Convex order for path-dependent derivatives: a dynamic programming approach

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Abstract

We investigate the (functional) convex order of for various continuous martingale processes, either with respect to their diffusions coefficients for Lévy-driven SDEs or their integrands for stochastic integrals. Main results are bordered by counterexamples. Various upper and lower bounds can be derived for pathwise European option prices in local volatility models. In view of numerical applications, we adopt a systematic (and symmetric) methodology: (a) propagate the convexity in a *simulatable* dominating/dominated discrete time model through a backward induction (or linear dynamical principle); (b) Apply functional weak convergence results to numerical schemes/time discretizations of the continuous time martingale satisfying (a) in order to transfer the convex order properties. Various bounds are derived for European options written on convex pathwise dependent payoffs. We retrieve and extend former results obtains by several authors ([8, 2, 15, 13]) since the seminal paper [10] by Hajek. In a second part, we extend this approach to Optimal Stopping problems using a that the Snell envelope satisfies (a') a Backward Dynamical Programming Principle to propagate convexity in discrete time; (b') satisfies abstract convergence results under non-degeneracy assumption on filtrations. Applications to the comparison of American option prices on convex pathwise payoff processes are given obtained by a purely probabilistic arguments.

Keywords. Convex order ; local volatility models ; Itô processes ; Lévy-Itô processes ; Laplace transform ; Lévy processes ; completely monotone functions ; pathwise European options ; pathwise American options ; comparison of option prices.

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1 Introduction

The first aim of this paper is to propose a systematic and unified approach to establish *functional convex order* results for discrete and continuous time martingale stochastic processes using the propagation of convexity through some kind of backward dynamic programing principles (in discrete time) and weak functional limit theorems (to switch to continuous time. The term "functional" mainly refers to the "parameter" we deal with: thus, for diffusions processes (possibly with jumps) this parameter is the diffusion coefficient or, for stochastic integrals, their integrand. Doing so we will retrieve, extend and sometimes establish new results on functional convex order. As a second step, we will tackle the same type of question in the framework of Optimal Stopping Theory for the Snell envelopes and

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their means, the so-called *réduites* (which maybe provides a better justification for the terminology "dynamic programming approach" used in the title).

Let us first briefly recall that if X and Y are two real-valued random variables, X is dominated by Y for the convex order – denoted $X \preceq_c Y$ – if, for every convex functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(X),\ f(Y) \in L^1(\mathbb{P})$,

$$\mathbb{E}\, f(X) \le \mathbb{E}\, f(Y).$$

Thus, if $(M_\lambda)_{\lambda>0}$ denotes a martingale indexed by a parameter λ, then $\lambda \mapsto M_\lambda$ is non-decreasing for the convex order as a straightforward consequence of Jensen's Inequality. The converse is clearly not true but, as first established by Kellerer in [22], whenever $\lambda \mapsto X_\lambda$ is non-decreasing for the convex order, there exists a martingale $(\widetilde{X}_\lambda)_{\lambda\ge 0}$ such that $X_\lambda \stackrel{d}{=} \widetilde{X}_\lambda$ for every $\lambda \ge 0$ (we will say that (X_λ) and $(\widetilde{X}_\lambda)$ coincide en 1-marginal distributions.

The connection with Finance and, to be more precise with the pricing and hedging of derivative products is straightforward : let $(X^{(\theta)}_t)_{t\in[0,T]}$ be a family of non-negative $\mathbb{P}$-martingales on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ indexed by a parameter θ. Such a family can be seen as possible models for the discounted price dynamics of a risky asset under its/a risk-neutral probability where θ (temporarily) is a real parameter (*e.g.* representative of the volatility). If $\theta \mapsto X^{(\theta)}_T$ is non-decreasing for the convex order, then for every convex *vanilla payoff* function $f : \mathbb{R}_+ \to \mathbb{R}_+$, the function $\theta \mapsto \mathbb{E}\, f(X^{(\theta)}_T)$ is non-decreasing or equivalently its *greek* $\frac{\partial}{\partial\theta}\mathbb{E} f(X^{(\theta)}_T)$ with respect to θ is non-negative. Typically, in a discounted Black-Scholes model

$$X^{\sigma,x}_t = x e^{\sigma W_t - \frac{\sigma^2}{2}t},\ x,\ \sigma > 0,$$

the function $\sigma \longmapsto \mathbb{E} f\big(x\, e^{\sigma W_T - \frac{\sigma^2 T}{2}}\big)$ since

$$\forall\, t\in [0,T], \quad x\, e^{\sigma W_T - \frac{\sigma^2 T}{2}} \stackrel{\mathcal{L}}{\sim} \Big[e^{B_u - \frac{u}{2}}\Big]_{|u=\sigma^2 T}$$

where $u \mapsto e^{B_u - \frac{u}{2}}$ is a martingale as as well as its composition with $\sigma \mapsto \sigma^2 T$. So $(X^{\sigma,x}_T)_{\sigma\ge 0}$coincides in 1-marginal distributions with a martingale. The same result holds true for the premium of convex *Asian payoff* functions of the form

$$\mathbb{E}\, f\left(\frac{1}{T}\int_0^T x e^{\sigma W_t - \frac{\sigma^2 t}{2}} dt\right)$$

but, by contrast, its proof is significantly more involved (see [6] or, more recently, the proof in [13] where an explicit martingale based on the Brownian sheet coinciding in 1-dimensional martingale is exhibited). Both results turn out to be examples of a general result dealing with convex pathwise dependent functionals (see *e.g.* [13] or [31] where a functional co-monotony argument is used).

A natural question at this stage is to try establishing a *functional version* of these results in terms of θ-parameter *i.e.* when θ is longer a real number or a vector but lives in a subset of a functional space or even of space of stochastic processes. A typical example where θ is a function is the case where $X^{(\theta)}$ is a diffusion process, (weak) solution to a Stochastic Differential equation (SDE) of the form

$$dX^{(\theta)}_t = \theta(t, X^{(\theta)}_{t-})dZ_t,\ X^{(\theta)}_0 = x,\ t\in [0,T],$$

with $Z = (Z_t)_{t\in[0,T]}$ a martingale Lévy process (having moments of order at least 1). The parameter θ can also be a (predictable) stochastic process when

$$X^{(\theta)}_t = \int_0^t \theta_s dZ_s,\ t\in [0,T].$$

When dealing with optimal stopping problems, *i.e.* with the *réduite* of a target process $Y_t = F(t, X^{(\theta),t})$, $t \in [0,T]$, (where $X_s^{(\theta),t} = X_{s\wedge t}^{(\theta)}$ is the stopped process $X^{(\theta)}$ at t) and all the functionals $F(t,.)$ are (continuous) convex functionals defined on the path space of the process X, the *functional convex order* as defined above amounts to determine the sign of the sensitivity with respect to the functional parameter θ of an American option with payoff functional $F(t,.)$ at time $t \in [0,T]$, "written" on $X^{(\theta)}$: if the holder of the American option contract exercises the option at time t, she receives $F(t, X^{(\theta),t})$.

More generally, various notions of convex order in Finance are closely related to risk modeling and come out in many other frameworks than the pricing and hedging of derivatives.

Many of these questions have already been investigated for a long time: thus, the first result known to us goes back to Hajek in [10] where convex order is established for Brownian martingale diffusions "parametrized" by their (convex) diffusion coefficients (with an extension to drifted diffusions with non-decreasing convex drifts but with a restriction to non-decreasing convex functions f of X_T). The first application to the sensitivity of (vanilla) options of both European and American style, is due to [8]. It is shown that the options with convex payoffs in a $[\sigma_{\min}, \sigma_{\max}]$-valued local volatility model with bounded volatility can be upper- and lower-bounded by the premium of the the same option contracts in a Black-Scholes model with volatilities $\sigma_{\min}$ and $\sigma_{\max}$ respectively (note however that a PDE approach relying on a maximal principle provides an alternative easier proof). See also [15] for a result on lookback options. More recently, in a series of papers (see [3, 2, 4]) Bergenthum and Rüschendorf extensively investigated the above mentioned problems (for both fixed maturity and for optimal stopping problemss) for various classes of continuous and jump processes, including general semi-martingales in [3] (where the comparison is carried out in terms of their predictable local characteristics, assuming one of them propagates convexity, then proving this last fact). In several of these papers, the convexity is – but not always (see [2]) – propagated directly in continuous time which is clearly an elegant way to proceed but also seems to more heavily rely on specific features of the investigated class of processes (see [13]). In this paper, we propose a generic and systematic systematic – but maybe also more "symmetric" – two-fold approach which turns out to be efficient for many classes of stochastic dynamics and processes which is based on a swathing from discrete times to continuous time using functional weak limit theorems "à la Jacod-Shyriaev" (see [18]). To be more precise:

– As first step, we study the propagation of convexity "through" a discrete time dynamics – typically a $GARCH$ model – in a very elementary way for path-dependent convex functionals relying on repeated elementary backward inductions and conditional Jensen's inequality. These inductions take advantage of the "linear" backward dynamical programming principle resulting from from a discrete time martingale property written in a step-by-step manner. This terminology borrowed from stochastic control can be viewed as a bit excessive but refers to a second aspect of the paper devoted to optimal stopping theory (see further on).

– As a second step, we use that these discrete time $GARCH$ model are discretization schemes for the "target" continuous time dynamics (typically the "Euler schemes as concerns diffusion processes) and we transfer to this target the searched functional convex order property by calling upon functional limit theorems for the convergence of stochastic integrals (typically borrowed form [20] and/or [23].

Our typical result for jump diffusions reads as follows (for a more complete and rigorous statements see *e.g.* Theorems 2.1 and 2.2 in Section 2). If $0 \le \kappa_1 \le \kappa \le \kappa_2$ are continuous functions with linear growth defined on $\mathbb{R}$ and κ is convex then the existing weak solutions $X^{(\kappa_i)}$, $i = 1, 2$, to the SDEs

$$X_t^{(\kappa_i)} = x + \int_{(0,t]} \kappa_i(X_{s-}^{(\kappa_i)})dZ_s$$

where $Z = (Z_t)_{t\in[0,T]}$ is a martingale Lévy process with Lévy measure ν satisfying $\nu(z^2) < +\infty$, then $X^{(\kappa_1)} \preceq_{fc} X^{(\kappa_2)}$ for the convex order defined on (continuous for the Skorokhod topology) convex functionals (with polynomial growth). Note that when Z is a Brownian motion, the continuity of the functional appears as a consequence of its convexity (under the polynomial growth assumption, see the remark in Section 2.1). Equivalently, we have $X^{(\kappa_1)} \preceq_{fc} X^{(\kappa)} \preceq_{fc} X^{(\kappa_2)}$ as soon as both functions κ_i are convex. Results in the same spirit are obtained for stochastic integrals, Doléans exponentials (which unfortunately requires one of the two integrands H_1 and H_2 to be deterministic). Counter-examples to put the main results in perspective are exhibited to prove the consistency of these assumptions in both settings.

We also deal with non-linear problems, typically optimal stopping problems, framework where we use the same approach from discrete to continuous time, taking advantage of the Backward Dynamic Programming Principle in the first framework and using various convergence results for the Snell envelope (see [25]). In fact, a similar approach in discrete time has already been been developed to solve the propagation of convexity in a stochastic control problem "through" the dynamic programming principle in a pioneering work by Hernández-Lerma and Runggaldier [11].

The main reason for developing in a systematic manner this approach is related with Numerical Probability: our discrete time models appear as *simulatable* discretization schemes of the continuous time dynamics of interest. It is important for applications, especially in Finance, to have at hand discretization schemes which *both* preserve the (functional) convex order and can be simulated at a reasonable cost. So is the case of the Euler scheme for Lévy driven diffusions (as soon as the underling Lévy processes is itself simulatable). This is not always the case: think *e.g.* to the (second order) Milstein scheme for Brownian diffusions, in spite of its better performances in term of strong convergence rate.

The paper is organized as follows. Section 2 is devoted to functional convex order for path-dependent functionals of Brownian and Lévy driven martingale diffusion processes. Section 3 is devoted to comparison results for Itô processes based on comparison of their integrand. Section 4 deals with *réduites*, Snell envelopes of path dependent obstacle processes (American options) in both Brownian and Lévy driven martingale diffusions. In the two-fold appendix, we provide short proofs of functional weak convergence of the Euler scheme toward a weak solution of *SDE*s in both Brownian and Lévy frameworks under natural continuity-linear growth assumptions on the diffusion coefficient.

NOTATION: • For every $T > 0$ and every integer $n \geq 1$, one denotes the uniform mesh of $[0,T]$ by $t_k^n = \frac{kT}{n}$, $k = 0,\ldots,n$. Then for every $t \in [\frac{kT}{n}, \frac{(k+1)T}{n})$, we set $\underline{t}_n = \frac{kT}{n}$ and $\overline{t}^n = \frac{(k+1)T}{n}$ with the convention $\underline{T}_n = T$. We also set $\underline{t}_{n-} = \lim_{s\to t} \underline{s}_n = \frac{kT}{n}$ if $t \in \big(\frac{kT}{n}, \frac{(k+1)T}{n}\big]$.

• For every $u = (u_1,\ldots,u_d)$, $v = (v_1,\ldots,v_d) \in \mathbb{R}^d$, $(u|v) = \sum_{i=1}^d u_i v_i$, $|u| = \sqrt{(u|u)}$ and $x_{m:n} = (x_m,\ldots,x_n)$ (where $m \leq n$, $m, n \in \mathbb{N}\setminus\{0\}$).

• $\mathcal{F}([0,T],\mathbb{R})$ denotes the $\mathbb{R}$-vector space of $\mathbb{R}$-valued functions $f : [0,T] \to \mathbb{R}$ and $\mathcal{C}([0,T],\mathbb{R})$ denotes the subspace of $\mathbb{R}$-valued continuous functions defined over $[0,T]$.

• For every $\alpha \in \mathcal{F}([0,T],\mathbb{R})$, we define $\mathrm{Cont}(\alpha) = \big\{t \in [0,T] \,:\, \alpha \text{ is continuous at } t\big\}$ with the usual left- and right- continuity conventions at 0 and T respectively. We also define the *uniform continuity modulus* of α by where $w(\alpha,\delta) = \sup\big\{|\alpha(u)-\alpha(v)|,\, u,v \in [0,T], |u-v| \leq \delta\big\}$ ($\delta \in [0,T]$).

• $L^p_T = L^p([0,T],dt)$, $1 \leq p \leq +\infty$, $|f|_{L^p_T} = \big(\int_0^T |f(t)|^p dt\big)^{\frac{1}{p}} \leq +\infty$, $1 \leq p < +\infty$ and $|f|_{L^\infty_T} = dt\text{-esssup}|f|$ where dt stands for the Lebesgue measure on $[0,T]$ equipped with its Borel σ-field.

• For a function $f : [0,T] \to \mathbb{R}$, we denote $\|f\|_{\sup} = \sup_{u\in[0,T]} |f(u)|$.

• Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space and let $p \in (0,+\infty)$. For every random vector $X : (\Omega,\mathcal{A}) \to \mathbb{R}^d$ we set $\|X\|_p = \big(\mathbb{E}|X|^p\big)^{\frac{1}{p}}$. $L^p_{\mathbb{R}^d}(\Omega,\mathcal{A},\mathbb{P})$ denotes the vector space of (classes) of $\mathbb{R}^d$-valued random

vectors X such that $\|X\|_p < +\infty$. $\|\,.\,\|_p$ is a norm on $L^p_{\mathbb{R}^d}(\Omega, \mathcal{A}, \mathbb{P})$ for $p \in [1, +\infty)$ (the mention of Ω, $\mathcal{A}$ and the subscript $_{\mathbb{R}^d}$ will be dropped when there is no ambiguity).
• If $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ denotes a filtration on $(\Omega, \mathcal{A}, \mathbb{P})$, let $\mathcal{T}^{\mathcal{F}}_{[0,T]} = \{\tau : \Omega \to [0,T], \mathcal{F}\text{-stopping time}\}$.
• $\mathcal{F}^Y = (\mathcal{F}^Y_t)_{t\in[0,T]}$ is the smallest right continuous filtration $(\mathcal{G}_t)_{t\in[0,T]}$ that makes the process $Y = (Y_t)_{t\in[0,T]}$ a $(\mathcal{G}_t)_{t\in[0,T]}$-adapted process.
• $I\!D([0,T], \mathbb{R}^d)$ denotes the set of $\mathbb{R}^d$-valued right continuous left limited (or càdlàg following the French acronym) function defined on the interval $[0,T]$, $T > 0$. It is usually endowed with the Skorokhod topology denoted Sk (see [17], chapter VI or [1], chapter 3, for an introduction to Skorokhod topology).
• If two random vectors U and V have the same distribution, we write $U \overset{d}{\sim} V$. If an (S, d_S)-valued sequence of random variable ((S,d) Polish space equipped with its Borel σ-field $\mathcal{B}or(S)$) *weakly converges* toward an (S,d)-valued random variable Y_∞ (we will also say *converge in distribution or in law*), we will denote $Y_n \overset{\mathcal{L}(S,d_S)}{\longrightarrow} Y_\infty$ or, if no ambiguity, $Y_n \overset{\mathcal{L}(d_S)}{\longrightarrow} Y_\infty$.

We will extensively make use the following classical result:

Let $(Y_n)_{n\geq 1}$be a sequence of tight random variables taking values in a Polish space (S, d_S) (see [1], Chapter 1). If Y_n weakly converges toward Y_∞ and $(\Phi(Y_n))_{n\geq 1}$ is uniformly integrable where $\Phi : S \to \mathbb{R}$ is a Borel function then, for every $\mathbb{P}_{Y_\infty}$-*a.s.* continuous Borel functional $F : S \to \mathbb{R}$ such that $|F(u)| \leq C(1 + \Phi(u))$ for every $u \in S$, one has $\mathbb{E}\, F(Y_n) \to \mathbb{E}\, F(Y_\infty)$.

2 Functional convex order

2.1 Brownian martingale diffusion

The main result of this section is the proposition below.

Theorem 2.1. *Let σ, $\theta : [0,T] \times \mathbb{R} \to \mathbb{R}$ be two continuous functions with linear growth in x uniformly in $t \in [0,T]$. Let $X^{(\sigma)}$ and $X^{(\theta)}$ be two Brownian martingale diffusions, supposed to be the unique* weak *solutions starting from x at time 0, to the stochastic differential equations (with 0 drift)*

$$dX^{(\sigma)}_t = \sigma(t, X^{(\sigma)}_t) dW^{(\sigma)}_t,\ X^{(\sigma)}_0 = x \quad \text{and} \quad dX^{(\theta)}_t = \theta(t, X^{(\theta)}_t) dW^{(\theta)}_t,\ X^{(\theta)}_0 = x \tag{2.1}$$

respectively, where $W^{(\sigma)}$ and $W^{(\theta)}$ are standard one dimensional Brownian motions.
(a) Partitioning assumption*: Let $\kappa : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ be a continuous function with (at most) linear growth in x uniformly in $t \in [0,T]$, satisfying*

$$\kappa(t, .) \ \text{ is convex for every } t \in [0,T] \text{ and } \ 0 \leq \sigma \leq \kappa \leq \theta.$$

Then, for every convex functional $F : \mathcal{C}([0,T], \mathbb{R}) \to \mathbb{R}$ with $(r, \|\,.\,\|_{\sup})$-polynomial growth, $r \geq 1$, in the following sense

$$\forall\, \alpha \in \mathcal{C}([0,T], \mathbb{R}), \quad |F(\alpha)| \leq C(1 + \|\alpha\|^r_{\sup})$$

one has

$$\mathbb{E}\, F(X^{(\sigma)}) \leq \mathbb{E}\, F(X^{(\theta)}).$$

From now on, the function κ is called a partitioning function*.*
(a') *Claim (a) can be reformulated equivalently as follows: if either $\sigma(t, .)$ is convex for every $t \in [0,T]$ or $\theta(t, .)$ is convex for every $t \in [0,T]$ and $0 \leq \sigma \leq \theta$, then the conclusion of (a) still holds true.*
(b) Domination assumption*: If $|\sigma| \leq \theta$ and θ is convex, then*

$$\mathbb{E}\, F(X^{(\sigma)}) \leq \mathbb{E}\, F(X^{(\theta)}).$$

Remarks. • The linear growth assumption on the convex functional F implies its everywhere local boundedness on the Banach space $\big(\mathcal{C}([0,T],\mathbb{R}), \|\,.\,\|_{\sup}\big)$, hence its $\|\,.\,\|_{\sup}$-continuity (see *e.g.* Lemma 2.1.1 in [27], p.22).

• The introduction of two standard Brownian motions $W^{(\sigma)}$ and $W^{(\theta)}$ in the above claim (a) is just a way to recall that the two diffusions processes can be defined on different probability spaces, although it may be considered as an abuse of notation. By "unique weak solutions", we mean classically that two such solutions (with respect to possibly different Brownian motions) share the same distribution on the Wiener space.

• Weak uniqueness holds true as soon as strong uniqueness holds *e.g.* as soon as σ and θ are Lipschitz continuous in x, uniformly in $t \in [0,T]$, (as it can easily be derived from Theorem A.3.3, p.271, in [5]).

The proof of this theorem can be decomposed in two main steps: the first one is a dynamic programming approach in discrete time detailed in Proposition 2.1 below which relies itself on a revisited version of Jensen's Inequality. The second one remiss on a functional weak approximation argument.

The first ingredient is a simple reinterpretation of the celebrated Jensen Lemma.

Lemma 2.1. *(Revisited Jensen's Lemma) Let $Z : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$ be an integrable centered $\mathbb{R}$-valued random vector.*

(a) *Assume that $Z \in L^r(\mathbb{P})$ for an $r \geq 1$. For every Borel function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $|\varphi(x)| \leq C(1+|x|^r)$, $x \in \mathbb{R}$, we define*

$$\forall\, u \in \mathbb{R},\; Q\varphi(u) = \mathbb{E}\,\varphi\big(uZ\big). \tag{2.2}$$

If φ is convex, then, $Q\varphi$ is convex and $u \mapsto Q\varphi(u)$ is non-decreasing on $\mathbb{R}_+$, non-increasing on $\mathbb{R}_-$.

(b) *If Z has exponential moments in the sense that*

$$\forall\, u \in \mathbb{R},\; \mathbb{E}(e^{uZ}) < +\infty$$

(or equivalently $\mathbb{E}(e^{a|Z|}) < +\infty$ for every $a \geq 0$), then claim (a) holds true for any convex function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying an exponential growth condition of the form $|\varphi(x)| \leq Ce^{C|x|}$, $x \in \mathbb{R}$, for a real constant $C \geq 0$.

(c) *If Z has a symmetric distribution (i.e. Z and $-Z$ have the same distribution) and $\varphi : \mathbb{R} \to \mathbb{R}$ is convex, then $Q\varphi$ is an even function, hence satisfying the following* maximum principle*:*

$$\forall\, a \in \mathbb{R}_+, \quad \sup_{|u| \leq a} Q\varphi(u) = Q\varphi(a).$$

Proof. (a)-(b) Existence and convexity of $Q\varphi$ are obvious. The function $Q\varphi$ is clearly finite on $\mathbb{R}$ and convex. Furthermore, Jensen's Inequality implies that

$$Q\varphi(u) = \mathbb{E}\,\varphi(u\,Z) \geq \varphi(\mathbb{E}\,u\,Z) = \varphi(0) = Q\varphi(0)$$

since Z is centered. Hence $Q\,\varphi$ is convex and minimum at $u = 0$ which implies that it is non-increasing on $\mathbb{R}_-$ and non-decreasing on $\mathbb{R}_+$.

(c) is obvious. □

Proposition 2.1. *Let $(Z_k)_{1 \leq k \leq n}$ be a sequence of independent, centered, $\mathbb{R}$-valued random vectors lying in $L^r(\Omega, \mathcal{A}, \mathbb{P})$, $r \geq 1$, and let $(\mathcal{F}^Z_k)_{0 \leq k \leq n}$ denote its natural filtration. Let $(X_k)_{0 \leq k \leq n}$ and $(Y_k)_{0 \leq k \leq n}$ be two sequences of random vectors recursively defined by*

$$X_{k+1} = X_k + \sigma_k(X_k)Z_{k+1},\; Y_{k+1} = Y_k + \theta_k(Y_k)Z_{k+1},\; 0 \leq k \leq n-1,\; X_0 = Y_0 = x \tag{2.3}$$

where σ_k, $\theta_k:\mathbb{R}\to\mathbb{R}$, $k=0,\ldots,n-1$, are Borel functions with linear growth i.e. $|\sigma_k(x)|+|\theta_k(x)|\le C(1+|x|)$, $x\in\mathbb{R}$, for a real constant $C\ge 0$.

(a) Assume that, either σ_k is convex for every $k\in\{0,\ldots,n-1\}$, or θ_k is convex for every $k\in\{0,\ldots,n-1\}$, and that

$$\forall\, k\in\{0,\ldots,n-1\},\ 0\le\sigma_k\le\theta_k.$$

Then, for every convex function $\Phi:\mathbb{R}^{n+1}\to\mathbb{R}$ with r-polynomial growth, $r\ge 1$, i.e. satisfying $|\Phi(x)|\le C(1+|x|^r)$, $x\in\mathbb{R}$, for a real constant $C\ge 0$,

$$\mathbb{E}\,\Phi(X_{0:n})\le\mathbb{E}\,\Phi(Y_{0:n}).$$

(b) If the random variables Z_k have symmetric distributions, if the functions θ_k are all convex and if

$$\forall\, k\in\{0,\ldots,n-1\},\ |\sigma_k|\le\theta_k,$$

then the conclusion of claim (a) remains valid.

Proof. (a) First one shows by an easy induction that the random variables X_k and Y_k all lie in L^r. Let Q_k, $k=1,\ldots,n$, denote the operator attached to Z_k by (2.2) in Lemma 2.1.

Then, one defines the following martingales

$$M_k=\mathbb{E}\big(\Phi(X_{0:n})\,|\,\mathcal{F}_k^Z\big)\quad\text{ and }\quad N_k=\mathbb{E}\big(\Phi(Y_{0:n})\,|\,\mathcal{F}_k^Z\big),\ 0\le k\le n.$$

Their existence follows from the growth assumptions on Φ, σ_k and θ_k, $k=1,\ldots,n$. Now we define recursively in a backward way two sequences of functions Φ_k and $\Psi_k:\mathbb{R}^{k+1}\to\mathbb{R}$, $k=0,\ldots,n$,

$$\Phi_n=\Phi\ \text{ and }\ \Phi_k(x_{0:k})=\big(Q_{k+1}\Phi_{k+1}(x_{0:k},x_k+.)\big)(\sigma_k(x_k)),\ x_{0:k}\in\mathbb{R}^{k+1},\ k=0,\ldots,n-1,$$

on the one hand and, on the other hand,

$$\Psi_n=\Phi\ \text{ and }\ \Psi_k(x_{0:k})=\big(Q_{k+1}\Psi_{k+1}(x_{0:k},x_k+.)\big)(\theta_k(x_k)),\ x_{0:k}\in\mathbb{R}^{k+1},\ k=0,\ldots,n-1.$$

This can be seen as a *linear Backward Dynamical Programming Principle*. It is clear by a (first) backward induction and the definition of the operators Q_k that, for every $k\in\{0,\ldots,n\}$,

$$M_k=\Phi_k(X_{0:k})\quad\text{ and }\quad N_k=\Psi(Y_{0:k}).$$

Let $k\in\{0,\ldots,n-1\}$. One derives from the properties of the operator Q_{k+1} (and the representation below as an expectation) that, for any convex function $G:\mathbb{R}^{k+2}\to\mathbb{R}$ with r-polynomial growth, $r\ge 0$, the function

$$\widetilde{G}:(x_{0:k},u)\longmapsto\big(Q_{k+1}G(x_{0:k},x_k+\ .)\big)(u)=\mathbb{E}\,G(x_{0:k},x_k+uZ_{k+1})\tag{2.4}$$

is convex. Moreover, owing to Lemma 2.1(a), for fixed $x_{0:k}$, $\widetilde{G}$ is non-increasing on $(-\infty,0)$, non-decreasing on $(0,+\infty)$ as a function of u. In turn, this implies that, if $\gamma:\mathbb{R}\to\mathbb{R}_+$ is convex (and non-negative), then $\xi\mapsto\widetilde{G}\circ\gamma(\xi)=Q_{k+1}G(x_{0:k},x_k+.)\big(\gamma(\xi)\big)$ is convex in ξ.

$\triangleright$ Assume all the functions σ_k, $k=0,\ldots,n-1$, are non-negative and convex. One shows by a (second) backward induction that the functions Φ_k are all convex.

Finally, we prove that $\Phi_k\le\Psi_k$ for every $k=0,\ldots,n-1$, using again a (third) backward induction on k. First note that $\Phi_n=\Psi_n=\Phi$. If $\Phi_{k+1}\le\Psi_{k+1}$, then

$$\begin{aligned}\Phi_k(x_{0:k})=\big(Q_{k+1}\Phi_{k+1}(x_{0:k},x_k+.)\big)(\sigma_k(x_k))&\le\big(Q_{k+1}\Phi_{k+1}(x_{0:k},x_k+.)\big)(\theta_k(x_k))\\&\le\big(Q_{k+1}\Psi_{k+1}(x_{0:k},x_k+.)\big)(\theta_k(x_k))=\Psi_k(x_{0:k}).\end{aligned}$$

In particular, when $k = 0$, we get $\Phi_0(x) \le \Psi_0(x)$ or, equivalently, taking advantage of the martingale property, $\mathbb{E}\,\Phi(X_{0:n}) \le \mathbb{E}\,\Phi(Y_{0:n})$.

$\triangleright$ If all the functions θ_k, $k = 0, \ldots, n-1$ are convex, then all functions Ψ_k, $k = 0, \ldots, n$, are convex and one shows like wise that $\Phi_k \le \Psi_k$ for every $k = 0, \ldots, n-1$.

(b) The proof follows the same lines as (a) calling upon Claim (c) of Lemma 2.1. In particular, the functions $u \mapsto \widetilde{G}(x_{0:k}, u)$ is also even so that $\sup_{u\in[-a,a]} \widetilde{G}(x_{0:k}, u) = \widetilde{G}(x_{0:k}, a)$ for any $a \ge 0$. $\square$

To prove Theorem 2.1 we need to transfer the above result into a continuous time setting by a functional weak approximation result. To this end, we introduce the notion of *piecewise affine interpolator* and recall an elementary weak convergence lemma.

Definition 2.1. (a) *For every integer $n \ge 1$, let $i_n : \mathbb{R}^{n+1} \to \mathcal{C}([0,T], \mathbb{R})$ denote the* piecewise affine interpolator *defined by*

$$\forall\, x_{0:n} \in \mathbb{R}^{n+1},\ \forall\, k = 0, \ldots, n-1,\ \forall\, t \in [t_k^n, t_{k+1}^n], \quad i_n(x_{0:n})(t) = \frac{n}{T}\big((t_{k+1}^n - t)x_k + (t - t_k^n)x_{k+1}\big).$$

(b) *For every integer $n \ge 1$, let $I_n : \mathcal{F}([0,T], \mathbb{R}) \to \mathcal{C}([0,T], \mathbb{R})$ denote the* functional interpolator *defined by*

$$\forall\, \alpha \in \mathcal{F}([0,T], \mathbb{R}), \quad I_n(\alpha) = i_n\big(\alpha(t_0^n), \ldots, \alpha(t_n^n)\big).$$

We will use extensively the following obvious fact

$$\sup_{t\in[0,T]} |I_n(\alpha)_t| \le \sup_{t\in[0,T]} |\alpha(t)|$$

in particular for uniform integrability purpose.

Lemma 2.2. *Let X^n, $n \ge 1$, be a sequence of continuous processes weakly converging towards X for the $\|\,.\,\|_{\sup}$-norm. Then the sequence of continuously interpolated processes $\widetilde{X}^n = I_n(X^n)$ of X^n, $n \ge 1$, is weakly converging toward X for the $\|\,.\,\|_{\sup}$-norm topology.*

Proof. For every integer $n \ge 1$ and every $\alpha \in \mathcal{F}([0,T], \mathbb{R}^d)$, the interpolation operators $I_n(\alpha)$ reads

$$I_n(\alpha) = \frac{n}{T}\big((t_{k+1}^n - t)\alpha(t_k^n) + (t - t_k^n)\alpha(t_{k+1}^n)\big),\ t \in [t_k^n, t_{k+1}^n],\ k = 0, \ldots, n-1.$$

Note that I_n maps $\mathcal{C}([0,T], \mathbb{R}^d)$ into itself. One easily checks that $\|I_n(\alpha) - \alpha\|_{\sup} \le w(\alpha, T/n)$ (keep in mind that w denotes the uniform continuity modulus of α) and $\|I_n(\alpha) - I_n(\beta)\|_{\sup} \le \|\alpha - \beta\|_{\sup}$. We use the standard distance d_{wk} for weak convergence on Polish metric spaces defined by

$$d_{wk}\big(\mathcal{L}(X), \mathcal{L}(Y)\big) = \sup\big\{|\mathbb{E}\,F(X) - \mathbb{E}\,F(Y)|,\ [F]_{\rm Lip} \le 1,\ \|F\|_{\sup} \le 1\big\}.$$

Then

$$\begin{aligned} d_{wk}\big(\mathcal{L}(I_n(X^n)), \mathcal{L}(X)\big) &\le d_{wk}\big(\mathcal{L}(I_n(X^n)), \mathcal{L}(I_n(X))\big) + d_{wk}\big(\mathcal{L}(I_n(X)), \mathcal{L}(X)\big) \\ &\le d_{wk}\big(\mathcal{L}(X^n), \mathcal{L}(X)\big) + \mathbb{E}\,\big(w(X, T/n) \wedge 2\big) \end{aligned}$$

which goes to 0 since X has continuous paths. $\square$

Proof of Theorem 2.1. We consider now for both SDEs (related to $X^{(\sigma)}$ and $X^{(\theta)}$) their continuous (also known as "genuine") Euler schemes with step $\frac{T}{n}$, starting at x with respect to a given standard

Brownian motion W defined on an appropriate probability space. *E.g.*, to be more precise, the Euler scheme related to $X^{(\sigma)}$ is defined by

$$\begin{aligned}\bar{X}^{(\sigma),n}_{t^n_{k+1}} &= \bar{X}^{(\sigma),n}_{t^n_k} + \sigma(t^n_k, \bar{X}^{(\sigma),n}_{t^n_k})\big(W_{t^n_{k+1}} - W_{t^n_k}\big),\ k = 0,\ldots,n-1,\ \bar{X}^{(\sigma),n}_0 = x\\ \bar{X}^{(\sigma),n}_t &= \bar{X}^{(\sigma),n}_{t^n_k} + \sigma(t^n_k, \bar{X}^{(\sigma),n}_{t^n_k})\big(W_t - W_{t^n_k}\big), \quad t\in [t^n_k, t^n_{k+1}).\end{aligned}$$

It is clear that both sequences $(\bar{X}^{(\sigma),n}_{t^n_k})_{k=0:n}$ and $(\bar{X}^{(\theta),n}_{t^n_k})_{k=0:n}$ are of the form (2.3) with the Gaussian white noise sequence $Z_k = W_{t^n_k} - W_{t^n_{k-1}}$, $k = 1,\ldots,n$. Furthermore, owing to the linear growth assumption made on σ and θ, the sup-norm of these Euler schemes of Brownian diffusions lie in $L^p(\mathbb{P})$ for any $p\in(0,+\infty)$, uniformly in n, (see *e.g.* Lemma B.1.2 p.275 in [5] or Proposition A.1 in Appendix A)

$$\sup_{n\geq 1}\Big\|\sup_{t\in[0,T]}|\bar{X}^{(\sigma),n}_t|\Big\|_p + \sup_{n\geq 1}\Big\|\sup_{t\in[0,T]}|\bar{X}^{(\theta),n}_t|\Big\|_p < +\infty.$$

Furthermore, $I_n(\bar{X}^{(\sigma),n}) = i_n\big((\bar{X}^{(\sigma),n})_{t^n_{0:n}}\big)$ is but the piecewise affine interpolated Euler scheme (which coincide with $\bar{X}^{(\sigma),n}$ at times t^n_k). Note that the sup-norm of $I_n(\bar{X}^{(\sigma),n})$ also has finite polynomial moments uniformly in n like the genuine Euler scheme.

Let $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$ be a convex functional with $(r, \|\,.\,\|_{\sup})$-polynomial growth. For every integer $n \geq 1$, we define on $\mathbb{R}^{n+1}$ the function F_n by

$$F_n(x_{0:n}) = F\big(i_n(x_{0:n})\big),\ x_{0:n}\in\mathbb{R}^{n+1}. \tag{2.5}$$

It is clear that the convexity of F on $\mathcal{C}([0,T],\mathbb{R})$ is transferred to the functions F_n, $n \geq 1$. So does the polynomial growth property. Moreover, F is $\|\,.\,\|_{\sup}$-continuous since it is convex with $\|\,.\,\|_{\sup}$-polynomial growth (see Lemma 2.1.1 in [27]). It follows from Proposition 2.1 applied with $\Phi = F_n$, $(Z_k)_{1\leq k\leq n} = (W_{t^n_k} - W_{t^n_{k-1}})_{1\leq k\leq n}$, $\sigma_k = \sigma(t^n_k, .)$ and $\theta_k = \theta(t^n_k, .)$, $k = 0,\ldots,n$ which obviously satisfy the required linear growth and integrability assumptions, that, for every $n \geq 1$,

$$\mathbb{E}\,F\big(I_n(\bar{X}^{(\sigma),n})\big) = \mathbb{E}\,F_n\big((\bar{X}^{(\sigma),n}_{t^n_k})_{k=0:n}\big) \leq \mathbb{E}\,F_n\big((\bar{X}^{(\theta),n}_{t^n_k}))_{k=0:n}\big) = \mathbb{E}\,F\big(I_n(\bar{X}^{(\theta),n})\big). \tag{2.6}$$

On the other hand, it is classical background that the *genuine (continuous) Euler schemes* $\bar{X}^{(\sigma),n}$ weakly converges for the $\|\,.\,\|_{\sup}$-norm topology toward $X^{(\sigma)}$, unique weak solution to the $SDE \equiv dX_t = \sigma(X_t)dW_t$, $X_0 = x$, as $n \to +\infty$. For a proof we refer *e.g.* to exercise 23 in [32], p.359 when σ is homogeneous in t, see also [20, 23]; we also provide a short self-contained proof in Proposition A.1 in Appendix A). The key in all them being the weak convergence theorem for stochastic integrals first established in [20].

It follows from Lemma 2.2 and the $L^p(\mathbb{P})$-boundedness of the sup-norm of the sequence $(I_n(\bar{X}^{(\sigma),n}))_{n\geq 1}$ for $p > r$ that

$$\mathbb{E}\,F(X^{(\sigma)}) = \lim_n \mathbb{E}\,F\big(I_n(\bar{X}^{(\sigma),n})\big) = \lim_n \mathbb{E}\,F_n\big((\bar{X}^{(\sigma),n}_{t^n_k})_{0\leq k\leq n}\big).$$

The same holds true for the diffusion $X^{(\theta)}$ and its Euler scheme. The conclusion follows.

(a) Applying successively what precedes to the couples (σ,κ) and (κ,θ) until Equation (2.6) respectively, we derive that for every $n \geq 1$,

$$\mathbb{E}\,F\big(I_n(\bar{X}^{(\sigma),n})\big) \leq \mathbb{E}\,F\big(I_n(\bar{X}^{(\kappa),n})\big) \leq \mathbb{E}\,F\big(I_n(\bar{X}^{(\theta),n})\big)$$

and one concludes likewise by letting n go to infinity in the resulting inequality

$$\mathbb{E}\,F\big(I_n(\bar{X}^{(\sigma),n})\big) \leq \mathbb{E}\,F\big(I_n(\bar{X}^{(\theta),n})\big).$$

(b) The proof follows the same lines by calling upon item (c) of the above Lemma 2.1, having in mind that the distribution of a standard Brownian increment is symmetric with polynomial moments at any order as a Gaussian random vector. $\square$

Remarks. • Note that no "weak uniqueness" assumption is requested for the function κ.

• The Euler scheme has already been successfully used to establish convex order in [2].

The following corollaries can be obtain with obvious adaptations of the above proof.

Corollary 2.1. *Under the above assumption of Claim (a), if, furthermore, the SDE*

$$dX_t^{(\kappa)} = \kappa(t, X_t^{(\kappa)})dW_t,\ X_0^{(\kappa)} = x$$

has a unique weak solution, then, for every convex functional $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$ with $(r, \|\,.\,\|_{\sup})$-polynomial growth,

$$\mathbb{E}\,F(X^{(\sigma)}) \le \mathbb{E}\,F(X^{(\kappa)}) \le \mathbb{E}\,F(X^{(\theta)}).$$

Corollary 2.2. *If $\sigma, \theta : [0,T] \times I \to \mathbb{R}$, where I is a nontrivial interval of $\mathbb{R}$, are continuous with polynomial growth and if the related Brownian SDEs satisfy a weak uniqueness assumption for every I-valued weak solution starting from $x \in I$, at time $t = 0$. Then the above Proposition remains true (the extension of the functional weak convergence of the Euler scheme established in Appendix A (Proposition A.1) under the assumption made on the drift b is left to the reader).*

This approach based on the combination of a (linear) dynamic programming principle and a functional weak approximation argument also allows us to retrieve Hajek's result for drifted diffusions.

Proposition 2.2 (Extension to drifted diffusions, see [10])**.** *Let σ and θ be two functions on $[0,T] \times \mathbb{R}$ satisfying the partitioning or the dominating assumptions (a) or (b) from Theorem 2.1 respectively. Let $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function with linear growth in x uniformly in t and such that $b(t,.)$ is convex for every $t \in [0,T]$. Let $Y^{(\sigma)}$ and $Y^{(\theta)}$ be the* weak *solutions, supposed to be unique, starting from x at time 0 to the SDEs $dY_t^{(\sigma)} = b(t, Y_t^{(\sigma)})dt + \sigma(t, Y_t^{(\sigma)})dW_t^{(\sigma)}$ and $dY_t^{(\theta)} = b(t, Y_t^{(\theta)})dt + \theta(t, Y_t^{(\theta)})dW_t^{(\theta)}$. Then, for every* non-decreasing *convex function $f : \mathbb{R} \to \mathbb{R}$,*

$$\mathbb{E}\,f(X^{(\sigma)}) \le \mathbb{E}\,f(X^{(\theta)}).$$

Proof. We have to introduce the operators $Q_{b,\gamma,t}$, $\gamma > 0$, $t \in [0,T]$, defined for every Borel function $f : \mathbb{R} \to \mathbb{R}$ (satisfying the appropriate polynomial growth assumption in accordance with the existing moments of Z) by

$$Q_{b,\gamma,t}(f)(x,u) = \mathbb{E}\,f\big(x + \gamma b(t,x) + uZ\big).$$

One shows like in Lemma 2.1 above that, if the function f is convex, $Q_{b,\gamma,t}f$ is convex in (x,u), non-decreasing in u on $\mathbb{R}_+$, non-increasing in $u \in \mathbb{R}_-$. $\square$

2.2 Applications to (Brownian) functional peacocks

We consider a local volatility model on the discounted risky asset dynamics given by

$$dS_t^{(\sigma)} = S_t^{(\sigma)}\sigma(t, S_t^{(\sigma)})dW_t^{(\sigma)},\ S_0^{(\sigma)} = s_0 > 0, \tag{2.7}$$

where $\sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a bounded continuous function so that the above equation has at least a weak solution $(S_t^{(\sigma)})_{t\in[0,T]}$ with distribution on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ on which lives a Brownian motion $(W_t^{(\sigma)})_{\in[0,T]}$ (with augmented filtration $(\mathcal{F}_t^{W^{(\sigma)}})_{t\in[0,T]}$). This follows from the proof of Proposition A.1 in Appendix A (see also [33], p. ??). Then, $(S_t^{(\sigma)})_{t\in[0,T]}$ is a true $(\mathcal{F}_t^{W^{(\sigma)}})_{t\in[0,T]}$-martingale satisfying

$$S_t^{(\sigma)} = s_0 \exp\Big(\int_0^t \sigma(s, S_s^{(\sigma)})dW_s^{(\sigma)} - \frac{1}{2}\int_0^t \sigma^2(s, S_s^{(\sigma)})ds\Big)$$

so that $S_t^{(\sigma)} > 0$ for every $t \in [0,T]$. One introduces likewise the local volatility model $(S_t^{(\theta)})_{t\in[0,T]}$ related to the bounded volatility function $\theta : [0,T]\times\mathbb{R}_+ \to \mathbb{R}$, still starting from $s_0 > 0$. Then, the following proposition holds which appears as a functional or non-parametric extension of the fact that $\Big(\int_0^T e^{\sigma B_t - \frac{\sigma^2 t}{2}}dt\Big)_{\sigma\geq 0}$ is a peacock (see *e.g.* [6, 13]).

Proposition 2.3 (Functional peacocks)**.** *Let σ and θ be two real valued bounded continuous functions defined on $[0,T]\times\mathbb{R}$. Assume that $S^{(\sigma)}$ is the unique weak solution to (2.7) as well as $S^{(\theta)}$ for its* mutatis mutandis *counterpart involving θ. If one of the following additional conditions holds*

(i) Partitioning function*: there exists a function $\kappa : [0,T]\times\mathbb{R}_+ \to \mathbb{R}_+$ such that, for every $t \in [0,T]$, $x \mapsto x\,\kappa(t,x)$ is convex on $\mathbb{R}_+$ and $0 \leq \sigma(t,.) \leq \kappa(t,.) \leq \theta(t,.)$ on $\mathbb{R}_+$,*

or

(ii) Domination property: for every $t \in [0,T]$ the function $x \mapsto x\,\theta(t,x)$ *is convex on $\mathbb{R}_+$ and*

$$|\sigma(t,.)| \leq \theta(t,.),$$

then, for every convex (hence continuous) function $f : \mathbb{R} \to \mathbb{R}$ with polynomial growth

$$\mathbb{E}\, f\left(\int_0^T S_s^{(\sigma)}\mu(ds)\right) \leq \mathbb{E}\, f\left(\int_0^T S_s^{(\theta)}\mu(ds)\right)$$

where μ is a signed (finite) measure on $([0,T], \mathcal{B}or([0,T]))$. More generally, for every convex functional $F : \mathcal{C}([0,T],\mathbb{R}_+) \to \mathbb{R}$ with $(r, \|\,.\,\|_{\sup})$-polynomial growth polynomial growth, one has,

$$\mathbb{E}\, F\big(S^{(\sigma)}\big) \leq \mathbb{E}\, F\big(S^{(\theta)}\big). \tag{2.8}$$

Proof. We focus on the first *partitioning* setting. The second one can be treated likewise. First note that κ is bounded since θ is. As a consequence, the function $x \mapsto x\,\kappa(t,x)$ is zero at $x = 0$ and can be extended into a convex function on the whole real line by setting $x\,\kappa(t,x) = 0$ if $x \leq 0$. One extends $x\,\sigma(t,x)$ and $x\,\theta(t,x)$ by zero on $\mathbb{R}_-$ likewise. Once this has been done, this claim appears as a straightforward consequence of Theorem 2.1 for the (martingale) diffusion processes whose diffusion coefficients are given by (the extension) of $x\,\sigma(t,x)$ and $x\,\theta(t,x)$ on the whole real line. As above, the sup-norm continuity follows from the convexity and polynomial growth. In the end we take advantage of the *a posteriori* positivity of $S^{(\theta)}$ and $S^{(\sigma)}$ when starting from $s_0 > 0$ to conclude. $\square$

Applications to volatility comparison results. The corollary below shows that comparison results for vanilla European options established in [8] appear as a special case of Proposition 2.3.

Corollary 2.3. *Let $\sigma : [0,T]\times\mathbb{R} \to \mathbb{R}_+$ be a bounded continuous function*

$$0 \leq \sigma_{\min}(t) \leq \sigma(t,.) \leq \sigma_{\max}(t),\ t \in [0,T],$$

then for every convex functional $F : \mathcal{C}([0,T],\mathbb{R}_+) \to \mathbb{R}$ with $(r, \|\,.\,\|_{\sup})$-polynomial growth $(r \geq 1)$,

$$\mathbb{E}\, F\left(S_s^{(\sigma_{min})}\right) \leq \mathbb{E}\, F\left(S_s^{(\sigma)}\right) \leq \mathbb{E}\, F\left(S_s^{(\sigma_{\max})}\right). \tag{2.9}$$

Proof. We successively apply the former Proposition 2.3 to the couple $(\sigma_{\min}, \sigma)$ and the partitioning function $\kappa(t, x) = \sigma_{\min}(t)$ to get the left inequality and to the couple $(\sigma, \sigma_{\max})$ with $\kappa = \sigma_{\max}$ to get the right inequality. $\square$

Note that the left and right hand side of the above inequality are usually considered as quasi-closed forms since they correspond to Hull-White model (or even to the regular Black-Scholes model if $\sigma_{\min}$, $\sigma_{\max}$ are constant). Moreover, it has to be emphasized that no convexity assumption on σ is requested.

2.3 Counter-example (discrete time setting)

The above comparison results for the convex order can fail when the assumptions of Theorem 2.1 are not satisfied by the diffusion coefficient. In fact, for simplicity, the counter-example below is developed in a discrete time framework corresponding to Proposition 2.1. We consider the 2-period dynamics $X = X^{\sigma,x} = (X^{\sigma,x}_{0:2})$ satisfying

$$X_1 = x + \sigma Z_1 \quad \text{and} \quad X_2 = X_1 + \sqrt{2v(X_1)}Z_2$$

where $Z_{1:2} \stackrel{\mathcal{L}}{\sim} \mathcal{N}(0; I_2)$, $\sigma \geq 0$, and $v : \mathbb{R} \to \mathbb{R}_+$ is a bounded $\mathcal{C}^2$-function such that v has a strict local maximum at x_0 satisfying $v'(x_0) = 0$ and $v''(x_0) < -1$ (so is the case if $v(x) = v(x_0) - \rho(x - x_0)^2 + o((x-x_0)^2)$, $0 < \rho < \frac{1}{2}$, in the neighbourhood of x_0). Of course this implies that $\sqrt{v}$ cannot be convex.

Let $f(x) = e^x$. It is clear that

$$\varphi(x, \sigma) := \mathbb{E} f(X_2) = e^x \mathbb{E}\big(e^{\sigma Z_1 + v(x+\sigma Z_1)}\big).$$

Elementary computations show that

$$\begin{aligned}
\varphi'_\sigma(x, \sigma) &= e^x \mathbb{E}\Big(e^{\sigma Z_1 + v(x+\sigma Z_1)}\big(1 + v'(x + \sigma Z_1)\big)Z_1\Big) \\
\varphi''_{\sigma^2}(x, \sigma) &= e^x \left(\mathbb{E}\Big(e^{\sigma Z_1 + v(x+\sigma Z_1)}\big(1 + v'(x + \sigma Z_1)\big)^2 Z_1^2\Big) + \mathbb{E}\Big(e^{\sigma Z_1 + v(x+\sigma Z_1)} v''(x + \sigma Z_1) Z_1^2\Big)\right).
\end{aligned}$$

In particular

$$\varphi'_\sigma(x, 0) = e^{x+v(x)}(1 + v'(x))\mathbb{E}\, Z_1 = 0 \quad \text{and} \quad \varphi''_{\sigma^2}(x, 0) = e^{x+v(x)}\Big((1 + v'(x))^2 + v''(x)\Big)$$

so that $\varphi''_{\sigma^2}(x_0, 0) < 0$ which implies that there exists a small enough $\sigma_0 > 0$ such that

$$\varphi'_\sigma(x_0, \sigma) < 0 \quad \text{for every} \quad \sigma \in (0, \sigma_0],$$

This clearly exhibits a counter-example to Proposition 2.1 when the convexity assumption is fulfilled neither by the functions $(\sigma_k)_{k=0:n}$ nor the functions $(\kappa_k)_{k=0:n}$ (here with $n = 1$).

2.4 Lévy driven diffusions

Let $Z = (Z_t)_{t\in[0,T]}$ be a Lévy process with Lévy measure ν satisfying $\int_{|z|\geq 1} |z|^p \nu(dz) < +\infty$, $p \in [1, +\infty)$. Then $Z_t \in L^1(\mathbb{P})$ for every $t \in [0, T]$. Assume furthermore that $\mathbb{E}\, Z_1 = 0$: then $(Z_t)_{t\in[0,T]}$ is an $\mathcal{F}^Z$-martingale.

Theorem 2.2. *Let $Z = (Z_t)_{t\in[0,T]}$ be a martingale Lévy process with Lévy measure ν satisfying $\nu(|z|^p) < +\infty$ for a $p \in (1, +\infty)$ if Z has no Brownian component and $\nu(z^2) < +\infty$ if Z does have a Brownian component. Let $\kappa_i : [0, T] \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, be continuous functions with linear growth in*

x uniformly in $t\in[0,T]$. For $i=1,2$, let $X^{(\kappa_i)}=(X_t^{(\kappa_i)})_{t\in[0,T]}$ be the weak solution, assumed to be unique, to

$$dX_t^{(\kappa_i)}=\kappa_i(t,X_{t-}^{(\kappa_i)})dZ_t^{(\kappa_i)},\quad X_0^{(\kappa_i)}=x\in\mathbb{R},\tag{2.10}$$

where $Z^{(\kappa_i)}$, $i=1,2$ have the same distribution as Z. Let $F:\mathbb{D}([0,T],\mathbb{R})\to\mathbb{R}$ be a Borel convex functional, $\mathbb{P}_{X^{(\kappa_i)}}$-a.s. Sk-continuous, $i=1,2$, with $(r,\|.\|_{\sup})$-polynomial growth for some $r\in[1,p)$ i.e.

$$\forall\,\alpha\in\mathbb{D}([0,T],\mathbb{R}),\quad |F(\alpha)|\le C(1+\|\alpha\|_{\sup}^r).$$

(a) Partitioning function*: If there exists a function $\kappa:[0,T]\times\mathbb{R}\to\mathbb{R}_+$ such that $\kappa(t,.)$ is convex for every $t\in[0,T]$ and $0\le\kappa_1\le\kappa\le\kappa_2$, then*

$$\mathbb{E}\,F(X^{(\kappa_1)})\le\mathbb{E}\,F(X^{(\kappa_2)}).$$

(a') An equivalent form for claim (a) is: if $0\le\kappa_1\le\kappa_2$ and, either $\kappa_1(t,.)$ is convex for every $t\in[0,T]$, or $\kappa_2(t,.)$ is convex for every $t\in[0,T]$, then the conclusion of (a) still holds true.

(b) Domination property*: If Z has a symmetric distribution, $|\kappa_1|\le\kappa_2$ and κ_2 is convex, then*

$$\mathbb{E}\,F(X^{(\kappa_1)})\le\mathbb{E}\,F(X^{(\kappa_2)}).$$

Remarks. • The $\mathbb{P}_{X^{(\kappa_i)}}$-*a.s.* Sk-continuity of the functional F, $i=1,2$, is now requested sinceSk-continuity no longer follows form the convexity $\big((\mathbb{D}([0,T],\mathbb{R}),Sk\big)$ is a Polish space but not even a topological vector space). Thus the function $\alpha\mapsto|\alpha(t_0)|$ for a fixed $t_0\in(0,T)$ is continuous at a given $\beta\in\mathbb{D}([0,T],\mathbb{R})$ if and only if β is continuous at t_0 (see [1], Chapter 3).

• The result remains true under the less stringent moment assumption on the Lévy measure ν: $\nu(|z|^p\mathbf{1}_{\{|z|\ge1\}}<+\infty$ but would require much more technicalities since one has to carry out the reasoning of the proof below between two large jumps of Z and "paste" these inter-jump results.

The following technical lemma is the key that solves the approximation part of the proof in this càdlàg setting.

Lemma 2.3. *Let $\alpha\in\mathbb{D}([0,T],\mathbb{R})$. The sequence of stepwise constant approximations defined by*

$$\alpha_n(t)=\alpha(\underline{t}_n),\ t\in[0,T],$$

converges toward α for the Skorokhod topology.

Proof. See [18]Proposition VI.6.37 p.387 (second edition). □

Proof of Theorem 2.2. Step 1. Let $(\bar X_t^n)_{t\in[0,T]}$ be the genuine Euler scheme defined by

$$\bar X_t^n=x+\int_{(0,t]}\kappa(\underline{s}_n,\bar X^n_{\underline{s}_{n-}})dZ_s$$

where $\kappa=\kappa_1$ or κ_2. Then, owing to the linear growth of κ, we derive (see*e.g.* Proposition B.2 in Appendix B) that

$$\Big\|\sup_{t\in[0,T]}|X_t|\Big\|_p+\sup_{n\ge1}\Big\|\sup_{t\in[0,T]}|\bar X_t^n|\Big\|_p<+\infty.$$

We know, *e.g.* from form Proposition B.1 in Appendix B, that $(\bar X^n)_{n\ge1}$ functionally weakly converges for the Skorokhod topology toward the unique weak solution X of the *SDE* $dX_k=\kappa(t,X_{t_-})dZ_t$, $X_0=x$. In turn, Lemma 2.3 implies that $(\bar X^n_{\underline{t}_n})_{t\in[0,T]}$ Sk-weakly converges toward X.

STEP 2. Let $F : \mathbb{D}([0,T],\mathbb{R}) \to \mathbb{R}$ be a $\mathbb{P}_X$-Sk-continuous convex functional. For every integer $n \geq 1$, we still define the sequence of convex functional $F_n : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$F_n(x_{0:n}) = F\Big(\sum_{k=0}^{n-1} x_k \mathbf{1}_{[t_k^n, t_{k+1}^n)} + x_n \mathbf{1}_{\{T\}}\Big) \text{ so that } F_n\big((\bar{X}^n_{t_k^n})_{0:n}\big) = F\big((\bar{X}^n_{\underline{t}_n})_{t\in[0,T]}\big).$$

Now, for every $n \geq 1$, the discrete time Euler schemes $\bar{X}^{(\kappa_i),n}$, $i=1,2$, related to the jump diffusions with diffusion coefficients κ_1 and κ_2 are of the form (2.3) and $|F_n(x_{0:n})| \leq C(1+\|x_{0:n}\|^r)$, $r \in [1,p)$.

(a) Assume $0 \leq \kappa_1 \leq \kappa_2$. Then, taking advantage of the partitioning function κ, it follows from Proposition 2.1(a) that, for every $n \geq 1$, $\mathbb{E}\, F_n\big((\bar{X}^{(\kappa_1),n}_{t_k^n})_{0:n}\big) \leq \mathbb{E}\, F_n\big((\bar{X}^{(\kappa_2),n}_{t_k^n})_{0:n}\big)$ *i.e.* $\mathbb{E}\, F\big((\bar{X}^{(\kappa_1),n}_{\underline{t}_n})_{t\in[0,T]}\big) \leq \mathbb{E}\, F\big((\bar{X}^{(\kappa_2),n}_{\underline{t}_n})_{t\in[0,T]}\big)$. Letting $n \to +\infty$ completes the proof like in that of Theorem 2.1 since F is $\mathbb{P}_X$-*a.s.* Sk-continuous. □

(b) is an easy consequence of Proposition 2.1(b). □

3 Convex order for non-Markovian Itô and Doléans martingales

The results of this section illustrates another aspects of our paradigm in order to establish functional convex order for various classes of continuous time stochastic processes. Here we deal with (couples of) Itô-intregrals with the restriction that one of the two integrands needs to be deterministic.

3.1 Itô martingales

Proposition 3.1. *Let $(H_t)_{t\in[0,T]}$ be an $(\mathcal{F}_t)$-progressively measurable process defined on a) filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions and let $h = (h_t)_{t\in[0,T]} \in L^2_T$. Let $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$ be a convex functional with $(r, \|.\|_{\sup})$-polynomial growth, $r \geq 1$.*

(a) If $|H_t| \leq h_t$ $\mathbb{P}$-a.s. for every $t \in [0,T]$, then

$$\mathbb{E}\, F\left(\int_0^\cdot H_s dW_s\right) \leq \mathbb{E}\, F\left(\int_0^\cdot h_s dW_s\right).$$

(b) If $H_t \geq h_t \geq 0$ $\mathbb{P}$-a.s. for every $t \in [0,T]$ and $\|H\|_{L^2_T} \in L^{r'}(\mathbb{P})$ for $r' > r$, then

$$\mathbb{E}\, F\left(\int_0^\cdot H_s dW_s\right) \geq \mathbb{E}\, F\left(\int_0^\cdot h_s dW_s\right).$$

Remarks. • In the "marginal" case where F is of the from $F(\alpha) = f(\alpha(T))$, it has been shown in [14] that the above assumptions on H and h in (a) and (b) are too stringent and can be relaxed into

$$\int_0^T \mathbb{E}\, H_t^2 dt \leq \int_0^T h_t^2 dt \quad \text{and} \quad \int_0^T \mathbb{E}\, H_t^2 dt \geq \int_0^T h_t^2 dt$$

respectively. The main ingredient of the proof is the Dambis-Dubins-Schwartz representation theorem for one-dimensional Brownian martingales (see *e.g.* Theorem 1.6 in [33], p.181).

• The first step of the proof below, can be compared to Proposition 2.1 in a Markov framework as an autonomous proposition devoted to discrete time setting.

Proof. STEP 1 (Discrete time). Let $(Z_k)_{1\leq k\leq n}$ be an n-tuple of independent symmetric (hence centered) $\mathbb{R}$-valued random variables satisfying $Z_k \in L^r(\Omega, \mathcal{A}, \mathbb{P})$, $r \geq 1$, and let $\mathcal{F}_0^Z = \{\emptyset, \Omega\}$, $\mathcal{F}_k^Z =$

$\sigma\big(Z_{,}\ldots,Z_k\big)$, $k=1,\ldots,n$ be its natural filtration. Let $(H_k)_{0\le k\le n}$ be an $(\mathcal{F}^Z_k)_{0\le k\le n}$-adapted sequence such that $H_k\in L^r(\mathbb{P})$, $k=1,\ldots,n$.

Let $X=(X_k)_{0\le k\le n}$ and $Y=(Y_k)_{0\le k\le n}$ be the two sequences of random variables recursively defined by

$$X_{k+1}=X_k+H_kZ_{k+1},\quad Y_{k+1}=Y_k+h_kZ_{k+1},\quad 0\le k\le n-1,\quad X_0=Y_0=x_0.$$

These are the discrete time stochastic integrals of (H_h) and (h_k) with respect to $(Z_k)_{1\le k\le n}$. It is clear by induction that X_k, $Y_k\in L^r(\mathbb{P})$ for every $k=0,\ldots,n$ since H_k is $\mathcal{F}^Z_k$-measurable and Z_{k+1} is independent of $\mathcal{F}^Z_k$.

Let $\Phi:\mathbb{R}^{n+1}\to\mathbb{R}$ be a convex function such that $|\Phi(x)|\le C(1+|x|^r)$ where $C\ge 0$ is a real constant. Let us focus on the first inequality (discrete time counterpart of claim (a)). One proceeds like in the proof Proposition 2.1 to prove by (three) backward induction(s) that if $|H_k|\le h_k$, $k=0\!:\!n$, then

$$\mathbb{E}\,\Phi(X)\le\mathbb{E}\,\Phi(Y).$$

To be more precise, let us introduce by analogy with this proposition the sequence $(\Psi_k)_{0\le k\le n}$ of functions recursively defined by

$$\Psi_n=\Phi,\ \Psi_k(x_{0:k})=\big(Q_{k+1}\Psi_{k+1}(x_{0:k},x_k+.)\big)(h_k),\ x_{0:k}\in\mathbb{R}^{k+1},\ k=0,\ldots,n-1.$$

First note that the functions Ψ_k satisfy a linear dynamic programing principle

$$\Psi_k(Y_{0:k})=\mathbb{E}\big(\Psi_{k+1}(Y_{0:k+1})\,|\,\mathcal{F}^Z_k\big),\ k=0,\ldots,n-1$$

so that by the chaining rule for conditional expectations, we have

$$\Phi_k(Y_{0:k})=\mathbb{E}\big(\Phi(Y_{0:n})\,|\,\mathcal{F}^Z_k\big),\ k=0,\ldots,n.$$

Furthermore, owing to the properties of the operator Q_{k+1}, we already proved that for any convex function $G:\mathbb{R}^{k+2}\to\mathbb{R}$ such that $|G(x)|\le C(1+|x|^r)$, the function

$$(x_{0:k},u)\mapsto\big(Q_{k+1}G(x_{0:k},x_k+.)\big)(u)=\mathbb{E}\,G(x_{0:k},x_k+uZ_{k+1})$$

is convex and even as a function of u for every fixed $x_{0:k}$. As a consequence, it also satisfies the maximum principle established in Lemma 2.1(c) since the random variables Z_k have symmetric distributions.

Now, let us introduce the martingale induced by $\Phi(X_{0,n})$, namely

$$M_k=\mathbb{E}\big(\Phi(X_{0:n})\,|\,\mathcal{F}^Z_k)\big),\ k\in\{0,\ldots,n\}.$$

We will show by a backward induction that $M_k\le\Psi_k(X_{0:k})$ for every $k\in\{0,\ldots,n\}$. If $k=n$, this is trivial. Assume now that $M_{k+1}\le\Psi_{k+1}(X_{0:k+1})$ for a $k\in\{0,\ldots,n-1\}$. Then we get the following string of inequalities

$$\begin{aligned}
M_k=\mathbb{E}(M_{k+1}\,|\,\mathcal{F}^Z_k)\ &\le\ \mathbb{E}(\Psi_{k+1}(X_{0:k+1})\,|\,\mathcal{F}^Z_k)\\
&=\ \mathbb{E}(\Psi_{k+1}(X_{0:k},X_k+H_kZ_{k+1})\,|\,\mathcal{F}^Z_k)\\
&=\ \Big(\mathbb{E}(\Psi_{k+1}(x_{0:k},x_k+uZ_{k+1})\,|\,\mathcal{F}^Z_k)\Big)_{|x_{0;k}=X_{0:k},u=H_k}\\
&=\ \Big(Q_{k+1}\Psi_{k+1}(x_{0:k},x_k+.)(H_k)\Big)_{|x_{0;k}=X_{0:k}}\\
&\le\ \Big(Q_{k+1}\Psi_{k+1)}(x_{0:k},x_k+.)(h_k)\Big)_{|x_{0;k}=X_{0:k}}=\Psi_k(X_{0:k})
\end{aligned}\tag{3.11}$$

where we used in the fourth line that Z_{k+1} is independent of $\mathcal{F}_k^Z$ and in the penultimate line the assumption $|H_k| \le h_k$ and the maximum principle. Finally, at $k = 0$, we get $\mathbb{E}\,\Phi(X_{0,n}) = M_0 \le \Phi_0(x_0) = \mathbb{E}\Phi(Y_{0:n})$ which is the announced conclusion.

Step 2 (Approximation-Regularization). We temporarily assume that the function h (has a modification which) is bounded by a real constant so that $\mathbb{P}(d\omega)$-*a.s.* $\|H(\omega)\|_{\sup} \vee \|h\|_{\sup} \le K$. We first need a technical lemma inspired by Lemma 2.4 in [21] (p.132, 2^{nd} edition) about approximation of progressively measurable processes by *simple* processes, with in mind the preservation of the domination property requested in our framework.

Lemma 3.1. (*a*) *For every* $\varepsilon \in (0, T)$ *and every* $g \in L^2([0, T], dt)$ *we define*

$$\Delta_\varepsilon g(t) \equiv t \longmapsto \frac{1}{\varepsilon}\int_{(t-\varepsilon)_+}^{t} g(s)ds \in \mathcal{C}([0, T], \mathbb{R}).$$

The operator $\Delta_\varepsilon : L^2_T \to \mathcal{C}([0, T], \mathbb{R})$ *is non-negative. In particular, if* $g,\ \gamma \in L^2_T$ *with* $|g| \le \gamma$ λ_1*-a.e., then* $|\Delta_\varepsilon g| \le \Delta_\varepsilon\gamma$ *and* $\|\Delta_\varepsilon g\|_{\sup} \le |g|_{L^\infty_T}$.

(*b*) *If* $g \in \mathcal{C}([0, T], \mathbb{R})$*, define for every integer* $m \ge 1$*, the stepwise constant càglàd (for left continuous right limited) approximation* $\tilde{g}_n$ *of* g *by*

$$\widetilde{g}^m(t) = g(0)\mathbf{1}_{\{0\}}(t) + \sum_{k=1}^{m} g\big(t_{k-1}^m\big)\mathbf{1}_{(t_{k-1}^m, t_k^m]}.$$

Then $\widetilde{g}^m \overset{\|\,.\,\|_{\sup}}{\longrightarrow} g$ *as* $m \to +\infty$*. Furthermore, if* $g,\ \gamma \in \mathcal{C}([0, T], \mathbb{R})$ *and* $|g| \le \gamma$*, then for every* $m \ge 1$*,* $|\widetilde{g}^m| \le \widetilde{\gamma}^m$.

The details of the proof are left to the reader.

By the Lebesgue fundamental Theorem of Calculus we know that

$$\big|\Delta_{\frac{1}{n}} H - H\big|_{L^2_T} \longrightarrow 0 \quad \mathbb{P}\text{-}a.s.$$

Since $|\Delta_{\frac{1}{n}} H - H|_{L^2_T} \le 2K$, the Lebesgue dominated convergence theorem implies that

$$\mathbb{E}\int_0^T |\Delta_{\frac{1}{n}} H_t - H_t|^2 dt \longrightarrow 0 \ \text{ as } \ n \to +\infty. \tag{3.12}$$

By construction, $\Delta_{\frac{1}{n}} H$ is an $(\mathcal{F}_t)_t$-adapted pathwise continuous process satisfying the domination property $|\Delta_{\frac{1}{n}} H| \le \Delta_{\frac{1}{n}} h$ so that, in turn, using this time claim (*b*) of the above lemma, for every $n,\ m \ge 1$,

$$|\widetilde{\Delta_{\frac{1}{n}} H_t}^m| \le \widetilde{\Delta_{\frac{1}{n}} h_t}^m.$$

On the other hand, for every $n \ge 1$, the *a.s.* uniform continuity of $\Delta_{\frac{1}{n}} H$ over $[0, T]$ implies

$$\int_0^T \big|\widetilde{\Delta_{\frac{1}{n}} H_t}^m - \Delta_{\frac{1}{n}} H_t\big|^2 dt \le \sup_{t\in[0,T]} |\widetilde{\Delta_{\frac{1}{n}} H_t}^m - \Delta_{\frac{1}{n}} H_t|^2 \to 0 \ \text{ as } \ m \to +\infty \ \mathbb{P}\text{-}a.s.$$

One concludes again by the Lebesgue dominated convergence theorem that, for every $n \ge 1$,

$$\mathbb{E}\int_0^T \Big|\widetilde{\Delta_{\frac{1}{n}} H_t}^m - \Delta_{\frac{1}{n}} H_t\Big|^2 dt \longrightarrow 0 \ \text{ as } \ m \to +\infty.$$

One shows likewise for the function h it self that

$$\left|\Delta_{\frac{1}{n}} h - h\right|_{L^2_T} \to 0 \;\text{ as }\; n \to +\infty$$

and, for every $n \geq 1$,

$$\left|\widetilde{\Delta_{\frac{1}{n}} h}^{m} - \Delta_{\frac{1}{n}} h\right|_{L^2_T} \to 0 \;\text{ as }\; m \to +\infty.$$

Consequently there exists an increasing subsequence $m(n) \uparrow +\infty$ such that

$$\mathbb{E}\int_0^T \left|\widetilde{\Delta_{\frac{1}{n}} H_t}^{m(n)} - \Delta_{\frac{1}{n}} H_t\right|^2 dt + \int_0^T \left|\widetilde{\Delta_{\frac{1}{n}} h_t}^{m(n)} - \Delta_{\frac{1}{n}} h_t\right|^2 dt \longrightarrow 0 \;\text{ as }\; n \to +\infty$$

which in turn implies, combined with (3.12) (and its deterministic counterpart for h),

$$\mathbb{E}\int_0^T \left|\widetilde{\Delta_{\frac{1}{n}} H_t}^{m(n)} - H_t\right|^2 dt + \int_0^T \left|\widetilde{\Delta_{\frac{1}{n}} h_t}^{m(n)} - h_t\right|^2 dt \longrightarrow 0 \;\text{ as }\; n \to +\infty.$$

At this stage, we set for every integer $n \geq 1$,

$$H^{(n)}_t = \widetilde{\Delta_{\frac{1}{n}} H_t}^{m(n)} \quad \text{ and } \quad h^{(n)}_t = \widetilde{\Delta_{\frac{1}{n}} h_t}^{m(n)} \tag{3.13}$$

which satisfy

$$\mathbb{E}|H - H^{(n)}|^2_{L^2_T} + |h - h^{(n)}|_{L^2_T} \longrightarrow 0 \;\text{ as }\; n \to +\infty. \tag{3.14}$$

It should be noted that these processes $H^{(n)}$, H and these functions $h^{(n)}$, h are all bounded by $2K$.

We consider now the continuous modifications of the four (square integrable) Brownian martingales associated to the integrands $H^{(n)}$, H, $h^{(n)}$ and h (the last two being of Wiener type in fact). It is clear by Doob's Inequality that

$$\sup_{t\in[0,T]}\left|\int_0^t H^{(n)}_s dW_s - \int_0^t H_s dW_s\right| + \sup_{t\in[0,T]}\left|\int_0^t h^{(n)}_s dW_s - \int_0^t h_s dW_s\right| \stackrel{L^2(\mathbb{P})}{\longrightarrow} 0 \;\text{ as }\; n \to +\infty.$$

In particular $\left(\int_0^{\cdot} H^{(n)}_s dW_s\right)_{t\in[0,T]}$ functionally weakly converges to $\left(\int_0^{\cdot} H_s dW_s\right)_{t\in[0,T]}$ for the $\|\,.\,\|_{\sup}$-norm topology. We also have, owing to the $B.D.G.$ Inequality, that for every $p \in (0, +\infty)$,

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t H^{(n)}_s dW_s\right|^p \leq c_p^p\, \mathbb{E}\|H^{(n)}\|^p_{L^2_T} \leq c_p K^p \tag{3.15}$$

where c_p is the universal constant involved in the $B.D.G.$ inequality. The same holds true for the three other integrals related to $h^{(n)}$, H, and h.

Let $n \geq 1$. Set $H^n_k = H^{(n)}_{t^{m(n)}_k}$, $h^n_k = h^{(n)}_{t^{m(n)}_k}$, $k = 0, \ldots, m(n)$ and $Z^n_k = W_{t^{m(n)}_k} - W_{t^{m(n)}_{k-1}}$, $k = 1, \ldots, n(m)$. One easily checks that $\displaystyle\int_0^{t^{m(n)}_k} H^{(n)}_s dW_s = \sum_{\ell=1}^k H^n_\ell Z^n_\ell$, $k = 0, \ldots, m(n)$ so that

$$I_{m(n)}\left(\int_0^{\cdot} H^{(n)}_s dW_s\right) = i_{m(n)}\left(\left(\sum_{\ell=1}^k H^n_\ell Z^n_\ell\right)_{k=0:m(n)}\right).$$

Let $F_{m(n)}$ be defined by (2.5) from the convex functional F (with $(r, \|\,.\,\|_{\sup})$-polynomial growth). It is clearly convex. One derives from Step 1 with applied with horizon $m(n)$ and discrete time random sequences $(Z^n_k)_{k=1:m(n)}$, $(H^n_k)_{k=0:m(n)}$, $(h_k)_{k=0:m(n)}$ that

$$\begin{aligned}\mathbb{E}\, F\circ I_{m(n)}\Big(\int_0^{\cdot} H^{(n)}_s dW_s\Big) &= \mathbb{E}\, F_{m(n)}\left(\Big(\sum_{\ell=1}^{k} H^n_\ell Z^n_\ell\Big)_{k=0:m(n)}\right)\\ &\le \mathbb{E}\, F_{m(n)}\left(\Big(\sum_{\ell=1}^{k} h^n_\ell Z^n_\ell\Big)_{k=0:m(n)}\right) = \mathbb{E}\, F\circ I_{m(n)}\Big(\int_0^{\cdot} h^{(n)}_s dW_s\Big).\end{aligned}$$

Combining the above functional weak convergence, Lemma 2.2 and the uniform integrability derived form (3.15) (with any $p > r$) yields the expected inequality by letting n go to infinity.

Step 3. (Second approximation) Let $K \in \mathbb{N}$ and $\chi_K : \mathbb{R} \to \mathbb{R}$ the thresholding function defined by $\chi_K(u) = (u \wedge K) \vee (-K)$. It follows from the $B.D.G.$ Inequality that for every $p\in (0, +\infty)$

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\Big|\int_0^t H_s dW_s - \int_0^t \chi_K(H_s)dW_s\Big|^p &\le c_p^p\, \mathbb{E}|H - \chi_K(H)|^p_{L^2_T}\\ &= c_p^p\, \mathbb{E}\big|\big(|H| - K\big)_+\big|^p_{L^2_T} \qquad (3.16)\\ &\le c_p^p\, \big|\big(|h| - K\big)_+\big|^p_{L^2_T} \qquad (3.17)\end{aligned}$$

where $u_+ = \max(u, 0)$, $u\in \mathbb{R}_+$. The same bound obviously holds when replacing H by h. This shows that the convergence holds in every $L^p(\mathbb{P})$ space, $p\in (0, +\infty)$ as $K \to +\infty$. Hence, one may let K go to infinity in the inequality

$$\mathbb{E}F\left(\int_0^{\cdot} \chi_K(H_s)dW_s\right) \le \mathbb{E}F\left(\int_0^{\cdot} \chi_K(h_s)dW_s\right) = \mathbb{E}F\left(\int_0^{\cdot} h_s \wedge K dW_s\right) \qquad (3.18)$$

which yield the expected inequality.

(b) We consider the same steps as for the upper-bound established in (a) with the same notations.

Step 1: First, in a discrete time setting, we assume that $0 \le h_k \le H_k\in L^r(\mathbb{P})$ and we aim at showing that by backward induction that $M_k \ge \Psi_k(X_{0:k})$ where $M_k = \mathbb{E}\big(\Phi(X_{0,n})\,|\,\mathcal{F}^Z_k\big)$.

If $k = n$, the inequality hold as an equality since $\Psi_n = \Phi$. Now assume $M_{k+1} \ge \Psi_{k+1}(X_{0:k+1})$. Then, like in (a),

$$\begin{aligned}M_k &= \mathbb{E}\big(M_{k+1}\,|\,\mathcal{F}^Z_k\big)\\ &\ge \mathbb{E}\big(\Phi(X_{0,k+1})\,|\,\mathcal{F}^Z_k\big) = \mathbb{E}\big(\Phi(X_{0,k}, X_k + H_k Z_{k+1})\,|\,\mathcal{F}^Z_k\big) = \Big(Q_k\Psi_{k+1}(x_{0:k}, x_k + .\,)(H_k)\Big)_{|x_{0:k}=X_{0:k}}\\ &\ge \Big(Q_k\Psi_{k+1}(x_{0:k}, x_k + .\,)(h_k)\Big)_{|x_{0:k}=X_{0:k}} = \Psi_k(X_{0:k}).\end{aligned}$$

Step 2. This step is devoted to approximation in a bounded setting where $0 \le h_t \le H_t \le K$. It follows the lines of its counterpart in claim (a) taking advantage of the global boundedness by K.

Step 3. This last step is devoted to the approximation procedure in the general setting. It differs from the above one since there is no longer a deterministic upper-bound provided by the function $h \in L^2_T$. Then, the key is to show that the process $\int_0^{\cdot} \chi_K(H_s)dW_s$ converges for the sup norm over $[0, T]$ in $L^{r'}(\mathbb{P})$ toward the process $\int_0^{\cdot} H_s dW_s$. In fact, it follows from (3.16) applied with $p = r'$ that

$$\mathbb{E}\sup_{t\in[0,T]}\Big|\int_0^t H_s dW_s - \int_0^t \chi_K(H_s)dW_s\Big|^{r'} \le c_p^p\, \mathbb{E}\big|\big(|H| - K\big)_+\big|^{r'}_{L^2_T}.$$

As $|H|_{L^2_T} \in L^{r'}(\mathbb{P})$, one concludes by the Lebesgue dominated convergence theorem by letting $K \to +\infty$. $\square$

Remarks. • Step 1 can be extended to non-symmetric, centered independent random variables $(Z_k)_{1\le k\le n}$ if the sequences $(H_k)_{0\le k\le n-1}$ and $(h_k)_{0\le k\le n-1}$ under consideration satisfy $0 \le H_k \le h_k$, $k = 0,\dots,n-1$.

• When H has left continuous paths, the proof can be significantly simplified by considering the simpler approximating sequence $H^{(n)}_t = \widetilde{H}^n_t$ which clearly converges toward H $d\mathbb{P}\otimes dt$-*a.e.* (and in the appropriate $L^p(dP\otimes dt)$-spaces as well).

3.2 Lévy-Itô martingales

Proposition 3.2. *Let $Z = (Z_t)_{t\in[0,T]}$ be an integrable centered Lévy process with Lévy measure ν satisfying $\nu(|x|^p \mathbf{1}_{\{|x|\ge 1\}}) < +\infty$ for a real exponent $p > 1$. Let $F : I\!D([0,T],\mathbb{R}) \to \mathbb{R}$ be a convex Skorokhod-continuous functional with $(p, \|\,.\,\|_{\sup})$-polynomial growth. Let $(H_t)_{t\in[0,T]}$ be an $(\mathcal{F}_t)$-predictable process and let $h = (h_t)_{t\in[0,T]} \in \|h\|_{L^{p\vee 2}_T} < +\infty$.*

(a) If $0 \le H_t \le h_t$ dt-a.e., $\mathbb{P}$-a.s. then

$$\mathbb{E}\, F\left(\int_0^{\cdot} H_s dZ_s\right) \le \mathbb{E}\, F\left(\int_0^{\cdot} h_s dZ_s\right).$$

If furthermore Z is symmetric, the result holds as soon as $|H_t| \le h_t$ dt-a.e., $\mathbb{P}$-a.s..

(b) If $H_t \ge h_t \ge 0$ dt-a.e., $\mathbb{P}$-a.s. and $|H|_{L^{p\vee 2}_T} \in L^p(\mathbb{P})$, then

$$\mathbb{E}\, F\left(\int_0^{\cdot} H_s dZ_s\right) \ge \mathbb{E}\, F\left(\int_0^{\cdot} h_s dZ_s\right).$$

(c) If the Lévy process Z has no Brownian component, the above claims claims (a) and (b) remain true if we only assume $h \in L^p_T$ and $|H|_{L^p_T} \in L^p(\mathbb{P})$ respectively.

Proof. (a) This proof follows the approach introduced for the is an extension of the Brownian-Itô case up to the technicalities induced by Lévy processes.

Step 1 (Discrete time). This step does not differ from that developed for Brownian-Itô martingales, except that in the the Lévy setting we rely on claim (a) of Lemma 2.1 since the marginal distribution of the increment of a Lévy process has no reason to be symmetric.

Step 2 (Approximation-Regularization). Temporarily assume that h is bounded. We consider the approximation procedure of H by stepwise constant càglàd $(\mathcal{F}_t)_t$-adapted (hence predictable) processes $H^{(n)}$ already defined by (3.13) in the proof of the previous proposition. Then, we first consider the Lévy-Khintchine decomposition of the Lévy martingale Z

$$\forall\, t \in [0,T], \qquad Z_t = a\, W_t + \widetilde{Z}^{\eta}_t + Z^{\eta}_t, \quad a \ge 0,$$

where $\widetilde{Z}^{\eta}$ is a martingale with jumps of size at most η and Lévy measure $\nu(\,.\cap\{|z|\le\eta\})$ and Z^{η} is a compensated Poisson process with (finite) Lévy measure $\nu(.\cap\{|z|>\eta\})$. Let n be a positive integer. We will perform a "cascade" procedure to make p decrease thanks to st the $B.D.G.$ Inequality. This – classical – method is more detailed in the proof of Proposition 3.1 in Appendix B (higher moments of Lévy driven diffusions).

We first assume assume that $p\in(1,2]$. Combining Minkowski's and $B.D.G.$'s Inequalities yields

$$\begin{aligned}\Big\|\sup_{t\in[0,T]}\Big|\int_0^t H_s dZ_s-\int_0^t H^{(n)}_s dZ_s\Big|\Big\|_p &\le c_p\, a\Big\||H-H^{(n)}|_{L^2_T}\Big\|_p\\ &+c_p\Big\|\sum_{0<s\le T}(H_s-H^{(n)}_s)^2(\Delta Z_s)^2\mathbf{1}_{\{|\Delta Z_s|>\eta\}}\Big\|^{\frac12}_{\frac p2}\\ &+c_p\Big\|\sum_{0<s\le T}(H_s-H^{(n)}_s)^2(\Delta Z_s)^2\mathbf{1}_{\{|\Delta Z_s|\le\eta\}}\Big\|^{\frac12}_{1}\end{aligned}$$

where we used in the last line the monotony of $L^p(\mathbb{P})$-norm $\frac p2\le 1$.

Using now the compensation formula and again that $\frac p2\in(0,1]$, it follows

$$\begin{aligned}\mathbb{E}\Big|\sum_{0<s\le T}(H_s-H^{(n)}_s)^2(\Delta Z_s)^2\mathbf{1}_{\{|\Delta Z_s|>\eta\}}\Big|^{\frac p2} &\le \mathbb{E}\sum_{0<s\le T}|H_s-H^{(n)}_s|^p|\Delta Z_s|^p\mathbf{1}_{\{|\Delta Z_s|>\eta\}}\\ &= \mathbb{E}|H-H^{(n)}|^p_{L^p_T}\,\nu(|z|^p\mathbf{1}_{\{|z|>\eta\}})\\ &\le T^{1-\frac p2}\mathbb{E}|H-H^{(n)}|^p_{L^2_T}\,\nu(|z|^p\mathbf{1}_{\{|z|>\eta\}})\\ &\le T^{1-\frac p2}\left(\mathbb{E}|H-H^{(n)}|^2_{L^2_T}\right)^{\frac p2}\nu(|z|^p\mathbf{1}_{\{|z|>\eta\}}).\end{aligned}$$

On the other hand,

$$\mathbb{E}\Big|\sum_{0<s\le T}(H_s-H^{(n)}_s)^2(\Delta Z_s)^2\mathbf{1}_{\{|\Delta Z_s|\le\eta\}}\Big|=\mathbb{E}|H-H^{(n)}|^2_{L^2_T}\,\nu(z^2\wedge\eta).$$

We derive from (3.14) that the above three terms go to 0 as n goes to infinity so that

$$\sup_{t\in[0,T]}\Big|\int_0^t H^{(n)}_s dZ_s-\int_0^t H_s dZ_s\Big|\xrightarrow{L^p(\mathbb{P})}0.$$

Then, Lemma 2.3 applied to the subsequence $(m(n))_{n\ge1}$ implies that the stepwise constant process $\left(\int_0^{\underline{t}_{m(n)}} H^{(n)}_s dZ_s\right)_{t\in[0,T]}$ satisfies

$$\mathrm{dist}_{Sk}\left(\int_0^{\underline{\cdot}_{m(n)}} H^{(n)}_s dZ_s,\int_0^{\cdot} H_s dZ_s\right)\xrightarrow{\mathbb{P}}0$$

which in turn implies the functional Sk-weak convergence. Furthermore, the above L^p-convergence implies that the sequence $\left(\sup_{t\in[0,T]}\Big|\int_0^t H^{(n)}_s dZ_s\Big|\right)_{n\ge1}$ is uniformly L^p-integrable which is also clearly true for $\left(\sup_{t\in[0,T]}\Big|\int_0^{\underline{t}_{m(n)}} H^{(n)}_s dZ_s\Big|\right)_{n\ge1}$. Following the same lines and still using Lemma 2.3, we get

$$\mathrm{dist}_{Sk}\left(\int_0^{\underline{\cdot}_{m(n)}} h^{(n)}_s dZ_s,\int_0^{\cdot} h_s dZ_s\right)\xrightarrow{\mathbb{P}\text{-}a.s.}0\text{ and }\left(\sup_{t\in[0,T]}\Big|\int_0^t h^{(n)}_s dZ_s\Big|\right)_{n\ge1}\text{ is uniformly }L^p\text{-integrable.}$$

Since $0 \leq H_t \leq h(t)$ dt-$a.e.$ $\mathbb{P}$-$a.s.$ (or $0 \leq |H_t| \leq h_t$ if Z is symmetric), for every fixed integer $n \geq 1$, we have, owing to Step 1 and following the lines of Step 3 of the proof of Proposition 3.1,

$$\mathbb{E}\left(F\Big(\int_0^{t_{m(n)}} H_s^{(n)} dZ_s\Big)_{t\in[0,T]}\right) \leq \mathbb{E}\left(F\Big(\int_0^{t_{m(n)}} h_s^{(n)} dZ_s\Big)_{t\in[0,T]}\right).$$

Letting $n \to +\infty$ yields the announced result since F is Sk-continuous with $(p, \|\,.\,\|_{\sup})$-polynomial growth (owing to the above uniform L^p-integrability results).

Assume now $p > 2$. First note that since h is bounded one can extend (3.14) as follows: there exists a sequence $m(n) \uparrow +\infty$ such that the processes $H^{(n)}$ and the functions $h^{(n)}$ defined by (3.13) satisfy

$$\mathbb{E}|H - H^{(n)}|^p_{L^p_T} + |h - h^{(n)}|_{L^p_T} \longrightarrow 0 \ \text{ as } \ n \to +\infty. \tag{3.19}$$

To this end, we introduce the dyadic logarithm m of p *i.e.* the integer ℓ_p such that where $2^{\ell_p} < p \leq 2^{\ell_p+1}$. Thus, if $p \in (2,4]$ *i.e.* $\ell_p = 1$,

$$\left\|\sup_{t\in[0,T]}\Big|\int_0^t H_s dZ_s - \int_0^t H_s^{(n)} dZ_s\Big|\right\|_p \leq c_p\left(\kappa\Big\||H - H^{(n)}|_{L^2_T}\Big\|_p + \Big\|\sum_{0<s\leq T}(H_s - H_s^{(n)})^2(\Delta Z_s)^2\Big\|^{\frac12}_{\frac p2}\right). \tag{3.20}$$

Now, Minkowski's Inequality applied with $\|.\|_{\frac p2}$ yields

$$\begin{aligned}\Big\|\sum_{0<s\leq T}(H_s - H_s^{(n)})^2(\Delta Z_s)^2\Big\|_{\frac p2} \ &\leq\ \Big\|\sum_{0<s\leq T}(H_s - H_s^{(n)})^2(\Delta Z_s)^2 - \nu(z^2)\int_0^T (H_s^{(n)} - H_s)^2 ds\Big\|_{\frac p2}\\ &\quad +\nu(z^2)\Big\||H^{(n)} - H|_{L^2_T}\Big\|^2_p.\end{aligned}$$

In turn, the $B.D.G.$ Inequality applied to the martingale

$$M_t^{(1)} = \sum_{0<s\leq t}(H_s - H_s^{(n)})^2(\Delta Z_s)^2 - \nu(z^2)\int_0^t (H_s^{(n)} - H_s)^2 ds,\ t \in [0,T],$$

yields

$$\begin{aligned}\Big\|\sum_{0<s\leq T}(H_s - H_s^{(n)})^2(\Delta Z_s)^2 - \nu(z^2)\int_0^T (H_s^{(n)} - H_s)^2 ds\Big)\Big\|_{\frac p2} \ &\leq\ c_{\frac p2}\Big\|\sum_{0<s\leq T}(H_s - H_s^{(n)})^4(\Delta Z_s)^4\Big\|^{\frac12}_{\frac p4}\\ &\leq\ c_{\frac p2}\Big(\mathbb{E}\sum_{0<s\leq T}(H_s - H_s^{(n)})^p|\Delta Z_s|^p\Big)^{\frac2p}\\ &=\ c_{\frac p2}\Big(\nu(|z|^p)\mathbb{E}\int_0^T |H_s - H_s^{(n)}|^p ds\Big)^{\frac2p}\\ &=\ c_{\frac p2}\Big(\nu(|z|^p)\Big)^{\frac2p}\Big\||H - H^{(n)}|_{L^p_T}\Big\|^2_p\end{aligned}$$

where we successively used that $\frac p4 \leq 1$ in the second line and the compensation formula in the third line. Finally, we note that, as $p \geq 2$,

$$\Big\||H^{(n)} - H|_{L^2_T}\Big\|_p \leq T^{\frac12 - \frac1p}\Big\||H^{(n)} - H|_{L^p_T}\Big\|_p \leq T^{\frac12 - \frac1p}\Big\||H^{(n)} - H|_{L^p_T}\Big\|_p \to 0 \text{ as } \ n \to +\infty$$

owing to (3.19). This shows that both terms in the right hand side of (3.20) converge to 0 as $n \to +\infty$, so that

$$\left\|\sup_{t\in[0,T]}\Big|\int_0^t H_s^{(n)}dZ_s - \int_0^t H_s dZ_s\Big|\right\|_p \longrightarrow 0 \;\text{ as } n\to+\infty.$$

We show likewise

$$\left\|\sup_{t\in[0,T]}\Big|\int_0^t h_s^{(n)}dZ_s - \int_0^t h_s dZ_s\Big|\right\|_p \longrightarrow 0 \;\text{ as } n\to+\infty.$$

These two convergences imply the $L^p(\mathbb{P})$-uniform integrability of both sequences $\left(\sup_{t\in[0,T]}\Big|\int_0^t H_s^{(n)}dZ_s\Big|\right)_{n\geq 1}$ and $\left(\sup_{t\in[0,T]}\Big|\int_0^t h_s^{(n)}dZ_s\Big|\right)_{n\geq 1}$. At this stage, one concludes like in the case $p\in(1,2]$.

In the general case, one proceeds by a classical "cascade" argument based on repeated applications of the $B.D.G.$ Inequality involving the martingales (see the proof of Proposition B.2 in Appendix B for a more detailed implementation this cascade procedure in a similar situation)

$$M_t^{(k)} = \sum_{0\leq s\leq t}(H_s^{(n)}-H_s)^{2^k}(\Delta Z_s)^{2^k} - \nu(|z|^{2^k})\int_0^t (H_s^{(n)}-H_s)^{2^k}ds,\; t\geq 0,\; k=1,\ldots,\ell_p.$$

We show by switching from p to $p/2$, $p/2^2,\ldots,p/2^k,\ldots$ until we get $p/2^{\ell_p}\in(1,2]$ when $k=\ell_p$, that

$$\begin{aligned}\left\|\sup_{t\in[0,T]}\Big|\int_0^t H_s dZ_s - \int_0^t H_s^{(n)}dZ_s\Big|\right\|_p &\leq c_p\,\kappa\Big\||H-H^{(n)}|_{L^2_T}\Big\|_p \\ &\quad +\kappa_{p,\nu}\sum_{\ell=1}^{\ell_p}\Big\||H^{(n)}-H|_{L^{2^\ell}_T}\Big\|_p^2+\Big\||H^{(n)}-H|_{L^p_T}\Big\|_p^2.\end{aligned}$$

One shows likewise the counterpart related to h and $h^{(n)}$.

Step 3 (Second approximation). Now we have to get rid of the boundedness of h. Like in the Brownian Itô case, we approximate h by $h\wedge K$ and H by $\chi_K(H)$ where the thresholding function χ_K have been introduced in Step 3 of the proof of Theorem 2.2 (to take into account at the same time the symmetric and the standard settings for the Lévy process Z). Let $p\in(1,+\infty)$.

$$\begin{aligned}\left\|\sup_{t\in[0,T]}\Big|\int_0^t H_s dZ_s - \int_0^t \chi_K(H_s)dZ_s\Big|\right\|_p &\leq c_p\left(\kappa\Big\||H-\chi_K(H)|_{L^2_T}\Big\|_p+\Big\|\sum_{0<s\leq T}(H_s-\chi_K(H_s))^2(\Delta Z_s)^2\Big\|_{\frac{p}{2}}\right)\\ &= c_p\left(\kappa\Big\||(|H|-K)_+|_{L^2_T}\Big\|_p+\Big\|\sum_{0<s\leq T}(|H_s|-K)_+^2(\Delta Z_s)^2\Big\|_{\frac{p}{2}}\right)\\ &\leq c_p\left(\kappa|(h-K)_+|_{L^2_T}+\Big\|\sum_{0<s\leq T}(h_s-K)_+^2(\Delta Z_s)^2\Big\|_{\frac{p}{2}}\right).\end{aligned}$$

We derive again by this cascade argument that $\Big\|\sum_{0<s\leq T}(h_s-K)_+^2(\Delta Z_s)^2\Big\|_{\frac{p}{2}}$ can be upper-bounded by linear combinations of quantities of the form

$$|(h-K)_+|_{L_T^{2^k}}\nu(z^{2^k}),\; 0\leq k\leq \ell_p$$

and

$$\mathbb{E}\sum_{0<s\leq T}(h_s-K)_+^p|\Delta Z_s|^p = |(h-K)_+|^p_{L^p_T}\nu(|z|^p).$$

Consequently, if $h\in L^p_T$, all these quantities go to zero as $K\to+\infty$n owing to the Lebesgue dominated convergence theorem. In turn this implies that

$$\Big\|\sup_{t\in[0,T]}\Big|\int_0^t H_s dZ_s - \int_0^t \chi_K(H_s)dZ_s\Big|\Big\|_p \longrightarrow 0 \;\text{ as } K\to+\infty.$$

The same holds with h and $h\wedge K$. So it is possible to let K go to infinity in the inequality

$$\mathbb{E}F\Big(\int_0^{\cdot}\chi_K(H_s)dZs\Big)\leq \mathbb{E}F\Big(\int_0^{\cdot}\chi_K(h_s)dZs\Big)$$

to get the expected result.

(b) is proved adapting the lines of the proof Proposition 3.1(b) as we did for (a). The main point is to get rid of the boundedness of h *i.e.* to obtain the conclusion of the above Step 3 without "domination property" of H by h. The additional assumption $|H|_{L^{p\vee 2}_T}\in L^p$ clearly yields to the expected conclusion.

(c) This follows from a careful reading of the proof, having in mind that terms of the form $\Big\||H-H^{(n)}|_{L^2_T}\Big\|_p$ vanish when $\kappa=0$. □

3.2.1 Doléans (Brownian) martingales

The dame methods applied to Doléans exponential yields similar result holds with direct applications to the robustness of Black-Scholes formula for option pricing. First we recall that the Doléans exponential of a continuous local martingale $(M_t)_{t\in[0,T}$ is continuous local martingale defined by

$$\mathcal{E}\big(M\big)_t = e^{M_t-\frac{1}{2}\langle M\rangle_t},\; t\in[0,T].$$

It is a martingale on $[0,T]$ if an sonly if $\mathbb{E}\,e^{M_t-\frac{1}{2}\langle M\rangle_t}=1$. A practical criterion, due to Novikov, says that, s a martingale on $[0,T]$ as soon as $\mathbb{E}\,e^{\frac{1}{2}\langle M\rangle_t}<+\infty$.

Proposition 3.3. *Let $(H_t)_{t\in[0,T]}$ and $h=(h_t)_{t\in[0,T]}$ be like in Proposition 3.1. Let $F:\mathcal{C}([0,T],\mathbb{R}_+)\to\mathbb{R}$ be a convex functional with $(r,\|\,.\,\|_{\sup})$-polynomial growth ($r\geq 1$).*
(a) If ($|H_t|\leq h_t$ dt-a.e.) $\mathbb{P}$-a.s., then

$$\mathbb{E}\,F\left(\mathcal{E}\Big(\int_0^{\cdot}H_s dW_s\Big)\right)\leq \mathbb{E}\,F\left(\mathcal{E}\Big(\int_0^{\cdot}h_s dW_s\Big)\right).$$

(b) If ($H_t\geq h_t\geq 0$ dt-a.e.) $\mathbb{P}$-a.s. and there exists $\varepsilon>0$ such that

$$\mathbb{E}\Big(e^{\frac{r^2+\varepsilon}{2}|H|_{L^2_T}}\Big)<+\infty,$$

then

$$\mathbb{E}\,F\left(\mathcal{E}\Big(\int_0^{\cdot}H_s dW_s\Big)\right)\geq \mathbb{E}\,F\left(\mathcal{E}\Big(\int_0^{\cdot}h_s dW_s\Big)\right).$$

Proof. (*a*) STEP 1: For a fixed integer $n \geq 1$, we consider the sequence of random variables $(\Xi^n_k)_{k=0:n}$ recursively defined in a forward way by

$$\Xi^n_0 = 1 \quad \text{and} \quad \Xi^n_k = \Xi^n_{k-1} \exp\Big(H_{t^n_{k-1}} \Delta W_{t^n_k} - \frac{T}{2n} H^2_{t^n_{k-1}}\Big), \quad k = 1, \ldots, n,$$

(where $\Delta W_{t^n_k} = W_{t^n_k} - W_{t^n_{k-1}}$) and the sequences $(\xi^{n,k}_\ell)_{\ell=k:n}$defined, still in a recursive forward way, by

$$\xi^{n,k}_k = 1,\ \xi^{n,k}_\ell = \xi^{n,k}_{\ell-1} \exp\Big(h_{t^n_{\ell-1}} \Delta W_{t^n_\ell} - \frac{T}{2n} h^2_{t^n_{\ell-1}}\Big),\ \ell = k+1, \ldots, n.$$

We denote by $\widetilde{Q}^{(n)}$ the operator defined on Borel functions $f : \mathbb{R}_+ \to \mathbb{R}$ with polynomial growth by

$$\forall\, x,\, h \in \mathbb{R}_+, \quad \widetilde{Q}^{(n)}(f)(x,h) = \mathbb{E}\, f\Big(x \exp\big(hW_{\frac{T}{n}} - \frac{T}{2n} h^2\big)\Big).$$

It is clear that $\Big(\exp\big(hW_{\frac{T}{n}} - \frac{T}{2n}h^2\big)\Big)_{h\geq 0}$ is increasing for the convex order (*i.e.* a peacock as already mentioned on the introduction since $\exp\big(hW_{\frac{T}{n}} - \frac{T}{2n}h^2\big) \overset{d}{\sim} \exp\big(W_{h^2\frac{T}{n}} - \frac{1}{2}\frac{T}{n}h^2\big)$ and $(e^{W_u - \frac{u}{2}})_{u\geq 0}$ is a martingale. Hence, as soon as f is convex,

$$h \mapsto \widetilde{Q}^{(n)}(f)(x,h) \text{ satisfies the maximum principle } i.e. \text{ is even and)non-decreasing on } \mathbb{R}_+. \tag{3.21}$$

In turn, it implies that the function $(x,h) \mapsto \widetilde{Q}^{(n)}(f)(x,h)$ is convex on $\mathbb{R} \times \mathbb{R}_+$ since for every $x, x' \in \mathbb{R}_+$, $h, h' \in \mathbb{R}$, $\lambda \in [0,1]$,

$$\begin{aligned}
&\mathbb{E}\, f\Big(\lambda x \exp\big(\lambda h W_{\frac{T}{n}} - \frac{T}{2n}(\lambda h)^2\big) + (1-\lambda)x' \exp\big((1-\lambda)h' W_{\frac{T}{n}} - \frac{T}{2n}((1-\lambda)h')^2\big)\Big) \\
&\qquad \leq \lambda \mathbb{E}\, f\Big(x \exp\big(\lambda h W_{\frac{T}{n}} - \frac{T}{2n}(\lambda h)^2\big)\Big) + (1-\lambda)\mathbb{E}\, f\Big(x' \exp\big((1-\lambda)h' W_{\frac{T}{n}} - \frac{T}{2n}((1-\lambda)h')^2\big)\Big) \\
&\qquad \leq \lambda \mathbb{E}\, f\Big(x \exp\big(|h| W_{\frac{T}{n}} - \frac{T}{2n}h^2\big)\Big) + (1-\lambda)\mathbb{E}\, f\Big(x' \exp\big(|h'| W_{\frac{T}{n}} - \frac{T}{2n}(h')^2\big)\Big) \\
&\qquad = \lambda \mathbb{E}\, f\Big(x \exp\big(h W_{\frac{T}{n}} - \frac{T}{2n}h^2\big)\Big) + (1-\lambda)\mathbb{E}\, f\Big(x' \exp\big(h' W_{\frac{T}{n}} - \frac{T}{2n}(h')^2\big)\Big)
\end{aligned}$$

where we used the convexity of f in the first inequality and (3.21) in the second one. From now on, we consider the discrete time filtration $\mathcal{G}^n_k = \mathcal{F}^W_{t^n_k}$ and set $\mathbb{E}_k = \mathbb{E}(\,.\,|\mathcal{G}^n_k)$.

We temporarily assume that for every $k = 0, \ldots, n$, $|H_{t^n_k}| \leq h_{t^n_k}$ $\mathbb{P}$-*a.s.*. Let $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$ be a (Borel) functional with $(r, \|\,.\,\|_{\sup})$-polynomial growth and let $F_n = F \circ i_n$. We will show by induction that, for every $k \in \{1, \ldots, n\}$,

$$\mathbb{E}_{k-1} F_n(\Xi^n_{0:k-1}, \Xi^n_k\, \xi^{n,k}_{k:n}) \leq \mathbb{E}_{k-1} F_n(\Xi^n_{0:k-2}, \Xi^n_{k-1} \xi^{n,k-1}_{k-1:n}) \tag{3.22}$$

with the obvious convention $\Xi^n_{0:-1} = \emptyset$. Starting from the identity

$$F_n(\Xi^n_{0:k-1}, \Xi^n_k\, \xi^{n,k}_{k:n}) \;=\; F_n\Big(\Xi^n_{0:k-1}, \Xi^n_{k-1} \exp\big(H_{t^n_{k-1}} \Delta W_{t^n_k} - \frac{T}{2n} H^2_{t^n_{k-1}}\big)\xi^{n,k}_{k:n}\Big),$$

we derive

$$\mathbb{E}_{k-1} F_n(\Xi^n_{0:k-1}, \Xi^n_k\, \xi^{n,k}_{k:n}) \;=\; \Big(\mathbb{E}\Big(F(x_{0:k-1}, x_{k-1} \exp\big(\eta \Delta W_{t^n_k} - \frac{T}{2n}\eta^2\big)\xi^{n,k}_{k:n})\Big)\Big)_{|x_{0:k-1} = \Xi^n_{0:k-1}, \eta = H_{t^n_{k-1}}}$$

since $(\Xi^n_{0:k-1}, H_{t^n_{k-1}})$ is $\mathcal{G}^n_{k-1}$-measurable and $(\Delta W_{t^n_k}, \xi^{n,k}_{k:n})$ is independent of $\mathcal{G}^n_{k-1}$. Now set, for every $x_{0:k-1} \in \mathbb{R}^k_+$, $\widetilde{x}_k \in \mathbb{R}_+$,

$$G_{n,k}(x_{0:k-1}, \widetilde{x}_k) = \mathbb{E} F_n\big(x_{0:k-1}, \widetilde{x}_k\, \xi^{n,k}_{k:n}\big)$$

so that

$$\widetilde{Q}^{(n)}(G_{n,k}(x_{0:k-1},.))(x_{k-1},\eta) = \mathbb{E}\,F_n\Big(x_{0:k-1}, x_{k-1}\exp\big(\eta\Delta W_{t^n_k} - \frac{T}{2n}\eta^2\big)\xi^{n,k}_{k:n}\Big).$$

The function F_n being convex on $\mathbb{R}^{n+1}_+$, it is clear that $G_{n,k}$ is convex on $\mathbb{R}^{k+1}_+$ as well. It is in particular convex in the variable $\widetilde{x}_k$ which in turn implies by (3.21) that $\eta \mapsto \widetilde{Q}^{(n)}(G_{n,k}(x_{0:k-1},.))(x_{k-1},\eta)$ satisfies the maximum principle *i.e.* is even and convex. As a consequence, $|H_{t^n_{k-1}}| \le h_{t^n_{k-1}}$ implies

$$\begin{aligned}
\mathbb{E}_{k-1}F_n(\Xi^n_{0:k-1}, \Xi^n_k\,\xi^{n,k}_{k:n}) &= \Big[\widetilde{Q}^{(n)}\big(G_{n,k}(x_{0:k-1},\,.\,)\big)(x_{k-1},\eta)\Big]_{|x_{0:k-1}=\Xi^n_{0:k-1},\eta=H_{t^n_{k-1}}}\\
&= \Big[\widetilde{Q}^{(n)}\big(G_{n,k}(x_{0:k-1},\,.\,)\big)(x_{k-1},\eta)\Big]_{|x_{0:k-1}=\Xi^n_{0:k-1},\eta=|H_{t^n_{k-1}}|}\\
&\le \Big[\widetilde{Q}^{(n)}\big(G_{n,k}(x_{0:k-1},\,.\,)\big)(x_{k-1},\eta)\Big]_{|x_{0:k-1}=\Xi^n_{0:k-1},\eta=h_{t^n_{k-1}}}\\
&= \mathbb{E}_{k-1}\Big(F_n\big(\Xi^n_{0:k-1}, \Xi^n_{k-1}\exp\big(h_{t^n_{k-1}}\Delta W_{t^n_k} - \frac{T}{2n}h^2_{t^n_{k-1}}\big)\xi^{n,k}_{k:n}\big)\Big)\\
&= \mathbb{E}_{k-1}\Big(F_n\big(\Xi^n_{0:k-2}, \Xi^n_{k-1}\xi^{n,k-1}_{k-1:n}\big)\Big)
\end{aligned}$$

where we used once again that $\xi^{n,k}_{k:n}$ is independent of $\mathcal{G}^n_{k-1}$ in the penultimate line.

One derives by taking expectation of the resulting inequality that the sequence $\mathbb{E}_{k-1}F_n(\Xi^n_{0:k-1}, \Xi^n_k\,\xi^{n,k}_{k:n})$, $k = 1:n$, is non-increasing. Finally, by comparing the terms for $k = n$ and $k = 0$, we get

$$\mathbb{E}F\big(X^{n,n}\big) = \mathbb{E}\,F_n(\Xi^n_{0:n}) \le \mathbb{E}\,F_n(\xi^{n,0}_{0:n}) = \mathbb{E}F\big(X^{n,0}\big).$$

Step 2 (Approximation-Regularization). We closely follow the approach developed in Steps 2 and 3 of Proposition 3.1. First, we temporarily assume that h is bounded by a real constant K and we introduce the stepwise constant càglàd processes $(H^{(n)})_{t\in[0,T]}$ and $(h^{(n)}_t)_{t\in[0,T]}$ defined by (3.13) (and satisfying (3.14)), namely

$$\Big\|\big|H^{(n)} - H\big|_{L^2_T}\Big\|_2 + \big|h^{(n)} - h\big|_{L^2_T} \longrightarrow 0 \;\text{ as }\; n \to +\infty.$$

In particular

$$\sup_{t\in[0,T]}\Big|\int_0^t (H^{(n)}_s)^2 ds - \int_0^t H^2_s ds\Big| \le 2K\big|H^{(n)} - H\big|_{L^1_T} \le 2K\sqrt{T}\big|H^{(n)} - H\big|_{L^2_T}.$$

As a consequence

$$\begin{aligned}
\sup_{t\in[0,T]}\Big|\int_0^t H^{(n)}_s dW_s - \frac12\int_0^t (H^{(n)}_s)^2 ds - \Big(\int_0^t H_s dW_s - \frac12\int_0^t H^2_s ds\Big)\Big| &\le \sup_{t\in[0,T]}\Big|\int_0^t (H^{(n)}_s - H_s)dW_s\Big|\\
&\quad + K\sqrt{T}\big|H^{(n)} - H\big|_{L^2_T}.
\end{aligned}$$

Set for notational convenience

$$X^{(n)}_t = \mathcal{E}\Big(\int_0^{\cdot} H^{(n)}_s dW_s\Big)_t \quad\text{ and }\quad X_t = \mathcal{E}\Big(\int_0^{\cdot} H_s dW_s\Big)_t,\; t\in[0,T].$$

which are both true martingales owing to Novikov's criterion. The above inequality combined with Doob's Inequality implies that

$$\sup_{t\in[0,T]}\Big|\log X^{(n)}_t - \log X_t\Big| \xrightarrow{L^2} 0 \;\text{ as }\; n \to +\infty.$$

As a consequence, $X^{(n)} \overset{\mathcal{L}(\|\,.\,\|_{\sup})}{\longrightarrow} X$ since the exponential function is continuous. Denoting by $x^{(n)}$ and x the counterpart of these processes for the functions $h^{(n)}$ and h, we get likewise $x^{(n)} \overset{\mathcal{L}(\|\,.\,\|_{\sup})}{\longrightarrow} x$. Owing once again to Lemma 2.2, the continuity of the exponential again, and the chain rule for weak convergence, we finally get

$$e^{I_{m(n)}(\log X^{(n)})} \overset{\mathcal{L}(\|\,.\,\|_{\sup})}{\longrightarrow} e^{\log X} = X \quad \text{and} \quad e^{I_{m(n)}(\log x^{(n)})} \overset{\mathcal{L}(\|\,.\,\|_{\sup})}{\longrightarrow} e^{\log x} = x \quad \text{as} \quad n \to +\infty.$$

Applying Step 1 with $X^{(n)}$ and $x^{(n)}$

$$\forall\, n \in \mathbb{N}, \quad \mathbb{E}\, F(X^{(n)}) \le \mathbb{E}\, F(x^{(n)}).$$

To let n go to infinity in this inequality, we again need a uniform integrability argument namely that $\|X^{(n)}\|_{\sup}$ and $\|x^{(n)}\|_{\sup}$ are both L^p-bounded for a $p > r$ since the functional F has at most a $(r, \|\,.\,\|_{\sup})$-polynomial growth. So, let $p > r \vee 1$. It follows from Doob's Inequality applied to the non-negative sub-martingale$(X^{(n)})^p$ that

$$\begin{aligned}
\mathbb{E}\Big(\sup_{t\in[0,T]} (X_t^{(n)})^p\Big) &\le \Big(\frac{p}{p-1}\Big)^p \mathbb{E}\big(X_T^{(n)}\big)^p \\
&\le \Big(\frac{p}{p-1}\Big)^p \mathbb{E}\left(\mathcal{E}\big(p\int_0^{\cdot} H_s^{(n)} dW_s\big)_T\right) e^{\frac{p(p-1)}{2}\int_0^T (H_s^{(n)})^2 ds} \\
&\le \Big(\frac{p}{p-1}\Big)^p e^{\frac{p(p-1)}{2}K^2 T}
\end{aligned}$$

where we used that $\left(\mathcal{E}\Big(p\int_0^{\cdot} H_s^{(n)} dW_s\Big)_t\right)_{t\ge 0}$ is a true martingale (owing to Novikov' criterion). The case of $F(x^{(n)})$ follows likewise.

Step 3: The extension to $h \in L_T^2$ is similar that performed in the former propositions: first note that

$$\mathcal{E}\Big(\int_0^{\cdot} \chi_K(H_s) dW_s\Big) \overset{\mathcal{L}(\|\,.\,\|_{\sup})}{\longrightarrow} \mathcal{E}\Big(\int_0^{\cdot} H_s dW_s\Big) \quad \text{as} \quad K \to +\infty.$$

The uniform integrality of $\sup_{t\in[0,T]} \mathcal{E}\Big(\int_0^{\cdot} \chi_K(H_s) dW_s\Big)_t$ as K grows to infinity follows form its $L^p(\mathbb{P})$-boundedness for a $p \in (1, +\infty)$ which in turn is a consequence of Doob's inequality:

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,T]} \left(\mathcal{E}\Big(\int_0^{\cdot} \chi_K(H_s) dW_s\Big)_t\right)^p &\le \Big(\frac{p}{p-1}\Big)^p \mathbb{E}\, \mathcal{E}\Big(\int_0^{\cdot} \chi_K(H_s) dW_s\Big)_T^p \\
&\le \Big(\frac{p}{p-1}\Big)^p e^{\frac{p(p-1)}{2}\int_0^T \chi_K^2(h_s) ds}\, \mathbb{E}\, \mathcal{E}\Big(p\int_0^{\cdot} \chi_k(H_s) dW_s\Big)_T \\
&= \Big(\frac{p}{p-1}\Big)^p e^{\frac{p(p-1)}{2}\int_0^T \chi_K^2(h_s) ds} \\
&\le \Big(\frac{p}{p-1}\Big)^p e^{\frac{p(p-1)}{2}|h|_{L_T^2}} < +\infty
\end{aligned}$$

which yields L^p -boundedness with respect to the threshold K.

(b) The discrete time part can be established by adapting item (a) in the spirit of Proposition 3.1(b). The approximation step follows like above as well, except for the final uniform integrability argument which needs specific care. It suffices to show that for an $r' > r$, $\sup_{t\in[0,T]} \mathcal{E}\Big(\int_0^{\cdot} \chi_K(H_s) dW_s\Big)_t$ is $L^{r'}$-bounded as $K \to +\infty$.

$\triangleright$ If $r \in (0,1)$, one may choose $r' \in (r,1)$. Then, one checks that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \mathcal{E}\Big(\int_0^{\cdot} \chi_K(H_s)dW_s\Big)_t^{r'}\right] \; le \mathbb{E}\left[\sup_{t\in[0,T]} \mathcal{E}\Big(\int_0^{\cdot} \chi_K(H_s)dW_s\Big)_t\right].$$

Now if $p > 1$, Doob's Inequality implies

$$\mathbb{E}\left[\sup_{t\in[0,T]} \mathcal{E}\Big(\int_0^{\cdot} \chi_K(H_s)dW_s\Big)_t^p\right] \le \Big(\frac{p}{p-1}\Big)^p \mathbb{E}\left[\sup_{t\in[0,T]} \mathcal{E}\Big(\int_0^{\cdot} \chi_K(H_s)dW_s\Big)_T^p\right] \le \Big(\frac{p}{p-1}\Big)^p.$$

which yields the announced result.

$\triangleright$ If $r \ge 1$, let $p > r$. Combining successively Doob's Inequality and Hölder's Inequality, for every $p > r \vee 1$ and every Hölder conjugate exponents $\lambda, \mu = \frac{\lambda}{\lambda-1} > 1$, leads to

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,T]} \left(\mathcal{E}\Big(\int_0^{\cdot} \chi_K(H_s)dW_s\Big)_t\right)^p &\le \Big(\frac{p}{p-1}\Big)^p \mathbb{E}\,\mathcal{E}\Big(\int_0^{\cdot} \chi_K(H_s)dW_s\Big)_T^p \\
&\le \Big(\frac{p}{p-1}\Big)^p \left[\underbrace{\mathbb{E}\,\mathcal{E}\Big(\lambda p \int_0^{\cdot} \chi_K(H_s)dW_s\Big)_T}_{=1}\right]^{\frac{1}{\lambda}} \left[\mathbb{E}\, e^{\frac{\lambda p^2}{2}\int_0^T \chi_K(H_s)^2 ds}\right]^{\frac{\lambda-1}{\lambda}} \\
&\le \Big(\frac{p}{p-1}\Big)^p \left[\mathbb{E}\, e^{\frac{\lambda p^2}{2}|H|^2_{L^2_T}}\right]^{\frac{\lambda-1}{\lambda}}.
\end{aligned}$$

Consequently, for λ close enough to 1 and p close enough to r, we have $\lambda p^2 \le r^2 + \varepsilon$ which ensures the $L^p(\mathbb{P})$-boundedness as $K \uparrow +\infty$. $\square$

3.2.2 A counter-example

The counter-example below shows that Theorem 3.1 is no longer true if we relax the assumption that the dominating process $(h_t)_{t\in[0,T]}$ is deterministic.

Let $X = X^\sigma = (X^\sigma_{0:2})$ be a two period process satisfying

$$X_0 = 0, \quad X_1 = \sigma Z_1 \quad \text{and} \quad X_2 = X_1 + \sqrt{2v(Z_1)}Z_2$$

where $Z_{1:2} \overset{\mathcal{L}}{\sim} \mathcal{N}(0; I_2)$, $\sigma \ge 0$, and $\varphi : \mathbb{R} \to \mathbb{R}_+$ is a bounded non-increasing function.

Let $f(x) = e^x$ and let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by

$$\varphi(\sigma) := \mathbb{E} f(X_2) = \mathbb{E}\big(e^{\sigma Z_1 + v(Z_1)}\big).$$

Differentiating φ yields

$$\varphi'(\sigma) = \mathbb{E}\big(e^{\sigma Z_1 + v(Z_1)} Z_1\big)$$

so that

$$\varphi'(0) = \mathbb{E}\big(e^{v(Z_1)} Z_1\big) < \mathbb{E}\, e^{v(Z_1)} \mathbb{E}\, Z_1 = 0$$

by a standard one-dimensional co-monotony argument: both functions $z \mapsto e^{v(z)}$, $z \mapsto z$ are non-decreasing which implies $\varphi'(0) \le 0$ but none of them are $\mathbb{P}_{Z_1}$-$a.s.$ constant, hence equality cannot hold. As a consequence, φ is (strictly) decreasing on a right neighbourhood $[0, \sigma_0]$, $\sigma_0 > 0$, of 0.

To include this into a Brownian stochastic integral framework, one proceeds as follows: let W be a standard Brownian motion and $\sigma, \widetilde{\sigma} \in (0, \sigma_0]$, $\sigma < \widetilde{\sigma}$.

$$H_t = \sigma \mathbf{1}_{[0,1]}(t) + \sqrt{2v(W_1)}\mathbf{1}_{(1,2]}(t), \; \widetilde{H}_t = \widetilde{\sigma}\mathbf{1}_{[0,1]}(t) + \sqrt{2v(W_1)}\mathbf{1}_{(1,2]}(t).$$

It is clear that $0 \le H_t \le \widetilde{H}_t$, $t \in [0,2]$, whereas

$$\mathbb{E}\Big(e^{\int_0^2 H_s dW_s}\Big) > \mathbb{E}\Big(e^{\int_0^2 \widetilde{H}_s dW_s}\Big).$$

This makes up a counter-example to the conclusion of Proposition 3.1.

It has to be noted that if the function v is non-decreasing, then choosing $f(x) = e^{-x}$ leads to a similar result since

$$\psi(\sigma) := \mathbb{E} f(X_2) = \mathbb{E}\big(e^{-\sigma Z_1 + v(Z_1)}\big)$$

satisfies $\Psi'(\sigma) = -\mathbb{E}\big(e^{-\sigma Z_1 + v(Z_1)}\big)$. In particular one still has by a co-monotony argument that $\psi'(0) < 0$ since v is not constant.

3.2.3 A comparison theorem for Laplace transforms of Brownian stochastic integrals

Applying our paradigm, we start by a discrete time result with its own interest for applications.

Proposition 3.4. *Let $(Z_k)_{1\le k\le n}$ be a sequence of $\mathcal{N}(0;1)$-random variables. We set $S_0 = 0$ and $S_k = Z_1 \cdots + Z_k$, $k = 1, \ldots, n$ (partial sums). We consider the two discrete time stochastic integrals*

$$X_k = \sum_{\ell=1}^{k} f_\ell(S_{\ell-1})Z_\ell \quad \textit{and} \quad Y_k = \sum_{\ell=1}^{k} g_\ell(S_{\ell-1})Z_\ell, \quad k = 1, \ldots, n, \quad X_0 = Y_0 = 0$$

where f_k, $g_k : \mathbb{R} \to \mathbb{R}_+$, $k = 1, \ldots, n$ are non-negative Borel functions satisfying:

either all f_k, $k = 1, \ldots, n$, are non-decreasing or all g_k, $k = 1, \ldots, n$, are non-decreasing.

If, furthermore, $0 \le f_k \le g_k$ for all $k = 1, \ldots, n$, then

$$\forall \lambda \ge 0, \quad \mathbb{E}\, e^{\lambda X_n} \le \mathbb{E}\, e^{\lambda Y_n}.$$

Proof. We start from the Cameron-Martin identity which reads on Borel function $\varphi : \mathbb{R} \to \mathbb{R}$

$$\forall \sigma \in \mathbb{R}, \quad \mathbb{E}\, e^{\sigma Z + \varphi(Z)} = e^{\frac{\sigma^2}{2}}\,\mathbb{E}\, e^{\varphi(Z+\sigma)} \le +\infty.$$

First, we define in a backward way functions $\widetilde{f}_k$ and $\widetilde{g}_k$, $k = 1, \ldots, n+1$ by $\widetilde{f}_{n+1} = \widetilde{g}_{n+1} \equiv 0$,

$$\widetilde{f}_k(x) = \frac{\lambda^2}{2} f_k^2(x) + \log \mathbb{E}\Big(e^{\widetilde{f}_{k+1}(x+\lambda f_k(x)+Z)}\Big), \; k = 0, \ldots, n, \tag{3.23}$$

where $Z \sim \mathcal{N}(0;1)$. The functions $\widetilde{g}_k$ are defined from the g_k the same way round. Then, relying on the chaining rule for conditional expectations, we check by a backward induction that

$$\mathbb{E}\, e^{\lambda X_n} = \mathbb{E}\, e^{\lambda X_k + \widetilde{f}_{k+1}(S_k)}, \; k = 1, \ldots, n.$$

In particular, when $k = 0$, we get

$$\mathbb{E}\, e^{\lambda X_n} = e^{\widetilde{f}_1(0)}.$$

It follows from (3.23) and a second backward induction that, if the functions f_k are non-decreasing for every $k = 1, \ldots, n$, so are the functions $\widetilde{f}_k$. The same holds for $\widetilde{g}_k$ with respect to the functions g_k. Assume *e.g.* that all the functions $\widetilde{f}_k$ are non-decreasing. Then, a (third) backward induction shows: $\widetilde{f}_k \le \widetilde{g}_k$ since $\widetilde{f}_n \le \widetilde{g}_n$, and, if $\widetilde{f}_{k+1} \le \widetilde{g}_{k+1}$, then for every $x \in \mathbb{R}$,

$$\widetilde{f}_{k+1}(x + \lambda f_k(x) + Z) \le \widetilde{f}_{k+1}(x + \lambda g_k(x) + Z) \le \widetilde{g}_{k+1}(x + \lambda g_k(x) + Z).$$

Plugging this inequality in (3.23) combined with $f_k^2 \le g_k^2$, one concludes that $\widetilde{f}_k \le \widetilde{g}_k$. A similar reasoning can be carried out if the functions $\widetilde{g}_k$ are non-decreasing. $\Box$

By the standard weak approximation method detailed in the former results, we derive the following continuous time version of this result involving (non-decreasing) *completely monotone* functions defined below.

Definition 3.1. *A* non-decreasing *function $\varphi : \mathbb{R} \to \mathbb{R}$ is completely monotone if it is the Laplace transform of a non-negative Borel measure μ supported by the non-negative real line, namely*

$$\forall\, x \in \mathbb{R}, \quad \varphi(x) = \int_{\mathbb{R}_+} e^{\lambda x} \mu(d\lambda).$$

Theorem 3.1. *Let $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ two bounded Borel functions such that*

$$\begin{cases} (i) & f,\ g \text{ are } dt \otimes dx\text{-a.e. continuous,} \\ (ii) & 0 \le f \le g, \\ (iii) & \Big(\forall\, t \in [0,T],\ f(t,.) \text{ is non-decreasing}\Big) \text{ or } \Big(\forall\, t \in [0,T],\ g(t,.) \text{ is non-decreasing}\Big). \end{cases} \tag{3.24}$$

Then,

$$\forall\, \lambda \ge 0, \quad \mathbb{E}\, e^{\lambda \int_0^T f(t,W_t)dW_t} \le \mathbb{E}\, e^{\lambda \int_0^T g(t,W_t)dW_t}$$

so that, for every non-decreasing completely monotone *function $\varphi : \mathbb{R} \to \mathbb{R}_+$*

$$\mathbb{E}\, \varphi\left(\int_0^T f(t,W_t)dW_t\right) \le \mathbb{E}\, \varphi\left(\int_0^T g(t,W_t)dW_t\right).$$

Remarks. • The finiteness of these integrals follows from Novikov's criterion.
• One derives from (3.24) the seemingly more general result

$$\begin{cases} (i) & f, g \text{ are } dt \otimes dx\text{-a.e. continuous,} \\ (ii) & \exists\, h : [0,T] \times \mathbb{R} \to \mathbb{R}_+ \text{ such that } \begin{cases} (a) & 0 \le f \le h \le g \text{ and} \\ (b) & \forall\, t \in [0,T],\ h(t,.) \text{ is non-decreasing.} \end{cases} \end{cases} \tag{3.25}$$

Proof. Assume *e.g.* that $f(t,.)$ is non-decreasing for every $t \in [0,T]$. First note that by Fubini's Theorem and Itô's isometry

$$\left\| \int_0^T f(s,W_s)dW_s - \int_0^T f(\underline{s}_n, W_{\underline{s}_n})dW_s \right\|_2^2 = \int_0^T \mathbb{E}\big(f(s,W_s) - f(\underline{s}_n, W_{\underline{s}_n})\big)^2 ds.$$

Now, if we denote $C_s = \{x \in \mathbb{R} \,|\, f \text{ is continuous at } (s,x)\}$ for every $t \in [0,T]$, it follows from Assumption (3.24)(i) that $\lambda({}^cC_s) = 0$ ds-*a.e.* still by Fubini's Theorem. As $\mathbb{P}_{X_s}$ is equivalent to the Lebesgue measure, one derives that $\mathbb{P}_s(C_s) = 1$ ds-*a.e.*. As a consequence, $\mathbb{E}\big(f(s,W_s) - f(\underline{s}_n, W_{\underline{s}_n})\big)^2 \to 0$ ds-*a.e.* as $n \to +\infty$ since $(\underline{s}_n, W_{\underline{s}_n}) \to (s, W_s)$. One concludes by the dominated Lebesgue theorem that $\left\| \int_0^T f(s,W_s)dW_s - \int_0^T f(\underline{s}_n, W_{\underline{s}_n})dW_s \right\|_2 \to 0$ since f is bounded.

Now, define for every $k = 1, \dots, n$,

$$X_k = \int_0^{t_k^n} f(\underline{s}_n, W_{\underline{s}_n})dW_s = \sum_{\ell=1}^{k} \sqrt{\frac{T}{n}} f(t_{\ell-1}^n, W_{t_{\ell-1}^n})U_\ell^n$$

where $U_\ell^n = \sqrt{\frac{n}{T}}(W_{t_\ell^n} - W_{t_{\ell-1}^n})$, $\ell = 1, \dots, n$. We define likewise $(Y_k)_{k=0:n}$ with respect to the function g. It is clear that both (X_k) and (Y_k) satisfy the assumptions of the above Proposition 3.4 so that

$$\forall\, \lambda \geq 0, \qquad \mathbb{E}\, e^{\lambda \int_0^T f(\underline{s}_n, W_{\underline{s}_n})dW_s} \leq \mathbb{E}\, e^{\lambda \int_0^T g(\underline{s}_n, W_{\underline{s}_n})dW_s}.$$

One concludes by combining the above quadratic (hence weak) convergence and the uniform integrability argument which follows from

$$\forall\, \lambda > 0, \qquad \sup_n \mathbb{E}\, e^{\lambda \int_0^T f(\underline{s}_n, W_{\underline{s}_n})dW_s} \leq e^{\frac{\lambda^2}{2}\|f\|_{\sup} T} < +\infty. \qquad \square$$

4 Convex order for the *réduite* and applications to path-dependent American options

In this section, we aim at applying the methodology developed in the former sections to Optimal Stopping Theory, *i.e.*, as far as financial applications are concerned, to Bermuda and American style options. For general background on Optimal Stopping theory, we refer to [29] (Chapter VI) and [7] (Chapter 5.1) in discrete time and, among others, to [9, 21, 35] in continuous time. For a discussion (and results) on comparison methods for American option prices, which usually includes an analytic component involving variational inequalities, we refer to [2] and the references therein.

4.1 Bermuda options

We start from the discrete time dynamics introduced in the "European" framework. Let $(Z_k)_{1\leq k\leq n}$ be a sequence of independent $\mathbb{R}^d$-valued random vectors satisfying $Z_k \in L^r(\Omega, \mathcal{A}, \mathbb{P})$, $r \geq 1$ and $\mathbb{E}\, Z_k = 0$, $k = 1, \dots, n$. Let $(X_k)_{0\leq k\leq n}$ and $(Y_k)_{0\leq k\leq n}$ be the two sequences of random vectors defined by (2.3) *i.e.*

$$X_{k+1}^x = X_k^x + \sigma_k(X_k^x)Z_{k+1}, \quad Y_{k+1}^x = Y_k^x + \theta_k(Y_k^x)Z_{k+1},\ 0 \leq k \leq n-1,\ X_0^x = Y_0^x = x$$

where σ_k, θ_k, $k = 0, \dots, n$ are functions from $\mathbb{R}$ to $\mathbb{R}$, all with linear growth. This implies by a straightforward induction that the random variables X_k^x and Y_k^x all lie in L^r since, *e.g.*, $\sigma_k(X_k^x)$ are adapted to $\mathcal{F}_k^Z$ hence independent of Z_{k+1}, $k = 0, \dots, n-1$.

Let $\mathcal{F} = (\mathcal{F}_k)_{0\leq k\leq n}$ and $\mathcal{G} = (\mathcal{G}_k)_{0\leq k\leq n}$ two filtrations on $(\Omega, \mathcal{A}, \mathbb{P})$ such that X^x is $\mathcal{F}$-adapted and Y^x is $\mathcal{G}$-adapted. Let $F_k : \mathbb{R}^{k+1} \to \mathbb{R}_+$, $k = 0, \dots, n$ be a sequence of non-negative functions with *r-polynomial growth* (*i.e.* $0 \leq F_k(x_{0:k}) \leq C(1 + |x_{0:k}|^r)$, $k = 0, \dots, n$), $r \geq 1$. Then the processes $\big(F_k(X_{0:k}^x)\big)_{0\leq k\leq n}$ and $(F_k(Y_{0:k}^x))_{0\leq k\leq n}$ are called *payoff* or *obstacle* processes ($\mathcal{F}$-adapted and $\mathcal{G}$-adapted respectively).

We define the $\mathcal{F}$- and $\mathcal{G}$-"*réduites*" associated to these payoff processes by

$$u_0(x) = \sup\big\{\mathbb{E}\, F_\tau(X_{0:\tau}^x),\ \tau\ \mathcal{F}\text{-stopping time}\big\} \quad \text{and} \quad v_0(x) = \sup\big\{\mathbb{E}\, F_\tau(Y_{0:\tau}^x),\ \tau\ \mathcal{G}\text{-stopping time}\big\}$$

respectively. These quantities are closely related to the optimal stopping problems attached to these dynamics since they represent the supremum of possible gains among "honest" stopping strategies

($i.e.$ non-anticipative with respect to the filtration) in a game where one wins $F_k(X^x_{0;k})$ ($F_k(Y^x_{0:k})$ respectively) when leaving the game at time k. Owing to the dynamic programing formula (see the proof of Proposition below) and the Markov property shared by both dynamics X^x and Y^x, it is clear that we may assume without loss of generality that $\mathcal{F} = \mathcal{F}^X$ (natural filtration of X^x) and $\mathcal{G} = \mathcal{F}^Y$ (natural filtration of Y^x) or even $\mathcal{F} = \mathcal{G} = \mathcal{F}^Z$ without changing the value of the *réduites*.

The proposition below is the counterpart of Proposition 2.1 in discrete time for "European" options.

Proposition 4.1. *Let $F_k : \mathbb{R}^{k+1} \to \mathbb{R}_+$, $k = 0, \ldots, n$, be a sequence of non-negative functions with r-polynomial growth ($r \geq 1$). Assume that all these functions F_k are* convex, *$k = 0, \ldots, n$.*

(a) Partitioning function*: If, for every $k \in \{0, \ldots, n-1\}$, there exists a convex function κ_k such that $0 \leq \sigma_k \leq \kappa_k \leq \theta_k$, then, for every $x \in \mathbb{R}$,*

$$u_0(x) \leq v_0(x).$$

(b) Dominating function*: If the random variable Z_k have symmetric distributions, the functions θ_k, $k = 1, \ldots, n$, are convex and $|\sigma_k| \leq \theta_k$, $k = 1, \ldots, n$, then the above inequality remains holds true.*

Remark. An equivalent formulation of claim (a) is: assume that both $(\sigma_k)_{0 \leq k \leq n}$ and $(\theta_k)_{0 \leq k \leq n}$ are non-negative convex functions with r-linear growth, then for every sequence $(\kappa_k)_{0 \leq k \leq n}$ of functions such that $\sigma_k \leq \kappa_k \leq \theta_k$, $k = 0, \ldots, n-1$,

$$u_0(x) \leq c_\kappa(x) \leq v_0(x)$$

where $c_\kappa(x)$ is the réduite of $(F_k(K^x_{0:k}))_{0 \leq k \leq n}$ where $(K^x_k)_{0 \leq k \leq n}$ satisfies the discrete time dynamics

$$K^x_{k+1} = K^x_k + \kappa_k(K^x_k)Z_{k+1},\ k = 0, \ldots, n-1,\ K^x_0 = x.$$

This follows from (a) applied successively to the pair $(\sigma_k, \kappa_k)_{0 \leq k \leq n}$ and $(\kappa_k, \theta_k)_{0 \leq k \leq n}$.

Proof. (a) It is clear that this claim is equivalent to proving the expected inequality either if all the functions $(\sigma_k)_{0 \leq k \leq n}$ or all the functions $(\theta_k)_{0 \leq k \leq n}$ are convex.

We introduce $U^x = (U^x_k)_{0 \leq k \leq n}$ and $V^x = (V^x_k)_{0 \leq k \leq n}$ the $(\mathbb{P}, \mathcal{F})$-Snell envelopes of $\big(F_k(X^x_{0:k})\big)_{0 \leq k \leq n}$ and $\big(F_k(Y^x_{0:k})\big)_{0 \leq k \leq n}$ respectively $i.e.$

$$U^x_k = \mathbb{P}\text{-supess}\Big\{\mathbb{E}\big(F_\tau(X^x_{0:\tau}) \,|\, \mathcal{F}_k\big),\ \tau\ \mathcal{F}\text{-stopping time}, \tau \geq k\Big\}$$

and

$$V^x_k = \mathbb{P}\text{-supess}\Big\{\mathbb{E}\big(F_\tau(Y^x_{0:\tau}) \,|\, \mathcal{F}_k\big),\ \tau\ \mathcal{G}\text{-stopping time}, \tau \geq k\Big\}$$

The connection between réduite and Snell envelope is a classical fact from Optimal Stopping Theory for which we refer $e.g.$ to $e.g.$ [29], Chapter VI), namely

$$u_0(x) = \mathbb{E}\, U^x_0$$

f (idem for v_0, V^x_0 for Y^x). It is also classical background on Optimal stopping theory (see again $e.g.$ [29], Chapter VI) that the $(\mathbb{P}, \mathcal{F})$-Snell envelope U^x satisfies the following Backward Dynamic Programming principle

$$U^x_n = F_n(X^x_{0:n}),\ U^x_k = \max\big(F_k(X^x_{0:k}), \mathbb{E}(U_{k+1} \,|\, \mathcal{F}_k)\big),\ k = 0, \ldots, n-1.$$

Then, we derive from the dynamics satisfied by the X^x_k and the independence of the random vectors Z_k that $(X^x_k)_{k=0:n}$ is a Markov chain. In turn a a first backward induction shows that $U^x_k = u_k(X^x_{0:k})$ *a.s.*, $k = 0,\ldots,n$, where the Borel functions $u_k : \mathbb{R}^{k+1} \to \mathbb{R}$, $k = 0,\ldots,n$, satisfy the backward induction

$$u_n = F_n,\ u_k(x_{0:k}) = \max\Big(F_k\big(x_{0:k}\big), Q_{k+1}u_{k+1}(x_{0:k}, x_k + .)\big)(\sigma_k(x_k))\big)\Big),\ k = 0,\ldots,n-1. \tag{4.26}$$

We define likewise the functions $v_k : \mathbb{R}^{k+1} \to \mathbb{R}$, $k = 0,\ldots,n$, related to the $(\mathbb{P},\mathcal{G})$-Snell envelopes of $\big(F_k(Y^x_{0:k})\big)_{0\le k\le n}$.

To emphasize the analogy with the proof of Proposition 2.1 we will detail the case where all the functions $\sigma_k = \kappa_k$ are convex, $k = 0,\ldots,n$ and satisfy $0 \le \sigma_k \le \theta k$. Following the lines of the proof of this proposition, we show, still by induction, that the functions $u_k : \mathbb{R}^{k+1} \to \mathbb{R}$ are convex by combining Lemma 2.1 and (4.26). The additional argument to ensure the propagation of convexity is to note that the function $(u, v) \mapsto \max(u, v)$ is convex and increasing in each of its variable u and v.

On the other hand, as $0 \le \sigma_k \le \theta_k$, $k = 0,\ldots,n$ and σ_k are all convex, we can show by a new backward induction that $u_k \le v_k$, $k = 0,\ldots,n$. If $k = n$ this is obvious. If it holds true with $k+1 \le n$, then for every $x_{0:k} \in \mathbb{R}^{k+1}$,

$$\begin{aligned} u_k(x_{0:k}) &\le \max\Big(F_k\big(x_{0:k}\big), \big(Q_{k+1}u_{k+1}(x_{0:k}, x_k + .)\big)(\theta_k(x_k))\big)\Big) \\ &\le \max\Big(F_k\big((x_{0:k}\big), \big(Q_{k+1}v_{k+1}(x_{0:k}, x_k + .)\big)(\theta_k(x_k))\big)\Big) = v_k(x_{0:k}) \end{aligned}$$

where we used successively that $u \mapsto \big(Q_{k+1}u_{k+1}(x_{0:k}, x_k + .)\big)(u)$ is non-decreasing on $\mathbb{R}_+$ since u_{k+1} is convex and that $u_{k+1} \le v_{k+1}$. Finally, the inequality for $k = 0$ reads

$$u_0(x) = \mathbb{E}\, U^x_0 \le \mathbb{E}\, V^x_0 = v_0(x)$$

which yields the announced result. Other cases follow the same way round following the lines of the proof of Proposition 4.26. □

4.2 Continuous time optimal stopping and American options

4.2.1 Brownian diffusions

In this section, we switch to the continuous time setting. We will investigate the (functional) convex order properties of the réduite (or the Snell envelope) of payoff processes obtained as adapted convex functionals of Brownian martingale diffusion processes *i.e.* of the form $(F(t, X^t))_{t\in[0,T]}$ where X^t denotes the stopped process $(X_s)_{s\in[0,T]}$ at time $t \in [0, T]$ where X itself is a martingale Brownian diffusion of type $X^{(\sigma)}$ as defined in (2.1). This embodies most pricing problems for American options in local volatility models.

In particular, the results of this section can be seen as an extension to path-dependent "payoff functionals" of El Karoui-Jeanblanc-Shreve's Theorem (see [8]) which mainly deals with convex functions of the marginal of the processes at time T (see also [15] devoted to parhwise-dependent lookback options). The proposition below is also very close to former results by Bergenthum and Rüschendorf by combining Theorems 3.2 and 3.6 from [2] with Theorem 4.1 from [4]. Here, we focus on the partitioning function.

Proposition 4.2. *Let σ, $\theta : [0, T] \times \mathbb{R}$ be two Lipschitz continuous functions in (t, x) and let W be a standard $\mathcal{F} = (\mathcal{F}_t)_{t\ge0}$-Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where $\mathcal{F}$ satisfies the usual conditions. Let $(X^{(\sigma),x}_t)_{t\in[0,T]}$ and $(X^{(\theta),x}_t)_{t\in[0,T]}$ be the martingale diffusions, unique strong solutions starting at $x \in \mathbb{R}$ to (2.1) (where $W^{(\sigma)} = W^{(\theta)} = W$).*

Assume that there exists a partitioning function $\kappa : [0,T]\times\mathbb{R}\to\mathbb{R}$ such that $\kappa(t,.)$ is convex for every $t\in[0,T]$ with linear growth in x uniformly in $t\in[0,T]$ and

$$0\le\sigma(t,.)\le\kappa(t,.)\le\theta(t,.),\ t\in[0,T].$$

Let $F:[0,T]\times\mathcal{C}([0,T],\mathbb{R})\to\mathbb{R}_+$ be a $\|\,.\,\|_{\sup}$-continuous functional with $(r,\|\,.\,\|_{\sup})$-polynomial growth ($r\ge 1$) in $\alpha\in\mathcal{C}([0,T],\mathbb{R})$, uniformly in $t\in[0,T]$. Moreover, assume that, for every $t\in[0,T]$, $F(t,.)$ is convex *on $\mathcal{C}([0,T],\mathbb{R})$. Let $u_0(x)$ and $v_0(x)$ denote the $\mathcal{F}$-*réduites *of $\big(F(t,(X^{(\sigma),x})^t)\big)_{t\in[0,T]}$ and $\big(F(t,(X^{(\theta),x})^t)\big)_{t\in[0,T]}$ respectively defined by*

$$u_0(x)=\sup\left\{\mathbb{E}\,F\big(\tau,(X^{(\sigma),x})^\tau\big),\ \tau\in\mathcal{T}^{\mathcal{F}}_{[0,T]}\right\}\quad\text{and}\quad v_0(x)=\sup\left\{\mathbb{E}\,F\big(\tau,(X^{(\theta),x})^\tau\big),\ \tau\in\mathcal{T}^{\mathcal{F}}_{[0,T]}\right\}$$

where $\mathcal{T}^{\mathcal{F}}_{[0,T]}=\{\tau:\Omega\to[0,T],\mathcal{F}\text{-stopping time}\}$. Then

$$u_0(x)\le v_0(x).$$

Remark. All the quantities involved in the above theorem do exist since the sup norm of $X^{(\sigma),x}$ and $X^{(\theta),x}$ have polynomial moments at any order. Moreover, the Lipschitz continuity assumption is too stringent but we adopt it to shorten the proof of the"approximation" step from discrete to continuous time dynamics.

Proof. STEP 1 (Euler schemes) We consider the Euler schemes $\bar X^{(\sigma),n}$ and $\bar X^{(\theta),n}$ (with step $\frac{T}{n}$) of both diffusions (we drop the dependence on the starting value x). Both schemes are adapted to the filtration $\mathcal{F}^{(n)}:=(\mathcal{F}_{t_k^n})_{0\le k\le n}$.

It follows from Proposition 4.1 that the $(\mathbb{P},\mathcal{F}^{(n)})$-Snell envelopes $\bar U^{(n)}=(\bar U^{(n)}_{t_k^n})_{0\le k\le n}$, $\bar K^{(n)}=(\bar K^{(n)}_{t_k^n})_{0\le k\le n}$ and $\bar V^{(n)}=(\bar V^{(n)}_{t_k^n})_{0\le k\le n}$ of the $\mathcal{F}^{(n)}$-adapted payoff processes $F\Big(t_k^n,\big[I_n\big(\bar X^{(\sigma),n}\big)\big]^{t_k^n}\Big)$, $k=0,\dots,n$, $F\Big(t_k^n,\big[I_n\big(\bar X^{(\kappa),n}\big)\big]^{t_k^n}\Big)$, $k=0,\dots,n$, and $F\Big(t_k^n,\big[I_n\big(\bar X^{(\theta),n}\big)\big]^{t_k^n}\Big)$, $k=0,\dots,n$, satisfy

$$\mathbb{E}\,\bar U_0^n\le\mathbb{E}\bar K_0^n\le\mathbb{E}\,\bar V_0^n. \tag{4.27}$$

Note that it is always possible to define the Euler scheme associated to the function κ regardless of its convergence toward the related SDE.

STEP 2 (Convergence) First, set for convenience $\bar Y^{(n)}_{t_k^n}=F\Big(t_k^n,\big[I_n\big(\bar X^{(\sigma),n}\big)\big]^{t_k^n}\Big)$, $k=0,\dots,n$, so that

$$\bar U^{(n)}_{t_k^n}=\mathbb{P}\text{-supess}\Big\{\mathbb{E}(\bar Y^{(n)}_\tau\,|\,\mathcal{F}_{t_k^n}),\ \tau\in\mathcal{T}^{(n)}_{t_k^n,T}\Big\},\ k=0,\dots,n,$$

where $\mathcal{T}^{(n)}_{t_k^n,T}=\Big\{\tau:\Omega\to\{t_k^n,\dots,t_\ell^n,\dots,t_n^n\},\ \mathcal{F}^{(n)}\text{-stopping time}\Big\}$; we also know that the $(\mathbb{P},\mathcal{F})$-Snell envelope of the process $Y_t=F(t,X^t)$, $t\in[0,T]$, is defined by

$$U_t=\mathbb{P}\text{-supess}\Big\{\mathbb{E}\big(Y_\tau\,|\,\mathcal{F}_t\big),\ \tau\in\mathcal{T}^{\mathcal{F}}_{t,T}\Big\},\ t\in[0,T],$$

where $\mathcal{T}^{\mathcal{F}}_{t,T}=\Big\{\tau:\Omega\to[t,T],\ \mathcal{F}\text{-stopping time}\Big\}$. This Snell envelope is well-defined since $\|X\|_{\sup}$ lies in every $L^p(\mathbb{P})$, $p\in(0,+\infty)$, which implies in turn that $\|Y\|_{\sup}$ lies in every $L^p(\mathbb{P})$. As the obstacle process $(F(t,X^t))_{t\in[0,T]}$ has continuous paths and is uniformly integrable, it is regular for optimal stopping and $t\mapsto\mathbb{E}\,U_t$ is continuous (see [9, 26]. Hence, the super-martingale $(U_t)_{t\in[0,T]}$ has a (non-negative) càdlàg modification whose compensator is continuous (and non-decreasing). More generally, if a sequence os stopping times $\tau_n \uparrow \tau < +\infty$ and $U_\tau \in L^1$, then $\mathbb{E}U_{\tau_n} \to \mathbb{E}U_\tau$. For technical purpose, we introduce an intermediate quantity defined by

$$\widetilde{U}_{t_k^n} = \mathbb{P}\text{-supess}\Big\{\mathbb{E}(Y_\tau \,|\, \mathcal{F}_{t_k^n}),\ \tau \in \mathcal{T}^{(n)}_{t_k^n,T}\Big\} \le U_{t_k^n},\ k = 0,\ldots,n.$$

Our aim is to prove, after having canonically extended $\bar{U}^{(n)}$ into a càdlàg stepwise constant process by setting $\bar{U}_t^{(n)} = \bar{U}_{t_k^n}^{(n)}$, $t \in [t_k^n, t_{k+1}^n)$, that $\bar{U}_t^{(n)}$ converges to U_t in L^p for every $t \in [0,T]$. We start from the fact that

$$|U_t - \bar{U}^{(n)}_{\underline{t}_n}| \le |U_t - U_{\underline{t}_n}| + U_{\underline{t}_n} - \widetilde{U}^{(n)}_{\underline{t}_n} + |\widetilde{U}^{(n)}_{\underline{t}_n} - \bar{U}^{(n)}_{\underline{t}_n}|. \tag{4.28}$$

Once again, regularity for optimal stopping of U implies in particular that $(U_t)_{t\in[0,T]}$ is L^1-left continuous in t. In particular $\mathbb{E}|U_t - U_{t_k^n}| \to 0$ as $n \to +\infty$.

As concerns the second term in the right hand side of (4.28), we proceed as follows

$$0 \le U_{t_k^n} - \widetilde{U}^{(n)}_{t_k^n} \le \mathbb{P}\text{-supess}\Big\{\mathbb{E}\big(Y_\tau - Y_{\tau^{(n)}} \,|\, \mathcal{F}^{(n)}_{t_k^n}\big),\ \tau \in \mathcal{T}_{t_k^n,T}\Big\}$$

where $\tau^{(n)} = \sum_{\ell=k}^n \frac{\ell T}{n}\mathbf{1}_{\{\frac{(\ell-1)T}{n} < \tau \le \frac{\ell T}{n}\}} = \sum_{\ell=k}^n \bar{t}^n \mathbf{1}_{\{t^n_{\ell-1} < \tau \le t^n_\ell\}} \in \mathcal{T}^{(n)}_{t_k^n,T} \subset \mathcal{T}_{t_k^n,T}$ so that

$$0 \le U_{t_k^n} - \widetilde{U}^{(n)}_{t_k^n} \le \mathbb{E}\big(\sup_{t\ge t_k^n} |Y_t - Y_{\bar{t}^n}| \,|\, \mathcal{F}_{t_k^n}\big) \le \mathbb{E}\big(\sup_{t\in[0,T]} |Y_t - Y_{\bar{t}^n}| \,|\, \mathcal{F}_{t_k^n}\big).$$

Doob's Inequality applied to the martingale $M_n = \mathbb{E}\big(\sup_{t\in[0,T]} |Y_t - Y_{\bar{t}^n}| \,|\, \mathcal{F}_{t_k^n}\big)$, $n \ge 1$, implies that for every $p \in (1,+\infty)$,

$$\Big\|\max_{0\le k\le n}(U_{t_k^n n} - \widetilde{U}^{(n)}_{t_k^n})\Big\|_p \le \frac{p}{p-1}\|M_n\|_p = \frac{p}{p-1}\Big\|\sup_{t\in[0,T]}|Y_t - Y_{\bar{t}_n}|\Big\|_p \quad\to\quad 0 \ \text{ as } n \to +\infty$$

since $X^{\bar{t}_n}$ $a.s.$ converges towards X^t for the sup-norm owing to the pathwise continuity of X. This in turn, implies that $F(\bar{t}^n, X^{\bar{t}^n})$ $a.s.$ converges toward $F(t, X^t)$ since F is continuous. The L^p-convergence follows by uniform integrability, still since $\|Y\|_{\sup}$ has polynomial moments at any order.

Now we investigate the second term in the right hand side of (4.28).

$$\begin{aligned}
|\widetilde{U}^{(n)}_{t_k^n} - \bar{U}^{(n)}_{t_k^n}| &\le \text{supess}\Big\{\mathbb{E}\big(|Y_\tau - \bar{Y}^{(n)}_\tau| \,|\, \mathcal{F}_{t_k^n}\big),\ \tau \in \mathcal{T}^{(n)}_{t_k^n,T}\Big\}\\
&\le \mathbb{E}\Big(\max_{0\le k\le n}|\bar{Y}^{(n)}_{t_k^n} - Y_{t_k^n}| \,|\, \mathcal{F}_{t_k^n}\Big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\max_{0\le k\le n}|\bar{Y}^{(n)}_{t_k^n} - Y_{t_k^n}| &\le \max_{0\le k\le n}\big|F\big(t_k^n, (I_n(\bar{X}^{(\sigma),n}))^{t_k^n}\big) - F\big(t_k^n, (X^{(\sigma)})^{t_k^n}\big)\big|\\
&\le \sup_{t\in[0,T]}\big|F\big(t, (I_n(\bar{X}^{(\sigma),n}))^t\big) - F\big(t, (X^{(\sigma)})^t\big)\big|.
\end{aligned} \tag{4.29}$$

Now, note that the functional $\alpha \mapsto \Big(t \mapsto F\big(t, \alpha^t\big)\Big)$ defined from $(\mathcal{C}([0,T],\mathbb{R}), \|\,.\,\|_{\sup})$ into itself is continuous: if $(t_n, \alpha_n) \to (t, \alpha)$ for the product topology on $[0,T] \times (\mathcal{C}([0,T],\mathbb{R})$, then

$$\|\alpha_n^{t_n} - \alpha^t\|_{\sup} \le \|\alpha_n - \alpha\|_{\sup} + w(\alpha, |t - t_n|)$$

so that $(t_n, \alpha^{t_n}) \to (t, \alpha^t)$. As a consequence, the functional F being continuous on $[0,T] \times \mathcal{C}([0,T],\mathbb{R})$, $F(t_n, \alpha^{t_n}) \to F(t, \alpha^t)$ which in turn implies that $\sup_{t\in[0,T]}|F(t, \alpha_n^t) - F(t, \alpha^t)| \to 0$. As $I_n(\alpha) \to \alpha$

for the sup norm as $n \to +\infty$, we derive that if $\alpha_n \to \alpha$ for the sup norm then $\sup_{t\in[0,T]} |F(t, I_n(\alpha_n)^t) - F(t,\alpha^t)| \to 0$ as $n \to +\infty$.

Then under the Lipschitz continuity assumption on σ, we know that the Euler scheme $\bar{X}^{(\sigma),x,n)} \to X^{(\sigma),x}$ $\mathbb{P}$-*a.s.* as $n \to +\infty$ *a.s.* (see *e.g.* [5], Theorem B.14, p.276). The $(r, \|\,.\,\|_{\sup})$-polynomial growth assumption made on F and the the fact that $\sup_{n\geq 1} \mathbb{E}\|\bar{X}^{(\sigma),x,n}\|^p_{\sup} < +\infty$ for any $p > r$ implies the L^1-convergence to 0 of the term in (4.29). Finally, this shows that

$$\mathbb{E}\,\bar{U}^n_0 \to u_0(x) \quad \text{as} \quad n \to +\infty.$$

The conclusion follows from (4.27) in Step 1 by letting $n \to +\infty$ in the resulting inequality $\mathbb{E}\,\bar{U}^n_0 \leq \mathbb{E}\,\bar{V}^n_0$. $\square$

Applications to comparison theorems for American options in local volatility models. By specifying our diffusion dynamics as a local volatility model as defined by (2.7), we can extend the comparison result (2.8) to path-dependent American options provided the "payoff" functionals $F(t,.)$ are convex with polynomial growth as specified in the above theorem.

4.2.2 The case of jump martingale diffusions

In what follows the product space $[0,T] \times I\!D([0,T],\mathbb{R})$ is endowed with the product topology $|\,.\,| \otimes Sk$. The notation $X_t(\alpha) = \alpha(t)$, $\alpha \in I\!D([0,T],\mathbb{R})$ still denotes the canonical process on $I\!D([0,T],\mathbb{R})$ and θ denotes the canonical random variable on $[0,T]$ (*i.e.* $\theta(t) = t$, $t \in [0,T]$).

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a right continuous filtration on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let Y be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted càdlàg process defined on this probability space. We introduce the so-called $(\mathcal{H})$-assumption (also known as filtration enlargement assumption) which reads as follows:

$$(\mathcal{H}) \equiv \forall\, H : \Omega \to \mathbb{R},\ \text{bounded and } \mathcal{F}^Y_T\text{-measurable},\ \mathbb{E}\big(H\,|\,\mathcal{F}_t\big) = \mathbb{E}\big(H\,|\,\mathcal{F}^Y_t\big)\ \mathbb{P}\text{-}a.s.$$

This filtration enlargement assumption is equivalent to the following more tractable condition: there exists $D \subset [0,T]$, everywhere dense in $[0,T]$, with $T \in D$, such that

$$\forall\, n \geq 1,\ \forall\, t_1,\ldots,t_n \in D,\ \forall\, h \in \mathcal{C}_0(\mathbb{R}^n,\mathbb{R}),\ \mathbb{E}\big(h(Y_{t_1},\ldots,Y_{t_n})\,|\,\mathcal{F}_t\big) = \mathbb{E}\big(h(Y_{t_1},\ldots,Y_{t_n})\,|\,\mathcal{F}^Y_t\big)\ \mathbb{P}\text{-}a.s.$$

where $\mathcal{C}_0(\mathbb{R}^n,\mathbb{R}) = \{f \in \mathcal{C}(\mathbb{R}^n,\mathbb{R})\,\text{such that}\ \lim_{|x|\to+\infty} f(x) = 0\}$. We still consider the jump diffusions of the form (2.10) *i.e.*

$$dX_t = \kappa(t, X_{t_-})dZ_t$$

where $\kappa : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, Lipschitz continuous in x uniformly in $t \in [0,T]$.

The aim of this section is to extend the result obtained for convex order for Brownian diffusions to such jump diffusions. We will rely on an abstract convergence result for *réduites* established in [25] (Theorem 3.7 and the remark that follows) that we recall below. To this end, we need to recall two classical definitions on stochastic processes.

Definition 4.1. (a) Class (D) processes: *A càdlàg process $(Y_t)_{t\in[0,T]}$ is of class (D) if*

$$\big\{Y_\tau,\ \tau \in \mathcal{T}_{[0,T]}\big\}\ \textit{is uniformly integrable.} \tag{4.30}$$

(b) Aldous's tightness criterion (see *e.g.* [18], Chapter VI, Theorem 4.5, p.356): *A sequence of $\mathcal{F}^n$-adapted càdlàg processes $Y^n = (Y^n_t)_{t\in[0,T]}$, $n \geq 1$, defined on filtered stochastic spaces $(\Omega^n, \mathcal{A}^n, \mathcal{F}^n, \mathbb{P}^n)$, $n \geq 1$, satisfies Aldous's tightness criterion with respect to the filtrations $\mathcal{F}^n$, $n \geq 1$, if*

$$\forall\, \eta > 0,\ \lim_{\delta\to 0} \limsup_n \sup_{\tau_n \leq \tau'_n \leq (\tau_n+\delta)\wedge T} \mathbb{P}^n\big(|Y^n_{\tau_n} - Y^n_{\tau'_n}| \geq \eta\big) = 0 \tag{4.31}$$

where τ_n *and* τ'_n *run over* $[0,T]$*-valued* $\mathcal{F}^{Y^n}$*-stopping times.*

Then, the sequence $(Y^n)_{n\geq 1}$ *is tight for the Skorokhod topology.*

Theorem 4.1. (a) *Let* $(X^n)_{n\geq1}$ *be a sequence of adapted quasi-left càdlàg processes* ([1]) *defined on a probability spaces* $(\Omega^n,\mathcal{F}^n,\mathbb{P}^n)$ *of class* (D) *and satisfying the above Aldous tightness criterion* (4.31). *For every* $n\geq 1$*, let*

$$u_0^n = \sup\{\mathbb{E}\,X^n_\tau,\ \tau\ [0,T]\text{-valued}\,\mathcal{F}^n\text{-stopping time}\}$$

denote the $\mathcal{F}^n$-réduite *of* X^n*. Let* $(\tau^*_n)_{n\geq1}$ *be a sequence of* $\big(\mathcal{F}^{X^n},\mathbb{P}^n\big)$*-optimal stopping times*([2])*. Assume furthermore that* $(X^n)_{n\geq1}$ *satisfies*

$$X^n \xrightarrow{\mathcal{L}} \mathbb{P},\ \mathbb{P}\ \text{probability measure on}\ (\mathbb{D}([0,T],\mathbb{R}),\mathcal{D}_T)\ \text{such that}\ \mathbb{E}_{\mathbb{P}}\sup_{t\in[0,T]}|X_t|<+\infty.$$

If every limiting value $\mathbb{Q}$ *of* $\mathcal{L}(X^n,\tau^*_n)$ *on* $\mathbb{D}([0,T],\mathbb{R})\times[0,T]$ *satisfies the* $(\mathcal{H})$ *property, then the* $(\mathcal{F}^n,\mathbb{P}^n)$-réduites u_0^n *of* X^n *converge toward the* $(\mathcal{D},\mathbb{P})$-réduites u_0 *of* X *i.e.*

$$\lim_n u_0^n = u_0.$$

Moreover, if the optimal stopping problem related to $(X,\mathbb{Q},\mathcal{D}^\theta)$ *has a unique solution in distribution, i.e.* $\theta \stackrel{d}{=} \mu^*_{\tau^*}$*, not depending on* $\mathbb{Q}$*, then* $\tau^*_n \xrightarrow{\mathcal{L}([0,T])} \mu^*_{\tau^*}$*.*
(b) *The same result holds when considering a sequence of* companion processes Y^n *having values in a Polish metric space* (E,d_E) *i.e. we consider that the filtration of interest at finite range n is now* $(\mathcal{F}_t^{(X^n,Y^n)})_{t\in[0,T]}$*. We assume that* X^n *is quasi-left continuous with respect to this enlarged filtration. We will only ask the couple* (X^n,Y^n) *to converge for the* product topology *i.e. on* $(\mathbb{D}([0,T],\mathbb{R}),Sk_{\mathbb{R}})\times(\mathbb{D}([0,T],E),Sk_E)$ *since this product topology spans the same Borel* σ*-field as the regular Skorokhod topology on* $\mathbb{D}([0,T],\mathbb{R}\times E)$*.*

The main result of this section is the following:

Theorem 4.2. *Let* $Z=(Z_t)_{t\in[0,T]}$ *be a martingale Lévy process with Lévy measure* ν *satisfying* $\nu(|z|^p)<+\infty$ *for* $p\in[2,+\infty)$*, so that the process* Z *is an* L^2*-martingale null at* 0*. Let* $X^{(\kappa_i,x)}$*,* $i=1,2$*, be the martingale jump diffusions driven by* Z *starting at (the same)* $x\in\mathbb{R}$*. Let* $F:[0,T]\times\mathbb{D}([0,T],\mathbb{R})\to\mathbb{R}_+$ *be a* convex *functional satisfying the following local Lipschitz assumption (w.r.t. to the sup norm) combined with a Skorokhod continuity assumption, namely*

$$\begin{cases} (i) & F:[0,T]\times\mathbb{D}([0,T],\mathbb{R})\to\mathbb{R}_+\ \text{is}\ Sk\text{-continuous},\\ (ii) & |F(t,\beta)-F(s,\alpha)|\leq C\Big(|t-s|^{\rho'}+\|\alpha-\beta\|^\rho_{\sup}\big(1+\|\alpha\|^{r-\rho}_{\sup}+\|\beta\|^{r-\rho}_{\sup}\big)\Big),\ \rho,\ \rho'\in(0,1],\ r\in[1,p),\end{cases} \tag{4.32}$$

Let $U^{(\kappa_i)}$ *denote the Snell envelopes of the processes* $\big(F(t,(X^{\kappa_i}))^t\big)_{t\in[0,T]}$*,* $i=1,2$ *respectively.*

If there exist $\kappa_i:[0,T]\times\mathbb{R}\to\mathbb{R}$*,* $i=1,2$*, two continuous functions with linear growth in* x*, uniformly in* $t\in[0,T]$*, and a* partitioning function $\kappa:[0,T]\times\mathbb{R}\to\mathbb{R}$*, convex in* x *for every* $t\in[0,T]$*, such that*

$$\kappa_1\leq\kappa\leq\kappa_2.$$

Then

$$U_0^{(\kappa_1)}\leq U_0^{(\kappa_2)}.$$

[1] A càdlàg $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $X=(X_t)_{t\in[0,T]}$ is *quasi-left continuous* with respect to the right continuous filtration $\mathcal{F}=(\mathcal{F}_t)_{t\in[0,T]}$ if for every $\mathcal{F}$-stopping time τ having values in $[0,T]\cup\{+\infty\}$ and every increasing sequence of $\mathcal{F}$-stopping times $(\tau_k)_{k\geq1}$ with limit τ, $\lim_k X_{\tau_k}=X_\tau$ on the event $\{\tau<+\infty\}$ (see *e.g.* [18], Chapter I.2.25, p.22).

[2] *i.e.* satisfying $\mathbb{E}\,X^n_{\tau^*_n}=u^n_0$.

Remarks. • Note that, since $p \geq 2$,

$$\nu(|z|^p) < +\infty \longleftrightarrow \nu(|z|^p \mathbf{1}_{\{|z|\geq 1\}}) < +\infty \longleftrightarrow Z_t \in L^p \longleftrightarrow \sup_{t\in[0,T]} |Z_t| \in L^p.$$

• One proves likewise that, for every $t \in [0,T]$,

$$\mathbb{E}(U_t^{(\kappa_1)}) \leq \mathbb{E}(U_t^{(\kappa_2)}).$$

• If the functions $\kappa(t,.)$, $t \in [0,T]$ are all convex (but possibly not the functions $\kappa_i(t,.)$) then the same proof shows by coupling (κ_1,κ) and (κ,κ_2) that

$$\mathbb{E}(U_0^{(\kappa_1)}) \leq \mathbb{E}(U_0^{(\kappa)}) \leq \mathbb{E}(U_0^{(\kappa_2)}).$$

Lemma 4.1. *Let $X = (X_t)_{t\in[0,T]}$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted càdlaàg process defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$ where $(\mathcal{F}_t)_{t\in[0,T]}$ is a càd filtration. Let $G : [0,T]\times I\!D([0,T],\mathbb{R}) \to \mathbb{R}_+$ be a Skorokhod continuous functional such that $|G(\alpha)| \leq C(1+\|\alpha\|^r_{\sup})$, $r \in (0,p)$. If X is quasi-left continuous and if $\|X\|_{\sup} \in L^p$, then the "obstacle process" $(G(t,X^t))_{t\in[0,T]}$ is* regular for optimal stopping *i.e. as soon as $\tau < +\infty$ $\mathbb{P}$-a.s. $\mathbb{E}G(\tau_n, X^{\tau_n}) \to \mathbb{E}\,G(\tau,X^\tau)$.*

Proof. First one easily proves by coming back to the very definition of Skorokhod topology that $\alpha_n \xrightarrow{Sk} \alpha$ and $t_n \to t \in \mathrm{Cont}(\alpha)$ then $\alpha_n^{t_n} \xrightarrow{Sk} \alpha^t$. Let $(\tau_n)_{n\geq 1}$ be a sequence of $\mathcal{F}_t$-stopping times satisfying $\tau_n \uparrow \tau < +\infty$ $\mathbb{P}$-*a.s.*, then $X_\tau = X_{\tau_-}$ $\mathbb{P}$-*a.s.* *i.e.* $\tau(\omega) \in \mathrm{Cont}(X(\omega))$ $\mathbb{P}(d\omega)$-*a.s.*. It follows that $(\tau_n X^{\tau_n}) \to (\tau, X^\tau)$ $\mathbb{P}$-*a.s.*. The continuity assumption made on G implies that $G(\tau_n, X^{\tau_n}) \xrightarrow{Sk} G(\tau,X^\tau)$. One concludes by a uniform integrability argument that $\mathbb{E}\,G(\tau_n,X^{\tau_n}) \to \mathbb{E}\,G(\tau,X^\tau)$ since $\|X\|_{\sup} \in L^p$ implies that $\big(G(\tau_n,X^{\tau_n})\big)_{n\geq 1}$ is $L^{\frac{p}{r}}$-bounded. $\square$

Proof. Step 1 *Aldous tightness criterion.* We still consider the stepwise constant Euler scheme $\bar{X}^n = (\bar{X}^n_t)_{t\in[0,T]}$ with step $\frac{T}{n}$defined by

$$\bar{X}^n_{t^n_k} = \bar{X}^n_{t^n_{k-1}} + \kappa(t^n_{k-1}, \bar{X}^n_{t^n_{k-1}})(Z_{t^n_k} - Z_{t^n_{k-1}}),\ k = 1,\dots,n,\ \bar{X}^n_0 = X_0$$

and *i.e.* $\bar{X}^n_t = \bar{X}^n_{\underline{t}_n}$. Let $\sigma_n, \tau_n \in \mathcal{T}^{\mathcal{F}^n}_{[0,T]}$, such that $\sigma_n \leq \tau_n \leq (\sigma_n+\delta)\wedge T$. In fact, following Lemma 2.3, we may assume without loss of generality that σ_n and τ_n take values in $\{t^n_k,\ k = 0,\dots,n\}$. Then, owing to (4.32),

$$\mathbb{E}\big|F(\tau_n,(\bar{X}^n)^{\tau_n}) - F(\sigma_n,(\bar{X}^n)^{\sigma_n})\big| \leq C\delta^{\rho'} + C\,\mathbb{E}\big(\|(\bar{X}^n)^{\tau_n} - (\bar{X}^n)^{\sigma_n}\|^\rho_{\sup}(1+2\|\bar{X}^n\|^{r-\rho}_{\sup})\big).$$

Hölder Inequality applied with the conjugate exponents $a = \frac{r}{\rho}$ and $b = \frac{r}{r-\rho}$ yields

$$\mathbb{E}\Big(\|(\bar{X}^n)^{\tau_n} - (\bar{X}^n)^{\sigma_n}\|^\rho_{\sup}\big(1+2\|\bar{X}^n\|^{r-\rho}_{\sup}\big)\Big) \quad\leq\quad \Big\|\sup_{\sigma_n\leq s\leq(\sigma_n+\delta)\wedge T} |\bar{X}^n_s - \bar{X}^n_{\sigma_n}|\Big\|^\rho_r \Big(1+2\big\|\sup_{t\in[0,T]}|\bar{X}^n_t|\big\|^{r-\rho}_r\Big).$$

As $\nu(z^2) < +\infty$, we can decompose the Lévy process Z into $Z_t = a\,W_t + \widetilde{Z}_t$, $a \geq 0$ where W is a standard Brownian motion and $\widetilde{Z}$ is a pure jump square integrable martingale Lévy process.

• If $r\in[1,2]$: it follows from the $B.D.G.$ Inequality applied to the local martingale $(\bar X_{\sigma_n+\frac{iT}{n}}-\bar X^n_{\sigma_n})_{i\ge 0}$ that

$$
\begin{aligned}
\Big\|\sup_{\sigma_n\le t^n_k\le(\sigma_n+\delta)\wedge T}|\bar X^n_{t^n_k}-\bar X^n_{\sigma_n}|\Big\|_r^r \;&\le\; c_r a^r\Big\|\sum_{\sigma_n<t^n_k\le(\sigma_n+\delta)\wedge T}\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})^2\big(W_{t^n_k}-W_{t^n_{k-1}}\big)^2\Big\|_{L^{\frac r2}}^{\frac r2}\\
&\quad+c_r\Big\|\sum_{\sigma_n<t^n_k\le(\sigma_n+\delta)\wedge T}\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})^2\big(\widetilde Z_{t^n_k}-\widetilde Z_{t^n_{k-1}}\big)^2\Big\|_{L^{\frac r2}}^{\frac r2}\\
&\le\; c_r a^r\Big\|\sum_{\sigma_n<t^n_k\le(\sigma_n+\delta)\wedge T}\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})^2\big(W_{t^n_k}-W_{t^n_{k-1}}\big)^2\Big\|_{L^2}^{\frac r2}\\
&\quad+c_r\mathbb{E}\left(\sum_k \mathbf{1}_{\{\sigma_n<t^n_k\le(\sigma_n+\delta)\wedge T\}}|\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})|^r|Z_{t^n_k}-Z_{t^n_{k-1}}|^r\right).
\end{aligned}
$$

Now

$$
\begin{aligned}
\mathbb{E}\Big[\sum_{\sigma_n<t^n_k\le(\sigma_n+\delta)\wedge T}\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})^2\big(W_{t^n_k}-W_{t^n_{k-1}}\big)^2\Big] \;&=\; \frac{T}{n}\mathbb{E}\Big[\sum_{\sigma_n<t^n_k\le(\sigma_n+\delta)\wedge T}\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})^2\Big]\\
&\le\; \frac{T}{n}\mathbb{E}|Z_{\frac Tn}|^r\mathbb{E}\Big[\max_{1\le k\le n}|\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})|^2\times\mathrm{card}\{k:\sigma_n<t^n_k\le(\sigma_n+\delta)\wedge T\}\Big]\\
&\le\; \frac{T}{n}\mathbb{E}\Big[\max_{1\le k\le n}|\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})|^r\Big]\frac{\delta n}{T}\\
&=\; \delta\Big\|\max_{1\le k\le n}|\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})|\Big\|_2^2
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\Big\|\sum_{\sigma_n<t^n_k\le(\sigma_n+\delta)\wedge T}\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})^2\big(\widetilde Z_{t^n_k}-\widetilde Z_{t^n_{k-1}}\big)^2\Big\|_{L^{\frac r2}}^{\frac r2} \;&\le\; \mathbb{E}\sum_k \mathbf{1}_{\{\sigma_n<t^n_k\le(\sigma_n+\delta)\wedge T\}}|\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})|^r|\widetilde Z_{t^n_k}-\widetilde Z_{t^n_{k-1}}|^r\\
&=\; \mathbb{E}|\widetilde Z_{t^n_k}-\widetilde Z_{t^n_{k-1}}|^r\mathbb{E}\Big[\sum_k \mathbf{1}_{\{\sigma_n<t^n_k\le(\sigma_n+\delta)\wedge T\}}|\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})|^r\Big]\\
&=\; \mathbb{E}|\widetilde Z_{\frac Tn}|^r\mathbb{E}\Big[\max_{1\le k\le n}|\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})|^r\times\mathrm{card}\{k:\sigma_n<t^n_k\le(\sigma_n+\delta)\wedge T\}\Big]\\
&\le\; \mathbb{E}|\widetilde Z_{\frac Tn}|^r\mathbb{E}\Big[\max_{1\le k\le n}|\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})|^r\Big]\frac{\delta n}{T}\\
&\le\; \delta\Big(\frac nT\mathbb{E}|\widetilde Z_{\frac Tn}|^r\Big)\mathbb{E}\Big[\max_{0\le k\le n-1}|\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})|^r\Big]\\
&\le\; C_{\kappa,\widetilde Z,T}\,\delta\Big\|\max_{1\le k\le n}|\kappa(t^n_{k-1},\bar X^n_{t^n_{k-1}})|\Big\|_r^r
\end{aligned}
$$

where we used that $t\mapsto\frac1t\mathbb{E}|\widetilde Z_t|^r$ remains bounded on the whole interval $(0,T]$.

Under the assumptions $\nu(z^2)<+\infty$ and κ with linear growth (in x uniformly in $t\in[0,T]$), it follows form Proposition B.2 in Appendix B that $\sup_{n\ge1}\big\|\sup_{0\le k\le n}|\kappa(t^n_k,\bar X^n_{t^n_k})|\big\|_r<+\infty$ since $r\le 2$ (see the first remark below the statement of the theorem), we get

$$
\Big\|\sup_{\sigma_n\le t^n_k\le(\sigma_n+\delta)\wedge T}|\bar X^n_{t^n_k}-\bar X^n_{\sigma_n}|\Big\|_r\le C_{\rho,r,\kappa,Z,T}\big(\delta^{\frac14}+\delta^{\frac1r}\big)
$$

where the real constant $C_{\rho,r,\kappa,Z,T}$ does not depend on n, σ_n, τ_n and δ. This implies in turn that

$$\lim_{\delta\to 0}\limsup_n \sup_{\sigma_n<\tau_n\le(\sigma_n+\delta)\wedge T}\mathbb{E}\big|F(\tau_n,\bar{X}^{n,\tau_n})-F(\sigma_n,\bar{X}^{n,\sigma_n})\big|=0$$

and the conclusion follows.

• If $r\in[2,4]$: One writes

$$\begin{aligned}\sum_{\sigma_n<t_k^n\le(\sigma_n+\delta)\wedge T}\kappa(t^n_{k-1},\bar{X}^n_{t^n_{k-1}})^2\big(Z_{t^n_k}-Z_{t^n_{k-1}}\big)^2 &= \sum_{\sigma_n<t_k^n\le(\sigma_n+\delta)\wedge T}\kappa(t^n_{k-1},\bar{X}^n_{t^n_{k-1}})^2\big(\big(Z_{t^n_k}-Z_{t^n_{k-1}}\big)^2-\mathbb{E}|Z_{\frac{T}{n}}|^2\big)\\ &\quad+\mathbb{E}|Z_{\frac{T}{n}}|^2\sum_{\sigma_n<t_k^n\le(\sigma_n+\delta)\wedge T}\kappa(t^n_{k-1},\bar{X}^n_{t^n_{k-1}})^2\big(Z_{t^n_k}-Z_{t^n_{k-1}}\big)^2.\end{aligned}$$

The second term of the sum in the right hand side of the above equality can be treated as above (it corresponds to $r=2$). As concerns the first one, note that the *i.i.d.* sequence $\big((Z_{t^n_k}-Z_{t^n_{k-1}})^2-\mathbb{E}|Z_{\frac{T}{n}}|^2\big)_{1\le k\le n}$ is centered and lies in $L^{\frac{r}{2}}(\mathbb{P})$ with $\frac{r}{2}\in[1,2]$. Hence it can be controlled like the former case. Carrying on the process by a cascade induction as detailed *e.g.* in the proof of Proposition B.2 in Appendix B, one can lower r to $r/2,\ldots,r/2^{\ell_r}\in(1,2]$ by induction, owing to *B.D.G.* inequality.

Step 2. It follows from Step 1 of Theorem 2.2 (adapted to a 2-dimensional framework with $(\kappa,\mathbf{1})$ as a drift) that

$$\Big(\bar{X}^n,I_n(Z)\Big)\stackrel{\mathcal{L}(Sk)}{\longrightarrow}(X,Z)\quad\text{as}\quad n\to+\infty.$$

If we consider the discrete time Optimal Stopping problem(s) related to the Euler schemes $\bar{X}^{(n,\kappa_i)}$, $i=1,2$, which turns out the be the same as in Step 1 of the proof of Proposition 4.2, the existence of optimal stopping times $\tau_n^{(i)}$, $i=1,2$, taking values in $\{t^n_k,\,k=0,\ldots,n\}$ is straightforward owing to the finite horizon of these problems (see [29], Chapter VI for more details).

Step 3: Let $\Omega_c=I\!D([0,T],\mathbb{R})^2\times[0,T]$ be the canonical space of the distribution of the sequence $(\bar{X}^n,I_n(Z),\tau_n^*)_{n\ge1}$. For every $(\alpha,u)\in I\!D([0,T],\mathbb{R})^2\times[0,T]$, the canonical process is defined by $\Xi_t(\alpha,u)=\alpha(t)=(\alpha^1(t),\alpha^2(t))\in\mathbb{R}^2$ and the canonical random times is given by $\theta(\alpha,u)=u$. Furthermore we will denote by $\Xi=(\Xi^1,\Xi^2)$ the two components of Ξ.

Let

$$\mathcal{D}^\theta_t=\cap_{s>t}\sigma(\Xi_u,\{\theta\le u\},\,0\le u\le s\}\ \text{ if }\ t\in[0,T)\ \text{ and }\ \mathcal{D}^\theta_T=\sigma\big(\Xi_s,\{\theta\le s\},\,0\le s\le T\big)$$

denote the canonical right-continuous filtration on Ω_c. This canonical space Ω_c is equipped with the product metric topology $Sk^{\otimes2}\otimes|.|$ where $|.|$ denotes the standard topology on $[0,T]$ induced by the absolute value.

In order to conclude to the convergence of the *réduites*, we need, following Theorem 4.1 established in [25], to show that any limiting distribution $\mathbb{Q}=\lim_n\mathbb{P}_{((\bar{X}^n,I_n(Z)),\tau_n^*)}$ on the canonical space $\big(I\!D([0,T],\mathbb{R}^2)\times[0,T],Sk^{\otimes2}\otimes|.|\big)$ satisfies the $(\mathcal{H})$-assumption, namely

$$\mathbb{E}_{\mathbb{Q}}\big(H\,|\,\mathcal{D}^\theta_t\big)=\mathbb{E}_{\mathbb{Q}}\big(H\,|\,\mathcal{D}_t\big)\ \mathbb{Q}\text{-}a.s.$$

for every random variable H defined on Ω_c.

Let $\mathrm{Atom}_{\mathbb{Q}}(\theta)=\{s\in[0,T],\,\mathbb{Q}_\theta(\{s\})>0\}$ be the set, possibly empty, of $\mathbb{Q}$-atoms of θ. Let $\Phi:I\!D([0,T],\mathbb{R}^2)\to\mathbb{R}$ and $\Psi:I\!D([0,T],\mathbb{R})\to\mathbb{R}$ two bounded functionals, $Sk^{\otimes2}$- and Sk-continuous respectively and let $u \notin \mathrm{Atom}_{\mathbb{Q}}(\theta)$, $u \le s \le T$. Noting that $\Psi(I_n(Z)^s)\mathbf{1}_{\{\tau_n^* \le u\}}$ is $\mathcal{F}_s^n$-measurable, we get

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\big(\Phi(\Xi)\Psi(\Xi^{2,s})\mathbf{1}_{\{\theta\le u\}}\big) &= \lim_n \mathbb{E}\Big(\Phi(\bar{X}^n, I_n(Z))\Psi(I_n(Z)^s)\mathbf{1}_{\{\tau_n^*\le u\}}\Big)\\ &= \lim_n \mathbb{E}\Big(\mathbb{E}\big[\Phi(\bar{X}^n, I_n(Z))|\mathcal{F}_s^Z\big]\Psi(I_n(Z)^s)\mathbf{1}_{\{\tau_n^*\le u\}}\Big).\end{aligned}$$

Up to an extraction (n'), we may assume that $\mathbb{E}\big[\Phi(\bar{X}^{n'}, I_{n'}(Z))|\mathcal{F}_s^Z\big]$ weakly converges to $\mathbb{E}\big[\Phi(X,Z)|\mathcal{F}_s^Z\big]$ since $\Phi(\bar{X}^{n'}, I_{n'}(Z))$ weakly converges toward $\Phi(X,Z)$. Up to a second extraction, still denoted (n'), we may assume that $\Psi(I_n(Z)^s)$ *a.s.* converges toward $\Psi(Z^s)$ for the Skorokhod topology since $\mathbb{P}(\Delta Z_s \ne 0) = 0$ (the stopping operator at time s, $\alpha \mapsto \alpha^s$, is Sk-continuous at functions α which are continuous at s).

Consequently, going back on the canonical space Ω_c, we obtain

$$\Big(\mathbb{E}\big[\Phi(\bar{X}^n, I_n(Z))|\mathcal{F}_s^Z\big],\ \Psi(I_n(Z)^s),\ \mathbf{1}_{\{\tau_n^*\le u\}}\Big) \xrightarrow{\mathcal{L}} \mathcal{L}_{\mathbb{Q}}\left(\mathbb{E}\big[\Phi(\Xi)|\mathcal{F}_s^{\Xi^2}\big], \Psi(\Xi^{2,s}), \mathbf{1}_{\{\theta\le u\}}\right).$$

which ensures that

$$\mathbb{E}_{\mathbb{Q}}\left(\Phi(\Xi)\psi(\Xi^{2,s})\mathbf{1}_{\{\theta\le u\}}\right) = \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\big[\Phi(\Xi)|\mathcal{D}_{s_-}\big]\Psi(\Xi^{2,s})\mathbf{1}_{\{\theta\le u\}}\right).$$

One concludes by standard functional monotone approximation arguments that the equality holds true for any bounded measurable functional Φ, Ψ and any $u \in [0,T]$. Then, by considering a sequence $s_n \downarrow s$, $s_n > s$, we derive that

$$\mathbb{E}_{\mathbb{Q}}\left(\Phi(\Xi)\,|\,\mathcal{D}_s^{\theta}\right) = \mathbb{E}_{\mathbb{Q}}\Big(\Phi(\Xi)\,|\,\mathcal{D}_s\Big).$$

This shows that the $(\mathcal{H})$-assumption is fulfilled so that by Theorem 4.1, U_0^n converges toward U_0. □

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A Appendix: Euler scheme for Brownian martingale diffusions

Proposition A.1. *Let* $(\bar X^n_t)_{t\in[0,T]}$ *be the genuine Euler scheme of step* $\frac{T}{n}$ *of the SDE* $dX_t = \sigma(t,X_t)dW_t$, $X_0 = x$ *defined as the solution to*

$$d\bar X^n_t = \sigma(\underline{t}_n, \bar X^n_{\underline{t}_n})dW_t,\ \bar X^n_0 = x.$$

If $\sigma : [0,T]\times\mathbb{R}\to\mathbb{R}$ *is continuous and satisfies the linear growth assumption*

$$\forall\, t\in[0,T],\ \forall\, x\in\mathbb{R},\quad |\sigma(t,x)|\le C_\sigma(1+|x|)$$

Then the sequence $(\bar X^n)_{n\ge1}$ *is* C*-tight on* $\mathcal{C}([0,T],\mathbb{R})$ *and any of its limiting distribution is a weak solution to the above SDE. In particular if a weak uniqueness assumption holds, then* $\bar X^n \stackrel{(\|\,.\,\|_{\sup})}{\longrightarrow} X$.

Following *e.g.* [5] (Lemma B.1.2, p.275, see also [24, 30]), we first show that, owing to the linear growth assumption $|\sigma(t,x)|\le C_\sigma(1+|x|)$ made on σ, the non-decreasing function $\varphi_{p,n}(t) = \mathbb{E}\sup_{s\in[0,t]}|\bar X^n_s|^p$, $p\in[1,+\infty)$ is finite for every $t\in[0,T]$. Using Doob's Inequality and Gronwall's Lemma, it follows that there exists a real constant $C = C'_{p,\sigma} > 0$ such that

$$\varphi_{p,n}(t)\le\varphi_p(t) := Ce^{Ct}(1+|x|^p).$$

Consequently, it follows from the the successive application of the L^p-B.D.G. and Hölder inequalities for $p\in(2,+\infty)$ that, for every $s,t\in[0,T]$, $s\le t$,

$$\begin{aligned}\mathbb{E}|\bar X^n_t-\bar X^n_s|^p &\le c_p^p\,\mathbb{E}\left(\int_s^t|\sigma(\underline{u}_n,\bar X^n_{\underline{u}_n})|^2du\right)^{\frac p2}\\ &\le c_p^p|t-s|^{\frac p2}\big(1+\varphi_p(T)\big).\end{aligned}$$

Kolmogorov's criterion (see [1], Theorem 12.3, p.95) implies that the sequence $M_n = (W_t,\bar X^n_t)_{t\in[0,T]}$ is C-tight (*i.e.* tight as $(\mathcal{C}([0,T],\mathbb{R}),\|\,.\,\|_{\sup})$-valued random variables). From now on, we mainly rely on the results established in [20]. Let n' be a subsequence such that $(\bar X^{n'},W)$ functionally weakly converges to a probability $\mathbb{Q}$ on $(\mathcal{C}([0,T],\mathbb{R}^2),\|\,.\,\|_{\sup})$, hence it satisfies the *U.T.* (for *Uniform Tightness*) assumption (see Proposition 3.2 in [20]). The function σ being continuous on $[0,T]\times\mathbb{R}$, the sequence $(\sigma(\underline{t}_n,\bar X^n_{\underline{t}_n}))_{n\ge1}$ is C-tight on the Skorokhod space since $\big((\underline{t}_n,\bar X^n_{\underline{t}_n})_{t\in[0,T]}\big)_{n\ge1}$ clearly is. One derives that, up to a new extraction still denoted (n'), we may assume that $\big(\sigma(\underline{t}_{n'}, \bar{X}^{n'}_{\underline{t}_{n'}})_{t\in[0,T]}, \bar{X}^{n'}, W\big)_{n\geq 1}$ functionally converges toward a probability $\mathbb{P}$ on $I\!D([0,T],\mathbb{R}^3)$. By Theorem 2.6 from [20] for the functional convergence of stochastic integrals, we know that

$$\left(\sigma(\underline{t}_{n'}, \bar{X}^{n'}_{\underline{t}_{n'}}), (\bar{X}^{n'}_t, W_t), \int_0^t \sigma(\underline{s}_{n'}, \bar{X}^{n'}_{\underline{s}_{n'}})dW_s\right)_{t\in[0,T]} \stackrel{\mathcal{L}(Sk)}{\longrightarrow} \mathbb{Q} \quad \text{as} \quad n\to+\infty$$

where $\mathbb{Q}$ is a probability distribution on $I\!D([0,T],\mathbb{R}^4)$ such that the canonical process $Y=(Y^i)_{i=1:4}$ satisfies $Y \stackrel{\mathcal{L}}{\sim} \big(Y^1, (Y^2, B), \int_0^{\cdot} Y^2_s dB_s\big)$ where $B : Y^3$ is a standard $\mathbb{Q}$-Brownian motion with respect to the $\mathbb{Q}$-completed right continuous canonical filtration $(\mathcal{D}^4_t)_{t\in[0,T]}$ on $I\!D([0,T],\mathbb{R}^4)$. Furthermore, we know that $Y^1=\sigma(.,Y^2)$ $\mathbb{Q}$-$a.s.$ since $\sup_{t\in[0,T]}|\sigma(\underline{t}_{n'}, \bar{X}^{n'}_{\underline{t}_{n'}}) - \sigma(t, \bar{X}^{n'}_t)|$ converges to 0 in probability. The former claim follows from the facts that $\sup_{t\in[0,T]}|\bar{X}^n_t|$ is tight and $\sigma(t,\xi)$ is uniformly continuous on every compact set of $[0,T]\times\mathbb{R}$ with linear growth in ξ uniformly in $t\in[0,T]$. On the other hand, we know that $\bar{X}^{n'} = x + \int_0^{\cdot}\sigma(\underline{s}_{n'}, \bar{X}^{n'}_{\underline{s}_{n'}})dW_s$ which in turn implies that $Y_{2,.} = x+\int_0^{\cdot}\sigma(s, Y_{2,s})dW_s$. This shows the existence of a weak solution to the SDE $X_t = x+\int_0^t \sigma(s,X_s)\,dW_s$, $t\in[0,T]$.

Under the weak uniqueness assumption, this distribution is unique hence is the only functional weak limiting distribution for the tight sequence $(\bar{X}^n)_{n\geq1}$, hence we get the convergence in distribution on $\mathcal{C}([0,T],\mathbb{R})$. ☐

Remark. If the original SDE has a unique strong solution, the same proof leads to the establish the convergence in probability of the Euler scheme toward X. One just has to add the process X itself to the sequence $\big((\sigma(\underline{t}_n, \bar{X}^n_{\underline{t}_n}))_{t\in[0,T]}, \bar{X}^n, W\big)_{n\geq1}$

B Appendix: Euler scheme for a Lévy driven martingale diffusion

We consider the following SDE driven by a martingale Lévy process Z with Lévy measure ν

$$X_t = x + \int_{(0,t]} \kappa(s, X_{s_-})dZ_s,\ X_0 = x \tag{B.33}$$

(where κ is a Borel function on $[0,T]\times\mathbb{R}$) and its *genuine* Euler scheme defined by

$$\bar{X}^n_{t_{k+1}} = \bar{X}^n_{t_k} + \kappa(t_k, \bar{X}_{t_k})(Z_{t_{k+1}} - Z_{t_k}),\ k=1,\dots,n,\ \bar{X}_0 = X_0 = x \tag{B.34}$$

at discrete times t^n_k and extended into a continuous time process by setting $\bar{X}^t = \bar{X}^n_{\underline{t}_n}$ so that

$$\bar{X}^n_t = x + \int_{(0,t]} \kappa(\underline{s}_{n-}, \bar{X}^n_{\underline{s}_n-})dZ_s. \tag{B.35}$$

B.1 Convergence of the Euler scheme toward a solution to Lévy driven SDE

Proposition B.1. (a) *Assume that $\nu(|z|^p)<+\infty$ for a $p\in(1,2]$, has no Brownian component and $\kappa(t,\xi)$ has linear growth in ξ uniformly in $t\in[0,T]$. Then*

$$\sup_{n\geq1}\big\|\sup_{t\in[0,T]}|\bar{X}^n_t|\big\|_p + \big\|\sup_{t\in[0,T]}|X_t|\big\|_p < +\infty.$$

If moreover κ is continuous, the SDE (2.10) has at least one weak solution and, as soon as weak uniqueness holds for (B.33),

$$\bar{X}^n_t \stackrel{\mathcal{L}(Sk)}{\longrightarrow} X.$$

(b) *If $\nu(z^2)<+\infty$, the same result remains true* mutatis mutandis *if Z has a non-zero Brownian component.*

Remark. In fact, as soon as (B.33) has a strong solution; one shows by the same argument as those developed below the stronger result

$$\sup_{t\in[0,T]}|\bar{X}^n_t - X_t| \stackrel{\mathbb{P}}{\longrightarrow} 0 \ \text{ as } \ n\to+\infty.$$

We refer to [16] for a proof when κ is homogeneous and C^3.

Proof. (a) We consider the Lévy-Khintchine decomposition of the Lévy process $Z = (Z_t)_{t\in[0,T]}$, namely

$$Z_t = \widetilde{Z}_t + Z^1,\ t\in[0,T],$$

where $\widetilde{Z}^\eta$ is a pure jump *square integrable* martingale with jumps of size at most 1 and Lévy measure $\nu(\,.\cap\{|z|\leq 1\})$ (having a second moment by construction) and Z^1 is a compensated (hence martingale) Poisson process with (finite) Lévy measure $\nu(\,.\cap\{|z|>1\})$.

It is clear from (B.34) that $\bar{X}^n_{t^n_k}\in L^p$ for every $k=0\dots,n$. Then, as $\nu(|z|^p)<+\infty$, it follows classically that $\sup_{u\in[t^n_k,t^n_{k+1}]}|Z_u| \stackrel{d}{\sim} \sup_{[0,\frac{T}{n}]}|Z_u|\in L^p$ (see $e.g.$ [34]). Combining these two results implies that $\varphi_{p,n}(t) := \big\|\sup_{s\in[0,t]}|\bar{X}^n_s|\big\|_p$ is finite for every $t\in[0,T]$.

It follows from Equation (B.35) satisfied by $\bar{X}$ that, for every $t\in[0,T]$,

$$\sup_{s\in[0,t]}|\bar{X}^n_s| \leq |x| + \sup_{s\in[0,t]}\left|\int_{(0,s]}\kappa(\underline{u}_{n-},\bar{X}^n_{\underline{u}_{n-}})dZ_u\right|.$$

Consequently

$$\varphi_{p,n}(t)\leq |x| + \left\|\sup_{s\in[0,t]}\left|\int_{(0,s]}\kappa(\underline{u}_{n-},\bar{X}^n_{\underline{u}_{n-}})dZ_u\right|\right\|_p$$

The L^p-B.D.G Inequality implies (since $p>1$)

$$\left\|\sup_{s\in[0,t]}\left|\int_{(0,s]}\kappa(\underline{u}_n,\bar{X}_{\underline{u}_n-})dZ_u\right|\right\|_p \leq c_p\left\|\sum_{0<s\leq t}\kappa(\underline{s}_n,\bar{X}_{\underline{s}_{n-}})^2(\Delta Z_s)^2\right\|_{\frac{p}{2}}^{\frac12}.$$

As $p\in(1,2]$, $\frac{p}{2}\leq 1$ which implies

$$\begin{aligned}\left\|\sum_{0<s\leq t}\kappa(\underline{s}_n,\bar{X}_{\underline{s}_{n-}})^2(\Delta Z_s)^2\right\|_{\frac{p}{2}}^{\frac12} &\leq \left(\mathbb{E}\sum_{0<s\leq t}|\kappa(\underline{s}_n,\bar{X}_{\underline{s}_{n-}})|^p|\Delta Z_s|^p\right)^{\frac1p} = \left(\nu(|z|^p)\,\mathbb{E}\int_0^t|\kappa(\underline{s}_n,\bar{X}_{\underline{s}_{n-}})|^p ds\right)^{\frac1p}\\ &\leq C^p_{\kappa,p}\nu(|z|^p)^{\frac1p}\left(\int_0^t(1+\varphi(s)^p)ds\right)^{\frac1p}\end{aligned}$$

where $C_{\kappa,p}$ is a real constant satisfying $|\kappa(s,\xi)|\leq C(1+|\xi|^p)^{\frac1p}$, $(s,\xi)\in[0,T]\times\mathbb{R}$.

Finally, there exists a real constant $C'=C'_{\kappa,p,\nu}$ such that the function $\varphi_{p,n}$ satisfies

$$\varphi_{p,n}(t)^p\leq C'\left(|x|^p+t+\int_0^t\varphi(s)^p ds\right)$$

which in turn implies by Gronwall's Lemma

$$\forall\, t\in[0,T],\quad \varphi_{p,n}(t)^p\leq e^{C't}C'(1+|x|^p)$$

or, equivalently,

$$\forall\, t\in[0,T],\quad \varphi_{p,n}(t)\leq\varphi(t) = e^{C''t}C''(1+|x|)\quad\text{where}\quad C''=C'/p.$$

To establish the Skorokhod tightness of the sequence $(\bar{X}^n)_{n\geq1}$, we rely on the Aldous tightness criterion (see Definition 4.1(b) or [18], Theorem 4.5, p.356). Let $\rho\in(0,1]$. Let σ and τ be two $[0,T]$-valued $\mathcal{F}^Z$-stopping

stopping times such that $\sigma \le \tau \le (\sigma+\delta)\wedge T$.

$$\begin{aligned}\mathbb{E}|\bar X^n_\tau - \bar X^n_\sigma|^\rho = \mathbb{E}\Big|\sum_{\sigma<u\le\tau}\kappa(\underline{u}_n, \bar X^n_{\underline{u}_{n-}})\Delta Z_u\Big|^\rho &\le \mathbb{E}\Big(\sum_{\sigma<u\le\tau}|\kappa(\underline{u}_n, \bar X^n_{\underline{u}_{n-}})|^\rho|\Delta Z_u|^\rho\Big)\\ &= \nu(|z|^\rho)\mathbb{E}\int_\sigma^{(\sigma+\delta)\wedge T}|\kappa(\underline{u}_n, \bar X^n_{\underline{u}_{n-}})|^\rho\\ &\le \delta\,\nu(|z|^\rho)\,\mathbb{E}\sup_{t\in[0,T]}|\kappa(t,\bar X^n_t)|^\rho\\ &\le \delta\,\nu(|z|^\rho)\,C_\kappa(1+\varphi_p(T))^{\frac{\rho}{p}}\end{aligned}$$

where we used that $\rho\le 1\le p$ and $\nu(|z|^\rho)\le\nu(|z|^2\wedge 1)+\nu(|z|^p)<+\infty$. Then

$$\sup\big\{\mathbb{E}|\bar X_\tau-\bar X_\sigma|^\rho+\mathbb{E}|Z_\tau-Z_\sigma|^\rho,\ \sigma\le\tau\le(\sigma+\delta)\wedge T,\ \mathcal{F}^Z\text{-stopping times}\big\}\le\nu(|z|^\rho)\big(1+C_\kappa(1+\varphi_p(T))^{\frac{\rho}{p}}\big)\delta$$

which goes to 0 as $\delta\to 0$. This implies that the sequence $M_n=(\bar X^n, Z)$, $n\ge 1$, is Sk-tight. Moreover, following Proposition 3.2 from [20], the sequence $(M_n)_{n\ge1}$ satisfies the $U.T.$ condition it is Sk-tight and

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\big(|\Delta\bar X^n|\vee|\Delta Z_t|\big) &\le \Big[\mathbb{E}\Big(\sum_{0<t\le T}|\Delta\bar X^n_t|^p+|\Delta Z_t|^p\Big)\Big]^{\frac1p}\\ &\le \Big[\nu(|z|^p)\mathbb{E}\int_0^T\big(1+|\kappa(\underline{t}_n,\bar X^n_{\underline{t}_n})|^p\big)dt\Big]^{\frac1p}\\ &\le \big(\nu(|z|^p)\big)^{\frac1p}\big(T+C^p_{\kappa,p}(1+\varphi_p(T))\big)^{\frac1p}<+\infty.\end{aligned}$$

On the other hand, the sequence $\Big((\kappa(\underline{t}_n,\bar X^n_{\underline{t}_n}))_{t\in[0,T]}, M_n\Big)_{n\ge1}$ is Sk-tight owing to the following lemma.

Lemma B.1. *Let $\mathcal{V}^+_{[0,T]}$ be the set of functions $\mu:[0,T]\to[0,T]$ such that $\mu(0)=0$ and $\mu(T)=T$ endowed with the sup norm. Assume $\kappa:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous. Then the mapping $\Psi:\mathcal{V}^+_{[0,T]}\times I\!D([0,T],\mathbb{R}^d)\to I\!D([0,T],\mathbb{R}^{1+d})$ defined by $\Psi(\mu,\alpha)=\big(\kappa(\mu(.),\alpha^1(.)),\alpha\big)$ is continuous ($\alpha=(\alpha^1,\dots,\alpha^d)$) for the product topology.*

Proof (of the lemma). Let $(\lambda_n)_{n\ge1}$ be a sequence of homeomorphisms of $[0,T]$ such that $\lambda_n\to Id_{[0,T]}$ and $\alpha_n\circ\lambda_n\to\alpha$ uniformly and let $\mu_n\to\mu$ in $\mathcal{V}^+_{[0,T]}$ where $Id_{[0,T]}$ denotes the identity of $[0,T]$ to $[0,T]$. Then the closure of $(\alpha_n\circ\lambda_n(t))_{n\ge1,t\in[0,T]}$ is a compact set K of $\mathbb{R}^d$ hence the function κ is uniformly continuous on $[0,T]\times K$. On the other hand

$$\|\mu_n\circ\lambda_n-Id_{[0,T]}\|_{\sup}\le\|\mu_n-Id_{[0,T]}\|_{\sup}+\|\lambda_n-Id_{[0,T]}\|_{\sup}\quad\text{as}\quad n\to+\infty$$

and $\|\alpha_n\circ\lambda_n-\alpha\|_{\sup}\to 0$ as $n\to+\infty$. The conclusion follows. ☐

Up to an extraction, we may assume that the triplet $\big((\kappa(\underline{t}_{n'},\bar X^{n'}_{\underline{t}_{n'}})_{t\in[0,T]}, M_{n'}\big)_{n\ge1}$ weakly converges for the Skorokhod topology toward a probability $\mathbb{P}$ on the canonical Skorokhod space $(I\!D([0,T],\mathbb{R}^3),(\mathcal{D}_t)_{t\in[0,T]})$.

By Theorem 2.6 from [20] for the functional convergence of stochastic integrals, we know that

$$\Big(\kappa(\underline{t}_{n'},\bar X^{n'}_{\underline{t}_{n'}}),(\bar X^{n'}_t,Z_t),\int_0^t\kappa(\underline{s}_{n'-},\bar X^{n'}_{\underline{s}_{n'-}})dZ_s\Big)_{t\in[0,T]}\overset{\mathcal{L}(Sk)}{\longrightarrow}\mathbb{Q}$$

probability distribution on $I\!D([0,T],\mathbb{R}^4)$ such that the canonical process $Y=(Y^i)_{i=1:4}$ satisfies $Y\overset{\mathcal{L}}{\sim}\big(Y^1,(Y^2,Z),\int_0^\cdot Y^2_s dZ_s\big)$ where Y^3 is a Lévy process with respect to the $\mathbb{Q}$-completed right continuous canonical filtration $(\mathcal{D}^{\mathbb{Q}}_t)_{t\in[0,T]}$ on $I\!D([0,T],\mathbb{R}^4)$ having the distribution of Z (*i.e.* $\mathbb{Q}_{Y^3}=\mathcal{L}(Z)$). Furthermore, we know that $Y^1=\kappa(.,Y^2.)$ $\mathbb{Q}$-*a.s.* since the mapping $(\mu,(\alpha^i)_{i=1:4})\mapsto\alpha^1-\kappa(\mu,\alpha^2)$ is continuous from $\mathcal{V}^+_{[0,T]}\times I\!D([0,T],\mathbb{R}^4)$ to $I\!D([0,T],\mathbb{R})$ (and $\underline{t}_n$ converges uniformly to $Id_{[0,T]}$).

On the other hand we know that $\bar X^{n'}_t=x+\int_0^t\kappa(\underline{s}_{n'-},\bar X^{n'}_{\underline{s}_{n'-}})dZ_s$, $t\in[0,T]$ which in turn implies that $(Y^2_t=x+\int_0^t\kappa(s,Y^2_{s_-})dZ_s,\ t\in[0,T])$ $\mathbb{Q}$-*a.s.*. This shows the existence of a weak solution to the SDE $X_t=x+\int_0^t\kappa(s,X_{s_-})dZ_s$, $t\in[0,T]$.

Under the weak uniqueness assumption, the distribution $\mathbb{Q}_{Y^2}$ of Y^2 is unique equal, say, to $\mathbb{P}_X$.

(b) We assume that the Lévy measure has a finite second moment $\nu(z^2) < +\infty$ on the whole real line. Then one can decompose Z as

$$Z_t = a\,W_t + \widetilde{Z}_t,\ t\in[0,T], \quad (a \geq 0)$$

where $\kappa \geq 0$ and $\widetilde{Z}$ is a pure jump martingale Lévy process with Lévy measure ν. Then one shows like in the Brownian case that $\varphi(t) = \mathbb{E}\sup_{s\in[0,t]}|\bar{X}^n_s|^2$ is finite over $[0,T]$ using that all $\bar{X}_{t_k}$ are square integrable and $\mathbb{E}\sup_{s\in[t_k,t_{l+1})}|Z_s - Z_{t_k}|^2 = \mathbb{E}\sup_{s\in[0,\frac{T}{n}]}|Z_s|^2 < +\infty$. Then, using Doob's Inequality, we show that

$$\varphi(t) \leq 4C_\kappa^2(a^2+\nu(z^2))\left(t + \int_0^t \varphi(s)ds\right)$$

where C_κ is a real constant satisfying $\kappa(t,\xi) \leq C_\kappa(1+|\xi|^2)^{\frac{1}{2}}$, $\xi\in\mathbb{R}$.

To establish the Skorokhod tightness of the sequence, we rely on the Aldous tightness criterion (see Definition 4.1(b) or[18], Theorem 4.5, p.356). Let σ and τ be two $[0,T]$-valued $\mathcal{F}^Z$-stopping stopping times such that $\sigma \leq \tau \leq (\sigma+\delta)\wedge T$. Then applying Doob's Inequality, this time to the martingale $\left(\int_\sigma^{\sigma+s}\kappa(\underline{u}_n, \bar{X}_{\underline{u}_{n-}})dZ_u\right)_{s\geq 0}$, we get

$$\begin{aligned}
\mathbb{E}|\bar{X}_\tau - \bar{X}_\sigma|^2 &\leq 4a^2\mathbb{E}\left(\int_\sigma^\tau |\kappa(\underline{u}_n, \bar{X}^n_{\underline{u}_{n-}})|^2 du\right) + 4\,\mathbb{E}\left(\sum_{\sigma<u\leq\tau}|\kappa(\underline{u}_n, \bar{X}^n_{\underline{u}_{n-}})|^2|\Delta Z_u|^2\right)\\
&= 4\big(a^2+\nu(z^2)\big)\mathbb{E}\left(\int_\sigma^\tau |\kappa(\underline{u}_n, \bar{X}^n_{\underline{u}_{n-}})|^2 du\right)\\
&\leq 4\big(a^2+\nu(z^2)\big)\mathbb{E}\left(\int_\sigma^{(\sigma+\delta)\wedge T} |\kappa(\underline{u}_n, \bar{X}^n_{\underline{u}_{n-}})|^2 du\right)\\
&\leq 4\big(a^2+\nu(z^2)\big)\delta\, C_\kappa^2(1+\varphi(T)).
\end{aligned}$$

Then $\mathbb{E}|\bar{X}_\tau - \bar{X}_\sigma|^2 + \mathbb{E}|\bar{Z}_\tau - \bar{Z}_\sigma|^2 \leq 4\big(a^2+\nu(z^2)\big)\nu(z^2)\big(1+\varphi(T)\big)\delta$ which clearly implies the Sk-tightness of the sequence $M_n = (\bar{X}^n, Z)$, $n\geq 1$.

The sequence satisfies the $U.T.$ condition from [20] since $(M_n)_{n\geq 1}$ is Sk-tight and (see Proposition 3.2 from [20])

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\big(|\Delta\bar{X}^n|\vee|\Delta Z_t|\big) &\leq \Big(\mathbb{E}\sum_{0<t\leq T}|\Delta\bar{X}^n_t|^2 + |\Delta Z_t|^2\Big)^{\frac{1}{2}}\\
&\leq \Big(\nu(z^2)\mathbb{E}\int_0^T(1+|\kappa(\underline{t}_n, \bar{X}^n_{\underline{t}_n})|^2dt\Big)^{\frac{1}{2}}\\
&\leq \nu(z^2)(1+C_\kappa(1+\varphi(T))) < +\infty.
\end{aligned}$$

From this point, the proof is quite similar to that of claim (a). $\square$

B.2 Higher moments

Let $Z_t = aW_t + \widetilde{Z}_t$, $t\in[0,T]$, be the decomposition of the Lévy process Z where W is a standard B.M. and $\widetilde{Z}$ is an independent pure jump Lévy process.

Proposition B.2. *If $\nu(|z|^p) < +\infty$ for $p\in[2,+\infty)$, then*

$$\sup_{n\geq 1}\Big\|\sup_{t\in[0,T]}|\bar{X}^n_t|\Big\|_p < +\infty.$$

Proof. If $p\in(1,2]$, the claim follows from the above Proposition B.1. Assume from now on $p\in[2,+\infty)$. Let $\varphi_{p,n}(t) = \mathbb{E}\big(\sup_{t\in[0,T]}|\bar{X}^n_t|^p\big)$. Let ℓ_p be the unique integer defined by the inequality $2^{\ell_p} < p \leq 2^{\ell_p+1}$. It is straightforward using the same arguments as above that $\varphi_{p,n}(T) < +\infty$ since $\sup_{t\in[0,T]}|Z_t|^p \in L^1$ (see [34], Theorem 25.18, p. 166) and $X_{t_k} \in L^p$ by induction using (B.34). For convenience, we denote $\kappa_{s_-} = \kappa(\underline{s}_n, \bar{X}^n_{\underline{s}_n-})$.

Now, combining the (integral and regular) Minkowski and the B.D.G. Inequalities implies

$$\begin{aligned}\varphi_{p,n}(t)^{\frac{1}{p}} &\leq |x| + c_p\Big\|a^2\int_0^t \kappa^2_{s_-}ds + \sum_{0<s\leq t}\kappa^2_{s_-}(\Delta Z_s)^2\Big\|^{\frac12}_{\frac{p}{2}} \\ &\leq |x| + c_p\Big(a\Big\|\int_0^t \kappa^2_{s_-}ds\Big\|^{\frac12}_{\frac{p}{2}} + \Big\|\sum_{0<s\leq t}\kappa^2_{s_-}(\Delta Z_s)^2\Big\|^{\frac12}_{\frac{p}{2}}\Big) \qquad \text{(B.36)}\end{aligned}$$

where we used in the second inequality that $\sqrt{u+v}\leq\sqrt{u}+\sqrt{v}$, $u,\,v\geq 0$. First note that by two successive applications of Hölder Inequality to dt and $d\mathbb{P}$, we obtain

$$\Big\|\int_0^t \kappa^2_{s_-}ds\Big\|^{\frac12}_{\frac{p}{2}} \leq T^{\frac12-\frac1p}\left(\int_0^t \mathbb{E}\,|\kappa_{s_-}|^p ds\right)^{\frac1p} \qquad \text{(B.37)}$$

Now using that for every $\ell\in\{1,\ldots,\ell_p\}$, $\displaystyle\sum_{0<s\leq t}|\kappa_{s_-}|^{2^\ell}|\Delta Z_s|^{2^\ell} - \int_0^t |\kappa_{s_-}|^{2^\ell}ds\,\nu(|z|^{2^\ell})$, $t\in[0,T]$, is a true martingale, we have by combining the Minkowski inequality, the B.D.G. Inequality applied with $\frac{p}{2^\ell}>1$ and the elementary inequality $(u+v)^r\leq u^r+v^r$, $u,\,v\geq 0$, $r\in(0,1]$, yield

$$\begin{aligned}\Big\|\sum_{0<s\leq t}|\kappa_{s_-}|^{2^\ell}(\Delta Z_s)^{2^\ell}\Big\|^{\frac{1}{2^\ell}}_{\frac{p}{2^\ell}} &\leq \Big\|\sum_{0<s\leq t}|\kappa_{s_-}|^{2^\ell}(\Delta Z_s)^{2^\ell} - \int_0^t|\kappa_{s_-}|^{2^\ell}ds\,\nu(|z|^{2^\ell})\Big\|^{\frac{1}{2^\ell}}_{\frac{p}{2^\ell}} + \Big\|\int_0^t|\kappa_{s_-}|^{2^\ell}ds\Big\|^{\frac{1}{2^\ell}}_{\frac{p}{2^\ell}}\nu(|z|^{2^\ell})^{\frac{1}{2^\ell}} \\ &\leq c^{\frac{1}{2^\ell}}_{\frac{p}{2^\ell}}\Big\|\sum_{0<s\leq t}|\kappa_{s_-}|^{2^{\ell+1}}(\Delta Z_s)^{2^{\ell+1}}\Big\|^{\frac{1}{2^{\ell+1}}}_{\frac{p}{2^{\ell+1}}} + \Big\|\int_0^t|\kappa_{s_-}|^{2^\ell}ds\Big\|^{\frac{1}{2^\ell}}_{\frac{p}{2^\ell}}\nu(|z|^{2^\ell})^{\frac{1}{2^\ell}}.\end{aligned}$$

Then two applications of Hölder Inequality applied to dt and $d\mathbb{P}$ successively imply

$$\Big\|\int_0^t|\kappa_{s_-}|^{2^\ell}ds\Big\|^{\frac{1}{2^\ell}}_{\frac{p}{2^\ell}} \leq T^{\frac{1}{2^\ell}-\frac1p}\left(\int_0^t\mathbb{E}\,|\kappa_{s_-}|^p ds\right)^{\frac1p}.$$

Summing up these inequalities in cascade finally yields a positive real constant $K^{(0)}=K_{p,\nu,a,T}$ such that

$$\Big\|\sum_{0<s\leq t}|\kappa_{s_-}|^2(\Delta Z_s)^2\Big\|^{\frac12}_{\frac{p}{2}} \leq K^{(0)}\left(\left(\int_0^t\mathbb{E}\,|\kappa_{s_-}|^p ds\right)^{\frac1p} + \Big\|\sum_{0<s\leq t}|\kappa_{s_-}|^{2^{\ell_p+1}}(\Delta Z_s)^{2^{\ell_p+1}}\Big\|^{\frac{1}{2^{\ell_p+1}}}_{\frac{p}{2^{\ell_p+1}}}\right).$$

Now, as $\frac{p}{2^{\ell_p+1}}\leq 1$, one gets by the compensation formula

$$\Big\|\sum_{0<s\leq t}|\kappa_{s_-}|^{2^{\ell_p+1}}|\Delta Z_s|^{2^{\ell_p+1}}\Big\|^{\frac{1}{2^{\ell_p+1}}}_{\frac{p}{2^{\ell_p+1}}} \leq \left(\mathbb{E}\sum_{0<s\leq t}|\kappa_{s_-}|^p(\Delta Z_s)^p\right)^{\frac1p} = \left(\int_0^t\mathbb{E}|\kappa_{s_-}|^p ds\right)^{\frac1p}\nu(|z|^p)^{\frac1p}.$$

Hence, there exists a real constant $K^{1)}=K^{(1)}_{p,\nu,a,T}>0$

$$\Big\|\sum_{0<s\leq t}|\kappa_{s_-}|^2(\Delta Z_s)^2\Big\|^{\frac12}_{\frac{p}{2}} \leq K^{(1)}_{p,\nu,a,T}\left(\int_0^t\mathbb{E}|\kappa_{s_-}|^p ds\right)^{\frac1p}. \qquad \text{(B.38)}$$

Finally, plugging (B.37) and (B.38) in (B.36), there exist positive real constants $K^{(\ell)} = K^{(\ell)}_{p,\nu,a,T}$, $\ell = 2, 3$, such that

$$\begin{aligned}\varphi_{p,n}(t)^{\frac{1}{p}} &\leq K^{(2)}_{p,\nu,a,T}\left(|x| + \left(\int_0^t \mathbb{E}|\kappa_{s_-}|^p ds\right)^{\frac{1}{p}}\right)\\ &\leq K^{(3)}_{p,\nu,a,T}\left(|x| + 1 + \left(\int_0^t \varphi_{p,n}(s)ds\right)^{\frac{1}{p}}\right)\end{aligned}$$

(where we used in the second inequality that κ has linear growth) so that

$$\varphi_{p,n}(t) \leq 2^{p-1}(K'^{(3)}_{p,\nu,a,T})^p\Big(\big(|x|+1\big)^p + \int_0^t \varphi_{p,n}(s)ds\Big).$$

Gronwall's lemma completes the proof since it implies that

$$\varphi_{p,n}(t) \leq e^{2^{p-1}(K'^{(3)}_{p,\nu,a,T})^p\, t}\, 2^{p-1}(K'^{(3)}_{p,\nu,a,T})^p(|x|+1)^p. \quad \square$$