in the Appendix D for details) so that we can show the following result:

Lemma 21. *Let $o \in \mathcal{O}_{\mathtt{J}}$. If $o \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$ then o is $\mathcal{S}$-typable.*

Proof. Using the equality $\mathcal{SN}(\mathtt{J}\backslash\mathtt{w}) = \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$ to reason by induction on $o \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$. The proof also uses Lemma 20. See Appendix D for details.

Lemma 22 (From $\mathtt{J}\backslash\mathtt{w}$ to $\mathtt{J}$). *Let $o \in \mathcal{O}_{\mathtt{J}}$. If $o \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$, then $o \in \mathcal{SN}(\mathtt{J})$.*

Proof. The proof proceeds as the one for Lemma 11, but uses Lemma 12 instead to achieve the postponement of $\mathtt{w}$-steps.

We can now conclude with the main result of this section.

Theorem 6. *Let $o \in \mathcal{O}_{\mathtt{J}}$. Then o is $\mathcal{S}$-typable iff $o \in \mathcal{SN}(\mathtt{J})$.*

Proof. Same scheme used in the proof of Theorem 3, but using Lemmas 19, 21 and 22 in place of Lemmas 7, 9 and 11 respectively.

8 Conclusion

This paper studies quantitative types for two intuitionistic term languages specified by natural deduction and sequent calculus. We characterize linear-head, weak and strongly normalizing sets of terms in each language. In particular, the correspondence between head β-normalization for λ-terms and linear-head $\mathtt{M}$-normalization for terms with ES can now be obtained by means of an *indirect* logical reasoning (*i.e.* the $\mathcal{HW}$-system), in contrast to the operational result in [4].

The type systems are given by simple formalisms: 1) intersection is represented by multisets so that no axioms for commutative and associative laws are needed, 2) the typing rules are syntax-oriented so that no generation lemmas are used, 3) the type systems used to characterize strongly normalizing terms are just based on a notion of witness sequent, no subtyping relation is used [10], and 4) the type systems are specified in an homogeneous way, so that the schemes of the proofs for the $\mathcal{HW}$-systems can be reused in the development of the proofs for the $\mathcal{S}$-systems. Similarly, the proofs for the $\mathtt{J}$-calculus follow the same schemes than those for the $\mathtt{M}$-calculus.

One current investigation line is the use of quantitative type systems to *directly* characterize *head* $\mathtt{M}$-normalization (instead of linear-head) for terms with ES, as already done with *head* β-normalization for λ-terms. This could be done by redefining the notion of redex typed occurrence and/or by considering some equivalence on terms with ES.

The relation between our type systems and the bounds for linear-head and head-reduction obtained in [4] should be studied. Recovering the bounds on the longest reduction sequences defined in [10] is another further interesting question. Last but not least, we would like to derive from each type system a notion of relational model [21] for the corresponding term calculus.

Although type inference is undecidable for any system characterizing termination properties, semi-decidable restrictions are expected to hold. Principal typing is a necessary property (*cf.* [21]) to obtain partial typing inference algorithms [42, 41, 32]. Moreover, relevance in the sense of [18] is a key property to obtain principal typings. Therefore semi-decidable typing inference algorithms are also expected to hold for our four non-idempotent type systems.

Neergard *et al.* [38] proved that type inference and execution of typed programs are in different classes of complexity in the idempotent case but in the same class in the the non-idempotent case. However, as noted there, the system introduced by Carlier *et al.* [14] allows to relax the notion of type linearity. An interesting challenge would be to understand how to use this relaxed notion of linear types in order to gain expressivity while staying in a different class.

Last but not least, the inhabitation problem for idempotent intersection types typing the λ-calculus is known to be undecidable [43], while the problem was recently shown to be decidable in the non-idempotent case [12]. An interesting question concerns the inhabitation problems for our non-idempotent type systems.

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References

1. M. Abadi, L. Cardelli, P-L. Curien, and J-J. Lévy. Explicit substitutions. *JFP*, 1(4):375–416, 1991.
2. B. Accattoli. Evaluating functions as processes. In *TERMGRAPH*, EPTCS 110:41–55, 2013.
3. B. Accattoli, E. Bonelli, D. Kesner, and C. Lombardi. A nonstandard standardization theorem. In *POPL*, ACM, 2014.
4. B. Accattoli and U. Dal Lago. On the invariance of the unitary cost model for head reduction. In *RTA*, LIPIcs 15:22–37, 2012.
5. B. Accattoli and D. Kesner. The structural λ-calculus. In *CSL*, LNCS 6247:381–395, 2010.
6. H. Barendregt. *The Lambda Calculus: Its Syntax and Semantics (revised edition)*. Elsevier Science, Amsterdan, The Netherlands, 1984.
7. H. Barendregt, M. Coppo, and M. Dezani-Ciancaglini. A filter lambda model and the completeness of type assignment. *Bulletin of Symbolic Logic*, 48:931–940, 1983.
8. A. Bernadet and S. Lengrand. Complexity of strongly normalising λ-terms via non-idempotent intersection types. In *FOSSACS*, LNCS 6604:88–107, 2011.
9. A. Bernadet and S. Lengrand. Filter models: non-idempotent intersection types, orthogonality and polymorphism. In *CSL*, LIPIcs 12:51–66, 2011.
10. A. Bernadet and S. Lengrand. Non-idempotent intersection types and strong normalisation. *LMCS*, 9(4), 2013.
11. G. Boudol, P-L. Curien, and C. Lavatelli. A semantics for lambda calculi with resources. *MSCS*, 9(4):437–482, 1999.
12. A. Bucciarelli, D. Kesner, and S. Ronchi Della Rocca. The inhabitation problem for non-idempotent intersection types. Submitted.
13. F. Cardone and M. Coppo. Two extension of Curry's type inference system. In *Logic and Computer Science*, *APIC Series* 31, pages 19–75. Academic Press, 1990.
14. S. Carlier, J. Polakow, J. B. Wells, and A. J. Kfoury. System E: Expansion variables for flexible typing with linear and non-linear types and intersection types. In *ESOP'04*, LNCS 2986:294-309. Springer-Verlag, 2004.
15. M. Coppo and M. Dezani-Ciancaglini. A new type-assignment for lambda terms. *Archiv für Mathematische Logik und Grundlagenforschung*, 19:139–156, 1978.
16. M. Coppo and M. Dezani-Ciancaglini. An extension of the basic functionality theory for the lambda-calculus *Notre Dame Journal of Formal Logic* 21(4):685–693, 1980.
17. M. Coppo, M. Dezani-Ciancaglini, and B. Venneri. Functional characters of solvable terms. *Mathematical Logic Quarterly*, 27(2-6):45–58, 1981.

18. F. Damiani and P. Giannini. A decidable intersection type system based on relevance. In *TACS'94*, LNCS 789:707–725. Springer, 1994.
19. V. Danos and L. Regnier. Head linear reduction, 2003. `iml.univ-mrs.fr/~regnier/articles/pam.ps.gz`.
20. E. De Benedetti and S. Ronchi Della Rocca. Bounding normalization time through intersection types. *ITRS*, EPTCS 121, pages 48-57, 2013.
21. D. de Carvalho. *Sémantiques de la logique linéaire et temps de calcul*. PhD Univ. Aix-Marseille II, 2007.
22. R. Di Cosmo, D. Kesner, and E. Polonovski. Proof nets and explicit substitutions. *MSCS*, 13(3):409–450, 2003.
23. D. Dougherty, S. Ghilezan, and P. Lescanne. Characterizing strong normalization in the curien-herbelin symmetric lambda calculus: Extending the Coppo-Dezani heritage. *TCS*, 398(1-3):114–128, 2008.
24. J. Espírito Santo, S. Ghilezan, and J. Ivetic. Characterising strongly normalising intuitionistic sequent terms. In *TYPES*, LNCS 4941:85–99, 2007.
25. J-Y. Girard. Linear logic. *TCS*, 50:1–102, 1987.
26. J-Y. Girard. Proof-nets: The parallel syntax for proof-theory. In *Logic and Algebra*, pages 97–124, 1996.
27. H. Herbelin. A lambda-calculus structure isomorphic to Gentzen-style sequent calculus structure. In *CSL*, LNCS 1092:61–75, 1995.
28. D. Kesner. The theory of calculi with explicit substitutions revisited. In *CSL*, LNCS 4646:238–252, 2007.
29. D. Kesner. A theory of explicit substitutions with safe and full composition. *LMCS*, 5(3), 2009.
30. D. Kesner and S. Lengrand. Resource operators for lambda-calculus. *IandC*, 205(4):419–473, 2007.
31. D. Kesner and S. Ó Conchúir. Milner's lambda calculus with partial substitutions, 2008. `http://www.pps.univ-paris-diderot.fr/~kesner/papers/shortpartial.pdf`.
32. A. J. Kfoury and J. B. Wells. Principality and type inference for intersection types using expansion variables, *TCS*, 311(1-3):1–70, 2004.
33. J.-L. Krivine. Lambda-calculus, types and models, Ellis Horwood, 1993.
34. S. Lengrand, P. Lescanne, D. Dougherty, M. Dezani-Ciancaglini, and S. van Bakel. Intersection types for explicit substitutions. *IandC*, 189:17–42, 2004.
35. J-M. Madiot. Embedding intersection types into multiplicative linear logic. TR, LIPN, Paris 13, 2010.
36. G. Mascari and M. Pedicini. Head linear reduction and pure proof net extraction. *TCS*, 135(1):111–137, 1994.
37. R. Milner. Local bigraphs and confluence: Two conjectures: (extended abstract). *ENTCS*, 175(3):65–73, 2007.
38. P. Neergaard and H. Mairson. Types, Potency, and Idempotency: Why Nonlinearity and Amnesia Make a Type System Work. In *ICFP ' 04*, SIGPLAN Not. vol. 39(9):138–149, 2004.
39. G. Pottinger. A type assignment for the strongly normalizable λ-terms. In *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 561–578. Academic Press, 1980.
40. L. Regnier. Une équivalence e sur les lambda-termes. *TCS*, 2(126):281292, 1994.
41. S. Ronchi Della Rocca. Principal Type scheme and unification for intersection type discipline. *TCS*, 59:1–29, 1988.
42. S. Ronchi Della Rocca and B. Venneri. Principal Type Scheme for an extended type theory. *TCS*, 28:151–169, 1984.
43. P. Urzyczyn. The emptiness problem for intersection types. *Journal of Symbolic Logic*, 64(3):1195–1215, 1999.

A Appendix: Characterization of Linear-Head and Weak M-Normalization

Lemma 23 (Partial Substitution I). *If* $\Phi_{\mathtt{C}[\![x]\!]} \triangleright x{:}[\sigma_i]_{i\in I}; \Gamma \vdash_{\mathcal{HW}} \mathtt{C}[\![x]\!]{:}\tau$ *and* $(\Phi^i_u \triangleright \Delta_i \vdash_{\mathcal{HW}} u{:}\sigma_i)_{i\in I}$ *then* $\Phi_{\mathtt{C}[\![u]\!]} \triangleright x{:}[\sigma_i]_{i\in I\setminus K}; \Gamma +_{k\in K} \Delta_k \vdash_{\mathcal{HW}} \mathtt{C}[\![u]\!]{:}\tau$*, for some* $K \subseteq I$ *where* $\mathtt{sz}(\Phi_{\mathtt{C}[\![u]\!]}) = \mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) +_{k\in K} \mathtt{sz}(\Phi^k_u) - |K|$*. Moreover, if* $C|_p = \Box$ *and* $p \in \mathtt{pos}(\mathtt{C}[\![x]\!])$ *is a T-occurrence of* $\mathtt{C}[\![x]\!]$ *in* $\Phi_{\mathtt{C}[\![x]\!]}$*, then* $K \neq \emptyset$*.*

Proof. By induction on the type derivation $\Phi_{\mathtt{C}[\![x]\!]}$.

- If $C = \Box$, then by construction $\Gamma = \emptyset$, $I = \{1\}$, $\rho_1 = \tau$ and Φ_x has the following form:

$$\Phi_x \triangleright \frac{}{x{:}[\tau] \vdash x{:}\tau}\ (\mathtt{ax})$$

  Thus, for $K = \{1\} \subseteq I$ we have $\mathtt{C}[\![s]\!] = s$ and $\Phi_s = \Phi^1_s$. Moreover, $\mathtt{sz}(\Phi_s) = \mathtt{sz}(\Phi^1_s) = 1 + \mathtt{sz}(\Phi^1_s) - 1 = \mathtt{sz}(\Phi_x) + \mathtt{sz}(\Phi^1_s) - 1$. Thus the statement holds.
- If $C = \lambda y.D$, then the property is straightforward by the *i.h.*
- If $C = Dt$, then by construction $x{:}[\rho_i]_{i\in I}; \Gamma = \Delta +_{j\in J} \Gamma_j$ and $\Phi_{\mathtt{C}[\![x]\!]}$ is of the following form

$$\Phi_{\mathtt{C}[\![x]\!]} \triangleright \frac{\Phi_{\mathtt{D}[\![x]\!]} \triangleright \dfrac{\nabla}{\Delta \vdash \mathtt{D}[\![x]\!]{:}[\sigma_j]_{j\in J} \to \tau} \qquad \left(\Phi^j_t \triangleright \dfrac{\nabla}{\Gamma_j \vdash t{:}\sigma_j}\right)_{j\in J}}{x{:}[\rho_i]_{i\in I}; \Gamma \vdash \mathtt{D}[\![x]\!]t{:}\tau}\ (\to \mathtt{e}_{\mathcal{HW}})$$

  Moreover, $\mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) = \mathtt{sz}(\Phi_{\mathtt{D}[\![x]\!]}) +_{j\in J} \mathtt{sz}(\Phi^j_t) + 1$.
  We necessarily have $\Delta = x{:}[\rho_i]_{i\in L}; \Delta'$, where $L \subseteq I$. The *i.h.* then holds for $\Phi_{\mathtt{D}[\![x]\!]}$ so that we get $\Phi_{\mathtt{D}[\![s]\!]} \triangleright x{:}[\rho_i]_{i\in L\setminus K}; \Delta' +_{i\in K} \Delta_i \vdash \mathtt{D}[\![s]\!]{:}[\sigma_j]_{j\in J} \to \tau$, for some $K \subseteq L$ where $\mathtt{sz}(\Phi_{\mathtt{D}[\![s]\!]}) = \mathtt{sz}(\Phi_{\mathtt{D}[\![x]\!]}) +_{i\in K} \mathtt{sz}(\Phi^i_s) - |K|$. Then we construct the following derivation

$$\Phi_{\mathtt{C}[\![s]\!]} \triangleright \frac{\Phi_{\mathtt{D}[\![s]\!]} \triangleright \dfrac{\nabla}{x{:}[\rho_i]_{i\in L\setminus K}; \Delta' +_{i\in K} \Delta_i \vdash \mathtt{D}[\![s]\!]{:}[\sigma_j]_{j\in J} \to \tau} \qquad \left(\Phi^j_t \triangleright \dfrac{\nabla}{\Gamma_j \vdash t{:}\sigma_j}\right)_{j\in J}}{(x{:}[\rho_i]_{i\in L\setminus K}; \Delta' +_{i\in K} \Delta_i) +_{j\in J} \Gamma_j \vdash \mathtt{D}[\![s]\!]t{:}\tau}\ (\to \mathtt{e}_{\mathcal{HW}})$$

  We thus have $x{:}[\rho_i]_{i\in I}; \Gamma = \Delta +_{j\in J} \Gamma_j$ implies $x{:}[\rho_i]_{i\in I}; \Gamma = x{:}[\rho_i]_{i\in L}; \Delta' +_{j\in J} \Gamma_j$ implies $x{:}[\rho_i]_{i\in I}; \Gamma = x{:}[\rho_i]_{i\in L\setminus K} + x{:}[\rho_k]_{k\in K}; \Delta' +_{j\in J} \Gamma_j$ implies $x{:}[\rho_i]_{i\in I\setminus K}; \Gamma = x{:}[\rho_i]_{i\in L\setminus K}; \Delta' +_{j\in J} \Gamma_j$ implies $x{:}[\rho_i]_{i\in I\setminus K}; \Gamma +_{i\in K} \Delta_i = (x{:}[\rho_i]_{i\in L\setminus K}; \Delta' +_{i\in K} \Delta_i) +_{j\in J} \Gamma_j$ and $\mathtt{sz}(\Phi_{\mathtt{C}[\![s]\!]}) = \mathtt{sz}(\Phi_{\mathtt{D}[\![s]\!]}) +_{j\in J} \mathtt{sz}(\Phi^j_t) + 1 =_{i.h.} \mathtt{sz}(\Phi_{\mathtt{D}[\![x]\!]}) +_{i\in K} \mathtt{sz}(\Phi^i_s) - |K| +_{j\in J} \mathtt{sz}(\Phi^j_t) + 1 = \mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) +_{i\in K} \mathtt{sz}(\Phi^i_s) - |K|$.
  We conclude the first part of the statement since $K \subseteq L \subseteq I$. Finally, if the position $p = 0p' \in \mathtt{pos}(\mathtt{C}[\![x]\!])$ is a T-occurrence of $\mathtt{C}[\![x]\!]$ $\Phi_{\mathtt{C}[\![x]\!]}$, then p' is a T-occurrence of $\mathtt{D}[\![x]\!]$ $\Phi_{\mathtt{D}[\![x]\!]}$ so that $K \neq \emptyset$ by the *i.h.*
- If $C = tD$, then by construction $x{:}[\rho_i]_{i\in I}; \Gamma = \Delta +_{j\in J} \Gamma_j$, and $\Phi_{\mathtt{C}[\![x]\!]}$ is of the form

$$\Phi_{\mathtt{C}[\![x]\!]} \triangleright \frac{\Phi_t \triangleright \dfrac{\nabla}{\Delta \vdash t{:}[\sigma_j]_{j\in J} \to \tau} \qquad \left(\Phi^j_{\mathtt{D}[\![x]\!]} \triangleright \dfrac{\nabla}{\Gamma_j \vdash \mathtt{D}[\![x]\!]{:}\sigma_j}\right)_{j\in J}}{x{:}[\rho_i]_{i\in I}; \Gamma \vdash t\mathtt{D}[\![x]\!]{:}\tau}\ (\to \mathtt{e}_{\mathcal{HW}})$$

  Moreover, $\mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) = \mathtt{sz}(\Phi_t) +_{j\in J} \mathtt{sz}(\Phi^j_{\mathtt{D}[\![x]\!]}) + 1$.

We necessarily have $\Gamma_j = x{:}[\rho_i]_{i\in L_j}; \Gamma'_j$, where $L_j \subseteq I$. The *i.h.* then holds for $\Phi^j_{\mathtt{D}[\![x]\!]} \triangleright x{:}[\rho_i]_{i\in L_j}; \Gamma'_j \vdash \mathtt{D}[\![x]\!]{:}\sigma_j$ so that $\Phi^j_{\mathtt{D}[\![s]\!]} \triangleright x{:}[\rho_i]_{i\in L_j \setminus K_j}; \Gamma'_j +_{i\in K_j} \Delta_i \vdash \mathtt{D}[\![s]\!]{:}\sigma_j$ for $K_j \subseteq L_j$ and $\mathtt{sz}(\Phi^j_{\mathtt{D}[\![s]\!]}) = \mathtt{sz}(\Phi^j_{\mathtt{D}[\![x]\!]}) +_{i\in K_j} \mathtt{sz}(\Phi^i_s) - |K_j|$. We then construct the following derivation:

$$\Phi_{\mathtt{C}[\![s]\!]} \triangleright \dfrac{\Phi_t \triangleright \dfrac{\nabla}{\Delta \vdash t{:}[\sigma_j]_{j\in J} \to \tau} \qquad \left(\Phi^j_{\mathtt{D}[\![s]\!]} \triangleright \dfrac{\nabla}{x{:}[\rho_i]_{i\in L_j \setminus K_j}; \Gamma'_j +_{i\in K_j} \Delta_i \vdash \mathtt{D}[\![s]\!]{:}\sigma_j}\right)_{j\in J}}{\Delta +_{j\in J} (x{:}[\rho_i]_{i\in L_j \setminus K_j}; \Gamma'_j +_{i\in K_j} \Delta_i) \vdash t\mathtt{D}[\![s]\!]{:}\tau} \ (\to \mathtt{e}_{\mathcal{HW}})$$

We have $\Delta +_{j\in J} (x{:}[\rho_i]_{i\in L_j \setminus K_j}; \Gamma'_j +_{i\in K_j} \Delta_i) = x{:}[\rho_i]_{i\in I \setminus \cup_{j\in J} K_j}; \Gamma +_{j\in J} +_{i\in K_j} \Delta_i$ so that let $K = \cup_{j\in J} K_j \subseteq \cup_{j\in J} L_j \subseteq I$. If the position $p = 1p' \in \mathtt{pos}(\mathtt{C}[\![x]\!])$ is a T-occurrence of $\mathtt{C}[\![x]\!]$ in $\Phi_{\mathtt{C}[\![x]\!]}$, then p' is a T-occurrence of $\mathtt{D}[\![x]\!]$ in some $\Phi^j_{\mathtt{D}[\![x]\!]}$ so that $K_j \neq \emptyset$ by the *i.h.* We thus have $\emptyset \neq K_j \subseteq K$ which concludes the first part of the statement.
Finally, $\mathtt{sz}(\Phi_{\mathtt{C}[\![s]\!]}) = \mathtt{sz}(\Phi_t) +_{j\in J} \mathtt{sz}(\Phi^j_{\mathtt{D}[\![s]\!]}) + 1 =_{i.h.} \mathtt{sz}(\Phi_t) +_{j\in J} (\mathtt{sz}(\Phi^j_{\mathtt{D}[\![x]\!]}) +_{i\in K_j} \mathtt{sz}(\Phi^i_s) - |K_j|) + 1 = \mathtt{sz}(\Phi_t) +_{j\in J} \mathtt{sz}(\Phi^j_{\mathtt{D}[\![x]\!]}) +_{i\in K} \mathtt{sz}(\Phi^i_s) - |K| + 1 = \mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) +_{i\in K} \mathtt{sz}(\Phi^i_s) - |K|$.

– If $C = D[x/u]$ or $C = t[x/D]$ we proceed similarly to the previous case.

Using Lemma 1 (Weak) and Lemma 23 we can show the Subject Reduction property.

Lemma 2 (SR I) Let $\Phi \triangleright \Gamma \vdash_{\mathcal{HW}} t{:}\tau$. If $t \to_{\mathtt{M}} t'$ reduces a redex T-occurrence of t in Φ then $\Phi' \triangleright \Gamma \vdash_{\mathcal{HW}} t'{:}\tau$ and $\mathtt{sz}(\Phi) > \mathtt{sz}(\Phi')$.

Proof. By induction on the reduction relation $\to_{\mathtt{M}}$.

– If $t = \mathtt{L}[\![\lambda x.v]\!]s \to \mathtt{L}[\![v[x/s]]\!] = t'$, then we proceed by induction on $\mathtt{L}$ and we show that in this case $\mathtt{sz}(\Phi) > \mathtt{sz}(\Phi')$. Let $\mathtt{L} = \Box$. By construction the derivation Φ is of the form:

$$\Phi \triangleright \dfrac{\dfrac{\Phi_v \triangleright \dfrac{\nabla}{x : [\rho_i]_{i\in I}; \Pi \vdash v{:}\sigma}}{\Pi \vdash \lambda x.v{:}[\rho_i]_{i\in I} \to \sigma} \qquad \left(\Phi^i_s \triangleright \dfrac{\nabla}{\Gamma_i \vdash s{:}\rho_i}\right)_{i\in I}}{\Pi +_{i\in I} \Gamma_i \vdash (\lambda x.v)s{:}\tau}$$

Moreover, $\mathtt{sz}(\Phi) = \mathtt{sz}(\Phi_v) +_{i\in I} \mathtt{sz}(\Phi^i_s) + 2$. Hence,

$$\Phi' \triangleright \dfrac{\Phi_v \triangleright \dfrac{\nabla}{x : [\rho_i]_{i\in I}; \Pi \vdash v{:}\sigma} \qquad \left(\Phi^i_s \triangleright \dfrac{\nabla}{\Gamma_i \vdash s{:}\rho_i}\right)_{i\in I}}{\Pi +_{i\in I} \Gamma_i \vdash v[x/s]{:}\sigma}$$

We have $\mathtt{sz}(\Phi') = \mathtt{sz}(\Phi_v) +_{i\in I} \mathtt{sz}(\Phi^i_s) + 1 < \mathtt{sz}(\Phi)$.
Let $\mathtt{L} = \mathtt{L}'[y/u]$. By construction the derivation has the following form:

$$\dfrac{\dfrac{\dfrac{\nabla}{\Gamma_0; y : [\rho_j]_{j\in J} \vdash \mathtt{L}'[\![\lambda x.v]\!]{:}[\sigma_i]_{i\in I} \to \tau} \qquad \left(\dfrac{\nabla}{\Pi_j \vdash u{:}\rho_j}\right)_{j\in J}}{\Gamma_0 +_{j\in J} \Pi_j \vdash \mathtt{L}'[\![\lambda x.v]\!][y/u]{:}[\sigma_i]_{i\in I} \to \tau} \qquad \left(\dfrac{\nabla}{\Delta_i \vdash s{:}\sigma_i}\right)_{i\in I}}{\Gamma_0 +_{j\in J} \Pi_j +_{i\in I} \Delta_i \vdash \mathtt{L}'[\![\lambda x.v]\!][y/u]s{:}\tau}$$

We can then construct the following derivation:

$$\Phi_{\mathtt{L}'[\![\lambda x.v]\!]s} \triangleright \dfrac{\dfrac{\nabla}{\Gamma_0; y : [\rho_j]_{j\in J} \vdash \mathtt{L}'[\![\lambda x.v]\!]{:}[\sigma_i]_{i\in I} \to \tau} \qquad \left(\dfrac{\nabla}{\Delta_i \vdash s{:}\sigma_i}\right)_{i\in I}}{\Gamma_0; y : [\rho_j]_{j\in J} +_{i\in I} \Delta_i \vdash \mathtt{L}'[\![\lambda x.v]\!]s{:}\tau}$$

By the *i.h.* there is a derivation ending with $\Phi_{\mathtt{L}'[\![v[x/s]]\!]} \triangleright \Gamma_0; y : [\rho_j]_{j\in J} +_{i\in I} \Delta_i \vdash \mathtt{L}'[\![v[x/s]]\!]{:}\tau$ such that $\mathtt{sz}(\Phi_{\mathtt{L}'[\![\lambda x.v]\!]s}) > \mathtt{sz}(\Phi_{\mathtt{L}'[\![v[x/s]]\!]})$. We thus conclude with the following derivation.

$$\dfrac{\dfrac{\nabla}{\Gamma_0; y : [\rho_j]_{j\in J} +_{i\in I} \Delta_i \vdash \mathtt{L}'[\![v[x/s]]\!]{:}\tau} \qquad \left(\dfrac{\nabla}{\Pi_j \vdash u{:}\rho_j}\right)_{j\in J}}{\Gamma_0 +_{j\in J} \Pi_j +_{i\in I} \Delta_i \vdash \mathtt{L}'[\![v[x/s]]\!][y/u]{:}\tau}$$

We have $\mathtt{sz}(\Phi_{\mathtt{L}'[\![v[x/s]]\!][y/u]}) = \mathtt{sz}(\Phi_{\mathtt{L}'[\![v[x/s]]\!]}) +_{j\in J} \mathtt{sz}(\Phi^j_u) + 1 <_{i.h.} \mathtt{sz}(\Phi_{\mathtt{L}'[\![\lambda x.v]\!]s}) +_{j\in J} \mathtt{sz}(\Phi^j_u) + 1 = \mathtt{sz}(\Phi_{\mathtt{L}'[\![\lambda x.v]\!]}) +_{i\in I} \mathtt{sz}(\Phi^i_s) +_{j\in J} \mathtt{sz}(\Phi^j_u) + 2 = \mathtt{sz}(\Phi_{\mathtt{L}'[\![\lambda x.v]\!][y/u]s})$.

– If $o = \mathtt{C}[\![x]\!][x/u] \to \mathtt{C}[\![u]\!][x/u] = o'$, where $|\mathtt{C}[\![x]\!]|_x \geq 1$, then by construction Φ is of the form

$$\Phi \triangleright \dfrac{\Phi_{\mathtt{C}[\![x]\!]} \triangleright \dfrac{\nabla}{x{:}[\rho_i]_{i\in I}; \Pi \vdash \mathtt{C}[\![x]\!]{:}\tau} \qquad \left(\Phi^i_u \triangleright \dfrac{\nabla}{\Delta_i \vdash u{:}\rho_i}\right)_{i\in I}}{\Pi +_{i\in I} \Delta_i \vdash \mathtt{C}[\![x]\!][x/u]{:}\tau}$$

Moreover, $\mathtt{sz}(\Phi) = \mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) +_{i\in I} \mathtt{sz}(\Phi^i_u) + 1$. By Lemma 23 we have $\Phi_{\mathtt{C}[\![u]\!]} \triangleright x{:}[\rho_i]_{i\in I\setminus K}; \Pi +_{i\in K} \Delta_i \vdash \mathtt{C}[\![u]\!]{:}\tau$, for some $K \subseteq I$ where $\mathtt{sz}(\Phi_{\mathtt{C}[\![u]\!]}) = \mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) +_{i\in K} \mathtt{sz}(\Phi^i_u) - |K|$. Hence

$$\Phi' \triangleright \dfrac{\Phi_{\mathtt{C}[\![u]\!]} \triangleright \dfrac{\nabla}{x{:}[\rho_i]_{i\in I\setminus K}; \Pi +_{i\in K} \Delta_i \vdash \mathtt{C}[\![u]\!]{:}\tau} \qquad \left(\Phi^i_u \triangleright \dfrac{\nabla}{\Delta_i \vdash u{:}\rho_i}\right)_{i\in I\setminus K}}{\Pi +_{i\in K} \Delta_i +_{i\in I\setminus K} \Delta_i \vdash \mathtt{C}[\![u]\!][x/u]{:}\tau}$$

We have $\mathtt{sz}(\Phi') = \mathtt{sz}(\Phi_{\mathtt{C}[\![u]\!]}) +_{i\in I\setminus K} \mathtt{sz}(\Phi^i_u) + 1 = \mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) +_{i\in K} \mathtt{sz}(\Phi^i_u) - |K| +_{i\in I\setminus K} \mathtt{sz}(\Phi^i_u) + 1 = \mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) +_{i\in I} \mathtt{sz}(\Phi^i_u) + 1 - |K| = \mathtt{sz}(\Phi) - |K| \leq \mathtt{sz}(\Phi)$.
By hypothesis, the hole of $\mathtt{C}$ is a T-occurrence in Φ, so that Lemma 23 guarantees $K \neq \emptyset$ and thus $\mathtt{sz}(\Phi') < \mathtt{sz}(\Phi)$.

– If $o = t[x/u] \to t = o'$ with $|t|_x = 0$, then by construction and Lemma 1 (Weak) Φ is of the form

$$\Phi \triangleright \dfrac{\Phi_t \triangleright \dfrac{\nabla}{x{:}[\,]; \Gamma \vdash t{:}\tau}}{\Gamma \vdash t[x/u]{:}\tau}$$

where $\mathtt{sz}(\Phi) = \mathtt{sz}(\Phi_t) + 1$. The result then holds for $\Phi' := \Phi_t$.

– All the inductive cases are straightforward.

The following Lemma is used in the Subject Expansion property for system $\mathcal{HW}$.

Lemma 24. *Let* $\mathtt{C}[\![x]\!], s$ *be* $\mathtt{M}$*-terms s.t.* $x \notin \mathtt{fv}(s)$ *and* $\Gamma \vdash_{\mathcal{HW}} \mathtt{C}[\![s]\!]{:}\tau$*. Then* $\exists \Gamma_0, \exists I, \exists (\Gamma_i)_{i\in I}, \exists (\sigma_i)_{i\in I}$ *such that* $\Gamma = \Gamma_0 +_{i\in I} \Gamma_i$*,* $\Gamma_0 + \{x : [\sigma_i]_{i\in I}\} \vdash_{\mathcal{HW}} \mathtt{C}[\![x]\!]{:}\tau$*, and* $(\Gamma_i \vdash_{\mathcal{HW}} s{:}\sigma_i)_{i\in I}$*.*

Proof. By induction on the structure of $\mathtt{C}[\![s]\!]$.

- If $C = \Box$ then $\mathtt{C}[\![s]\!] = s$ and the result holds, for $\Gamma_0 = \emptyset$, $|I| = 1$ and $\sigma_i = \tau$.
- If $C = \lambda y.D$ then the property is straightforward by the *i.h.* (since $y \notin \mathtt{fv}(s)$ by α-conversion).
- If $C = D\,r$ then $\mathtt{C}[\![s]\!] = \mathtt{D}[\![s]\!]\,r$ and by construction $\Gamma = \Pi +_{j\in J} \Gamma_j$ and $\Phi_{\mathtt{D}[\![s]\!]} \triangleright \Pi \vdash \mathtt{D}[\![s]\!]{:}[\rho_j]_{j\in J} \to \tau$ and $(\Phi_r^j \triangleright \Gamma_j \vdash r{:}\rho_j)_{j\in J}$. By the *i.h.* $\Pi = \Pi_0 +_{i\in I} \Gamma_i$ where $x{:}[\sigma_i]_{i\in I} + \Pi_0 \vdash \mathtt{D}[\![x]\!]{:}[\rho_j]_{j\in J} \to \tau$ and $(\Gamma_i \vdash s{:}\sigma_i)_{i\in I}$. Then, by the rule $(\to \mathtt{e}_{\mathcal{HW}})$, $(x{:}[\sigma_i]_{i\in I} + \Pi_0) +_{j\in J} \Gamma_j \vdash \mathtt{D}[\![x]\!]\,r{:}\tau$. The result then holds for $\Gamma_0 := \Pi_0 +_{j\in J} \Gamma_j$.
- If $C = r\,D$ then $\mathtt{C}[\![s]\!] = r\,\mathtt{D}[\![s]\!]$ and by construction $\Gamma = \Pi +_{j\in J} \Gamma_j$ and $\Phi_r \triangleright \Pi \vdash r{:}[\rho_j]_{j\in J} \to \tau$ and $(\Phi^j_{\mathtt{D}[\![s]\!]} \triangleright \Gamma_j \vdash \mathtt{D}[\![s]\!]{:}\rho_j)_{j\in J}$. By the *i.h.* for each $j \in J$, $\Gamma_j = \Gamma_j^0 +_{i\in I_j} \Gamma_i$ where $x{:}[\sigma_i]_{i\in I_j} + \Gamma_j^0 \vdash \mathtt{D}[\![x]\!]{:}\rho_j$ and $(\Gamma_i \vdash s{:}\sigma_i)_{i\in I_j}$. Let $I := \cup_{j\in J} I_j$. Then, by the rule $(\to \mathtt{e}_{\mathcal{HW}})$, $\Pi +_{j\in J} (x{:}[\sigma_i]_{i\in I_j} + \Gamma_j^0) \vdash r\,\mathtt{D}[\![x]\!]{:}\tau$. Note that $\Pi +_{j\in J} (x{:}[\sigma_i]_{i\in I_j} + \Gamma_j^0) = x{:}[\sigma_i]_{i\in I} + \Pi +_{j\in J} \Gamma_j^0$. The result then holds for $\Gamma_0 := \Pi +_{j\in J} \Gamma_j^0$.
- All the remaining cases are similar to the previous ones.

Lemma 3 (SE I). If $\Gamma \vdash_{\mathcal{HW}} t'{:}\tau$ and $t \to_{\mathtt{M}} t'$ then $\Gamma \vdash_{\mathcal{HW}} t{:}\tau$.

Proof. Let $\Gamma \vdash_{\mathcal{HW}} t'{:}\tau$. The proof is by induction on $t \to_{\mathtt{M}} t'$.

- If $t = \mathtt{L}[\![\lambda x.p]\!]s \to \mathtt{L}[\![p[x/s]]\!] = t'$, then we proceed by induction on $\mathtt{L}$. Let $\mathtt{L} = \Box$, then by construction $\Gamma = \Delta +_{i\in I} \Gamma_i$ and we have the following derivation:

$$\dfrac{\dfrac{\nabla}{x{:}[\sigma_i]_{i\in I}; \Delta \vdash p{:}\tau} \qquad \left(\dfrac{\nabla}{\Gamma_i \vdash s{:}\sigma_i}\right)_{i\in I}}{\Gamma \vdash p[x/s]{:}\tau}$$

We then construct the following derivation

$$\dfrac{\dfrac{\dfrac{\nabla}{x{:}[\sigma_i]_{i\in I}; \Delta \vdash p{:}\tau}}{\Delta \vdash \lambda x.p{:}[\sigma_i]_{i\in I} \to \tau}\ (\to \mathtt{i}) \qquad \left(\dfrac{\nabla}{\Gamma_i \vdash s{:}\sigma_i}\right)_{i\in I}}{\Gamma \vdash (\lambda x.p)\,s{:}\tau}\ (\to \mathtt{e}_{\mathcal{HW}})$$

Let $\mathtt{L} = \mathtt{L}'[y/u]$ so that $\mathtt{L}'[y/u][\![p[x/s]]\!] = \mathtt{L}'[\![p[x/s]]\!][y/u]$. Then by construction there is a derivation of the following form:

$$\dfrac{\dfrac{\nabla}{\Gamma_0; y : [\rho_j]_{j\in J} +_{i\in I} \Delta_i \vdash \mathtt{L}'[\![p[x/s]]\!]{:}\tau} \qquad \left(\dfrac{\nabla}{\Pi_j \vdash u{:}\rho_j}\right)_{j\in J}}{\Gamma_0 +_{j\in J} \Pi_j +_{i\in I} \Delta_i \vdash \mathtt{L}'[\![p[x/s]]\!][y/u]{:}\tau}$$

By the *i.h.* there is a derivation ending with $\Phi_{\mathtt{L}'[\![\lambda x.p]\!]s} \triangleright \Gamma_0; y : [\rho_j]_{j\in J} +_{i\in I} \Delta_i \vdash \mathtt{L}'[\![\lambda x.p]\!]s{:}\tau$ so that by construction there is a derivation of the following form:

$$\dfrac{\dfrac{\nabla}{\Gamma_0; y : [\rho_j]_{j\in J} \vdash \mathtt{L}'[\![\lambda x.p]\!]{:}[\sigma_i]_{i\in I} \to \tau} \qquad \left(\dfrac{\nabla}{\Delta_i \vdash s{:}\sigma_i}\right)_{i\in I}}{\Gamma_0; y : [\rho_j]_{j\in J} +_{i\in I} \Delta_i \vdash \mathtt{L}'[\![\lambda x.p]\!]s{:}\tau}$$

We can then conclude by the following derivation:

$$\dfrac{\dfrac{\dfrac{\nabla}{\Gamma_0; y:[\rho_j]_{j\in J} \vdash \mathtt{L}'[\![\lambda x.p]\!]:[\sigma_i]_{i\in I}\to\tau} \quad \left(\dfrac{\nabla}{\Pi_j \vdash u:\rho_j}\right)_{j\in J}}{\Gamma_0 +_{j\in J} \Pi_j \vdash \mathtt{L}'[\![\lambda x.p]\!][y/u]:[\sigma_i]_{i\in I}\to\tau} \quad \left(\dfrac{\nabla}{\Delta_i \vdash s:\sigma_i}\right)_{i\in I}}{\Gamma_0 +_{j\in J} \Pi_j +_{i\in I} \Delta_i \vdash \mathtt{L}'[\![\lambda x.p]\!][y/u]s:\tau}$$

- $t = p[x/s] \to p$, where $|p|_x = 0$. Since $x \notin \mathtt{fv}(p)$ one has $\Gamma = x:[\,];\Gamma$. Therefore, one can construct the following derivation

$$\dfrac{x:[\,];\Gamma \vdash p:\tau}{\Gamma \vdash p[x/s]:\tau}\ (\mathtt{cut}_{\mathcal{HW}})$$

- If $t = \mathtt{C}[\![x]\!][x/s] \to \mathtt{C}[\![s]\!][x/s] = t'$, then by construction $\Gamma = \Delta +_{i\in I} \Gamma_i$ and the type derivation of t' has the following form:

$$\dfrac{\dfrac{\nabla}{x:[\sigma_i]_{i\in I};\Delta \vdash \mathtt{C}[\![s]\!] : \tau} \quad \left(\dfrac{\nabla}{\Gamma_i \vdash s:\sigma_i}\right)_{i\in I}}{\Delta +_{i\in I} \Gamma_i \vdash \mathtt{C}[\![s]\!][x/s] : \tau}\ (\mathtt{cut}_{\mathcal{HW}})$$

By Lemma 24 $x:[\sigma_i]_{i\in I};\Delta = \Gamma_0 +_{j\in J} \Gamma_j$ s.t. $x:[\sigma_j]_{j\in J} + \Gamma_0 \vdash \mathtt{C}[\![x]\!]:\tau$ and $(\Gamma_j \vdash s:\sigma_j)_{j\in J}$. Note that $x \notin \mathtt{fv}(s)$ hence $\Gamma_0 = x:[\sigma_i]_{i\in I};\Gamma'_0$ where $\Gamma'_0 +_{j\in J} \Gamma_j = \Delta$.
Let $K := I\cup J$. Then $x:[\sigma_j]_{j\in J}+\Gamma_0 = x:[\sigma_k]_{k\in K};\Gamma'_0$ and we can construct the following derivation

$$\dfrac{\dfrac{\nabla}{x:[\sigma_k]_{k\in K};\Gamma'_0 \vdash \mathtt{C}[\![x]\!] : \tau} \quad \left(\dfrac{\nabla}{\Gamma_k \vdash s:\sigma_k}\right)_{k\in K}}{\Gamma'_0 +_{k\in K} \Gamma_k \vdash \mathtt{C}[\![x]\!][x/s] : \tau}\ (\mathtt{cut}_{\mathcal{HW}})$$

We conclude since $\Gamma'_0 +_{k\in K} \Gamma_k = \Gamma'_0 +_{j\in J} \Gamma_j +_{i\in I} \Gamma_i = \Delta +_{i\in I} \Gamma_i = \Gamma$ as expected.
- All the inductive cases are straightforward.

Lemma 4. If $\Phi \triangleright \Gamma \vdash_{\mathcal{HW}} t:\tau$ and t has no $(\mathtt{dB}, \mathtt{c}|_{\mathtt{L_H}})$-redex T-occurrences in Φ then $t \in \mathtt{L_H M}$-nf.

Proof. Suppose t is not a $\mathtt{L_H M}$-nf. Then it is not difficult to show that $t \in B \cup C$, where B and C are defined as follows :

- $\mathtt{L}[\![\lambda x.u]\!]v \in B$.
- If $u \in B$, then $uv \in B$, $u[x/v] \in B$, $\lambda x.u \in B$.
- $y \in A_y$.
- If $u \in A_y$, then $uv \in A_y$, $\lambda x.u \in A_y$, $u[x/v] \in A_y$ for $x \neq y$.
- If $u \in A_y$, then $u[y/v] \in C$.
- If $u \in C$, then $uv \in C$, $\lambda x.u \in C$, $u[x/v] \in C$.

We show that $t \in B \cup C$ implies that t has a redex T-occurrence in Φ, which leads to a contradiction. We proceed by induction on the definitions of B and C. But first of all we show that $t \in A_y$ implies y has a T-occurrence in Φ.

If $t = y \in A_y$, then the property is straightforward. If $t = uv \in A_y$ or $t = \lambda x.u \in A_y$ or $t = u[x/v] \in A_y$ for $x \neq y$, where $u \in A_y$, then by the *i.h.* the variable y has a T-occurrence in the corresponding subderivation of Φ so that y has a T-occurrence in Φ.

If $t = \mathtt{L}[\![\lambda x.u]\!]v \in B$, then ϵ is a redex T-occurrence in Φ. If $t = uv \in B$ or $t = u[x/v] \in B$ or $t = \lambda x.u \in B$, where $u \in B$, then by the *i.h.* the subterm u has a redex T-occurrence in the corresponding subderivation of Φ so that also t has a redex T-occurrence in Φ. Exactly the same reasoning applies for $t = uv$, or $t = u[x/v]$ or $t = \lambda x.u$ belonging to C where $u \in C$. Finally, if $t = u[y/v]$, where $u \in A_y$, then by the first property shown before we know that y has a T-occurrence in the corresponding subderivation of Φ so that the redex $u[y/v]$ has a T-occurrence in Φ. This concludes the proof.

Lemma 25. *If t is linear-head* M*-normalizing then t is $\mathcal{HW}$-typable.*

Proof. By induction on the length of the linear-head M-normalizing reduction. Let $t \to^k_{\mathtt{L_H M}} t'$, where $t' \in \mathtt{L_H M}$-nf. If $k = 0$ (*i.e.* $t = t'$), then it is not difficult to prove that $t \in \mathtt{A}(n, y, m)$, for some variable y and some $n, m \geq 0$, where $\mathtt{A}(n, y, m)$ is defined as follows:

- If $t \in \mathtt{B}(n, y, m)$, then $t \in \mathtt{A}(n, y, m)$.
- If $t \in \mathtt{A}(n, y, m)$, then $\lambda x.t \in \mathtt{A}(n, y, m)$.
- If $t \in \mathtt{A}(n, y, m)$, then $t[x/u] \in \mathtt{A}(n, y, m)$ for any M-term u and $x \neq y$.
- $y \in \mathtt{B}(n, y, n)$ for any $n \geq 0$.
- If $t \in \mathtt{B}(n, y, m)$ and $m > 0$, then $tu \in \mathtt{B}(n, y, m-1)$ for any M-term u.
- If $t \in \mathtt{B}(n, y, m)$, then $t[x/u] \in \mathtt{B}(n, y, m)$ for any M-term u and $x \neq y$.

Let $\tau^n = \mathcal{M}_1 \to \cdots \to \mathcal{M}_n \to \tau$ $(n \geq 0)$ such that $\mathcal{M}_i = [\,]$ $(1 \leq i \leq n)$. We first prove by induction on $\mathtt{B}(n, y, m)$ that $t \in \mathtt{B}(n, y, m)$ implies $y{:}[\tau^n] \vdash_{\mathcal{HW}} t{:}\tau^m$.

- If $y \in \mathtt{B}(n, y, n)$, then $y{:}[\tau^n] \vdash_{\mathcal{HW}} y{:}\tau^n$ by the typing rule $(\mathtt{ax})$.
- If $tu \in \mathtt{B}(n, y, m)$ comes from $t \in \mathtt{B}(n, y, m+1)$, then $y{:}[\tau^n] \vdash_{\mathcal{HW}} t{:}\tau^{m+1}$ holds by the *i.h.* so that $y{:}[\tau^n] \vdash_{\mathcal{HW}} tu{:}\tau^m$ holds by application of the typing rule $(\to \mathtt{e}_{\mathcal{HW}})$.
- If $t[x/u] \in \mathtt{B}(n, y, m)$ comes from $t \in \mathtt{B}(n, y, m)$, then $y{:}[\tau^n] \vdash_{\mathcal{HW}} t{:}\tau^m$ by holds the *i.h.* so that $y{:}[\tau^n] \vdash_{\mathcal{HW}} t[x/u]{:}\tau^m$ holds by application of the typing rule $(\mathtt{cut}_{\mathcal{HW}})$.

Now, we prove by induction on $\mathtt{A}(n, y, m)$ that $t \in \mathtt{A}(n, y, m)$ implies $\Gamma \vdash_{\mathcal{HW}} t{:}\sigma$ where the domain of Γ has at most the variable y.

- If $t \in \mathtt{A}(n, y, m)$, where $t \in \mathtt{B}(n, y, m)$, then the property follows by the previous point.
- If $\lambda x.t \in \mathtt{A}(n, y, m)$, where $t \in \mathtt{A}(n, y, m)$, then $\Gamma \vdash_{\mathcal{HW}} t{:}\sigma$ by the *i.h.* so that $\Gamma \setminus\!\!\setminus x \vdash_{\mathcal{HW}} \lambda x.t{:}\Gamma(x) \to \sigma$ by application of the typing rule $(\to \mathtt{i})$. If Γ has at most y, then also does $\Gamma \setminus\!\!\setminus x$.
- If $t[x/u] \in \mathtt{A}(n, y, m)$, where $t \in \mathtt{A}(n, y, m)$, then $\Gamma \vdash_{\mathcal{HW}} t{:}\sigma$ by the *i.h.* so that $\Gamma \vdash_{\mathcal{HW}} t[x/u]{:}\sigma$ by application of the typing rule $(\mathtt{cut}_{\mathcal{HW}})$. Since Γ has at most y, then we are done.

Otherwise, let $t \to_{\mathtt{L_H M}} u \to^k_{\mathtt{L_H M}} t'$. By the *i.h.* the term u is $\mathcal{HW}$-typable and thus by Lemma 3 the same holds for t.

Lemma 26. *Let $\Phi \triangleright \Gamma \vdash_{\mathcal{HW}} t{:}\tau$, where $t = \mathtt{L}_n[\ldots \mathtt{L}_1[\mathtt{L}_0[y]\, t_1] \cdots t_n]$, then $y{:}\mathcal{N}_1 \to \cdots \to \mathcal{N}_n \to \tau \in \Gamma$, i.e. $\Gamma = \Gamma' + \{y{:}\mathcal{N}_1 \to \cdots \to \mathcal{N}_n \to \tau\}$.*

Proof. By induction on n.

Let $\Phi \triangleright \Gamma \vdash t{:}\tau$. We define the predicate $A(t, \Phi) :=$ t has no $(\mathtt{dB}, \mathtt{c}, \mathtt{w})$-redex T-occurrences in Φ.

Lemma 27. *Let $\Phi \triangleright \Gamma \vdash_{\mathcal{HW}} t{:}\tau$ such that $A(t,\Phi)$.*

1. *If $[\,]\notin\mathcal{P}(\Gamma)$, and $t = \mathtt{L}_n[\ldots\ \mathtt{L}_1[\mathtt{L}_0[y]\, t_1]\cdots t_n]$, then t has no substitution and $x\in\mathtt{fv}(t)$ implies x has some T-occurrence in Φ.*
2. *If $[\,]\notin\mathcal{P}(\langle\Gamma,\tau\rangle)$, then t has no substitution and $x\in\mathtt{fv}(t)$ implies x has some T-occurrence in Φ.*

Proof. We proceed by induction on Φ.

- $t = y$. Then t has no substitution. We have that $x \in \mathtt{fv}(t)$ implies $x = y$ so that x trivially has some T-occurrence in Φ.
- $t = u[z/v]$. We have $\Gamma = \Gamma_0 +_{i\in I} \Delta_i$ and Φ has necessarily the following form:

$$\Phi \triangleright \dfrac{\Phi_u \triangleright \dfrac{\nabla}{\Gamma_0; z{:}[\sigma_i]_{i\in I} \vdash u{:}\tau} \qquad \left(\Phi_v^i \triangleright \dfrac{\nabla}{\Delta_i \vdash v{:}\sigma_i}\right)_{i\in I}}{\Gamma_0 +_{i\in I} \Delta_i \vdash u[z/v]{:}\tau}$$

  Moreover, $A(t,\Phi)$ implies $A(u,\Phi_u)$. We consider two cases.
  - $z \notin \mathtt{fv}(u)$. Then t has a $\mathtt{w}$-redex T-occurrence which contradicts $A(t,\Phi)$.
  - $z \in \mathtt{fv}(u)$. If z has some T-occurrence in Φ_u, then t has a $\mathtt{c}$-redex T-occurrence which contradicts $A(t,\Phi)$. Therefore, z only has untyped occurrences of Φ_u and thus $I = \emptyset$. We have $\Gamma_0; z{:}[\sigma_i]_{i\in I} = \Gamma_0; z{:}[\,] = \Gamma_0$. We consider again two cases:
    1. If t is of the form of item 1, the hyp $[\,] \notin \mathcal{P}(\Gamma)$ implies $[\,] \notin \mathcal{P}(\Gamma_0)$. Therefore, the *i.h.* on (1) (from right to left) allows to conclude that $z \notin \mathtt{fv}(u)$ which leads to a contradiction.
    2. Otherwise, the hyp $[\,] \notin \mathcal{P}(\langle\Gamma,\tau\rangle)$ implies $[\,] \notin \mathcal{P}(\langle\Gamma_0,\tau\rangle)$. Therefore the *i.h.* on (2) (from right to left) then allows to conclude that $z \notin \mathtt{fv}(u)$ which leads to a contradiction.

  We then conclude that t cannot be a substitution.
- $t = uv$. Then t (and u) is necessarily of the form of item 1, otherwise there is a $\mathtt{dB}$-redex T-occurrence which contradicts the hyp. Then $\Gamma = \Gamma_0 +_{i\in I} \Delta_i$ and Φ has the following form

$$\Phi \triangleright \dfrac{\Phi_u \triangleright \dfrac{\nabla}{\Gamma_0 \vdash u{:}[\sigma_i]_{i\in I} \to \tau} \qquad \left(\Phi_v^i \triangleright \dfrac{\nabla}{\Delta_i \vdash v{:}\sigma_i}\right)_{i\in I}}{\Gamma_0 +_{i\in I} \Delta_i \vdash uv{:}\tau}$$

  By Lemma 26 we have that $\Gamma = \Gamma' + \{y : \mathcal{M}_1 \to \ldots \to \mathcal{M}_{n-1} \to [\sigma_i]_{i\in I} \to \tau\}$ for some y being the head of the term u. Moreover $A(t,\Phi)$ implies $A(u,\Phi_u)$.
  Let us suppose $[\,] \in \mathcal{P}(\Gamma_0)$, then $[\,] \in \mathcal{P}(\Gamma)$, which leads to a contradiction. Therefore $[\,] \notin \mathcal{P}(\Gamma_0)$. By the *i.h.* on (1) u has no substitution and $x \in \mathtt{fv}(u)$ implies x has some T-occurrence in Φ_u, which also means some T-occurrence in Φ. Now we consider two cases:
  1. If $I = \emptyset$, then $[\,] \in \mathcal{N}([\,] \to \tau)$ implies $[\,] \in \mathcal{N}(\mathcal{M}_1 \to \ldots \to \mathcal{M}_{n-1} \to [\,] \to \tau)$, which in turn implies $[\,] \in \mathcal{P}(y : \mathcal{M}_1 \to \ldots \to \mathcal{M}_{n-1} \to [\,] \to \tau)$ and $[\,] \in \mathcal{P}(\Gamma)$. This contradicts the hyp.
  2. If $I \neq \emptyset$, suppose $[\,] \in \mathcal{P}(\Delta_i)$ for some $i \in I$. Then $[\,] \in \mathcal{P}(\Gamma_0 +_{i\in I} \Delta_i)$ which contradicts the hyp. Therefore, $([\,] \notin \mathcal{P}(\Delta_i))_{i\in I}$. Moreover, if $[\,] \in \mathcal{P}(\sigma_i)$, then $[\,] \in \mathcal{P}([\sigma_i]_{i\in I})$, which implies $[\,] \in \mathcal{N}([\sigma_i]_{i\in I} \to \tau)$, which implies $[\,] \in \mathcal{N}(\mathcal{M}_1 \to \ldots \to \mathcal{M}_{n-1} \to [\sigma_i]_{i\in I} \to \tau)$ and thus $[\,] \in \mathcal{P}(\Gamma)$. This leads to a contradiction with the hyp. Therefore $([\,] \notin \mathcal{P}(\sigma_i))_{i\in I}$ and thus $([\,] \notin \mathcal{P}(\langle\Delta_i,\sigma_i\rangle))_{i\in I}$. Since $A(t,\Phi)$ implies $A(v,\Phi_v^i)$, then we can now apply the *i.h.* on (2) so that we obtain that v has no substitution. Therefore t has no substitution.
     If $x \in \mathtt{fv}(v)$, the *i.h.* on (2) also gives that x has a T-occurrence in Φ_v, thus a T-occurrence in Φ. If $x \in \mathtt{fv}(u)$, then x has a T-occurrence in Φ_u as explained, thus a T-occurrence in Φ.

– $t = \lambda y.u$. Then $A(t, \Phi)$ implies $A(u, \Phi_u)$ and we are necessarily in case 2. By construction we have $\tau = \mathcal{M} \to \sigma$ and $\Phi_u \triangleright \Gamma; y{:}\mathcal{M} \vdash_{\mathcal{HW}} u{:}\sigma$. If $[\,] \in \mathcal{P}(\langle \Gamma; y{:}\mathcal{M}, \sigma\rangle)$ then either $[\,] \in \mathcal{P}(\Gamma)$, $[\,] \in \mathcal{N}(\mathcal{M})$ or $[\,] \in \mathcal{P}(\sigma)$, which leads to a contradiction with the hyp. Hence, $[\,] \notin \mathcal{P}(\langle \Gamma; y{:}\mathcal{M}, \sigma\rangle)$ and the property is straightforward by the *i.h.*
– There is no other possible case.

Lemma 5. Let $\Phi \triangleright \Gamma \vdash_{\mathcal{HW}} t{:}\tau$ s.t. $[\,] \notin \mathcal{P}(\langle \Gamma, \tau\rangle)$. If t has no $(\mathtt{dB}, \mathtt{c}, \mathtt{w})$-redex T-occurrences in Φ, then t is in $\mathtt{M}$-nf.

Proof. We proceed by induction on Φ.

– If $t = y$, the two statements are trivial.
– Let $t = \lambda x.u$. By construction we have $\tau = \mathcal{M} \to \sigma$ and $\Phi_u \triangleright \Gamma; x{:}\mathcal{M} \vdash_{\mathcal{HW}} u{:}\sigma$. If $[\,] \in \mathcal{P}(\langle \Gamma; x{:}\mathcal{M}, \sigma\rangle)$ then either $[\,] \in \mathcal{P}(\Gamma)$, $[\,] \in \mathcal{N}(\mathcal{M})$ or $[\,] \in \mathcal{P}(\sigma)$, leading to a contradiction with the hyp. Hence, $[\,] \notin \mathcal{P}(\langle \Gamma; x{:}\mathcal{M}, \sigma\rangle)$ and the property is straightforward by the *i.h.*
– Let $t = u[x/v]$. Since $[\,] \notin \mathcal{P}(\langle \Gamma, \tau\rangle)$ and $A(t, \Phi)$, then t has no substitution by Lemma 27, thus t cannot be of this form.
– If t is an application, then $t = \mathtt{L}_n[\ldots\ \mathtt{L}_1[\mathtt{L}_0[y]\, t_1] \cdots t_n]$, for $n > 0$, otherwise t would have a $\mathtt{dB}$-redex T-occurrence in Φ. Moreover, $A(t, \Phi)$ and $[\,] \notin \mathcal{P}(\langle \Gamma, \tau\rangle)$ hold by hypothesis, so that t has no substitution, thus $(\mathtt{L}_i = \Box)_{0 \le i \le n}$, by Lemma 27. Therefore $t = yt_1 \ldots t_n$.
By construction $(\Gamma_0^{(j+1)} = \Gamma_0^j +_{i \in I_j} \Gamma_i^j)_{1 \le j \le n}$, $(\sigma_j = \mathcal{N}_j \to \sigma_{(j+1)})_{1 \le j \le n}$ and $(\mathcal{N}_j = [\rho_i]_{i \in I_j})_{1 \le j \le n}$, where $\Phi = \Phi_0^{n+1}$, $\Gamma = \Gamma_0^{n+1}$ and $\tau = \sigma_{n+1}$. Moreover, each Φ_0^{j+1} has the following form

$$\Phi_0^{j+1} \triangleright \frac{\Phi_0^j \triangleright \dfrac{\nabla}{\Gamma_0^j \vdash y\, t_1 \cdots t_{(j-1)}{:}[\rho_i]_{i \in I_j} \to \sigma_{(j+1)}} \qquad \left(\Phi_{t_j}^i \triangleright \dfrac{\nabla}{\Gamma_i^j \vdash t_j{:}\rho_i}\right)_{i \in I_j}}{\Gamma_0^j +_{i \in I_j} \Gamma_i^j \vdash y\, t_1 \cdots t_j{:}\sigma_{(j+1)}}\ (\to \mathtt{e}_{\mathcal{HW}})$$

Suppose $[\,] \in \mathcal{P}(\Gamma_0^j)$ for some $j \in \{1 \ldots n\}$. Then $[\,] \in \mathcal{P}(\Gamma)$ which leads to a contradiction. Therefore, $([\,] \notin \mathcal{P}(\Gamma_0^j))_{1 \le j \le n}$.
Suppose $[\,] \in \mathcal{P}(\Gamma_i^j)$ for some $i \in I_j$ and some $1 \le j \le n$. Then $[\,] \in \mathcal{P}(\Gamma)$ which leads to a contradiction. Therefore, $([\,] \notin \mathcal{P}(\Gamma_i^j))_{i \in I_j, 1 \le j \le n}$.
In particular $\Phi_0^1 \triangleright \Gamma_0^1 \vdash_{\mathcal{HW}} y{:}\sigma_1 = \mathcal{N}_1 \to \cdots \to \mathcal{N}_n \to \tau$ thus, by construction, $\Gamma_0^1 = \{y : [\mathcal{N}_1 \to \cdots \to \mathcal{N}_n \to \sigma]\}$. Since $[\,] \notin \mathcal{P}(\Gamma_0^1)$ then $(I_j \neq \emptyset)_{1 \le j \le n}$ and $([\,] \notin \mathcal{P}(\rho_i))_{i \in I_j, 1 \le j \le n}$.
Therefore $(\Phi_{t_j}^i \triangleright \Gamma_i^j \vdash t_j{:}\rho_i$ and $[\,] \notin \mathcal{P}(\langle \Gamma_i^j, \rho_i\rangle))_{i \in I_j, 1 \le j \le n}$. Note also that, any redex T-occurrence of t_j in some $\Phi_{t_j}^i$, for some $i \in I_j$, would also be a redex T-occurrence of t in in Φ. Since $A(t, \Phi)$, then also $A(t_j, \Phi_{t_j}^i)_{i \in I_j, 1 \le j \le n}$. Finally, $(t_j)_{1 \le j \le n}$ is a $\mathtt{M}$-nf by the *i.h.* so that $t = yt_1 \ldots t_n$ is a $\mathtt{M}$-nf too.

Lemma 28. *Let $t \in \mathcal{T}_{\mathtt{M}}$. If $t \in \mathcal{WN}(\mathtt{M})$ then $\Gamma \vdash_{\mathcal{HW}} t{:}\tau$ and $[\,] \notin \mathcal{P}(\langle \Gamma, \tau\rangle)$.*

Proof. Let $t \to_{\mathtt{M}}^k t'$, where $t' \in \mathtt{M}$-nf. We proceed by induction on k. If $k = 0$ (*i.e.* $t = t'$), then $t = \lambda x_1 \ldots \lambda x_m.yt_1 \ldots t_n$, where $m, n \ge 0$ and $t_i \in \mathtt{M}$-nf for $1 \le i \le n$. We then proceed by induction on $\mathtt{M}$-nf.

The *i.h.* gives $\Gamma_i \vdash t_i{:}\sigma_i$, s.t. $[\,] \notin \mathcal{P}(\langle \Gamma_i, \sigma_i\rangle)$ for $1 \le i \le n$. Let $\tau = [\sigma_1] \to \cdots \to [\sigma_n] \to \alpha$, for α a base type, and let $\Gamma = \{y{:}[\tau]\} +_{1 \le i \le n} \Gamma_i$. Note that $[\,] \notin \mathcal{N}(\tau)$ thus $[\,] \notin \mathcal{P}(\Gamma)$ and $[\,] \notin \mathcal{P}(\langle \Gamma, \alpha\rangle)$. By the typing rule $(\mathtt{ax})$ and n applications of $(\to \mathtt{e}_{\mathcal{HW}})$ we obtain $\Gamma \vdash_{\mathcal{HW}} yt_1 \ldots t_n{:}\alpha$ where $[\,] \notin \mathcal{P}(\langle \Gamma, \alpha\rangle)$. Therefore, by m applications of the rule $(\to \mathtt{i})$ we obtain $\Gamma \setminus\!\!\setminus (x_m, \ldots, x_1) \vdash_{\mathcal{HW}}$

$t{:}\Gamma(x_1)\to\cdots\to\Gamma(x_m)\to\alpha$. We have in particular $[\,]\notin\mathcal{N}(\Gamma(x_i))$ since $[\,]\notin\mathcal{P}(\Gamma)$ and $[\,]\notin\mathcal{N}([\,])$. Therefore we conclude $[\,]\notin\mathcal{P}(\langle\Gamma\setminus\!\!\setminus(x_m,\ldots,x_1),\Gamma(x_1)\to\cdots\to\Gamma(x_m)\to\alpha\rangle)$.

Otherwise, let $t\to_{\mathtt{M}} u\to^k_{\mathtt{M}} t'$. By the *i.h.* we have $\Gamma\vdash_{\mathcal{HW}} u{:}\tau$ and $[\,]\notin\mathcal{P}(\langle\Gamma,\tau\rangle)$. Thus by Lemma 3 the same holds form t.

B Appendix: Characterization of Strong M-Normalization

As for the $\mathcal{HW}$ typing system, the $\mathcal{S}$ system verifies the following partial substitution lemma.

Lemma 29 (Partial Substitution II). *If $\Phi_{\mathtt{C}[\![x]\!]}\rhd x{:}[\sigma_i]_{i\in I};\Gamma\vdash_{\mathcal{S}}\mathtt{C}[\![x]\!]{:}\tau$ and $(\Phi^i_u\rhd\Delta_i\vdash_{\mathcal{S}} u{:}\sigma_i)_{i\in I}$ then $\Phi_{\mathtt{C}[\![u]\!]}\rhd x{:}[\sigma_i]_{i\in I\setminus K};\Gamma+_{k\in K}\Delta_k\vdash_{\mathcal{S}}\mathtt{C}[\![u]\!]{:}\tau$, for some $\emptyset\neq K\subseteq I$ where $\mathtt{sz}(\Phi_{\mathtt{C}[\![u]\!]})=\mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]})+_{k\in K}\mathtt{sz}(\Phi^k_u)-|K|$.*

Proof. By induction on the type derivation $\Phi_{\mathtt{C}[\![x]\!]}$. We only show here the two application cases, the cases with explicit substitutions follow the same scheme of this one, and all the other ones are very similar to those in the proof of Lemma 23.

- If $C=\mathtt{D}t$, then by construction $x{:}[\rho_i]_{i\in I};\Gamma=\Delta+_{j\in J\cup\{w\}}\Gamma_j$ and $\Phi_{\mathtt{C}[\![x]\!]}$ is of the following form

$$\Phi_{\mathtt{C}[\![x]\!]}\rhd\dfrac{\Phi_{\mathtt{D}[\![x]\!]}\rhd\dfrac{\nabla}{\Delta\vdash\mathtt{D}[\![x]\!]{:}[\sigma_j]_{j\in J}\to\tau}\qquad\left(\Phi^j_t\rhd\dfrac{\nabla}{\Gamma_j\vdash t{:}\sigma_j}\right)_{j\in J\cup\{w\}}}{x{:}[\rho_i]_{i\in I};\Gamma\vdash\mathtt{D}[\![x]\!]t{:}\tau}\ (\to\mathtt{e}_{\mathcal{S}})$$

  and $\mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]})=\mathtt{sz}(\Phi_{\mathtt{D}[\![x]\!]})+_{j\in J\cup\{w\}}\mathtt{sz}(\Phi^j_t)+1$.
  By Lemma 1 we necessarily have $\Delta=x{:}[\rho_i]_{i\in L};\Delta'$, where $\emptyset\neq L\subseteq I$. The *i.h.* then holds for $\Phi_{\mathtt{D}[\![x]\!]}\rhd x{:}[\rho_i]_{i\in L};\Delta'\vdash\mathtt{D}[\![x]\!]{:}[\sigma_j]_{j\in J}\to\tau$ so that $\Phi_{\mathtt{D}[\![s]\!]}\rhd x{:}[\rho_i]_{i\in L\setminus K};\Delta'+_{k\in K}\Delta_k\vdash\mathtt{D}[\![s]\!]{:}[\sigma_j]_{j\in J}\to\tau$, for some $\emptyset\neq K\subseteq L$ where $\mathtt{sz}(\Phi_{\mathtt{D}[\![s]\!]})=\mathtt{sz}(\Phi_{\mathtt{D}[\![x]\!]})+_{k\in K}\mathtt{sz}(\Phi^k_s)-|K|$. Then we construct the following derivation

$$\Phi_{\mathtt{C}[\![s]\!]}\rhd\dfrac{\Phi_{\mathtt{D}[\![s]\!]}\rhd\dfrac{\nabla}{x{:}[\rho_i]_{i\in L\setminus K};\Delta'+_{k\in K}\Delta_k\vdash\mathtt{D}[\![s]\!]{:}[\sigma_j]_{j\in J}\to\tau}\qquad\left(\Phi^j_t\rhd\dfrac{\nabla}{\Gamma_j\vdash t{:}\sigma_j}\right)_{j\in J\cup\{w\}}}{(x{:}[\rho_i]_{i\in L\setminus K};\Delta'+_{k\in K}\Delta_k)+_{j\in J\cup\{w\}}\Gamma_j\vdash\mathtt{D}[\![s]\!]t{:}\tau}\ (\to\mathtt{e}_{\mathcal{S}})$$

  We thus have $x{:}[\rho_i]_{i\in I};\Gamma=\Delta+_{j\in J\cup\{w\}}\Gamma_j$ implies $x{:}[\rho_i]_{i\in I};\Gamma=x{:}[\rho_i]_{i\in L};\Delta'+_{j\in J\cup\{w\}}\Gamma_j$ implies $x{:}[\rho_i]_{i\in I};\Gamma=x{:}[\rho_i]_{i\in L\setminus K}+x{:}[\rho_k]_{k\in K};\Delta'+_{j\in J\cup\{w\}}\Gamma_j$ implies $x{:}[\rho_i]_{i\in I\setminus K};\Gamma=x{:}[\rho_i]_{i\in L\setminus K};\Delta'+_{j\in J\cup\{w\}}\Gamma_j$ implies $x{:}[\rho_i]_{i\in I\setminus K};\Gamma+_{i\in K}\Delta_i=(x{:}[\rho_i]_{i\in L\setminus K};\Delta'+_{i\in K}\Delta_i)+_{j\in J\cup\{w\}}\Gamma_j$ and $\mathtt{sz}(\Phi_{\mathtt{C}[\![s]\!]})=\mathtt{sz}(\Phi_{\mathtt{D}[\![s]\!]})+_{j\in J\cup\{w\}}\mathtt{sz}(\Phi^j_t)+1=_{i.h.}\mathtt{sz}(\Phi_{\mathtt{D}[\![x]\!]})+_{k\in K}\mathtt{sz}(\Phi^k_s)-|K|+_{j\in J\cup\{w\}}\mathtt{sz}(\Phi^j_t)+1=\mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]})+_{k\in K}\mathtt{sz}(\Phi^k_s)-|K|$.
  We conclude since $K\subseteq L\subseteq I$.
- If $C=t\mathtt{D}$, then by construction $x{:}[\rho_i]_{i\in I};\Gamma=\Delta+_{j\in J\cup\{w\}}\Gamma_j$, and $\Phi_{\mathtt{C}[\![x]\!]}$ is of the form

$$\Phi_{\mathtt{C}[\![x]\!]}\rhd\dfrac{\Phi_t\rhd\dfrac{\nabla}{\Delta\vdash t{:}[\sigma_j]_{j\in J}\to\tau}\qquad\left(\Phi^j_{\mathtt{D}[\![x]\!]}\rhd\dfrac{\nabla}{\Gamma_j\vdash\mathtt{D}[\![x]\!]{:}\sigma_j}\right)_{j\in J\cup\{w\}}}{x{:}[\rho_i]_{i\in I};\Gamma\vdash t\mathtt{D}[\![x]\!]{:}\tau}\ (\to\mathtt{e}_{\mathcal{S}})$$

and $\mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) = \mathtt{sz}(\Phi_t) +_{j \in J \cup \{w\}} \mathtt{sz}(\Phi^j_{\mathtt{D}[\![x]\!]}) + 1$.

By Lemma 1 we necessarily have $\Gamma_j = x{:}[\rho_i]_{i \in L_j}; \Gamma'_j$, where $\emptyset \neq L_j \subseteq I$. The *i.h.* then holds for $\Phi^j_{\mathtt{D}[\![x]\!]} \triangleright x{:}[\rho_i]_{i \in L_j}; \Gamma'_j \vdash \mathtt{D}[\![x]\!]{:}\sigma_j$ so that $\Phi^j_{\mathtt{D}[\![s]\!]} \triangleright x{:}[\rho_i]_{i \in L_j \setminus K_j}; \Gamma'_j +_{i \in K_j} \Delta_i \vdash \mathtt{D}[\![s]\!]{:}\sigma_j$ for $\emptyset \neq K_j \subseteq L_j$ and $\mathtt{sz}(\Phi^j_{\mathtt{D}[\![s]\!]}) = \mathtt{sz}(\Phi^j_{\mathtt{D}[\![x]\!]}) +_{i \in K_j} \mathtt{sz}(\Phi^i_s) - |K_j|$.

We then construct the following derivation:

$$\Phi_{\mathtt{C}[\![s]\!]} \triangleright \dfrac{\Phi_t \triangleright \dfrac{\nabla}{\Delta \vdash t{:}[\sigma_j]_{j \in J} \rightarrow \tau} \qquad \left(\Phi^j_{\mathtt{D}[\![s]\!]} \triangleright \dfrac{\nabla}{x{:}[\rho_i]_{i \in L_j \setminus K_j}; \Gamma'_j +_{i \in K_j} \Delta_i \vdash \mathtt{D}[\![s]\!]{:}\sigma_j} \right)_{j \in J \cup \{w\}}}{\Delta +_{j \in J \cup \{w\}} (x{:}[\rho_i]_{i \in L_j \setminus K_j}; \Gamma'_j +_{i \in K_j} \Delta_i) \vdash t\mathtt{D}[\![s]\!]{:}\tau} \ (\rightarrow \mathtt{e}_{\mathcal{S}})$$

We have $\Delta +_{j \in J \cup \{w\}} (x{:}[\rho_i]_{i \in L_j \setminus K_j}; \Gamma'_j +_{i \in K_j} \Delta_i) = x{:}[\rho_i]_{i \in I \setminus \cup_{j \in J} K_j}; \Gamma +_{j \in J \cup \{w\}} +_{i \in K_j} \Delta_i$ so that we can conclude with $K = \cup_{j \in J \cup \{w\}} K_j$.

Finally, $\mathtt{sz}(\Phi_{\mathtt{C}[\![s]\!]}) = \mathtt{sz}(\Phi_t) +_{j \in J \cup \{w\}} \mathtt{sz}(\Phi^j_{\mathtt{D}[\![s]\!]}) + 1 =_{i.h.} \mathtt{sz}(\Phi_t) +_{j \in J \cup \{w\}} (\mathtt{sz}(\Phi^j_{\mathtt{D}[\![x]\!]}) +_{i \in K_j} \mathtt{sz}(\Phi^i_s) - |K_j|) + 1 = \mathtt{sz}(\Phi_t) +_{j \in J \cup \{w\}} \mathtt{sz}(\Phi^j_{\mathtt{D}[\![x]\!]}) +_{i \in K} \mathtt{sz}(\Phi^i_s) - |K| + 1 = \mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) +_{i \in K} \mathtt{sz}(\Phi^i_s) - |K|$.

We conclude since $K = \cup_{j \in J \cup \{w\}} K_j \subseteq \cup_{j \in J \cup \{w\}} L_j \subseteq I$. Finally, $K_j \neq \emptyset$ by the *i.h.* so that $\emptyset \neq K_j \subseteq K$ as required.

Using Lemma 1 (Strong) and Lemmas 29 we can show the Subject Reduction property.

Lemma 7 (SR II). Let $\Phi \triangleright \Gamma \vdash_{\mathcal{S}} t{:}\tau$. If $t \rightarrow_{\mathtt{M}\setminus\mathtt{w}} t'$ then $\Phi' \triangleright \Gamma \vdash_{\mathcal{S}} t'{:}\tau$ and $\mathtt{sz}(\Phi) > \mathtt{sz}(\Phi')$.

Proof. By induction on the reduction relation $\rightarrow_{\mathtt{M}\setminus\mathtt{w}}$. We only show here the most interesting case as the other ones are similar to the those of the proof of Lemma 2. Remark that the case $\rightarrow_{\mathtt{w}}$ is not treated since the statement only concerns the non-erasing reduction $\rightarrow_{\mathtt{M}\setminus\mathtt{w}}$.

Let $t = \mathtt{C}[\![x]\!][x/u] \rightarrow \mathtt{C}[\![u]\!][x/u] = t'$, where $|\mathtt{C}[\![x]\!]|_x \geq 1$, then by construction Φ is of the form

$$\Phi \triangleright \dfrac{\Phi_{\mathtt{C}[\![x]\!]} \triangleright \dfrac{\nabla}{x{:}[\rho_i]_{i \in I}; \Pi \vdash \mathtt{C}[\![x]\!]{:}\tau} \qquad \left(\Phi^i_u \triangleright \dfrac{\nabla}{\Delta_i \vdash u{:}\rho_i} \right)_{i \in I \cup \{w\}}}{\Pi +_{i \in I \cup \{w\}} \Delta_i \vdash \mathtt{C}[\![x]\!][x/u]{:}\tau}$$

where $\mathtt{sz}(\Phi) = \mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) +_{i \in I \cup \{w\}} \mathtt{sz}(\Phi^i_u) + 1$. By Lemma 29, $\Phi_{\mathtt{C}[\![u]\!]} \triangleright x{:}[\rho_i]_{i \in I \setminus K}; \Pi +_{k \in K} \Delta_k \vdash \mathtt{C}[\![u]\!]{:}\tau$, for some $\emptyset \neq K \subseteq I$ where $\mathtt{sz}(\Phi_{\mathtt{C}[\![u]\!]}) = \mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) +_{k \in K} \mathtt{sz}(\Phi^k_u) - |K|$. Hence $I \cup \{w\} \setminus K = (I \setminus K) \cup \{w\}$ and

$$\Phi' \triangleright \dfrac{\Phi_{\mathtt{C}[\![u]\!]} \triangleright \dfrac{\nabla}{x{:}[\rho_i]_{i \in I \setminus K}; \Pi +_{i \in K} \Delta_i \vdash \mathtt{C}[\![u]\!]{:}\tau} \qquad \left(\Phi^i_u \triangleright \dfrac{\nabla}{\Delta_i \vdash u{:}\rho_i} \right)_{i \in I \cup \{w\} \setminus K}}{\Pi +_{i \in K} \Delta_i +_{i \in I \cup \{w\} \setminus K} \Delta_j \vdash \mathtt{C}[\![u]\!][x/u]{:}\tau}$$

where $\mathtt{sz}(\Phi') = \mathtt{sz}(\Phi_{\mathtt{C}[\![u]\!]}) +_{i \in I \cup \{w\} \setminus K} \mathtt{sz}(\Phi^i_u) + 1 = \mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) +_{k \in K} \mathtt{sz}(\Phi^k_u) - |K| +_{i \in I \cup \{w\} \setminus K} \mathtt{sz}(\Phi^i_u) + 1 = \mathtt{sz}(\Phi_{\mathtt{C}[\![x]\!]}) +_{i \in I \cup \{w\}} \mathtt{sz}(\Phi^i_u) + 1 - |K| = \mathtt{sz}(\Phi) - |K| \leq \mathtt{sz}(\Phi)$.

Since $K \neq \emptyset$, then $\mathtt{sz}(\Phi') < \mathtt{sz}(\Phi)$.

The following Lemma is used in the Subject Expansion property for system $\mathcal{S}$.

Lemma 30. *If $\Gamma \vdash_{\mathcal{S}} C[\![s]\!]{:}\tau$ and $x \notin \mathtt{fv}(s)$ then $\exists \Delta, \exists I, \exists (\Pi_i)_{i\in I}, \exists (\sigma)_{i\in I}$ s.t. $\Gamma = \Delta +_{i\in I} \Pi_i$ and $\{x{:}[\sigma_i]_{i\in I}\} + \Delta \vdash_{\mathcal{S}} C[\![x]\!]{:}\tau$ and $(\Pi_i \vdash_{\mathcal{S}} s{:}\sigma_i)_{i\in I}$.*

Proof. By induction on the structure of $C[\![s]\!]$. The proof is very similar to that of Lemma 24.

Lemma 8 (SE II). Let $\Gamma \vdash_{\mathcal{S}} t'{:}\tau$. If $t \to_{\mathtt{M}\setminus\mathtt{w}} t'$ then $\Gamma \vdash_{\mathcal{S}} t{:}\tau$.

Proof. The proof is by induction on the reduction relation and proceeds similarly to that of Lemma 3. We only show here the two most interesting cases.

- If $t = \mathtt{L}[\![(\lambda x.p)]\!]s \to \mathtt{L}[\![p[x/s]]\!] = t'$, then we proceed by induction on $\mathtt{L}$. If $\mathtt{L} = \Box$, then by construction $\Gamma = \Delta +_{i\in I\cup\{w\}} \Gamma_i$ and we have the following derivation:

$$\dfrac{\dfrac{\nabla}{x{:}[\sigma_i]_{i\in I}; \Delta \vdash p{:}\tau} \qquad \left(\dfrac{\nabla}{\Gamma_i \vdash s{:}\sigma_d ij}\right)_{i\in I\cup\{w\}}}{\Gamma \vdash p[x/s]{:}\tau}\ (\mathtt{cut}_{\mathcal{S}})$$

  We then construct the following derivation

$$\dfrac{\dfrac{\dfrac{\nabla}{x{:}[\sigma_i]_{i\in I}; \Delta \vdash p{:}\tau}}{\Delta \vdash \lambda x.p{:}[\sigma_i]_{i\in I} \to \tau}\ (\to \mathtt{i}) \qquad \left(\dfrac{\nabla}{\Gamma_i \vdash s{:}\sigma_i}\right)_{i\in I\cup\{w\}}}{\Gamma \vdash (\lambda x.p)\, s{:}\tau}\ (\to \mathtt{e}_{\mathcal{S}})$$

  If $\mathtt{L} = \mathtt{L}'[y/u]$, then we proceed exactly as in Lemma 3.
- If $t = C[\![x]\!][x/s] \to C[\![s]\!][x/s] = t'$, where $|C[\![x]\!]|_x \geq 1$, then by construction $\Gamma = \Delta +_{i\in I\cup\{w\}} \Pi_i$ and the typing derivation of t' has the following form:

$$\dfrac{\dfrac{\nabla}{x{:}[\sigma_i]_{i\in I}; \Delta \vdash C[\![s]\!] : \tau} \qquad \left(\Phi_s^i \triangleright \dfrac{\nabla}{\Pi_i \vdash s : \sigma_i}\right)_{i\in I\cup\{w\}}}{\Delta +_{i\in I\cup\{w\}} \Pi_i \vdash C[\![s]\!][x/s] : \tau}\ (\mathtt{cut}_{\mathcal{S}})$$

  By Lemma 30 $x{:}[\sigma_i]_{i\in I}; \Delta = \Lambda +_{l\in L} \Pi'_l$ and $\exists (\sigma_l)_{l\in L}$ s.t. $\{x{:}[\sigma_l]_{l\in L}\} + \Lambda \vdash C[\![x]\!]{:}\tau$ and $(\Pi'_l \vdash s{:}\sigma_l)_{l\in L}$. Note that $x \notin \mathtt{fv}(s)$ by α-conversion hence by Lemma 1 $\Lambda = x{:}[\sigma_i]_{i\in I}; \Lambda'$ where $\Lambda' +_{l\in L} \Pi'_l = \Delta$. Let $K := I \cup L$. Then $x{:}[\sigma_l]_{l\in L} + \Lambda = x{:}[\sigma_k]_{k\in K}; \Lambda'$ and we can construct the following derivation

$$\dfrac{\dfrac{\nabla}{x{:}[\sigma_k]_{k\in K}; \Lambda' \vdash C[\![x]\!] : \tau} \qquad \left(\Phi_s^k \triangleright \dfrac{\nabla}{\Pi_k \vdash s : \sigma_k}\right)_{k\in K\cup\{w\}}}{\Lambda' +_{k\in K\cup\{w\}} \Pi_k \vdash C[\![x]\!][x/s] : \tau}\ (\mathtt{cut}_{\mathcal{S}})$$

  We conclude since $\Lambda' +_{k\in K\cup\{w\}} \Pi_k = \Lambda' +_{i\in I} \Pi_i +_{l\in L} \Pi_l + \Pi_w = \Delta +_{i\in I\cup\{w\}} \Pi_j = \Gamma$ as expected.

Let $t \in \mathcal{T}_{\mathtt{M}}$ such that $|t|_x = n$. If $y \notin \mathtt{fv}(t)$, then we write $t_{[x/y]}$ to denote an arbitrary nondeterministic replacement of i ($0 \leq i \leq n$) occurrences of x by the variable y. Thus for example if $t = xx$, then $t_{[x/y]}$ may denote one of the terms xx, yx, xy or yy.

Lemma 31. *Let $t \in \mathcal{T}_{\mathtt{M}}$ s.t. $y \notin \mathtt{fv}(t)$. If $t \to_{\mathtt{M}\backslash\mathtt{w}} t'$, then $t_{[x/y]} \to_{\mathtt{M}\backslash\mathtt{w}} t'_{[x/y]}$ and $t_{[x/y]}\{y/v\} \to^{+}_{\mathtt{M}\backslash\mathtt{w}} t'_{[x/y]}\{y/v\}$.*

A consequence of the previous lemma is that $t \to t'$ implies $t\{x/v\} \to^{+}_{\mathtt{M}\backslash\mathtt{w}} t'\{x/v\}$.

Corollary 3. *Let $t, u \in \mathcal{T}_{\mathtt{M}}$ s.t. $y \notin \mathtt{fv}(t)$. If $t_{[x/y]}\{y/u\} \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ then $t \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$.*

A consequence of the previous corollary is that $C[\![u]\!] \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ implies $C[\![x]\!] \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$.

Lemma 32. $\mathcal{SN}(\mathtt{M}\backslash\mathtt{w}) = \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$.

Proof. If $t \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$, then we show $t \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ by induction on $\langle \eta_{\mathtt{M}\backslash\mathtt{w}}(t), t\rangle$ w.r.t. the lexicographic order. We reason by cases. If $t = x$ or $t = \lambda x.u$ or $t = u[x/v]$ with $|u|_x = 0$, then the property is straightforward. Otherwise,

- If $t = u[x/v]$ with $|u|_x > 0$, *i.e.* $u = \mathtt{C}[\![x]\!]$, then every t s.t. $t \to t'$ verifies $t' \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ and in particular $t' = \mathtt{C}[\![v]\!][x/v]$. Since $\eta_{\mathtt{M}\backslash\mathtt{w}}(t') < \eta_{\mathtt{M}\backslash\mathtt{w}}(t)$, then $t' \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, so that we can conclude $t \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ by (C).
- If t is an application, then we reason by cases.
  - If $t = u_0[x/u_1]t_1 \dots t_n$, with $n \geq 1$, then $\eta_{\mathtt{M}\backslash\mathtt{w}}(u_0[x/u_1]t_1 \dots t_n) = \eta_{\mathtt{M}\backslash\mathtt{w}}((u_0 t_1 \dots t_n)[x/u_1])$ by Lemma 6. Therefore $(u_0 t_1 \dots t_n)[x/u_1] \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ and in particular $\eta_{\mathtt{M}\backslash\mathtt{w}}(u_0 t_1 \dots t_n), \eta_{\mathtt{M}\backslash\mathtt{w}}(u_1) \leq \eta_{\mathtt{M}\backslash\mathtt{w}}(u_0[x/u_1]t_1 \dots t_n)$. Thus $u_0 t_1 \dots t_n, u_1 \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ holds by the *i.h.* If $|u_0 t_1 \dots t_n|_x = 0$, then $(u_0 t_1 \dots t_n)[x/u_1] \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ by (W) and $u_0[x/u_1]t_1 \dots t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ by several applications of (E). If $|u_0 t_1 \dots t_n|_x > 0$, then $(u_0 t_1 \dots t_n)[x/u_1] = C[\![x]\!][x/u_1] \to_{\mathtt{c}} C[\![u_1]\!][x/u_1]$ and thus $\eta_{\mathtt{M}\backslash\mathtt{w}}(C[\![u_1]\!][x/u_1]) < \eta_{\mathtt{M}\backslash\mathtt{w}}(C[\![x]\!][x/u_1])$. The *i.h.* gives $C[\![u_1]\!][x/u_1] \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ so that $(u_0 t_1 \dots t_n)[x/u_1] = C[\![x]\!][x/u_1] \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ holds by (C) and $u_0[x/u_1]t_1 \dots t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ by several applications of (E).
  - If $t = x t_1 \dots t_n$, with $n \geq 1$, then $t \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ implies $t_i \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$. Moreover $\langle \eta_{\mathtt{M}\backslash\mathtt{w}}(t_i), t_i\rangle <_{lex} \langle \eta_{\mathtt{M}\backslash\mathtt{w}}(t), t\rangle$ so the *i.h.* gives $t_i \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ and thus $x t_1 \dots t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ by (V).
  - If $t = (\lambda x.u_0)t_1 \dots t_n$, with $n \geq 1$, then $u_0[x/t_1]t_2 \dots t_n \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$. Moreover we ave that $\langle \eta_{\mathtt{M}\backslash\mathtt{w}}(u_0[x/t_1]t_2 \dots t_n), u_0[x/t_1]t_2 \dots t_n\rangle <_{lex} \langle \eta_{\mathtt{M}\backslash\mathtt{w}}(t), t\rangle$ so the *i.h.* gives $u_0[x/t_1]t_2 \dots t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ and thus $(\lambda x.u_0)t_1 \dots t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ by (B).

For the converse we reason by induction on the definition of $t \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$.

- If $t = x t_1 \dots t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, where $t_1, \dots, t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, then the *i.h.* gives $t_1, \dots, t_n \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ so that the term $x t_1 \dots t_n$ is trivially in $\mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$.
- If $t = \lambda x.v \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, where $v \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, then the *i.h.* gives $v \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ so that the term $\lambda x.v$ is trivially in $\mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$.
- If $t = (\lambda x.v)u t_1 \dots t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, where $v[x/u]t_1 \dots t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, then the *i.h.* gives $v[x/u]t_1 \dots t_n \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ so that in particular $v, u, t_1 \dots t_n \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$. We show that $t \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ by a second induction on $\eta_{\mathtt{M}\backslash\mathtt{w}}(v) + \eta_{\mathtt{M}\backslash\mathtt{w}}(u) + \sum \eta_{\mathtt{M}\backslash\mathtt{w}}(t_i)$.
  Let us see how are all the reducts of t.
  If $t \to (\lambda x.v')u t_1 \dots t_n = t'$, where $v \to v'$ or $t \to (\lambda x.v)u' t_1 \dots t_n = t'$, where $u \to u'$, or $t \to (\lambda x.v)u t_1 \dots t'_i \dots t_n = t'$, where $t_i \to t'_i$, then $t' \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ by the second *i.h.*
  If $t \to v[x/u]t_1 \dots t_n = t'$, then $t' \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ by the first *i.h.*
  Since all reducts of t are in $\mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$, then $t \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$.
- If $t = v[x/u] \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, where $|v|_x = 0$ and $v, u \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, then the *i.h.* gives $u, v \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ so that the term $v[x/u] \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ is trivial.

– If $C[\![x]\!][x/u] \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ where $C[\![u]\!][x/u] \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, then the *i.h.* gives $C[\![u]\!][x/u] \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ so in particular $C[\![u]\!], u \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$. By Corollary 3 we also have $C[\![x]\!], u \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$. We show that $t \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ by induction on $\eta_{\mathtt{M}\backslash\mathtt{w}}(C[\![x]\!]) + \eta_{\mathtt{M}\backslash\mathtt{w}}(u)$.
  Let us see how are all the reducts of t.
  If $t \to C'[x/u] = t'$, where $C[\![x]\!] \to C'$ or $t \to C[\![x]\!][x/u'] = t'$, where $u \to u'$, then $t' \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ by the second *i.h.*
  If $t \to C[\![u]\!][x/u] = t'$, then $t' \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ by the first *i.h.*
  Since all reducts of t are in $\mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$, then $t \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$.
– If $v[x/u]s \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ where $(vs)[x/s] \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, then $(vs)[x/u] \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ by the *i.h.* so that $v[x/u]s \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ since $\eta_{\mathtt{M}\backslash\mathtt{w}}(v[x/u]s) = \eta_{\mathtt{M}\backslash\mathtt{w}}((vs)[x/u])$ by Lemma 6.

Lemma 9. Let t be a $\mathtt{M}$-term. If $t \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w})$ then t is $\mathcal{S}$-typable.

Proof. By induction on the structure of $t \in \mathcal{SN}(\mathtt{M}\backslash\mathtt{w}) =_{L.\ 32} \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$.

– If $t = x\, t_1 \cdots t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ with $n \geq 0$ and $t_1, \ldots, t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ then $(\Gamma_i \vdash t_i{:}\sigma_i)_{i=1\ldots n}$ by the *i.h.* Let $\tau = [\sigma_1] \to \cdots [\sigma_n] \to \alpha$, where α is a base type, and $\Gamma = x{:}[\tau] + \Gamma_1 + \Gamma_1 + \cdots + \Gamma_n + \Gamma_n$. Then, $x{:}[\tau] \vdash x{:}\tau$ by the typing rule $(\mathtt{ax})$ and, by n applications of the typing rule $(\to \mathtt{e}_{\mathcal{S}})$, $\Gamma \vdash x\, t_1 \cdots t_n{:}\alpha$.
– If $t = \lambda x.u \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ with $u \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ then, by the *i.h.* $\Gamma \vdash u{:}\tau$ thus, by the rule $(\to \mathtt{i})$, $\Gamma \setminus\!\!\setminus x \vdash \lambda x.u{:}\Gamma(x) \to \tau$.
– If $t = (\lambda x.u)vt_1, \ldots, t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ with $u[x/v]t_1, \ldots, t_n \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ then, by the *i.h.* $\Gamma \vdash u[x/v]t_1, \ldots, t_n{:}\tau$. We conclude by Lemma 8.
– If $C[\![x]\!][x/s] \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ with $C[\![s]\!][x/s] \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, we conclude by the *i.h.* and Lemma 8.
– If $t[x/s]u \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, with $(tu)[x/s] \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$, then we conclude by the *i.h.* and the Invariance Lemma 6.
– If $t = u[x/v]$ where $|u|_x = 0$ and $u, v \in \mathcal{ISN}(\mathtt{M}\backslash\mathtt{w})$ then, by the *i.h.* $\Gamma_0 \vdash u{:}\tau$ and $\Gamma_1 \vdash v{:}\sigma$ thus, by the rule $(\mathtt{cut}_{\mathcal{S}})$, $\Gamma_0 + \Gamma_1 \vdash u[x/v]{:}\tau$.

C Appendix: Characterization of Linear-Head and Weak J-Normalization

Lemma 33 (Partial Substitution III). *If* $\Phi_{\mathtt{O}[\![xl]\!]} \rhd x{:}[\rho_i]_{i\in I}; \Gamma \mid \Sigma \vdash_{\mathcal{HW}} \mathtt{O}[\![xl]\!]{:}\tau$ *and* $(\Phi^i_u \rhd \Delta_i \mid \_ \vdash_{\mathcal{HW}} u{:}\rho_i)_{i\in I}$ *then* $\Phi_{\mathtt{O}[\![ul]\!]} \rhd x{:}[\rho_i]_{i\in I\setminus K}; \Gamma +_{i\in K} \Delta_i \mid \Sigma \vdash_{\mathcal{HW}} \mathtt{O}[\![ul]\!]{:}\tau$*, for some* $K \subseteq I$ *where* $\mathtt{sz2}(\Phi_{\mathtt{O}[\![ul]\!]}) = \mathtt{sz2}(\Phi_{\mathtt{O}[\![xl]\!]}) +_{i\in K} \mathtt{sz2}(\Phi^i_u) - |K|$*. Moreover, if* $\mathtt{O}|_p = \Box$ *and* $p \in \mathtt{pos}(\mathtt{O}[\![xl]\!])$ *is a T-occurrence of* $\mathtt{O}[\![xl]\!]$ *in* $\Phi_{\mathtt{O}[\![xl]\!]}$*, then* $K \neq \emptyset$.

Proof. By induction on the typing derivation $\Phi_{\mathtt{O}[\![xl]\!]} \rhd x{:}[\rho_i]_{i\in I}; \Gamma \mid \Sigma \vdash_{\mathcal{HW}} \mathtt{O}[\![xl]\!]{:}\tau$.

– If $\mathtt{O} = \Box$ then, by construction, $\Sigma = \_$, $x{:}[\rho_i]_{i\in I}; \Gamma = \Gamma_l + \{x{:}[\sigma]\}$ and Φ_{xl} is of the form:

$$\Phi_{xl} \rhd \dfrac{\Phi_l \rhd \dfrac{\nabla}{\Gamma_l \mid \sigma \vdash l{:}\tau}}{\Gamma_l + \{x{:}[\sigma]\} \mid \_ \vdash xl{:}\tau}\ (\mathtt{hlist})$$

Therefore, $\sigma = \rho_k$ and $\Gamma_l = x{:}[\rho_i]_{i\in I\setminus\{k\}}; \Gamma$ for some $k \in I$. Moreover, $\mathtt{sz2}(\Phi_{xl}) = \mathtt{sz2}(\Phi_l) + 2$. Let $K := \{k\}$. Hence,

$$\Phi_{sl} \rhd \dfrac{\Phi_s \rhd \dfrac{\nabla}{\Delta_k \mid \_ \vdash s{:}\rho_k} \qquad \Phi_l \rhd \dfrac{\nabla}{x{:}[\rho_i]_{i\in I\setminus\{k\}}; \Gamma \mid \rho_k \vdash l{:}\tau}}{x{:}[\rho_i]_{i\in I\setminus\{k\}}; \Gamma \mid \_ \vdash sl{:}\tau}$$

Moreover, $\mathtt{sz2}(\Phi_{sl}) = \mathtt{sz2}(\Phi_s) + \mathtt{sz2}(\Phi_l) + 1 = \mathtt{sz2}(\Phi_s) + \mathtt{sz2}(\Phi_{xl}) - 1$. Thus the statement holds.

- If $\mathtt{O} = y\mathtt{U}$, then by construction $\Sigma = \_$ and $\Phi_{y\mathtt{U}[\![xl]\!]}$ is of the form

$$\Phi_{y\mathtt{U}[\![xl]\!]} \triangleright \frac{\Phi_{\mathtt{U}[\![xl]\!]} \triangleright \dfrac{\nabla}{\Gamma' \mid \sigma \vdash \mathtt{U}[\![xl]\!]{:}\tau}}{\Gamma' + \{y{:}[\sigma]\} \mid \_ \vdash y\mathtt{U}[\![xl]\!]{:}\tau}$$

where $\Gamma' + \{y{:}[\sigma]\} = x{:}[\rho_i]_{i\in I}; \Gamma$ and $\mathtt{sz2}(\Phi_{y\mathtt{U}[\![xl]\!]}) = \mathtt{sz2}(\Phi_{\mathtt{U}[\![xl]\!]}) + 2$. W.l.o.g. let $y \neq x$. Then $\Gamma' = x{:}[\rho_i]_{i\in I}; \Gamma''$ where $\Gamma'' + \{y{:}[\sigma]\} = \Gamma$. By *i.h.* we have that $\Phi_{\mathtt{U}[\![sl]\!]} \triangleright x{:}[\rho_i]_{i\in I\setminus K}; \Gamma'' +_{i\in K} \Delta_i \mid \sigma \vdash \mathtt{U}[\![sl]\!]{:}\tau$ for some $K \subseteq I$ where $\mathtt{sz2}(\Phi_{\mathtt{U}[\![sl]\!]}) = \mathtt{sz2}(\Phi_{\mathtt{U}[\![xl]\!]}) +_{i\in K} \mathtt{sz2}(\Phi^i_s) - |K|$. Note that if $p \in \mathtt{pos}(y\mathtt{U}[\![xl]\!])$ is a T-occurrence of $y\mathtt{U}[\![xl]\!]$ in $\Phi_{y\mathtt{U}[\![xl]\!]}$ s.t. $(y\mathtt{U})|_p = \Box$ then $p = 0p'$ where $p' \in \mathtt{pos}(\mathtt{U}[\![xl]\!])$ is a T-occurrence of $\mathtt{U}[\![xl]\!]$ in $\Phi_{\mathtt{U}[\![xl]\!]}$. In this case, $K \neq \emptyset$ by the *i.h.* Therefore,

$$\Phi_{y\mathtt{U}[\![sl]\!]} \triangleright \frac{\Phi_{\mathtt{U}[\![sl]\!]} \triangleright \dfrac{\nabla}{x{:}[\rho_i]_{i\in I\setminus K}; \Gamma'' +_{i\in K} \Delta_i \mid \sigma \vdash \mathtt{U}[\![sl]\!]{:}\tau}}{x{:}[\rho_i]_{i\in I\setminus K}; (\Gamma'' +_{i\in K} \Delta_i) + \{y{:}[\sigma]\} \mid \_ \vdash y\mathtt{U}[\![sl]\!]{:}\tau}$$

where $(\Gamma'' +_{i\in K} \Delta_i) + \{y{:}[\sigma]\} = \Gamma +_{i\in K} \Delta_i$ and $\mathtt{sz2}(\Phi_{y\mathtt{U}[\![sl]\!]}) = \mathtt{sz2}(\Phi_{\mathtt{U}[\![sl]\!]}) + 2 =_{i.h.} \mathtt{sz2}(\Phi_{\mathtt{U}[\![xl]\!]}) +_{i\in K} \mathtt{sz2}(\Phi^i_s) - |K| + 2 = \mathtt{sz2}(\Phi_{y\mathtt{U}[\![xl]\!]}) +_{i\in K} \mathtt{sz2}(\Phi^i_s) - |K|$.
- $\mathtt{O} = t\mathtt{U}$, $\mathtt{O} = \mathtt{D}m$ and $\mathtt{O} = \lambda z.\mathtt{D}$ are similar.
- If $\mathtt{O} = \mathtt{D}; m$, then by construction $\Sigma = [\sigma_j]_{j\in J} \to \varphi$ and $\Phi_{\mathtt{D}[\![xl]\!];m}$ is of the form

$$\Phi_{\mathtt{D}[\![xl]\!];m} \triangleright \frac{\left(\Phi^j_{\mathtt{D}[\![xl]\!]} \triangleright \dfrac{\nabla}{\Gamma_j \mid \_ \vdash \mathtt{D}[\![xl]\!]{:}\sigma_j}\right)_{j\in J} \qquad \Phi_m \triangleright \dfrac{\nabla}{\Pi \mid \varphi \vdash m{:}\tau}}{\Pi +_{j\in J} \Gamma_j \mid [\sigma_j]_{j\in J} \to \varphi \vdash \mathtt{D}[\![xl]\!]; m{:}\tau}$$

where $\Pi +_{j\in J} \Gamma_j = x{:}[\rho_i]_{i\in I}; \Gamma$ thus $\Pi = x{:}[\rho_i]_{i\in I_m}; \Pi'$ and $(\Gamma_j = x{:}[\rho_i]_{i\in I_j}; \Gamma'_j)_{j\in J}$ s.t. $I = I_m \cup_{j\in J} I_j$ and $\Gamma = \Pi' +_{j\in J} \Gamma'_j$. Moreover $\mathtt{sz2}(\Phi_{\mathtt{D}[\![xl]\!];m}) = \mathtt{sz2}(\Phi_m) +_{j\in J} \mathtt{sz2}(\Phi^j_{\mathtt{D}[\![xl]\!]}) + 1$. By *i.h.* for each $j \in J$, $\Phi^j_{\mathtt{D}[\![sl]\!]} \triangleright x{:}[\rho_i]_{i\in I_j\setminus K_j}; \Gamma'_j +_{i\in K_j} \Delta_i \mid \_ \vdash \mathtt{D}[\![sl]\!]{:}\sigma_j$ for some $K_j \subseteq I_j$ where $\mathtt{sz2}(\Phi^j_{\mathtt{D}[\![sl]\!]}) = \mathtt{sz2}(\Phi^j_{\mathtt{D}[\![xl]\!]}) +_{i\in K_j} \mathtt{sz2}(\Phi^i_s) - |K_j|$. If $p \in \mathtt{pos}(\mathtt{D}[\![xl]\!]; m)$ is a T-occurrence of $\mathtt{D}[\![xl]\!]; m$ s.t. $(\mathtt{D}; m)|_p = \Box$ then $p = 0p'$ where $p' \in \mathtt{pos}(\mathtt{D}[\![xl]\!])$ is a T-occurrence in $\Phi^j_{\mathtt{D}[\![xl]\!]}$ for some $j \in J$. In this case, $K_j \neq \emptyset$ by the *i.h.* Let $K := \cup_{j\in J} K_j$ so that $K \neq \emptyset$.
Therefore,

$$\Phi_{\mathtt{D}[\![sl]\!];m} \triangleright \frac{\left(\Phi^j_{\mathtt{D}[\![sl]\!]} \triangleright \dfrac{\nabla}{x{:}[\rho_i]_{i\in I_j\setminus K_j}; \Gamma'_j +_{i\in K_j} \Delta_i \mid \_ \vdash \mathtt{D}[\![sl]\!]{:}\sigma_j}\right)_{j\in J} \qquad \Phi_m \triangleright \dfrac{\nabla}{x{:}[\rho_i]_{i\in I_m}; \Pi' \mid \varphi \vdash m{:}\tau}}{x{:}[\rho_i]_{i\in I\setminus K}; \Pi' +_{j\in J} \Gamma'_j +_{i\in K} \Delta_i \mid [\sigma_j]_{j\in J} \to \varphi \vdash \mathtt{D}[\![sl]\!]; m{:}\tau}$$

where $\mathtt{sz2}(\Phi_{\mathtt{D}[\![sl]\!];m}) = \mathtt{sz2}(\Phi_m) +_{j\in J} \mathtt{sz2}(\Phi^j_{\mathtt{D}[\![sl]\!]}) + 1 =_{i.h.} \mathtt{sz2}(\Phi_m) +_{j\in J} (\mathtt{sz2}(\Phi^j_{\mathtt{D}[\![xl]\!]}) +_{i\in K_j} \mathtt{sz2}(\Phi^i_s) - |K_j|) + 1 = \mathtt{sz2}(\Phi_m) +_{j\in J} \mathtt{sz2}(\Phi^j_{\mathtt{D}[\![xl]\!]}) +_{i\in K} \mathtt{sz2}(\Phi^i_s) - |K| + 1 = \mathtt{sz2}(\Phi_{\mathtt{D}[\![xl]\!];m}) +_{i\in K} \mathtt{sz2}(\Phi^i_s) - |K|$.
- $\mathtt{O} = t; \mathtt{U}$, $\mathtt{O} = \mathtt{D}[z/u]$ or $\mathtt{O} = t[z/\mathtt{D}]$ are similar.

Lemma 34. *If $\Phi_l \triangleright \Gamma \mid \delta \vdash_{\mathcal{HW}} l{:}\sigma$ and $\Phi_m \triangleright \Delta \mid \sigma \vdash_{\mathcal{HW}} m{:}\tau$, then $\Phi_{l@m} \triangleright \Gamma + \Delta \mid \delta \vdash_{\mathcal{HW}} l@m{:}\tau$ and* $\mathtt{sz2}(\Phi_{l@m}) = \mathtt{sz2}(\Phi_l) + \mathtt{sz2}(\Phi_m) - 1$.

Proof. By induction on the type derivation $\Phi_l \triangleright \Gamma \mid \delta \vdash l{:}\sigma$.

If $l = \mathtt{nil}$, then $\Gamma = \emptyset$, $\delta = \sigma$ and $\mathtt{nil}@m = m$. Moreover, $\mathtt{sz2}(\Phi_l) = 1$. We have $\Phi_{l@m} = \Phi_m$ so that $\mathtt{sz2}(\Phi_m) = 1 + \mathtt{sz2}(\Phi_m) - 1$.

If $l = t; r$, then $\delta = [\sigma_i]_{i\in I} \to \delta'$, $\Gamma = \Gamma_0 +_{i\in I} \Gamma_i$ and Φ_l is of the form

$$\Phi_l \triangleright \dfrac{\left(\Phi_t^i \triangleright \dfrac{\nabla}{\Gamma_i \mid \_ \vdash t{:}\sigma_i}\right)_{i\in I} \qquad \Phi_r \triangleright \dfrac{\nabla}{\Gamma_0 \mid \delta' \vdash r{:}\sigma}}{\Gamma_0 +_{i\in I} \Gamma_i \mid [\sigma_i]_{i\in I} \to \delta' \vdash t; r{:}\sigma}\ (\to l)$$

and $\mathtt{sz2}(\Phi_l) = \mathtt{sz2}(\Phi_r) + \sum_{i\in I} \mathtt{sz2}(\Phi_t^i) + 1$. By the *i.h.* $\Phi_{r@m} \triangleright \Gamma_0 + \Delta \mid \delta' \vdash r@m{:}\tau$ and $\mathtt{sz2}(\Phi_{r@m}) = \mathtt{sz2}(\Phi_r) + \mathtt{sz2}(\Phi_m) - 1$. Hence,

$$\Phi_{l@m} \triangleright \dfrac{\left(\Phi_t^i \triangleright \dfrac{\nabla}{\Gamma_i \mid \_ \vdash t{:}\sigma_i}\right)_{i\in I} \qquad \Phi_{r@m} \triangleright \dfrac{\nabla}{\Gamma_0 + \Delta \mid \delta' \vdash r@m{:}\delta}}{\Gamma_0 + \Delta +_{i\in I} \Gamma_i \mid [\sigma_i]_{i\in I} \to \delta' \vdash t; (r@m){:}\tau}$$

We have $\mathtt{sz2}(\Phi_{l@m}) = \mathtt{sz2}(\Phi_{r@m}) + \sum_{i\in I} \mathtt{sz2}(\Phi_t^i) + 1 =_{i.h.} \mathtt{sz2}(\Phi_r) + \mathtt{sz2}(\Phi_m) - 1 + \sum_{i\in I} \mathtt{sz2}(\Phi_t^i) + 1 = \mathtt{sz2}(\Phi_l) + \mathtt{sz2}(\Phi_m) - 1$.

Lemma 14 (SR III). Let $\Phi \triangleright \Gamma \mid \Sigma \vdash o{:}\tau$. If $o \to_{\mathtt{J}} o'$ reduces a redex T-occurrence of o in Φ then $\Phi' \triangleright \Gamma \vdash o'{:}\tau$ and $\mathtt{sz2}(\Phi) > \mathtt{sz2}(\Phi')$.

Proof. By induction on the reduction relation $\to_{\mathtt{J}}$.

- If $o = \mathtt{L}[\![\lambda x.v]\!]\mathtt{nil} \to \mathtt{L}[\![\lambda x.v]\!] = o'$, then by construction we have $\Sigma = \_$ and Φ is of the form

$$\Phi \triangleright \dfrac{\Phi_{\mathtt{L}[\![\lambda x.v]\!]} \triangleright \dfrac{\nabla}{\Gamma \mid \_ \vdash \mathtt{L}[\![\lambda x.v]\!]{:}\tau} \qquad \Phi_{\mathtt{nil}} \triangleright \dfrac{}{\emptyset \mid \tau \vdash \mathtt{nil}{:}\tau}}{\Gamma \mid \_ \vdash \mathtt{L}[\![\lambda x.v]\!]\mathtt{nil}{:}\tau}$$

  We have $\Phi' = \Phi_{\mathtt{L}[\![\lambda x.v]\!]}$ and $\mathtt{sz2}(\Phi) = \mathtt{sz2}(\Phi_{\mathtt{L}[\![\lambda x.v]\!]}) + \mathtt{sz2}(\Phi_{\mathtt{nil}}) + 1 > \mathtt{sz2}(\Phi_{\mathtt{L}[\![\lambda x.v]\!]}) = \mathtt{sz2}(\Phi')$.
- If $o = \mathtt{L}[\![\lambda x.v]\!](s; l) \to \mathtt{L}[\![v[x/s]l]\!] = o'$, then we show $\mathtt{sz2}(\Phi) > \mathtt{sz2}(\Phi')$ by induction on $\mathtt{L}$. Let $\mathtt{L} = \Box$. By construction we have that $\Sigma = \_$ and Φ is of the form:

$$\Phi \triangleright \dfrac{\dfrac{\Phi_v \triangleright \dfrac{\nabla}{x : [\rho_i]_{i\in I}; \Pi \mid \_ \vdash v{:}\sigma}}{\Pi \mid \_ \vdash \lambda x.v{:}[\rho_i]_{i\in I} \to \sigma} \qquad \Phi_{s;l} \triangleright \dfrac{\left(\Phi_s^i \triangleright \dfrac{\nabla}{\Gamma_i \mid \_ \vdash s{:}\rho_i}\right)_{i\in I} \qquad \Phi_l \triangleright \dfrac{\nabla}{\Delta \mid \sigma \vdash l{:}\tau}}{\Delta +_{i\in I} \Gamma_i \mid [\rho_i]_{i\in I} \to \sigma \vdash s; l{:}\tau}}{\Pi + \Delta +_{i\in I} \Gamma_i \mid \_ \vdash (\lambda x.v)(s; l){:}\tau}$$

  Moreover, $\mathtt{sz2}(\Phi) = \mathtt{sz2}(\Phi_v) +_{i\in I} \mathtt{sz2}(\Phi_s^i) + \mathtt{sz2}(\Phi_l) + 3$. Hence,

$$\Phi' \triangleright \dfrac{\Phi_{v[x/s]} \triangleright \dfrac{\Phi_v \triangleright \dfrac{\nabla}{x : [\rho_i]_{i\in I}; \Pi \mid \_ \vdash v{:}\sigma} \qquad \left(\Phi_s^i \triangleright \dfrac{\nabla}{\Gamma_i \mid \_ \vdash s{:}\rho_i}\right)_{i\in I}}{\Pi +_{i\in I} \Gamma_i \mid \_ \vdash v[x/s]{:}\sigma} \qquad \Phi_l \triangleright \dfrac{\nabla}{\Delta \mid \sigma \vdash l{:}\tau}}{\Pi + \Delta +_{i\in I} \Gamma_i \mid \_ \vdash v[x/s]l{:}\tau}$$

We have $\mathtt{sz2}(\Phi') = \mathtt{sz2}(\Phi_{v[x/s]}) + \mathtt{sz2}(\Phi_l) + 1 = \mathtt{sz2}(\Phi_v) +_{i\in I} \mathtt{sz2}(\Phi_s^i) + \mathtt{sz2}(\Phi_l) + 2 < \mathtt{sz2}(\Phi)$.
Let $\mathtt{L} = \mathtt{L}'[y/u]$, so that $\mathtt{L}\llbracket t\rrbracket = \mathtt{L}'\llbracket t\rrbracket[y/u]$ for any t. By construction Φ is of the form:

$$\dfrac{\dfrac{\dfrac{\nabla}{\Gamma_0; y : [\rho_j]_{j\in J} \vdash \mathtt{L}'\llbracket \lambda x.v\rrbracket{:}[\sigma_i]_{i\in I} \to \sigma} \qquad \left(\Phi_u^j \triangleright \dfrac{\nabla}{\Pi_j \mid \_ \vdash u{:}\rho_j}\right)_{j\in J}}{\Gamma_0 +_{j\in J} \Pi_j \vdash \mathtt{L}'\llbracket \lambda x.v\rrbracket[y/u]{:}[\sigma_i]_{i\in I} \to \sigma} \qquad \dfrac{\left(\Phi_s^i \triangleright \dfrac{\nabla}{\Gamma_i \mid \_ \vdash s{:}\sigma_i}\right)_{i\in I} \qquad \Phi_l \triangleright \dfrac{\nabla}{\Delta \mid \sigma \vdash l{:}\tau}}{\Delta +_{i\in I} \Gamma_i \mid [\sigma_i]_{i\in I} \to \sigma \vdash s; l{:}\tau}}{\Gamma_0 +_{j\in J} \Pi_j + \Delta +_{i\in I} \Gamma_i \vdash \mathtt{L}'\llbracket \lambda x.v\rrbracket[y/u](s;l){:}\tau}$$

We can then construct the following derivation

$$\Phi_{\mathtt{L}'\llbracket \lambda x.v\rrbracket(s;l)} \triangleright \dfrac{\dfrac{\nabla}{\Gamma_0; y : [\rho_j]_{j\in J} \vdash \mathtt{L}'\llbracket \lambda x.v\rrbracket{:}[\sigma_i]_{i\in I} \to \sigma} \qquad \dfrac{\left(\Phi_s^i \triangleright \dfrac{\nabla}{\Gamma_i \mid \_ \vdash s{:}\sigma_i}\right)_{i\in I} \qquad \Phi_l \triangleright \dfrac{\nabla}{\Delta \mid \sigma \vdash l{:}\tau}}{\Delta +_{i\in I} \Gamma_i \mid [\sigma_i]_{i\in I} \to \sigma \vdash s; l{:}\tau}}{\Gamma_0; y : [\rho_j]_{j\in J} + \Delta +_{i\in I} \Gamma_i \vdash \mathtt{L}'\llbracket \lambda x.v\rrbracket(s;l){:}\tau}$$

By the *i.h.* there is a derivation ending with $\Phi_{\mathtt{L}'\llbracket v[x/s]l\rrbracket} \triangleright \Gamma_0; y : [\rho_j]_{j\in J} + \Delta +_{i\in I} \Gamma_i \vdash \mathtt{L}'\llbracket v[x/s]l\rrbracket{:}\tau$ such that $\mathtt{sz2}(\Phi_{\mathtt{L}'\llbracket \lambda x.v\rrbracket(s;l)}) > \mathtt{sz2}(\Phi_{\mathtt{L}'\llbracket v[x/s]l\rrbracket})$.
We thus conclude by the existence of the following derivation.

$$\dfrac{\dfrac{\nabla}{\Gamma_0; y : [\rho_j]_{j\in J} + \Delta +_{i\in I} \Gamma_i \vdash \mathtt{L}'\llbracket v[x/s]l\rrbracket{:}\tau} \qquad \left(\Phi_u^j \triangleright \dfrac{\nabla}{\Pi_j \mid \_ \vdash u{:}\rho_j}\right)_{j\in J}}{\Gamma_0 +_{j\in J} \Pi_j + \Delta +_{i\in I} \Gamma_i \vdash \mathtt{L}'\llbracket v[x/s]l\rrbracket[y/u]{:}\tau}$$

We have $\mathtt{sz2}(\Phi_{\mathtt{L}'\llbracket v[x/s]l\rrbracket[y/u]}) = \mathtt{sz2}(\Phi_{\mathtt{L}'\llbracket v[x/s]l\rrbracket}) +_{j\in J} \mathtt{sz2}(\Phi_u^j) + 1 <_{i.h.} \mathtt{sz2}(\Phi_{\mathtt{L}'\llbracket \lambda x.v\rrbracket(s;l)}) +_{j\in J} \mathtt{sz2}(\Phi_u^j) + 1 = \mathtt{sz2}(\Phi_{\mathtt{L}'\llbracket \lambda x.v\rrbracket}) +_{j\in J} \mathtt{sz2}(\Phi_u^j) + \mathtt{sz2}(\Phi_{s;l}) + 2 = \mathtt{sz2}(\Phi_{\mathtt{L}'\llbracket \lambda x.v\rrbracket[y/u](s;l)})$.

– If $o = \mathtt{C}\llbracket xl\rrbracket[x/u] \to \mathtt{C}\llbracket ul\rrbracket[x/u] = o'$ then, by construction, $\Sigma = \_$ and Φ is of the form

$$\Phi \triangleright \dfrac{\Phi_{\mathtt{C}\llbracket xl\rrbracket} \triangleright \dfrac{\nabla}{x{:}[\rho_i]_{i\in I}; \Pi \mid \_ \vdash \mathtt{C}\llbracket xl\rrbracket{:}\tau} \qquad \left(\Phi_u^i \triangleright \dfrac{\nabla}{\Delta_i \mid \_ \vdash u{:}\rho_i}\right)_{i\in I}}{\Pi +_{i\in I} \Delta_i \mid \_ \vdash \mathtt{C}\llbracket xl\rrbracket[x/u]{:}\tau}$$

where $\mathtt{sz2}(\Phi) = \mathtt{sz2}(\Phi_{\mathtt{C}[\![xl]\!]}) +_{i\in I} \mathtt{sz2}(\Phi^i_u)+1$. By Lemma 33 $\Phi_{\mathtt{C}[\![ul]\!]} \triangleright x{:}[\rho_i]_{i\in I\setminus K}; \Pi +_{i\in K} \Delta_i \mid \_ \vdash \mathtt{C}[\![ul]\!]{:}\tau$ for some $K \subseteq I$ where $\mathtt{sz2}(\Phi_{\mathtt{C}[\![ul]\!]}) = \mathtt{sz2}(\Phi_{\mathtt{C}[\![xl]\!]}) +_{i\in K} \mathtt{sz2}(\Phi^i_u) - |K|$. Hence

$$\Phi' \triangleright \dfrac{\Phi_{\mathtt{C}[\![ul]\!]} \triangleright \dfrac{\nabla}{x{:}[\rho_i]_{i\in I\setminus K}; \Pi +_{i\in K} \Delta_i \mid \_ \vdash \mathtt{C}[\![ul]\!]{:}\tau} \qquad \left(\Phi^i_u \triangleright \dfrac{\nabla}{\Delta_i \mid \_ \vdash u{:}\rho_i}\right)_{i\in I\setminus K}}{\Pi +_{i\in I} \Delta_i \mid \_ \vdash \mathtt{C}[\![ul]\!][x/u]{:}\tau}$$

where $\mathtt{sz2}(\Phi') = \mathtt{sz2}(\Phi_{\mathtt{C}[\![ul]\!]}) +_{i\in I\setminus K} \mathtt{sz2}(\Phi^i_u) + 1 =_{L.33} \mathtt{sz2}(\Phi_{\mathtt{C}[\![xl]\!]}) +_{i\in I} \mathtt{sz2}(\Phi^i_u) - |K| + 1 \leq \mathtt{sz2}(\Phi)$. By hypothesis, the hole of $\mathtt{C}$ is a T-occurrence in Φ, so that Lemma 33 guarantees $K \neq \emptyset$ and thus $\mathtt{sz}(\Phi') < \mathtt{sz}(\Phi)$.

- If $o = t[x/u] \to t = o'$ where $|t|_x = 0$, then by construction and Lemma 13 we have that $\Sigma = \_$ and Φ is of the form

$$\Phi \triangleright \dfrac{\Phi_t \triangleright \dfrac{\nabla}{x{:}[\,]; \Gamma \mid \_ \vdash t{:}\tau}}{\Gamma \mid \_ \vdash t[x/u]{:}\tau}$$

where $\mathtt{sz2}(\Phi) = \mathtt{sz2}(\Phi_t) + 1$. The result then holds for $\Phi' := \Phi_t$.
- If $o = \mathtt{L}[\![xl]\!]\ m \to \mathtt{L}[\![x(l@m)]\!] = o'$, then we proceed by induction on $\mathtt{L}$ by showing in particular that $\mathtt{sz2}(\Phi) > \mathtt{sz2}(\Phi')$. If $\mathtt{L} = \Box$, then by construction we have $\Sigma = \_$ and $\Gamma = \Gamma_0 + \{x : [\pi]\} + \Gamma_1$ and Φ is of the form

$$\Phi \triangleright \dfrac{\dfrac{\Phi_l \triangleright \dfrac{\nabla}{\Gamma_0 \mid \pi \vdash l{:}\delta}}{\Gamma_0 + \{x : [\pi]\} \mid \_ \vdash xl{:}\delta} \qquad \Phi_m \triangleright \dfrac{\nabla}{\Gamma_1 \mid \delta \vdash m{:}\tau}}{\Gamma_0 + \{x : [\pi]\} + \Gamma_1 \mid \_ \vdash (xl)m{:}\tau}$$

Moreover, $\mathtt{sz2}(\Phi) = \mathtt{sz2}(\Phi_l) + \mathtt{sz2}(\Phi_m) + 3$.
By Lemma 34 we have $\Phi_{l@m} \triangleright \Gamma_0 + \Gamma_1 \mid \pi \vdash l@m{:}\tau$, where $\mathtt{sz2}(\Phi_{l@m}) = \mathtt{sz2}(\Phi_l) + \mathtt{sz2}(\Phi_m) - 1$.
We construct the following derivation

$$\Phi_{x(l@m)} \triangleright \dfrac{\Phi_{l@m} \triangleright \dfrac{\nabla}{\Gamma_0 + \Gamma_1 \mid \pi \vdash l@m{:}\tau}}{\Gamma_0 + \Gamma_1 + \{x : [\pi]\} \mid \_ \vdash x(l@m){:}\tau}$$

Hence, $\mathtt{sz2}(\Phi_{x(l@m)}) = \mathtt{sz2}(\Phi_{l@m}) + 2 = \mathtt{sz2}(\Phi_l) + \mathtt{sz2}(\Phi_m) + 1 < \mathtt{sz2}(\Phi)$.
Let $\mathtt{L} = \mathtt{L}'[y/u]$ so that $\mathtt{L}[\![xl]\!] = \mathtt{L}'[\![xl]\!][y/u]$. By construction Φ is of the following form:

$$\Phi \triangleright \dfrac{\dfrac{\Phi_{\mathtt{L}'[\![xl]\!]} \triangleright \dfrac{\nabla}{\Gamma_0; y{:}[\rho_i]_{i\in I} \mid \_ \vdash \mathtt{L}'[\![xl]\!]{:}\delta} \qquad \left(\Phi^i_u \triangleright \dfrac{\nabla}{\Delta_i \mid \_ \vdash u{:}\rho_i}\right)_{i\in I}}{\Gamma_0 +_{i\in I} \Delta_i \mid \_ \vdash \mathtt{L}'[\![xl]\!][y/u]{:}\delta} \qquad \Phi_m \triangleright \dfrac{\nabla}{\Gamma_1 \mid \delta \vdash m{:}\tau}}{\Gamma_0 + \Gamma_1 +_{i\in I} \Delta_i \mid \_ \vdash \mathtt{L}'[\![xl]\!][y/u]m{:}\tau}$$

Moreover, $\mathtt{sz2}(\Phi) = \mathtt{sz2}(\Phi_{\mathtt{L}'[\![xl]\!]}) +_{i\in I} \mathtt{sz2}(\Phi^i_u) + \mathtt{sz2}(\Phi_m) + 2$.
We can then construct the following derivation:

$$\Phi_{\mathtt{L}'[\![xl]\!]m} \triangleright \frac{\Phi_{\mathtt{L}'[\![xl]\!]} \triangleright \dfrac{\nabla}{\Gamma_0; y{:}[\rho_i]_{i\in I} \mid \_ \vdash \mathtt{L}'[\![xl]\!]{:}\delta} \qquad \Phi_m \triangleright \dfrac{\nabla}{\Gamma_1 \mid \delta \vdash m{:}\tau}}{\Gamma_0; y{:}[\rho_i]_{i\in I} + \Gamma_1 \mid \_ \vdash \mathtt{L}'[\![xl]\!]m{:}\tau}$$

By the *i.h.* there is a derivation $\Phi_{\mathtt{L}'[\![x(l@m)]\!]} \triangleright \Gamma_0; y{:}[\rho_i]_{i\in I} + \Gamma_1 \mid \_ \vdash \mathtt{L}'[\![x(l@m)]\!]{:}\tau$ such that $\mathtt{sz2}(\Phi_{\mathtt{L}'[\![xl]\!]m}) > \mathtt{sz2}(\Phi_{\mathtt{L}'[\![x(l@m)]\!]})$. We thus conclude with the following derivation:

$$\Phi' \triangleright \frac{\Phi_{\mathtt{L}'[\![x(l@m)]\!]} \triangleright \dfrac{\nabla}{\Gamma_0; y{:}[\rho_i]_{i\in I} + \Gamma_1 \mid \_ \vdash \mathtt{L}'[\![x(l@m)]\!]{:}\tau} \qquad \left(\Phi_u^i \triangleright \dfrac{\nabla}{\Delta_i \mid \_ \vdash u{:}\rho_i}\right)_{i\in I}}{\Gamma_0 + \Gamma_1 +_{i\in I} \Delta_i \mid \_ \vdash \mathtt{L}[\![x(l@m)]\!]{:}\tau}$$

We have $\mathtt{sz2}(\Phi) = \mathtt{sz2}(\Phi_{\mathtt{L}'[\![xl]\!]}) + \mathtt{sz2}(\Phi_m) + 1 +_{i\in I} \mathtt{sz2}(\Phi_u^i) + 1 = \mathtt{sz2}(\Phi_{\mathtt{L}'[\![xl]\!]m}) +_{i\in I} \mathtt{sz2}(\Phi_u^i) + 1 >_{i.h.} \mathtt{sz2}(\Phi_{\mathtt{L}'[\![x(l@m)]\!]}) +_{i\in I} \mathtt{sz2}(\Phi_u^i) + 1 = \mathtt{sz2}(\Phi')$.

– If $o = \mathtt{L}[\![tl]\!]\ m \to \mathtt{L}[\![t(l@m)]\!] = o'$, then we proceed by induction on $\mathtt{L}$ as the previous case. We only show here the base case where $\mathtt{L} = \Box$. In that case, by construction, we have $\Sigma = \_$ and $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2$ and Φ is of the form

$$\Phi \triangleright \frac{\dfrac{\Phi_t \triangleright \dfrac{\nabla}{\Gamma_0 \mid \_ \vdash t{:}\sigma} \qquad \Phi_l \triangleright \dfrac{\nabla}{\Gamma_1 \mid \sigma \vdash l{:}\delta}}{\Gamma_0 + \Gamma_1 \mid \_ \vdash tl{:}\delta} \qquad \Phi_m \triangleright \dfrac{\nabla}{\Gamma_2 \mid \delta \vdash m{:}\tau}}{\Gamma_0 + \Gamma_1 + \Gamma_2 \mid \_ \vdash (tl)m{:}\tau}$$

Moreover, $\mathtt{sz2}(\Phi) = \mathtt{sz2}(\Phi_t) + \mathtt{sz2}(\Phi_l) + \mathtt{sz2}(\Phi_m) + 2$.
By Lemma 34 we have $\Phi_{l@m} \triangleright \Gamma_1 + \Gamma_2 \mid \sigma \vdash l@m{:}\tau$, where $\mathtt{sz2}(\Phi_{l@m}) = \mathtt{sz2}(\Phi_l) + \mathtt{sz2}(\Phi_m) - 1$.
Then we can construct the following typing derivation:

$$\Phi' \triangleright \frac{\Phi_t \triangleright \dfrac{\nabla}{\Gamma_0 \mid \_ \vdash t{:}\sigma} \qquad \Phi_{l@m} \triangleright \dfrac{\nabla}{\Gamma_1 + \Gamma_2 \mid \sigma \vdash l@m{:}\tau}}{\Gamma_0 + \Gamma_1 + \Gamma_2 \mid \_ \vdash t(l@m){:}\tau}$$

Hence, $\mathtt{sz2}(\Phi') = \mathtt{sz2}(\Phi_t) + \mathtt{sz2}(\Phi_{l@m}) + 1 = \mathtt{sz2}(\Phi_t) + \mathtt{sz2}(\Phi_l) + \mathtt{sz2}(\Phi_m) < \mathtt{sz2}(\Phi)$.

– All the inductive cases are straightforward.

We need to prove a couple of technical results in order to prove the Subject Expansion property.

Lemma 35. *If* $\Gamma|\delta \vdash_{\mathcal{HW}} l@m{:}\tau$ *then* $\exists\Gamma_1, \exists\Gamma_2, \exists\sigma$ *s.t.* $\Gamma{=}\Gamma_1{+}\Gamma_2$, $\Gamma_1|\delta \vdash_{\mathcal{HW}} l{:}\sigma$ *and* $\Gamma_2|\sigma \vdash_{\mathcal{HW}} m{:}\tau$

Proof. By induction on the type derivation $\Phi \triangleright \Gamma \mid \Sigma \vdash_{\mathcal{HW}} l@m{:}\tau$, reasoning by cases on l.

Lemma 36. *Let* $\mathtt{O}[\![xl]\!] \in \mathcal{O}_{\mathtt{J}}, u \in \mathcal{T}_{\mathtt{J}}$ *s.t.* $|u|_x = 0$ *and* $\Gamma \mid \Sigma \vdash_{\mathcal{HW}} \mathtt{O}[\![ul]\!]{:}\tau$. *Then* $\exists\Gamma_0, \exists I, \exists(\Gamma_i)_{i\in I}, \exists(\sigma_i)_{i\in I}$ *s.t.* $\Gamma = \Gamma_0 +_{i\in I} \Gamma_i$, $\Gamma_0 + \{x : [\sigma_i]_{i\in I}\} \mid \Sigma \vdash_{\mathcal{HW}} \mathtt{O}[\![xl]\!]{:}\tau$, *and* $(\Gamma_i \mid \_ \vdash_{\mathcal{HW}} u{:}\sigma_i)_{i\in I}$.

Proof. By induction on the typing derivation $\Phi_{\mathtt{O}[\![ul]\!]} \triangleright \Gamma \mid \Sigma \vdash \mathtt{O}[\![ul]\!]{:}\tau$.

– If $\mathtt{O} = \Box$, then by construction we know that $\Sigma = \_$, $\Gamma = \Gamma_u + \Gamma_l$ and Φ_{ul} is of the form:

$$\Phi_{ul} \triangleright \frac{\dfrac{\nabla}{\Gamma_u \mid \_ \vdash u{:}\sigma} \qquad \dfrac{\nabla}{\Gamma_l \mid \sigma \vdash l{:}\tau}}{\Gamma_u + \Gamma_l \mid \_ \vdash ul{:}\tau}$$

Therefore, $\Gamma_l + \{x{:}[\sigma]\} \mid \_ \vdash_{\mathcal{HW}} xl{:}\tau$ by the rule (hlist). We then conclude with $\Gamma_0 := \Gamma_l$, $I = \{u\}$ and $\sigma_u := \sigma$.

- If $\mathtt{O} = y\mathtt{U}$, then by construction $\Sigma = \_$ and $\Phi_{y\mathtt{U}[\![ul]\!]}$ is of the form

$$\Phi_{y\mathtt{U}[\![ul]\!]} \triangleright \frac{\Gamma_{\mathtt{U}[\![ul]\!]} \mid \sigma \vdash \mathtt{U}[\![ul]\!]{:}\tau}{\Gamma_{\mathtt{U}[\![ul]\!]} + \{y{:}[\sigma]\} \mid \_ \vdash y\mathtt{U}[\![ul]\!]{:}\tau}$$

By the *i.h.* $\Gamma_{\mathtt{U}[\![ul]\!]} = \Gamma'_0 +_{i\in I} \Gamma_i$ s.t. $\Gamma'_0 + \{x{:}[\sigma_i]_{i\in I}\} \mid \_ \vdash \mathtt{U}[\![xl]\!]{:}\tau$ and $(\Gamma_i \mid \_ \vdash u{:}\sigma_i)_{i\in I}$. We have that $\Gamma'_0 + \{x{:}[\sigma_i]_{i\in I}\} + \{y{:}[\sigma]\} \mid \_ \vdash y\mathtt{U}[\![xl]\!]{:}\tau$ by the rule (hlist) and we conclude with $\Gamma_0 := \Gamma'_0 + \{y{:}[\sigma]\}$ since $\Gamma_0 +_{i\in I} \Gamma_i = \Gamma'_0 + \{y{:}[\sigma]\} +_{i\in I} \Gamma_i = \Gamma_{\mathtt{U}[\![ul]\!]} + \{y{:}[\sigma]\}$.
- $\mathtt{O} = t\mathtt{U}$, $\mathtt{O} = \mathtt{D}m$ and $\mathtt{O} = \lambda z.\mathtt{D}$ are similar.
- If $\mathtt{O} = \mathtt{D}; m$, then by construction $\Sigma = [\rho_j]_{j\in J} \to \varphi$ and $\Phi_{\mathtt{D}[\![ul]\!];m}$ is of the form

$$\Phi_{\mathtt{D}[\![ul]\!];m} \triangleright \frac{\left(\frac{\nabla}{\Delta_j \vdash \mathtt{D}[\![ul]\!]{:}\rho_j}\right)_{j\in J} \qquad \frac{\nabla}{\Gamma_m \mid \varphi \vdash m{:}\tau}}{\Gamma_m +_{j\in J} \Delta_j \mid [\rho_j]_{j\in J} \to \varphi \vdash \mathtt{D}[\![ul]\!]; m{:}\tau}$$

By the *i.h.* for each $j \in J$, $\Delta_j = \Delta^j_0 +_{i\in I_j} \Gamma_i$ and $\Delta^j_0 + \{x{:}[\sigma_i]_{i\in I_j}\} \mid \_ \vdash_{\mathcal{HW}} \mathtt{D}[\![xl]\!]{:}\rho_j$ and $(\Gamma_i \mid \_ \vdash_{\mathcal{HW}} u{:}\sigma_i)_{i\in I_j}$. Let $I := \cup_{j\in J} I_j$. Hence,

$$\frac{\left(\frac{\nabla}{\Delta^j_0 + \{x{:}[\sigma_i]_{i\in I_j}\} \mid \_ \vdash \mathtt{D}[\![xl]\!]{:}\rho_j}\right)_{j\in J} \qquad \frac{\nabla}{\Gamma_m \mid \varphi \vdash m{:}\tau}}{\Gamma_m +_{j\in J} \Delta^0_j + \{x{:}[\sigma_i]_{i\in I}\} \mid [\rho_j]_{j\in J} \to \varphi \vdash \mathtt{D}[\![xl]\!]; m{:}\tau}$$

We then conclude with $\Gamma_0 := \Gamma_m +_{j\in J} \Delta^0_j$ since $\Gamma_0 +_{i\in I} \Gamma_i = \Gamma_m +_{j\in J} \Delta^0_j +_{j\in J} (+_{i\in I_j} \Gamma_i) = \Gamma_m +_{j\in J} (\Delta^0_j +_{i\in I_j} \Gamma_i) = \Gamma_m +_{j\in J} \Delta_j = \Gamma$.
- $\mathtt{O} = t; \mathtt{U}$, $\mathtt{O} = \mathtt{D}[z/u]$ and $\mathtt{O} = t[z/\mathtt{D}]$ are similar.

Lemma 15 (SE III). If $\Gamma \mid \Sigma \vdash_{\mathcal{HW}} o'{:}\tau$ and $o \to_{\mathtt{J}} o'$ then $\Gamma \mid \Sigma \vdash_{\mathcal{HW}} o{:}\tau$.

Proof. The proof is by induction on $o \to_{\mathtt{J}} o'$.

- If $o = \mathtt{L}[\![\lambda x.v]\!]\mathtt{nil} \to \mathtt{L}[\![\lambda x.v]\!] = o'$, then $\Sigma = \_$ and $\emptyset \mid \tau \vdash \mathtt{nil}{:}\tau$ and $\Gamma \mid \_ \vdash \mathtt{L}[\![\lambda x.v]\!]\mathtt{nil}{:}\tau$ by the rules (ax) and (app).
- If $o = \mathtt{L}[\![\lambda x.v]\!](u; l) \to \mathtt{L}[\![v[x/u]l]\!] = o'$, then we proceed by induction on $\mathtt{L}$. Let $\mathtt{L} = \Box$. In that case, by construction, we have $\Sigma = \_$ and $\Gamma = \Gamma_0 +_{i\in I} \Gamma_i + \Delta$ and Φ is of the form

$$\Phi \triangleright \frac{\dfrac{\frac{\nabla}{\Gamma_0; x{:}[\rho_i]_{i\in I} \mid \_ \vdash v{:}\sigma} \qquad \left(\frac{\nabla}{\Gamma_i \mid \_ \vdash u{:}\rho_i}\right)_{i\in I}}{\Gamma_0 +_{i\in I} \Gamma_i \mid \_ \vdash v[x/u]{:}\sigma} \qquad \frac{\nabla}{\Delta \mid \sigma \vdash l{:}\tau}}{\Gamma_0 +_{i\in I} \Gamma_i + \Delta \mid \_ \vdash v[x/u]l{:}\tau}$$

Therefore, we can construct the derivation Φ' below:

$$\Phi' \triangleright \frac{\dfrac{\frac{\nabla}{\Gamma_0; x{:}[\rho_i]_{i\in I} \mid \_ \vdash v{:}\sigma}}{\Gamma_0 \mid \_ \vdash \lambda x.v{:}[\rho_i]_{i\in I} \to \sigma} \qquad \dfrac{\left(\frac{\nabla}{\Gamma_i \mid \_ \vdash u{:}\rho_i}\right)_{i\in I} \qquad \frac{\nabla}{\Delta \mid \sigma \vdash l{:}\tau}}{\Delta +_{i\in I} \Gamma_i \mid [\rho_i]_{i\in I} \to \sigma \vdash u; l{:}\tau}}{\Gamma_0 +_{i\in I} \Gamma_i + \Delta \mid \_ \vdash (\lambda x.v)(u; l){:}\tau}$$

Let $\mathtt{L} = \mathtt{L}'[y/s]$ so that $\mathtt{L}[\![v[x/u]l]\!] = \mathtt{L}'[\![v[x/u]l]\!][y/s]$. By construction we have a derivation of the following form:

$$\Phi \triangleright \frac{\dfrac{\nabla}{\Gamma_0; y{:}[\rho_i]_{i\in I} \mid \_ \vdash \mathtt{L}'[\![v[x/u]l]\!]{:}\tau} \qquad \left(\dfrac{\nabla}{\Gamma_i \mid \_ \vdash s{:}\rho_i}\right)_{i\in I}}{\Gamma_0 +_{i\in I} \Gamma_i \mid \_ \vdash \mathtt{L}'[\![v[x/u]l]\!][y/s]{:}\tau}$$

By *i.h.* from $\Gamma_0; y{:}[\rho_i]_{i\in I} \mid \_ \vdash \mathtt{L}'[\![v[x/u]l]\!]{:}\tau$ we have that $\Gamma_0; y{:}[\rho_i]_{i\in I} \mid \_ \vdash \mathtt{L}'[\![\lambda x.v]\!](u;l){:}\tau$ and, by construction, $\Gamma_0; y{:}[\rho_i]_{i\in I} = \Delta_1 + \Delta_2$ s.t. $\Delta_1 \mid \_ \vdash \mathtt{L}'[\![\lambda x.v]\!]{:}\sigma$ and $\Delta_2 \mid \sigma \vdash (u;l){:}\tau$. We can assume by α-conversion that $|u;l|_y = 0$ thus, by Lemma 13, we necessarily have that $\Delta_1 = \Delta'; y{:}[\rho_i]_{i\in I}$ and $\Gamma_0 = \Delta' + \Delta_2$. Therefore, we can construct the following derivation:

$$\Phi' \triangleright \frac{\dfrac{\dfrac{\nabla}{\Delta'; y{:}[\rho_i]_{i\in I} \mid \_ \vdash \mathtt{L}'[\![\lambda x.v]\!]{:}\sigma} \qquad \left(\dfrac{\nabla}{\Gamma_i \mid \_ \vdash s{:}\rho_i}\right)_{i\in I}}{\Delta' +_{i\in I} \Gamma_i \mid \_ \vdash \mathtt{L}'[\![\lambda x.v]\!][y/s]{:}\sigma} \qquad \dfrac{\nabla}{\Delta_2 \mid \sigma \vdash (u;l){:}\tau}}{\Gamma_0 +_{i\in I} \Gamma_i \mid \_ \vdash \mathtt{L}'[\![\lambda x.v]\!][y/s](u;l){:}\tau}$$

– If $o = \mathtt{C}[\![xl]\!][x/u] \to \mathtt{C}[\![ul]\!][x/u] = o'$, then by construction we have that $\Sigma = \_$ and Φ is of the form

$$\Phi \triangleright \frac{\dfrac{\nabla}{x{:}[\rho_j]_{j\in J}; \Pi \mid \_ \vdash \mathtt{C}[\![ul]\!]{:}\tau} \qquad \left(\dfrac{\nabla}{\Gamma_j \mid \_ \vdash u{:}\rho_j}\right)_{j\in J}}{\Pi +_{j\in J} \Gamma_j \mid \_ \vdash \mathtt{C}[\![ul]\!][x/u]{:}\tau}$$

By Lemma 36, $\exists \Gamma_0, \exists I, \exists (\Gamma_i)_{i\in I}, \exists (\rho_i)_{i\in I}$ s.t. $x{:}[\rho_j]_{j\in J}; \Pi = \Gamma_0 +_{i\in I} \Gamma_i$, $\Gamma_0 + \{x{:}[\rho_i]_{i\in I}\} \mid \_ \vdash \mathtt{C}[\![xl]\!]{:}\tau$ and $(\Gamma_i \mid \_ \vdash u{:}\rho_i)_{i\in I}$. By Lemma 13 and α-conversion we necessarily have that $\Gamma_0 = x{:}[\rho_j]_{j\in J}; \Pi'$ s.t. $\Pi = \Pi' +_{i\in I} \Gamma_i$ thus $\Gamma_0 + \{x{:}[\rho_i]_{i\in I}\} = x{:}[\rho_k]_{k\in I\cup J}; \Pi'$. Let $K := I \cup J$.
Hence

$$\Phi' \triangleright \frac{\dfrac{\nabla}{x{:}[\rho_k]_{k\in K}; \Pi' \mid \_ \vdash \mathtt{C}[\![xl]\!]{:}\tau} \qquad \left(\dfrac{\nabla}{\Gamma_k \mid \_ \vdash u{:}\rho_k}\right)_{k\in K}}{\Pi' +_{k\in K} \Gamma_k \mid \_ \vdash \mathtt{C}[\![xl]\!][x/u]{:}\tau}$$

Observe that $\Pi' +_{k\in K} \Gamma_k = \Pi' +_{i\in I} \Gamma_i +_{j\in J} \Gamma_j = \Pi +_{j\in J} \Gamma_j$.

– If $o = t[x/u] \to t = o'$ where $|t|_x = 0$, then by construction and Lemma 13 we have that $\Sigma = \_$ and $x \notin \mathtt{dom}(\Gamma)$. Therefore,

$$\frac{\dfrac{\nabla}{x{:}[\,]; \Gamma \mid \_ \vdash t{:}\tau}}{\Gamma \mid \_ \vdash t[x/u]{:}\tau}\ (\mathtt{es}_{\mathcal{HW}})$$

– If $o = \mathtt{L}[\![xl]\!]m \to \mathtt{L}[\![x(l@m)]\!] = o'$, then we proceed by induction on $\mathtt{L}$. If $\mathtt{L} = \Box$, then by construction we have $\Sigma = \_$ and $\Gamma = \Delta + \{x : [\pi]\}$ and Φ is of the form

$$\Phi \triangleright \frac{\dfrac{\nabla}{\Delta \mid \pi \vdash l@m{:}\tau}}{\Delta + \{x{:}[\pi]\} \mid \_ \vdash x(l@m){:}\tau}$$

By Lemma 35, $\exists \Delta_1, \exists \Delta_2, \exists \sigma$ s.t. $\Delta = \Delta_1 + \Delta_2$, $\Delta_1 \mid \pi \vdash l{:}\sigma$ and $\Delta_2 \mid \sigma \vdash m{:}\tau$. Therefore,

$$\Phi' \triangleright \dfrac{\dfrac{\dfrac{\nabla}{\Delta_1 \mid \pi \vdash l{:}\sigma}}{\Delta_1 + \{x{:}[\pi]\} \mid \_ \vdash xl{:}\sigma} \qquad \dfrac{\nabla}{\Delta_2 \mid \sigma \vdash m{:}\tau}}{\Delta_1 + \{x{:}[\pi]\} + \Delta_2 \mid \_ \vdash (xl)m{:}\tau}$$

Let $\mathtt{L} = \mathtt{L}'[y/u]$ so that $\mathtt{L}[\![x(l@m)]\!] = \mathtt{L}'[\![x(l@m)]\!][y/u]$. By construction we have a derivation of the following form:

$$\Phi \triangleright \dfrac{\dfrac{\nabla}{\Gamma_0; y{:}[\rho_i]_{i\in I} \mid \_ \vdash \mathtt{L}'[\![x(l@m)]\!]{:}\tau} \qquad \left(\dfrac{\nabla}{\Delta_i \mid \_ \vdash u{:}\rho_i}\right)_{i\in I}}{\Gamma_0 +_{i\in I} \Delta_i \mid \_ \vdash \mathtt{L}'[\![x(l@m)]\!][y/u]{:}\tau}$$

By the *i.h.* there is a derivation $\Gamma_0; y{:}[\rho_i]_{i\in I} \mid \_ \vdash \mathtt{L}'[\![xl]\!]m{:}\tau$ and, by construction we have that $\Gamma_0; y{:}[\rho_i]_{i\in I} = \Delta_1 + \Delta_2$ s.t. $\Delta_1 \mid \_ \vdash \mathtt{L}'[\![xl]\!]{:}\sigma$ and $\Delta_2 \mid \sigma \vdash m{:}\tau$.
We can assume by α-conversion that $|m|_y = 0$ thus, by Lemma 13, we necessarily have that $\Delta_1 = \Delta'; y{:}[\rho_i]_{i\in I}$ s.t. $\Gamma_0 = \Delta' + \Delta_2$. We can then construct the following derivation:

$$\Phi' \triangleright \dfrac{\dfrac{\dfrac{\nabla}{\Delta'; y{:}[\rho_i]_{i\in I} \mid \_ \vdash \mathtt{L}'[\![xl]\!]{:}\sigma} \qquad \left(\dfrac{\nabla}{\Delta_i \mid \_ \vdash u{:}\rho_i}\right)_{i\in I}}{\Delta' +_{i\in I} \Delta_i \mid \_ \vdash \mathtt{L}[\![xl]\!]{:}\sigma} \qquad \dfrac{\nabla}{\Delta_2 \mid \sigma \vdash m{:}\tau}}{\Delta' +_{i\in I} \Delta_i + \Delta_2 \mid \_ \vdash \mathtt{L}[\![xl]\!]m{:}\tau}$$

- If $o = \mathtt{L}[\![tl]\!]m \to \mathtt{L}[\![t(l@m)]\!] = o'$, then we proceed by induction on $\mathtt{L}$ as the previous case. We only show here the base case where $\mathtt{L} = \Box$. In that case, by construction, we have $\Sigma = \_$ and $\Gamma = \Gamma_0 + \Gamma_1$ and Φ is of the form

$$\Phi \triangleright \dfrac{\dfrac{\nabla}{\Gamma_0 \mid \_ \vdash t{:}\sigma} \qquad \dfrac{\nabla}{\Gamma_1 \mid \sigma \vdash l@m{:}\tau}}{\Gamma_0 + \Gamma_1 \vdash t(l@m){:}\tau}$$

By Lemma 35, $\exists \Delta_1, \exists \Delta_2, \exists \delta$ s.t. $\Gamma_1 = \Delta_1 + \Delta_2$, $\Delta_1 \mid \sigma \vdash l{:}\delta$ and $\Delta_2 \mid \delta \vdash m{:}\tau$. Therefore,

$$\Phi' \triangleright \dfrac{\dfrac{\dfrac{\nabla}{\Gamma_0 \mid \_ \vdash t{:}\sigma} \qquad \dfrac{\nabla}{\Delta_1 \mid \sigma \vdash l{:}\delta}}{\Gamma_0 + \Delta_1 \mid \_ \vdash tl{:}\delta} \qquad \dfrac{\nabla}{\Delta_2 \mid \delta \vdash m{:}\tau}}{\Gamma_0 + \Delta_1 + \Delta_2 \mid \_ \vdash (tl)m{:}\tau}$$

- All the inductive cases are straightforward.

Lemma 16 If $\Phi \triangleright \Gamma \mid \_ \vdash_{\mathcal{HW}} u{:}\tau$ and u has no $(\mathtt{B}@, \mathtt{c}|_{\mathtt{L_H}})$-redex T-occurrences in Φ then $u \in \mathtt{L_H}\mathtt{J}$-nf.

Proof. Suppose that u is not a $\mathtt{L_H}\mathtt{M}$-nf. Then it is not difficult to show that $u \in B \cup C$, where B and C are defined as follows :

- $\mathtt{L}[\![\lambda x.u]\!]\mathtt{nil} \in B$.
- $\mathtt{L}[\![\lambda x.u]\!](v; l) \in B$.
- $\mathtt{L}[\![xl]\!]m \in B$.
- $\mathtt{L}[\![tl]\!]m \in B$.
- If $u \in B$, then $ul \in B$, $u[x/v] \in B$, $\lambda x.u \in B$.
- $yl \in A_{yl}$.
- If $u \in A_{yl}$, then $ul \in A_{yl}$, $\lambda x.u \in A_{yl}$, $u[x/v] \in A_{yl}$ for $x \neq y$.
- If $u \in A_{yl}$, then $u[y/v] \in C$.
- If $u \in C$, then $ul \in C$, $\lambda x.u \in C$, $u[x/v] \in C$.

Let $\Phi \triangleright \Gamma \mid \_ \vdash u{:}\tau$. We then show that $u \in B \cup C$ implies that u has a redex T-occurrence in Φ, reasoning by induction on the definitions of B and C. First, we prove that for any $u \in A_{yl}$, yl has a T-occurrence in Φ.

If $u = yl \in A_{yl}$, then the property is straightforward. If $u = vm \in A_{yl}$ or $u = \lambda x.v \in A_{yl}$ or $u = v[x/v'] \in A_{yl}$ for $x \neq y$, where $v \in A_{yl}$, then by the *i.h.* yl has a T-occurrence in the corresponding subderivation of Φ so that yl has a T-occurrence in Φ.

If $u = \mathtt{L}[\![\lambda x.v]\!]\mathtt{nil} \in B$, $u = \mathtt{L}[\![\lambda x.u']\!](v; m) \in B$, $u = \mathtt{L}[\![xl]\!]m \in B$ or $u = \mathtt{L}[\![vl]\!]m \in B$ then ϵ is a redex T-occurrence in Φ. If $u = vm \in B$ or $u = v[x/v'] \in B$ or $u = \lambda x.v \in B$, where $v \in B$, then by the *i.h.* the subterm v has a redex T-occurrence in the corresponding subderivation of Φ so that also u has a redex T-occurrence in Φ. Exactly the same reasoning applies for $u = vm$, or $u = v[x/v']$ or $u = \lambda x.v$ belonging to C where $v \in C$. Finally, if $u = v[y/v']$, where $v \in A_{yl}$, then by the first property shown before we know that yl has a T-occurrence in the corresponding subderivation of Φ so that the redex $v[y/v']$ has a T-occurrence in Φ. This concludes the proof.

Lemma 37. *If u is linear-head* J*-normalizing then u is $\mathcal{HW}$-typable.*

Proof. By induction on the length of the linear-head J-normalizing reduction. Let $u \to^{k}_{\mathtt{L_H J}} u'$, where $u' \in \mathtt{L_H}$J-nf. If $k = 0$ (*i.e.* $u = u'$), then it is not difficult to prove that $u \in A^n_y$, for some symbol y, where A^n_y is defined as follows:

- If $u \in B^n_y$, then $u \in A^n_y$.
- If $u \in A^n_y$, then $\lambda x.u \in A^n_y$.
- If $u \in A^n_y$, then $u[x/v] \in A^n_y$ for any J-term v and $x \neq y$.
- $yl \in B^n_y$, where $|l|_; = n$ (the number of ";" in l).
- If $u \in B^n_y$, then $u[x/v] \in B^n_y$ for any J-term v and $x \neq y$.

Let $\tau^n = \mathcal{M}_1 \to \cdots \to \mathcal{M}_n \to \tau$ ($n \geq 0$) such that $\mathcal{M}_i = [\,]$ ($1 \leq i \leq n$). We first prove by induction on $|l|_; = n$ that $\emptyset \mid \tau^n \vdash_{\mathcal{HW}} l{:}\tau$. If $n = 0$ then $l = \mathtt{nil}$ and $\emptyset \mid \tau \vdash_{\mathcal{HW}} \mathtt{nil}{:}\tau$ by the typing rule ($\mathtt{ax}$). If $l = v; m$ then $\emptyset \mid \tau^n \vdash_{\mathcal{HW}} m{:}\tau$ by the *i.h.* and $\emptyset \mid [\,] \to \tau^n \vdash_{\mathcal{HW}} v; m{:}\tau$ by the rule ($\to \mathtt{l}_{\mathcal{HW}}$).

Secondly, we prove by induction on B^n_y that $u \in B^n_y$ implies $y{:}[\tau^n] \mid \_ \vdash_{\mathcal{HW}} u{:}\tau$:

- If $yl \in B^n_y$ then $\emptyset \mid \tau^n \vdash_{\mathcal{HW}} l{:}\tau$ by the previous proof and $y{:}[\tau^n] \mid \_ \vdash_{\mathcal{HW}} yl{:}\tau$ by the rule ($\mathtt{hlist}$).
- If $u[x/v] \in B^n_y$ comes from $u \in B^n_y$ then $y{:}[\tau^n] \mid \_ \vdash_{\mathcal{HW}} u{:}\tau$ holds by the *i.h.* thus $y{:}[\tau^n] \mid \_ \vdash_{\mathcal{HW}} u[x/v]{:}\tau$ holds by the typing rule ($\mathtt{es}_{\mathcal{HW}}$).

Now, we prove by induction on A^n_y that $u \in A^n_y$ implies $\Gamma \mid \_ \vdash_{\mathcal{HW}} u{:}\sigma$ where the domain of Γ has at most the symbol y.

- If $u \in A^n_y$, where $u \in B^n_y$, then the property follows by the previous point.

– If $\lambda x.u \in A^n_y$, where $u \in A^n_y$, then $\Gamma \mid \_ \vdash_{\mathcal{HW}} u{:}\sigma$ by the *i.h.* so that $\Gamma \setminus\!\!\setminus x \mid \_ \vdash_{\mathcal{HW}} \lambda x.t{:}\Gamma(x) \rightarrow \sigma$ by application of the typing rule $(\rightarrow \mathtt{r})$. If Γ has at most y, then also does $\Gamma \setminus\!\!\setminus x$.
– If $u[x/v] \in A^n_y$, where $u \in A^n_y$, then $\Gamma \mid \_ \vdash_{\mathcal{HW}} u{:}\sigma$ by the *i.h.* so that $\Gamma \mid \_ \vdash_{\mathcal{HW}} u[x/v]{:}\sigma$ by application of the typing rule $(\mathtt{es}_{\mathcal{HW}})$.

Otherwise, let $u \rightarrow_{\mathtt{L_HJ}} v \rightarrow^k_{\mathtt{L_HJ}} u'$. By the *i.h.* the term v is $\mathcal{HW}$-typable and thus by Lemma 15 the same holds for u.

Given $\Phi \triangleright \Gamma \mid \Sigma \vdash o{:}\tau$, we define $A(o, \Phi) :=$ o has no $(\mathtt{B@}, \mathtt{c}, \mathtt{w})$-redex T-occurrences in Φ.

Lemma 38. *Let $\Phi \triangleright \Gamma \mid \_ \vdash_{\mathcal{HW}} o{:}\tau$ s.t. $A(o, \Phi)$.*

1. *If $[\,] \notin \mathcal{P}(\Gamma \mid \Sigma)$ and $o = \mathtt{L}[\![yl]\!]$ or $o = l$, then o has no substitutions and $|o|_x > 0$ implies xl' has a T-occurrence in Φ, for some $l' \in \mathcal{L}_{\mathtt{J}}$.*
2. *If $[\,] \notin \mathcal{P}(\langle \Gamma \mid \Sigma, \tau \rangle)$, then o has no substitutions and $|o|_x > 0$ implies xl has a T-occurrence in Φ, for some $l \in \mathcal{L}_{\mathtt{J}}$.*

Proof. The proof is by induction on Φ.

– $o = yl$. By construction Φ is of the form:

$$\Phi \triangleright \dfrac{\Phi_l \triangleright \dfrac{\nabla}{\Gamma \mid \sigma \vdash l{:}\tau}}{\Gamma + \{y{:}[\sigma]\} \mid \_ \vdash yl{:}\tau}$$

Moreover, $A(o, \Phi)$ implies $A(l, \Phi_l)$ and $|yl|_x > 0$ implies either $x = y$ or $|l|_x > 0$.
Suppose that $[\,] \in \mathcal{P}(\Gamma \mid \sigma)$. Then either $[\,] \in \mathcal{P}(\Gamma)$ or $[\,] \in \mathcal{N}(\sigma)$, leading to a contradiction with the hyp in (1). Hence, $[\,] \notin \mathcal{P}(\Gamma \mid \sigma)$ and l has no substitution by the *i.h.* on (1). We reason by cases.

- If $x = y$ then we trivially have that xl has a T-occurrence in Φ.
- If $|l|_x > 0$ then by the *i.h.* on (1) xl' has some T-occurrence in Φ_l thus in Φ.

We can then conclude since l without substitutions implies xl has no substitution.
– $o = u[z/v]$. Then $\Gamma = \Gamma_0 +_{i \in I} \Delta_i$ and Φ has necessarily the following form:

$$\Phi \triangleright \dfrac{\Phi_u \triangleright \dfrac{\nabla}{\Gamma_0; z{:}[\sigma_i]_{i \in I} \mid \_ \vdash u{:}\tau} \qquad \left(\Phi^i_v \triangleright \dfrac{\nabla}{\Delta_i \mid \_ \vdash v{:}\sigma_i} \right)_{i \in I}}{\Gamma_0 +_{i \in I} \Delta_i \mid \_ \vdash u[z/v]{:}\tau}$$

Moreover, $A(o, \Phi)$ implies $A(u, \Phi_u)$. We consider two cases.

- Suppose $|u|_z = 0$. Then o has a $\mathtt{w}$-redex T-occurrence which contradicts $A(o, \Phi)$.
- Suppose $|u|_z > 0$. If zl' has some T-occurrence in Φ_u, then o has a $\mathtt{c}$-redex T-occurrence which contradicts $A(o, \Phi)$. Therefore, zl' only has untyped occurrences in Φ_u, for any $l' \in \mathcal{L}_{\mathtt{J}}$, and thus $I = \emptyset$. We have $\Gamma_0; z{:}[\sigma_i]_{i \in I} = \Gamma_0; z{:}[\,] = \Gamma_0$.
  We consider again two cases.
  1. If o is of the form of item 1, the hypothesis $[\,] \notin \mathcal{P}(\Gamma \mid \_)$ implies $[\,] \notin \mathcal{P}(\Gamma_0 \mid \_)$. Therefore, the *i.h.* on (1) (from right to left) allows to conclude that $|u|_z = 0$ which leads to a contradiction.

2. Otherwise, the hypothesis $[] \notin \mathcal{P}(\langle \Gamma \mid \_, \tau\rangle)$ implies $[] \notin \mathcal{P}(\langle \Gamma_0 \mid \_, \tau\rangle)$. Therefore the *i.h.* on (2) (from right to left) then allows to conclude that $|u|_z = 0$ which leads to a contradiction.

We conclude that o cannot be a substitution.

- $o = \lambda y.u$. Then we are necessarily in case (2). By construction $\Sigma = \_$, $\tau = \mathcal{M} \to \sigma$ and Φ is obtained applying the rule $(\to \mathtt{r})$ on $\Phi_u \triangleright \Gamma; y{:}\mathcal{M} \mid \_ \vdash_{\mathcal{HW}} u{:}\sigma$. Moreover $A(o,\Phi)$ implies $A(u,\Phi_u)$. Suppose $[] \in \mathcal{P}(\langle \Gamma; y{:}\mathcal{M} \mid \_, \sigma\rangle)$. Then either $[] \in \mathcal{P}(\Gamma)$, $[] \in \mathcal{N}(\mathcal{M})$ or $[] \in \mathcal{P}(\sigma)$, leading to a contradiction with the hyp in (2). Therefore, $[] \notin \mathcal{P}(\langle \Gamma; y{:}\mathcal{M} \mid \_, \sigma\rangle)$ and the result is straightforward by the *i.h.* on (2).
- If $o = \mathtt{nil}$, the two statements are trivial.
- $o = u; l$. Then, by construction we have $\Sigma = [\sigma_i]_{i\in I} \to \sigma$, and Φ has the following form.

$$\Phi \triangleright \frac{\left(\Phi_u^i \triangleright \dfrac{\nabla}{\Gamma_i \mid \_ \vdash u{:}\sigma_i}\right)_{i\in I} \qquad \Phi_l \triangleright \dfrac{\nabla}{\Gamma_0 \mid \sigma \vdash l{:}\tau}}{\Gamma_0 +_{i\in I} \Gamma_i \mid [\sigma_i]_{i\in I} \to \sigma \vdash u; l{:}\tau}$$

  Moreover $A(o,\Phi)$ implies $(A(u,\Phi_u^i))_{i\in I}$ and $A(l,\Phi_l)$.
  Suppose that $[] \in \mathcal{P}(\Gamma_0 \mid \sigma)$. Then either $[] \in \mathcal{P}(\Gamma_0)$ or $[] \in \mathcal{N}(\sigma)$, which implies $[] \in \mathcal{P}(\Gamma_0 +_{i\in I} \Gamma_i \mid [\sigma_i]_{i\in I} \to \sigma)$, contradicting the hyp in (2), and thus also in (1). Hence, $[] \notin \mathcal{P}(\Gamma_0 \mid \sigma)$. Thus, by the *i.h.* on (1) l has no substitution and if $|l|_x > 0$ then xl' has some T-occurrence in Φ_l; thus xl' has some T-occurrence in Φ.
  We now consider two cases.

  1. $I = \emptyset$. Then $[] \in \mathcal{N}([] \to \sigma)$, leading to a contradiction with the hyp in (1) and (2).
  2. $I \neq \emptyset$. Suppose that $[] \in \mathcal{P}(\langle \Gamma_i \mid \_, \sigma_i\rangle)$ for some $i \in I$. Then either $[] \in \mathcal{P}(\Gamma_i)$ or $[] \in \mathcal{P}(\sigma_i)$, which implies $[] \in \mathcal{P}(\Gamma_0 +_{i\in I} \Gamma_i \mid [\sigma_i]_{i\in I} \to \sigma)$, contradicting the hyp in (2), and thus also in (1). Hence, $([] \notin \mathcal{P}(\langle \Gamma_i \mid \_, \sigma_i\rangle))_{i\in I}$ so by the *i.h.* on (2) u has no substitution and if $|u|_x > 0$ then xl' has some T-occurrence in Φ_u^i for each i, thus xl' has some T-occurrence in Φ.

  Therefore, o has no substitution and if $|o|_x > 0$ then xl' has some T-occurrence of o in Φ.
- There is no other possible case.

Lemma 17. Let $\Phi \triangleright \Gamma \mid \Sigma \vdash_{\mathcal{HW}} o{:}\tau$ s.t. $[] \notin \mathcal{P}(\langle \Gamma \mid \Sigma, \tau\rangle)$. If o has no $(\mathtt{B@}, \mathtt{c}, \mathtt{w})$-redex T-occurrence in Φ, then $o \in \mathtt{J}$-nf.

Proof. We proceed by induction on Φ.

- $o = \mathtt{nil}$. Then the statement is trivial.
- $o = yl$. By construction $\Sigma = \_$, $\Gamma = \Gamma_0 + \{y{:}[\sigma]\}$ and Φ is obtained by applying the rule $(\mathtt{hlist})$ on $\Phi_l \triangleright \Gamma_0 \mid \sigma \vdash l{:}\tau$. Moreover, $A(yl,\Phi)$ implies $A(l,\Phi_l)$. Suppose $[] \in \mathcal{P}(\langle \Gamma_0 \mid \sigma, \tau\rangle)$. Then either $[] \in \mathcal{P}(\Gamma_0)$ or $[] \in \mathcal{P}(\tau)$ or $[] \in \mathcal{N}(\sigma)$, which leads to a contradiction with the hyp. Hence, $[] \notin \mathcal{P}(\langle \Gamma_0 \mid \sigma, \tau\rangle)$. By the *i.h.* l is a $\mathtt{J}$-nf thus yl is a $\mathtt{J}$-nf.
- $o = \lambda x.u$. By construction $\Sigma = \_$, $\tau = \mathcal{M} \to \sigma$ and Φ is obtained applying the rule $(\to \mathtt{r})$ on $\Phi_u \triangleright \Gamma; x{:}\mathcal{M} \mid \_ \vdash u{:}\sigma$. Moreover, $A(o,\Phi)$ implies $A(u,\Phi_u)$. Suppose $[] \in \mathcal{P}(\langle \Gamma; x{:}\mathcal{M} \mid \_, \sigma\rangle)$. Then either $[] \in \mathcal{P}(\Gamma)$ or $[] \in \mathcal{N}(\mathcal{M})$ or $[] \in \mathcal{P}(\sigma)$, which leads to a contradiction with the hyp. Therefore $[] \notin \mathcal{P}(\langle \Gamma; x{:}\mathcal{M} \mid \_, \sigma\rangle)$ and the result is straightforward by the *i.h.*
- $o = u[x/v]$. By Lemma 38 o has no susbtitutions. We then conclude that o cannot be of the form $u[x/v]$.

– o is an application. Then $o = \mathtt{L}[\![yl]\!]$, otherwise o would have a $\mathtt{B}$@-redex T-occurrence in Φ. Moreover, o has no substitution by Lemma 38 so that $o = yl$ and the result is already proved to hold in the first point.
– $o = u;l$. By construction we have $\Sigma = [\sigma_i]_{i\in I} \to \sigma$, and Φ has the following form.

$$\Phi \rhd \frac{\left(\Phi_u^i \rhd \dfrac{\nabla}{\Gamma_i \mid \_ \vdash u{:}\sigma_i}\right)_{i\in I} \qquad \Phi_l \rhd \dfrac{\nabla}{\Gamma_0 \mid \sigma \vdash l{:}\tau}}{\Gamma_0 +_{i\in I} \Gamma_i \mid [\sigma_i]_{i\in I} \to \sigma \vdash u;l{:}\tau}$$

Moreover $A(o,\Phi)$ implies $(A(u,\Phi_u^i))_{i\in I}$ and $A(l,\Phi_l)$.
Suppose that $[\,] \in \mathcal{P}(\langle \Gamma_0 \mid \sigma,\tau\rangle)$. Then either $[\,] \in \mathcal{P}(\Gamma_0)$ or $[\,] \in \mathcal{N}(\sigma)$ or $[\,] \in \mathcal{P}(\tau)$, which implies $[\,] \in \mathcal{P}(\langle \Gamma_0 +_{i\in I} \Gamma_i \mid [\sigma_i]_{i\in I} \to \sigma,\tau\rangle)$, contradicting the hyp. Hence, $[\,] \notin \mathcal{P}(\langle \Gamma_0 \mid \sigma,\tau\rangle)$. Thus, by *i.h.* $l \in \mathtt{J}$-nf.
We now consider two cases.

1. $I = \emptyset$. Then $[\,] \in \mathcal{N}([\,]\to\sigma)$, leading to a contradiction with the hyp.
2. $I \neq \emptyset$. Suppose that $[\,] \in \mathcal{P}(\langle \Gamma_i \mid \_,\sigma_i\rangle)$ for some $i \in I$. Then either $[\,] \in \mathcal{P}(\Gamma_i)$ or $[\,] \in \mathcal{P}(\sigma_i)$, which implies $[\,] \in \mathcal{P}(\langle \Gamma_0 +_{i\in I} \Gamma_i \mid [\sigma_i]_{i\in I} \to \sigma,\tau\rangle)$, contradicting the hyp. Hence, $([\,] \notin \mathcal{P}(\langle \Gamma_i \mid \_,\sigma_i\rangle))_{i\in I}$ so that by the *i.h.* $u \in \mathtt{J}$-nf.

Therefore, $o = u;l \in \mathtt{J}$-nf.

Lemma 39. *Let $o \in \mathcal{O}_{\mathtt{J}}$. If $o \in \mathcal{WN}(\mathtt{J})$ then $\Gamma \mid \Sigma \vdash_{\mathcal{HW}} o{:}\tau$ and $[\,] \notin \mathcal{P}(\langle \Gamma \mid \Sigma,\tau\rangle)$.*

Proof. By induction on the length of the $\mathtt{J}$-normalizing reduction. Let $o \to_{\mathtt{J}}^{k} o'$, where $o' \in \mathtt{J}$-nf. If $k = 0$ (*i.e.* $o = o'$). Then we proceed by induction on the structure of $\mathtt{J}$-nfs (cf. [27]):

– If $o = \mathtt{nil}$ then $\emptyset \mid \alpha \vdash \mathtt{nil}{:}\alpha$ by the rule $(\mathtt{ax})$, where α is a base type, so that $[\,] \notin \mathcal{P}(\langle \emptyset \mid \alpha,\alpha\rangle)$.
– If $o = xl$, where l is a $\mathtt{J}$-nf, then by the *i.h.* $\Gamma \mid \sigma \vdash l{:}\tau$, where $[\,] \notin \mathcal{P}(\langle \Gamma \mid \sigma,\tau\rangle)$. Hence, by the rule $(\mathtt{hlist})$ we have $\Gamma + \{x{:}[\sigma]\} \mid \_ \vdash xl{:}\tau$ so that $[\,] \notin \mathcal{P}(\langle \Gamma + \{x{:}[\sigma]\} \mid \_,\tau\rangle)$.
– If $o = \lambda x.t$, where t is a $\mathtt{J}$-nf, then by the *i.h.* $\Gamma \mid \_ \vdash t{:}\tau$, where $[\,] \notin \mathcal{P}(\langle \Gamma \mid \_,\tau\rangle)$. Hence, by the rule $(\to \mathtt{r})$, $\Gamma \setminus\!\!\setminus x \mid \_ \vdash \lambda x.t{:}\Gamma(x) \to \tau$. Note that, since $[\,] \notin \mathcal{P}(\Gamma)$, $[\,] \notin \mathcal{N}(\Gamma(x))$. Therefore, $[\,] \notin \mathcal{P}(\Gamma(x) \to \tau)$ thus $[\,] \notin \mathcal{P}(\langle \Gamma \setminus\!\!\setminus x \mid \_,\Gamma(x) \to \tau\rangle)$.
– If $o = t;l$, where t and l are $\mathtt{J}$-nf, then, by the *i.h.* $\Gamma \mid \_ \vdash t{:}\rho$ and $\Delta \mid \sigma \vdash l{:}\tau$ where $[\,] \notin \mathcal{P}(\langle \Gamma \mid \_,\rho\rangle)$ and $[\,] \notin \mathcal{P}(\langle \Delta \mid \sigma,\tau\rangle)$. Hence, by the rule $(\to \mathtt{l}_{\mathcal{HW}})$, $\Delta + \Gamma \mid [\rho] \to \sigma \vdash t;l{:}\tau$. Note that $[\,] \notin \mathcal{P}(\Delta + \Gamma)$. Moreover, since $[\,] \notin \mathcal{P}(\rho)$ and $[\,] \notin \mathcal{N}(\sigma)$, then $[\,] \notin \mathcal{N}([\rho] \to \sigma)$. Therefore, $[\,] \notin \mathcal{P}(\langle \Delta + \Gamma \mid [\rho] \to \sigma,\tau\rangle)$.

Otherwise, let $o \to_{\mathtt{J}} p \to_{\mathtt{J}}^{k} o'$. By the *i.h.* the term p is $\mathcal{HW}$-typable and thus by Lemma 15 the same holds for o.

D Appendix: Characterization of Strong J-Normalization

Lemma 40 (Partial Substitution IV). *If $\Phi_{\mathtt{O}[\![xl]\!]} \rhd x{:}[\rho_i]_{i\in I};\Gamma \mid \Sigma \vdash_{\mathcal{S}} \mathtt{O}[\![xl]\!]{:}\tau$ and $(\Phi_s^i \rhd \Delta_i \mid \_ \vdash_{\mathcal{S}} s{:}\rho_i)_{i\in I}$ then $\Phi_{\mathtt{O}[\![sl]\!]} \rhd x{:}[\rho_i]_{i\in I\setminus K};\Gamma +_{i\in K} \Delta_i \mid \Sigma \vdash_{\mathcal{S}} \mathtt{O}[\![sl]\!]{:}\tau$, for some $\emptyset \neq K \subseteq I$ where $\mathtt{sz2}(\Phi_{\mathtt{O}[\![sl]\!]}) = \mathtt{sz2}(\Phi_{\mathtt{O}[\![xl]\!]}) +_{i\in K} \mathtt{sz2}(\Phi_s^i) - |K|$.*

Proof. By induction on the typing derivation $\Phi_{\mathtt{O}[\![xl]\!]} \triangleright x{:}[\rho_i]_{i\in I};\Gamma \mid \Sigma \vdash_{\mathcal{S}} \mathtt{O}[\![xl]\!]{:}\tau$. We only show here the case $\mathtt{O} = \mathtt{D}; m$, the cases with explicit substitutions follow the same scheme of this one, and all the other ones are very similar to those in the proof of Lemma 33.

So let $\mathtt{O} = \mathtt{D}; m$. By construction $\Sigma = [\sigma_j]_{j\in J} \to \varphi$ and $\Phi_{\mathtt{D}[\![xl]\!];m}$ is of the form

$$\Phi_{\mathtt{D}[\![xl]\!];m} \triangleright \frac{\left(\Phi^j_{\mathtt{D}[\![xl]\!]} \triangleright \frac{\nabla}{\Gamma_j \mid \_ \vdash \mathtt{D}[\![xl]\!]{:}\sigma_j}\right)_{j\in J\cup\{w\}} \qquad \Phi_m \triangleright \frac{\nabla}{\Pi \mid \varphi \vdash m{:}\tau}}{\Pi +_{j\in J\cup\{w\}} \Gamma_j \mid [\sigma_j]_{j\in J} \to \varphi \vdash \mathtt{D}[\![xl]\!]; m{:}\tau}$$

where $\Pi +_{j\in J\cup\{w\}} \Gamma_j = x{:}[\rho_i]_{i\in I};\Gamma$ thus $\Pi = x{:}[\rho_i]_{i\in I_m};\Pi'$ and $(\Gamma_j = x{:}[\rho_i]_{i\in I_j};\Gamma'_j)_{j\in J\cup\{w\}}$ where $I = I_m \cup_{j\in J\cup\{w\}} I_j$ and $\Gamma = \Pi' +_{j\in J\cup\{w\}} \Gamma'_j$. Moreover $\mathtt{sz2}(\Phi_{\mathtt{D}[\![xl]\!];m}) = \mathtt{sz2}(\Phi_m) +_{j\in J\cup\{w\}} \mathtt{sz2}(\Phi^j_{\mathtt{D}[\![xl]\!]}) + 1$. By *i.h.* for each $j \in J\cup\{w\}$, $\Phi^j_{\mathtt{D}[\![sl]\!]} \triangleright x{:}[\rho_i]_{i\in I_j\setminus K_j};\Gamma'_j +_{i\in K_j} \Delta_i \mid \_ \vdash \mathtt{D}[\![sl]\!]{:}\sigma_j$ for some $\emptyset \neq K_j \subseteq I_j$ where $\mathtt{sz2}(\Phi^j_{\mathtt{D}[\![sl]\!]}) = \mathtt{sz2}(\Phi^j_{\mathtt{D}[\![xl]\!]}) +_{i\in K_j} \mathtt{sz2}(\Phi^i_s) - |K_j|$. Let $K := \cup_{j\in J\cup\{w\}} K_j$ so that $K \neq \emptyset$. Therefore,

$$\Phi_{\mathtt{D}[\![sl]\!];m} \triangleright \frac{\left(\Phi^j_{\mathtt{D}[\![sl]\!]} \triangleright \frac{\nabla}{x{:}[\rho_i]_{i\in I_j\setminus K_j};\Gamma'_j +_{i\in K_j} \Delta_i \mid \_ \vdash \mathtt{D}[\![sl]\!]{:}\sigma_j}\right)_{j\in J\cup\{w\}} \qquad \Phi_m \triangleright \frac{\nabla}{x{:}[\rho_i]_{i\in I_m};\Pi' \mid \varphi \vdash m{:}\tau}}{x{:}[\rho_i]_{i\in I\setminus K};\Pi' +_{j\in J\cup\{w\}} \Gamma'_j +_{i\in K} \Delta_i \mid [\sigma_j]_{j\in J} \to \varphi \vdash \mathtt{D}[\![sl]\!]; m{:}\tau}$$

where $\mathtt{sz2}(\Phi_{\mathtt{D}[\![sl]\!];m}) = \mathtt{sz2}(\Phi_m) +_{j\in J\cup\{w\}} \mathtt{sz2}(\Phi^j_{\mathtt{D}[\![sl]\!]}) + 1 =_{i.h.} \mathtt{sz2}(\Phi_m) +_{j\in J\cup\{w\}} (\mathtt{sz2}(\Phi^j_{\mathtt{D}[\![xl]\!]}) +_{i\in K_j} \mathtt{sz2}(\Phi^i_s) - |K_j|) + 1 = \mathtt{sz2}(\Phi_m) +_{j\in J\cup\{w\}} \mathtt{sz2}(\Phi^j_{\mathtt{D}[\![xl]\!]}) +_{i\in K} \mathtt{sz2}(\Phi^i_s) - |K| + 1 = \mathtt{sz2}(\Phi_{\mathtt{D}[\![xl]\!];m}) +_{i\in K} \mathtt{sz2}(\Phi^i_s) - |K|$.

Lemma 41. *If $\Phi_l \triangleright \Gamma \mid \delta \vdash_{\mathcal{S}} l{:}\sigma$ and $\Phi_m \triangleright \Delta \mid \sigma \vdash_{\mathcal{S}} m{:}\tau$, then $\Phi_{l@m} \triangleright \Gamma + \Delta \mid \delta \vdash_{\mathcal{S}} l@m{:}\tau$ and* $\mathtt{sz2}(\Phi_{l@m}) = \mathtt{sz2}(\Phi_l) + \mathtt{sz2}(\Phi_m) - 1$.

Proof. By induction on the type derivation $\Phi_l \triangleright \Gamma \mid \delta \vdash_{\mathcal{S}} l{:}\sigma$. The proof is similar to that of Lemma 34.

Lemma 19 (SR IV). Let $\Phi \triangleright \Gamma \mid \Sigma \vdash_{\mathcal{S}} o{:}\tau$. If $o \to_{\mathtt{J}\setminus\mathtt{W}} o'$, then $\Phi' \triangleright \Gamma \mid \Sigma \vdash_{\mathcal{S}} o'{:}\tau$ and $\mathtt{sz2}(\Phi) > \mathtt{sz2}(\Phi')$.

Proof. By induction on the reduction relation $\to_{\mathtt{J}}$. We only show the most interesting case as the other ones are similar to the those of the proof of Lemma 14. Remark that the case $\to_{\mathtt{W}}$ is not treated since the statement only concerns the non-erasing reduction $\to_{\mathtt{J}\setminus\mathtt{W}}$.

Let $o = \mathtt{C}[\![xl]\!][x/u] \to \mathtt{C}[\![ul]\!][x/u] = o'$, so that, by construction, $\Sigma = \_$ and Φ is of the form

$$\Phi \triangleright \frac{\Phi_{\mathtt{C}[\![xl]\!]} \triangleright \frac{\nabla}{x{:}[\rho_i]_{i\in I};\Pi \mid \_ \vdash \mathtt{C}[\![xl]\!]{:}\tau} \qquad \left(\Phi^i_u \triangleright \frac{\nabla}{\Delta_i \mid \_ \vdash u{:}\rho_i}\right)_{i\in I\cup\{w\}}}{\Pi +_{i\in I\cup\{w\}} \Delta_i \mid \_ \vdash \mathtt{C}[\![xl]\!][x/u]{:}\tau}$$

where $\mathtt{sz2}(\Phi) = \mathtt{sz2}(\Phi_{\mathtt{C}[\![xl]\!]}) +_{i\in I\cup\{w\}} \mathtt{sz2}(\Phi^i_u) + 1$. By Lemma 40 $\Phi_{\mathtt{C}[\![ul]\!]} \triangleright x{:}[\rho_i]_{i\in I\setminus K};\Pi +_{i\in K} \Delta_i \mid \_ \vdash \mathtt{C}[\![ul]\!]{:}\tau$ for some $\emptyset \neq K \subseteq I$ s.t. $\mathtt{sz2}(\Phi_{\mathtt{C}[\![ul]\!]}) = \mathtt{sz2}(\Phi_{\mathtt{C}[\![xl]\!]}) +_{i\in K} \mathtt{sz2}(\Phi^i_u) - |K|$. Hence

$$\Phi' \triangleright \frac{\Phi_{\mathtt{C}[\![ul]\!]} \triangleright \frac{\nabla}{x{:}[\rho_i]_{i\in I\setminus K};\Pi +_{i\in K} \Delta_i \mid \_ \vdash \mathtt{C}[\![ul]\!]{:}\tau} \qquad \left(\Phi^i_u \triangleright \frac{\nabla}{\Delta_i \mid \_ \vdash u{:}\rho_i}\right)_{i\in I\setminus K\cup\{w\}}}{\Pi +_{i\in I\cup\{w\}} \Delta_i \mid \_ \vdash \mathtt{C}[\![ul]\!][x/u]{:}\tau}$$

where $\mathtt{sz2}(\Phi') = \mathtt{sz2}(\Phi_{\mathtt{C}[\![ul]\!]}) +_{i\in I\setminus K\cup\{w\}} \mathtt{sz2}(\Phi^i_u) + 1 =_{L.40} \mathtt{sz2}(\Phi_{\mathtt{C}[\![xl]\!]}) +_{i\in I\cup\{w\}} \mathtt{sz2}(\Phi^i_u) - |K| + 1$. Since $K \neq \emptyset$, then $\mathtt{sz}(\Phi) > \mathtt{sz}(\Phi')$.

We need to prove a couple of technical results in order to prove the Subject Expansion property.

Lemma 42. *If $\Gamma|\delta \vdash_{\mathcal{S}} l@m{:}\tau$ then $\exists\Gamma_1, \exists\Gamma_2, \exists\sigma$ s.t. $\Gamma = \Gamma_1 + \Gamma_2$, $\Gamma_1|\delta \vdash_{\mathcal{S}} l{:}\sigma$ and $\Gamma_2|\sigma \vdash_{\mathcal{HW}} m{:}\tau$*

Proof. The proof is by induction on the type derivation $\Phi \triangleright \Gamma \mid \Sigma \vdash_{\mathcal{S}} l@m{:}\tau$, and is similar to that of Lemma 35.

Lemma 43. *Let $\mathtt{O}[\![xl]\!] \in \mathcal{O}_{\mathtt{J}}, u \in \mathcal{T}_{\mathtt{J}}$ s.t. $|u|_x = 0$ and $\Gamma \mid \Sigma \vdash_{\mathcal{S}} \mathtt{O}[\![ul]\!]{:}\tau$. Then $\exists\Gamma_0, \exists I, \exists(\Gamma_i)_{i\in I}, \exists(\sigma_i)_{i\in I}$ s.t. $\Gamma = \Gamma_0 +_{i\in I} \Gamma_i$, $\Gamma_0 + \{x : [\sigma_i]_{i\in I}\} \mid \Sigma \vdash_{\mathcal{S}} \mathtt{O}[\![xl]\!]{:}\tau$, and $(\Gamma_i \mid \_ \vdash_{\mathcal{S}} u{:}\sigma_i)_{i\in I}$.*

Proof. The proof is by induction on the typing derivation $\Gamma \mid \Sigma \vdash \mathtt{O}[\![ul]\!]{:}\tau$ and is similar to that of Lemma 36. We only show here the most interesting case.

Let $\mathtt{O} = \mathtt{D}; m$, so that $\Sigma = [\rho_j]_{j\in J} \to \varphi$ and the type derivation is of the following form

$$\dfrac{\left(\dfrac{\nabla}{\Delta_j \vdash \mathtt{D}[\![ul]\!]{:}\rho_j}\right)_{j\in J\cup\{w\}} \qquad \dfrac{\nabla}{\Gamma_m \mid \varphi \vdash m{:}\tau}}{\Gamma_m +_{j\in J\cup\{w\}} \Delta_j \mid [\rho_j]_{j\in J} \to \varphi \vdash \mathtt{D}[\![ul]\!]; m{:}\tau}$$

By the *i.h.* for each $j \in J \cup \{w\}$, $\Delta_j = \Delta^j_0 +_{i\in I_j} \Gamma_i$ and $\Delta^j_0 + \{x{:}[\sigma_i]_{i\in I_j}\} \mid \_ \vdash \mathtt{D}[\![xl]\!]{:}\rho_j$ and $(\Gamma_i \mid \_ \vdash u{:}\sigma_i)_{i\in I_j}$. Let $I := \cup_{j\in J\cup\{w\}} I_j$. Hence,

$$\dfrac{\left(\dfrac{\nabla}{\Delta^j_0 + \{x{:}[\sigma_i]_{i\in I_j}\} \mid \_ \vdash \mathtt{D}[\![xl]\!]{:}\rho_j}\right)_{j\in J\cup\{w\}} \qquad \dfrac{\nabla}{\Gamma_m \mid \varphi \vdash m{:}\tau}}{\Gamma_m +_{j\in J\cup\{w\}} \Delta^j_0 + \{x{:}[\sigma_i]_{i\in I}\} \mid [\rho_j]_{j\in J} \to \varphi \vdash \mathtt{D}[\![xl]\!]; m{:}\tau}$$

We then conclude with $\Gamma_0 := \Gamma_m +_{j\in J\cup\{w\}} \Delta^j_0$ since $\Gamma_0 +_{i\in I} \Gamma_i = \Gamma_m +_{j\in J\cup\{w\}} \Delta^j_0 +_{j\in J\cup\{w\}} (+_{i\in I_j}\Gamma_i) = \Gamma_m +_{j\in J\cup\{w\}} (\Delta^j_0 +_{i\in I_j} \Gamma_i) = \Gamma_m +_{j\in J\cup\{w\}} \Delta_j = \Gamma$.

Lemma 20 (SE IV). Let $\Gamma \mid \Sigma \vdash_{\mathcal{S}} o'{:}\tau$. If $o \to_{\mathtt{J}\setminus\mathtt{w}} o'$, then $\Gamma \mid \Sigma \vdash_{\mathcal{S}} o{:}\tau$.

Proof. The proof is by induction on $o \to_{\mathtt{J}\setminus\mathtt{w}} o'$ and proceeds similarly to that of Lemma 15. We only show here the two most interesting cases.

- If $o = \mathtt{L}[\![\lambda x.v]\!](u; l) \to \mathtt{L}[\![v[x/u]l]\!] = o'$, then we proceed by induction on $\mathtt{L}$. Let $\mathtt{L} = \Box$. By construction, we have $\Sigma = \_$ and $\Gamma = \Gamma_0 +_{i\in I} \Gamma_i + \Delta$ and Φ is of the form

$$\Phi \triangleright \dfrac{\dfrac{\dfrac{\nabla}{\Gamma_0; x{:}[\rho_i]_{i\in I} \mid \_ \vdash v{:}\sigma} \qquad \left(\dfrac{\nabla}{\Gamma_i \mid \_ \vdash u{:}\rho_i}\right)_{i\in I\cup\{w\}}}{\Gamma_0 +_{i\in I\cup\{w\}} \Gamma_i \mid \_ \vdash v[x/u]{:}\sigma} \qquad \dfrac{\nabla}{\Delta \mid \sigma \vdash l{:}\tau}}{\Gamma_0 +_{i\in I\cup\{w\}} \Gamma_i + \Delta \mid \_ \vdash v[x/u]l{:}\tau}$$

Therefore, we can construct the derivation Φ' below:

$$\Phi' \triangleright \dfrac{\dfrac{\dfrac{\nabla}{\Gamma_0; x{:}[\rho_i]_{i\in I} \mid \_ \vdash v{:}\sigma}}{\Gamma_0 \mid \_ \vdash \lambda x.v{:}[\rho_i]_{i\in I} \to \sigma} \qquad \dfrac{\left(\dfrac{\nabla}{\Gamma_i \mid \_ \vdash u{:}\rho_i}\right)_{i\in I\cup\{w\}} \qquad \dfrac{\nabla}{\Delta \mid \sigma \vdash l{:}\tau}}{\Delta +_{i\in I\cup\{w\}} \Gamma_i \mid [\rho_i]_{i\in I} \to \sigma \vdash u;l{:}\tau}}{\Gamma_0 +_{i\in I\cup\{w\}} \Gamma_i + \Delta \mid \_ \vdash (\lambda x.v)(u;l){:}\tau}$$

Let $\mathtt{L} = \mathtt{L}'[y/s]$ so that $\mathtt{L}[\![v[x/u]l]\!] = \mathtt{L}'[\![v[x/u]l]\!][y/s]$. By construction we have a derivation of the following form:

$$\Phi \triangleright \dfrac{\dfrac{\nabla}{\Gamma_0; y{:}[\rho_i]_{i\in I} \mid \_ \vdash \mathtt{L}'[\![v[x/u]l]\!]{:}\tau} \qquad \left(\dfrac{\nabla}{\Gamma_i \mid \_ \vdash s{:}\rho_i}\right)_{i\in I\cup\{w\}}}{\Gamma_0 +_{i\in I\cup\{w\}} \Gamma_i \mid \_ \vdash \mathtt{L}'[\![v[x/u]l]\!][y/s]{:}\tau}$$

By *i.h.* from $\Gamma_0; y{:}[\rho_i]_{i\in I} \mid \_ \vdash \mathtt{L}'[\![v[x/u]l]\!]{:}\tau$ we have that $\Gamma_0; y{:}[\rho_i]_{i\in I} \mid \_ \vdash \mathtt{L}'[\![\lambda x.v]\!](u;l){:}\tau$ and, by construction, $\Gamma_0; y{:}[\rho_i]_{i\in I} = \Delta_1 + \Delta_2$ s.t. $\Delta_1 \mid \_ \vdash \mathtt{L}'[\![\lambda x.v]\!]{:}\sigma$ and $\Delta_2 \mid \sigma \vdash (u;l){:}\tau$. We can assume by α-conversion that $|u;l|_y = 0$ thus, by Lemma 13, we necessarily have that $\Delta_1 = \Delta'; y{:}[\rho_i]_{i\in I}$ and $\Gamma_0 = \Delta' + \Delta_2$. Therefore, we can construct the following derivation:

$$\Phi' \triangleright \dfrac{\dfrac{\dfrac{\nabla}{\Delta'; y{:}[\rho_i]_{i\in I} \mid \_ \vdash \mathtt{L}'[\![\lambda x.v]\!]{:}\sigma} \qquad \left(\dfrac{\nabla}{\Gamma_i \mid \_ \vdash s{:}\rho_i}\right)_{i\in I\cup\{w\}}}{\Delta' +_{i\in I\cup\{w\}} \Gamma_i \mid \_ \vdash \mathtt{L}'[\![\lambda x.v]\!][y/s]{:}\sigma} \qquad \dfrac{\nabla}{\Delta_2 \mid \sigma \vdash (u;l){:}\tau}}{\Gamma_0 +_{i\in I\cup\{w\}} \Gamma_i \mid \_ \vdash \mathtt{L}'[\![\lambda x.v]\!][y/s](u;l){:}\tau}$$

– If $o = \mathtt{C}[\![xl]\!][x/u] \to \mathtt{C}[\![ul]\!][x/u] = o'$, then by construction we have that $\Sigma = \_$ and Φ is of the form

$$\Phi \triangleright \dfrac{\dfrac{\nabla}{x{:}[\rho_j]_{j\in J}; \Pi \mid \_ \vdash \mathtt{C}[\![ul]\!]{:}\tau} \qquad \left(\dfrac{\nabla}{\Gamma_j \mid \_ \vdash u{:}\rho_j}\right)_{j\in J\cup\{w\}}}{\Pi +_{j\in J\cup\{w\}} \Gamma_j \mid \_ \vdash \mathtt{C}[\![ul]\!][x/u]{:}\tau}$$

By Lemma 43, $\exists \Gamma_0, \exists I, \exists(\Gamma_i)_{i\in I}, \exists(\rho_i)_{i\in I}$ s.t. $x{:}[\rho_j]_{j\in J}; \Pi = \Gamma_0 +_{i\in I} \Gamma_i$, $\Gamma_0 + \{x{:}[\rho_i]_{i\in I}\} \mid \_ \vdash \mathtt{C}[\![xl]\!]{:}\tau$ and $(\Gamma_i \mid \_ \vdash u{:}\rho_i)_{i\in I}$. By Lemma 13 and α-conversion we necessarily have that $\Gamma_0 = x{:}[\rho_j]_{j\in J}; \Pi'$ s.t. $\Pi = \Pi' +_{i\in I} \Gamma_i$ thus $\Gamma_0 + \{x{:}[\rho_i]_{i\in I}\} = x{:}[\rho_k]_{k\in I\cup J}; \Pi'$. Let $K := I \cup J$. Hence

$$\Phi' \triangleright \dfrac{\dfrac{\nabla}{x{:}[\rho_k]_{k\in K}; \Pi' \mid \_ \vdash \mathtt{C}[\![xl]\!]{:}\tau} \qquad \left(\dfrac{\nabla}{\Gamma_k \mid \_ \vdash u{:}\rho_k}\right)_{k\in K\cup\{w\}}}{\Pi' +_{k\in K\cup\{w\}} \Gamma_k \mid \_ \vdash \mathtt{C}[\![xl]\!][x/u]{:}\tau}$$

Observe that $\Pi' +_{k\in K\cup\{w\}} \Gamma_k = \Pi' +_{i\in I} \Gamma_i +_{j\in J\cup\{w\}} \Gamma_j = \Pi +_{j\in J\cup\{w\}} \Gamma_j$.

Lemma 44. *Let $n, l \in \mathcal{L}_{\mathtt{J}}$. Then $n, l \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$ iff $n@l \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$.*

Proof. By induction on n using the fact that $n = t;m \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$ iff $t, m \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$.

Lemma 45. *Let $\mathtt{M}$ be a list context, $t \in \mathcal{T}_{\mathtt{J}}$ and $m, l \in \mathcal{L}_{\mathtt{J}}$.*

1. $\mathtt{M}[\![tl]\!] \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$ *implies* $\mathtt{M}[\![t]\!] \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$.
2. $\mathtt{M}[\![t(m@l)]\!] \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$ *implies* $\mathtt{M}[\![tm]\!] \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$.

Proof. The proof of the first point is by induction on $\eta_{\mathtt{J}}(\mathtt{M}[\![tl]\!])$ and the proof of the second one is by induction on $\eta_{\mathtt{J}}(\mathtt{M}[\![t(m@l)]\!])$ and uses the first point.

Let $o \in \mathcal{O}_{\mathtt{J}}$ s.t. $|o|_x = n$. If $|o|_y = 0$, then we write $o_{[x/y]}$ to denote an arbitrary nondeterministic replacement of i $(0 \leq i \leq n)$ occurrences of x by the fresh symbol y. Thus for example, if $o = x_{\mathtt{nil}}[z/x_{\mathtt{nil}}]$, then $o_{[x/y]}$ may denote one of the terms $x_{\mathtt{nil}}[z/x_{\mathtt{nil}}]$, $y_{\mathtt{nil}}[z/x_{\mathtt{nil}}]$, $x_{\mathtt{nil}}[z/y_{\mathtt{nil}}]$, or $y_{\mathtt{nil}}[z/y_{\mathtt{nil}}]$.

Lemma 46. *Let* $o{\in}\mathcal{O}_{\mathtt{J}}$ *and* $v{\in}\mathcal{T}_{\mathtt{J}}$. *If* $o \rightarrow_{\mathtt{J}\backslash\mathtt{w}} o'$, *then* $o_{[x/y]} \rightarrow_{\mathtt{J}\backslash\mathtt{w}} o'_{[x/y]}$ *and* $o_{[x/y]}\{y/v\} \rightarrow^{+}_{\mathtt{J}\backslash\mathtt{w}} o'_{[x/y]}\{y/v\}$.

A consequence of the previous lemma is that $o \rightarrow_{\mathtt{J}\backslash\mathtt{w}} o'$ implies $o\{x/v\} \rightarrow^{+}_{\mathtt{J}\backslash\mathtt{w}} o'\{x/v\}$.

Corollary 4. *Let* $o \in \mathcal{O}_{\mathtt{J}}$ *and* $v \in \mathcal{T}_{\mathtt{J}}$. *If* $o_{[x/y]}\{y/v\} \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$ *then* $o \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$.

A consequence of the previous Corollary is that $C[\![vl]\!] \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$ implies $C[\![xl]\!] \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$.

Lemma 47. $\mathcal{SN}(\mathtt{J}\backslash\mathtt{w}) = \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$.

Proof. If $o \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$, then we show $o \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$ by induction on $\langle \eta_{\mathtt{J}\backslash\mathtt{w}}(o), |o| \rangle$. We reason by cases. If $o = \mathtt{nil}$, $o = t;l$, $o = \lambda x.t$, $o = xl$ or $o = t[x/u]$ with $|t|_x = 0$, then the property is straightforward. Otherwise,

- If $o = t[x/v]$ with $|t|_x > 0$, *i.e.* $t = \mathtt{C}[\![xl]\!]$, then every o' s.t. $o \rightarrow o'$ verifies $o' \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$ and in particular $o' = \mathtt{C}[\![vl]\!][x/v]$. Since $\eta_{\mathtt{J}\backslash\mathtt{w}}(o') < \eta_{\mathtt{J}\backslash\mathtt{w}}(o)$, then $o' \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$, so that we can conclude $o \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$ by rule (C).
- If $o = vl$ is an application, then we reason by cases on v.
  - If $o = u_0[x/u_1]l_0 \dots l_n$, with $n \geq 0$, then $\eta_{\mathtt{J}\backslash\mathtt{w}}(u_0[x/u_1]l_0 \dots l_n) = \eta_{\mathtt{J}\backslash\mathtt{w}}((u_0 l_0 \dots l_n)[x/u_1])$ by Lemma 18. Therefore $o' \in \mathcal{SN}(\mathtt{J}\backslash\mathtt{w})$ and in particular $\eta_{\mathtt{J}\backslash\mathtt{w}}(u_0 l_0 \dots l_n), \eta_{\mathtt{J}\backslash\mathtt{w}}(u_1) \leq \eta_{\mathtt{J}\backslash\mathtt{w}}(o)$. Thus, since $|u_0 l_0 \dots l_n|, |u_1| < |o|$, $u_0 l_0 \dots l_n, u_1 \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$ holds by the *i.h.*
    If $|u_0 l_0 \dots l_n|_x = 0$, then $(u_0 l_0 \dots l_n)[x/u_1] \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$ by rule (W) and thus $o \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$ by several applications of rule (E).
    If $|u_0 l_0 \dots l_n|_x > 0$, then $(u_0 l_0 \dots l_n)[x/u_1] = C[\![xl]\!][x/u_1] \rightarrow_{\mathtt{c}} C[\![u_1 l]\!][x/u_1]$ and thus we have that $\eta_{\mathtt{J}\backslash\mathtt{w}}(C[\![u_1 l]\!][x/u_1]) < \eta_{\mathtt{J}\backslash\mathtt{w}}(C[\![xl]\!][x/u_1])$. The *i.h.* gives $C[\![u_1 l]\!][x/u_1] \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$ so that $(u_0 l_0 \dots l_n)[x/u_1] = C[\![xl]\!][x/u_1] \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$ holds by (C) and $o \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$ holds by several applications of rule (E).
  - If $o = x l_0 l_1 \dots l_n$, with $n \geq 1$, then $o \rightarrow_{@_{\mathtt{var}}} x(l_0 @ l_1) \dots l_n = o'$. Moreover $\eta_{\mathtt{J}\backslash\mathtt{w}}(o') < \eta_{\mathtt{J}\backslash\mathtt{w}}(o)$ so that the *i.h.* gives $o' \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$. We conclude by rule $(@_{\mathtt{var}})$.
  - If $o = t l_0 \dots l_n$, with $n \geq 0$, we reason similarly to the previous case but we conclude with rule $(@_{\mathtt{app}})$.
  - If $o = (\lambda x.t) l_0 \dots l_n$, with $n \geq 0$, then we reason by cases on l_0. In every case we can conclude as before but using rules $(\mathtt{dB}_{\mathtt{nil}})$ and $(\mathtt{dB}_{\mathtt{cons}})$.

For the converse we reason by induction on the definition of $o \in \mathcal{ISN}(\mathtt{J}\backslash\mathtt{w})$. If $o = \mathtt{nil}$, $o = t;l$, $o = \lambda x.t$, $o = xl$, or $o = t[x/v]$ with $|t|_x = 0$, then the property is straightforward by using the *i.h.* The remaining cases are:

- If $o = (\lambda x.t)(u; l_0)l_1 \dots l_n \in \mathcal{ISN}(\mathtt{J\backslash w})$ where $o' = t[x/u]l_0 l_1 \dots l_n \in \mathcal{ISN}(\mathtt{J\backslash w})$, then the *i.h.* gives $o' \in \mathcal{SN}(\mathtt{J\backslash w})$ so that in particular $t, u, l_i \in \mathcal{SN}(\mathtt{J\backslash w})$.
  To prove $o \in \mathcal{SN}(\mathtt{J\backslash w})$ it is is sufficient to prove that all its immediate reducts are in $\mathcal{SN}(\mathtt{J\backslash w})$. This can be done by another induction on $\eta_{\mathtt{J\backslash w}}(t) + \eta_{\mathtt{J\backslash w}}(u) + \sum \eta_{\mathtt{J\backslash w}}(l_i)$.
  - If $o \to o'$, then the property holds by the first *i.h.* as already mentioned.
  - If o reduces some subterm t, u or l_i, then we conclude by the second *i.h.*
- If $o = (\lambda x.t)\mathtt{nil} l_1 \dots l_n \in \mathcal{ISN}(\mathtt{J\backslash w})$ where $o' = (\lambda x.t)l_1 \dots l_n \in \mathcal{ISN}(\mathtt{J\backslash w})$, then the *i.h.* gives $o' \in \mathcal{ISN}(\mathtt{J\backslash w})$ and we proceed similarly to the previous case.
- If $o = (xm)l_0 l_1 \dots l_n \in \mathcal{ISN}(\mathtt{J\backslash w})$ where $o' = x(m@l_0)l_1 \dots l_n \in \mathcal{ISN}(\mathtt{J\backslash w})$, then $o' \in \mathcal{SN}(\mathtt{J\backslash w})$ by the *i.h.* so that in particular $m@l_0, l_1, \dots, l_n \in \mathcal{SN}(\mathtt{J\backslash w})$. Lemma 44 also gives $m, l_0 \in \mathcal{SN}(\mathtt{J\backslash w})$. To prove $o \in \mathcal{SN}(\mathtt{J\backslash w})$ it is is sufficient to prove that all its immediate reducts are in $\mathcal{SN}(\mathtt{J\backslash w})$. This can be done by second induction on $\eta_{\mathtt{J\backslash w}}(m) + \sum \eta_{\mathtt{J\backslash w}}(l_i)$.
  - If $o \to o'$, then the property holds by the first *i.h.* as already mentioned.
  - If o reduces some subterm m, l_i, then we conclude by the second *i.h.*
- If $o = (tm)l_0 l_1 \dots l_n \in \mathcal{ISN}(\mathtt{J\backslash w})$ where $o' = (t(m@l_0))l_1 \dots l_n \in \mathcal{ISN}(\mathtt{J\backslash w})$, then $o' \in \mathcal{SN}(\mathtt{J\backslash w})$ by the *i.h.* so that in particular $t(m@l_0), l_i \in \mathcal{SN}(\mathtt{J\backslash w})$. Lemma 45 also gives $tm, l_0 \in \mathcal{SN}(\mathtt{J\backslash w})$. To prove $o \in \mathcal{SN}(\mathtt{J\backslash w})$ it is sufficient to prove that all its immediate reducts are in $\mathcal{SN}(\mathtt{J\backslash w})$. This can be done by a second induction on $\eta_{\mathtt{J\backslash w}}(tm) + \sum \eta_{\mathtt{J\backslash w}}(l_i)$.
  - If $o \to o'$, then the property holds by the first *i.h.* as already mentioned.
  - If o reduces some subterm tm or l_i, then we conclude by the second *i.h.*
- If $o = C[\![xl]\!][x/v] \in \mathcal{ISN}(\mathtt{J\backslash w})$ where $o' = C[\![vl]\!][x/v] \in \mathcal{ISN}(\mathtt{J\backslash w})$, then $o' \in \mathcal{SN}(\mathtt{J\backslash w})$ by the *i.h.* so that in particular $v, C[\![vl]\!] \in \mathcal{SN}(\mathtt{J\backslash w})$. By Corollary 4, $C[\![xl]\!] \in \mathcal{SN}(\mathtt{J\backslash w})$. To prove that $o \in \mathcal{SN}(\mathtt{J\backslash w})$ it is sufficient to prove that all its immediate reducts are in $\mathcal{SN}(\mathtt{J\backslash w})$. This can be done by induction on $\eta_{\mathtt{J\backslash w}}(v) + \eta_{\mathtt{J\backslash w}}(C[\![xl]\!])$.
  - If $o \to o'$, then we conclude by the first *i.h.* as already mentioned.
  - If o reduces some subterm $C[\![xl]\!]$ or v, then we conclude by the second *i.h.*
- If $v[x/u]l \in \mathcal{ISN}(\mathtt{J\backslash w})$ where $(vl)[x/u] \in \mathcal{ISN}(\mathtt{J\backslash w})$, then $(vl)[x/u] \in \mathcal{SN}(\mathtt{J\backslash w})$ by the *i.h.* so that $v[x/u]l \in \mathcal{SN}(\mathtt{J\backslash w})$ since $\eta_{\mathtt{J\backslash w}}(v[x/u]l) =_{L.\ 18} \eta_{\mathtt{J\backslash w}}((vl)[x/u])$.

Lemma 21. Let $o \in \mathcal{O}_{\mathtt{J}}$. If $o \in \mathcal{SN}(\mathtt{J\backslash w})$ then o is $\mathcal{S}$-typable.

Proof. By induction on the structure of $o \in \mathcal{SN}(\mathtt{J\backslash w}) =_{L.\ 47} \mathcal{ISN}(\mathtt{J\backslash w})$.

- If $o = \mathtt{nil}$, $o = t; l$, $o = \lambda x.t$, $o = xl$, or $o = u[x/v]$ with $|u|_x = 0$, then the proof is straightforward by using the *i.h.*
- If $o \in \mathcal{ISN}(\mathtt{J\backslash w})$ comes from one of the rules $(\mathtt{dB_{nil}})$, $(\mathtt{dB_{cons}})$, (C), $(@_{\mathtt{var}})$ or $(@_{\mathtt{app}})$, then the property holds by the *i.h.* and the Subject Expansion Lemma 20.
- If $t[x/s]l \in \mathcal{ISN}(\mathtt{J\backslash w})$ comes from the rule (E), then $(tl)[x/s] \in \mathcal{ISN}(\mathtt{J\backslash w})$, so that $(tl)[x/s]$ is $\mathcal{S}$-typable by the *i.h.* and the property holds by Lemma 18.