What is known about the Value 1 Problem for Probabilistic Automata?

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Abstract. The value 1 problem is a decision problem for probabilistic automata over finite words: are there words accepted by the automaton with arbitrarily high probability? Although undecidable, this problem attracted a lot of attention over the last few years. The aim of this paper is to review and relate the results pertaining to the value 1 problem.

In particular, several algorithms have been proposed to partially solve this problem. We show the relations between them, leading to the following conclusion: the Markov Monoid Algorithm is the *most correct* algorithm known to (partially) solve the value 1 problem.

1 Introduction

In 1963 Rabin [Rab63] introduced the notion of probabilistic automata, which are finite automata with randomized transitions. This powerful model has been widely studied and has applications in many fields like image processing [CK97], computational biology [DEKM99] and speech processing [Moh97]. Several algorithmic properties of probabilistic automata have been considered in the litterature. For instance, Schützenberger [Sch61] proved in 1961 that *functional equivalence* is decidable in polynomial time (see also [Tze92]), and even faster with randomized algorithms, which led to applications in software verification [KMO$^+$11].

However, many natural decision problems are undecidable, and part of the literature on probabilistic automata is about *undecidability results*. For example the *emptiness*, the *isolation* and the *value* 1 problems are undecidable, as shown in [Paz71,BMT77,GO10]. To overcome untractability results, a lot of effort went into finding subclasses of probabilistic automata for which natural decision problems become decidable. For instance, the papers [KVAK10,CKV$^+$11] look at restrictions implying a decidable model-checking problem against ω-regular specifications, and the paper [CSV13] investigates whether assuming isolated cut-points leads to decidability for the emptiness problem.

We focus here on the efforts made to understand the value 1 problem. The aim of this paper is to review and relate the attempts made in this direction over the last few years [GO10,CSV11,FGO12,CT12,BBG12,FGKO14].

2 Definitions

Let Q be a finite set of states. A probability distribution over Q is a function $\delta : Q \to [0,1]$ such that $\sum_{q \in Q} \delta(q) = 1$.

Let A be a finite alphabet. The transitions of a probabilistic automaton are given by a function $\Delta : Q \times A \to \mathcal{D}(Q)$; equivalently, for each letter $a \in A$ we consider a probabilistic transition matrix M_a, which is a square matrix in $[0,1]^{Q \times Q}$ such that every row of M_a is a probability distribution over Q. The value of $M_a(s,t)$ is the probability to go from state s to state t when reading the letter a.

Given an input word $w \in A^*$, we denote $\mathbb{P}_{\mathcal{A}}(s \xrightarrow{w} t)$ the probability to go from state s to state t when reading the word w. Formally, if $w = a_1 a_2 \cdots a_n$ then $\mathbb{P}_{\mathcal{A}}(s \xrightarrow{w} t) = (M_{a_1} M_{a_2} \cdots M_{a_n})(s,t)$.

Definition 1 (Probabilistic automaton). *A tuple $\mathcal{A} = (Q, A, q_0, \Delta, F)$ represents a probabilistic automaton, where Q is a finite set of states, A is the finite input alphabet, $q_0 \in Q$ is the initial state, Δ define the transitions and $F \subseteq Q$ is the set of accepting states.*

Definition 2 (Acceptance probability). *The* acceptance probability *of a word $w \in A^*$ by $\mathcal{A}$ is $\sum_{f \in F} \mathbb{P}_{\mathcal{A}}(q_0 \xrightarrow{w} f)$, denoted $\mathbb{P}_{\mathcal{A}}(w)$.*

Definition 3 (Value). *The* value *of $\mathcal{A}$, denoted* $\mathrm{val}(\mathcal{A})$*, is the supremum acceptance probability over all possible input words:*

$$\mathrm{val}(\mathcal{A}) = \sup_{w \in A^*} \mathbb{P}_{\mathcal{A}}(w) \ . \tag{1}$$

We are interested in the following decision problem:

Given a probabilistic automaton $\mathcal{A}$, decide whether $\mathrm{val}(\mathcal{A}) = 1$.

3 An Equivalent Formulation and the Exact Computational Complexity

The first result about the value 1 problem is its surprising undecidability, obtained with an elementary proof by Hugo Gimbert and Youssouf Oualhadj in [GO10].

In a related yet seemingly different line of work, Christel Baier, Marcus Größer and Nathalie Bertrand undertook a thorough study of probabilistic Büchi automata [BG05,BBG08,BBG09,BBG12]. One of the results obtained there is the undecidability of the emptiness problem for probabilistic Büchi automata with probable semantics. It turns out that the two problems are actually Turing-equivalent:

- the value 1 problem for probabilistic automata over finite words,
- the emptiness problem for probabilistic Büchi automata with probable semantics.

A first (very simple) reduction has been explained in [BBG12]: from a probabilistic automaton $\mathcal{A}$ over finite words, one can construct a probabilistic Büchi automaton $\mathcal{A}'$ of linear size, such that $\mathrm{val}(\mathcal{A}) = 1$ if and only if $\mathcal{A}'$ is non-empty for the probable semantics. The converse reduction is more involved, and follows from [CSV13], but here the constructed automaton is of exponential size.

Even better, the exact computational complexity has been given in [CSV13]: both problems are Σ_2^0-complete.

Theorem 1 ([BBG12,CSV13]). *The value* 1 *problem for probabilistic automata over finite words and the emptiness problem for probabilistic Büchi automata with probable semantics are Turing-equivalent and* Σ_2^0*-complete.*

4 Decidable Subclasses of Probabilistic Automata

Several subclasses of probabilistic automata were constructed in order to decide the value 1 problem on such instances.

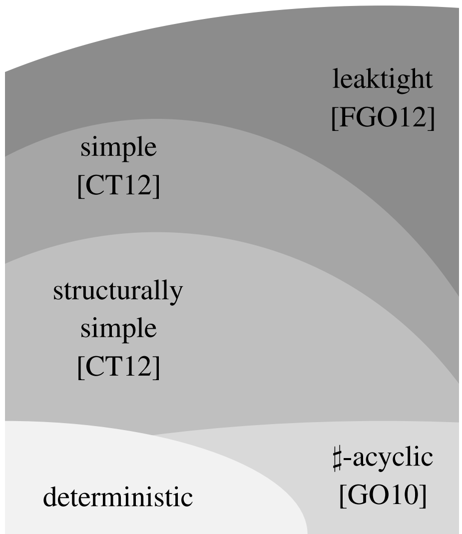
The first class was the $\sharp$-acyclic automata by Gimbert and Oualhadj [GO10].

Later but concurrently, two different works have been published in the very same conference. The first one introduces simple automata and structurally simple automata, by Krishnendu Chatterjee and Mathieu Tracol [CT12]. The second, by Hugo Gimbert, Youssouf Oualhadj and the author introduces leaktight automata [FGO12].

Although geared towards the same goal (deciding the value 1 problem), the two classes came from different perspectives. The paper of Krishnendu Chatterjee and Mathieu Tracol relies on a theorem from Probability Theory, called the jet decompositions of (infinite) Markov Chains. The paper of Hugo Gimbert, Youssouf Oualhadj and the author relies on a theorem from Algebra, called Simon's theorem, asserting the existence of factorization trees of bounded height.

Subsequent studies [FGKO14] showed that the class of leaktight automata actually strictly contains all the other classes, implying that the Markov Monoid

Algorithm used to decide the value 1 problem for leaktight automata actually decides the value 1 problem for all cases where it is known to be decidable.



Conclusion and Perspectives

In this paper, we discussed some recent developments about the value 1 problem. We first gathered some results from the literature, explaining that it is actually Turing-equivalent to the emptiness for probabilistic Büchi automata with the probable semantics, and Σ_2^0-complete. Then we presented the different attempts to decide the value 1 problems on subclasses of probabilistic automata. As a conclusion, the Markov Monoid Algorithm introduced in [FGO12], used to decide the value 1 problem for leaktight automata, is actually the *most correct* algorithm known so far, as the class of leaktight automata strictly contains all other classes for which the value 1 problem is known to be decidable.

This motivates a deeper understanding of this algorithm. We know that the Markov Monoid Algorithm cannot solve the value 1 problem, as this problem is undecidable, but then what is the problem solved by this algorithm? In other words, can we characterize for which probabilistic automata the Markov Monoid Algorithm finds a value 1 witness?

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