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Abstract

We develop in this Note a homogenization method to tackle the problem of a diffusion process through a cracked medium. We assume that the cracks are orthogonal to the surface of the material, where an incoming heat flux is applied. The cracks are supposed to be of depth 1, of small width, and periodically arranged. We show that the cracked surface of the domain induces a volume source term in the homogenized equation.

Résumé

Equation de la chaleur dans un milieu fracturé Nous présentons dans cette Note une méthode originale pour traiter la propagation de la chaleur dans un milieu fracturé. Nous considérons ici le cas de fractures perpendiculaires à l'axe du matériau, de profondeur unité, et disposées périodiquement. Nous montrons que la perturbation du flux induite par la fracture peut être redistribuée en un terme source en volume dans l'équation homogénéisée.

Version française abrégée

Dans cette Note, on traite le cas de la diffusion linéaire dans un milieu périodique fracturé. Nous montrons que l'effet des fractures sur l'énergie du système peut être modélisé par un terme source en volume au sein du milieu homogénéisé. L'analyse asymptotique du problème de départ (1)-(2) conduit à la formulation du modèle homogénéisé (10)-(11) pour lequel il est nécessaire d'introduire des conditions de transmission à l'interface correspondant à la singularité en pointe des fissures. Nous montrons en particulier que la solution u, et son gradient $\partial_n u$, y subissent un saut.

Nous mettons au point une approche par point fixe consistant à résoudre successivement (13-gauche) dans le sous-domaine Ω^+ intact , et (13-droite) dans le sous-domaine Ω^- contenant les fractures (voir

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figure (1)). Nous montrons ainsi que le problème (10)-(11) est bien posé. Cela nous donne au passage une méthode de construction de la solution, que nous exploiterons à des fins numériques.

Nous énonçons alors la proposition 2.2 où l'on établit que la solution du problème exact (1)-(2) défini dans la géométrie fracturée Ω_ε converge faiblement vers la solution de (10)-(11) défini dans le domaine homogénéisé Ω, ne contenant plus la description des fractures (voir figure (1)). La preuve rigoureuse de la proposition 2.2 sera exposée dans une publication ultérieure [6].

Enfin, nous appliquons la méthode du point fixe précédemment décrite pour résoudre numériquement (10)-(11). Nous développons une approche équivalente en écrivant la formulation faible du problème écrite dans tout le domaine homogénéisé conduisant à l'équation (14), caractérisée par la présence d'une masse de Dirac localisée en pointe de fissure. Ces deux méthodes donnent des résultats cohérents avec le calcul direct de la fracture.

1. Motivation and setting of the problem

We consider the propagation of heat through a cracked medium, exposed to an incoming energy flux.

Physically, the exchange surface between the medium and the source may be greatly modified by the fractures. This may have a significant impact on the energy balance of the considered system. In many situations, the geometry of the cracked media is too intricate to be described precisely. Thus, we cannot model the surface of the cracked medium directly. Besides, the shape of the fractures may have a stochastic feature and it may involve many spatial scales. Full numerical simulations of such multi-scaled media become hence infeasible.

That is why we have been looking for an **average** approach, to capture the effects of cracks in a homogenized medium. The model presented here is simple enough to be coupled to standard Finite Element codes. The physical idea behind the method developed in this Note, called "MOSAIC" (Method Of Sinks Averaging Inhomogeneous behavior of Cracked media), is to treat the flux enhancement induced by the crack as a volume **source term** in the homogenized energy equation. We will show that this can be justified rigorously by homogenization theory.

For the sake of simplicity, we shall assume a linear behavior law of the material [1].

The linear diffusion problem can thus be modeled by:

$$\begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \partial_n u_\varepsilon = 0 & \text{on } \Gamma_\varepsilon^0 \\ \partial_n u_\varepsilon = 1 & \text{on } \Gamma_\varepsilon^1, \\ \partial_n u_\varepsilon = \dfrac{\alpha-\beta}{2}\varepsilon & \text{on } \Gamma_\varepsilon^\alpha, \\ \partial_n u_\varepsilon = \dfrac{\beta}{\alpha} & \text{on } \Gamma_\varepsilon^\beta. \end{cases} \tag{1}$$

We also need an initial condition:

$$u_\varepsilon(x, y, t = 0) = u^0(x, y), \tag{2}$$

so that problem (1)-(2) is well-posed.

We impose $u_\varepsilon(x, y, t)$ to be periodic of period ε with respect to the variable y. The domain Ω_ε, as well as the boundaries $\Gamma_\varepsilon^0, \Gamma_\varepsilon^1, \Gamma_\varepsilon^\alpha, \Gamma_\varepsilon^\beta$ are defined on Figure 1. As suggested by the red lines, the left part of the figure is a zoom in the y variable of the small shaded area in Ω.

The period $\varepsilon > 0$ is supposed to be small and will tend to 0, whereas $\alpha \in [0, 1)$ is a fixed parameter related to the width of the crack. The parameter $\beta \in [0, \alpha)$ measures the portion of the flux which,

1. But the results given here could be extended to the non-linear (power law) case.

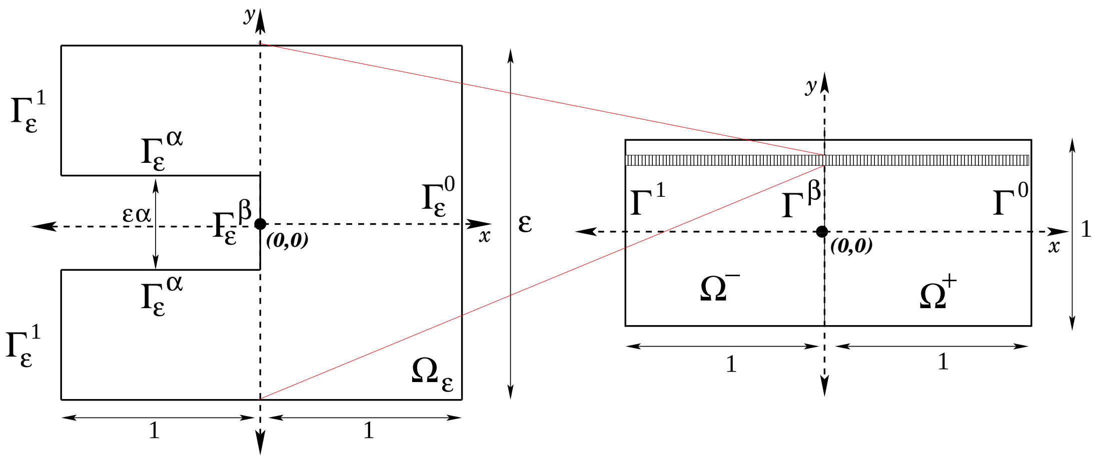


Figure 1. The cracked domain Ω_ε and the homogenized domain $\Omega = \Omega^+ \cup \Omega^-$.

coming through the segment $\{x = -1, -\alpha\varepsilon/2 < y < \alpha\varepsilon/2\}$, reaches the bottom Γ_ε^β of the crack. The remaining part of the incoming flux is distributed on the horizontal part of the boundary, namely $\Gamma_\varepsilon^\alpha$. The parameter β is supposed to be fixed. The boundary conditions in (1) are defined in such a way that the total incoming flux is exactly equal to 1, which is the value we impose on the left boundary in the case $\alpha = 0$ (no crack). A space-time dependence of the flux applied on the boundaries $\Gamma_\varepsilon^\alpha$ may be introduced but it does not affect the homogenization process that we describe here.

Similar homogeneization problems have already been tackled in [2], [3] or [4] but the geometry, the equation type as well as the boudary conditions were not the same as the one considered here.

2. Asymptotic expansion and homogenized equation

To carry out an asymptotic expansion of the solution u_ε of (1) in powers of ε, we "scale" the variable y, in the spirit of [1]. Actually, 2 scales describe the model: the variable y is the macroscopic one, whereas $\dfrac{y}{\varepsilon}$ represents the "microscopic geometry". Thus, we define:

$$u_\varepsilon(x, y, t) = v_\varepsilon\left(x, \frac{y}{\varepsilon}, t\right),$$

so that v_ε is periodic of period 1 in y. Applying the change of variable in (1), we can write the system satisfied by v_ε:

$$\begin{cases} -\partial_x^2 v_\varepsilon - \dfrac{1}{\varepsilon^2}\partial_y^2 v_\varepsilon + \partial_t v_\varepsilon = 0 & \text{in } \Omega_1, \\ \partial_n v_\varepsilon = 0 & \text{on } \Gamma_1^0, \\ \partial_n v_\varepsilon = 1 & \text{on } \Gamma_1^1, \\ \dfrac{1}{\varepsilon^2}\partial_n v_\varepsilon = \dfrac{\alpha-\beta}{2} & \text{on } \Gamma_1^\alpha, \\ \partial_n v_\varepsilon = \dfrac{\beta}{\alpha} & \text{on } \Gamma_1^\beta. \end{cases} \tag{3}$$

Firstly, we notice that in the system (3) the domain Ω_1 does not depend on ε anymore. We have to study an equation depending on ε in a fixed domain. Secondly, the parameter ε appears in the equation (3) only as ε^2, which means that ε^2 is a good parameter for an asymptotic expansion. Thus, it seems natural to look for v_ε as follows:

$$v_\varepsilon(x,y,t) = v_0(x,y,t) + \varepsilon^2 v_1(x,y,t) + \varepsilon^4 v_2(x,y,t) + \dots \tag{4}$$

Hence, we insert the ansatz (4) into the system (3) and identify the different powers of ε^2. We obtain:

– <u>At the order ε^{-2} :</u>

$$\partial_y^2 v_0 = 0,$$

and the condition of periodicity in y verified by v: $v_0(x,1/2,t) = v_0(x,-1/2,t)$, implies that v_0 does not depend on y :

$$v_0(x,y,t) = v_0(x,t). \tag{5}$$

– <u>At the order ε^0 :</u>

$$-\partial_x^2 v_0 - \partial_y^2 v_1 + \partial_t v_0 = 0. \tag{6}$$

The boundary conditions on v_1 give

$$\partial_n v_1 = 0 \text{ on } \Gamma_1^0, \quad \partial_n v_1 = 0 \text{ on } \Gamma_1^1, \quad \partial_n v_1 = \frac{\alpha-\beta}{2} \text{ on } \Gamma_1^\alpha, \quad \partial_n v_1 = 0 \text{ on } \Gamma_1^\beta.$$

Integrating (6) with respect to y and using the boundary value for $\partial_n v_1$ as well as the periodicity in y, we get:

$$\begin{cases} -\partial_x^2 v_0 + \partial_t v_0 = \dfrac{\alpha-\beta}{1-\alpha} & \text{in } \Omega_1 \cap \{x<0\}, \\ \partial_n v_0 = 1 & \text{on } \Gamma_1^1, \\ \partial_n v_0 = 0 & \text{on } \Gamma_1^\alpha, \end{cases} \qquad \begin{cases} -\partial_x^2 v_0 + \partial_t v_0 = 0 & \text{in } \Omega_1 \cap \{x>0\}, \\ \partial_n v_0 = 0 & \text{on } \Gamma_1^0, \\ \partial_n v_0 = \dfrac{\beta}{\alpha} & \text{on } \Gamma_1^\beta. \end{cases} \tag{7}$$

This system is not well-posed, since boundary conditions are missing at the interface $\{x=0\}$. Actually, we can show that the flux $\partial_x v_0$ is not continuous accross the interface $\{x=0\}$, so that we need to introduce a boundary layer at this interface. This is done by changing variables once again and defining $v_\varepsilon(x,y,t) = w_\varepsilon\left(\frac{x}{\varepsilon}, y, t\right)$. This function w_ε is thus defined on the set

$$\widetilde{\Omega}_\varepsilon := \left\{(x,y) \in \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right), \quad |y| > \frac{\alpha}{2} \text{ if } x<0\right\}.$$

On this set, w_ε satisfies the equation

$$\varepsilon^2 \partial_t w_\varepsilon - \partial_x^2 w_\varepsilon - \partial_y^2 w_\varepsilon = 0,$$

with the boundary conditions $\partial_x w_\varepsilon = -\frac{\varepsilon\beta}{\alpha}$ on $\{x=0\} \cap \{|y| < \alpha/2\}$. Integrating this equation in the domain $\widetilde{\Omega}_\varepsilon \cap \{|x| < \delta\}$, letting ε, then δ, go to zero, we find that

$$\lim_{\varepsilon\to 0}\left(\int_{\{x=0^+\}} \partial_x w_\varepsilon\right) = \lim_{\varepsilon\to 0}\left(\int_{\{x=0^-, \alpha/2<|y|<1/2\}} \partial_x w_\varepsilon - \varepsilon\beta\right).$$

Recalling the link between w_ε and v_ε, this leads to the so-called transmission conditions (8):

$$v_0(x=0^-) = v_0(x=0^+), \quad (1-\alpha)\partial_x v_0(x=0^-) = \partial_x v_0(x=0^+) + \beta. \tag{8}$$

on $\{x=0\} \setminus \Gamma_1^\beta = \{x=0\} \cap \{|y| > \alpha/2\}$.

Making use of:

$$v\left(\frac{y}{\varepsilon}\right) \overset{*}{\rightharpoonup} \int_{-1/2}^{1/2} v(y)dy, \tag{9}$$

in L^∞ for any periodic function v, together with the fact that

$$u_\varepsilon(x,y,t) \approx v_0\left(x, \frac{y}{\varepsilon}, t\right),$$

we find that, in the limit $\varepsilon \to 0$, u_ε is well approximated (in the weak sense) by the solution u of

$$\begin{cases} -\Delta u + \partial_t u = \alpha - \beta & \text{in } \Omega^-, \\ \partial_n u = 1-\alpha & \text{on } \Gamma^1, \\ u \text{ is } 1-\text{periodic in } y. \end{cases} \qquad \begin{cases} -\Delta u + \partial_t u = 0 & \text{in } \Omega^+, \\ \partial_n u = 0 & \text{on } \Gamma^0, \\ u \text{ is } 1-\text{periodic in } y. \end{cases} \tag{10}$$

We also have the corresponding transmission condition inherited from (8):

$$u(x=0^-) = (1-\alpha)u(x=0^+), \quad \partial_x u(x=0^-) = \partial_x u(x=0^+) + \beta. \tag{11}$$

u evolves in the homogenized domain $\Omega = \Omega^+ \cup \Omega^-$ with the boundaries $\Gamma^1, \Gamma^\beta, \Gamma^0$ defined on Figure 1.

Remark 1 *Of course, if* u_0 *does not depend on* y*, the solution of* (10)-(11) *is constant with respect to* y*. However, it is also possible to apply an incoming flux wich depends on* y*, that is, impose* $\partial_n u_\varepsilon = F(y)$ *on* Γ^1_ε*, in system* (1)*. In such a case, the above analysis is still valid, and the solution a priori depends on* y*, even if* u_0 *does not.*

Proposition 2.1 *Assume that* $\alpha \in (0,1)$*. Then for any* $T > 0$*, problem (10)-(11) has a unique solution* $(u_-, u_+) \in X$*, where*

$$X = C^0\left([0,T], H^1(\Omega^-)\right) \cap C^1\left([0,T], L^2(\Omega^-)\right) \times C^0\left([0,T], H^1(\Omega^+)\right) \cap C^1\left([0,T], L^2(\Omega^+)\right).$$

Proposition 2.2 *Let* u_ε *be the unique solution to (1). We extend it by* 0 *outside* Ω_ε*, and assume that the initial data* $u_\varepsilon(t=0)$ *is such that* $u_\varepsilon(t=0) \rightharpoonup u_0$ *in* $L^2(\Omega)$*. Then, for any* $T > 0$*, we have*

$$u_\varepsilon \underset{\varepsilon \to 0}{\rightharpoonup} u \text{ in } L^2(\Omega \times [0,T]), \tag{12}$$

where u *is the unique solution to (10)-(11).*

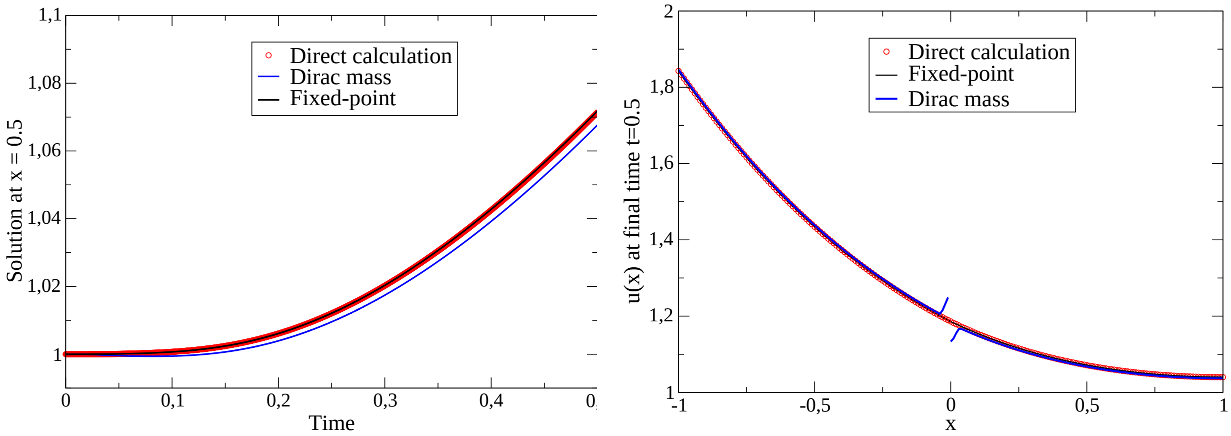
3. Building of the solution

3.1. *Fixed-point approach*

To compute the solution of the homogenized problem (10)-(11) , we firstly devise a fixed-point approach connecting the sub-domains Ω^+ and Ω^-:

$$\begin{cases} -\Delta u + \partial_t u = 0 & \text{in } \Omega^+, \\ \partial_n u = 0 & \text{on } \Gamma^0, \\ \partial_n u = F & \text{on } \Gamma^\beta, \\ u \text{ is } 1-\text{periodic in } y, \\ u(t=0) = u_0 & \text{in } \Omega^+. \end{cases} \qquad \begin{cases} -\Delta u + \partial_t u = \alpha - \beta & \text{in } \Omega^-, \\ \partial_n u = 1-\alpha & \text{on } \Gamma^1, \\ u = g & \text{on } \Gamma^\beta, \\ u \text{ is } 1-\text{periodic in } y, \\ u(t=0) = u_0 & \text{in } \Omega^-. \end{cases} \tag{13}$$

We start with an initial guess for the flux $F^0 \in L^2(\Gamma^\beta)$, to which we associate the solution $u^{+,0}$ of (13-left). Then, define g^0 as the trace of $(1-\alpha)u^{+,0}$ on Γ^β, and solve (13-right) with data $g = g^0$: this defines $u^{-,0}$, and a new flux $F^1 = \beta - \partial_n u^{-,0}$ in $L^2(\Gamma^\beta)$. Repeating this procedure, we build a sequence converging towards the solution of (10)-(11). This scheme is actually used to prove Proposition 2.1.



(a) **Time evolution of** $u(0.5, 0)$.　　(b) **Spatial profile at time** $t_1 = 0.5$.

Figure 2. **Comparison of the direct and homogenized approaches for** $\alpha = 0.1$. *The red curve is the direct calculation of the crack, i.e the limit solution of (1) as* $\varepsilon \to 0$. *The two other curves correspond to the solution of the homogenized problem (10) computed by two approaches: in black, it is the fixed-point method whereas the blue curve represents the numerical solution of the weak formulation (14) involving a Dirac mass at* $x = 0$. *Computations presented here are performed using a* P^1 *finite element approximation on triangular meshes. It has been implemented using the software FreeFem++ [5].*

3.2. *Weak formulation approach*

Another possible way to solve (10)-(11) is to compute the weak formulation of the problem in the whole domain Ω corresponding to equation (14) :

$$\begin{cases} -\Delta u + \partial_t u = (\alpha - \beta)\mathbf{1}_{\{x<0\}} - \alpha \partial_x \left(u(x = 0^+)\delta_{x=0} \right) + \beta \delta_{x=0} & \text{in } \Omega, \\ \partial_n u = 1 - \alpha & \text{on } \Gamma^1, \\ \partial_n u = 0 & \text{on } \Gamma^0, \\ u \text{ est } 1 - \text{periodic in } y. \end{cases} \tag{14}$$

A Dirac mass at $\{x = 0\}$ has appeared in (14). This represents the singularity at the bottom of the crack.

Those two methods are illustrated on figure (2) and compared with the direct calculation. We can note that the fixed-point method appears to be more accurate than the solution of the weak formulation (14) especially as α increases. This is due to the fact that (14) involves a Dirac mass at $x = 0$, proportional to the width of the crack α. This term is treated approximately in our finite element simulation and leads to more significant errors for greater α. In some way, the fixed-point method amounts to treat the Dirac mass at $x = 0$ exactly.

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